Pricing barrier and lookback options using finite difference numerical methods

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Declaration

I hereby declare that the research work `Pricing barrier and lookback options using finite difference numerical methods` is of my own originality. It had not been submitted to any other university or institution for examination purposes. All the sources consulted in this work are fully acknowledged and are completely found in the reference section. Furthermore, this research work is submitted in partial fulfilment of the requirements for the degree Master of Science at the Centre for Business Mathematics and Informatics, North-West University (Potchefstroom campus).

21 November 2016

Umeorah Nneka Ozioma

Date

Dedication

To all who love what is right.

"I raise my eyes to the mountains. From where will my help come? My help comes from Jehovah, The maker of heaven and earth."

Psalms 121 vs 1-2.

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Abstract

This research work focuses on the estimation of barrier and lookback option prices using finite difference numerical methods. Here, we aim at approximating the fair prices of the zero rebate up-and-out and down-and-out knock out barrier options, as well as the fixed strike lookback options. Simulation and finite difference techniques will be used to approximate these prices. The Monte-Carlo simulation, the antithetic Monte-Carlo simulations and the Crank-Nicolson approach will be specifically employed on the barrier options. Other finite difference methods like the implicit and the explicit method will be discussed but the Crank-Nicolson method will be employed in the numerical valuations owing to its accuracy in comparison to others. Next, the fixed strike lookback option prices will be estimated using the Monte-Carlo and the antithetic Monte-Carlo simulation methods. An extended version of the Black-Scholes model will be used in the valuation of their exact prices owing to their exotic nature. The Monte-Carlo and the antithetic Monte-Carlo methods are next employed to simulate the values of these option prices. The resulting prices will be compared to the exact fair prices and this will be followed by some error analysis.

From the findings, the antithetic method gave the best option price estimate in comparison to the ordinary Monte-Carlo method when the simulation approach was used. It will also be observed that the Monte-Carlo simulation had a slow rate of convergence as a result of higher variances of the estimate from the true solution. Hence, such inefficiency was curbed by the introduction of antithetic Monte-Carlo simulation which had smaller variances of the estimate, and this in turn gave a better estimate. Furthermore, it will also be observed that the Crank-Nicolson method converged faster with increase in the discretisation steps of the underlying asset and the time.

Keywords: Black-Scholes model, Lookback options, Barrier options, Finite difference methods, Monte-Carlo simulation, Antithetic Monte-Carlo simulation.

List of Acronyms

PDE - Partial differential equations

FDM - Finite difference methods

ATM - At-the-money

ITM - In-the-money

OTM - Out-of-the-money

MCS - Monte-Carlo simulations

AMCS - Antithetic Monte-Carlo simulations

SDE - Stochastic differential equations

DOBO - Down-and-out barrier options

UOBO - Up-and-out barrier options

DIBO - Down-and-in barrier options

UIBO - Up-and-in barrier options

AON - Asset-or-nothing

CON - Cash-or-nothing

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1. Introduction

1.1 Background

A financial agreement which gives the option holder the right to trade a specific underlying in the future at a specified price (known as the strike price) on or before the expiration of the contract is referred to as an Option. The option writer sells the contract and he is obligated to perform the transaction to the option holder. The option holder buys the contract and decides whether or not to exercise it. This is one of the features that distinguish options trading from futures or forwards trading, where the holder must exercise the contract. Options are generally classified into plain vanilla options and exotic options. The plain vanilla options are the American options and the European options. American options give the holder the right to exercise the contract on or before its expiration, whereas the European options are exercised at the end of the contract. The plain vanilla options are by far the most traded options in the financial market. The exotic options are options that cannot be classified as plain vanilla, owing to the additional features they possess. With respect to the classification according to their rights of exercise, we have the call and the put options. A call option is the right to purchase a particular underlying for an agreeable amount at a specified time in the future, whereas the put option deals with the right to sell a particular underlying at a fixed amount in the future. A call (put) option is classified as **in-the-money (ITM)** if the underlying price is greater (lesser) than the strike price; **out-of-the-money (OTM)** if the underlying price is lesser (greater) than the strike price and **at-the-money (ATM)** if the underlying price is equal to the strike price.

According to Snyder (1969), exotic options had been in existence prior to the establishment of the Chicago Board of Exchange in 1973. Later around late 1980 and early 1990, with the evolution of certain features which the plain vanilla lack, the pricing of exotic options came to limelight. Zhang (1998) further explained that investors sought ways to price non-standardized option contract and as a result, such quest favoured the introduction of exotic options in the market. Any option that have complex features compared to the plain vanilla counterpart, in terms of its valuation is referred to as the exotic option. They are characterized by their unusual payoffs and their path-dependency which give rise to their complex nature of pricing. Exotic options when compared to the plain vanilla options in terms of their valuations are not straight-forward. There are many types of exotic options according to their classifications.

Path-dependent exotic options: The payoffs of path-dependent exotic options consist of functions of the continuous paths which the underlying follows during the life of the contract. They include:

- Asian option: The payoff of this option depends on the average value of the underlying over the lifetime of the contract.
- Lookback option: The payoff of the lookback option depends on the optimal value of the asset price path.
- Barrier option: The option's payoff is dependent on whether the price of the underlying breaches a certain barrier on or before the contract's expiration.

Exotic options with unusual payoffs include:

- Binary options: The payoffs of binary options are discontinuous in nature and are dependent on the terminal underlying price. They either pay a fixed amount or nothing at all, if and only if some conditions stipulated initially are met.
- Chooser options: These options allow the option holder to choose at a specific point in time before the contract's expiration, whether the option should be a put or a call.

- Basket options: These are option contracts which involves two or more risky underlying assets.

Some other examples include compound options which are options whose values are options (Geske 1978), forward-start options which are paid for at present but come into existence at some time in the future (Musiela & Rutkowski 2006) and the spread options which are based on differences between two indices, rates or prices.

In the valuation of exotic options, it is very pertinent to know where the associated risks lies. In other words, what point of the option's contract does the option value have the highest greek?. With regards to the hedging of exotic options using their specific underlying, some of them are easier to hedge whereas others are difficult. The barrier options are difficult to hedge because once the barrier level is breached, the delta of the option tends to be discontinuous. The Asian option on the other hand is easier to hedge because the averaging of the prices of the underlying posses a greater advantage and as the option gets to its expiration, the payoff becomes more certain. Wilmott (2006) explained that in pricing and hedging exotic options, these six features are very important: path-dependence, time-dependence, order, dimensionality, cash flows, embedded decisions.

1.2 Problem Statement

Option pricing presents a significant role in risk management as financial investors apply its concept in hedging and speculation. In the theory of option pricing, prior knowledge of the current price of the asset, the inherent volatility, the dividend yield, the time to expiration, the risk-free interest rate, as well as the strike price are very essential in order to determine the fair price of an option. My recent research work in option pricing considered the American call options which allow the flexibility of early exercise from the option holder (see Umeorah (2015)). The finite difference method was employed to numerically approximate the Black-Scholes partial differential equation (PDE) which describes the linear complementary problem of American call options.

In this research, we try to extend numerical methods to more complicated class of options, namely the exotic options. Exotic options are options which are traded mainly over the counter and they possess complex features when compared to the plain vanilla options. Exotic options are preferred to the vanilla options when considering options which are duly characterized by their unusual payoffs, path-dependency formats and other structures which are designed and tailored to meet the specific needs of their investors. Some research has been done on exotic option pricing. Hongbin (2009) applied the concept of Monte-Carlo simulation to value arithmetic Asian options. Numerical PDE approach using higher order finite difference method was conducted by Kumar, Waikos & Chakrabarty (2011), where the value of average strike Asian call option was obtained. Numerical implementation of barrier option can also be found in Goldman, Sosin & Gatto (1979), Reiner & Rubinstein (1991) and Merton (1973). Explicit formula for obtaining the knock-out discount for barrier options can be found in Musiela & Rutkowski (2006). They also developed the arbitrage prices for the floating strike lookback options, together with their pricing method. More recent works on double barrier option pricing were done by Farnoosh, Sobhani, Rezazadeh & Beheshti (2015) and Chen, Xu & Zhu (2015). In this research, special interest would be channelled to zero rebate knock-out option with barrier features and the fixed strike lookback options. For the knock-out barrier options, when the barrier level is reached by the underlying price, whether from below or above, the option is extinguished or knocked out. Thus, the option becomes worthless. The barrier option holder might receive a rebate (positive discount) when the option is being knocked-out before expiry and thus in this work, we consider a situation where there is no such rebate. These price options were chosen based on the fact that relatively, not much work has been done on their numerical computation. Hence, this research seeks to address the existing gap, thereby implementing the Monte-Carlo simulations and the finite difference methods to exotic option pricing.

1.3 Motivation

In the field of mathematical finance, the advantages of option pricing can never be underestimated. Option pricing serves as a tool for investors in determining how to invest their wealth, as it offers them higher potential returns and less risk depending on the usage. Having worked on the vanilla option pricing, this research serves to consider the applications of numerical methods and approximations to exotic option pricing. The motive of this work is to algebraically compute the numerical values of the prices of some exotic options, and hence compare them to their exact prices, with the intention of improving the existing work done on the exotic-type of options.

1.4 Aim of the Study

To price zero rebate knock-out barrier options and the fixed strike lookback options using finite difference numerical methods and the simulation methods.

1.5 Objectives of the Study

The specific objectives of this research include:

- To investigate the concept of Black-Scholes model, as well as its extension in the pricing of exotic options.

- To solve the corresponding Black-Scholes PDE using the finite difference approximations and hence obtain the option values.
- To employ the Monte-Carlo and the antithetic Monte-Carlo simulations in the estimation of the option prices.

1.6 Method of Investigation:

With regards to the designs and methodology, this research seeks to:

- Consider the mathematical framework. This consists of introducing the concept of Black-Scholes pricing model which has significant applications in the theory of derivative pricing. The pricing model would be extended to the exotic option since most of the options classified as exotic are priced using the same assumptions and the risk-neutrality concept of the Black-Scholes model.
- Computationally analyze the solution of option pricing PDE using some necessary boundary and terminal value conditions. Here, we would introduce the concept of finite difference methods (FDM) which consists of the Implicit, Explicit and Crank-Nicolson method. The Crank-Nicolson, out of all the FDM would be applied computationally. Also, possible asset price movements based on the Monte-Carlo simulations would be modelled using some computer programs.
- Convert the PDEs into a set of difference equations, apply their boundary conditions and solve their equations using backward iteration.
- Implement the scheme numerically. Here, the research would consider some computing methods where IPython notebook or Matlab programming language would be used in the quantitative analysis of the problem.

1.7 Dissertation Overview

Chapter 1: Provides an introduction to the research. It explains the problem statement in detail and states the motivation for carrying out the research. It highlights the aims and objectives. The chapter further mentions the methodologies employed in the research and it finally provides an overview of the study.

Chapter 2: Introduces some financial and mathematical preliminaries where some theorems, definitions, corollaries and lemmas used in the work will be stated. Furthermore, this chapter will describe the concept of the geometric Brownian motion and its relation to option pricing. Finally, the Black-Scholes model will be explained in detail. Here, the derivation of its PDE, the solution and its applicability to option pricing will be analyzed.

Chapter 3: Explains some detailed overview of different types of exotic options, like the Asian options, lookback options, barrier options and the binary options. Their payoff structures and valuations will also be considered.

Chapter 4 & 5: Investigates how the fixed strike lookback and the zero-rebate knock-out barrier options (down-and-out and up-and-out) can be valuated numerically. Their mathematical background will be analyzed in detail. Furthermore, Chapter 5 would discuss the concept of valuating the options using the FDM and the Monte-Carlo simulations. The implicit, the explicit and the Crank-Nicolson finite difference methods will be discussed in detail.

Chapter 6: Compares different results obtained from estimating the options using the methods explained in Chapter 5. This would be followed by computational analysis of the results. Furthermore, inference would be made on which method provides the fair approximated value when compared to the exact Black-Scholes price.

Chapter 7: Concludes the study and makes some recommendations.

2. Technical Preliminaries

In this chapter, we consider some mathematical definitions, lemmas and some financial definitions. Financial background of the study, which includes the Black-Scholes model, its PDE, as well as the solution will be discussed.

2.1 Mathematical and Financial Preliminaries

In this section, we consider some mathematical and financial definitions so as to reduce possible ambiguity and hence familiarize the reader with the concept.

2.1.1 Definition. $\sigma-$algebra

Let X be a non-empty set. $\mathbb{F}$ which consists of collection of subsets of X is a $\sigma-$algebra on X, provided the following conditions are satisfied: (Shreve 2004).

- $\emptyset \in \mathbb{F}$.
- If $\lambda \in \mathbb{F}$, then $\lambda^{\mathsf{C}} \in \mathbb{F}$ (That is, closed under complementation).
- If $\lambda_1, \lambda_2, \cdots$ is a sequence of elements in $\mathbb{F}$, then $\cup_{n=1}^{\infty}\lambda_n \in \mathbb{F}$.

2.1.2 Definition. Probability space

A probability space is a set consisting of the triplet $(X, \mathbb{F}, \mathbb{P})$, where X is the set of all outcomes, $\mathbb{F}$ is a $\sigma-$algebra on X and $\mathbb{P}$ is the given probability measure, also known as the real-world probability measure (Shreve 2004).

2.1.3 Definition. Filtered Probability space

A filtered probability space is a set consisting of $(X, \mathbb{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t$ is a filtration which refers to the collection of information for all times up to and including t.

Mathematically, $\mathcal{F}_t$ refers to an increasing sequence of $\sigma-$algebras which is defined on a given measurable space $(X, \mathbb{F})$. Thus, we have: (Shreve 2004).

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathbb{F}\,, \quad \text{for all} \quad 0 < s < t < T\,.$$

2.1.4 Definition. Stopping time

A random variable τ with values on the interval $[0, \infty]$ defined on a given probability space with filtration $(\mathcal{F}_t : t \geq 0)$ is called a stopping time with respect to the filtration if for all $t \geq 0$, we have $\{\tau \leq t\} \in \mathcal{F}_t$ (Mörters & Peres 2010).

2.1.5 Definition. Adapted stochastic process

A stochastic process is a collection of random variables $\{X(t), t \in T\}$ defined on a given filtered probability space. We consider a continuous time process, hence $T = \mathbb{R}_+ = [0, \infty)$. A collection of $X(t)$ is said to be an adapted process if the random variable $X(t)$ is $\mathcal{F}_t$-measurable. Hence, the value of the random variable $X(t)$ can be completely observed by the information at time t (Buchen 2012).

2.1.6 Definition. Brownian motion

A Brownian motion $B(t)$ is a collection of random processes indexed for all times $t \geq 0$, defined on a state space $S = \mathbb{R}$, having the following properties (Wiersema 2008):

- The process has continuous sample paths.
- At the initial time $t = 0$, the process is at rest. That is, $B(0) = 0$.
- The process has independent increments over non-overlapping time intervals. That is, $B(t) - B(s)$ in independent of $\{B(k) : k \leq s\}$, whenever $s < t$.
- The increment $B(t) - B(s)$, where $s < t$, has normal probability distribution with zero mean and variance, the length of the increment. That is, $B(t)-B(s) \sim N(0, t-s)$.

2.1.7 Definition. Stochastic differential equation

Let $S(t)$ be a stochastic process. A stochastic differential equation (SDE) is an equation which consists of combination of deterministic term and a stochastic term (white noise). The equation of the form describes it: (Buchen 2012)

$$\mathrm{d}S(t) = \mu(t, S(t))\,\mathrm{d}t + \sigma(t, S(t))\mathrm{d}B(t)\,, \tag{2.1.1}$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are adapted processes. Also, let $B(t)$ denote the Brownian motion defined in the real world probability measure. In integral form, it is represented as

$$S(t) = S(0) + \int_0^t \mu(s, X(s))\mathrm{d}s \;+ \int_0^t \sigma(s, X(s))\mathrm{d}B(s)\,.$$

2.1.8 Lemma. Ito's Lemma (one dimension process)

Let $S(t)$ be a stochastic process which follows an Ito process defined in equation (2.1.1). Suppose there exists a function $f \in C^{1,2}$ and define the process Z by $Z(t) = f(t, S(t)),\ \mu(t, S(t)) = \mu$ and $\sigma(t, S(t)) = \sigma$. Then Z follows the stochastic differential equation given by: (Björk 2009).

$$\mathrm{d}f(t, S(t)) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right)\mathrm{d}t + \frac{\partial f}{\partial S}\sigma\,\mathrm{d}B(t)\,.$$

2.1.9 Lemma. Ito's Lemma ($N-$dimension process)

Let an N-dimensional process S have dynamics given in equation (2.1.1) and let $f \in C^{1,2}$. Then the process $f(t, S(t))$ satisfy a stochastic differential equation given by: (Björk 2009).

$$\mathrm{d}f(t, S(t)) = \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{i=1}^{n}\frac{\partial f}{\partial s_i}\mathrm{d}S_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial s_i \partial s_j}\mathrm{d}S_i\mathrm{d}S_j\,,$$

with the following box calculus properties:

$$\begin{cases} (\mathrm{d}t)^2 & = 0\,, \\ \mathrm{d}t.\mathrm{d}B & = 0\,, \\ (\mathrm{d}B(i))^2 & = \mathrm{d}t\,, \quad \text{for } i = 1, \cdots, d \\ \mathrm{d}B(i).\mathrm{d}B(j) & = 0\,, \quad \text{for } i \neq j\,. \end{cases}$$

2.1.10 Definition. Normal distribution

Y is a normal random variable defined by $N(\mu, \sigma^2)$ if the probability density function of Y is given by: (Björk 2009).

$$f(Y) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(Y-\mu)^2}{2\sigma^2}}, \quad \text{for} \quad Y \in (-\infty, \infty)\,.$$

The cumulative density function of a standard normal random variable Y defined by $N(0,1)$ is given by

$$N(Y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{Y} \mathrm{e}^{\frac{-z^2}{2}} \mathrm{d}z\,.$$

2.1.11 Definition. Log-Normal distribution

Let Y be a normal distribution with mean μ and variance σ^2. A random variable $X = \mathrm{e}^Y$ is said to be lognormal variate with parameters (μ, σ) if the probability density function of X is given by: (Björk 2009).

$$f(X; \mu, \sigma^2) = \frac{1}{X\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(\ln X-\mu)^2}{2\sigma^2}}\,.$$

2.2 Geometric Brownian Motion

This is a model that measures the change in the random process $(\mathrm{d}S(t))$ with respect to the current underlying value $(S(t))$. It is an exponentiated form of the Brownian motion. The SDE below defines the motion:

$$\mathrm{d}S(t) = S(t)\{\mu \mathrm{d}t + \sigma \mathrm{d}B(t)\}\,,$$

where $\mu, \sigma \in \mathbb{R}$.

The SDE is composed of a deterministic and the stochastic part. To solve the SDE, we apply the concept of Ito's lemma. Consider the function $f \in \mathcal{C}^{1,2}$ defined by

$$f(t, S(t)) = \log S(t)\,.$$

The Taylor expansion of the function f is

$$\mathrm{d}f(t, S(t)) = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S(t)}\mathrm{d}S(t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2(t)}\mathrm{d}S^2(t)\,.$$

Substituting for the first and second partial derivatives with respect to $S(t)$, we have

$$\mathrm{d}f(t, S(t)) = \frac{1}{S(t)}\mathrm{d}S(t) - \frac{1}{2}\frac{1}{S^2(t)}\mathrm{d}S^2(t)\,.$$

Further application of Ito's lemma results to

$$\mathrm{d}f(t, S(t)) = \mu \mathrm{d}t + \sigma \mathrm{d}B(t) - \frac{\sigma^2}{2}\mathrm{d}t\,,$$

which becomes

$$\mathrm{d}\log S(t) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}B(t)\,.$$

Taking the integrals of both sides from 0 to t, the following results

$$\int_0^t \log S(s) = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}s + \sigma \int_0^t \mathrm{d}B(s)$$

$$\log S(t) = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B(t)$$

Finally, we have the solution to be

$$S(t) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)}. \tag{2.2.1}$$

Properties of geometric Brownian motion

The first and second moment of $S(t)$ which follows a log-normal distribution is given by

1. $\mathbb{E}[S(t)] = S(0)\mathrm{e}^{\mu t}$.
2. $\mathrm{Var}[S(t)] = (S(0))^2\mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma^2 t} - 1)$.
3. The probability density function of $S(t)$ is given by

$$f(S : \mu, \sigma, t) = \frac{1}{S\sigma\sqrt{2\pi t}} \exp\left(-\frac{(\log S - \log S(0) - \left(\mu - \frac{\sigma^2}{2}\right) t)^2}{2\sigma^2 t}\right).$$

Figure 2.1 below depicts a computer simulation of an underlying asset price which is defined based on geometric Brownian motion.

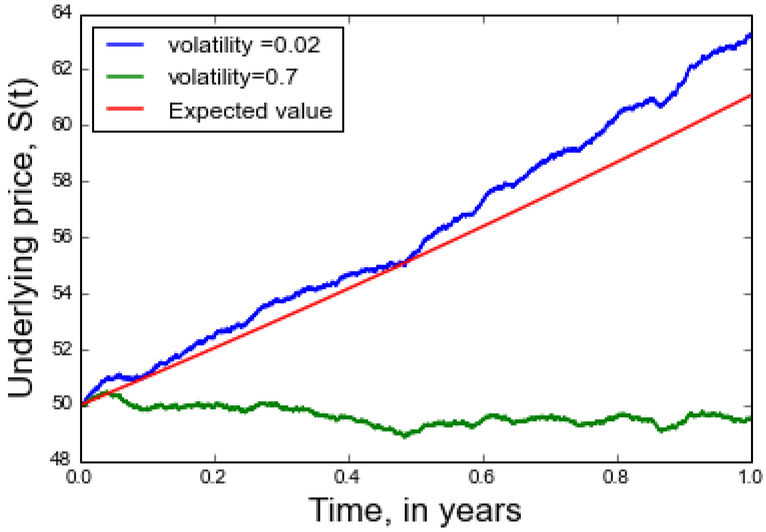

Figure 2.1: Geometric Brownian motion simulation

Figure 2.1 was obtained using the parameters: $S(0) = 50, r = 0.2, \sigma_1 = 0.02, \sigma_2 = 0.7$ and $T = 1$.

The smooth line denotes the graph of the expected value of the underlying. The asset path remains close to the expected value provided that the value of σ remains small. A larger value of the standard deviation forces the trajectory to move away from the expected value function. This results to large random variations.

2.3 The Black-Scholes Pricing Model for Options

Fisher Black and Myron Scholes in 1973 propounded a mathematical model that has been used for many years in the field of option pricing. The model is also referred to as Black-Scholes-Merton model because of the contribution of Robert Merton in the latter part of the work. Below are the assumptions that led to the model (Black &

Scholes 1973):

(i) The underlying price follows a log-normal random walk in a continuous time framework with constant drift and constant volatility.

(ii) The interest rate is constant and its value is known.

(iii) The stock pays no dividend throughout its history.

(iv) There exist a frictionless market, meaning that transactions incur no costs.

(v) Short selling is permitted.

(vi) Assets are divisible, as one can borrow any fraction of the share price at a short-term interest rate.

(vii) The European option is considered and can be exercised only at the expiry.

The assumptions above were used to formulate a mathematical model that computes the price of European options and the value obtained is very close to the observed market value.

Notations Used

- $S(t)-$ the current price of the stock at time, t.
- $\sigma-$ the volatility or the standard deviation of the underlying's return.
- $K-$ the strike price.
- $r-$ the risk-free interest rate which is continuously compounded.
- $T-$ the time to expiry.

- $\mu-$ the drift term on the stock.
- $\Pi-$ the value of the portfolio.
- $v(t, S)-$ the option value.
- $\tau-$ the remaining time to expiry and is denoted by $\tau = T - t$. At expiry, $\tau = 0$.
- $N()-$ the cumulative distribution function of a standard normal distribution and it is defined as
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-z^2}{2}} \mathrm{d}z\,.$$
- $N'(x)-$ refers to the standard normal probability density function and it is defined as
$$N'(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\,.$$
- $B(t)$ is a standard Brownian motion defined on a real world filtered probability space.

2.3.1 Derivation of the Black Scholes pricing PDE

One of the Black-Scholes assumptions is that underlying prices follow a geometric Brownian motion with constant drift and constant volatility. Precisely, we consider underlying asset prices as stochastic processes which satisfy the stochastic differential equation:
$$\mathrm{d}S(t) = S(t)(\mu \mathrm{d}t + \sigma \mathrm{d}B(t))\,. \tag{2.3.1}$$
The solution to equation (2.3.1) can also be written as
$$\ln S(t) - \ln S(0) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B(t)\,.$$

Thus, we can have that

$$\ln S(T) - \ln S(0) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right].$$

It is observed that $\ln S(T)$ has a normal distribution and hence $S(T)$ is log-normally distributed. Thus, we have

$$\ln S(T) \sim N\left[\ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right].$$

Let $v(t,S)$ be the value of non-dividend paying European call option, then the derivative of the function is

$$\begin{aligned} \mathrm{d}v(t,S) &= \frac{\partial v(t,S)}{\partial t}\mathrm{d}t + \frac{\partial v(t,S)}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 v(t,S)}{\partial S^2}\mathrm{d}S^2 \\ &= \frac{\partial v(t,S)}{\partial t}\mathrm{d}t + \frac{\partial v(t,S)}{\partial S}S(\mu\mathrm{d}t + \sigma\mathrm{d}B(t)) + \frac{1}{2}\frac{\partial^2 v(t,S)}{\partial S^2}(S^2\sigma^2\mathrm{d}t), \end{aligned}$$

Since from Ito's calculus, we have $\mathrm{d}S^2 = S^2(\mu\mathrm{d}t+\sigma\mathrm{d}B)^2 = S^2(\mu^2 \underbrace{\mathrm{d}t^2}_{=0} + 2\mu\sigma \underbrace{\mathrm{d}t\mathrm{d}B}_{=0} + \sigma^2 \underbrace{\mathrm{d}B^2}_{=\mathrm{d}t})$.

