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FUBINI'S THEOREM FOR A VECTOR-VALUED DANIELL INTEGRAL

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Abstract. We define the product integral for two Riesz space-valued Daniell integrals and prove Fubini's theorem establishing the relation between product integrals and iterated integrals. The properties of the Fremlin tensor product of two Archimedean Riesz spaces enable one to do this in an elegant way.

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Key words: Fubini's theorem, product integral, Riesz space, Archimedean tensor product.

1. Introduction. Protter showed in [4] how useful the L_p-valued Daniell integral is in the study of Stochastic Analysis. The abstract Riesz space-valued Daniell integral is easily defined following the ideas of Protter and the original ideas of Daniell (see [3]). Given two such integrals I and J, we are interested in defining a primitive product Daniell integral on some Riesz space of functions of two variables which can then be extended by the well-known Daniell procedure to an integral on the class of Daniell-summable functions.

If $\mathcal{L}_I$ and $\mathcal{L}_J$ are the Riesz spaces of I and J summable functions, one has to find a Riesz space to act as the initial domain on which the primitive product integral can be defined. It is easy to define the product integral K on the space $\mathcal{L}_I \otimes \mathcal{L}_J$, by putting $K(X \otimes Y) := I(X)J(Y)$ and extending it linearly. However, this space is not a Riesz space (see Rompf and Kersting [5]) but only an ordered vector space.

Fremlin solved this problem by defining and proving the existence of the so-called Fremlin tensor product (see [2]).

Definition 1.1. Let $\mathfrak{E}$ and $\mathfrak{F}$ be Archimedean Riesz spaces. An Archimedean Riesz space H that contains the algebraic tensor product $\mathfrak{E}\otimes\mathfrak{F}$ as a vector subspace is called a Riesz tensor product of $\mathfrak{E}$ and $\mathfrak{F}$, if it has the following properties:

1. The embedding map $\sigma : \mathfrak{E} \times \mathfrak{F} \to H$ is a Riesz bimorphism, i.e., $\sigma(\cdot, y)$ and $\sigma(x, \cdot)$ are Riesz homomorphisms.
2. If $\mathfrak{G}$ is an Archimedean Riesz space and if $\psi : \mathfrak{E} \times \mathfrak{F} \to \mathfrak{G}$ is a Riesz bimorphism, then there exists a unique Riesz homomorphism $\tau : H \to \mathfrak{G}$ such that $\psi = \tau \circ \sigma$.

As mentioned above, Fremlin proved the existence and uniqueness of H and it is often referred to as the Fremlin tensor product of $\mathfrak{E}$ and $\mathfrak{F}$, denoted by $\mathfrak{E}\overline{\otimes}\mathfrak{F}$.

We shall use the following facts that were proved by Fremlin [2]:

THEOREM 1.2. *1. The Riesz subspace generated by $\mathfrak{E}\otimes\mathfrak{F}$ in $\mathfrak{E}\overline{\otimes}\mathfrak{F}$ is equal to $\mathfrak{E}\overline{\otimes}\mathfrak{F}$.*

2. *If $\mathfrak{G}$ is an Archimedean Riesz space and if $\psi : \mathfrak{E}\times\mathfrak{F}\to\mathfrak{G}$ is a Riesz bimorphism satisfying $\psi(x,y) > 0$ if $x > 0$ and $y > 0$ then the Riesz subspace generated by $\psi(\mathfrak{E}\times\mathfrak{F})$ in $\mathfrak{G}$, is (Riesz isomorphic to) the Fremlin tensor product $\mathfrak{E}\overline{\otimes}\mathfrak{F}$.*

3. *If $\mathfrak{G}$ is a relatively uniformly complete Riesz space, and if $\psi : \mathfrak{E}\times\mathfrak{F}\to\mathfrak{G}$ is a positive bilinear operator, then there exists a unique positive linear operator $S : \mathfrak{E}\overline{\otimes}\mathfrak{F}\to\mathfrak{G}$ such that $\psi = S\circ\sigma$, with σ as before.*

2. The Riesz space-valued Daniell integral. Let $\mathfrak{E}$ be a universally complete Riesz space with weak order unit E, that is separated by its order continuous dual $\mathfrak{E}^\sim_{00}$ and that is perfect, i.e., that satisfies the condition $\mathfrak{E} = (\mathfrak{E}^\sim_{00})^\sim_{00}$. We denote its sup-completion by $\mathfrak{E}^s$. Replacing the set of real numbers in Daniell's integration theory by $\mathfrak{E}$, we get a Daniell integral for $\mathfrak{E}$-valued functions (see [3]). We provide some of the detail for the convenience of the reader.

