Solutions of two (3+1)-dimensional Gardner-type equations using Lie symmetry analysis

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Dissertation accepted in fulfilment of the requirements for the degree Master of Science in Applied Mathematics at the North West University

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Graduation: July 2023

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SOLUTIONS OF TWO (3+1)-DIMENSIONAL GARDNER-TYPE EQUATIONS USING LIE SYMMETRY ANALYSIS

by

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Dissertation submitted in fulfilment of the requirements for the degree of Master in Applied Mathematics in the Department of Mathematical Sciences in the Faculty of Natural and Agricultural Sciences at North-West University, Mafikeng Campus

November 2022

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Declaration

I, Masego Mafora (31179797), declare that the dissertation for the degree of Master of Science at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: .....

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Date: .....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

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Dedication

To my family, Tshireletso Mpolokeng and everyone who contributed to my entire studies.

Acknowledgements

Thank you so much, Dr L.D. Moleleki and Dr G. Magalakwe, for the wonderful support and guidance throughout this entire project.

Thank you so much, Dr K. Plaatjie and Mr M.Y.T. Lephoko, for mentoring and supporting me with everything I struggled with.

All thanks go to the National Research Foundation (NRF) and North-West University (Mafikeng Campus) for the financial support they showered me with.

Finally, thanks to my family for their everlasting love.

Abstract

This research project seeks to investigate the two (3+1)-dimensional Gardner-type equations that fall under the nonlinear partial differential equations (NPDEs), which appear in sound wave nonlinear physical phenomena. The study uses Lie symmetry to obtain the class of exact solutions of NPDE models under investigation. Exact solutions are obtained from transformed solvable nonlinear ordinary differential equations (NODEs) representing the models obtained from Lie point symmetries without changing the dynamics of the models. The obtained solutions are represented graphically and analysed to study momentum (velocity) and energy variation in order to explore the sound wave dynamics modelled by the Gardner-type equations studied in this project. In addition to exact solutions, conservation laws that represent conserved quantities of the models under investigation are obtained by invoking Ibragimovs theorem and multipliers approach.

Introduction

Various nonlinear phenomena have received much attention due to their industrial importance and the applications addressed in nonlinear sciences. These nonlinear phenomena that constitute real-international conditions are defined through NPDEs. Models of NPDEs in arithmetic and physics, play a critical function in theoretical science. Understanding these NPDEs is also important for many applications, such as meteorology and the space industry to name a few. NPDEs are the most basic models for studying nonlinear phenomena of practical importance. Finding solutions for such NPDEs that represent real-world situations helps us understand the dynamics modelled by these NPDEs. Solving such NPDEs helps engineers, physicists and scientists to accurately analyse such nonlinear phenomena.

Finding the exact solutions of such NPDEs is difficult, if not impossible. Mathematicians have proposed various efficient and practical methods to solve NPDEs analytically in recent years. There is absolutely no unique way to obtain the exact solutions of such NPDEs. Some of the practical methods include the nonclassical Lie group approach [1], Lie symmetry method [2], sine-cosine method [3], Clarkson-Kruskals method [4], Deformation mapping method [5], transformed rational function method [6], and the simplest equation method [7].

In addition to solutions of NPDEs, one other important analysis is to find conservation laws of such NPDEs, which represent conserved quantities such as mass, momentum, energy and concentration of the case study under investigation. The obtained conserved quantities provide ways to study mass flow, momentum, energy and concentration variation in a system. Conservation laws are one of the important

concepts in physics. In mathematics, conservation laws provide one of the fundamental principles of modelling and research. For example, the existence of a large set of conservation laws for partial differential equations (PDEs) can be a powerful indicator of their integrability. Conservation laws are utilised to find the existence, stability, and uniqueness of NPDEs solutions. The methods that are mostly used to derive the conservation laws are the multiplier method [8], Noethers theorem [9], and the conservation theorem by Ibragimov [10]. This makes conservation laws essential in the study and solution process of NPDEs.

One such important phenomena which is modelled by NPDEs is wave motions. Wave motion describes a wide range of phenomena across many diverse applications. Such wave phenomena are defined by various properties, one of them being their ability to travel. Travelling waves and plane waves are two basic categories of waves. The travelling waves, which are real-valued, keep their shape while moving at a constant speed. On the other hand, plane waves are a complex-valued form of travelling waves. As mentioned in the study of Degon and Chowdhury [11], a special kind of wave known as solitary waves were discovered in the 19th century. They (solitary waves) are considered unique because they can propagate without change on the surface of shallow inviscid fluid layers and retain their shape upon interaction. In a sense, they have earned the name soliton due to this ability to maintain the same shape. While regular waves usually collapse or spread out because of different speeds, solitons can travel long distances without collapsing or spreading out.

However, the governing equations for solitons are, as a rule, nonlinear. In the study of Korteweg and de Vries [12], one fundamental equation which describes the propagation of nonlinear waves in a one-dimensional case in a weakly dispersive medium is a famous Korteweg-de Vries (KdV) equation. One of the critical physical properties of such nonlinear equations that admit soliton solutions is that they have infinite conservation laws and associated symmetries strongly related to their integrability.

Ito [13] introduced another equation which describes the propagation of nonlinear waves named the modified Korteweg-de Vries (mKdV) equation by employing Miura

transformation to the famous KdV equation. Also, mKdV equation was used to understand the process of the role of the nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compactons (solitons with compact support) as mentioned in [14]. Different techniques have been used to construct solutions of mKdV equation. In 1972, exact solutions of mKdV were constructed by Hirota [15], thereafter Tanaka employed the inverse scattering technique to solve the mKdV equation [16].

Previously, back in 1968, there was a mathematician named Clifford Gardner who introduced a NPDE called the Gardner equation, which is a generalization of KdV and mKdV to learn more about wave phenomena. The investigation of the (1+1)-dimensional Gardner equation given below

$$v_t + 6\,\beta\,v\,v_x + v_{xxx} - \frac{3}{2}\,\alpha^2\,v^2\,v_x = 0, \tag{1}$$

where β, α are arbitrary constants which models sound wave propagation, was done by Alfred Osborne [17]. The well-known KdV equation is the special case of the above equation (1) when $\alpha = 0$, thus the KdV equation is complemented with a higher-order cubic nonlinear term of the form $v^2\,v_x$ known as the dispersion term to obtain the Gardner equation (1), which is completely integrable like the KdV equation and the mKdV equation as mentioned in [18].

Equation (1) has been studied by various scientists to find the solutions which they used to learn more about the dynamics of this equation. Yu et al. [19] solved Gardner's equation using the Riccati expansion method and obtained soliton-cnoidal wave interaction solutions. Ren [20] solved Clifford Gardner's equation (1) by applying the Lie symmetry method to construct different types of solutions such as solitary waves, rational solutions, Painlevé II solutions. The exact solution of (1+1)-dimensional Gardner equation (1) was obtained using the improved $tan(\Theta(\vartheta))$-expansion method and the wave ansatz method by Gunerhan [21].

To investigate how the dynamics modelled by Gardner equations evolved within a plane led to the study of (2+1)-dimensional Gardner equations. The study of how sound wave propagation evolves in a plane over time was studied in [18] by Wazwaz.

Wazwaz, in his study considered the integro-differential form of (2+1)-dimensional Gardner equation given by

$$v_t + 6\,\beta\,v\,v_x + v_{xxx} - \frac{3}{2}\,\alpha^2\,v^2\,v_x + 3\,\sigma^2\,\partial_x^{-1}\,v_{yy} - 3\,\alpha\,\sigma\,\partial_x^{-1}\,v_y = 0, \tag{2}$$

where ∂_x^{-1} as the inverse of ∂_x such that $\partial_x^{-1}\partial_x = 1$, and solved equation (2) using Hirota's direct method. The author obtained multiple kink and multiple singular kink solutions of the integro-differential form of the above (2+1)-dimensional Gardner equation. The (2+1)-dimensional Gardner equation (2) was also studied by Konopelchenko using the inverse spectral transform method in [22]. Konopelchenko, in his study, used the δ-dressing method to construct classes of exact solutions for the (2+1)-dimensional Gardner equation (2). In [23], the authors presented the Casorati and Grammian determinant solutions of the (2+1)-dimensional Gardner equation (2). The authors Chen and Yan [24] utilized a method based on the Sine-Gordon equation expansion to construct precise solutions of equation (2). Kumar and Tanwar [25], on the other hand, used the similarity transformation method to obtain invariant solutions of equation (2).