Thus, we have

$$\mathrm{d}v(t,S) = \left(\frac{\partial v(t,S)}{\partial t} + \frac{\partial v(t,S)}{\partial S}\mu S + \frac{S^2\sigma^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2}\right)\mathrm{d}t + \sigma S\frac{\partial v(t,S)}{\partial S}\mathrm{d}B(t).$$

Next, a risk-less self-financing portfolio is constructed which consists of short one derivative and long Δ shares of the underlying asset. Hence, we have

$$\Pi = -v(t,S) + \Delta S.$$

At any time interval dt, the change in the portfolio value is given as

$$\mathrm{d}\Pi = -\mathrm{d}v(t,S) + \Delta \mathrm{d}S\,.$$

Substituting for the value of d$v(t,S)$ and choosing $\Delta = \frac{\partial v(t,S)}{\partial S}$, the random component is eliminated and this gives rise to a risk-less portfolio. That is,

$$\mathrm{d}\Pi = \left(-\frac{\partial v(t,S)}{\partial t} - \frac{S^2\sigma^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2}\right)\mathrm{d}t\,.$$

The change in the portfolio value must be the same as the rate of return on any other riskless variables and thus to avoid arbitrage, we have the expression

$$\mathrm{d}\Pi = r\Pi \mathrm{d}t\,,$$

so that

$$\left(-\frac{\partial v(t,S)}{\partial t} - \frac{S^2\sigma^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2}\right)\mathrm{d}t = r\left(-v(t,S) + \frac{\partial v(t,S)}{\partial S}S\right)\mathrm{d}t\,.$$

Hence, the Black-Scholes PDE is given by

$$\boxed{\frac{\partial v(t,S)}{\partial t} + rS\frac{\partial v(t,S)}{\partial S} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2} = rv(t,S)} \tag{2.3.2}$$

The Black-Scholes PDE is a parabolic PDE which can be solved backwards to obtain the present value of the option [1]. The value of the option at expiry is known upfront and thus, the PDE can be solved either analytically or numerically using the terminal

[1]See Appendix A for the solution of the Black-Scholes PDE.

and the boundary conditions given below:

$$\begin{aligned} v(T,S) &= \max\{S(T)-K,0\}\,, \\ v(t,0) &= 0\,, \\ v(t,\infty) &\sim S\,. \end{aligned}$$

The above conditions hold for call options. For put options, we have the following

$$\begin{aligned} v(T,S) &= \max\{K-S(T),0\}\,, \\ v(t,0) &= K\mathrm{e}^{-r(T-t)}\,, \\ v(t,\infty) &\sim 0\,. \end{aligned}$$

2.3.2 Hedging and Greeks

Hedging is the act of using financial instruments to eliminate the risks that can be encountered as a result of adverse movements of the underlying prices. Hedging can be referred to as insurance, as investors use options to hedge against possible risks. A simple way to hedge is delta-hedging which ensures that the portfolio is delta-neutral. An example can be seen in the derivation of the Black-Scholes PDE which resulted in the elimination of the random component of the portfolio. Greeks are referred to as risk or hedge parameters. Examples which appeared in the Black-Scholes PDE are given below:

- **Delta:** This measures the sensitivity of the option value with respect to the underlying price. Delta-neutral ensures setting the delta to be equal to zero, that is, $\Delta = 0$. They are positive for calls and negative for puts. Delta measures the amount of shares needed to ensure delta-neutral, as well as, the chances of the option to expire in the money. It can be derived as follows: The value of a non-dividend paying European call option is

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2)\,.$$

$$\begin{aligned}
\frac{\partial v}{\partial S} = \Delta_C =&N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}\\
=&N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\frac{N'(d_2)}{S\sigma\sqrt{T-t}}\\
=&N(d_1) + \frac{1}{\sigma\sqrt{T-t}}\frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{Ke^{-r(T-t)}}{S\sigma\sqrt{T-t}}\frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\\
=&N(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sigma\sqrt{2\pi(T-t)}} - \frac{Ke^{-r(T-t)}}{S\sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}(d_1-\sigma\sqrt{T-t})^2}\\
=&N(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sigma\sqrt{2\pi(T-t)}} - \frac{S}{S\sigma\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}\\
=&N(d_1)\,.
\end{aligned}$$

The put can also be obtained in a similar way, as well as all the greeks. Thus, the delta of non-dividend paying European call and put options are given respectively as,

$$\Delta_C = N(d_1) \qquad \text{and} \qquad \Delta_P = N(d_1) - 1 \equiv -N(-d_1)\,.$$

- **Gamma:** They are second order Greeks, as they measure the rate of change of delta with respect to the underlying price. Gammas are positive for long options and negative for short options. They are the same for both calls and puts. For non-dividend paying European options, they are denoted mathematically by

$$\Gamma_C = \Gamma_P = \frac{1}{S\sigma\sqrt{T-t}} N'(d_1) \,.$$

If the option is at the money, gamma becomes very high and it progressively lowers when the option becomes out of the money or in the money. High gamma implies high variation in delta and the function becomes more convex.

- **Theta:** Theta also known as time decay, measures the sensitivity of option price with respect to time. The theta of a non-dividend paying European option is:

$$\Theta = \frac{-S\sigma}{2\sqrt{T-t}} N'(d_1) - \lambda r K e^{-r(T-t)} N(\lambda d_2) \,,$$

where $\lambda = 1$, for call option and $\lambda = -1$, for put option.

When the option is at the money, the value of theta becomes large and negative. For the call option, when the underlying price assumes the largest value, theta approaches $-rKe^{-r(T-t)}$ and theta approaches zero, as the stock price goes to zero.

Thus, the Black-Scholes PDE can be written in their greek term as

$$\Theta + rS\Delta + \frac{S^2\sigma^2}{2}\Gamma - rv = 0 \,.$$

3. The Concept of Exotic Options

3.1 Asian Options

Asian options are options whose payoffs are dependent on the average price of the underlying over a specified time interval. This is in contrast to the plain vanilla options whose payoffs are dependent on the price of the underlying at maturity time. Asian options are path-dependent since their payoffs depend on the path taken by the underlying. Zhang (1998) described Asian options as 'the natural development of vanilla options to capture path-dependence'. Asian options were first priced successfully in 1987 by David Spaughton and Mark Standish of the Banker's Trust. They developed the pricing formula in an attempt to deducing the average price of crude oil (Wilmott 2006). According to Kemna & Vorst (1990), such options are important for thinly-traded asset like crude oil, since price manipulations are avoided and volatility inherent in such options are totally reduced. Buchen (2012) also explained that Asian option was introduced to discourage market manipulation.

Consider a European call option which is currently in the money, the option holder is optimistic that the option ends in the money, so that the option would be exercised with a view to profit making. At expiry, the underlying price crashes, leaving the option out of the money and the option holder is left with the choice of not exercising the option. But Asian options consider some sort of averaging of the underlying value and hence, the effect of price movements of the underlying near the expiration of the contract is totally reduced. Asian options are not traded on standardized exchange but over the counter and very interesting for familiarity due to the fact that they are highly economical and can be used for hedging cash flows. Asian options are cheaper to price, and in fact their values are always less than or equal to the standard European options (Kemna & Vorst 1990).

A lot of research has been carried out on Asian options. Kemna & Vorst (1990) priced options on average asset values using Monte Carlo simulation with variance reduction elements. Shi & Yang (2014) used numerical methods to price arithmetic Asian options in a stochastic volatility with jumps. German & Yor (1993) used stochastic analysis and the Bessel process for the Laplace transform in time to obtain analytic solution of arithmetic Asian options. Cruz-Báez & González-Rodriguez (2008) extended the work of German & Yor without using any previous results obtained from the Bessel process.

3.1.1 Types of averaging in Asian options

With regards to the styles of averaging, Asian options are classified into arithmetic average and geometric average. The arithmetic average deals with the mean of the underlying, whereas the geometric average considers the exponential form of the mean of the underlying price, using its logarithmic form. Consider the following:

Let A be the average and $S(t)$ be the price of the underlying at time t. In the continuous monitoring process, the arithmetic average and the geometric average can be expressed respectively as

$$A = \frac{1}{T}\int_0^T S(t)\mathrm{d}t \quad \text{and} \quad A = \exp\left[\frac{1}{T}\int_0^T \ln(S(t))\mathrm{d}t\right].$$

In discrete form, for dates $t_1, t_2, \cdots, t_N$, the arithmetic and the geometric average are written respectively as

$$A = \frac{1}{N}\sum_{i=1}^{N} S(t_i) \quad \text{and} \quad A = \left[\prod_{i=1}^{N} S(t_i)\right]^{\frac{1}{N}}.$$

3.1.2 Asian options based on their payoff

The payoffs for the Asian options are classified into fixed strike or average rate and floating strike or average strike Asian options. Hence, we have:

(1) Floating strike Asian options: The payoffs for the call and put are denoted respectively as

$$\max\{S(T) - A, 0\} \quad \text{and} \quad \max\{A - S(T), 0\}\,.$$

(2) Fixed strike Asian options: The payoffs for the call and put are denoted respectively as

$$\max\{A - K, 0\} \quad \text{and} \quad \max\{K - A, 0\}\,.$$

We observe that the payoffs of both options are similar to the payoffs of the European calls and puts. The difference being that for floating strike, the strike price K is replaced with the average of the underlying (be it arithmetic or geometric); whereas in the fixed strike, the strike price K is fixed but the underlying price is replaced by the average of the underlying price. Floating Asian options guarantee that for call options, the final price of an underlying asset is not less than the average price of an underlying which is paid at a specific time interval. Furthermore, when the put option is considered, the final price of an underlying asset is always less than the average amount of the underlying asset which is received over the given period of time.

3.1.3 Valuation of Asian options

In obtaining the analytic solution of Asian option, Buchen (2012) explained that a closed form solution in the Black-Scholes model exists for the geometric average but this is not true for the arithmetic average. He pointed out that when the underlying price is log-normal for the geometric average, the geometric mean (either discrete or continuous) is log-normal but when it is arithmetic, it would not be log-normal. Hence, the Black-Scholes model is applicable for pricing the geometric Asian options. Closed form solutions for the European-style Asian options with fixed strike prices based on the discrete and the continuous geometric average prices are known (Zhang 1998). For the arithmetic Asian options, numerical approximation are used. Mudzimbabwe, Patidar & Witbooi (2012) used the explicit and the implicit finite difference approach to numerically price the arithmetic Asian options. Also, the pricing of arithmetic Asian options under hybrid stochastic and local volatility was done by Lee, Kim & Jang (2014).

The value of Asian option is a function of three independent variables, written as $V(t, S, I)$ and this is in contrast to the vanilla option which depends only on S and t. The term I is the historical integral of the underlying (the averaging). The first closed form solution for the approximation of geometric Asian option prices was given by Kemna & Vorst (1990). According to their work, the analytic closed form solution for the geometric averaging of the Asian option is similar to that obtained by Black and Scholes. The exceptions are that the volatility is replaced by $\frac{\sigma}{\sqrt{3}}$ and the dividend yield by $\frac{1}{2}\left(r + \frac{\sigma^2}{2}\right)$. Hence, in valuing Asian option either arithmetically or geometrically, the PDE obtained is similar to the Black-Scholes PDE but with an additional term (see equations (3.1.4) and (3.1.8) respectively).

Consider the fixed strike arithmetic Asian option on a continuous state space. The path-dependent parameter I can be denoted by

$$I = \int_0^T S(t)\mathrm{d}t\,. \tag{3.1.1}$$

For the call option, the payoff is given by $V(T) = (A - K)^+$. The pair (S, I) constitutes a Markov process and hence under the risk neutral pricing measure, there must be a function $v(t, S, I)$ that computes the option value at time $t \in [0, T]$ which is denoted by

$$v(t, S, I) = V(t) = \mathbb{E}_Q[e^{-r(T-t)}V(T)|\mathcal{F}_t]\,.$$

The Black-Scholes PDE for pricing Asian options

From the path-dependent quality defined in equation (3.1.1) above, the stochastic differential for I is given by $\mathrm{d}I = S(t)\mathrm{d}t$. We assume that the underlying price $S(t)$ follows a geometric Brownian motion in a continuous state space, defined in the real world probability measure. Thus, the SDE given below is satisfied:

$$\mathrm{d}S = S(\mu \mathrm{d}t + \sigma \mathrm{d}B)\,. \tag{3.1.2}$$

Consider the fixed strike arithmetic Asian option whose value is $v(t, S, I)$ and by Ito's lemma, we have:

$$\begin{aligned}
\mathrm{d}v &= \frac{\partial v}{\partial t}\mathrm{d}t + \frac{\partial v}{\partial S}\mathrm{d}S + \frac{\partial v}{\partial I}\mathrm{d}I + \frac{1}{2}\frac{\partial^2 v}{\partial S^2}\mathrm{d}S^2 \\
&= \frac{\partial v}{\partial t}\mathrm{d}t + \frac{\partial v}{\partial S}S(\mu \mathrm{d}t + \sigma \mathrm{d}B) + \frac{\partial v}{\partial I}\mathrm{d}I + \frac{1}{2}\frac{\partial^2 v}{\partial S^2}(S^2\sigma^2\mathrm{d}t)\,.
\end{aligned}$$

Next, we set up a riskless portfolio Π, at each time step which consists of long an Asian option and short Δ underlying units. We have $\Pi = v - \Delta S$. Thus, the pure investment acquired over the time interval $[t, t + \mathrm{d}t]$ becomes $\mathrm{d}\Pi = \mathrm{d}v - \Delta \mathrm{d}S$. This becomes,

$$\mathrm{d}\Pi = \left(\frac{\partial v}{\partial t} + \mu S\frac{\partial v}{\partial S} + S\frac{\partial v}{\partial I} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v}{\partial S^2}\right)\mathrm{d}t + S\sigma\frac{\partial v}{\partial S}\mathrm{d}B - \Delta(\mu S\mathrm{d}t + \sigma S\mathrm{d}B)\,. \tag{3.1.3}$$

To eliminate the random part of the process, we choose $\Delta = \frac{\partial v}{\partial S}$ and 3.1.3 reduces to

$$\mathrm{d}\Pi = \left(\frac{\partial v}{\partial t} + S\frac{\partial v}{\partial I} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v}{\partial S^2}\right)\mathrm{d}t\,.$$

The no-arbitrage principle implies that $\mathrm{d}\Pi = r\Pi\mathrm{d}t$. Thus, we have

$$\left(\frac{\partial v}{\partial t} + S\frac{\partial v}{\partial I} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v}{\partial S^2}\right)\mathrm{d}t = r\left(v - S\frac{\partial v}{\partial S}\right)\mathrm{d}t\,.$$

Finally, the function $v(t,S,I)$ satisfies the Black-Scholes PDE below:

$$\frac{\partial v(t,S,I)}{\partial t} + rS\frac{\partial v(t,S,I)}{\partial S} + S\frac{\partial v(t,S,I)}{\partial I} + \frac{\sigma^2}{2}S^2\frac{\partial^2 v(t,S,I)}{\partial S^2} - rv(t,S,I) = 0\,, \quad (3.1.4)$$

with terminal and boundary conditions below. For $I \in \mathbb{R}, S \geq 0, t \in [0,T)$,

$$v(T,S,I) = (A-K)^+\,; \quad (3.1.5)$$

$$\lim_{I\to-\infty} v(t,S,I) = 0\,; \quad (3.1.6)$$

$$v(t,0,I) = e^{-r(T-t)}V(T)\,. \quad (3.1.7)$$

Similarly, the geometric Asian average PDE for continuous time is given by

$$\frac{\partial v(t,S,I)}{\partial t} + rS\frac{\partial v(t,S,I)}{\partial S} + \log S\frac{\partial v(t,S,I)}{\partial I} + \frac{\sigma^2}{2}S^2\frac{\partial^2 v(t,S,I)}{\partial S^2} - rv(t,S,I) = 0\,, \quad (3.1.8)$$

where

$$I = \int_0^T \log S(t)\mathrm{d}t\,.$$

The terminal and the boundary conditions of the geometric Asian average PDE, as well as, its solution are found in Kemna & Vorst (1990). There is no explicit form for the arithmetic Asian average based on the concept of risk-neutrality and the lognormal nature of the asset price, its solution is computed numerically. Also, approximation of arithmetic Asian options using the corresponding geometric Asian options can be obtained (See (Zhang 1998)). Thus, the value of the non-dividend geometric Asian call option defined under the Black-Scholes framework, is given by

$$C_{geo} = Se^{-\left(r+\frac{\sigma^2}{6}\right)\frac{1}{2}(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)\,, \tag{3.1.9}$$

where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)(T-t)}{\sigma\sqrt{\frac{1}{3}(T-t)}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{\frac{1}{3}(T-t)}\,.$$

Figure 3.1 below shows the non-dividend Asian option values with geometric call features. We consider the parameters: $K = 75, r = 0.05, T = 1.0$ and $\sigma = 0.2$.

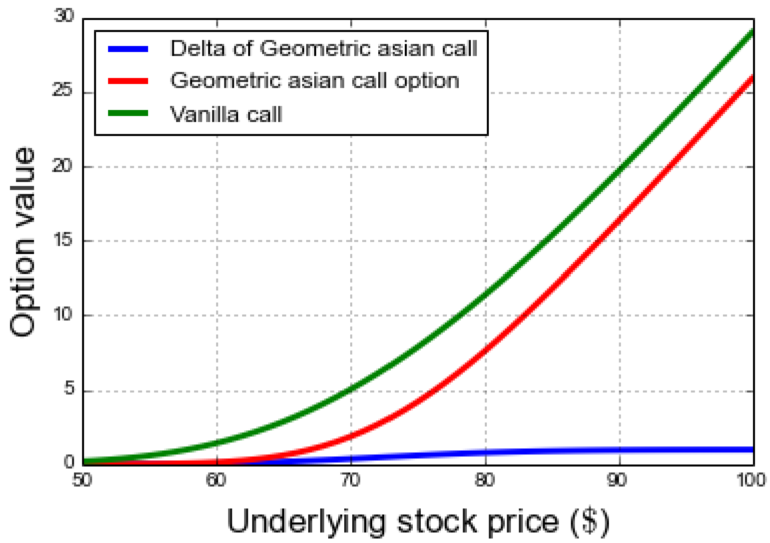

Figure 3.1: Geometric Asian call with zero dividend

We observe that the values of the Asian call option are lesser than that of the standard vanilla call as depicted in Figure 3.1 above. With regards to the delta, it is observed that delta is bounded between 0 and 1 for the call option. When the option is OTM, the delta of the geometric Asian call becomes very small but increases exponentially with an upper bound of 1, when the option is deep ITM.

3.2 Lookback Options

Lookback options are path-dependent exotic options whose payoffs depend on the extremum of the underlying asset over the duration of the contract. It considers the history of the behaviour of the underlying prices which spans the contract's lifetime. The term 'lookback' is used by the option holder to actually look back hindsight to determine the payoff of the contract (Bouzoubaa & Osseiran 2010, p.227). This type of option contract tends to be expensive because the option can be tailored in such a way that it minimizes the chance of expiring OTM. The holder of the option can make maximum profit which comes in the form of buying at a cheapest rate and selling at the highest rate. They help investors to minimize regrets and provide them with essential information on stock's behaviour over time, with the exception of the information on terminal stock (See Goldman et al. (1979) and Buchen (2012)). Despite the fact that lookback option provides a great advantage to its investors, the "no-free-lunch" principle still applies to it and this is why it is expensive in nature. Thus, most investors are saddened by this disadvantage.

3.2.1 Types of lookback options

Lookback options are characterized with regards to the nature of their strike prices, just like the Asian options. They include:

(1) Standard lookback or floating lookback options: Here, the strike price is yet to be determined during the life of the contract, but at maturity, the strike price is obtained. This is in contrast to the standard European call option where the

strike price had been fixed at the onset of the contract. For convenience, let $S_{\max}$ be the maximum value the underlying price had attained to, $S_{\min}$ be the minimum value and $S(T)$ be the value of the underlying at the maturity time T. Then, the payoffs for the call and put options are respectively given by:

$$\max\{S(T) - S_{\min}, 0\} \equiv S(T) - S_{\min} \qquad \text{and} \qquad \max\{S_{\max} - S_T, 0\} \equiv S_{\max} - S(T)\,.$$

The standard lookback option payoffs are always non-negative and can never be left to expiry without being exercised. This is because the option provides the holder with the right to purchase an underlying asset at the expiry using the lowest value the asset had attained to (call).

At time $t = 0$ and using the geometric Brownian features, the analytic solution for the standard lookback options is given as the discounted expectation value of the payoff defined under the risk-neutral measure Q. Thus, the values for the standard lookback calls and puts are respectively:

$$e^{-rT} E_Q[S(T) - S_{\min}|\mathcal{F}_t] \quad \text{and} \quad e^{-rT} E_Q[S_{\max} - S(T)|\mathcal{F}_t]\,.$$

(2) Fixed strike lookback options: Similar to the standard European options, the strike price is fixed. The payoffs for the fixed strike lookback call option and put option are respectively given by

$$\max\{S_{\max} - K, 0\} \qquad \text{and} \qquad \max\{K - S_{\min}, 0\}\,.$$

(3) Another type of lookback option is the partial lookback options. This was discovered to curb the expensive nature of the standard lookback options, while maintaining similar characteristics exhibited by the standard lookback options. Here, the features are restricted to the early parts or the latter parts of the option's life. As such, it considers only a sub-interval of $[0, T]$ in the calculation of its extremum. A partial lookback call gives the holder the right "to buy at some

percentage over the minimum" (Conze 1991). The payoffs for the call and put options are respectively given by

$$\max\{S(T) - \lambda S_{\min}, 0\} \qquad \text{and} \qquad \max\{\lambda S_{\max} - S(T), 0\}\,,$$

where λ is the degree of partiality with $\lambda > 1$ for calls and $0 < \lambda < 1$ for puts. Also, if $\lambda = 1$, the partial lookback option becomes standard lookback option.

3.2.2 Put-Call Parity

The put-call parity explains the relationship between the fixed strike lookback options and the floating strike lookback options. For convenience, let C_{fl}, C_{fix},P_{fl}, and P_{fix} represent the values of the floating lookback call, fixed lookback call, floating lookback put and the fixed lookback put options respectively. Consider the non-dividend paying option and let the real maximum of the underlying price $S_{\max}$ be denoted by $S'_{\max}$, which is defined as

$$S'_{\max} = \max\{S_{\max}, K\}\,.$$

Then, we can have

$$C_{fix} = P'_{fl} + S(0) - Ke^{-rT}\,,$$

provided that the time to expiration of C_{fix} and P_{fl} are the same. Similarly, for the fixed lookback put and the floating lookback call defined under

$$S'_{\min} = \min\{S_{\min}, K\}\,.$$

Then, we can have

$$P_{fix} = C'_{fl} + Ke^{-rT} - S(0)\,.$$

Observe that P'_{fl} has a payoff of $\max\{S'_{max} - S(T), 0\}$ and C'_{fl} has a payoff of $\max\{S(T) - S'_{min}\}$.

3.2.3 The valuation of lookback options

One of the earliest works on lookback options was found in Goldman et al. (1979) when they obtained the closed form solution of the standard lookback options with European features priced under the Black-Scholes framework. They further explained that a put (call) option on the maximum (minimum) value of an asset can be perfectly hedged. Conze (1991) computed the explicit formulas for different European lookback options and also introduced the concept of probabilisitic tools (Snell envelope) to obtain results for their American counterparts. The value of the floating strike lookback options were valued using Monte Carlo method, as shown in Kyprianou, Schoutens & Wilmott (2006). For the American lookback types, Zhang, Zhang & Zhu (2009) used the finite difference approximation to value the options and Lai & Lim (2004) established methods that compute the American fixed lookback options. Musiela & Rutkowski (2006) developed the arbitrage prices for the floating strike lookback options, together with their pricing method.

Under the Black-Scholes model, Zhang (1998) generalized the values of the floating and the fixed strike lookback options. He used density function for the maximum and the minimum values of the underlying prices during the lifetime of the contract to determine the values for the puts and the calls respectively.

Consider the standard lookback option. Let $S_{\max} = \max_{t\in[0,T]} S(t)$ and $S_{\min} = \min_{t\in[0,T]} S(t)$ be denoted by x and y respectively. The pair $(S(t), x)$ and $(S(t), y)$ are both Markov processes. According to Shreve (2004, p.309), there must be a function $v(t, S, x)$ which calculates the value of a standard lookback put option defined by

$$v(t, S, x) = \mathbb{E}_Q[e^{-r(T-t)}(x - S(T))|\mathcal{F}_t] \, .$$

The PDE for pricing the lookback options can be obtained the same way as the Asian options. This involves setting up a risk-less, self financing portfolio that consists of long position in one lookback option and short position in Δ units of the underlying.

Based on the no-arbitrage principle and applying the Ito's lemma, the stochastic part is eliminated. This gives rise to a function v which satisfies the PDE below:

$$\frac{\partial v(t,S,x)}{\partial t} + rS\frac{\partial v(t,S,x)}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 v(t,S,x)}{\partial S^2} - rv(t,S,x) = 0\,, \tag{3.2.1}$$

in the region $\{(t,S,x) : t \in [0,T), 0 \leq S \leq x\}$. The boundary and the terminal conditions for the PDE in equation (3.2.1) are given below:

$$v_x(t,S,x) = 0, \quad \text{where } S = x,\ t \in [0,T),\ x > 0 \tag{3.2.2}$$

$$v(t,0,x) = xe^{-r(T-t)}, \quad \text{where } t \in [0,T),\ x > 0 \tag{3.2.3}$$

$$v(T,S,x) = x - S(T), \quad \text{where } 0 \leq S < x\,. \tag{3.2.4}$$

In equation (3.2.2), when the asset price is maximum, the option price becomes insensitive to any changes with respect to the maximum. This is because the current maximum of the asset assuming the maximum at expiry has zero probability. Thus, $\max\{S_{\max} - S(T), 0\} = \max\{S_{\max} - S_{\max}\} = 0$. Hence, any small change in the option price is insignificant. Equation (3.2.3) occurs when the underlying price is 0, the option value becomes the discounted value of maximum underlying asset price. For the call, we consider the situation when the underlying price is very large. Thus the boundary condition for the call becomes $v(t,S,y) \approx S$. Finally, equation (3.2.4) gives the payoff for the put. For the call, the payoff becomes $v(t,S,y) = S - y$.

Equation (3.2.1), with conditions (3.2.2),(3.2.3) and (3.2.4) is of 3-dimension and its solution would be complex and numerically ambiguous. Hence, it suffices to transform it into a 2-dimensional problem which encompasses the use of similarity reduction where the linearity scaling property of $v(t,S,x)$ would be introduced. The function $v(t,S,x)$ can be written as $v(t,\beta S,\beta x) = \beta v(t,S,x)$, for $\beta > 0$ (Shreve 2004).

Let $u(t,m) = v(t,m,1)$, then

$$v(t,S,x) = xv\left(t, \frac{S}{x}, 1\right) = xu\left(t, \frac{S}{x}\right)\,, \tag{3.2.5}$$

where $t \in [0, T],\ m \in [0, 1],\ 0 \leq S \leq x$ and $x > 0$.