Let T be an arbitrary set and let $\mathfrak{E}^T$ be the set of all functions defined on T with values in $\mathfrak{E}$. We note that $\mathfrak{E}$ is an f-algebra with E as algebraic unit.

By defining the operations point wise, i.e., for all $t\in T$,

$$(X+Y)(t) := X(t)+Y(t), \quad (cX)(t) := cX(t), \quad X\le Y \iff X(t)\le Y(t),$$

it follows that $\mathfrak{E}^T$ is a Dedekind complete Riesz space with weak order unit $(E(t))$, where $E(t) = E$ for all $t\in T$.

Let $\mathcal{L}_0$ be a Riesz subspace of $\mathfrak{E}^T$ contained in the ideal generated in $\mathfrak{E}^T$ by the weak order unit $E = E(t)$ for all $t\in T$.

DEFINITION 2.1. A positive linear operator $I : \mathcal{L}_0\to\mathfrak{E}$ is called a *vector-valued I-integral* whenever, for every sequence (X_n) in $\mathcal{L}_0$ that satisfies $X_n(t)\downarrow 0$ for every $t\in\mathfrak{E}^T$, it follows that $I(X_n)\downarrow 0$.

We refer to $\mathcal{L}_0$ as the *initial domain* of the primitive Daniell integral I. Thus, the integral is a positive σ-order continuous linear operator mapping $\mathcal{L}_0$ into $\mathfrak{E}$.

In [3] a complete description can be found of the well-known Daniell extension procedure by which the primitive I-integral is extended to a positive integral defined on a Riesz space $\mathcal{L}_I\supset\mathcal{L}_0$ of all summable functions. It starts by considering positive functions and shows how I can be extended from the positive cone of $\mathcal{L}_0$ to its upper envelope

$$\mathcal{L}_0^\uparrow := \{0\le X\in(\mathfrak{E}^s)^T : \exists 0\le X_n\in\mathcal{L}_0,\ n\in\mathbb{N}, X_n\uparrow X\}.$$

If $0\le X\in\mathcal{L}_0^\uparrow$ and if $0\le X_1\le X_2\le\dots$ is an increasing sequence with $X=\sup X_n$, we define

$$I(X) = \sup\{I(X_n) : X_n\in\mathcal{L}_0, X_n\uparrow X\}.$$

This supremum always exists in the sup-completion $\mathfrak{E}^s$ of $\mathfrak{E}$ (see [1]). The set $\mathcal{L}_0^{\uparrow\Downarrow}$ is set to be:

$$\mathcal{L}_0^{\uparrow\Downarrow} = \{X \in (\mathfrak{E}^s)^T \,:\, X = \inf X_\alpha \in \mathcal{L}_0^{\uparrow}, X_\alpha \geq X\}.$$

Equivalently,

$$\mathcal{L}_0^{\uparrow\Downarrow} = \{X \in (\mathfrak{E}^s)^T \,:\, \exists X_\alpha \in \mathcal{L}_0^{\uparrow},\ X_\alpha \downarrow X\}.$$

The *upper semi-integral* of $X \in \mathcal{L}_0^{\uparrow\Downarrow}$ is defined to be

$$\dot{I}(X) := \inf_\alpha \{I(X_\alpha) \,:\, X_\alpha \in \mathcal{L}_0^{\uparrow}, X_\alpha \downarrow X\}.$$

The upper semi-integral is positive homogeneous, sub-additive and monotone. The *lower semi-integral of* X is defined as $\underset{\cdot}{I}(X) := -\dot{I}(-X)$ for X such that $-X \in \mathcal{L}_0^{\uparrow\Downarrow}$.

It follows that $\underset{\cdot}{I}(X) \leq \dot{I}(X)$ and that

$$\dot{I}(|X|) - \underset{\cdot}{I}(|X|) \leq \dot{I}(X) - \underset{\cdot}{I}(X).$$

The lower semi-integral is positive homogenuous, super-additive and monotone.

If $\dot{I}(X) = \underset{\cdot}{I}(X) \in \mathfrak{E}$, we say that X is *summable* and we define

$$I(X) := \dot{I}X = \underset{\cdot}{I}(X).$$

If X^+ and X^- are summable, we define $I(X) := I(X^+) - I(X^-)$.