The analysis of (3+1)-dimensional Gardner-type equations was done by Wazwaz [26], from which the inclusion of the two terms $3\,\sigma^2\,\partial_x^{-1}\,v_{zz}$ and $3\,\sigma^2\,\partial_x^{-1}\,v_{zz} - 3\,\alpha\,\sigma\,v_x\partial_x^{-1}\,v_z$, respectively to the (2+1)-dimensional Gardner equation (2), lead to the study of the following two (3+1)-dimensional Gardner-type equations in an integro-differential form given by

$$v_t + 6\,\beta\,v\,v_x + v_{xxx} - \frac{3}{2}\,\alpha^2\,v^2\,v_x + 3\,\sigma^2\,\partial_x^{-1}\,v_{yy} - 3\,\alpha\,\sigma\,\partial_x^{-1}\,v_y + 3\,\sigma^2\,\partial_x^{-1}\,v_{zz} = 0, \tag{3}$$

and

$$\begin{aligned} &v_t + 6\,\beta\,v\,v_x + v_{xxx} - \frac{3}{2}\,\alpha^2\,v^2\,v_x + 3\,\sigma^2\,\partial_x^{-1}\,v_{yy} - 3\,\alpha\,\sigma\,\partial_x^{-1}\,v_y + 3\,\sigma^2\,\partial_x^{-1}\,v_{zz} \\ &+ 3\,\sigma^2\,\partial_x^{-1}\,v_{zz} - 3\,\alpha\,\sigma\,v_x\partial_x^{-1}\,v_z = 0, \end{aligned} \tag{4}$$

respectively, which model ion plasma sound wave propagation. To eliminate the integrals from (3) and (4), Wazwaz [26] used the following potential transformation

$$v(t,x,y,z) = u_x(t,x,y,z),$$

which transformed equations (3) and (4) into the following differential equations

$$u_{xt} + 6\,\beta\,u_x u_{xx} + u_{xxxx} - \frac{3}{2}\,\alpha^2\,u_x^2 u_{xx} + 3\,\sigma^2\,u_{yy} - 3\,\alpha\,\sigma\,u_{xx}u_y + 3\,\sigma^2\,u_{zz} = 0, \quad (5)$$

and

$$u_{xt} + 6\,\beta\,u_x u_{xx} + u_{xxxx} - \frac{3}{2}\,\alpha^2\,u_x^2 u_{xx} + 3\,\sigma^2\,u_{yy} - 3\,\alpha\,\sigma\,u_{xx}u_y + 3\,\sigma^2\,u_{zz} - 3\,\alpha\,\sigma\,u_{xx}u_z = 0, \quad (6)$$

where t is an independent variable representing time, x, y and z are independent variables that represent spatial coordinates and $u(t, x, y, z)$ denotes the elongation/propagation of the wave at any point (x, y, z). Thereafter, Wazwaz [26] obtained multiple kink solutions of the above two (3+1)-dimensional Gardner-type equations (5) and (6) using the Hirota bilinear method and defined the necessary conditions for which multiple kink solutions exist. Recently, the (3+1)-dimensional integro-differential form of the Gardner equation given by equation (3) was solved using Hirotas bilinear method along with the Cole-Hopf transformation to derive exact travelling solutions of the (3+1)-dimensional Gardner-type equation in the works of A Aslanov [27]. The author illustrated how to obtain travelling wave solutions from double-kink solutions with the necessary conditions for which the double-kink solutions exist.

Most real-world applications are modelled by nonlinear differential equations, which are sometimes difficult to solve analytically and numerically, if not impossible. Nonlinear phenomena have gained a lot of attention which led to various researchers investigating such models experimentally and numerically. Based on the literature above, none of the studies has directly managed to integrate (3+1)-dimensional Gardner-type equations. The current solutions, according to the literature, exist under certain conditions, see [26] for such conditions. Thus, this current study seeks to find exact solutions of (3+1)-dimensional Gardner-type equations which exist without conditions like the one in the [26]. To obtain the exact solutions of a NPDE models under investigation, the study aims to transform the PDEs into a simpler model that can be solved directly without assuming the solution, such as letting the solution to be of

a certain form like in the case of using simplest equation method [7] and sine-cosine method [3] to name a few.

The analysis of sound waves in three-dimensional space and time, the (3+1)-dimensional Gardner-type equations (5) and (6) studied by Wazwaz [26] and the difficulty to integrate these equations directly, inspired further analysis of these equations in order to learn more about the (3+1)-dimensional Gardner-type equations and to find a simpler way of solving these models. Thus, solve these equations using standard integration techniques and also have a better understanding of how sound wave evolves in space, thus study the wave movement along the x, y and z, respectively as time evolves. The literature shows that the integration of such models directly is still a big challenge, hence the study aims to find a way to solve equations (5) and (6) directly. The above two equations (5) and (6) are the two equations of interest in this dissertation since their solutions lead to better understanding of sound wave dynamics along all directions (space).

In this study, we seek to directly integrate the two (3+1)-dimensional Gardner-type equations (5) and (6) for the first time and obtain exact solutions that exists without any conditions. Lie symmetry analysis will be used to transform both equation (5) and (6) into solvable ODEs without changing the physics modelled by these equations. The transformed ODEs will be solved using direct integration to find elongation solutions (velocity). Also, the obtained solutions will be used to study velocity, momentum and energy variation during wave elongation. Graphical representations will be included to portray the dynamics of the solutions obtained. Lastly, by making use of the Ibragimov's theorem and the help of the multiplier method, conservation laws are then obtained.

Motivated by the studies mentioned above and the task to get an optimal way to solve systems of NPDEs, the current research project seeks to advance further ways to solve NPDEs analytically by investigating the two (3+1)-dimensional Gardner-type equations. These two (3+1)-dimensional Gardner-type equations model ion plasma sound waves propagation. In addition to solutions of the (3+1)-dimensional

Gardner-type equations, the study will further compute conserved quantities that represent mass flow, momentum and energy variation during the propagation of sound waves [18, 22, 28–32].

The layout of this dissertation is as follows:

Chapter 1, theorems and definitions covering the one-parameter groups of transformations are introduced. Fundamental operators and their connection for the conservation laws are then given. Also, the variational method for a system and its adjoint is presented.

Chapter 2 generates, Lie point symmetries of the first (3+1)-dimensional Gardner-type equation by the help of Lie group theorem. The Lie point symmetries generated are then utilized for symmetry reduction to obtain different ordinary differential equations (ODEs). Solving those ODEs, we get different exact solutions and analyse them. Lastly, by the help of Ibragimovs theorem and multiplier's approach, we acquire conservation laws.

Chapter 3, Lie point symmetries of the second (3+1)-dimensional Gardner-type equation are obtained as well. By using those Lie point symmetries obtained, we transform the (3+1)-dimensional Gardner-type equation to an ODE, then use the ODE to obtain the exact solution of the second (3+1)-dimensional Gardner-type equation. Exact solutions are also represented graphically and are analysed to illustrate the dynamic of the model equation under investigation. Finally, conservation laws are then generated by using Ibragimov theorem and Multiplier's technique.

In Chapter 4, we summarise the investigation of this research project as well as proposed future work.

Bibliography at the end is shown. The purpose of this is to show all the references we used throughout this research project.

Chapter 1

Lie symmetry methods for differential equations and the conservation theorems

Some fundamental techniques of the well known Lie symmetry analysis of DEs are introduced in this Chapter as well as the algorithm to find the Lie point symmetries of PDEs. Also, we give the fundamental operators and their relationships and the variational approaches to construct conservation laws for a system of PDEs.

1.1 Introduction

At the end of the 19th century, the eminent mathematician called Sophus Lie (1842-1899) came up with a new technique, named Lie group analysis, for working out DEs and revealed that most of the adhoc techniques of integration of DEs could be described, and deduced simply by means of his theory. Lately, most excellent books have emerged in the writings in this field. We mention a few of them here, Bluman and Kumei [33], Ovsiannikov [34], Olver [35], Stephani [36], Ibragimov [37, 38], Cantwell [39], see also Mahomed [40].

This Chapter has definitions and outcomes which are taken from the books hinted

above.

1.2 Local one-parameter Lie group

Here a change may be understood to intend an invertible transformation, i.e., a bijective map. By deciding to let u be our new dependent variable and then afterwards take t, x, y and z to be also our new independent variables. Considering a small change of the variables t, x, y, z and u:

$$\begin{aligned} T_a: \quad & \bar{t} = f(t,x,y,z,u,a),\ \bar{x} = g(t,x,y,z,u,a), \\ & \bar{y} = l(t,x,y,z,u,a),\ \bar{z} = k(t,x,y,z,u,a) \\ & \bar{u} = h(t,x,y,z,u,a). \end{aligned} \tag{1.1}$$

This time we choose a to be a real parameter, which continuously differs in value from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f, g, k, l and h are differentiable functions.