Taking the partial derivatives and substituting into (3.2.1), the following ensues:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left[xu\left(t, \frac{S}{x}\right)\right] = xu_t\left(t, \frac{S}{x}\right)\,; \tag{3.2.6}$$

$$\frac{\partial v}{\partial S} = \frac{\partial}{\partial S}\left[xu\left(t, \frac{S}{x}\right)\right] = xu_m\left(t, \frac{S}{x}\right).\frac{\partial}{\partial S}\left(\frac{S}{x}\right) = u_m\left(t, \frac{S}{x}\right)\,; \tag{3.2.7}$$

$$\frac{\partial^2 v}{\partial S^2} = \frac{\partial}{\partial S}\left[u_m\left(t, \frac{S}{x}\right)\right] = u_{mm}\left(t, \frac{S}{x}\right).\frac{\partial}{\partial S}\left(\frac{S}{x}\right) = \frac{1}{x}u_{mm}\left(t, \frac{S}{x}\right)\,; \tag{3.2.8}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[xu\left(t, \frac{S}{x}\right)\right] = xu_m\left(t, \frac{S}{x}\right).\frac{\partial}{\partial x}\left(\frac{S}{x}\right) + u\left(t, \frac{S}{x}\right)$$

$$= u\left(t, \frac{S}{x}\right) - \frac{S}{x}u_m\left(t, \frac{S}{x}\right)\,. \tag{3.2.9}$$

Substitute equations (3.2.5), (3.2.6), (3.2.7) and (3.2.8) into equation (3.2.1) to have

$$xu_t\left(t, \frac{S}{x}\right) + rSu_m\left(t, \frac{S}{x}\right) + \frac{\sigma^2 S^2}{2}.\frac{1}{x}u_{mm}\left(t, \frac{S}{x}\right) - rxu\left(t, \frac{S}{x}\right) = 0\,.$$

Factorizing x out, we have

$$x\left[u_t\left(t, \frac{S}{x}\right) + r\frac{S}{x}u_m\left(t, \frac{S}{x}\right) + \frac{\sigma^2 S^2}{2}.\frac{1}{x^2}u_{mm}\left(t, \frac{S}{x}\right) - ru\left(t, \frac{S}{x}\right)\right] = 0\,. \tag{3.2.10}$$

From the condition $x > 0$, we see that $x \neq 0$ and thus the equation (3.2.10) becomes

$$u_t(t, m) + rmu_m(t, m) + \frac{\sigma^2}{2}m^2 u_{mm}(t, m) - ru(t, m) = 0\,, \tag{3.2.11}$$

where $m = \frac{S}{x}$, for $t \in [0, T]$ and $m \in [0, 1]$.

We obtain the terminal and the boundary conditions:

(1) Recall that $v(t, S, x) = xu\left(t, \frac{S}{x}\right)$. But from equation (3.2.3), $v(t, 0, x) = xe^{-r(T-t)}$. Thus, we have $u(t, 0) = e^{-r(T-t)}$.

(2) From equation (3.2.9), we have $v_x(t, S, x) = u(t, m) - mu_m(t, m)$ and from equation (3.2.2),

we have $v_x(t, S, x) = 0$ for $S = x$. This shows that $m = \frac{S}{x} = 1$ and thus, we have $0 = u(t, 1) - u_m(t, 1) \implies u(t, 1) = u_m(t, 1)$.

(3) From equation (3.2.4), we have $v(T, S, x) = x - S(T) \implies x - S(T) = xu\left(T, \frac{S}{x}\right)$ and thus, $u(T, m) = 1 - \frac{S(T)}{x}$.

For $m \in [0, 1]$ and $0 \leq t \leq T$, the 3-dimensional PDE has been reduced to its 2-dimensional form with the boundary and terminal conditions given below as:

$$
\begin{aligned}
&u_t(t, m) + rmu_m(t, m) + \frac{\sigma^2}{2} m^2 u_{mm}(t, m) - ru(t, m) = 0 \quad \text{and} \\
&u(t, 0) = e^{-r(T-t)}\,, \\
&u(t, 1) = u_m(t, 1)\,, \\
&u(T, m) = 1 - \frac{S(T)}{x}\,.
\end{aligned}
$$

According to Musiela & Rutkowski (2006), the price of standard European lookback call option is given by

$$C_1 = SN\left(\frac{\ln\left(\frac{S}{m}\right)+r_1\tau}{\sigma\sqrt{\tau}}\right) - me^{-r\tau}N\left(\frac{\ln\left(\frac{S}{m}\right)+r_2\tau}{\sigma\sqrt{\tau}}\right) - \frac{S\sigma^2}{2r}N\left(\frac{\ln\left(\frac{m}{S}\right)-r_1\tau}{\sigma\sqrt{\tau}}\right) + e^{-r\tau}\frac{S\sigma^2}{2r}\left(\frac{m}{S}\right)^{\frac{2r}{\sigma^2}}N\left(\frac{\ln\left(\frac{m}{S}\right)+r_2\tau}{\sigma\sqrt{\tau}}\right), \tag{3.2.12}$$

where $m = \min S(t),\ \forall\, t \in [0,T],\ \ \tau = T - t$ and $r_{1,2} = r \pm \frac{\sigma^2}{2}$.

The Put equivalent is given also by

$$P_1 = -SN\left(-\frac{\ln\left(\frac{S}{M}\right)+r_1\tau}{\sigma\sqrt{\tau}}\right) + Me^{-r\tau}N\left(-\frac{\ln\left(\frac{S}{M}\right)+r_2\tau}{\sigma\sqrt{\tau}}\right) + \frac{S\sigma^2}{2r}N\left(\frac{\ln\left(\frac{S}{M}\right)+r_1\tau}{\sigma\sqrt{\tau}}\right) - e^{-r\tau}\frac{S\sigma^2}{2r}\left(\frac{M}{S}\right)^{\frac{2r}{\sigma^2}}N\left(\frac{\ln\left(\frac{S}{M}\right)-r_2\tau}{\sigma\sqrt{\tau}}\right), \tag{3.2.13}$$

where $M = \max S(t),\ \forall\, t \in [0,T],\ \ \tau = T - t$ and $r_{1,2} = r \pm \frac{\sigma^2}{2}$.

3.2.4 Option values with lookback features

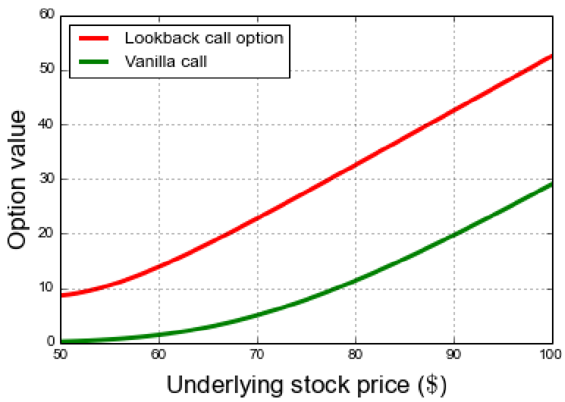

Figure 3.2: Lookback call & Vanilla call

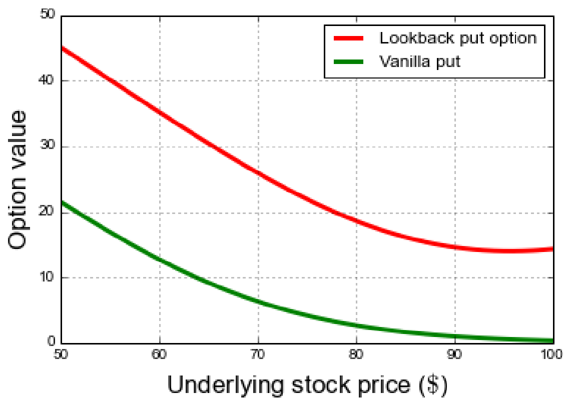

Figure 3.3: Lookback put & Vanilla put

Figure 3.2 above compares the standard lookback call to the plain vanilla call option and Figure 3.3 above compares the standard lookback put to the plain vanilla put option. Here, we consider the parameters: $K = 75, T = 1, r = 0.05$ and $\sigma = 0.20$ for both figures. We observe that the values of the floating lookback call option always exceed that of the plain vanilla call option and thus, they are never out of money. This is because the strike price for the put option is the maximum value that the stock achieved before its expiration and the call is the minimum value of the asset.

3.3 Barrier Options

Barrier options are example of path-dependent exotic options that are traded both on the standardized exchange and at over-the-counter market. According to Luenberger & Luenberger (1999), cited by Ilhan & Sircar (2006), barrier option trading accounts for "50% of the volume of all exotic options and 10% volume of all traded securities". The payoffs of barrier options depend on the barrier level that the underlying price attains to during the life of the contract. There is a presence of rebate (positive discount, often a small percentage of the option value) on the option holder and this increases the value of the barrier option, even though it has no effect on its payoff. According to Zhang (1998), barrier options are classified into vanilla barrier and exotic barrier options. Barrier options considered in this research work are of the vanilla class.

3.3.1 Classification of vanilla barrier options

Generally, vanilla barrier options are classified into knock-in (*Lightable*) and knock-out (*Extinguishable*) options. A barrier option is referred to as up (down) option if the barrier level is positioned above (below) the underlying's initial price. Let B be the barrier level, $S_M = \max_{0\leq t\leq T} S(t)$ and $S_m = \min_{0\leq t\leq T} S(t)$. Thus, we have the following classes:

(1) Knock-in barrier options: These are options that come into existence or lighted once the underlying price reaches the barrier level. Hence, they possess the European features with lower premium once they are activated. Prior to its hitting the level, the payoff equals zero but if this level is not breached before the contract expires, then the holder may enjoy a rebate. Examples are down-and-in options and up-and-in options.

   (i) Down-and-in barrier options (DIBO): Here, the barrier level is situated below the current price of the underlying. Thus, the option comes alive only if the underlying falls below the barrier during the contract's lifetime. The payoff for the DIBO (call) is given by

$$v(S(T),T) = \begin{cases} 0 & \text{if} \quad S_m > B, \quad \forall\, 0 \le t \le T \\ S(T) - K & \text{if} \quad S_m \le B, \quad \text{for at least one } t \le T \end{cases}$$

   (ii) Up-and-in barrier options (UIBO): The option is valuable if the underlying price reaches the barrier from below before the contract expires. The payoff for the UIBO (call) is given by

$$v(S(T),T) = \begin{cases} 0 & \text{if} \quad S_M < B, \quad \forall\, 0 \le t \le T \\ S(T) - K & \text{if} \quad S_M \ge B, \quad \text{for at least one } t \le T \end{cases}$$

(2) Knock-out barrier options: These options are extinguished or knocked out once the barrier level is reached by the price of the underlying. They expire worthlessly if this barrier is breached and the holder enjoys a rebate once it occurs before the contract expires. Even if the price of the underlying should move back within this barrier level before the expiration of the contract, the option is already knocked-out. Barrier options are highly sensitive with regards to the position of the barrier. The knock-out for example has an expected payoff of near-zero value

when the asset price becomes very close to the barrier and it outputs the value of the standard vanilla options as the asset price move increasingly away from the barrier (see Table 3.2). Examples of knock-out barrier options are down-and-out options and up-and-out options.

(i) Down-and-out barrier options (DOBO): This type of option creates a limited profit potential for the option holder since the holder is permitted to make profit provided the underlying price does not reach the specified level. Nevertheless, a down-and-out call can be purchased by an investor who expects an exponential increase in the underlying asset price and as such, unlimited profit just like the standard call can be made. The barrier level is positioned below the current price of the underlying. Thus, if the underlying price falls below the level, the option expires worthlessly. The payoff for the DOBO (call) is given by

$$v(S(T),T) = \begin{cases} S(T) - K & \text{if} \quad S_m > B, \quad \forall\, 0 \leq t \leq T \\ 0 & \text{if} \quad S_m \leq B, \quad \text{for at least one } t \leq T \end{cases}$$

(ii) Up-and-out barrier options (UOBO): The barrier level is positioned above the current price of the underlying asset. The option expires worthlessly if the barrier is breached from below by the price of the underlying before the expiration of the contract. The payoff for the UOBO (call) option is given by

$$v(S(T),T) = \begin{cases} S(T) - K & \text{if} \quad S_M < B, \quad \forall\, 0 \geq t \leq T \\ 0 & \text{if} \quad S_M \geq B, \quad \text{for at least one } t \leq T \end{cases}$$

Barrier options are summarized in Table 3.1 below. Let P be the payoff, C_T and P_T, the values of the plain vanilla calls and puts respectively:

Table 3.1: A summary of the barrier options

Option Rights	**Types**	**Barrier level position**	**Effect on payoff** $P(S,T)$
Call	DOBO	$B < S$	$B = S \implies P = 0$, $B \neq S \implies P = C_T$
	UOBO	$B > S$	$B = S \implies P = 0$, $B \neq S \implies P = C_T$
	DIBO	$B < S$	$B = S \implies P = C_T$, $B \neq S \implies P = 0$
	UIBO	$B > S$	$B = S \implies P = C_T$, $B \neq S \implies P = 0$
Put	DOBO	$B < S$	$B = S \implies P = 0$, $B \neq S \implies P = P_T$
	UOBO	$B > S$	$B = S \implies P = 0$, $B \neq S \implies P = P_T$
	DIBO	$B < S$	$B = S \implies P = P_T$, $B \neq S \implies P = 0$
	UIBO	$B > S$	$B = S \implies P = P_T$, $B \neq S \implies P = 0$

3.3.2 Input-parity

This is the combination of an ‘in’ and ‘out’ barrier option to yield a plain vanilla option, provided that both options possess the same expiration time and strike price. It is most suitable for European option without rebate. This explains better why barrier options are less expensive compared to the plain vanilla options. Table 3.2 below verifies that the input-parity of barrier options holds. For the up options, we choose a range for the underlying asset, $S = 100, 110, 120, 130, 140, 150$ respectively, $B = 150, K = 110, r = 5\%, \sigma = 45\%$ and $T = 2$. Next, we position the barrier above each of the underlying asset prices. As the underlying price increases, the values of the UOBO reduce and knocks out as soon as the barrier level is reached. This in turn makes the UIBO to be increasingly ITM. Also for the vanilla option, increase in the underlying price results in a linear increase in the value of the vanilla call option.

For the down options, we choose range for the asset, $S = 160, 150, 140, 130, 125, 120$ respectively, $B = 120, K = 125, r = 6\%, \sigma = 50\%$ and $T = 2$. We position the barrier below each of the underlying prices. Decrease in the underlying prices reduce the values of the vanilla call. As the underlying price tends towards the barrier, the DOBO reduces and becomes worthless, whereas the DIBO increases without bound.

Table 3.2: Up and Down barrier call options

UIBO	**UOBO**	**Vanilla**
24.6494	0.4933	25.1427
31.2819	0.4197	31.7016
38.4438	0.3265	38.7703
46.0517	0.2213	46.2730
54.0330	0.1108	54.1438
62.3262	0.0000	62.3262

DIBO	**DOBO**	**Vanilla**
21.4425	45.2082	66.6507
24.3851	34.3070	58.6921
27.8270	23.1841	51.0111
31.8732	11.7765	43.6497
34.1629	5.9407	40.1036
36.6560	0.0000	36.6560

3.3.3 Valuation of barrier options

The path-dependent nature of barrier options are of the weak form, since the values depend not just on the path taken by the underlying but on the breaching of the specified barrier level. This is in contrast to the Asian options whose path-dependency are strong. One of the earliest research on barrier option pricing can be found in Snyder (1969). He described the down-and-out options as limited risk special options, in which the holder agrees (with a favorable price in return) to limit the risk experienced by the writer by making the option void supposing “the price of the stock declines during the life of the contract to a specific point below the striking price called the expiration price”. Furthermore, Merton (1973) priced the down-and-out barrier call options using the PDE approach. Boyle & Tian (1998) applied the concept of modified explicit finite difference approach to obtain the price of barrier options. Pricing discrete barrier options are also found in Kou (2003) and Broadie, Glasserman & Kou (1997). Furthermore, Zvan, Vetzal & Forsyth (2000) presented an implicit method for solving PDE models in relation to barrier options. Guardasoni & Sanfelici (2006) applied the concept of boundary element method on barrier options. Explicit formula for obtaining the knock-out discount for barrier options can be found in Musiela & Rutkowski (2006). Recent works on double barrier option pricing can be found in Farnoosh et al. (2015) and Chen et al. (2015).

The call and put values of the dividend paying European options at time $t = 0$ are

$$C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) \qquad \text{and} \qquad P = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

respectively, where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T}\,.$$

The following closed form formulas for barrier call options which are monitored continuously are obtained using the extended Black-Scholes formula and can be found in Hull (2006, p.579-581) and Bouzoubaa & Osseiran (2010, p.153). [1]

The value of the **DIBO** when the barrier level $B \leq K$ is

$$C_{di} = S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2\lambda} N(y) - K e^{-rT} \left(\frac{B}{S_0}\right)^{2\lambda - 2} N(y - \sigma\sqrt{T})\,, \tag{3.3.1}$$

where

$$\lambda = \frac{r - q + \frac{\sigma^2}{2}}{\sigma^2} \qquad \text{and} \qquad y = \frac{\ln\left(\frac{B^2}{S_0 K}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}\,.$$

The value for the corresponding **DOBO** is obtained using the input-parity. That is,

[1] The call values follow the assumption that the underlying prices are lognormal, pay dividend q and are considered at time $t = 0$. The puts are obtained similarly. See Hull (2006).

$C_{do} = C - C_{di}$ and thus, we have:

$$\begin{aligned} C_{do} = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) - \Bigg[& S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2\lambda} N(y) \\ & - K e^{-rT} \left(\frac{B}{S_0}\right)^{2\lambda-2} N(y - \sigma\sqrt{T}) \Bigg] . \end{aligned} \tag{3.3.2}$$

Consider also the situation where $B > K$, we have the value of the **DOBO** to be

$$\begin{aligned} C_{do} = & S_0 e^{-qT} N(x_1) - K e^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2\lambda} N(y_1) \\ & + K e^{-rT} \left(\frac{B}{S_0}\right)^{2\lambda-2} N(y_1 - \sigma\sqrt{T}) , \end{aligned} \tag{3.3.3}$$

where

$$x_1 = \frac{\ln\left(\frac{S_0}{B}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \qquad \text{and} \qquad y_1 = \frac{\ln\left(\frac{B}{S_0}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} .$$

The value for the corresponding **DIBO** is obtained using the input-parity. Thus,

$$\begin{aligned} C_{di} = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) - \Bigg[& S_0 e^{-qT} N(x_1) - K e^{-rT} N(x_1 - \sigma\sqrt{T}) \\ & - S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2\lambda} N(y_1) + K e^{-rT} \left(\frac{B}{S_0}\right)^{2\lambda-2} N(y_1 - \sigma\sqrt{T}) \Bigg] . \end{aligned} \tag{3.3.4}$$

For $B > K$, the value of the **UIBO** is given by

$$\begin{aligned} C_{ui} = & S_0 e^{-qT} N(x_1) - K e^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2\lambda} [N(-y) - N(-y_1)] \\ & + K e^{-rT} \left(\frac{B}{S_0}\right)^{2\lambda-2} [N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})] . \end{aligned} \tag{3.3.5}$$

The corresponding value of the **UOBO** is obtained by

$$C_{uo} = C - C_{ui}\,. \tag{3.3.6}$$

For the value of the **UOBO** when $B \leq K$, the call option becomes $C_{uo} = 0$. The corresponding **UIBO** is given by

$$C_{ui} = C - 0 = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)\,. \tag{3.3.7}$$

3.3.4 Non-dividend option values with barrier call features

Figures 3.4 & 3.5 describe the 'down' options in comparison to the vanilla call values.

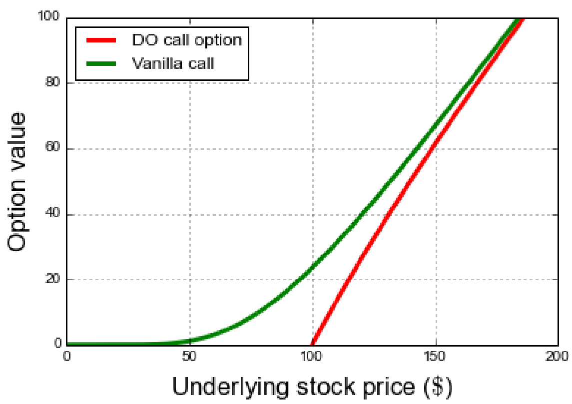

Figure 3.4: DOBO call & Vanilla call

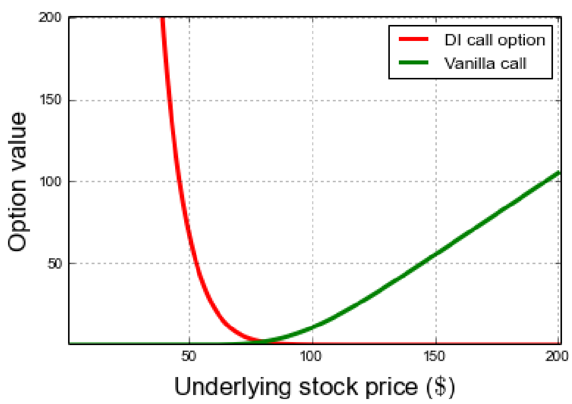

Figure 3.5: DIBO call & Vanilla call

In Figure 3.4, the barrier is at $B = 100$, $K = 90, \sigma = 0.4, r = 0.06$ and $T = 1.0$. We observe that decreasing the underlying prices reduce the values of the DOBO call and the option pays nothing once the barrier is reached. Figure 3.5 sets the barrier level at $B = 80$, $r = 0.05, K = 100, T = 1.0$ and $\sigma = 0.2$. We observe that prior to the barrier being breached, the option pays nothing. The option becomes active as the barrier is triggered and its values start to increase.

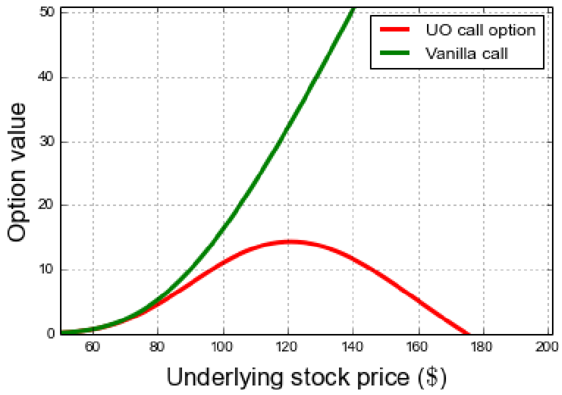

Figure 3.6: UOBO call & Vanilla call

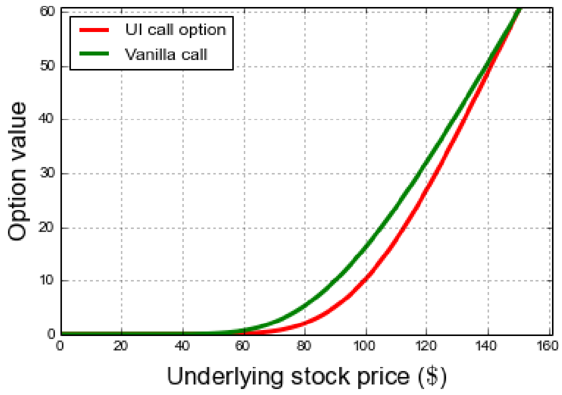

Figure 3.7: UIBO call & Vanilla call

In Figure 3.6, the barrier is at $B = 175$, $K = 100, \sigma = 0.2, r = 0.05$ and $T = 2.0$. The value for the up-and-out barrier call option is always less than the difference between the barrier level and the strike price, and hence it has limited up-side potential. It was also observed that increasing the underlying prices resulted to an increase in the option values. But the values reduced as the chances of the option being knocked out increased. Figure 3.7 sets the barrier level at $B = 150$, $r = 0.05, K = 100, T = 2.0$ and $\sigma = 0.2$. We observed that the value of the UIBO increased once it is ATM and rose sharply when the barrier is triggered, resulting to a deep ITM knock-in call value. Also, it was observed that at the barrier, the UIBO and the plain vanilla call intersected.

3.4 Binary Options

These are classes of exotic options whose payoffs are discontinuous. They are also known as bet options or digital options. The options are binary in nature because the two outcomes are dependent on their payoffs, that is, either a fixed amount is paid or nothing at all. According to Wilmott (2006), an investor can choose to long a position in binary call options if a less dramatic increment in the prices of the underlying over the strike price is expected. But if a more spontaneous increase is expected in the underlying price, then a plain vanilla call option is advisable. This is due to the fact that a binary option cannot pay more than the specified amount initiated at the onset of the contract but a plain vanilla possesses the best upside potential.

3.4.1 Types of binary options

Binary options are generally classified into gap options, asset-or-nothing and cash-or-nothing options.

(1) Gap options: The payoff depends on the difference between the underlying price and a specific price (different from the strike price), which is also known as the 'gap parameter'. Plain vanilla options are obtained from the gap options if the gap parameter equals the strike price (Zhang 1998). The payoff is given below [2]

$$P(S,T) = \begin{cases} \lambda(S_t - X), & \text{if } \lambda S > \lambda K \\ 0 & \text{otherwise} \end{cases}.$$

(2) Asset-or-nothing options (AON): This type of option gives the holder the right but not the obligation to own a particular underlying if the option expires in-the-money. When the gap parameter is zero, we have the AON. The payoff for the dividend paying AON options is given by

$$P(S,T) = \begin{cases} Se^{-q(T-t)}N(\lambda d_1), & \text{if } \lambda S > \lambda K \\ 0 & \text{otherwise} \end{cases}$$

where d_1 follows from the Black-Scholes formula.

[2] Note that $\lambda = \pm 1$ for calls or puts respectively.

The Figures (3.8 & 3.9) below explain the payoff structure when the strike price is $K = 50$. Also, consider when the AON binary option pays a fixed amount, say $R = 30$ if the option is ITM and nothing when the option is both ATM and OTM.

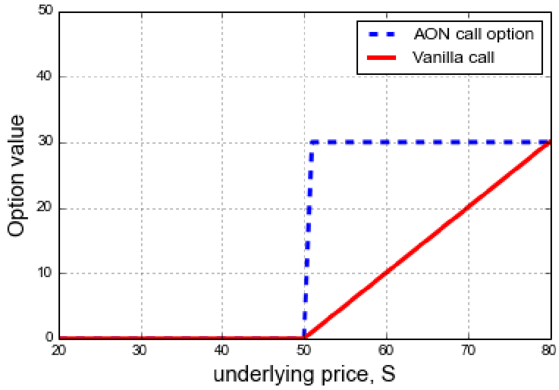

Figure 3.8: AON call & Vanilla call

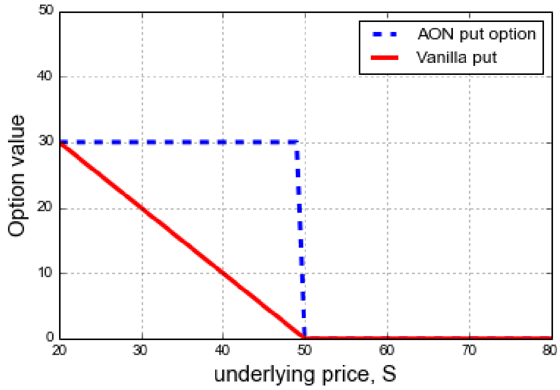

Figure 3.9: AON put & Vanilla put

(3) Cash-or-nothing (CON): Whenever the underlying price ends above the strike price, a specific amount is paid for the option. But the option has zero payoff if the strike price exceeds the underlying price at the end of the contract. The payoff for the dividend paying CON options is given by

$$P(S,T) = \begin{cases} e^{-r(T-t)}N(\lambda d_2), & \text{if } \lambda S > \lambda K \\ 0 & \text{otherwise} \end{cases}$$

An example of a binary option is the European call option on a long forward contract. The sum of the binary call and a binary put gives the discount factor $e^{-r(T-t)}$. Generally, the Black-Scholes pricing formula is decomposed into the CON and the AON binary options. For a dividend paying call option,

$$\text{Black-Scholes Value} = \underbrace{Se^{-q(T-t)}N(d_1)}_{CON} - K\,\underbrace{e^{-r(T-t)}N(d_2)}_{AON}.$$

4. Valuation of Fixed Strike Lookback Options and Zero-Rebate Knock-Out Barrier Options

In this chapter, we shall consider the mathematical formulations of these exotic option prices. For the fixed strike lookback options, we shall employ the concept of reflection principle and joint probability distribution in deriving the pricing formula for the call and put options. Next, we employ the concept of method of images and reduction to heat equation to obtain the pricing formula for the barrier options.