We recall that the following theorem holds ([3, Theorem 3.7 and Theorem 3.10]).

THEOREM 2.2. *The set $\mathcal{L}$ of all summable vector valued functions in $\mathfrak{E}^T$ is a Riesz space and the integral I defined on $\mathcal{L}$ is a positive linear operator mapping $\mathcal{L}$ into $\mathfrak{E}$.*

If there exists a sequence of functions U_n in $\mathcal{L}_0$ such that, for some $U \in \mathfrak{E}^T$ we have

$$\dot{I}(|U - U_n|) \to 0,$$

Then $U \in \mathcal{L}$, i.e., U is summable.

The last assertion of the theorem shows that $\mathcal{L}_0$ is dense in $\mathcal{L}$ relative to a vector-seminorm.

DEFINITION 2.3. 1. We call I a *stochastic integral* if $I(E(t)) = E$.

2. The integral is called $\mathfrak{E}$- *homogeneous*, if

$$I(XY(t)) = XI(Y(t)) \text{ for all } X \in \mathfrak{E},\ Y(t) \in \mathcal{L}_I.$$

If I is $\mathfrak{E}$-homogeneous, then every constant vector X in $\mathfrak{E}$ is summable because

$$I(X) = I(XE(t)) = XI(E(t)) = XE = X \in \mathfrak{E}.$$

We observe that if B is a Banach algebra, then the Bochner integral for B-valued functions is readily seen to be B-homogeneous.

3. The product integral. Let I and J denote two extended positive vector-valued Daniell integrals each of which is stochastic and $\mathfrak{E}$-homogeneous with values in $\mathfrak{E}$, the latter being a universally complete Riesz space. The spaces $\mathcal{L}_I$ and $\mathcal{L}_J$ of summable functions are subspaces of $\mathfrak{E}^S$ and $\mathfrak{E}^T$ respectively.

The space $\mathfrak{E}^{S\times T}$ is a Dedekind complete Riesz space. Recalling that $\mathfrak{E}$ is an f-algebra, we consider the bilinear map $\psi : \mathcal{L}_I \times \mathcal{L}_J \to \mathfrak{E}^{S\times T}$ with $\psi(X,Y) = X(s)Y(t) \in \mathfrak{E}^{S\times T}$.

PROPOSITION 3.1. *1. The bilinear map ψ is a Riesz bimorphism.*

2. The Riesz subspace generated by the elements $X(s)Y(t)$ in $\mathfrak{E}^{S\times T}$ is the Fremlin tensor product $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$.

Proof. 1. By a result of Schaefer [6], we have that, since $|\psi(X,Y)| = |X(s)Y(t)| = |X(s)|\,|Y(t)| = \psi(|X|,|Y|)$, the bilinear map ψ is a Riesz bimorphism.

2. If $X(s) > 0$ and $Y(t) > 0$ then $X(s)Y(t) > 0$ and so it follows from Theorem 1.2(2) that the Riesz subspace generated by the elements $X(s)Y(t)$ in $\mathfrak{E}^{S\times T}$ is the Fremlin tensor product $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$. $\square$

The algebraic tensor product $\mathcal{L}_I \otimes \mathcal{L}_J$ is the vector subspace of $\mathfrak{E}^{S\times T}$ with elements

$$\sum_{i=1}^{n} X_i(s)Y_i(t),\ X_i \in \mathcal{L}_I,\ Y_i \in \mathcal{L}_J.$$

and the Fremlin tensor product $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$ is the Riesz subspace generated by $\mathcal{L}_I \otimes \mathcal{L}_J$ in the Riesz space $\mathfrak{E}^{S\times T}$.

We now define a bilinear operator $b : \mathcal{L}_I \times \mathcal{L}_J \to \mathfrak{E}$ by

$$b(X,Y) = I(X)J(Y) \in \mathfrak{E},\ \text{for all } X \in \mathcal{L}_I, Y \in \mathcal{L}_J, \tag{3.1}$$

This is a positive bilinear operator defined on $\mathcal{L}_I \times \mathcal{L}_J$ with values in the f-algebra $\mathfrak{E}$.