Definition 1.1 A *continuous one-parameter (local) Lie group of transformations* is a set G of transformations (1.1) satisfying three of these conditions:

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then T_b, $T_a = T_c \in G$, $c = \phi(a,b) \in \mathcal{D}$ (Closure),

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\, T_a = T_a\, T_0 = T_a$ (Identity),

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\, T_{a^{-1}} = T_{a^{-1}}\, T_a = T_0$ (Inverse).

From (i) associativity property is satisfied. On top of that, if it happens that the identity transformation occurs at $a = a_0 \neq 0$ i.e., $T_{a_0}$is the identity, then a shift of

the parameter $a = \bar{a} + a_0$ will give T_0 as above. The property (i) can be written as

$$\begin{aligned}
\bar{\bar{t}} &\equiv f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, b) = f(t, x, y, z, u, \phi(a, b)), \\
\bar{\bar{x}} &\equiv g(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, b) = g(t, x, y, z, u, \phi(a, b)), \\
\bar{\bar{y}} &\equiv k(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, b) = k(t, x, y, z, u, \phi(a, b)), \\
\bar{\bar{z}} &\equiv l(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, b) = l(t, x, y, z, u, \phi(a, b)), \\
\bar{\bar{u}} &\equiv h(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, b) = h(t, x, y, z, u, \phi(a, b)).
\end{aligned} \tag{1.2}$$

The function ϕ is termed as the *group composition law*. A group parameter a is named *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial\, \phi(s, b)}{\partial b} \right|_{b=0}.$$

Symmetry group definition for the 4th-order partial DE is provided

$$u_t = F(t, x, y, u, u_t, u_x, u_y, u_z, u_{xx}, u_{xt}, u_{yy}, u_{zz}, ...), \quad \frac{\partial F}{\partial u_{xxxx}} \neq 0. \tag{1.3}$$

Definition 1.2 (Symmetry group) A one-parameter group G of transformations (1.1) is called a *symmetry group* of (1.3) if it is form-invariant (has the same form) in the new variables $\bar{t}, \bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{u}$, i.e. ,

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}}, \bar{u}_{\bar{z}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{t}}, \bar{u}_{\bar{y}\bar{y}}, \bar{u}_{\bar{z}\bar{z}}, \bar{u}_{\bar{x}\bar{y}\bar{y}}, ...), \tag{1.4}$$

where the function F is the same as in (1.3).

1.3 Infinitesimal transformations

According to Lie's theory, the construction of the symmetry group G is identical to the determination of the corresponding *infinitesimal transformations* :

$$\begin{aligned}
&\bar{t} \approx t + a\, \xi^1(t, x, y, z, u), \;\; \bar{x} \approx x + a\, \xi^2(t, x, y, z, u), \\
&\bar{y} \approx y + a\, \xi^3(t, x, y, z, u), \;\; \bar{z} \approx z + a\, \xi^4(t, x, y, z, u) \\
&\bar{u} \approx u + a\, \eta(t, x, y, z, u)
\end{aligned} \tag{1.5}$$

generated from (1.1) by expanding the following functions f, g, k, l and h into Taylor series in a about $a = 0$, not forgetting the initial conditions

$$f|_{a=0} = t, \quad g|_{a=0} = x, \quad l|_{a=0} = y, \quad k|_{a=0} = z, \quad h|_{a=0} = u$$

Therefore, we get

$$\begin{aligned}
\xi^1(t,x,y,z,u) &= \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \xi^2(t,x,y,z,u) = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \\
\xi^3(t,x,y,z,u) &= \left.\frac{\partial l}{\partial a}\right|_{a=0}, \quad \xi^4(t,x,y,z,u) = \left.\frac{\partial k}{\partial a}\right|_{a=0} \\
\eta(t,x,y,z,u) &= \left.\frac{\partial h}{\partial a}\right|_{a=0}
\end{aligned} \tag{1.6}$$

Finally, (1.5) can be written as

$$\bar{t} \approx (1 + a\,R)t, \quad \bar{x} \approx (1 + a\,R)x, \quad \bar{y} \approx (1 + a\,R)y, \quad \bar{z} \approx (1 + a\,R)z, \quad \bar{u} \approx (1 + a\,R)u,$$

where,

$$\begin{aligned}
R &= \xi^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,u)\frac{\partial}{\partial z} \\
&\quad + \eta(t,x,y,z,u)\frac{\partial}{\partial u}.
\end{aligned} \tag{1.7}$$

R is the well known operator mostly called the *infinitesimal operator (generator)* of the group G. Whenever group G is declared by (1.3), R then is an *admitted operator* of (1.3) or R becomes an *infinitesimal symmetry* of (1.3).

1.4 Group invariants

Definition 1.3 A function $F(t,x,y,z,u)$ is called an *invariant of the group of transformation* (1.1) if

$$\begin{aligned}
F(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) &\equiv F(f(t,x,y,z,u,a), g(t,x,y,z,u,a), k(t,x,y,z,u,a), l(t,x,y,z,u,a), \\
&\quad h(t,x,y,z,u,a)) \\
&= F(t,x,y,z,u),
\end{aligned} \tag{1.8}$$

identically in t, x, y, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(t,x,y,z,u)$ to be an invariant is that

$$\begin{aligned} R\,F &= \xi^1(t,x,y,z,u)\frac{\partial F}{\partial t}+\xi^2(t,x,y,z,u)\frac{\partial F}{\partial x}+\xi^3(t,x,y,z,u)\frac{\partial F}{\partial y}+\xi^4(t,x,y,z,u)\frac{\partial F}{\partial z} \\ &\quad +\eta(t,x,y,z,u)\frac{\partial F}{\partial u}. \end{aligned} \tag{1.9}$$

From Theorem 1.2, it follows that every one-parameter group of point transformations (1.1) contains four independent invariants, taken to be the left-hand side of any first integrals

$$J_1(t,x,y,z,u)=c_1,\;\; J_2(t,x,y,z,u)=c_2,\;\; J_3(t,x,y,z,u)=c_3,\;\; J_4(t,x,y,z,u)=c_4$$

of the characteristic equations

$$\frac{dt}{\xi^1(t,x,y,z,u)}=\frac{dx}{\xi^2(t,x,y,z,u)}=\frac{dx}{\xi^3(t,x,y,z,u)}=\frac{dy}{\xi^4(t,x,y,z,u)}=\frac{du}{\eta(t,x,y,z,u)}.$$

Theorem 1.3 infinitesimal transformation (1.5) or its symbol R, the corresponding one-parameter group G is generated by working out the Lie equations

$$\begin{aligned} &\frac{d\bar{t}}{da}=\xi^1(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}), \quad \frac{d\bar{x}}{da}=\xi^2(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}), \\ &\frac{d\bar{y}}{da}=\xi^3(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}), \quad \frac{d\bar{z}}{da}=\xi^4(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \\ &\frac{d\bar{u}}{da}=\eta(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \end{aligned} \tag{1.10}$$

subject to the initial conditions

$$\bar{t}|_{a=0}=t, \quad \bar{x}|_{a=0}=x, \quad \bar{y}|_{a=0}=y, \quad \bar{z}|_{a=0}=z, \quad \bar{u}|_{a=0}=u\,.$$

1.5 Construction of a symmetry group

Definitions are given before we express the algorithm to dictate a symmetry group for a given PDE

1.5.1 Prolongation of point transformations

Let us consider a fourth-order PDE

$$E(t, x, y, u, u_t, u_x, u_y, u_z, u_{xx}, u_{xt}, u_{yy}, u_{zz}, u_{xxxx}, ...) = 0, \tag{1.11}$$

having t, x, y and z as the four independent variables as well as taking u as a dependent variable. By letting

$$R = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \tag{1.12}$$

be named the infinitesimal generator of the one-parameter group G of transformation (1.1) where the coefficients η and ξ^i, where $i = 1, 2, 3, 4$ are functions of t, x, y, z and u. The *fourth prolongation* of R is given by $R^{[4]}$ and is denoted by

$$\begin{aligned} R^{[4]} = \quad & R + \zeta^2 \frac{\partial}{\partial u_x} + \zeta^3 \frac{\partial}{\partial u_y} + \zeta^{12} \frac{\partial}{\partial u_{tx}} + \zeta^{22} \frac{\partial}{\partial u_{xx}} + \zeta^{33} \frac{\partial}{\partial u_{yy}} \\ & + \zeta^{44} \frac{\partial}{\partial u_{zz}} + \zeta^{2222} \frac{\partial}{\partial u_{xxxx}} + ... \quad , \end{aligned} \tag{1.13}$$

where,

$$\begin{aligned} \zeta^2 &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4), \\ \zeta^3 &= D_y(\eta) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3) - u_z D_y(\xi^4), \\ \zeta^{12} &= D_x(\zeta^1) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - u_{ty} D_x(\xi^3) - u_{tz} D_x(\xi^4), \\ \zeta^{22} &= D_2(\zeta^2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4), \\ \zeta^{33} &= D_y(\zeta^3) - u_{ty} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{yy} D_y(\xi^3) - u_{yz} D_y(\xi^4), \\ \zeta^{44} &= D_z(\zeta^4) - u_{tz} D_z(\xi^1) - u_{xz} D_z(\xi^2) - u_{yz} D_z(\xi^3) - u_{zz} D_z(\xi^4), \\ \zeta^{2222} &= D_x(\zeta^{222}) - u_{txxx} D_x(\xi^1) - u_{xxxx} D_x(\xi^2) - u_{xxxy} D_x(\xi^3) - u_{xxxz} D_x(\xi^4), \end{aligned}$$