4.1 Fixed Strike Lookback Options

Suppose an investor makes a speculation that the price of an underlying asset would rise tremendously within a period of T years, then he decides to buy a plain vanilla European call option with expiration time T. After purchasing the contract, the underlying price rose as expected but due to some circumstances, the price fell down to the point that it gets below the agreed strike price. The investor will now be forced to terminate the contract with the loss of premium paid upfront. Thus, the investor will receive a lesser payoff than when the underlying price was significantly high. Hence, the introduction of fixed strike lookback call option limits such hindrance. In this contract, the investor would rather use the maximum value that the underlying had achieved before the expiration of the contract instead of the final asset value as used by the plain vanilla options. Thus, the fixed strike lookback call option promises a non-negative payoff (to the option holder) which measures the excess of the maximum asset price over the strike price. Fixed strike put on the other hand, measures the excess of the strike price over the minimum asset price.

4.1.1 Definition. Fixed-Strike lookback options

Fixed strike call option grants the option holder the right but not the obligation to purchase an underlying asset for a fixed amount K in the future and to sell at the maximum price the underlying achieved before the expiration of the contract. For the put option, the holder buys at the minimum price and sells at a fixed strike, K.

For the fixed strike lookback call option, we assume that the asset dynamics is based on geometric Brownian motion with constant variance σ^2 and drift $\left(r - \frac{\sigma^2}{2}\right)$. Let T be the expiration of the option contract and the interval $[0, T]$ be the duration of the lookback period. Define the stochastic variables under the risk neutral measure as

$$U_\lambda = \ln \frac{S_\lambda}{S} = \ln S_\lambda - \ln S, \quad \text{for } \lambda \in [t, T]\,,$$

$$Y_T = \ln \frac{M_t^T}{S} = \max\{U_\lambda : \lambda \in [t, T]\}\,,$$

$$X_T = \ln \frac{m_t^T}{S} = \min\{U_\lambda : \lambda \in [t, T]\}\,,$$

where M_t^T and m_t^T refer to the maximum and the minimum value of the underlying from time t to T respectively.

Before proceeding, we derive the joint probability density law for the maximum value over the interval $[0, T]$ and over the terminal value of the Brownian motion B_T^μ with drift μ. This will be done by applying the reflection principle on the Brownian motion.

4.1.2 Lemma. Reflection principle

Let $B(t)$ be a Brownian motion and $\{\mathcal{F}_t : t \geq 0\}$ be the usual filtration. For a fixed $\alpha > 0$, let the stopping time of the process be

$$\tau_\alpha = \mathrm{Inf}\{t \geq 0 : B(t) = \alpha\}\,,$$

and define the process $\{\widetilde{B}(t) : t \geq 0\}$ by

$$\widetilde{B}(t) = \begin{cases} B(t) & t < \tau_\alpha \\ 2\alpha - B(t) & t \geq \tau_\alpha \end{cases}.$$

Then $\{\widetilde{B}(t) : t \geq 0\}$ is a Brownian motion. (Yue-Kuen 1998). The reflection principle asserts that for $\omega \leq \alpha$,

$$\mathbb{P}[\tau_\alpha \leq t, B(t) \leq \omega] = \mathbb{P}[B(t) \geq 2\alpha - \omega].$$

Consider a zero drift Brownian motion B_t^0 starting at time zero with volatility σ. Let m denote a downstream barrier. We aim at obtaining the probability $\mathbb{P}[m_0^T < m, B_T^\mu > x]$, where $m \leq x$ and $m \leq 0$. Assuming that $m_0^T < m$ for the process B_t^0, then there exist a specific time point λ where

$$\lambda = \text{Inf}\{\lambda \in [0, T] : B_\lambda^0 = m\}.$$

Also since the Brownian motion consists of continuous sample paths, there must exist times when $B_t^0 < m$. Thus, we can say that the zero drift Brownian motion B_t^0 reduces to at least below the point m and equally increases up or above another point x at the terminal point T. Let $\widetilde{B}_t^0$ be the mirror reflection of the process B_t^0, and from the reflection principle above, we can have the reflected random process as

$$\widetilde{B}_t^0 = \begin{cases} B_t^0 & t < \lambda \\ 2m - B_t^0 & t \in [\lambda, T] \end{cases}.$$

Also, suppose that the process B_T^0 stops at a value which is higher than x, then the reflection path has a value which is lower than $2m - x$ at time T. Thus, we have that $B_T^0 > x = \widetilde{B}_T^0 < 2m - x$ and the reflection principle further assert that

$$\widetilde{B}_{\lambda+u}^0 - \widetilde{B}_\lambda^0 \equiv -(B_{\lambda+u}^0 - B_\lambda^0) \qquad \text{for } u > 0. \tag{4.1.1}$$

We can infer that the two Brownian increments above have the same distribution parameters, i.e, $N(0, \sigma^2 u)$ following the Markov property of Brownian motion.

4.1.3 Theorem. ***Strong Markov Property***

Let $\{B_u : u \geq 0\}$ be a Brownian motion started at $x \in \mathbb{R}^d$ and define a bounded measurable function $f : C([0, \infty], \mathbb{R}^d) \to \mathbb{R}$. Let λ be the stopping time of the process where $\lambda > 0$ and $\mathcal{F}_\lambda$ be the filtration, then the process $\{B_{\lambda+u} - B_\lambda : \lambda \geq 0\}$ is a Brownian motion started in the origin and it is independent of the process $\{B_u : 0 \leq \lambda \leq u\}$. (Mörters & Peres 2010). Alternatively,

$$\mathbb{E}_x[f(B_{\lambda+u})|\mathcal{F}_\lambda] = \mathbb{E}[f(B_\lambda)|B_\lambda] \,.$$

Note: The function $C(J)$ refers to the topological space which defines all continuous functions on $K \subset \mathbb{R}_d$ and has the norm property $||f|| = \sup_{x \in J} |f(x)|$.

If $B_T^0 > x$, then $B_T^0 < 2m - x$ and together with the extension of the reflection principle at equation (4.1.1), the joint probability distribution function can be defined as

$$\begin{aligned} \mathbb{P}[B_T^0 > x, m_0^T < m] &= \mathbb{P}[\widetilde{B}_T^0 < 2m - x] \\ &= \mathbb{P}[B_T^0 < 2m - x] \\ \\ &= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right) \qquad \text{for } m \leq \min(x, 0)\,. \end{aligned}$$

The last value follows from the definition of the cumulative standard normal distribution [1]. The joint distribution above is for Brownian motion with zero drift. Next,

[1]Recall from the definition of cumulative standard normal distribution, if Z is a random variable with parameters $N(\mu, \sigma^2)$, then

$$\mathbb{P}[Z \leq z] = N\left(\frac{z - \mu}{\sigma}\right) .$$

Thus, we see that $z = 2m - x, \mu = 0$ and $\sigma = \sigma\sqrt{T}$.

using the Girsanov change of measure, we upgrade it to have the non-zero drift, i.e $\left(r - \frac{\sigma^2}{2}\right)$. Under the measure $\mathbb{Q}$, assume B_t^μ to be a Brownian motion with constant drift μ. We aim at transforming the process B_t^μ to have a zero drift under the new measure $\widetilde{\mathbb{Q}}$, where B_t^μ becomes a Brownian motion.

4.1.4 Theorem. ***Girsanov Transform: one-dim Integral problem***

Let $B_t, t \in [0, T]$ be a Brownian motion with probability measure $\mathbb{Q}$ and let $\{\mathcal{F}_t, t \in [0, T]\}$ be the usual filtration of this Brownian motion. Define $Z(t)$ as the Radon-Nikodým derivative of the new measure $\widetilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ as: (Shreve 2004).

$$Z(t) = \exp\left\{-\int_0^t \phi_u \mathrm{d}B_u - \frac{1}{2}\int_0^t \phi_u^2 \mathrm{d}u\right\} \qquad \textit{and}$$

$$\widetilde{B}_t = B_t + \int_0^t \phi_u \mathrm{d}u\,,$$

where ϕ_u is an adapted process. Then under the measure $\widetilde{\mathbb{Q}}$, the process $\widetilde{B}_t$ is a Brownian motion.

Considering the probability distribution and applying the Girsanov transform, the following holds:

$$\begin{aligned}
\mathbb{P}[B_T^\mu > x, m_0^T < m] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\{B_T^\mu > x\}}\mathbb{I}_{\{m_T^0 < m\}}] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{B_T^\mu > x\}}\mathbb{I}_{\{m_T^0 < m\}}.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right] \\
&= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathbb{I}_{\{B_T^\mu > x\}}\mathbb{I}_{\{m_T^0 < m\}}.\exp\left(\frac{\mu B_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right]\,.
\end{aligned}$$

The indicator or step function $\mathbb{I}_{\{B_T^\mu > x\}}$ is defined as

$$\mathbb{I}_{\{B_T^\mu > x\}} = \begin{cases} 1 & \text{if} \quad B_T^\mu > x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Furthermore, applying Theorem 4.1.4 transforms the process B_T^μ from a zero drift process under $\mathbb{Q}$ to a zero drift process under $\widetilde{Q}$[2]. By reflection principle, we have

$$\begin{aligned} \mathbb{P}[B_T^\mu > x, m_0^T < m] &= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathbb{I}_{\{2m - B_T^\mu > x\}}.\exp\left(\frac{\mu(2m - B_T^\mu)}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\ &= \exp\left[\frac{2\mu m}{\sigma^2}\right]\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathbb{I}_{\{2m - B_T^\mu > x\}}.\exp\left(\frac{-\mu B_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\ &= \exp\left[\frac{2\mu m}{\sigma^2}\right]\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathbb{I}_{\{B_T^\mu < 2m - x\}}.\exp\left(\frac{-\mu B_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right]. \end{aligned}$$

Introducing the normal probability density function, consider $q \sim N(0, \sigma^2)$, we have

$$\begin{aligned} \mathbb{P}[B_T^\mu > x, m_0^T < m] &= \exp\left[\frac{2\mu m}{\sigma^2}\right]\int_{-\infty}^{2m-x}\frac{1}{\sqrt{2\pi\sigma^2 T}}\exp\left[\frac{-q^2}{2\sigma^2 T}\right].\exp\left(\frac{-\mu q}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\mathrm{d}q \\ &= \exp\left[\frac{2\mu m}{\sigma^2}\right]\int_{-\infty}^{2m-x}\frac{1}{\sqrt{2\pi\sigma^2 T}}\exp\left[\frac{-1}{2\sigma^2 T}(q^2 + 2\mu q T + \mu^2 T^2)\right]\mathrm{d}q \\ &= \exp\left[\frac{2\mu m}{\sigma^2}\right]\int_{-\infty}^{2m-x}\frac{1}{\sqrt{2\pi\sigma^2 T}}\exp\left[\frac{-1}{2\sigma^2 T}(q + \mu T)^2\right]\mathrm{d}q\,. \end{aligned}$$

[2] More details on the value of the Radon-Nikodým derivative denoted by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}$ can be found in Yue-Kuen (1998, p.205) and Baz & Chacko (2004, p.74).

This becomes

$$\mathbb{P}[B_T^\mu > x, m_0^T < m] = \exp\left[\frac{2\mu m}{\sigma^2}\right] N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right) \qquad m \leq \min(x, 0)\,.$$

Introducing the concept of total probability law, we have

$$\mathbb{P}[B_T^\mu > x, m_0^T = 0] = \mathbb{P}[B_T^\mu > x, m_0^T > m] + \mathbb{P}[B_T^\mu > x, m_0^T < m]\,.$$

Suppose we seek for the situation where $m_0^T > m$ together with $B_T^\mu > x$, we have

$$\mathbb{P}[B_T^\mu > x, m_0^T > m] = \mathbb{P}[B_T^\mu > x, m_0^T = 0] - \mathbb{P}[B_T^\mu > x, m_0^T < m] \tag{4.1.2}$$

$$= N\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) - \mathrm{e}^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right)\,. \tag{4.1.3}$$

The fixed strike lookback put option can be priced using the distribution in (4.1.3) above where m_0^T is the minimum value the underlying attained to over the period $[0, T]$, with m as the downstream barrier. Furthermore, for $x = m$, the distribution becomes

$$\mathbb{P}[B_T^\mu > m, m_0^T > m] = N\left(\frac{-m + \mu T}{\sigma\sqrt{T}}\right) - \mathrm{e}^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m + \mu T}{\sigma\sqrt{T}}\right)\,. \tag{4.1.4}$$

In valuing the fixed strike call option, we use M_0^T which is the maximum price of the underlying for the time period $[0, T]$. The joint PDF can be obtained the same way as that of the put above. Thus, we have that

$$\mathbb{P}[B_T^\mu > x, M_0^T < M] = N\left(\frac{M - \mu T}{\sigma\sqrt{T}}\right) - \mathrm{e}^{\frac{2\mu M}{\sigma^2}} N\left(\frac{-M - \mu T}{\sigma\sqrt{T}}\right)\,, \tag{4.1.5}$$

where M is an upstream barrier.

The payoff structure for the call option is given by $\max\{M_0^T - N, 0\}$, where N is the strike price and set $M = M_0^t$. Applying the risk neutral pricing measure, the value of the option is given by:

$$v(t, S, M) = \mathrm{e}^{-r\tau}\mathbb{E}_{\mathbb{Q}}[\max\{\{\max(M_0^t, M_t^T) - N\}, 0\}]\,.$$

Two conditions exist for the payoff structure:

(a) $M \leq N$: The payoff becomes $\max\{M_t^T - N, 0\}$.

(b) $M > N$: The payoff becomes $(M - N) + \max\{M_t^T - M, 0\}$.

Define a function f by

$$f(t, S; K) = \mathrm{e}^{-r\tau}\max\{M_t^T - K, 0\}\,, \qquad \text{for } K \in \mathbb{R}_+\,. \tag{4.1.6}$$

Thus, the call option value becomes

$$v(t, S, M) = \begin{cases} f(t, S; N) & M \leq N \\ \\ \mathrm{e}^{-r\tau}(M - N) + f(t, S; M) & M > N \end{cases} \tag{4.1.7}$$

From the function in (4.1.7) above, we observe that whenever $M \leq N$, the payoff does not depend on the value of M. Moreover if $M > N$, the payoff assumes the floor value of $M - N$. When this value is deducted from the call option price, we are left with $f(t, S; M)$ and this has a strike price which had been increased from K to M. Thus

$$\begin{aligned} f(t, S; M) &= \mathrm{e}^{-r\tau}\max\{M_t^T - K, 0\} \\ &= \mathrm{e}^{-r\tau}\int_0^\infty \mathbb{P}[M_t^T - K \geq b]\mathrm{d}b \qquad \text{(since the payoffs are non-negative values)}\,. \end{aligned}$$

Introducing log and the change of variables $v = K + b$, the above becomes

$$f(t, S; M) = \mathrm{e}^{-r\tau} \int_K^{\infty} \mathbb{P}\left[\log \frac{M_t^T}{S} \geq \log \frac{v}{S}\right] \mathrm{d}v$$

$$= \mathrm{e}^{-r\tau} \int_{\log \frac{K}{S}}^{\infty} \mathbb{P}\left[Y_T \geq y\right] S\mathrm{e}^{y} \mathrm{d}y\,. \tag{4.1.8}$$

The last result follows from $y = \log \frac{v}{S} \Rightarrow \mathrm{d}v = S\mathrm{e}^{y}\mathrm{d}y$ and $\log \frac{M_t^T}{S} = Y_T$.

From equation (4.1.5), we consider some change in variables. Let $M = y$, $x = u$, $U_T = B_T^{\mu}$, $Y_T = M_0^T$ and $\tau = T - t$, then the joint distribution of the non-zero drift probability function in the presence of a downstream barrier for the period $[t, T]$ is

$$\mathbb{P}[U_T \geq u, Y_T < y] = N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right). \tag{4.1.9}$$

This is equivalent to

$$\mathbb{P}[U_T \geq u, Y_T \geq y] = N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + \mathrm{e}^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right). \tag{4.1.10}$$

Thus, equation (4.1.8) can be written as:

$$f(t, S; M) = \mathrm{e}^{-r\tau} \int_{\log \frac{K}{S}}^{\infty} S\mathrm{e}^{y} \left[N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + \mathrm{e}^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right)\right] \mathrm{d}y\,.$$

Integrating yields the value of the fixed strike call option with $M \leq K$ (see Yue-Kuen (1998)). Let the value $f(t, S; K) = C$ and we have the value:

$$C = SN(d_1) - Ke^{-r\tau}N(d_2) + e^{-r\tau}\frac{S\sigma^2}{2r}\left(e^{r\tau}N(d_1) - \left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N\left(d_1 - \frac{2r\sqrt{\tau}}{\sigma}\right)\right), \tag{4.1.11}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r_1\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = \frac{\ln\left(\frac{S}{K}\right) + r_2\tau}{\sigma\sqrt{\tau}}, \tag{4.1.12}$$

$\tau = T - t$ and $r_{1,2} = r \pm \frac{\sigma^2}{2}$.

Similarly, the value for the condition when $K < M$ is given as

$$C = e^{-r\tau}(M(1 - N(d_2)) - K) + e^{-r\tau}\frac{S\sigma^2}{2r}\left(e^{r\tau}N(d_1) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}} N\left(d_1 - \frac{2r\sqrt{\tau}}{\sigma}\right)\right), \tag{4.1.13}$$

where d_1 and d_2 are the same with equations (4.1.12) but with $K = M$.

In equation (4.1.11), we observe that the value of the call option with fixed strike lookback features consists of the value of the plain vanilla call option with an extra feature. This extra feature gives the extra price incurred for exercising the option at the maximum asset price. Hence, the price of the lookback option always exceed that of the plain vanilla option.

Put: The payoff for the fixed strike put with European feature is given by

$$\max\{X - m_0^T, 0\},$$

where m_0^T is the minimum of all asset prices for the time interval $[0, T]$. Set $m_0^t = \mathbf{m}$. Using the same notion as from the valuation of the call options, the price of the fixed strike lookback put option is

$$P = Ke^{-r\tau}N(-d_2) - SN(-d_1) + e^{-r\tau}\frac{S\sigma^2}{2r}\left(-e^{r\tau}N(-d_1) + \left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N\left(-d_1 + \frac{2r\sqrt{\tau}}{\sigma}\right)\right), \tag{4.1.14}$$

where d_1 and d_2 are the same with equations (4.1.12), $\tau = T - t$ and $r_{1,2} = r \pm \frac{\sigma^2}{2}$.

Equation (4.1.14) occurs when $K \leq \mathbf{m}$. For $K > \mathbf{m}$, we have the value to be

$$P = e^{-r\tau}(K + \mathbf{m}(N(-d_2) - 1))) - SN(-d_1) + e^{-r\tau}\frac{S\sigma^2}{2r}\left(e^{r\tau}N(-d_1)\right.$$

$$\left. + \left(\frac{S}{\mathbf{m}}\right)^{\frac{-2r}{\sigma^2}} N\left(-d_1 + \frac{2r\sqrt{\tau}}{\sigma}\right)\right), \tag{4.1.15}$$

where d_1 and d_2 are the same with equations (4.1.12) but with $K = \mathbf{m}$.

4.1.5 Example. Consider a fixed strike ATM lookback option defined on a non-dividend paying underlying asset with volatility 40% per annum. The initial asset price is 100 and the option lasts for 3 months at a risk-free rate of 20% per annum.

Here, at $t = 0$, the option just originated and thus $M = S(0) = 100$ for call and $\mathbf{m} = S(0) = 100$ for put. The other parameters are $K = 100, r = 0.2, \sigma = 0.4$ and $T = 0.25$. The prices of the fixed strike call using equation (4.1.11) and the put using equation (4.1.14) are $C = 19.1676$ and $P = 12.3398$ respectively.

4.1.6 Example. Suppose the example above holds for the floating strike lookback options. Ignoring the strike price K, we have the values of the call using equation (3.2.12) and put using equation (3.2.13) as $C_1 = 17.2168$ and $P_1 = 14.2906$ respectively.

4.1.7 Example. Consider parity condition given in Section 3.2.2. We observe that

$$\begin{aligned} C &= P_1 + S_0 - K\mathrm{e}^{-rT} \\ 19.1676 &= 14.2906 + 100 - 100\mathrm{e}^{-0.2\times 0.25} \end{aligned}$$

$$\begin{aligned} P &= C_1 - S_0 + K\mathrm{e}^{-rT} \\ 12.3398 &= 17.2168 - 100 + 100\mathrm{e}^{-0.2\times 0.25} \end{aligned}$$

4.2 Down-and-Out Barrier Options

In this section, we consider the non-dividend down-and-out barrier options with European features. Recall that this option becomes worthless whenever the underlying price S reaches the barrier level B, but pays the value of the vanilla option if the barrier is not breached. Though this option is cheaper than the vanilla counterpart, its major disadvantage is that the option does not protect the holder once the underlying price falls below the barrier and rises drastically before the contract's expiration.

We consider the zero-rebate situation where the option holder receives nothing if the barrier is breached before expiry. Suppose that the barrier is not breached at an infinitesimal time step and that $B < S$ at time t, then it can be shown that the value of the down-and-out call option $v_{do}(t, S)$ satisfies the Black-Scholes PDE:

$$\frac{\partial v_{do}(t,S)}{\partial t} + rS\frac{\partial v_{do}(t,S)}{\partial S} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v_{do}(t,S)}{\partial S^2} = rv_{do}(t,S)\,. \tag{4.2.1}$$

Thus, we recall that at $B > S$, the option does not exist and hence, we consider the situation where $B < S < \infty$. The boundary conditions for the call are given by:

$$v_{do}(T,S) = \max\{S - K, 0\}\,; \tag{4.2.2}$$

$$v_{do}(t,B) = 0\,; \tag{4.2.3}$$

$$v_{do}(t,\infty) \sim S - Ke^{-r(T-t)}\,. \tag{4.2.4}$$

Equation (4.2.2) gives the payoff and (4.2.3) occurs when the barrier level is breached. Equation (4.2.4) occurs when the underlying price is sufficiently large and thus the probability of the option being knocked out is highly reduced. Also, observe that in case of a non-zero rebate option, equation (4.2.3) becomes

$$v_{do}(t,B) = R\,,$$

where R is the rebate value defined in the domain $(0, \infty)$.

The PDE in equation (4.2.1) can be solved using the hedging analysis of Black and Scholes. The equation is converted to heat equation in which the valuations of vanilla options can be related to the flow of heat in an infinite bar. This is solved by method of images where a mirror is positioned at the logarithm of the barrier (Buchen 2012).

4.2.1 Corollary. Let $\mathcal{L}v(t,S)$ denote the Black-Scholes PDE and $f(T,S)$ be the payoff function. Suppose there exists a terminal boundary value problem defined as: (Buchen 2012).

$$\begin{aligned}\mathcal{L}v(t,S) &= 0\,, \qquad \text{for}\\ v(T,S) &= f(T,S)\,,\\ v(t,B) &= 0\,,\end{aligned}$$

then the method of images can be used to solve this problem in relation to the standard European options.

4.2.2 Corollary. Let $v(t,S)$ be the value of a European option, then the image of the function $v(t,S)$ with respect to $S=B$ and the Black-Scholes differential operator is given by: (Buchen 2012).

$$v^*(t,S) = \left(\frac{B}{S}\right)^{\alpha} v\left(t, \frac{B^2}{S}\right)\,,$$

where $\alpha = \frac{2r}{\sigma^2} - 1$.

4.2.3 Theorem. *Let $v(t,S)$ be the solution of the terminal value problem described as: (Buchen 2012).*

$$\begin{aligned}\mathcal{L}v(t,S) &= 0\,,\\ v(T,S) &= f(T,S)\mathbb{I}(S>B)\,,\end{aligned}$$

in the domain $\{S>0, t<T\}$. Then $v_{do} = v(t,S) - v^*(t,S)$ solves the terminal boundary value problem for the down and out barrier option in the domain $\{S >$

$B, t < T\}$.

Thus, we can have that

$$v_{do} = SN(d_1) - Ke^{-r(T-t)}N(d_2) - \left(\frac{B}{S}\right)^{\alpha}\left[\frac{B^2}{S}N(y) - Ke^{-r(T-t)}N(y - \sigma\sqrt{T-t})\right], \tag{4.2.5}$$

where d_1, d_2 follows from Appendix A in equations (A.0.11) and (A.0.12) respectively, and also where

$$y = \frac{\log\left(\frac{B^2}{SK}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{4.2.6}$$

4.2.4 Valuation of down-and-out barrier options

In pricing the down-and-out barrier call options, the corresponding Black-Scholes equation (4.2.1) is reduced to heat equation, with the additional feature of the option being knocked out if the barrier is triggered. To reduce to heat equation, we consider some change of variables.