Recalling that $\mathfrak{E}$ is universally complete and therefore also relatively uniformly complete, it follows from Theorem 1.2(3) that there exists a unique positive linear operator $K : \mathcal{L}_I\overline{\otimes}\mathcal{L}_J \to \mathfrak{E}$ with the property that

$$K(X(s)Y(t)) = I(X)J(Y),\ \text{for all } X \in \mathcal{L}_I,\ Y \in \mathcal{L}_J.$$

Thus, the restriction of K to the algebraic tensor product $\mathcal{L}_I \otimes \mathcal{L}_J$ is the linearization of the bilinear form b.

We show in the next section that K is a Daniell integral with initial domain

$$\mathcal{L}_I\overline{\otimes}\mathcal{L}_J \subset \mathfrak{E}^{S\times T},$$

i.e., we show that K is sequentially pointwise order continuous. In exactly the same manner, by interchanging the variable in the above definition of K, we find a unique extension $\tilde{K}$ of the bilinear form $\tilde{b}(Y,X) := J(Y)I(X)$ satisfying

$$\tilde{K}(Y \otimes X) = J(Y)I(X) = I(X)J(Y) = K(X \otimes Y),$$

since the f-algebra $\mathfrak{E}$ is commutative.

4. The iterated integrals. Following Zaanen [7], we denote by $\mathcal{L}_I * \mathcal{L}_J$ the collection of all elements $U(s,t) \in \mathfrak{E}^{S\times T}$ with the property that for fixed s we have that $U^s(t) := U(s,t) \in \mathcal{L}_J$ and $J(U(s,t)) \in \mathcal{L}_I$.

For every element $U(s,t)$ in the algebraic tensor product $\mathcal{L}_I \otimes \mathcal{L}_J$ of the form

$$U(s,t) = \sum_{i=1}^{n} X_i(s)Y_i(t),\ X_i \in \mathcal{L}_I,\ Y_i \in \mathcal{L}_J.$$

we have that

$$\begin{aligned} I(J(U(s,t)) &= \sum_{i=1}^{n} I(J(X_i(s))Y_i(t)) \\ &= \sum_{i=1}^{n} I(X_i(s)J(Y_i(t)))\ (J \text{ is } \mathfrak{E}\text{-homogeneous}) \\ &= \sum_{i=1}^{n} J(Y_i(t))I(X_i(s))\ (I \text{ is } \mathfrak{E}\text{-homogeneous}) \\ &= \sum_{i=1}^{n} I(X_i(s))J(Y_i(t))\ (\ f\text{-algebra is commutative}). \\ &= K(U). \end{aligned} \tag{4.1}$$

This shows that $\mathcal{L}_I \otimes \mathcal{L}_J$ is contained in $\mathcal{L}_I * \mathcal{L}_J$ and that $K(U) = I(J(U(s,t))$. By the commutativity of $\mathfrak{E}$ it also follows that $\mathcal{L}_I \otimes \mathcal{L}_J$ is contained in $\mathcal{L}_J * \mathcal{L}_I$ and that

$$K(U) = I(J(U(s,t))) = J(I(U(s,t))) \text{ for all } U \in \mathcal{L}_I \otimes \mathcal{L}_J.$$

LEMMA 4.1.

$$\mathcal{L}_I \overline{\otimes} \mathcal{L}_J \subset \mathcal{L}_I * \mathcal{L}_J.$$

Proof. Let $U \in \mathcal{L}_I \overline{\otimes} \mathcal{L}_J$. By the properties of the Fremlin tensor product, there exist an element $(V,W) \in \mathcal{L}_I \times \mathcal{L}_J$ such that $U \le V \otimes W$ and a sequence $Z_n \in \mathcal{L}_I \otimes \mathcal{L}_J$ that converges $V \otimes W$-uniformly to U. Then, for the upper semi-integral $\dot{J}$, we have for $\epsilon > 0$ an index n_0 such that for each fixed s and $n \ge n_0$,

$$\dot{J}\left(|U(s,t) - Z_n(s,t)|\right) \le \epsilon \dot{J}(V(s)W(t)) = \epsilon J(V(s)W(t)) = \epsilon V(s)J(W).$$

Hence, $\dot{J}\left(|U(s,t) - Z_n(s,t)|\right)$ converges to 0 for each fixed s. It follows by Theorem 2.2 that $U(s,t)$ is a J-summable function of t for each fixed s.