Here, the total derivatives D_t, D_x,D_y and D_z are given by, respectively

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{1.14}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + \cdots, \tag{1.15}$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \cdots, \tag{1.16}$$

$$D_z = \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{xz} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_t} + \cdots \tag{1.17}$$

By making use of the definitions of D_t, D_x,D_y and D_z given above, we obtain

$$\begin{aligned}
\zeta_1 =& \eta_t + \eta_u\, u_t - u_t\, \xi^1_t - \xi^1_u\, u^2_t - u_x\, \xi^2_t - u_x\, \xi^2_u\, u_t - u_y\, \xi^3_t - u_y\, \xi^3_u\, u_t \\
& - u_z\, \xi^4_t - u_z\, \xi^4_u\, u_t, \\
\zeta_2 =& \eta_x + \eta_u\, u_x - u_t\, \xi^1_x - u_t\, \xi^1_u\, u_x - u_x\, \xi^2_x - \xi^2_u\, u^2_x - u_y\, \xi^3_x - u_y\, \xi^3_u\, u_x \\
& - u_z\, \xi^4_x - u_z\, \xi^4_u\, u_x, \\
\zeta_3 =& \eta_y + \eta_u\, u_y - u_t\, \xi^1_y - u_t\, \xi^1_u\, u_y - u_x\, \xi^2_y - u_x\, \xi^2_u\, u_y - u_y\, \xi^3_y - \xi^3_u\, u^2_y \\
& - u_z\, \xi^4_y - u_z\, \xi^4_u\, u_y, \\
\zeta_4 =& \eta_z + \eta_u\, u_z - u_t\, \xi^1_z - u_t\, \xi^1_u\, u_z - u_x\, \xi^2_z - u_x\, \xi^2_u\, u_z - u_y\, \xi^3_z - u_y\, \xi^3_u\, u_z - \\
& u_z\, \xi^4_z - \xi^4_u\, u^2_z, \\
\zeta_{22} =& \eta_{xx} - u_y\, \xi^3_u\, u_{xx} - u_z\, \xi^4_u\, u_{xx} - u_t\, \xi^1_u\, u_{xx} - u_t\, u^2_x\, \xi^1_{uu} - 2\, u_t\, \xi^1_{xu}\, u_x \\
& - 2\, u_y\, \xi^3_{xu}\, u_x - 2\, u_z\, \xi^4_{xu}\, u_x - u_y\, u^2_x\, \xi^3_{uu} - u_z\, u^2_x\, \xi^4_{uu} - u_y\, \xi^3_{xx} \\
& - u_z\, \xi^4_{xx} - u^3_x\, \xi^2_{uu} - u_t\, \xi^1_{xx} - u_x\, \xi^2_{xx} + \eta_{uu}\, u^2_x + 2\, \eta_{xu}\, u_x \\
& - 2\, \xi^2_{xu}\, u^2_x - 3\, \xi^2_u\, u_x\, u_{xx} - 2\, u_{xz}\, \xi^4_u\, u_x - 2\, u_{tx}\, \xi^1_u\, u_x - 2\, u_{xy}\, \xi^3_u\, u_x \\
& + \eta_u\, u_{xx} - 2\, u_{xy}\, \xi^3_x - 2\, u_{xz}\, \xi^4_x - 2\, u_{tx}\, \xi^1_x - 2\, u_{xx}\, \xi^2_x, \\
\zeta_{12} =& \eta_{xt} - u^2_t\, \xi^1_{uu}\, u_x - u_y\, \xi^3_{ut}\, u_x - u_z\, \xi^4_{xt} - u_y\, u_t\, \xi^3_{uu}\, u_x - u_z\, u_t\, \xi^4_{uu}\, u_x - u_{xz}\, \xi^4_u\, u_t \\
& - u_{tz}\, \xi^4_u\, u_x - u_z\, \xi^4_u\, u_{tx} - u_{tt}\, \xi^1_u\, u_x - u_{xx}\, \xi^2_u\, u_t - u_y\, \xi^3_u\, u_{tx} - u_{xy}\, \xi^3_u\, u_t \\
& - u_{ty}\, \xi^3_u\, u_x - u^2_x\, u_t\, \xi^2_{uu} - 2\, u_x\, \xi^2_u\, u_{tx} - 2\, \xi^1_u\, u_t\, u_{tx} + u_t\, \eta_{xu} - u^2_t\, \xi^1_{xu} \\
& - u_t\, \xi^1_{xt} - u_y\, \xi^3_{xt} + \eta_u\, u_{tx} - u_{xy}\, \xi^3_t - u_{xz}\, \xi^4_t - u_{tx}\, \xi^1_t - u_{ty}\, \xi^3_x - u_{tz}\, \xi^4_x - u_{tt}\, \xi^1_x \\
& - u_{xx}\, \xi^2_t - u_{tx}\, \xi^2_x - u_z\, \xi^4_{ut}\, u_x - u_t\, \xi^1_{ut}\, u_x - u_z\, u_t\, \xi^4_{xu} - u_y\, u_t\, \xi^3_{xu} - u_x\, u_t\, \xi^2_{xu} \\
& - u_x\, \xi^2_{xt} + \eta_{ut}\, u_x + u_t\, \eta_{uu}\, u_x - \xi^2_{ut}\, u^2_x, \\
\zeta_{33} =& \eta_{yy} - u_z\, \xi^4_u\, u_{yy} - u_t\, \xi^1_u\, u_{yy} - u_x\, \xi^2_u\, u_{yy} - 2\, u_{xy}\, \xi^2_u\, u_y - 2\, u_{yz}\, \xi^4_u\, u_y - 3\, \xi^3_u\, u_y\, u_{yy} \\
& - 2\, u_{ty}\, \xi^1_u\, u_y - u_t\, \xi^1_{yy} - u_x\, \xi^2_{yy} - u_y\, \xi^3_{yy} - u_z\, \xi^4_{yy} - u^3_y\, \xi^3_{uu} + \eta_{uu}\, u^2_y + \eta_u\, u_{yy} \\
& - 2\, u_{yy}\, \xi^3_y - 2\, u_{yz}\, \xi^4_y - 2\, u_{ty}\, \xi^1_y - 2\, u_{xy}\, \xi^2_y + 2\, \eta_{yu}\, u_y - u_x\, u^2_y\, \xi^2_{uu} - 2\, u_z\, u_y\, \xi^4_{yu} \\
& - u_t\, u^2_y\, \xi^1_{uu} - u_z\, u^2_y\, \xi^4_{uu} - 2\, \xi^3_{yu}\, u^2_y - 2\, u_t\, u_y\, \xi^1_{yu} - 2\, u_x\, u_y\, \xi^2_{yu}, \\
\zeta_{44} =& \eta_{zz} - 2\, u_y\, \xi^3_{zu}\, u_z - 3\, \xi^4_u\, u_z\, u_{zz} - 2\, u_{yz}\, \xi^3_z + \eta_{uu}\, u^2_z - 2\, u_{tz}\, \xi^1_z - 2\, u_{xz}\, \xi^2_z - 2\, u_{zz}\, \xi^4_z \\
& + \eta_u\, u_{zz} - u_x\, \xi^2_{zz} - u_t\, \xi^1_{zz} - u_z\, \xi^4_{zz} + 2\, \eta_{zu}\, u_z - u^3_z\, \xi^4_{uu} - u_y\, \xi^3_{zz} - u_x\, u^2_z\, \xi^2_{uu} \\
& - u_y\, u^2_z\, \xi^3_{uu} - u_y\, \xi^3_u\, u_{zz} - 2\, u_{yz}\, \xi^3_u\, u_z - 2\, u_x\, \xi^2_{zu}\, u_z - 2\, u_{xz}\, \xi^2_u\, u_z - u_x\, \xi^2_u\, u_{zz} \\
& - 2\, u_{tz}\, \xi^1_u\, u_z - u_t\, \xi^1_u\, u_{zz} - u_t\, u^2_z\, \xi^1_{uu} - 2\, u_t\, \xi^1_{zu}\, u_z - 2\, \xi^4_{zu}\, u^2_z.
\end{aligned}$$