Let

$$S = Be^x \implies x = \ln\left(\frac{S}{B}\right) \quad \text{and} \quad \tau = \frac{(T-t)\sigma^2}{2}.$$

Taking the derivatives, we have,

$$\frac{\partial v_{do}}{\partial t} = \frac{\partial v_{do}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{-\sigma^2}{2}\frac{\partial v_{do}}{\partial \tau}$$

$$\frac{\partial v_{do}}{\partial x} = \frac{\partial v_{do}}{\partial S}\frac{\partial S}{\partial x} = S\frac{\partial v_{do}}{\partial S}$$

$$\frac{\partial^2 v_{do}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v_{do}}{\partial x}\right) = S^2 \frac{\partial^2 v_{do}}{\partial S^2} + S\frac{\partial v_{do}}{\partial S}\,.$$

Re-arranging and substituting into the PDE in equation (4.2.1), we have

$$\frac{-\sigma^2}{2}\frac{\partial v_{do}}{\partial \tau} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial v_{do}}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v_{do}}{\partial x^2} = r v_{do}\,, \tag{4.2.7}$$

with the corresponding boundary conditions in equations (4.2.2) and (4.2.3) to be

$$v_{do}(0, x) = \max\{B\mathrm{e}^x - K, 0\} \tag{4.2.8}$$

$$v_{do}(\tau, 0) = 0\,. \tag{4.2.9}$$

Also, let

$$v_{do}(\tau, x) = B\mathrm{e}^{\alpha x + \beta\tau} u(\tau, x)\,. \tag{4.2.10}$$

Taking derivatives, we have,

$$\frac{\partial v_{do}}{\partial \tau} = B\mathrm{e}^{\alpha x + \beta\tau}\left[\frac{\partial u}{\partial \tau} + \beta u\right]$$

$$\frac{\partial v_{do}}{\partial x} = B\mathrm{e}^{\alpha x + \beta\tau}\left[\frac{\partial u}{\partial x} + \alpha u\right]$$

$$\frac{\partial v_{do}^2}{\partial x^2} = B\mathrm{e}^{\alpha x + \beta\tau}\left[\frac{\partial^2 u}{\partial x^2} + 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u\right]\,.$$

Re-arranging and substituting into the PDE at equation (4.2.7), we have

$$\frac{-\sigma^2}{2}\frac{\partial u}{\partial \tau} + \left[r - \frac{\sigma^2}{2} + \alpha\sigma^2\right]\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left[\frac{\sigma^2}{2}\left(\alpha^2 - \alpha - \beta\right) + r(\alpha - 1)\right] u = 0\,. \tag{4.2.11}$$

We seek to eliminate the u and the $\frac{\partial u}{\partial x}$ term and thus we equate their coefficients to zero. We have the following values for α and β,

$$\alpha = \frac{1}{2} - \frac{r}{\sigma^2} \quad \text{and} \quad \beta = \frac{-r}{\sigma^2} - \frac{r^2}{\sigma^4} - \frac{1}{4}\,.$$

The resulting PDE reduces to a heat equation which describes the problem of heat flow in an infinite bar. Thus, we have

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}\,, \quad \text{for} \quad x \in (0, \infty), \tau > 0 \tag{4.2.12}$$

and the boundary conditions [3]

$$\begin{aligned} u(0,x) &= \max\{e^{x-\alpha x} - \frac{K}{B}e^{-\alpha x}, 0\} \\ u(\tau, 0) &= 0\,. \end{aligned}$$

Solving the above problem involves not only solving for the positive real parts but for all x. This was adopted owing to the invariant nature of the heat equation under reflection and thus the solution involves $u(\tau, x)$ and $u(\tau, -x)$. Hence, we have

$$u(0,x) = \begin{cases} \max\{e^{x-\alpha x} - \frac{K}{B}e^{-\alpha x}, 0\} & x > 0 \\ \\ -\max\{e^{\alpha x - x} - \frac{K}{B}e^{\alpha x}, 0\} & x < 0 \end{cases}\,.$$

We now proceed to obtaining the price of the zero rebate knock-out barrier options, specifically the call value of DOBO. Assume that $v(t,S)$ is the price of the vanilla European call options with the same strike price and expiry time, without barrier. Also, assume that $U(\tau, x)$ is the solution to the heat equation defined in equation (4.2.12). Consider also the following:

If $S < K$, then $v(T,S) = 0$. This is due to the fact that the option is OTM.

[3] Boundary conditions deduced from equation (4.2.10)

Also $U(\tau, x) = 0$ since $S = Be^x$ when $x < \ln\left(\frac{K}{B}\right)$. We equally have that $\ln\left(\frac{K}{B}\right) > 0$ since the strike price K is higher than the barrier. Else, the option is knocked out. Suppose for $x < 0$, we let $u(0, x) = 0$. Then we consider the function $u(0, x)$ for all x and it is observed that $u(0, x) = U(\tau, x)$. Thus we have,

$$u(0, x) = U(\tau, x) - U(\tau, -x)\,,$$

and for all x,

$$u(\tau, x) = U(\tau, x) - U(\tau, -x)\,.$$

Then the value of the plain vanilla European call can be written as

$$v(t, S) = v(t(\tau), Be^x) = Be^{\alpha x + \beta\tau} U(\tau, x)\,. \tag{4.2.13}$$

From equation (4.2.13), we can deduce that

$$U(\tau, x) = \frac{e^{-\alpha x - \beta\tau} v(t(\tau), Be^x)}{B} \quad \text{and}$$

$$U(\tau, -x) = \frac{e^{\alpha x - \beta\tau} v(t(\tau), Be^{-x})}{B}\,.$$

Recall the option value is given in equation (4.2.10) as

$$\begin{aligned}
v_{do}(t, S) &= Be^{\alpha x + \beta\tau} u(\tau, x) \\
&= Be^{\alpha x + \beta\tau} \left[U(\tau, x) - U(\tau, -x)\right] \\
&= Be^{\alpha x + \beta\tau}.e^{-\alpha x - \beta\tau} \left[\frac{1}{B}\left(v(t(\tau), Be^x) - e^{2\alpha x} v(t(\tau), Be^{-x})\right)\right] \\
&= v(t(\tau), Be^x) - e^{2\alpha x} v(t(\tau), Be^{-x})
\end{aligned}$$

$$v_{do}(t,S) = v(t,S) - \left(\frac{S}{B}\right)^{2\alpha} v\left(t, \frac{B^2}{S}\right).$$

This is consistent with the value found in Theorem 4.2.3. The DIBO counterpart for the call can be obtained using the input-parity found in Subsection 3.3.2.

4.3 Up-and-Out Barrier Options

For the call feature of the DOBO, the risk neutral valuation is given as

$$v_{do} = \mathrm{e}^{-r\tau}\mathbb{E}_Q[(S(T)-K)\mathbb{I}_{\{K<S(T)<B\}}\mathbb{I}_{\{m_0^T>B\}}].$$

This can be re-written as

$$v_{do} = \mathrm{e}^{-r\tau}\int_{\ln\frac{K}{S}}^{\infty}(S\mathrm{e}^x - K)f_d(x,B,\tau)\mathrm{d}x, \tag{4.3.1}$$

where $f_d(x,m,\tau)$ refers to the joint probability density function of the Brownian motion B_T^μ (non-zero drift process). The Brownian process has a downstream barrier m such that $m \leq \min(x,0)$.

Define $m_0^T = \min S(u)$, where $u \in [0,t]$, then we have the function:

$$f_d(x,m,\tau)\mathrm{d}x = \mathbb{P}(B_T^\mu \in \mathrm{d}x, m_0^T > m).$$

The probability density function is obtained by taking the derivative of the equation (4.1.3) with respect to x. This gives the density function of the log-normal return on the underlying asset provided that the barrier level is not triggered. Thus, we have

$$f_d(x,m,\tau) = \frac{1}{\sigma\sqrt{\tau}}\left[N\left(\frac{x-\mu\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{\frac{2\mu m}{\sigma^2}}N\left(\frac{x-2m-\mu\tau}{\sigma\sqrt{\tau}}\right)\right].$$

Evaluating the integral in equation (4.3.1), the value for the down-and-out call for $B < K$ is written as follows:

$$v_{do} = \mathrm{e}^{-r\tau} \int_{\ln \frac{K}{S}}^{\infty} (S\mathrm{e}^{x} - K) f_d(x, B, T) \mathrm{d}x = v(t, S; K) - \left(\frac{B}{S}\right)^{2\alpha} v\left(t, \frac{B^2}{S}; K\right), \tag{4.3.2}$$

where v is the price of the plain vanilla call option and $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$.

Let v_{uo} denote the price of the up-and-out barrier call defined in the domain below:

$$D = \{(t, S) : 0 \leq S \leq B,\ t \in [0, T]\}.$$

The boundary conditions exist:

$$v_{uo}(T, S) = v(T, S) = \max\{S(T) - K, 0\} \tag{4.3.3}$$

$$v_{uo}(t, B) = 0 \tag{4.3.4}$$

$$v_{uo}(t, \infty) = 0 \tag{4.3.5}$$

We observe that the strike price is normally positioned below the barrier. But suppose that it is fixed above the barrier level, the option becomes worthless by the time it hits the barrier. Also, the maximum value of the underlying asset can attain is assumed to be below the barrier, B. Thus, the chances of the option expiring in-the-money become negligible.

Using the risk-neutral valuation, the value of the zero rebate up-and-out call option is given as

$$v_{uo} = \mathrm{e}^{-r\tau} \mathbb{E}_Q[(S(T) - K)\mathbb{I}_{\{K < S(T) < B\}} \mathbb{I}_{\{M_0^T < B\}}],$$

which can be written as

$$v_{uo} = \mathrm{e}^{-r\tau} \int_{\ln \frac{K}{S}}^{\ln \frac{B}{S}} (S\mathrm{e}^{y} - K) f_u(y, M, \tau) \mathrm{d}y, \tag{4.3.6}$$

where $f_u(y, M, \tau)$ refers to the joint probability density function of the Brownian motion B_T^μ (non-zero drift process) defined with an upstream barrier M, for $M > \max(y, 0)$. Define $M_0^T = \max S(u)$, for $u \in [0, t]$, then we have the function

$$f_u(y, M, \tau)\mathrm{d}y = \mathbb{P}(B_T^\mu \in \mathrm{d}y, M_0^T < M)\,.$$

The analytical form of f_u is the same as f_d and it is given as

$$f_u(y, M, \tau) = \frac{1}{\sigma\sqrt{\tau}}\left[N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu\tau}{\sigma\sqrt{\tau}}\right)\right].$$

From equation (4.3.6), we have that the price of an up-and-out call is

$$\begin{aligned} v_{uo} =& \mathrm{e}^{-r\tau}\int_{\ln\frac{K}{S}}^{\ln\frac{B}{S}} (S\mathrm{e}^y - K) f_u(y, M, \tau)\mathrm{d}y \\ =& \mathrm{e}^{-r\tau}\left[\int_{\ln\frac{K}{S}}^{\infty} (S\mathrm{e}^y - K) f_u(y, B, \tau)\mathrm{d}y - \int_{\ln\frac{B}{S}}^{\infty} (S\mathrm{e}^y - K) f_u(y, B, \tau)\mathrm{d}y\right]. \end{aligned}$$

According to Yue-Kuen (1998), we have that since the functions f_d and f_u have the same analytical function, then the following result holds:

$$v_{uo} = \mathrm{e}^{-r\tau}\left[\int_{\ln\frac{K}{S}}^{\infty} (S\mathrm{e}^x - K) f_d(x, B, \tau)\mathrm{d}x - \int_{\ln\frac{B}{S}}^{\infty} (S\mathrm{e}^x - K) f_u(x, B, \tau)\mathrm{d}x\right].$$

Thus, we have from equation (4.3.2) that

$$v_{uo} = v(t, S; K) - \left(\frac{B}{S}\right)^{2\alpha} v\left(t, \frac{B^2}{S}; K\right) - \left[v(t, S; B) - \left(\frac{B}{S}\right)^{2\alpha} v\left(t, \frac{B^2}{S}; B\right)\right], \tag{4.3.7}$$

where v is the value of the plain vanilla call options and $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$.

5. Numerical Approximations

In pricing derivatives using numerical methods, the PDE approach (finite difference method), the binomial or the trinomial approach and the Monte-Carlo simulation are basically employed (Brandimarte 2013). This chapter considers the finite difference approximations and the Monte-Carlo simulations. For the finite difference approximations, we discuss the explicit, the implicit and the Crank-Nicolson method. But in the numerical valuation of the options, we implement only the Crank-Nicolson method, owing to the fact that it is more accurate when compared to the other two (see Appendix B).

5.1 Finite Difference Methods

The finite difference methods (FDM) for solving the Black-Scholes PDE which describe the exotic option pricing involves solving the associated PDE on a discrete space-time grid. The computational domain is $[0, \mathrm{S}_{max}] \times [0, T]$ and this domain is discretized by the uniform asset and time mesh with steps ΔS and ΔT.

Consider Figure 5.1 below:

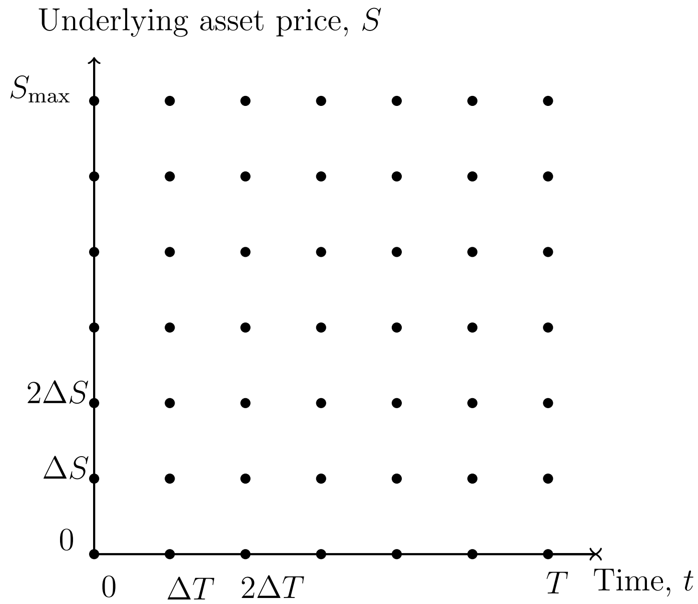

Figure 5.1: Asset-time discretization

The payoff at time T is known and hence the solution involves applying the concept of backward iteration on the square or rectangular grid up till time $t = 0$. With regards to the Black-Scholes formula, the option price is a function of the underlying price and time.

Consider also the discretizations below

$$
\begin{aligned}
&S = 0, \Delta S, 2\Delta S, \cdots, (m-1)\Delta S, m\Delta S = S_{\max} \quad \text{and} \\
&T = 0, \Delta T, \Delta T, \cdots, (n-1)\Delta T, n\Delta T = T
\end{aligned}
$$

The option price $v(T, S)$ can be denoted in grid form by $v_{i,k} = v(i\Delta t, k\Delta S)$, where $k = 0, 1, 2, \cdots, m$ and $i = 0, 1, \cdots, n$. Let $S_{\max}$ be the largest value that the underlying can possibly have. The corresponding terminal and boundary conditions of the PDE which give the values of the option prices at time $t = T$, $S = 0$ and $S = S_{\max}$ are known. Thus, it suffices to use the known values at the extreme end of the nodes to calculate the values for the other interior nodes.

Terminal and Boundary conditions

Zero-rebate knock out barrier options: The terminal and boundary conditions of the down-and-out barrier call options represented in equations (4.2.2), (4.2.3) and (4.2.4) can be written in discrete form as follows:

$$v_{n,k} = \max\{k\Delta S - K, 0\}\,, \tag{5.1.1}$$

$$v_{i,0} = 0\,, \tag{5.1.2}$$

$$v_{i,m\Delta S} = m\Delta S - Ke^{-r(n-i)\Delta T}\,. \tag{5.1.3}$$

The following finite difference methods can be employed in the approximation of the option pricing PDE:

(i) Forward difference: In time and in the underlying are given respectively as:

$$\frac{\partial v}{\partial t} = \frac{v_{i+1,k} - v_{i,k}}{\Delta T} \quad \text{and} \quad \frac{\partial v}{\partial S} = \frac{v_{i,k+1} - v_{i,k}}{\Delta S}\,.$$

(ii) Backward difference: In time and in the underlying are given respectively as:

$$\frac{\partial v}{\partial t} = \frac{v_{i,k} - v_{i-1,k}}{\Delta T} \quad \text{and} \quad \frac{\partial v}{\partial S} = \frac{v_{i,k} - v_{i,k-1}}{\Delta S}\,.$$

(iii) Central difference: In time and in the underlying are given respectively as:

$$\frac{\partial v}{\partial t} = \frac{v_{i+1,k} - v_{i-1,k}}{2\Delta T} \quad \text{and} \quad \frac{\partial v}{\partial S} = \frac{v_{i,k+1} - v_{i,k-1}}{2\Delta S}\,.$$

(iv) Second derivative with respect to the underlying:

$$\frac{\partial^2 v}{\partial S^2} = \frac{v_{i,k+1} - 2v_{i,k} + v_{i,k-1}}{\Delta S^2}\,.$$

5.1.1 Implicit FDM

Consider again the Black-Scholes PDE defined below:

$$\frac{\partial v}{\partial t} + rS\frac{\partial v}{\partial S} + \frac{S^2\sigma^2}{2}\frac{\partial^2 v}{\partial S^2} = rv\,. \tag{5.1.4}$$

The implicit FDM considers forward difference in time. For the first derivative with respect to the underlying, it considers the central difference approximation. For the second derivative with respect to the underlying, it considers the standard approximation. Hence, we have the following numerical approximations:

$$\frac{\partial v}{\partial t} = \frac{v_{i+1,k} - v_{i,k}}{\Delta T}$$

$$\frac{\partial v}{\partial S} = \frac{v_{i,k+1} - v_{i,k-1}}{2\Delta S}$$

$$\frac{\partial^2 v}{\partial S^2} = \frac{v_{i,k+1} - 2v_{i,k} + v_{i,k-1}}{\Delta S^2}\,.$$

Substitute into equation (5.1.4), we have

$$\frac{v_{i+1,k} - v_{i,k}}{\Delta T} + rk\Delta S\left[\frac{v_{i,k+1} - v_{i,k-1}}{2\Delta S}\right] + \frac{(\sigma k\Delta S)^2}{2}\left[\frac{v_{i,k+1} - 2v_{i,k} + v_{i,k-1}}{\Delta S^2}\right] = rv_{i,k}\,. \tag{5.1.5}$$

Rearranging the above gives

$$v_{i+1,k} = v_{i,k-1}\left[\frac{rk\Delta T}{2} - \frac{\sigma^2k^2\Delta T}{2}\right] + v_{i,k}\left[1 + \Delta T(r + \sigma^2k^2)\right] + v_{i,k+1}\left[\frac{-rk\Delta T}{2} - \frac{\sigma^2k^2\Delta T}{2}\right]\,.$$

$$v_{i+1,k} = A_k^* v_{i,k-1} + B_k^* v_{i,k} + C_k^* v_{i,k+1}\,, \tag{5.1.6}$$

where

$$A_k^* = \frac{\Delta T}{2}[rk - \sigma^2 k^2], \qquad B_k^* = [1 + \Delta T(r + \sigma^2 k^2)] \qquad \text{and} \qquad C_k^* = \frac{-\Delta T}{2}[rk + \sigma^2 k^2],$$

for $i = n-1, n-2, \cdots, 1, 0$ and $k = 1, 2, \cdots, m-1$.

Equation (5.1.6) when expanded yields

$$\begin{aligned}
v_{i+1,1} &= A_1^* v_{i,0} + B_1^* v_{i,1} + C_1^* v_{i,2}\\
v_{i+1,2} &= A_2^* v_{i,1} + B_2^* v_{i,2} + C_2^* v_{i,3}\\
v_{i+1,3} &= A_3^* v_{i,2} + B_3^* v_{i,3} + C_3^* v_{i,4}\\
&\vdots\\
v_{i+1,m-2} &= A_{m-2}^* v_{i,m-3} + B_{m-2}^* v_{i,m-2} + C_{m-2}^* v_{i,m-1}\\
v_{i+1,m-1} &= A_{m-1}^* v_{i,m-2} + B_{m-1}^* v_{i,m-1} + C_{m-1}^* v_{i,m}
\end{aligned}$$

It can be expressed further in form of matrix notation as

$$\begin{bmatrix} v_{i+1,1}\\ v_{i+1,2}\\ v_{i+1,3}\\ \\ \\ v_{i+1,m-2}\\ v_{i+1,m-1}\end{bmatrix} = \begin{bmatrix} B_1^* & C_1^* & & & & \\ A_2^* & B_2^* & C_2^* & & & \\ & A_3^* & B_3^* & C_3^* & & \\ & & & & & \\ & & & & & \\ & & & A_{m-2}^* & B_{m-2}^* & C_{m-2}^*\\ & & & & A_{m-1}^* & B_{m-1}^*\end{bmatrix} \begin{bmatrix} v_{i,1}\\ v_{i,2}\\ v_{i,3}\\ \\ \\ v_{i,m-2}\\ v_{i,m-1}\end{bmatrix} + \begin{bmatrix} A_1^* v_{i,0}\\ \\ \\ \vdots\\ \\ \\ C_{m-1}^* v_{i,m}\end{bmatrix}$$

In compact form, we can re-write the system of equations above as

$$v_{i+1,k} = D^* v_{i,k} + F_{i,k}^*. \tag{5.1.7}$$

The values $v_{i,k}$ are implicitly immersed in equation (5.1.7), where D^* is an $m-1$ tridiagonal matrix. The inversion of D^* is one possible method to solve the system above. The values at time $t = i+1 = n\Delta T$, together with the values at nodes $i, 0$ and i, m are known from the boundary conditions (5.1.1, 5.1.2 and 5.1.3). The solution is finally obtained using backward iteration. Figure 5.2 below describes the implicit method (Hull 2006).

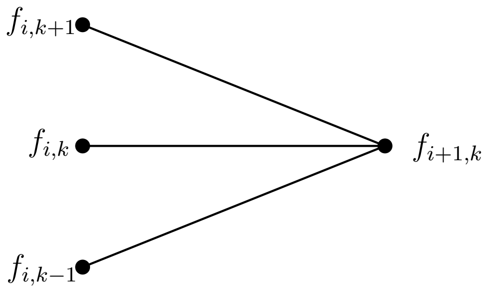

Figure 5.2: Implicit finite difference discretization

5.1.2 Explicit FDM

The explicit FDM considers the backward difference in time. For the first derivative with respect to the underlying, it considers the central difference. For the second derivatives with respect to the underlying, it considers the standard approximations. The approximations of the underlying at node (i, k) and the node $(i+1, k)$ are assumed to be the same (Hull 2006). The numerical approximations using the explicit FDM are given below:

$$\frac{\partial v}{\partial t} = \frac{v_{i+1,k} - v_{i,k}}{\Delta T}$$

$$\frac{\partial v}{\partial S} = \frac{v_{i+1,k+1} - v_{i+1,k-1}}{2\Delta S}$$

$$\frac{\partial^2 v}{\partial S^2} = \frac{v_{i+1,k+1} - 2v_{i+1,k} + v_{i+1,k-1}}{\Delta S^2}$$

Substitute into equation (5.1.4), we have

$$\frac{v_{i+1,k} - v_{i,k}}{\Delta T} + rk\Delta S\left[\frac{v_{i+1,k+1} - v_{i+1,k-1}}{2\Delta S}\right] + \frac{(\sigma k \Delta S)^2}{2}\left[\frac{v_{i+1,k+1} - 2v_{i+1,k} + v_{i+1,k-1}}{\Delta S^2}\right] = rv_{i,k}\,. \tag{5.1.8}$$

Rearranging equation (5.1.8) above gives

$$v_{i,k} = \frac{1}{1 + r\Delta T}[A_k v_{i+1,k-1} + B_k v_{i+1,k} + C_k v_{i+1,k+1}]\,, \tag{5.1.9}$$

where

$$A_k = \frac{\Delta T}{2(1 + r\Delta T)}[-rk + \sigma^2 k^2], \quad B_k = \frac{1}{1 + r\Delta T}[1 - \sigma^2 k^2 \Delta T] \quad \text{and} \quad C_k = \frac{\Delta T}{2(1 + r\Delta T)}[rk + \sigma^2 k^2]\,,$$

for $i = n-1, n-2, \cdots, 1, 0$ and $k = 1, 2, \cdots, m-1$.

After discretization, the next step is to apply the terminal and boundary conditions. This is dependent on the class of options being considered. Expanding equation (5.1.9) further yields:

$$\begin{aligned}
v_{i,1} &= A_1 v_{i+1,0} + B_1 v_{i+1,1} + C_1 v_{i+1,2} \\
v_{i,2} &= A_2 v_{i+1,1} + B_2 v_{i+1,2} + C_2 v_{i+1,3} \\
v_{i,3} &= A_3 v_{i+1,2} + B_3 v_{i+1,3} + C_3 v_{i+1,4} \\
&\vdots \\
v_{i,m-2} &= A_{m-2} v_{i+1,m-3} + B_{m-2} v_{i+1,m-2} + C_{m-2} v_{i+1,m-1} \\
v_{i,m-1} &= A_{m-1} v_{i+1,m-2} + B_{m-1} v_{i+1,m-1} + C_{m-1} v_{i+1,m}
\end{aligned}$$

The above can be expressed in form of matrix notation as

$$\begin{bmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \\ \\ v_{i,m-2} \\ v_{i,m-1} \end{bmatrix} = \begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & A_3 & B_3 & C_3 & & & \\ & & & & & & \\ & & & & & & \\ & & & & A_{m-2} & B_{m-2} & C_{m-2} \\ & & & & & A_{m-1} & B_{m-1} \end{bmatrix} \begin{bmatrix} v_{i+1,1} \\ v_{i+1,2} \\ v_{i+1,3} \\ \\ \\ v_{i+1,m-2} \\ v_{i+1,m-1} \end{bmatrix} + \begin{bmatrix} A_1 v_{i+1,0} \\ \\ \\ \vdots \\ \\ \\ C_{m-1} v_{i+1,m} \end{bmatrix}$$

Thus for $k = 1, 2, \cdots, m-1$, the option value $v_{i,k}$ is expressed explicitly as

$$v_{i,k} = Dv_{i+1,k} + F_{i+1,k} \, ,$$

where F is the column vector with values at nodes $v_{i+1,m}$ and $v_{i+1,0}$ known from their boundary conditions. The solution is obtained via backward iteration (using $i = n-1, n-2, \cdots, 1, 0$) to get the present value of the option value. The implementation of the computer algorithm is straightforward and the stability is conditional but if

$$\frac{\Delta T}{(\Delta S)^2} \leq \frac{1}{2} \, ,$$

then the solution converges. Also for accuracy, the step size of the time has to be reduced by a factor of 4 (Wilmott, Howison & Dewynne 1995, p.145). Figure 5.3 below describes the explicit method (Hull 2006).