Similarly,

$$\dot{I}\left(|J(U(s,t) - Z_n(s,t))|\right) \le \epsilon \dot{I}((V)(s)JW(t) = \epsilon I((V(s)JW(t)) = \epsilon I(V)J(W).$$

So, again by Theorem 2.2, $J(U(s,t)$ is I-summable.

This completes the proof. □

It follows by (4.1) that IJ, JI and K are equal on $\mathcal{L}_I \otimes \mathcal{L}_J$, and they all induce the positive bilinear operator b on $\mathcal{L}_I \times \mathcal{L}_J$. Hence, by the uniqueness of the extension of the linearization of the positive bilinear operator b from the algebraic tensor product to the Fremlin tensor product (Theorem 1.2(3)), $K = IJ = JI$ on $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$.

THEOREM 4.2. *Let K be the unique extension of the linearization of b, defined on $\mathcal{L}_I \otimes \mathcal{L}_J$, to $\mathcal{L}_I \overline{\otimes} L_J$. Then K is a Daniell integral on $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$.*

Proof. Let $Z_n(s,t) \downarrow 0$ for every $(s,t) \in S \times T$, with $Z_n \in \mathcal{L}_I \overline{\otimes} \mathcal{L}_Y$. Then for each fixed s, $Z_n^s(t) := Z_n(s,t) \downarrow 0$, and so, $J(Z_n^s(t)) \downarrow 0$ for each fixed s, implying that $I(J(Z_n(s,t))) \downarrow 0$. But, on $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$, IJ is equal to K. Thus $K(Z_n) \downarrow 0$. It follows that K is a primitive Daniell integral on the initial domain $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$. □

Using Daniell's procedure as explained in Section 2, K can be extended to the Riesz space $\mathcal{L}_K$ of all Daniell summable functions in $\mathfrak{E}^{S \times T}$.

The relation established between K, and IJ and JI still holds true for the extended integral. We get the following theorem that contains both the theorems of Fubini and Tonelli.

THEOREM 4.3. *The sets of K-summable, IJ-summable and JI-summable functions are equal, i.e.,*

$$\mathcal{L}_K = \mathcal{L}_{IJ} = \mathcal{L}_{JI}$$

and for all $U \in \mathcal{L}_K$ we have

$$K(U) = IJ(U) = JI(U).$$

Proof. Extending the three primitive Daniell integrals K, IJ and JI that are equal on the common initial domain $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$, yields the same integrals and the same sets of summable functions. Therefore, $\mathcal{L}_K = \mathcal{L}_{IJ} = \mathcal{L}_{JI}$ and $K(U) = IJ(U) = JI(U)$ for all elements U of $\mathcal{L}_K$. □

This theorem was traditionally split in two parts, which we state as corollaries.

COROLLARY 4.4. (Theorem of Fubini) *$\mathcal{L}_K \subset \mathcal{L}_{IJ} = \mathcal{L}_{JI}$ and for every $K \in \mathcal{L}_K$, we have that*

$$K(U(s,t)) = I(J(U^s(t)) = J(I(U^t(s)).$$

Thus, the integral of every summable function of two variables can be calculated as an iterated integral with the integration performed in any order.

COROLLARY 4.5. (Tonelli) *If the iterated integral $IJ(U)$ exists and if $IJ(|U|) \in \mathfrak{E}$, then $IJ(U) = JI(U) \subset \mathcal{L}_K$. Thus, the condition $IJ(|U|) \in \mathfrak{E}$ implies that the order of integration in the integral $IJ(U)$ can be reversed and that the function U is K-summable.*

Proof. Our assumption implies that $U \in \mathcal{L}_{IJ} = \mathcal{L}_K = \mathcal{L}_{JI}$. $\square$

We note that examples show that in general $IJ(U)$ and $JI(U)$ may exist without being equal. Such a function U is then obviously not summable.

Our final corollary is often stated in the context of Fubini's theorem.

THEOREM 4.6. *If $U(s,t) \geq 0$ and if $IJ(U) \in \mathfrak{E}$, then U is K-summable and $K(U(s,t) = I(J(U^s(t)) = J(I(U^t(s))$.*

We conclude with the remark that the Fremlin tensor product and Fremlin's extension theorem for positive operators play a crucial role in the proof of this theorem. Even in the scalar case, the theorem that we proved yields a generalization of the Fubini theorem as proved by Zaanen in [7], because we do not assume the Stone condition ($\min(X, 1) \in \mathcal{L}_0$ for all $X \in \mathcal{L}_0$) to hold.

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