$$
\begin{aligned}
\zeta_{2222} = {} & \eta_{xxxx} - 6\,u_{xxx}\,\xi^2_{xx} + 6\,\eta_{xxuu}\,u_x^2 - u_x\,\xi^2_{xxxx} - 4\,\xi^2_{xxxu}\,u_x^2 - 6\,u_{txx}\,\xi^1_{xx} - 6\,u_{xxz}\,\xi^4_{xx} - 6\,u_{xxy}\,\xi^3_{xx} z \\
& - 10\,u_x^2\,\xi^2_{uu}\,u_{xxx} - 4\,u_y\,\xi^3_{xu}\,u_{xxx} - 4\,u_z\,\xi^4_{xu}\,u_{xxx} - 12\,u_{xy}\,\xi^3_{xu}\,u_{xx} - 12\,u_{xxy}\,\xi^3_{xu}\,u_x \\
& - 12\,u_{xz}\,\xi^4_{xu}\,u_{xx} - 12\,u_{xxz}\,\xi^4_{xu}\,u_x - 4\,u_t\,\xi^1_{xu}\,u_{xxx} - 12\,u_{txx}\,\xi^1_{xu}\,u_x - 12\,u_{tx}\,\xi^1_{xu}\,u_{xx} - 16\,\xi^2_{xu}\,u_x\,u_{xxx} \\
& - 4\,u_z\,u_x\,\xi^4_{xxxu} - 4\,u_{tx}\,u_x^3\,\xi^1_{uuu} - u_y\,\xi^3_{xxxx} - 4\,u_x^4\,\xi^2_{xuuu} - u_x^5\,\xi^2_{uuuu} + u_x^4\,\eta_{uuuu} - u_y\,\xi^3_u\,u_{xxxx} \\
& - u_z\,\xi^4_u\,u_{xxxx} - u_t\,\xi^1_u\,u_{xxxx} - 12\,u_{tx}\,u_x\,\xi^1_{xxu} + 4\,u_x^3\,\eta_{xuuu} - u_t\,\xi^1_{xxxx} - 4\,u_{xxxy}\,\xi^3_x - 4\,u_{xxxz}\,\xi^4_x \\
& - 4\,u_{txxx}\,\xi^1_x - 4\,u_{xxxx}\,\xi^2_x + 4\,\eta_{uu}\,u_x\,u_{xxx} - 6\,u_x^3\,\xi^2_{xxuu} + 4\,\eta_{xu}\,u_{xxx} - 12\,u_y\,u_x\,\xi^3_{xuu}\,u_{xx} \\
& - 4\,u_t\,u_x\,\xi^1_{uu}\,u_{xxx} - 6\,u_t\,u_x^2\,\xi^1_{uuu}\,u_{xx} - 12\,u_{tx}\,u_x^2\,\xi^1_{xuu} - 4\,u_{xx}\,\xi^2_{xxx} + 3\,\eta_{uu}\,u_{xx}^2 - 6\,u_y\,u_{xx}\,\xi^3_{xxu} \\
& - u_t\,u_x^4\,\xi^1_{uuuu} - 4\,u_y\,u_x\,\xi^3_{uu}\,u_{xxx} - 12\,u_{xz}\,u_x^2\,\xi^4_{xuu} - 6\,u_z\,u_{xx}\,\xi^4_{xxu} + 6\,\eta_{xxu}\,u_{xx} - 4\,u_{xy}\,\xi^3_{xxx} \\
& - 4\,u_{xz}\,\xi^4_{xxx} - 4\,u_{xy}\,u_x^3\,\xi^3_{uuu} - 4\,u_{tx}\,\xi^1_{xxx} - 6\,u_y\,u_x^2\,\xi^3_{uuu}\,u_{xx} - 6\,u_z\,u_x^2\,\xi^4_{uuu}\,u_{xx} \\
& + 4\,\eta_{xxxu}\,u_x - 12\,u_{xy}\,u_x^2\,\xi^3_{xuu} - u_z\,\xi^4_{xxxx} - 12\,u_{xy}\,u_x\,\xi^3_{uu}\,u_{xx} - 4\,u_y\,u_x^3\,\xi^3_{xuuu} - 12\,u_{xy}\,u_x\,\xi^3_{xxu} \\
& - 10\,u_x^3\,\xi^2_{uuu}\,u_{xx} - 12\,u_{tx}\,u_x\,\xi^1_{uu}\,u_{xx} + 12\,\eta_{xuu}\,u_x\,u_{xx} + \eta_u\,u_{xxxx} - 6\,u_z\,u_x^2\,\xi^4_{xxuu} - 6\,u_{xxz}\,u_x^2\,\xi^4_{uu} \\
& - 6\,u_y\,u_x^2\,\xi^3_{xxuu} - 6\,u_{xxy}\,u_x^2\,\xi^3_{uu} - 18\,\xi^2_{xxu}\,u_x\,u_{xx} - 12\,u_{xz}\,u_x\,\xi^4_{uu}\,u_{xx} - 3\,u_z\,u_{xx}^2\,\xi^4_{uu} - 6\,u_{txx}\,u_x^2\,\xi^1_{uu} \\
& - 3\,u_t\,u_{xx}^2\,\xi^1_{uu} - 15\,u_x\,\xi^2_{uu}\,u_{xx}^2 - 12\,u_z\,u_x\,\xi^4_{xuu}\,u_{xx} - 12\,u_t\,u_x\,\xi^1_{xuu}\,u_{xx} - 24\,u_x^2\,\xi^2_{xuu}\,u_{xx} \\
& - 4\,u_z\,u_x\,\xi^4_{uu}\,u_{xxx} - 3\,u_y\,u_{xx}^2\,\xi^3_{uu} - 12\,u_{xz}\,u_x\,\xi^4_{xxu} - 6\,u_t\,u_{xx}\,\xi^1_{xxu} - 12\,\xi^2_{xu}\,u_{xx}^2 - 4\,u_y\,u_x\,\xi^3_{xxxu} \\
& - u_z\,u_x^4\,\xi^4_{uuuu} - 4\,u_z\,u_x^3\,\xi^4_{xuuu} - 4\,u_t\,u_x^3\,\xi^1_{xuuu} - 5\,\xi^2_u\,u_x\,u_{xxxx} - 6\,u_{xxy}\,\xi^3_u\,u_{xx} - 6\,u_{xxz}\,\xi^4_u\,u_{xx} \\
& - 4\,u_{tx}\,\xi^1_u\,u_{xxx} - 10\,\xi^2_u\,u_{xx}\,u_{xxx} - 4\,u_{txxx}\,\xi^1_u\,u_x - 6\,u_{txx}\,\xi^1_u\,u_{xx} - 4\,u_{xy}\,\xi^3_u\,u_{xxx} - 4\,u_{xxxy}\,\xi^3_u\,u_x \\
& - u_y\,u_x^4\,\xi^3_{uuuu} - 4\,u_{xxxz}\,\xi^4_u\,u_x - 4\,u_{xz}\,\xi^4_u\,u_{xxx} - 6\,u_t\,u_x^2\,\xi^1_{xxuu} + 6\,u_x^2\,\eta_{uuu}\,u_{xx} - 4\,u_t\,u_x\,\xi^1_{xxxu} \\
& - 4\,u_{xz}\,u_x^3\,\xi^4_{uuu}
\end{aligned}
$$

1.5.2 Group admitted by a PDE

The operator

$$
\begin{aligned}
R \;=\; & \xi^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,u)\frac{\partial}{\partial z} \\
& + \eta(t,x,y,z,u)\frac{\partial}{\partial u} \qquad (1.18)
\end{aligned}
$$

is a *point symmetry* of the four-order PDE (1.11) if

$$
R^{[4]}(E) = 0 \qquad (1.19)
$$

everytime when $E = 0$. Another way of writing it as (symmetry condition)

$$R^{[4]}(E)\big|_{E=0} = 0, \tag{1.20}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.4 Equation (1.20) is called the *determining equation* of (1.11), because it determines all the infinitesimal symmetries of (1.11).

The theorem 1.4 enables us to construct some solutions of (1.11) from the known one.

Theorem 1.4 A symmetry of (1.11) transforms any solution of (1.11) into another solution of the same equation.