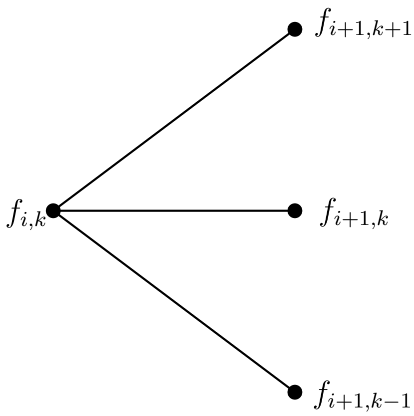

Figure 5.3: Explicit Finite difference discretization

5.1.3 Crank-Nicolson FDM

The Crank-Nicolson method was introduced to curb the instability, as well as to increase the efficiency and the accuracy of the implicit and the explicit method. This is achieved by combining and averaging the implicit and the explicit method, using the same boundary conditions. Consider Figure 5.4 below which describes the discretization using the Crank-Nicolson method:

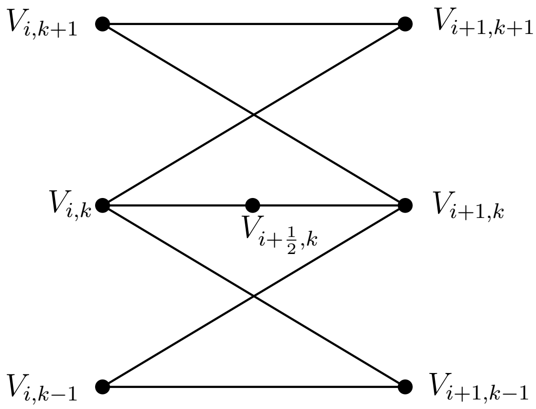

Figure 5.4: Crank Nicolson discretization

The implicit FDM can be written as

$$v_{i,k}-v_{i-1,k}+\frac{rk\Delta T}{2}[v_{i-1,k+1}-v_{i-1,k-1}]+\frac{\sigma^2k^2\Delta T}{2}[v_{i-1,k+1}-2v_{i-1,k}+v_{i-1,k-1}] = r\Delta T v_{i-1,k}\,. \tag{5.1.10}$$

The explicit FDM can be written as

$$v_{i,k}-v_{i-1,k}+\frac{rk\Delta T}{2}[v_{i,k+1}-v_{i,k-1}]+\frac{\sigma^2k^2\Delta T}{2}[v_{i,k+1}-2v_{i,k}+v_{i,k-1}] = r\Delta T v_{i,k}\,. \tag{5.1.11}$$

Taking the average and re-arranging, we have

$$\begin{aligned} &v_{i-1,k-1}\left[\frac{-rk\Delta T}{4}+\frac{\sigma^2k^2\Delta T}{4}\right]+v_{i-1,k}\left[-1-\frac{\Delta T}{2}(\sigma^2k^2+r)\right]+v_{i-1,k+1}\left[\frac{rk\Delta T}{4}+\frac{\sigma^2k^2\Delta T}{4}\right] \\ =&v_{i,k-1}\left[\frac{rk\Delta T}{4}-\frac{\sigma^2k^2\Delta T}{4}\right]+v_{i,k}\left[-1+\frac{\Delta T}{2}(\sigma^2k^2+r)\right]+v_{i,k+1}\left[\frac{-rk\Delta T}{4}-\frac{\sigma^2k^2\Delta T}{4}\right] \end{aligned} \tag{5.1.12}$$

Equation (5.1.12) can be written as

$$-\lambda_k v_{i-1,k-1}+(-1-\beta_k)v_{i-1,k}-\eta_k v_{i-1,k+1} = \lambda_k v_{i,k-1}+(-1+\beta_k)v_{i,k}+\eta_k v_{i,k+1} \tag{5.1.13}$$

for $i = n-1, n-2, \cdots, 1, 0$ and $k = 1, 2, \cdots, m-1$, where

$$\lambda_k = \frac{\Delta T}{4}[rk-\sigma^2k^2], \qquad \beta_k = \frac{\Delta T}{2}(\sigma^2k^2+r) \qquad \text{and} \qquad \eta_k = \frac{-\Delta T}{4}[rk+\sigma^2k^2]\,.$$

Let $\omega_k = 1 + \beta_k$ and $\zeta_k = -1 + \beta_k$. Then equation (5.1.13) when expanded yields

$$
\begin{aligned}
-\lambda_1 v_{i-1,0} - \omega_1 v_{i-1,1} - \eta_1 v_{i-1,2} &= \lambda_1 v_{i,0} + \zeta_1 v_{i,1} + \eta_1 v_{i,2} \\
-\lambda_2 v_{i-1,1} - \omega_2 v_{i-1,2} - \eta_2 v_{i-1,3} &= \lambda_2 v_{i,1} + \zeta_2 v_{i,2} + \eta_2 v_{i,3} \\
-\lambda_3 v_{i-1,2} - \omega_3 v_{i-1,3} - \eta_3 v_{i-1,4} &= \lambda_3 v_{i,2} + \zeta_3 v_{i,3} + \eta_3 v_{i,4} \\
&\vdots \\
-\lambda_{m-2} v_{i-1,m-3} - \omega_{m-2} v_{i-1,m-2} - \eta_{m-2} v_{i-1,m-1} &= \lambda_{m-2} v_{i,m-3} + \zeta_{m-2} v_{i,m-2} + \eta_{m-2} v_{i,m-1} \\
-\lambda_{m-1} v_{i-1,m-2} - \omega_{m-1} v_{i-1,m-1} - \eta_{m-1} v_{i-1,m} &= \lambda_{m-1} v_{i,m-2} + \zeta_{m-1} v_{i,m-1} + \eta_{m-1} v_{i,m} \,.
\end{aligned}
$$

It can be expressed in matrix form as

$$
\begin{bmatrix}
-\omega_1 & -\eta_1 & & & & \\
-\lambda_2 & -\omega_2 & -\eta_2 & & & \\
 & -\lambda_3 & -\omega_3 & -\eta_3 & & \\
 & & & & & \\
 & & & -\lambda_{m-2} & -\omega_{m-2} & -\eta_{m-2} \\
 & & & & -\lambda_{m-1} & -\omega_{m-1}
\end{bmatrix}
\begin{bmatrix}
v_{i-1,1} \\ v_{i-1,2} \\ v_{i-1,3} \\ \\ v_{i-1,m-2} \\ v_{i-1,m-1}
\end{bmatrix}
+
\begin{bmatrix}
-\lambda_1 v_{i-1,0} \\ \\ \vdots \\ \\ -\eta_{m-1} v_{i-1,m}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
\zeta_1 & \eta_1 & & & & \\
\lambda_2 & \zeta_2 & \eta_2 & & & \\
 & \lambda_3 & \zeta_3 & \eta_3 & & \\
 & & & & & \\
 & & & \lambda_{m-2} & \zeta_{m-2} & \eta_{m-2} \\
 & & & & \lambda_{m-1} & \zeta_{m-1}
\end{bmatrix}
\begin{bmatrix}
v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \\ v_{i,m-2} \\ v_{i,m-1}
\end{bmatrix}
+
\begin{bmatrix}
\lambda_1 v_{i,0} \\ \\ \vdots \\ \\ \eta_{m-1} v_{i,m}
\end{bmatrix}
$$

Representing the matrix in a more compact form gives

$$
A v_{i-1,k} = B v_{i,k} + F \,,
$$

where $F = [\lambda_1(v_{i,0} + v_{i-1,0}), \cdots, \eta_{m-1}(v_{i,m} + v_{i-1,m})]^T$.

The Crank-Nicolson method of option pricing can be solved the same way as implicit method. Both involve the inversion of the $m-1$ diagonal matrix. The option values are obtained via iteration. The Crank-Nicolson method provides the best approximate value in comparison to other finite difference methods. (See Appendix B).

5.2 Monte-Carlo Simulations

Monte-Carlo simulation (MCS) is a statistical estimation method which is based on the generation of random numbers. In this work, we use the computer program (`ipython notebook`) which contains an in-built function that is capable of generating normally distributed random numbers. Running the program severally results to different values and this is due to the presence of the random terms there. However, quite a large number of simulations are essential to obtaining fairly accurate, if not accurate results and this is a major drawback in using the method. The attractiveness of the MCS over other numerical methods owes to the fact that its implementation is very flexible and easy.

5.2.1 Basics of Monte-Carlo simulations

Let $P(X)$ be some arbitrary function and ω be a fixed parameter that needs to be estimated. Define

$$\omega = \mathbb{E}[P(X)] .$$

From the probability density function of $P(X)$, we can generate n independent random values $P_1, P_2, \cdots, P_n$. The estimator of ω is then given by

$$\hat{\omega} = \frac{1}{n}\sum_{i=1}^{n} P(X_i)\,.$$

5.2.2 Theorem. ***Law of Large numbers***

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed random variables (iidrv) with finite mean μ and finite variance σ^2. Define $X = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. Then for any $\epsilon > 0$, we have: (Feller 1968).

$$\mathbb{P}[|X-\mu| \geq \epsilon] \longrightarrow 0 \quad \textit{as} \quad n \to \infty\,.$$

Equivalently,

$$\mathbb{P}[|X-\mu| < \epsilon] \longrightarrow 1 \quad \textit{as} \quad n \to \infty\,.$$

5.2.3 Theorem. ***Central Limit Theorem***

Let $X_1, X_2, \cdots, X_n$ be a sequence of iidrv with parameters $\mu-$ mean and $\sigma-$ variance. Then the central limit theorem states that: (Feller 1968).

$$\textit{sample mean:} \qquad \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \longrightarrow \mathcal{N}(0,1) \quad \textit{as} \quad n \to \infty\,,$$

$$\textit{sample sum:} \qquad \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}} \longrightarrow \mathcal{N}(0,1) \quad \textit{as} \quad n \to \infty\,.$$

Hence applying the law of large numbers, the following ensues:

$$\omega \to \hat{\omega} \quad \text{or} \quad \mathbb{E}[P(X)] \to \frac{1}{n}\sum_{i=1}^{n} P(X_i) \quad \text{as} \quad n \to \infty\,.$$

The sample variance is given by

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}[P(X_i) - \hat{\omega}]^2 .$$

The central limit theorem (CLT) equally ensure that

$$\frac{\hat{\omega} - \omega}{\sigma/\sqrt{n}} \to \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty .$$

Also for large n, we have that

$$\mathbb{P}\left[\hat{\omega} - Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq \omega \leq \hat{\omega} + Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right] \approx 1 - \alpha$$

where α is the significance or probability level.

5.2.4 Option pricing using Monte-Carlo methods

The MCS had proved to be a promising numerical method for pricing complex derivative structures, especially when dealing with multi-dimensional option pricing. Boyle, Broadie & Glasserman (1997) first used MCS to price the European options under the assumptions of Black and Scholes. In order to price options using the MCS, we first convert the continuous time process of the extended Black-Scholes model to discrete time step. The MCS of option pricing under the Black-Scholes framework involves the generation of sample asset price movements and then estimating the payoffs. These payoffs are averaged and then discounted at a risk-free interest rate. Figure 5.5 below shows one of the simulations of asset price movement with $S_0 = 150, \mu = 0.05, \sigma = 0.3, T = 2$ and 25 simulations.

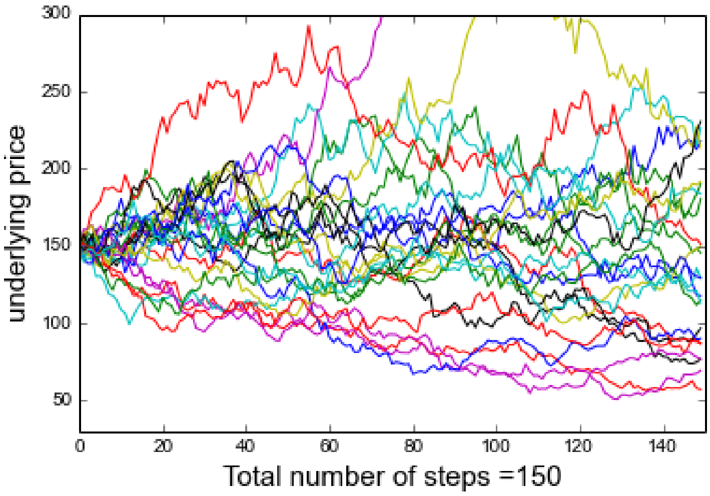

Figure 5.5: Asset price simulation

Thus, the following steps are essential for option price valuation using MCS method:

- Discretize the time period $[0, T]$ into subintervals such that $T_i = i\Delta t$. That is

$$\Delta t = \frac{T}{N}, \quad \text{for} \quad i = 0, 1, \cdots, N\,.$$

- Under the risk-neutrality assumptions, perform random simulations on the asset price movement based on the specific time interval. Choosing an M independent paths and thus, calculate the future prices of the specific underlying asset.

- Obtain the payoffs for each of the potential asset paths exhibited by the underlying. Discount the payoffs at a risk-free interest rate.

- Repeat the above for a large number of simulated asset paths.

- Take the average of the discounted payoffs over the number of the sample paths in order to get the value of the option.

Consider the asset price dynamics described by the SDE in equation (2.3.1), where the drift term $\mu = r$, the risk-free interest rate. The solution using Ito's lemma is:

$$S(t) = S(0)\text{exp}\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right) . \tag{5.2.1}$$

In generating the sample paths, we re-write equation (5.2.1) as follows

$$S(t+\Delta t) = S(t)\text{exp}\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma(\sqrt{\Delta t})\epsilon\right) , \tag{5.2.2}$$

since $B(t) \sim \sqrt{t}\epsilon \implies B(\Delta t) \sim \sqrt{\Delta t}\epsilon$ and $\epsilon \sim \mathcal{N}(0,1)$.

The payoffs are introduced next and will be discounted at a risk-free interest rate and this depends on the type of option being considered. For example, the discounted payoff for the zero rebate down-and-out barrier call option which has not been knocked out is given by

$$v^{+}(t+\Delta t) = \mathrm{e}^{-r\Delta t}\max\{S^{+}(t+\Delta t) - K, 0\} . \tag{5.2.3}$$

Finally, the option value is constructed by dividing the sum of the discounted payoff by the number of simulations. Thus, the value of the Monte-Carlo simulated value is given by:

$$V_M = \frac{1}{M}\sum_{i=1}^{M} v^{+}(t+\Delta t) . \tag{5.2.4}$$

The following pseudo-code explains how the down-and-out barrier call option can be implemented numerically using the standard MCS:

for k=1,$\cdots$,M

for i=0,1,$\cdots$,N-1

generate a $\mathcal{N}(0,1)$ sample ϵ

set $S(t+\Delta t) = S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma(\sqrt{\Delta t})\epsilon\right)$

end

if $\max_{0\leq i\leq N}\{S(t+\Delta t)\} > B$ then

$v_k = \mathrm{e}^{-r\Delta t}\max\{S(T)-K, 0\}$

else $v_k = 0$

end end set $\mathbf{v} = \frac{1}{M}\sum_{k=1}^{M} v_k$

set $\sigma^2 = \frac{1}{M-1}\sum_{k=1}^{M}(v_k - \mathbf{v})^2$

set $\tilde{v} = \left[\mathbf{v} - \frac{1.96\sigma}{\sqrt{M}}, \mathbf{v} + \frac{1.96\sigma}{\sqrt{M}}\right]$

Implementing the algorithm above displays three outputs. First the option value $\mathbf{v}$, second the variance σ and finally, an approximate 95% CI.

5.2.5 Definition. 95% Confidence Interval

Suppose that ω is a parameter to be estimated. A 95% confidence interval on ω is an interval $[a, b]$ such that the probability $\mathbb{P}[a \leq \omega \leq b] = 0.95$, where a and b are random endpoints.

From the definition above, if an experiment is to be repeated M times with 95% confidence interval, then approximately 95% of the constructed intervals in each case would have the true solution. Moreover, based on the law of large numbers, the convergence of the average of these discounted payoffs to the actual option price is made feasible. Also, the central limit theorem implies that the standard error obtained from the simulation tends to zero, as the rate of convergence of $\frac{1}{\sqrt{n}}$ increases. In order to calibrate the degree of precision achieved, we calculate the standard deviation of the discounted payoffs the same way the average is computed and this should be based on a given number of trials. From the CLT, the statistical error obtained during the simulation is proportional to $\frac{\sigma}{\sqrt{M}}$ and its boundedness is given by

$$|\mathcal{E}| \leq \frac{\epsilon\sigma}{\sqrt{M}},$$

where ϵ is a positive constant related to the confidence interval. Also, the option value $v(T, S)$ can be defined in a confidence interval bound. This interval can be reduced to get accurate price if the variance of the payoffs is reduced or if the number of iterations are increased. In valuing knock-out barrier options, the whole asset price is observed in order to determine if the option would be knocked out at some point in time before the expiration of the contract. This observation can pose an extreme intensive computation and this is quite a disadvantage. In other to improve the efficiency of the MCS to option pricing, the introduction of the variance reduction techniques which includes antithetic variables, control variates, etc (see Glasserman (2003) and Boyle et al. (1997)) had proved helpful.

5.3 Antithetic Monte-Carlo Simulation

A major setback in using the MCS is the slow rate of convergence at which the estimated values tend to the true solution. This can be explained by the large variances obtained during the simulation. It is equally observed that the confidence interval obtained using the MCS is greatly influenced by the ratio of the standard deviation to the square root of the number of simulations. To further reduce the width of the interval, the variance has to be reduced and this in turn gives a better estimate. The antithetic Monte-Carlo simulation method (AMCS) was introduced in this research to improve the flaws of the MCS. According to Glasserman (2003), AMCS focuses on the symmetric properties of the normal distribution to reduce the variance of the results being simulated. The aim was to introduce negatively correlated random variables. Let X be a random variable whose estimate is unknown. Also, let v and w be two negatively correlated variables with the same mean μ and variance σ^2. Define

$$X = \frac{1}{2}(v + w)\,,$$

then the mean is given by:

$$\mathbb{E}[X] = \mathbb{E}\left[\frac{1}{2}(v + w)\right] = \frac{1}{2}(\mathbb{E}[v] + \mathbb{E}[w]) = \mu\,.$$

Also, the variance,

$$\begin{aligned}\text{var}[X] = \text{var}\left[\frac{1}{2}(v + w)\right] &= \frac{1}{4}\left(\text{var}[v] + \text{var}[w] + 2\text{cov}[v, w]\right) \\ &= \frac{1}{2}\left(\text{var}[v] + \text{cov}[v, w]\right)\,.\end{aligned}$$

Thus, we have that

$$\text{var}[X] \quad \begin{cases} = \frac{\sigma^2}{2} & \text{if } v \text{ and } w \text{ are identically independent (cov=0)} \\ \\ < \frac{\sigma^2}{2} & \text{if} \quad \text{cov}[v, w] < 0 \end{cases}$$

It is obvious that if the $\text{cov}[v, w] < 0$, then the variance is reduced. For the outputs of the antithetic variates to be negatively correlated, it is essential that the inputs are negatively correlated. The mapping between them should be monotone. Non-monotonic functions result in a non-negative correlation and this could increase the variance, instead of reducing as expected.

5.3.1 Definition. Monotone Functions.

Let A be a subset of $\mathbb{R}$ and define a function $f : A \to \mathbb{R}$ and $x, y \in A$. f is monotonic if $x < y$ implies $f(x) \leq f(y)$ or $x < y$ implies $f(x) \geq f(y)$.

The following corollaries help in the generation of negatively correlated for uniformly and normally distributed random numbers .

5.3.2 Corollary. (Chan & Wong 2015). If $f(X_1, X_2, \cdots, X_n)$ is a monotone function of each of its argument. Then the following holds for a set $\eta_1, \cdots, \eta_m$ of independent and identically distributed uniformly random numbers on $(0, 1)$:

$$\mathrm{cov}[f(\eta_1, \cdots, \eta_m), f(1 - \eta_1, 1 - \eta_2, \cdots, 1 - \eta_m)] \le 0\,.$$

With respect to uniformly random numbers, let $\eta_1, \cdots, \eta_m$ and $\beta_1 = 1 - \eta_1, \cdots, \beta_m = 1 - \eta_m$ be uniformly random numbers with the properties that the pair (η_1, β_1) are negatively correlated. Define a monotone function f. If $X_1 = f(\eta_1, \cdots, \eta_m)$, then $X_2 = f(\beta_1, \cdots, \beta_m)$ must be defined with the same distribution as X_1. Thus, from Corollary 5.3.2, $\mathrm{cov}(X_1, X_2) \le 0$ and hence, X_1 and X_2 are negatively correlated.

5.3.3 Corollary. (Chan & Wong 2015). If $f(X_1, X_2, \cdots, X_n)$ is a monotone function of each of its argument. Then the following holds for a set $\eta_1, \cdots, \eta_m$ of independent and identically distributed normal random numbers on $(0, 1)$:

$$\mathrm{cov}[f(\eta_1, \cdots, \eta_m), f(-\eta_1, -\eta_2, \cdots, -\eta_m)] \le 0\,.$$

Suppose that $X_i \sim N(\mu, \sigma^2)$ and $Y_i = 2\mu - X_i$. Then from the linearity condition of the normal random variable, we see that $Y_i \sim N(\mu, \sigma^2)$. Thus, X_i and Y_i are negatively correlated. This follows from the definition of covariance:

$$\begin{aligned}
\mathrm{cov}[X_i, (2\mu - X_i)] &= \mathbb{E}[X_i(2\mu - X_i)] - \mathbb{E}[X_i]\mathbb{E}[2\mu - X_i] \\
&= \mathbb{E}[X_i](\mathbb{E}[2\mu - X_i] - \mathbb{E}[2\mu - X_i]) \\
&= 0\,.
\end{aligned}$$

5.3.4 Option pricing using AMCS method

Applying the concept of AMCS method to option pricing, the simulated asset prices assume the random variable. Another set of normally distributed random variables is equally simulated so that both would be negatively correlated. Taking an average, the final simulated asset price is obtained. Also, the generation of the pairs is computationally cheaper because instead of N as employed by the MCS, the AMCS generates $\frac{N}{2}$ pair of values.

Consider two discretized underlying asset processes defined by

$$S^{+}(t+\Delta t) = S(t)\exp\left(\left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma(\sqrt{\Delta t})\epsilon\right) \tag{5.3.1}$$

and

$$S^{-}(t+\Delta t) = S(t)\exp\left(\left(r-\frac{\sigma^2}{2}\right)\Delta t - \sigma(\sqrt{\Delta t})\epsilon\right). \tag{5.3.2}$$

Next, we employ the concept of discounted payoffs and this depends on the type of option being considered. Hence the discounted payoffs for the zero rebate down-and-out barrier call option for the two asset paths, which had not been knocked out are given by

$$v^{+}(t+\Delta t) = \mathrm{e}^{-r\Delta t}\max\{S^{+}(t+\Delta t)-K, 0\} \quad \text{and} \tag{5.3.3}$$

$$v^{-}(t+\Delta t) = \mathrm{e}^{-r\Delta t}\max\{S^{-}(t+\Delta t)-K, 0\}. \tag{5.3.4}$$

Finally, the mean estimator which is the required option value is constructed by taking the average of the discounted payoffs. Hence, the pricing formula using the AMCS is given by:

$$V_A = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{2}\left(v^{+}(t+\Delta t) + v^{-}(t+\Delta t)\right). \tag{5.3.5}$$

6. Computational Results and Analysis

This chapter considers some of the findings observed during the implementation of the methodologies found in Chapter 5. For the barrier options, we consider the zero-rebate non-dividend knock-out call options. Next, we consider the fixed-strike lookback options. In the numerical computation, we obtain our results using the program `ipython notebook` and all the codes used can be found in Appendix C.

6.1 Results on Down-and-Out Barrier Options

We consider the results obtained for the down-and-out barrier options. The MCS values are obtained using equation (5.2.4) and the AMCS values are valued using equation (5.3.5).

6.1.1 Down-and-out barrier options with ATM features

(A) We consider both the MCS and the AMCS of the above named option using the parameters: $S = K = 150, B = 125, r = 0.05, \sigma = 0.25$ and $T = 0.5$. The extended Black-Scholes formula in equation (3.3.2) is used to obtain the exact price. We also consider the time step to be $\Delta T = \frac{T}{N}$, where $N = 100$. Let $\mathbf{M}$ denote the number of simulations. Table 6.1 below shows the results of the simulated values in comparison to the exact values:

Table 6.1: Simulated values for ATM down-and-out barrier options

Exact value	**M**	**MCS value**	**AMCS value**
12.1861	10^1	15.5931	14.4021
12.1861	10^2	14.7494	12.7432
12.1861	10^3	13.0073	12.4896
12.1861	10^4	12.5972	12.4322
12.1861	10^5	12.4563	12.3939
12.1861	10^6	12.4241	12.3676

Table 6.1 compares the values obtained from the MCS and the AMCS. With the discretization time step of $N = 100$, we observe that the rate of convergence for both the MCS and the AMCS methods is slow. Furthermore, increasing the number of simulation and the time step would make a significant impact on the rate of convergence of the simulated values to the exact values. Also, as observed in Table 6.1, the values from the AMCS have the tendency to converge faster to the exact value in comparison to the ordinary MCS.

Next, we consider the variances. Table 6.2 shows the variances obtained from the results in Table 6.1, as it compares the variances of the MCS and the AMCS values from the exact values. The variances reduce with increase in the number of simulations. A higher discrepancy of the simulated value from the true value is obtained when the variance of the estimate is very large. The AMCS aims at reducing the variance of the estimate and thus, this account for it having a better estimate. Thus, the AMCS achieves its accuracy and high speed of convergence by reducing the variances of the simulation. We equally observed that the use of AMCS method reduce the variance of the MCS by two, or little above two.

Table 6.2: Variances for ATM down-and-out barrier options

Exact value	M	Variance (MCS value)	Variance (AMCS value)
12.1861	10^1	496.6263	243.3820
12.1861	10^2	454.2562	197.4071
12.1861	10^3	390.4154	179.8217
12.1861	10^4	362.5314	170.4178
12.1861	10^5	343.2436	169.4160
12.1861	10^6	339.4910	169.1509

(B) We next consider the relative and the standard errors obtained from the above simulations. Let SD, EBS and SIV denote the values of the standard deviation, the exact Black-Scholes values and the simulated values respectively. Also, let M denote the number of simulations. Then, the standard error is calculated by

$$\mathrm{SE} = \frac{\mathrm{SD}}{\sqrt{M}}\,, \tag{6.1.1}$$

whereas the relative error is calculated by

$$\mathrm{RE} = 100\% \frac{|\mathrm{SIV} - \mathrm{EBS}|}{\mathrm{EBS}}\,. \tag{6.1.2}$$

Table 6.3 shows the observed errors encountered by using the MCS and the AMCS method to value the down-and-out call options with the ATM features. Here, we observe the behaviour of both the standard and the relative error as given explicitly by the formulas in equations (6.1.1) and (6.1.2). With the increase in the number of simulations, both the relative and the standard error gradually converge to zero and this in turn, explains that the simulated values would converge to the true solution. The errors from the AMCS are seen to be lesser than that of the MCS method, hence providing more accurate result.

Table 6.3: Errors from the simulated values for ATM down-and-out barrier options

M	**MCS value (RE %)**	**AMCS value (RE %)**	**MCS value (SE)**	**AMCS value (SE)**
10^1	27.9581	18.1847	7.0472	4.9334
10^2	21.0346	4.5716	2.1313	1.4050
10^3	6.7388	2.4905	0.6248	0.4241
10^4	3.3735	2.0195	0.1904	0.1305
10^5	2.2173	1.7052	0.0586	0.0412
10^6	1.9531	1.4894	0.0184	0.0130

(C) Next, we output the computation time (CPU time) of the above simulations, as well as, the observed 95% confidence interval (CI). Table 6.4 provides the output.

With regards to the 95% confidence interval of the estimate, we observe a wide disparity when the number of simulations is very small. Thus, with increase in the simulation numbers, the range or the interval of the estimate becomes smaller as depicted in Table 6.4. It is also observed that the width of the interval for the AMCS is small compared to that of the MCS. Furthermore, comparing the MCS and the AMCS with respect to their computation time, the time for the AMCS is close to twice or a bit more of the time used by the standard MCS. Tavella (2003) explained that the computation time for the AMCS to output its results would approximately double and this is one of the major drawbacks of using the AMCS method.

Table 6.4: CPU time and 95% CI for ATM down-and-out barrier options

M	CPU time (MCS)	CPU time (AMCS)	CI (MCS)	CI (AMCS)
10^1	0.0034	0.0049	$(1.781, 29.405)$	$(4.733, 24.071)$
10^2	0.0035	0.0052	$(10.572, 18.927)$	$(9.990, 15.497)$
10^3	0.0183	0.0324	$(11.783, 14.232)$	$(11.659, 13.321)$
10^4	0.1470	0.2771	$(12.224, 12.970)$	$(12.176, 12.688)$
10^5	1.3518	2.6824	$(12.342, 12.571)$	$(12.313, 12.475)$
10^6	12.2476	24.4510	$(12.388, 12.460)$	$(12.342, 12.393)$

6.1.2 Down-and-out barrier options with OTM features

(A) We consider the MCS method for the down-and-out barrier option with OTM features. Here, the parameters to be considered are $S = 80, K = 100, B = 60, r = 0.08, \sigma = 0.3$ and $T = 0.5$. The extended Black-Scholes formula in equation (3.3.2) is used to obtain the exact price. We also consider the time step to be $\Delta T = \frac{T}{N}$, where $N = 1000$. Table 6.5 below shows the results obtained.