1.6 Lie algebras

Let us consider two operators R_1, and R_2 defined by

$$\begin{aligned} R_1 &= \xi_1^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi_1^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi_1^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi_1^4(t,x,y,z,u)\frac{\partial}{\partial z} \\ &\quad +\eta_1(t,x,y,z,u)\frac{\partial}{\partial u}, \end{aligned}$$

and

$$\begin{aligned} R_2 &= \xi_2^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi_2^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi_2^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi_2^4(t,x,y,z,u)\frac{\partial}{\partial z} \\ &\quad +\eta_2(t,x,y,z,u)\frac{\partial}{\partial u}. \end{aligned}$$

Definition 1.5 (Commutator) The *commutator* of R_1 and R_2, written as $[R_1, R_2]$, is defined by $[R_1, R_2] = R_1(R_2) - R_2(R_1)$.

Definition 1.6 (Lie algebra) A Lie algebra is a vector space L of operators with the following property: For all $R_1, R_2 \in L$, the commutator $[R_1, R_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

Theorem 1.5 The set of all solutions of any determining equation forms a Lie algebra.

1.7 Fundamental operators and their relationship

In this section, we briefly present the notation and pertinent results, which we utilise later in this dissertation.

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.21}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u^\alpha_i = D_i(u^\alpha), u^\alpha_{ij} = D_j D_i(u^\alpha), \ldots$, respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \ldots, \quad i = 1, \ldots, n, \tag{1.22}$$

and the summation convention is used whenever appropriate.

The following are known (see for e.g., [41] and the references therein).

The *Euler-Lagrange operator* for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m, \tag{1.23}$$

and the *Lie-Bäcklund operator* is

$$R = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \ \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.24}$$

where $\mathcal{A}$ is the space of *differential functions*.

The operator (1.24) is an abbreviated form of an infinite formal sum

$$R = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.25}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1\ldots i_s}^\alpha &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u_{ji_1\ldots i_s}^\alpha, \quad s>1, \end{aligned} \tag{1.26}$$

in which W^α is the *Lie characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{1.27}$$

One can write the Lie-Bäcklund operator (1.25) in characteristic form as

$$R = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\partial}{\partial u_{i_1 i_2\ldots i_s}^\alpha}. \tag{1.28}$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\delta}{\delta u_{ii_1 i_2\ldots i_s}^\alpha}, \quad i=1,\ldots,n, \tag{1.29}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.23) by replacing u^α with the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\ldots D_{j_s}\frac{\partial}{\partial u_{ij_1 j_2\ldots j_s}^\alpha}, \quad i=1,\ldots,n, \quad \alpha=1,\ldots,m, \tag{1.30}$$

and the Euler-Lagrange , Lie-Bäcklund and Noether operators are connected by the operator identity

$$R + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.31}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n), \quad T^j \in \mathcal{A}, \quad j=1,\ldots,n$, is a *conserved vector* of (1.21) if T^i satisfies

$$D_i T^i|_{(E_\alpha(x,u,u_{(1)},\ldots,u_{(k)})=0)} = 0. \tag{1.32}$$

The equation (1.32) defines a *local conservation law* of system (1.21).

1.8 Variational method for a system and its adjoint

The system of *adjoint equations* to the system of kth-order differential equations (1.21) is defined by [42]

$$E^*_\alpha(x, u, v, \ldots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.33}$$

where

$$E^*_\alpha(x, u, v, \ldots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha, \beta = 1, \ldots, m, \ v = v(x) \tag{1.34}$$

and $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables.

We recall here the following results as given in Ibragimov [10].

A system of equations (1.21) is said to be *self-adjoint* if the substitution of $v = u$ into the system of adjoint equations (1.33) yields the same system (1.21).

Assume the system of equations (1.21) admits the symmetry generator

$$R = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.35}$$

Then the system of adjoint equations (1.33) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha}, \quad \eta^\alpha_* = -[\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)], \tag{1.36}$$

where the operator (1.36) is an extension of (1.35) to the variable v^α, and the λ^α_β are obtainable from

$$R(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.37}$$

Theorem 3.1. We now state the new conservation theorem due to Ibragimov. Every Lie point, Lie-Bäcklund and non local symmetry (1.35) admitted by the system of equations (1.21) gives rise to a conservation law for the system consisting of the equation (1.21) and the adjoint equation (1.33), where the components T^i of the conserved vector $T = (T^1, \ldots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u^\alpha_i} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{ii_1 i_2 \ldots i_s}}, \ i = 1, \ldots, n, \tag{1.38}$$

with Lagrangian given by

$$L = v^{\alpha} E_{\alpha}(x, u, \ldots, u_{(k)}). \tag{1.39}$$

Remark: If we consider the differentiation of L up to third-order derivative only, then the equation (1.38) can be written as

$$\begin{aligned} T^i &= \xi^i L + W^{\alpha}\left\{ \frac{\partial L}{\partial u^{\alpha}_i} - D_j\left(\frac{\partial L}{\partial u^{\alpha}_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial u^{\alpha}_{ijk}}\right) \right\} \\ &\quad + D_j(W^{\alpha})\left\{ \left(\frac{\partial L}{\partial u^{\alpha}_{ij}}\right) - D_k\left(\frac{\partial L}{\partial u^{\alpha}_{ijk}}\right) \right\} + D_j D_k(W^{\alpha})\left(\frac{\partial L}{\partial u^{\alpha}_{ijk}}\right). \end{aligned} \tag{1.40}$$

1.9 The multiplier approach

The multiplier method is known to be one of the most powerful and preferred methods for deriving conservation laws as it is applies to both equations that are variational and not variational. This approach involves the variational derivative. See [35]. A multiplier $E_{\alpha}(x, u, u_{(1)}, \ldots, u_{(k)})$ contains the property that [35]

$$\Lambda_{\alpha} E_{\alpha} = D_i T^i \tag{1.41}$$

holds identically. The determining equation for the multiplier Λ_{α} is

$$\frac{\delta(\Lambda_{\alpha} E_{\alpha})}{\delta u^{\alpha}} = 0. \tag{1.42}$$

which holds for arbitrary functions $u(r^1, r^2, \ldots, r^n)$. Once we get the multipliers from (1.42), we may compute conserved vectors by invoking equation (1.41), See for example, reference [10, 38].

1.10 Conclusion

This chapter briefly introduced Lie group analysis of partial differential equations and presented some of the outcomes utilised in this project. We also presented an

algorithm for determining the Lie-point symmetry of partial differential equations. The basic operators of the conservation laws and their relationships are given, as well as the variational technique of the system and its adjoint.

Chapter 2

Symmetry analysis and conservation laws of the first (3+1)-dimensional Gardner-type equation

2.1 Exact solutions for (5)

For simplicity of constants found in equation (5), we let $a = 6\,\beta$, $b = \frac{3}{2}\,\alpha^2$, $c = 3\,\sigma^2$, $d = 3\,\alpha\,\sigma$ and $e = 3\,\sigma^2$. Thus, obtain the new form of equation (5) which will be our main focus given by

$$u_{xt} + a u_x u_{xx} + u_{xxxx} - b u_x^2 u_{xx} + c u_{yy} - d u_{xx} u_y + e u_{zz} = 0. \tag{2.1}$$

2.1.1 Lie Symmetries

This subsection generates Lie symmetries of (2.1) and makes use of the one-parameter group of transformations with an infinitesimal generator given in (1.7)

if and only if the following

$$R^{[4]}(u_{xt} + au_xu_{xx} + u_{xxxx} - bu_x^2u_{xx} + cu_{yy} - du_{xx}u_y + eu_{zz})\Big|_{(2.1)} = 0, \tag{2.2}$$

where $R^{[4]}$ is the 4th prolongation of R given by (1.13). Expanding equation (2.2) becomes

$$\begin{aligned}&\big[\zeta_{12} + \zeta_{2222} - du_{xx}\zeta_3 + e\zeta_{44} + c\zeta_{33} + \zeta_2(au_{xx} - 2bu_x) + \\ &+ \zeta_{22}(au_x - bu_x^2 - du_y)\big]\Big|_{(2.1)} = 0.\end{aligned} \tag{2.3}$$

Expanding equation (2.3) and equating all derivatives of u to zero, yields the following set of PDEs:

$$\xi^1_x = 0, \tag{2.4}$$

$$\xi^1_y = 0, \tag{2.5}$$

$$\xi^1_u = 0, \tag{2.6}$$

$$\xi^2_y = -\frac{\xi^3_t}{2\,c}, \tag{2.7}$$

$$\xi^2_u = 0, \tag{2.8}$$

$$\xi^3_x = 0, \tag{2.9}$$

$$\xi^3_u = 0, \tag{2.10}$$

$$\xi^4_x = 0, \tag{2.11}$$

$$\xi^4_y = 0, \tag{2.12}$$

$$\xi^4_u = 0, \tag{2.13}$$

$$\eta_u = 0, \tag{2.14}$$

$$
\begin{aligned}
\xi^1_t &= \frac{3\,\xi^4_z}{2}, && (2.15)\\
\xi^1_z &= 0, && (2.16)\\
\xi^2_x &= \frac{\xi^4_z}{2}, && (2.17)\\
\xi^2_z &= -\frac{\xi^4_t}{2\,e}, && (2.18)\\
\xi^3_y &= \xi^4_z, && (2.19)\\
\xi^3_z &= 0, && (2.20)\\
\xi^4_{tz} &= 0, && (2.21)\\
\xi^4_{zz} &= 0, && (2.22)\\
\eta_x &= \frac{a\,\xi^4_z}{4\,b} - \frac{d\,\xi^3_t}{4\,b\,c}, && (2.23)\\
\eta_y &= \frac{a^2\xi^4_z}{4\,d\,b} - \frac{a\,\xi^3_t}{4\,b\,c} - \frac{\xi^2_t}{d}, && (2.24)\\
\eta_{zz} &= -\frac{\xi^3_{tt}}{2\,d\,e} + \frac{d\,\xi^3_{tt}}{4\,b\,e\,c}. && (2.25)
\end{aligned}
$$

Working out the systems above of PDEs, we get the following

$$
\begin{aligned}
\xi^1 &= \frac{3}{2}k_1\,t + k_2,\\
\xi^2 &= \frac{1}{2}k_1\,x - \frac{y\,F_3'(t)}{2\,c} - \frac{z\,F_2'(t)}{2\,e} - d\,F_5(t) + \frac{a^2\,k_1\,t}{4\,b} - \frac{d\,a\,F_3(t)}{4\,b\,c} + k_3,\\
\xi^3 &= y\,k_1 + F_3(t),\\
\xi^4 &= z\,k_1 + F_2(t),\\
\eta &= -\frac{z^2\,F_3''(t)}{4\,d\,e} + \frac{y^2\,F_3''(t)}{4\,d\,c} + \frac{d\,z^2\,F_3''(t)}{8\,b\,c\,e} + \frac{z\,y\,F_2''(t)}{2\,d\,e} - \frac{d\,x\,F_3'(t)}{4\,b\,c}\\
&\quad + \frac{a\,x\,k_1}{4\,b} + y\,F_5'(t) + z\,F_4(t) + F_1(t), \qquad (2.26)
\end{aligned}
$$

where k_1, k_2, k_3 are constants and $F_1(t)$ to $F_5(t)$ are arbitrary functions with respect to t. To acquire physically interesting similarity transformation that yields travelling wave solutions of Eq. (2.1), we take $F_1(t)$ to $F_5(t)$ to be equal to 1. Thus we get

seven Lie symmetries of (2.1) as follows:

$$\begin{aligned}
R_1 &= \frac{\partial}{\partial t}, \quad \text{time translation} \\
R_2 &= \frac{\partial}{\partial x}, \quad \text{space translation} \\
R_3 &= \frac{\partial}{\partial y}, \quad \text{space translation} \\
R_4 &= \frac{\partial}{\partial z}, \quad \text{space translation} \\
R_5 &= \frac{\partial}{\partial u}, \quad \text{translation} \\
R_6 &= z\frac{\partial}{\partial u}, \quad \text{inversion} \\
R_7 &= 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x + a^2\,t)\frac{\partial}{\partial x} \\
&\quad +4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} + a\,x\,\frac{\partial}{\partial u}, \quad \text{inversion.}
\end{aligned} \tag{2.27}$$

2.1.2 Commutator table for (2.1)

In this particular subsection, we work out the commutation relations for all obtained Lie symmetries. First of all, we execute $[R_4, R_6]$ as an example

$$\begin{aligned}
[R_4, R_6] &= R_4\,R_6 - R_6\,R_4 \\
&= \frac{\partial}{\partial z}\left(z\frac{\partial}{\partial u}\right) - z\frac{\partial}{\partial u}\left(\frac{\partial}{\partial z}\right) \\
&= \frac{\partial}{\partial u} \\
&= R_5.
\end{aligned}$$

Similarly the commutation relations of the remaining symmetries were obtained, and the commutator relations of all the symmetries are presented in the tabular form as follows.

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_1	0	0	0	0	0	0	$6\,b\,R_1 + a^2\,R_2$
R_2	0	0	0	0	0	0	$2\,b\,R_2 + a\,R_5$
R_3	0	0	0	0	0	0	$4\,b\,R_3$
R_4	0	0	0	0	0	R_5	$4\,b\,R_4$
R_5	0	0	0	0	0	0	0
R_6	0	0	0	$-R_5$	0	0	$-4\,b\,R_6$
R_7	$-6\,b\,R_1 - a^2\,R_2$	$-2\,b\,R_2 - a\,R_5$	$-4\,b\,R_3$	$-4\,b\,R_4$	0	$4\,b\,R_6$	0

Table 1. Commutator table of (2.1)

2.1.3 Symmetry reductions

For this subsection, we are performing symmetry reductions using the Lie point symmetries that we obtained for (2.1), which will then lead us to the NODEs.

Case 1:

Firstly, we consider four translation symmetries R_1, R_2, R_3 and R_4. By taking a linear combination of these four translation symmetries, $R = R_1 + R_2 + R_3 + \alpha R_4$, having α as a constant. Working out the Lagrange system associated with this operator, invariants below are then generated

$$f = t - y, \quad g = t - x, \quad h = \alpha t - z, \quad u = \theta(f, g, h). \tag{2.28}$$

The above invariants reduce the first (3+1)-dimensional Gardner-type equation (2.1) to a NPDE

$$\theta_{gggg} - \theta_{gg} - \theta_{gf} - \alpha\theta_{gh} - a\theta_g\theta_{gg} - b\theta_g^2\theta_{gg} + c\theta_{ff} + d\theta_{gg}\theta_f + e\theta_{hh} = 0, \tag{2.29}$$

which similarly admits six Lie point symmetries below

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial f},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = \frac{\partial}{\partial h},\ \Gamma_4 = h\frac{\partial}{\partial \theta},\ \Gamma_5 = \frac{\partial}{\partial \theta},\\
&\Gamma_6 = -8bdech\frac{\partial}{\partial h} - 8bdecf\frac{\partial}{\partial f} + (2\alpha bcdh - 4bcdeg + 2bdef)\frac{\partial}{\partial g}\\
&\quad + (2a^2cef + 2acdeg - 2\alpha^2bcf - adef - 8bcef - d^2eg - 2bef)\frac{\partial}{\partial \theta}.
\end{aligned} \tag{2.30}$$

By using Γ_1, Γ_2 and Γ_3, we then take yet another linear combination of the form $\Gamma = \Gamma_1 + \Gamma_2 + \beta\,\Gamma_3$, here β is a constant, we reduce (2.29) to a simpler PDE. Working out the Lagrange system for Γ, three invariants are then generated:

$$w = \beta f - h, \quad r = f - g, \quad \theta = \phi(r, w). \tag{2.31}$$

Using the above three invariants transforms equation (2.29) to

$$\begin{aligned}&\phi_{rrrr} + (\beta - \alpha + 2\,c\,\beta)\phi_{rw} + (a+d)\phi_r\phi_{rr} - b\phi_r^2\phi_{rr} + c\phi_{rr} + (c\,\beta^2 + e)\phi_{ww}\\ &+ d\,\beta\,\phi_{rr}\phi_w = 0,\end{aligned} \tag{2.32}$$

where ϕ depends on w and r, which yields the following four symmetries

$$\begin{aligned}&\Sigma_1 = \frac{\partial}{\partial w},\ \Sigma_2 = \frac{\partial}{\partial r},\ \Sigma_3 = \frac{\partial}{\partial \phi},\\ &\Sigma_4 = (-8bcd\beta^3 w - 8bde\beta w)\frac{\partial}{\partial w} + (2bd\beta\alpha w - 4bcdr\beta^3 - 4bcd\beta^2 w - 4bder\beta - 2bd\beta^2 w)\frac{\partial}{\partial r}\\ &\quad + (-2acdr\beta^3 + 2a^2c\beta^2 w + 2acd\beta^2 w + d^2 r\beta^3 - d^2 r\beta^2\alpha - 2ader\beta - ad\beta^2 w + ad\beta\alpha w\\ &\quad - 8bc\beta^2 w + 8bc\beta\alpha w - 2d^2 er\beta - d^2\beta^2 w + d^2\beta\alpha w + 2a^2 ew + 4adew\\ &\quad + 8bcew - 2b\beta^2 w + 4b\beta\alpha w - 2b\alpha^2 w + 2d^2 ew)\frac{\partial}{\partial \phi},\end{aligned} \tag{2.33}$$

which leave equation (2.32) unchanged. Combining two translational symmetries, $\Sigma = \Sigma_1 + \delta\Sigma_2$, taking δ as a constant, two invariants are produced

$$k = \delta w - r, \quad \phi = \psi,$$

which yield invariant solution $\phi = \psi(k)$ and utilizing the above invariants, (2.32) yields the following fourth-order NODE

$$\begin{aligned}&\psi'''' - b(\psi')^2\psi'' + [-\delta(\beta - \alpha + 2c\beta) + c + \delta^2(e + c\beta^2)]\psi''\\ &+ (d\beta\delta - a - d)\psi'\psi'' = 0.\end{aligned} \tag{2.34}$$