Table 6.5: MCS values for OTM down-and-out barrier options

Exact value	**M**	**MCS value**	**Variance**	**RE (%)**	**SE**
1.9894	10^1	3.2034	73.2414	61.0234	2.7063
1.9894	10^2	2.7071	49.5335	36.0762	0.7038
1.9894	10^3	2.2711	46.2171	14.1601	0.2150
1.9894	10^4	2.0684	44.2774	3.9711	0.0665
1.9894	10^5	2.0189	40.3216	1.4829	0.0201
1.9894	10^6	1.9906	40.0428	0.0603	0.0063

(B) Consider the AMCS method on the down-and-out options with the OTM features using the same parameters above. Table 6.6 gives the following results:

Table 6.6: AMCS values for OTM down-and-out barrier options

Exact value	**M**	**AMCS value**	**Variance**	**RE (%)**	**SE**
1.9894	10^1	2.6194	28.1932	31.6678	1.6791
1.9894	10^2	2.4431	26.9841	22.8059	0.5195
1.9894	10^3	2.1808	21.2137	9.6210	0.1457
1.9894	10^4	2.0149	20.6777	1.2818	0.0455
1.9894	10^5	1.9991	20.3425	0.4876	0.0143
1.9894	10^6	1.9897	20.0641	0.0151	0.0045

Tables 6.5 and 6.6 further display the outputs obtained by comparing the estimated prices from the MCS and the AMCS methods. They highlight the fact that the AMCS provides the best estimate compared to the MCS counterpart.

6.1.3 Finite difference methods on down-and-out call options

Using Crank-Nicolson FDM, we consider the results obtained for the non-dividend zero-rebate down-and-out call option. Consider the parameters: $S = 50, K = 40, B = 20, r = 0.04, \sigma = 0.3, T = 1.0$ and $\mathrm{S}_{\mathrm{max}} = 225$. The exact value of the option using the extended Black-Scholes pricing formula in equation (3.3.2) is **12.9360**. Let N be the discretization steps of the time and M denote the discretization steps for the underlying asset. Also, let CNV be the Crank-Nicolson values obtained and thus, we obtain the following results:

Table 6.7: Effect of increasing discretization steps on Crank-Nicolson values

N	**M**	**CNV**	**N**	**M = 2N**	**CNV**
60	60	12.9439	60	120	12.9424
80	80	12.9447	80	160	12.9401
100	100	12.9437	100	200	12.9383
200	200	12.9383	200	400	**12.9360**
300	300	12.9363	300	600	12.9360
400	400	**12.9360**	400	800	12.9360
500	500	12.9360	500	1000	12.9360

Table 6.7 depicts the effect of increasing the space-time discretization steps on the values obtained using the Crank-Nicolson method. When the space and time steps are the same, we observed that the rate of convergence is slow. But when the space step is doubled with respect to the time step, the rate of convergence increased faster as shown in the table.

In the FDM, the choice of S_{max}, (maximum underlying price) being an artificial limit is yet to be known. Table 6.8 shows the behaviour of the Crank-Nicolson values for different values of S_{max}, using the following parameters below: $S = 80, K = 100, B = 60, r = 0.08, \sigma = 0.3$ and $T = 0.5$. The exact value of the option using equation (3.3.2) is **1.9894**.

Table 6.8: Effect of different choices of S_{max} on the Crank-Nicolson values

N = M	**Crank-Nicolson values**				
	$S_{max} = 2S$	$S_{max} = (2S + 50)$	$S_{max} = (2S + 100)$	$S_{max} = (2S + 150)$	$S_{max} = (2S + 200)$
100	1.8197	1.9927	1.9855	1.9833	2.0090
200	1.8211	1.9890	1.9884	1.9879	1.9945
300	1.8214	1.9875	1.9889	1.9887	1.9884
400	1.8215	1.9881	1.9891	1.9890	1.9906
500	1.8215	1.9880	1.9892	1.9891	1.9902
600	1.8215	1.9877	1.9893	1.9892	1.9891

From Table 6.8, we observe that the rate of convergence when the S_{max} is chosen to be $2S$ is very slow, though it would eventually converge at some point. Inconsistencies of the option values are observed when S_{max} is chosen to be $2S + 50$ and $2S + 200$

since, the values increased and decreased randomly. But when S_{max} is chosen to be $2S + 100$ and $2S + 150$, the option values maintain a regular pattern of increasing towards the exact option value. The rate of convergence when S_{max} is $2S + 100$ is faster than that of $2S + 150$. This is evident with the result obtained in Table 6.9.

Finally consider the parameters below: $K = 70, B = 55, r = 0.05, \sigma = 0.4, T = 0.25, N = 250$ and $M = 500$ for Table 6.9. The following outputs were obtained.

Table 6.9: Effect of different choices of S_{max} on the Crank-Nicolson values with increasing underlying prices

S	**Exact Values**	**CNV** ($\mathrm{S}_{\mathrm{max}} = 2S + 100$)	**CNV** ($\mathrm{S}_{\mathrm{max}} = 2S + 150$)
55	0.0000	0.0000	0.0000
60	1.5125	1.5125	1.5125
65	3.4193	3.4194	3.4195
70	5.9491	5.9495	5.9487
75	9.1306	9.1309	9.1310
80	12.8782	12.8783	12.8784

6.2 Results of the Up-and-Out Barrier Options

We consider the implementations of the MCS, AMCS and the FDM on the up-and-out call options. The exact values for the up-and-out call is found in equation (3.3.6).

6.2.1 Results using the MCS and the AMCS methods

Consider the up-and-out call option with parameters: $S = 50, K = 60, B = 80, r = 0.05, \sigma = 0.45, T = 0.5$. The exact value of the option using the extended Black-Scholes pricing formula is **0.8657**. The following results were obtained using points $N = 10^1, N = 10^2, N = 10^3$ and $N = 10^4$ depicted below on Tables 6.10, 6.11, 6.12 and 6.13 respectively. As usual, MCS and AMCS denote the values obtained using the Monte-Carlo and the antithetic Monte-Carlo simulations respectively.

Table 6.10: Simulation values for OTM up-and-out call option ($\Delta T = 10^{-1}$)

M	MCS values	AMCS values	MCS values (RE %)	AMCS values (RE %)
10^1	2.2362	1.4039	158.3112	62.1693
10^2	1.3701	1.2645	58.2650	46.0668
10^3	1.2558	1.2205	45.0618	40.9842
10^4	1.2353	1.2079	42.6938	39.5056
10^5	1.2311	1.2011	42.2086	38.7432

Table 6.11: Simulation values for OTM up-and-out call option ($\Delta T = 10^{-2}$)

M	MCS values	AMCS values	MCS values (RE %)	AMCS values (RE %)
10^1	1.3424	1.2220	55.0653	41.1575
10^2	1.2393	1.0885	43.1558	25.7364
10^3	1.0505	1.0403	21.3469	20.1850
10^4	1.0401	1.0010	20.1456	15.6290
10^5	1.0038	0.9982	15.9524	15.3180

Table 6.12: Simulation values for OTM up-and-out call option ($\Delta T = 10^{-3}$)

M	MCS values	AMCS values	MCS values (RE %)	AMCS values (RE %)
10^1	1.2592	1.0289	45.4546	18.8518
10^2	1.1698	0.9256	35.1276	6.9193
10^3	0.9864	0.9001	13.9425	3.9737
10^4	0.9426	0.8905	8.8830	2.8647
10^5	0.9295	0.8812	7.3698	1.7905

Table 6.13: Simulation values for OTM up-and-out call option ($\Delta T = 10^{-4}$)

M	MCS values	AMCS values	MCS values (RE %)	AMCS values (RE %)
10^1	1.0597	0.9634	22.4096	11.2857
10^2	0.9780	0.9011	12.9722	4.0892
10^3	0.9141	0.8845	5.5909	2.1717
10^4	0.8991	0.8670	3.8582	0.1502
10^5	0.8914	**0.8657**	2.9687	**0.0000**

It is obvious that the increase in the number of simulations would result to the high rate of convergence for both the Monte-Carlo and the antithetic simulated values. The relative errors from both simulations are reducing as well. Increasing the points N makes the simulation to be computationally costly, as the CPU time increased. The exact extended Black-Scholes price is obtained based on continuous monitoring of the time interval. This price can be achieved when the time steps (ΔT) is reduced drastically and this follows from the increment in the number of the discretization points N. Hence, the discreet points $\max_{0\leq j\leq N} S(j) < B$ tends towards the continuous points $\max_{0\leq t\leq T} S(t) < B$. At $M = 10^5$, the AMCS value converged as depicted in Table 6.13. We can thus conclude that the increasing the discretization time step helps in obtaining a high rate of convergence.

6.2.2 Effect of increase in volatility on the up-and-out call options

Table 6.14 shows the effect of increase in volatility on the ITM up-and-out call options. We display the outputs of the plain vanilla European call values, the up-and-out call values and the antithetic Monte-Carlo simulations for the up-and-out call. Here, all other pricing parameters are assumed to be constant but with varying volatility. Choosing $N = 100, S = 60, K = 50, B = 100, r = 0.04$ and $T = 0.5$, the following results were observed:

Table 6.14: Simulation values for ITM up-and-out call option with increasing volatility

Volatility (σ)		**Option values**	
	Vanilla call (v)	**Up-and-out call**	**AMCS value**
0.05	10.9907	10.9907	11.1862
0.1	10.9924	10.9924	11.2166
0.2	11.2524	11.2331	11.5034
0.35	12.5471	10.7812	11.1981
0.5	14.3434	8.1668	8.8923
0.65	16.3261	5.5958	6.2948
0.8	18.3782	3.7655	4.4464

From Table 6.14 above, a linear increase is observed on the plain vanilla option, as increasing the volatility led to increase in the option value with every other parameters kept constant. However the up-and-out call resulted in a non-linear function, as the option increased and declined at some points. The reduction in the option values is often due to the fact that the volatility is sufficiently large and thus increases the chances of the option being knocked out. This increased probability resulted in the decline in the option value. The antithetic MCS depicted above shows similar characteristics.

6.2.3 Finite difference methods on up-and-out call options

For the first table, we consider the parameters below: $S = 50, B = 80, K = 60, r = 0.02, \sigma = 0.5, T = 0.5$ and $\mathrm{S}_{\max} = 160$. The second table places the strike price at $K = 45$ using the same parameters as the first. Using the extended Black-Scholes pricing formula, the exact values of the up-and-out OTM and the ITM call options are $\mathbf{0.7360}$ and $\mathbf{6.6193}$ respectively. Below shows the outputs obtained:

Table 6.15: OTM and ITM for up-and-out call option valuations using Crank-Nicolson FDM

N=M	CNV	M=2N	CNV
100	0.7029	200	0.7357
200	0.7357	400	0.7359
300	0.7358	600	**0.7360**
400	**0.7360**	800	0.7360
500	0.7360	1000	0.7360
600	0.7360	1200	0.7360

N=M	CNV	M=2N	CNV
100	6.4212	200	6.6189
200	6.6189	400	6.6190
300	6.6190	600	**6.6193**
400	6.6190	800	6.6193
500	6.6192	1000	6.6193
600	**6.6193**	1200	6.6193

The tables in 6.15 display similar characteristics. The simulated values in bold denote the values when the convergence started. It was observed that increasing the discretization sizes for both the underlying asset and time resulted to a faster rate of convergence. Consider the OTM up-and-out call for example, the convergence started at $\mathbf{N} = \mathbf{M} = \mathbf{400}$ but when the asset steps are doubled, the convergence started at $\mathbf{N} = \mathbf{300}, \mathbf{M} = \mathbf{600}$. Thus, we can say that if the discretization asset step is twice the time steps, then there is higher convergence rate.

6.3 Results on Fixed Strike Lookback Options

Table 6.16: Simulated values for the fixed strike call options with different maturities

T	σ	**K**	**Exact value**	**MCS value**	**AMCS value**
0.25	0.5	100	36.87750	34.63185	34.86544
0.5	0.5	100	48.65638	46.66056	47.25615
0.75	0.5	100	57.50865	56.17422	56.52953
0.25	0.5	110	27.74725	25.31519	25.60858
0.5	0.5	110	41.76453	38.36640	38.69462
0.75	0.5	110	52.97611	48.75581	49.17202
0.25	0.5	120	19.58322	17.83340	17.94008
0.5	0.5	120	33.94013	30.27140	31.21146
0.75	0.5	120	45.66973	42.31194	42.86572
1.0	0.5	100	64.84088	64.36417	64.63686
1.0	0.5	110	62.49656	57.70139	58.45105
1.0	0.5	120	55.72884	51.15095	51.30543

Table 6.16 above was obtained at time $t = 0$ when the option had just been initiated. Here, we considered the combination of ITM, ATM and the OTM options using the following parameters: $S = 110, S_{\max} = 110$ and $r = 0.35$. Also, the number of simulations using the MCS and the AMCS methods was $M = 10000$ and the time step was $N = 100$.

We observed that increasing the strike price while keeping every other stochastic variables of the option constant led to a decrease in the value of the fixed strike lookback call price. Also, similar increase was observed when the time to expiry of the option was being increased, with every other variables kept constant. Furthermore, the antithetic Monte-Carlo price is significantly close to the exact observed price, as compared to the standard Monte-Carlo price.

For the put counterpart, we considered the combination of ITM, ATM and the OTM options using the following parameters: $S = 110, S_{\min} = 110$ and $r = 0.35$. Also, the number of simulations using the MCS and the AMCS methods is $M = 10000$ and the time step $N = 1000$. The results are depicted in Table 6.17 below.

We observed that increasing the time step of the simulation resulted to a faster rate of convergence, as simulated values obtained using a larger M (i.e $M = 1000$) were closer to the exact price, than when $M = 100$. Also, increasing the value of the strike price made the option to be increasingly in the money, as observed in Table 6.17 below. We equally observed that extending the time to expiry forward with every other variables constant increased the value of the put option price.

Table 6.17: Simulated values for the fixed strike put options with different maturities

T	σ	K	Exact value	MCS value	AMCS value
0.25	0.5	90	3.2895	3.1672	3.2087
0.5	0.5	90	5.6786	5.5093	5.5282
0.75	0.5	90	6.9211	6.6095	6.8369
0.25	0.5	100	7.7582	7.4642	7.5485
0.5	0.5	100	10.6159	10.2342	10.2816
0.75	0.5	100	11.7991	11.3204	11.3909
0.25	0.5	110	15.2399	14.7219	14.8855
0.5	0.5	110	17.7978	17.2727	17.3601
0.75	0.5	110	18.5010	17.8741	17.9960
1.0	0.5	90	7.5298	7.2052	7.25705
1.0	0.5	100	12.1921	11.7672	11.8436
1.0	0.5	110	18.4107	17.8286	17.8518

7. Conclusion and Recommendation

In this work, we have considered the numerical valuations of barrier options and the lookback options. The research focused on the non-dividend zero-rebate knock-out barrier options and the non-dividend fixed strike lookback options. We used three main numerical methods to estimate the values of these options and they were compared to their exact values which follow the assumptions of the Black-Scholes pricing model. As for the fixed strike lookback options, we only employed the MCS and the AMCS to estimate the price of the options and the obtained approximated values were compared to their exact values. For the zero-rebate knock-out options, we used the concept of Crank-Nicolson finite difference method, the MCS and the AMCS methods to value the option. Further comparison were also done to measure the error estimate observed in the simulations.

Using the simulation approach to value the options, It was observed that the MCS method involved the generation of a sample of N independent paths which were used to estimate their respective payoffs. This was dependent on the type of option being considered. Next, the payoffs were averaged and discounted at a risk-free interest rate, so as to obtain the present option value. It was also observed that increasing the number of simulations made the approximated values to converge to the true solution but their convergence rate were slow. The reason was attributed to the fact that there were lots of uncertainties in the estimate, owing to the larger variances obtained. The AMCS was introduced next, which is an improved version of the standard MCS method. The aim of the AMCS was to generate smaller variances which in turn yield a more favourable estimate. The AMCS achieved this by generating a sample of $\frac{N}{2}$ pair of independent underlying paths which were negatively correlated in contrast to the N independent paths generated by the MCS methods. From the result section of this research work, we observed intently that the AMCS method provides the best option estimate in contrast to the MCS method.

With regards to the Crank-Nicolson finite difference method, we observed that the rate of convergence was dependent on the choice of S_{max} and increasing the discretisation steps of the space and time resulted to a higher rate of convergence. The options we considered had closed form formulas which were derived by extending the Black-Scholes formulas for the plain vanilla options. This was based on the Black-Scholes assumptions. Hence, this research aimed at implementing the numerical PDE methods like the finite difference numerical method to value these options. To further show the indispensable nature of these numerical methods, the estimated values obtained were compared to the exact values and little discrepancies were observed. Hence, numerical approaches can equally be used to approximate options whose close form solution does not exist.

Furthermore, part of this research used the second order differential equation based on the Euler scheme to approximate option prices. To ensure accuracy and a higher convergence rate to the true solution, a higher order approach need to be used. Thus, in the future work, a higher order differential approach would be applied in the estimation of these exotic option prices.

Appendix A. Solution to the Black-Scholes PDE

The Black-Scholes PDE in equation (2.3.2) is of the parabolic linear form with non-constant coefficients. It can be converted to a constant coefficient PDE as:

$$\boxed{\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} + cv} \tag{A.0.1}$$

where v is a function of t and x; $b, c \in \mathbb{R}$ and $a = 1$. The parameters used to reduce equation (2.3.2) to a dimensionless form as found in equation (A.0.1) are:

$$S = Ke^x \implies x = \log\left(\frac{S}{K}\right), \quad v(t, S) = Km(n, x), \quad n = \frac{(T-t)\sigma^2}{2}.$$

The Black-Scholes equation arose as a result of diffusion problem which are constrained by time. It can be solved by reducing it to the heat equation whose solution exists. First, we take the derivatives of $v(t, S)$ with respect to t, S and S^2, we have

$$\frac{\partial v}{\partial t} = -\frac{\sigma^2 K}{2}\frac{\partial m}{\partial n},$$

$$\frac{\partial v}{\partial S} = \frac{K}{S}\frac{\partial m}{\partial x},$$

$$\frac{\partial^2 v}{\partial S^2} = \frac{-K}{S^2}\frac{\partial m}{\partial x} + \frac{K}{S^2}\frac{\partial^2 m}{\partial x^2}.$$

Substituting into equation (2.3.2), the equation reduces to

$$\frac{\partial m}{\partial n} = \frac{\partial^2 m}{\partial x^2} + (\omega - 1)\frac{\partial m}{\partial x} - \omega m, \qquad \text{where} \qquad \omega = \frac{2r}{\sigma^2}. \tag{A.0.2}$$

The initial condition $v(t,S) = \max\{S-K,0\}$ becomes $m(0,x) = \max\{e^x - 1, 0\}$ at $t = 0$. Consider yet another change of variable. Suppose

$$\lambda, \eta \in \mathbb{R} \quad \text{and} \quad m(n,x) = e^{\lambda x + \eta n} u(n,x)\,.$$

We seek for the derivatives of $m(n,x)$ with respect to x, xx and n. We have

$$\frac{\partial m}{\partial x} = e^{\lambda x + \eta n}\left[\frac{\partial u}{\partial x} + \lambda u\right],$$

$$\frac{\partial^2 m}{\partial x^2} = e^{\lambda x + \eta n}\left[\frac{\partial^2 u}{\partial x^2} + 2\lambda\frac{\partial u}{\partial x} + \lambda^2 u\right],$$

$$\frac{\partial m}{\partial n} = e^{\lambda x + \eta n}\left[\frac{\partial u}{\partial n} + \eta u\right].$$

Substituting into equation (A.0.2), the equation reduces to

$$\frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial x^2} + (2\lambda + \omega - 1)\frac{\partial u}{\partial x} + [\lambda^2 + (\omega - 1)\lambda - \omega - \eta]u\,. \tag{A.0.3}$$

The aim is to reduce the whole equation to a heat equation and thus, we choose the constants λ and η so that the coefficients of u and $\frac{\partial u}{\partial x}$ equal zero. That is,

$$0 = \lambda^2 + (\omega - 1)\lambda - \omega - \eta\,, \tag{A.0.4}$$

$$0 = 2\lambda + \omega - 1\,. \tag{A.0.5}$$

Solving equations (A.0.4) and (A.0.5) simultaneously, we have

$$\lambda = \frac{1}{2}(1-\omega) \quad \text{and} \quad \eta = \frac{-1}{4}(\omega + 1)^2\,.$$

Thus, the PDE in equation (A.0.3) reduces to

$$\frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial x^2}\,, \tag{A.0.6}$$

where $n > 0$ and $-\infty < x < \infty$. For the initial boundary condition. Recall that

$$m(n,x) = e^{\lambda x + \eta n} u(n,x)\,,$$

so that,

$$m(0,x) = e^{\lambda x} u(0,x) \implies u(0,x) = e^{-\lambda x} m(0,x)\,.$$

Substituting for λ and the value for $m(0,x)$, we have

$$u(0,x) = e^{\frac{-x}{2}(1-\omega)}[\max\{e^x - 1, 0\}] \tag{A.0.7}$$

$$= \max\{e^{\frac{x}{2}(\omega+1)} - e^{\frac{x}{2}(\omega-1)}, 0\}\,. \tag{A.0.8}$$

According to Wilmott et al. (1995), The solution to the heat equation (A.0.6) is:

$$u(n,x) = \frac{1}{2\sqrt{\pi n}} \int_{-\infty}^{\infty} u_0(S) e^{\frac{-(x-S)^2}{4n}} \mathrm{d}S\,,$$

where $u_0(S) = u(0,S)$. Consider yet another change in variable

$$y = \frac{S-x}{\sqrt{2n}}, \implies S = x + y\sqrt{2n} \quad \text{and} \quad \mathrm{d}y = \frac{\mathrm{d}S}{\sqrt{2n}}\,.$$

Thus,

$$u(n,x) = \frac{1}{2\sqrt{\pi n}} \int_{-\infty}^{\infty} u_0(x + y\sqrt{2n}) e^{\frac{-y^2}{2}} \sqrt{2n}\, \mathrm{d}y$$

$$u(n,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(0, x + y\sqrt{2n}) e^{\frac{-y^2}{2}} \, \mathrm{d}y \,.$$

Applying equation (A.0.8), we have

$$\begin{aligned}
u(n,x) &= \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{1}{2}(\omega+1)(x+y\sqrt{2n})} e^{\frac{-y^2}{2}} \, \mathrm{d}y - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{1}{2}(\omega-1)(x+y\sqrt{2n})} e^{\frac{-y^2}{2}} \, \mathrm{d}y \\
&= \frac{e^{\frac{x}{2}(\omega+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{1}{2}(\omega+1)(y\sqrt{2n}) - \frac{y^2}{2}} \, \mathrm{d}y - \frac{e^{\frac{x}{2}(\omega-1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{1}{2}(\omega-1)(y\sqrt{2n}) - \frac{y^2}{2}} \, \mathrm{d}y \\
&= \frac{e^{\frac{x}{2}(\omega+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y^2 - \sqrt{2n}(\omega+1)y]} \, \mathrm{d}y - \frac{e^{\frac{x}{2}(\omega-1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y^2 - \sqrt{2n}(\omega-1)y]} \, \mathrm{d}y \\
&= \frac{e^{\frac{x}{2}(\omega+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega+1)\sqrt{2n}]^2 + \frac{n}{4}(\omega+1)^2} \, \mathrm{d}y - \frac{e^{\frac{x}{2}(\omega-1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega-1)\sqrt{2n}]^2 + \frac{n}{4}(\omega-1)^2} \, \mathrm{d}y \\
&= \frac{e^{\frac{x}{2}(\omega+1) + \frac{n}{4}(\omega+1)^2}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega+1)\sqrt{2n}]^2} \, \mathrm{d}y - \frac{e^{\frac{x}{2}(\omega-1) + \frac{n}{4}(\omega-1)^2}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega-1)\sqrt{2n}]^2} \, \mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{\frac{x}{2}(\omega+1) + \frac{n}{4}(\omega+1)^2} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega+1)\sqrt{2n}]^2} \, \mathrm{d}y - e^{\frac{x}{2}(\omega-1) + \frac{n}{4}(\omega-1)^2} \int_{\frac{-x}{\sqrt{2n}}}^{\infty} e^{\frac{-1}{2}[y - \frac{1}{2}(\omega-1)\sqrt{2n}]^2} \, \mathrm{d}y \right]
\end{aligned}$$

Applying the definition of normal probability distribution function, we have

$$u(n,x) = e^{\frac{x}{2}(\omega+1) + \frac{n}{4}(\omega+1)^2} N\left(\frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega+1)\sqrt{2n}\right) - e^{\frac{x}{2}(\omega-1) + \frac{n}{4}(\omega-1)^2} N\left(\frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega-1)\sqrt{2n}\right) . \tag{A.0.9}$$

We observe that

$$\begin{aligned}
e^{\frac{x}{2}(\omega+1)+\frac{n}{4}(\omega+1)^2} \times e^{\frac{-x}{2}(\omega-1)-\frac{n}{4}(\omega+1)^2} &= e^{x} \\
e^{\frac{x}{2}(\omega-1)+\frac{n}{4}(\omega-1)^2} \times e^{\frac{-x}{2}(\omega-1)-\frac{n}{4}(\omega+1)^2} &= e^{-\omega n}\,.
\end{aligned}$$

Recall also that

$$m(n,x) = e^{\lambda x+\eta n} u(x,n)\,,$$

which implies that

$$m(n,x) = e^{\frac{-x}{2}(\omega-1)-\frac{n}{4}(\omega+1)^2} u(x,n)\,.$$

Thus, we have that

$$m(n,x) = e^{x} N\left(\frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega+1)\sqrt{2n}\right) - e^{-\omega n} N\left(\frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega-1)\sqrt{2n}\right)\,. \tag{A.0.10}$$

Let

$$d_1 = \frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega+1)\sqrt{2n} \quad \text{and} \quad d_2 = \frac{x}{\sqrt{2n}} + \frac{1}{2}(\omega-1)\sqrt{2n}\,.$$

But recall that

$$x = \log\left(\frac{S}{K}\right), \qquad n = \frac{(T-t)\sigma^2}{2} \qquad \text{and} \qquad \omega = \frac{2r}{\sigma^2}\,.$$

Substituting and evaluating for x, n and ω into d_1 and d_2, we have

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,, \tag{A.0.11}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) - \sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}\,. \tag{A.0.12}$$

From equation (A.0.10), we have that

$$m(n,x) = e^{x}N(d_1) - e^{-\omega n}N(d_2)\,.$$

Substituting also for x, n and ω, we have

$$m(n,x) = \frac{S}{K}N(d_1) - e^{-r(T-t)}KN(d_2)\,.$$

Also, recall that the option value $v(t,S) = Km(n,x)$.

Finally, substituting becomes

$$v(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2)\,, \tag{A.0.13}$$

where d_1 and d_2 are given in equations (A.0.11) and (A.0.12). Equation (A.0.13) gives the price of a non-dividend paying European call option. For the put options, the derivation follows above and the value is given by

$$v(t,S) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\,. \tag{A.0.14}$$

Appendix B. Truncation Errors for the Finite Difference Methods

Let $V(t,S)$ be represented in grid form by $V_{i,k}$. Then the following Taylor series expansion holds:

$$V(t+\Delta t, S) = V(t,S) + \frac{\partial V(t,S)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V(t,S)}{\partial t^2}\Delta t^2 + \mathcal{O}(\Delta t^3)\,, \tag{B.0.1}$$

$$V(t-\Delta t, S) = V(t,S) - \frac{\partial V(t,S)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V(t,S)}{\partial t^2}\Delta t^2 - \mathcal{O}(\Delta t^3)\,. \tag{B.0.2}$$

B.1 Explicit FDM

TIME: The explicit method assumes backward approximation in time. Hence, we use the Taylor series expansion,

$$V(t-\Delta t, S) = V(t,S) - \frac{\partial V(t,S)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V(t,S)}{\partial t^2}\Delta t^2 - \mathcal{O}(\Delta t^3)\,,$$

$$\frac{V(t-\Delta t, S)}{\Delta t} = \frac{V(t,S)}{\Delta t} - \frac{\partial V(t,S)}{\partial t} + \mathcal{O}(\Delta t)\,,$$

$$\frac{\partial V(t,S)}{\partial t} = \frac{V(t,S) - V(t-\Delta t, S)}{\Delta t} + \mathcal{O}(\Delta t)\,.$$

Let $V(t,S) = V_{i,k}$. Then we have

$$\frac{\partial V(t,S)}{\partial t} \approx \frac{V_{i,k} - V_{i-1,k}}{\Delta t} + \mathcal{O}(\Delta t)\,.$$

The explicit method assumes backward approximation in time and the truncation error is up to $\mathcal{O}(\Delta t)$.

STOCK: For the central difference approximation of the stock, we take the difference between the forward and the backward difference approximation with respect to the stock. Hence, we would have

$$\frac{\partial V(t,S)}{\partial S} = \frac{V(t,S+\Delta S) - V(t,S-\Delta S)}{2\Delta S} + \mathcal{O}(\Delta S^2)\,,$$

$$\approx \frac{V_{i,k+1} - V_{i,k-1}}{2\Delta S}\,.$$

For the standard approximation of the first derivative of the stock prices, we take the sum of the the forward and the backward difference approximation with respect to the stock. Hence, we have

$$V(t,S+\Delta S) + V(t,S-\Delta S) = 2V(t,S) + \frac{\partial^2 V(t,S)}{\partial S^2}\Delta S^2 + \mathcal{O}(\Delta S^4)\,,$$

$$\frac{\partial^2 V(t,S)}{\partial S^2} = \frac{V(t,S+\Delta S) + V(t,S-\Delta S) - 2V(t,S)}{\Delta S^2} + \mathcal{O}(\Delta S^2)\,,$$

$$\approx \frac{V_{i,k+1} - 2V_{i,k} + V_{i,k-1}}{\partial S^2} + \mathcal{O}(\Delta S^2)\,.$$

The truncation error of the stock prices for the explicit method is up to the order $\mathcal{O}(\Delta S^2)$.

B.2 Implicit FDM

TIME: The implicit method assumes forward approximation in time. Hence, we use the Taylor series expansion,

$$V(t+\Delta t, S) = V(t,S) + \frac{\partial V(t,S)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V(t,S)}{\partial t^2}\Delta t^2 + \mathcal{O}(\Delta t^3)\,,$$

$$\frac{V(t+\Delta t, S)}{\Delta t} = \frac{V(t,S)}{\Delta t} + \frac{\partial V(t,S)}{\partial t} + \mathcal{O}(\Delta t)\,,$$

$$\frac{\partial V(t,S)}{\partial t} = \frac{V(t+\Delta t, S) - V(t,S)}{\Delta t} + \mathcal{O}(\Delta t)\,.$$

Let $V(t,S) = V_{i,k}$. Then we have

$$\frac{\partial V(t,S)}{\partial t} \approx \frac{V_{i+1,k} - V_{i,k}}{\Delta t} + \mathcal{O}(\Delta t)\,.$$

The implicit method assumes forward approximation in time, and the truncation error is up to $\mathcal{O}(\Delta t)$.

For the stock, the implicit and the explicit assumes the same truncation error, that is $\mathcal{O}(\Delta S^2)$.

B.3 Crank Nicolson Method

The Crank Nicolson averages the implicit and the explicit method. Fadugba & Nwozo (2013) explained that the method makes use of the center difference approximation for $V_{i,k}$ and the symmetric central difference at the $V_{i+\frac{1}{2},k}$ term. This is given by

$$V_{i,k} \approx \frac{V_{i+1,k} - V_{i-1,k}}{2\Delta t} + \mathcal{O}(\Delta t^2)\,.$$

Expanding $V_{i+1,k}$ and $V_{i,k}$ at $V_{i+\frac{1}{2},k}$ using Taylor series, we have

$$V_{i+1,k} = V_{i+\frac{1}{2},k} + \frac{1}{2}\frac{\partial V}{\partial t}\Delta t + \mathcal{O}(\Delta t^2)\,,$$

$$V_{i,k} = V_{i+\frac{1}{2},k} - \frac{1}{2}\frac{\partial V}{\partial t}\Delta t + \mathcal{O}(\Delta t^2)\,.$$

Taking their average gives

$$V_{i,j+\frac{1}{2}} \approx \frac{1}{2}(V_{i,j+1} + V_{i,j}) + \mathcal{O}(\Delta t^2)\,.$$

Thus, we have that the Crank Nicolson method is correct up to the order of Δt^2 and ΔS^2. The same explanation for obtaining ΔS^2 for the Crank Nicolson is the same with the explicit FDM.

Appendix C. Python Codes

C.1 Python Codes for the Graphs Used

```
Initializing all used variables:
S0    = Current underlying asset price
K     = Strike price
r     = Risk-free interest rate
sig   = Volatility
T     = Time to expiry
N     = Time steps
dT    = T/float(N)      #time increment
n     = Number of different simulations

Packages imported:
import matplotlib.pyplot as plt
import numpy as np
import math

### (1) PLOT FOR THE GEOMETRIC BROWNIAN MOTION SIMULATION WITH N=5000 POINTS ###
time = np.linspace(0, T, N)
W    = np.random.standard_normal(size = N)
W    = np.cumsum(W)*np.sqrt(dT) # for the standard brownian motion
V    = (r-0.5*sig**2)*time + sig*W
S    = S0*np.exp(V) # Calculates the asset simulation with smaller variance
V1   = (r-0.5*sig1**2)*time + sig*W
S1   = S0*np.exp(V1)  # Calculates the asset simulation with larger variance
```

```
def expectedvalue(S0, r, time):
    mean = S0*np.exp(r*time)
    return mean
#Next plot S, S1 and expectedvalue all against time #

### (2) PLOT FOR THE GEOMETRIC ASIAN CALL, ITS DELTA AND THE PLAIN VANILLA CALL ###
def phi(x):
    return (1.0/2)*(1+math.erf(x/math.sqrt(2)))   #CDF

def Asiancall(S,K,r,sig,T):
    d  = math.exp(-r*T)
    Q1 = (np.log(S/float(K))+0.5*(r+ sig**2/6.0)*T)/float(sig*np.sqrt(T/3.0))
    Q2 = Q1-sig*np.sqrt(T/3.0)
    Call = S*math.exp(-(r+sig**2/6.0)*(0.5*T))*phi(Q1)-K*d*phi(Q2)
    return Call

def BScall(S,K,r,sig,T): #vanilla call#
    d1 = (np.log(S/float(K))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    d2 = (np.log(S/float(K))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Van_calloption = S*phi(d1)-K*math.exp(-r*T)*phi(d2)
    return Van_calloption

def DeltAgeocall(S,K,r,sig,T):
    Q1 = (np.log(S/float(K))+0.5*(r+ sig**2/6.0)*T)/float(sig*np.sqrt(T/3.0))
    Q2 = Q1-sig*np.sqrt(T/3.0)
    A = math.exp(-(r+sig**2/6.0)*(0.5*T))*phi(Q1)
    B = 1/float(sig*np.sqrt(2*math.pi*T/3.0))
    C = math.exp(-(0.5*r*T + T*sig**2/12.0 + 0.5*Q1**2 ))
    Delta = A+B*(C-K/float(S)*math.exp(-(r*T+0.5*Q2**2)))
    return Delta
```

```
#Next plot the Asiancall, BScall & DeltAgeocall all against the interval S=[50,100]

### (3) PLOT FOR THE STANDARD LOOKBACK CALL OPTION ###
def Lookbackcall(S,M,r,sig,T):
    d  = math.exp(-r*T)
    Q1 = (np.log(S/float(M))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q2 = (np.log(S/float(M))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q3 = (np.log(M/float(S))-(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q4 = (np.log(M/float(S))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q5 = ((S*sig**2)/float(2*r))*(M/float(S))**(2*r/float(sig**2))
    Call = S* phi(Q1)-M*d*phi(Q2)-(S*sig**2/float(2*r))*phi(Q3)+d*Q5*phi(Q4)
    return Call
#Next plot BScall & Lookbackcall against the interval S=[50,100] with M= min(S)#

### (4) PLOT FOR THE STANDARD LOOKBACK PUT OPTION ###
def Lookbackput(S,M,r,sig,T):
    d  = math.exp(-r*T)
    Q1 = (np.log(S/float(M))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q2 = (np.log(S/float(M))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Q5 = ((S*sig**2)/float(2*r))*(M/float(S))**(2*r/float(sig**2))
    Q6 = (np.log(S/float(M))-(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Put = -S* phi(-Q1)+M*d*phi(-Q2)+(S*sig**2/float(2*r))*phi(Q1)-d*Q5*phi(Q6)
    return Put

def BSput(S,K,r,sig,T):
    d1 = (np.log(S/float(K))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    d2 = (np.log(S/float(K))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    Van_putoption = -S*phi(-d1)+K*math.exp(-r*T)*phi(-d2)
    return Van_putoption
#Next plot BSput & Lookbackput against the interval S=[50,100] with M= max(S)#
```

```
### (5) PLOT FOR THE DOWN-AND-OUT CALL OPTION ###
def DOcall(S,B,K,r,sig,T):
    d     = math.exp(-r*T)
    alp   = (2*r)/float(sig**2)-1
    d1    = (np.log(S/float(K))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    d2    = (np.log(S/float(K))+(r-0.5*sig**2)*T)/float(sig*np.sqrt(T))
    y     = (np.log(B**2/float(S*K))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    x     = y-sig*np.sqrt(T)
    C     = S*phi(d1)-K*d*phi(d2)
    Call = C-(B/float(S))**alp*((B**2/float(S))*phi(y)-K*d*phi(x))
    return Call
#Next plot BScall and DOcall against the interval S=[1,200] #

### (6) PLOT FOR THE DOWN-AND-IN CALL OPTION ###
def DIcall(S,B,K,r,sig,T):
    d     = math.exp(-r*T)
    alp   = (2*r)/float(sig**2)-1
    y     = (np.log(B**2/float(S*K))+(r+0.5*sig**2)*T)/float(sig*np.sqrt(T))
    x     = y-sig*np.sqrt(T)
    Call = (B/float(S))**alp*((B**2/float(S))*phi(y)-K*d*phi(x))
    return Call
#Next plot BScall and DIcall against the interval S=[1,200] #

### (7) PLOT FOR THE UP-AND-OUT CALL OPTION ###
def UOcall(S,B,K,r,sig,T):
    lam   = (r+0.5*sig**2)/float(sig**2)
    x1    = (np.log(S/float(B))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
    x2    = x1-sig*np.sqrt(T)
    y1    = (np.log(B/float(S))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
```

```
    y     = (np.log(B**2/float(S*K))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
    A     = S*phi(x1)-K*math.exp(-r*T)*phi(x2)
    X     = (B/float(S))**(2*lam)*(phi(-y)-phi(-y1))
    Y     = K*math.exp(-r*T)*(B/float(S))**(2*lam-2)
    Z     = phi(-y+sig*np.sqrt(T))-phi(-y1+sig*np.sqrt(T))
    if B <=K:
        Call = 0
    else:
        Call = BScall(S,K,r,sig,T)-(A-S*X + Y*Z)
    return Call
#Next plot BScall and UOcall against the interval S=[50,200] #

### (8) PLOT FOR THE UP-AND-IN CALL OPTION ###
def UIcall(S,B,K,r,sig,T):
    lam   = (r+0.5*sig**2)/float(sig**2)
    x1    = (np.log(S/float(B))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
    x2    = x1-sig*np.sqrt(T)
    y1    = (np.log(B/float(S))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
    y     = (np.log(B**2/float(S*K))/float(sig*np.sqrt(T)) )+ lam*sig*np.sqrt(T)
    A     = S*phi(x1)-K*math.exp(-r*T)*phi(x2)
    X     = (B/float(S))**(2*lam)*(phi(-y)-phi(-y1))
    Y     = K*math.exp(-r*T)*(B/float(S))**(2*lam-2)
    Z     = phi(-y+sig*np.sqrt(T))-phi(-y1+sig*np.sqrt(T))
    if B <=K:
        Call = Van_calloption
    else:
        Call = A-S*X + Y*Z
    return Call
#Next plot BScall and UIcall against the interval S=[1,160] #
```

```
### (9) PLOT FOR THE ASSET ON NOTHING CALL OPTION ###
K=50
def aoncall(S,K):
    if S<=K:
        value= 0
    else:
        value = 30
    return value

def Vancall(S,K):
    V = max(S-K,0)
    return V
#Next plot Vancall and aoncall against the interval S=[20,80] #

### (10) PLOT FOR THE ASSET ON NOTHING PUT OPTION ###
def aonput(S,K):
    if S>=K:
        value= 0
    else:
        value = 30
    return value

def Vanput(S,K):
    V = max(K-S,0)
    return V
#Next plot Vanput and aonput against the interval S=[20,80] #

### (11) PLOT FOR THE ASSET PRICE SIMULATION WITH N=150 POINTS ###
S = np.zeros([n, N], dtype=float)
x = range(0, N, 1)
```

```
for j in range(0, n,1):
    S[j,0]= S0
    for i in x[:-1]:
         W=np.random.normal()
         S[j,i+1]=S[j,i]+S[j,i]*(r-(sig**2)/2.0)*dT+sig*S[j,i]*np.sqrt(dT)*W;
    plt.plot(x, S[j])
#Thus, the asset prices are plotted against the interval points [0,150] #
```

C.2 Python Codes for Results Displayed on Tables

```
Initializing all the variables used
R     = Exact value of the option being valuated
S     = Current underlying value
Smax  = Maximum value of the underlying asset
B     = Barrier level
K     = Strike price
MM    = Number of simulations
M     = Number of asset steps
N     = Number of time steps
T     = Time to expiry
r     = Risk-free interest rate
sig = Volatility
dS    = (Smax-B)/float(M) #step size for the stock
dT    = T/float(N)  #step size for time

Packages imported
import time, math, from scipy import stats, import numpy as np, import
scipy.linalg as linalg, from scipy.interpolate import interp1d
start_time=time.time()
```

```
### (1) MONTE-CARLO FOR DOWN-AND-OUT CALL ###
def sim_value(S,sig,r,T):
    e = np.random.normal()
    return S*np.exp((r-0.5*sig**2)*T+sig*e*np.sqrt(T))

def callpayoff(S_T,K):
    return np.exp(-r*T)*max(S_T-K,0)

S     = 200
B     = 175
K     = 150
r     = 0.05
sig = 0.2
T     = 0.5
MM    = 1000
R     = 18.8103
N     = 100.
vi = []
for i in xrange(MM):
    S_T = sim_value(S,sig,r,T)
    if vi <= B and S==B:
        S_T == 0
    else:
        vi.append(callpayoff(S_T,K))
price = 1/float(MM)*sum(vi)
Variance = 1/float(MM-1)*sum((vi-price)**2)
SE=np.sqrt(Variance/float(MM))
CI = stats.norm.interval(0.95, loc=price, scale= np.sqrt(Variance/float(MM)))
```

```
### (2) ANTITHETIC MONTE-CARLO FOR DOWN-AND-OUT CALL ###
def Asset(S,sig,r,T):
    e = np.random.normal()
    return S*np.exp((r-0.5*sig**2)*T+sig*e*np.sqrt(T))
def Asset1(S,sig,r,T):
    e = np.random.normal()
    return S*np.exp((r-0.5*sig**2)*T-sig*e*np.sqrt(T))

def Payoff(S_T,K):
    return np.exp(-r*T)*max(S_T-K,0)
def Payoff1(S_T1,K):
    return np.exp(-r*T)*max(S_T1-K,0)
S     = 200
B     = 175
r     = 0.05
sig = 0.20
T     = 0.5
MM    = 1000
R     = 18.81033
N     = 100.
vi = []
vj = []
for i in xrange(MM):
    S_T = Asset(S,sig,r,T)
    if vi <= B:
        S_T == 0
    else:
        vi.append(Payoff(S_T,K))
for j in xrange(MM):
    S_T1 = Asset1(S,sig,r,T)
```

```
    if vj <= B:
        S_T1 == 0
    else:
        vj.append(Payoff1(S_T1,K))
A = [a+b for a, b in zip(vi,vj)]
BB = [0.5*c for c in A]
price = 1/float(MM)*sum(BB)
Variance = 1/float(MM-1)*sum((BB-price)**2)
SE=np.sqrt(Variance/float(MM))
CI = stats.norm.interval(0.95, loc=price, scale= np.sqrt(Variance/float(MM)))

### (3) CRANK-NICOLSON FDM FOR DOWN-AND-OUT CALL ###
def Price(S0,B,K,r,sig,T,Smax,M,N):
    F  = np.zeros((M+1,N+1)) #setting up the matrix
    SS =  np.linspace(B,Smax,M+1)
    i=SS/float(dS)
    j=np.arange(1,N+1,dtype=np.float)
    #Terminal and boundary conditions
    F[:,N] =[np.maximum(SS[p]-K,0) for p in xrange(M+1)]
    F[0,:] = 0
    F[M,:] =[Smax * np.exp(-r*( N - j)*dT) for j in xrange(N+1)]
    F=np.matrix(np.array(F))
    #the coefficient matrices
    A   = -0.25*dT*(sig**2*i**2-r*i)
    BB  = dT*0.5*(sig**2*i**2+r)
    C   = -0.25*dT*(sig**2*i**2+r*i)
    #the two diagonal matrices
    Y = -np.diag(A[2:M], k=-1)+np.diag(-1-BB[1:M])-np.diag(C[1:M-1], k=1)
    Z =  np.diag(A[2:M], k=-1)+np.diag(-1+BB[1:M])+np.diag(C[1:M-1], k=1)
    #solving the linear system
```

```
    for j in range(N-1,-1,-1):
       d=np.zeros((M-1,1))  #computes the matrix d
        #inserts the first and the last element
       d[0]=(0.25*sig**2*1**2*dT-0.25*r*1*dT)*(F[0,j]+F[0,j+1])
       d[M-2]=(0.25*dT*(sig**2*(M-1)**2+r*(M-1)))*(F[M,j]+F[M,j+1])
       LU = linalg.lu_factor(Y)
       b = Z*(F[1:M,j+1]) + d
       F[1:M,j]=linalg.lu_solve(LU,b)
       price = interp1d(SS, F[:,0].squeeze())
    return price(S0)
print Price(80,55,70,0.05,0.4,0.25,310,500,250)
print('time: %.5f' %(time.time()-start_time))

### (4) MONTE-CARLO FOR UP-AND-OUT CALL ###
def SMCupandout(S,K,B,r,sig,T,MM):
    N=10.
    f=np.zeros((MM,1))
    for i in range(0,MM):
        e = np.random.randn(N,1)
        ST = S*np.cumprod(np.exp((r-0.5*sig**2)*dT+sig*np.sqrt(dT)*e))
        Smax=max(ST)
        if Smax>=B:
            f[i] = 0
        else:
            f[i] = np.exp(-r*dT)*max(ST[N-1]-K,0)
            price=np.mean(f)
    return price
print ('Option price: %.5f' % SMCupandout(60,50,100,0.35,0.25,0.5,10000))
print('time: %.5f' %(time.time()-start_time))
```

```
### (5) ANTITHETIC MONTE-CARLO FOR UP-AND-OUT CALL ###
def AMCupandout(S,K,B,r,sig,T,MM):
    N=100.
    f=np.zeros((MM,1))
    g=np.zeros((MM,1))
    h=np.zeros((MM,1))
    for i in range(MM):
        e = np.random.randn(N,1)
        ST1 = S*np.cumprod(np.exp((r-0.5*sig**2)*DT+sig*np.sqrt(DT)*e))
        Smax=max(ST1)
        if Smax>=B:
            f[i] = 0
        else:
            f[i] = np.exp(-r*DT)*max(ST1[N-1]-K,0)
    for i in range(MM):
        d = np.random.randn(N,1)
        ST2 = S*np.cumprod(np.exp((r-0.5*sig**2)*DT-sig*np.sqrt(DT)*d))
        Smax2=max(ST2)
        if Smax2>=B:
            g[i] = 0
        else:
            g[i] = np.exp(-r*DT)*max(ST2[N-1]-K,0)
        h[i]=0.5*(f[i]+g[i])
        price=np.mean(h)
    return price
print ('Option price: %.5f' % AMCupandout(60,50,100,0.35,0.25,0.5,10000))
print('time: %.5f' %(time.time()-start_time))

### (6) CRANK-NICOLSON FDM FOR UP-AND-OUT CALL ###
def Price(S0,B,K,r,sig,T,Smax,M,N):
```

```
F  = np.zeros((M+1,N+1)) #set up a matrix
SS =  np.linspace(0,B,M+1)
i=SS/float(dS)
j=np.arange(1,N+1,dtype=np.float)
#Terminal and boundary conditions
F[:,N] =[np.maximum(SS[p]-K,0) for p in xrange(M+1)]
F[B,:] = 0
if Smax>=B:
    F[M,:]=0
else:
    F[M,:] =[Smax * np.exp(-r*( N - j)*dT) for j in xrange(N+1)]
F=np.matrix(np.array(F))
#the coefficient matrix
A   = -0.25*dT*(sig**2*i**2-r*i)
BB  = dT*0.5*(sig**2*i**2+r)
C   = -0.25*dT*(sig**2*i**2+r*i)
#the two matrices
M1 = -np.diag(A[2:M], k=-1)+np.diag(-1-BB[1:M])-np.diag(C[1:M-1], k=1)
M2 =  np.diag(A[2:M], k=-1)+np.diag(-1+BB[1:M])+np.diag(C[1:M-1], k=1)
#solving the linear system
for j in range(N-1,-1,-1):
   d=np.zeros((M-1,1))  #computes the matrix d
    #inserts the first and the last element
   d[0]=(0.25*sig**2*1**2*dT-0.25*r*1*dT)*(F[0,j]+F[0,j+1])
   d[M-2]=(0.25*dT*(sig**2*(M-1)**2+r*(M-1)))*(F[M,j]+F[M,j+1])
   LU = linalg.lu_factor(M1)
   b = M2*(F[1:M,j+1]) + d
   F[1:M,j]=linalg.lu_solve(LU,b)
   price = interp1d(SS, F[:,0].squeeze())
return price(S0)
```

```
print Price(50,80,45,0.06,0.35,0.25,160,800,400)
print('time: %.5f' %(time.time()-start_time))

### (5) MONTE-CARLO FOR FIXED STRIKE LOOKBACK CALL ###
def MCLookcall(S0,K,r,sig,T,MM):
    V = []
    for k in range(0,MM):
        N=100
        St = []
        St.append(S0)
        DT=T/float(N)
        for i in range(1,N-1):
            e = random.gauss(0,1)
            St.append(St[i-1]*np.exp((r-0.5*sig**2)*DT+sig*np.sqrt(DT)*e))
        Smax=max(St)
        V.append(max(0, Smax-K)*np.exp(-r*T))
    price = sum(V)/float(MM)
    return price
print ('Option price: %.5f' %MCLookcall(110,120,0.35,0.5,1,10000))

### (6) ANTITHETIC MONTE-CARLO FOR FIXED STRIKE LOOKBACK CALL ###
def AMCLookcall(S0,K,r,sig,T,MM):
    V1 = []
    V2 = []
    for p in range(0,MM):
        N=100
        St1 = []
        St2 = []
        St1.append(S0)
        St2.append(S0)
```

```
        DT=T/float(N)
        for i in range(1,N-1):
            e = random.gauss(0,1)
            St1.append(St1[i-1]*np.exp((r-0.5*sig**2)*DT+sig*np.sqrt(DT)*e))
            St2.append(St2[i-1]*np.exp((r-0.5*sig**2)*DT-sig*np.sqrt(DT)*e))
        Smax1=max(St1)
        Smax2=max(St2)
        V1.append(max(0, Smax1-K)*np.exp(-r*T))
        V2.append(max(0, Smax2-K)*np.exp(-r*T))
    A=[a+b for a, b in zip(V1,V2)]
    BB=[0.5*c for c in A]
    price = sum(BB)/float(MM)
    return price
print ('Option price: %.5f' %AMCLookcall(110,120,0.35,0.5,1,100000))

### (7) MONTE-CARLO FOR FIXED STRIKE LOOKBACK PUT ###
def MCLookput(S0,K,r,sig,T,MM):
    V = []
    for k in range(0,MM):
        N=1000
        St = []
        St.append(S0)
        DT=T/float(N)
        for i in range(1,N-1):
            e = random.gauss(0,1)
            St.append(St[i-1]*np.exp((r-0.5*sig**2)*DT+sig*np.sqrt(DT)*e))
        Smin=min(St)
        V.append(max(0, K-Smin)*np.exp(-r*T))
    price = sum(V)/float(MM)
    return price
```

```
print ('Option price: %.5f' %MCLookput(110,110,0.35,0.5,0.25,10000))

### (8) ANTITHETIC MONTE-CARLO FOR FIXED STRIKE LOOKBACK PUT ###
def AMCLookput(S0,K,r,sig,T,MM):
    V1 = []
    V2 = []
    for p in range(0,MM):
        N=1000
        St1 = []
        St2 = []
        St1.append(S0)
        St2.append(S0)
        DT=T/float(N)
        for i in range(1,N-1):
            e = random.gauss(0,1)
            St1.append(St1[i-1]*np.exp((r-0.5*sig**2)*DT+sig*np.sqrt(DT)*e))
            St2.append(St2[i-1]*np.exp((r-0.5*sig**2)*DT-sig*np.sqrt(DT)*e))
        Smin1=min(St1)
        Smin2=min(St2)
        V1.append(max(0, K-Smin1)*np.exp(-r*T))
        V2.append(max(0, K-Smin2)*np.exp(-r*T))
    A=[a+b for a, b in zip(V1,V2)]
    BB=[0.5*c for c in A]
    price = sum(BB)/float(MM)
    return price
print ('Option price: %.5f' %AMCLookput(110,90,0.35,0.5,1,10000))
```

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