The above NODE may be rewritten as

$$\psi'''' - b\psi'^2\psi'' + A\psi'' + B\psi'\psi'' = 0, \tag{2.35}$$

where $A = -\delta(\beta - \alpha + 2c\beta) + c + \delta^2(e + c\beta^2)$ and $B = d\beta\delta - a - d$. Integrating (2.35), we get the following

$$\psi''' - \frac{b}{3}\psi'^3 + \frac{1}{2}B\psi'^2 + A\psi' + c_1 = 0, \tag{2.36}$$

treating c_1 as a constant. Multiplying equation (2.36) by ψ'' and integrating the resulting equation yields the following ODE

$$\frac{1}{2}\psi''^2 - \frac{b}{12}\psi'^4 + \frac{1}{6}B\psi'^3 + \frac{1}{2}A\psi'^2 + c_1\psi' + c_2 = 0, \tag{2.37}$$

where c_2 is the constant of integration. By letting $\psi'(k) = G(k)$ and solving (2.37) gives a new ODE in terms of $G(k)$

$$\frac{1}{2}G'^2 - \frac{b}{12}G^4 + \frac{1}{6}BG^3 + \frac{1}{2}AG^2 + c_1G + c_2 = 0. \tag{2.38}$$

Case i: Solution in terms of incomplete elliptic integral.

In order to obtain an elliptical integral solution we let $b = 0$ for our first case, equation (2.38) becomes

$$\frac{1}{2}G'^2 + \frac{1}{6}BG^3 + \frac{1}{2}AG^2 + c_1G + c_2 = 0. \tag{2.39}$$

For in-depth details on the method we refer the reader to the books [43,44]. Assuming that the cubic equation

$$G^3 + \frac{3\,A}{B}G^2 + \frac{6\,c_1}{B}G + \frac{6\,c_2}{B} = 0.$$

This cubic equation has the real roots $\sigma_1 - \sigma_3$ such that $\sigma_1 > \sigma_2 > \sigma_3$, then (2.39) becomes

$$G'^2 = -\frac{B}{3}(G - \sigma_1)(G - \sigma_2)(G - \sigma_3). \tag{2.40}$$

The general solution of (2.39), is then given in form of Jacobi elliptic function as [44, 45]

$$G(k) = \sigma_2 - (\sigma_1 - \sigma_2)\mathrm{cn}^2\left\{\sqrt{\frac{B\,(\sigma_1 - \sigma_3)}{12}}(k - k_0), H^2\right\},\ H^2 = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3},$$

where cn represents Jacobi cosine function. Integrating G yields ψ, from which equation (2.1) obtains the solution

$$u(t,x,y,z) = \sqrt{\frac{12\left(\sigma_1-\sigma_2\right)^2}{B\left(\sigma_1-\sigma_3\right)H^8}}\left[\text{EllipticE}\left\{\text{sn}\left(\sqrt{\frac{B\left(\sigma_1-\sigma_3\right)}{12}}\left(k-k_0\right),H^2\right),H^2\right\}\right] + \left\{\sigma_2-\left(\sigma_1-\sigma_2\right)\frac{1-H^4}{H^4}\right\}\left(k-k_0\right)+c_3, \tag{2.41}$$

where $k=(\delta\beta-\delta\alpha)t-y(\delta\beta-1)+\delta z-x$, c_3 and k_0 are constants of integration, sn is an elliptic sine function and EllipticE$[g,t]$ being the incomplete elliptic integral given as [43]

$$\text{EllipticE}[g,t]=\int_0^g\sqrt{\frac{1-t^2s^2}{1-s^2}}ds. \tag{2.42}$$

Plots representing the solution (2.41) graphically when parameter values $t=1\,, a=-4$, $\alpha=0.1$, $\delta=0.2$, $z=70$, $B=0.2$, $\beta=0.1$, $\alpha_1=100$, $\alpha_2=50.05$, $\alpha_3=-60$ and $c_1=1$ are as follows

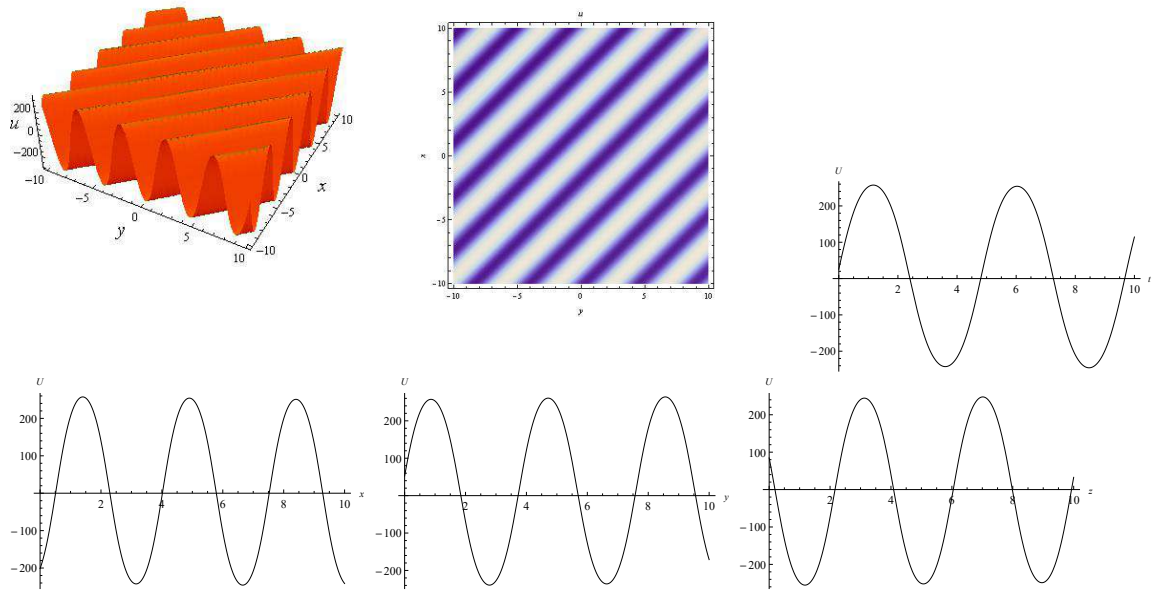


Figure 2.1: The 3D, Density plot and 2D illustrations of velocity solution (2.41)

Analysis of figure 2.1.

As seen from Figure 2.1, the wave velocity increases when space along x and y increases until the maximum velocity is reached and thereafter decreases until minimum velocity before increasing again. The velocity decreases along the normal direction z until minimum velocity and increases from there till maximum velocity is reached and thereafter decreases again. Since the increase in velocity enhances momentum and energy, this implies that when the momentum and energy increase along the surface ($x-y$ plane), they (momentum and energy) decrease along the z direction (away from the surface). Also, the decrease in momentum and energy along the surface happens when there is an increase along the normal direction z. As time evolves, the velocity increases until it reaches maximum velocity and thereafter decreases to its lowest velocity before increasing again. These dynamics indicate that the increases in sound projection along the surface lead to a decrease along the z direction and vice versa.

Case ii : Weierstrass elliptic function solution

This technique provides the most common solution of NODE on a complex level. Our second case begins by letting $b = 0$, $B = -12$ and $A = 0$, thus equation (2.38) becomes

$$G'^2 - 4G^3 + 2c_1G + 2c_2 = 0. \tag{2.43}$$

The above equation (2.43) has the following Weierstrass solution

$$G(k) = \wp\left(k - k_0; g_1, g_2\right), \tag{2.44}$$

where the invariants g_1, g_2 are given by $g_1 = 2c_1$ and $g_2 = 2c_2$. Thus,

$$u(t, x, y, z) = -\zeta\left(k - k_0; g_1, g_2\right) + c_4, \tag{2.45}$$

where c_4, k_0 are constant, ζ is the Weierstrass zeta function and $k = (\delta\beta - \delta\alpha)t - y(\delta\beta - 1) + \delta z - x$.

The graphical representation solution (2.45) obtained when parameter values: $\delta = 0$, $\beta = -0.6$, $c_4 = 0.7$, $t = 5$, $\alpha = 0.7$, $c_1 = 0.7$, $c_2 = 0.7$ and $z = 5.628$, are as follows

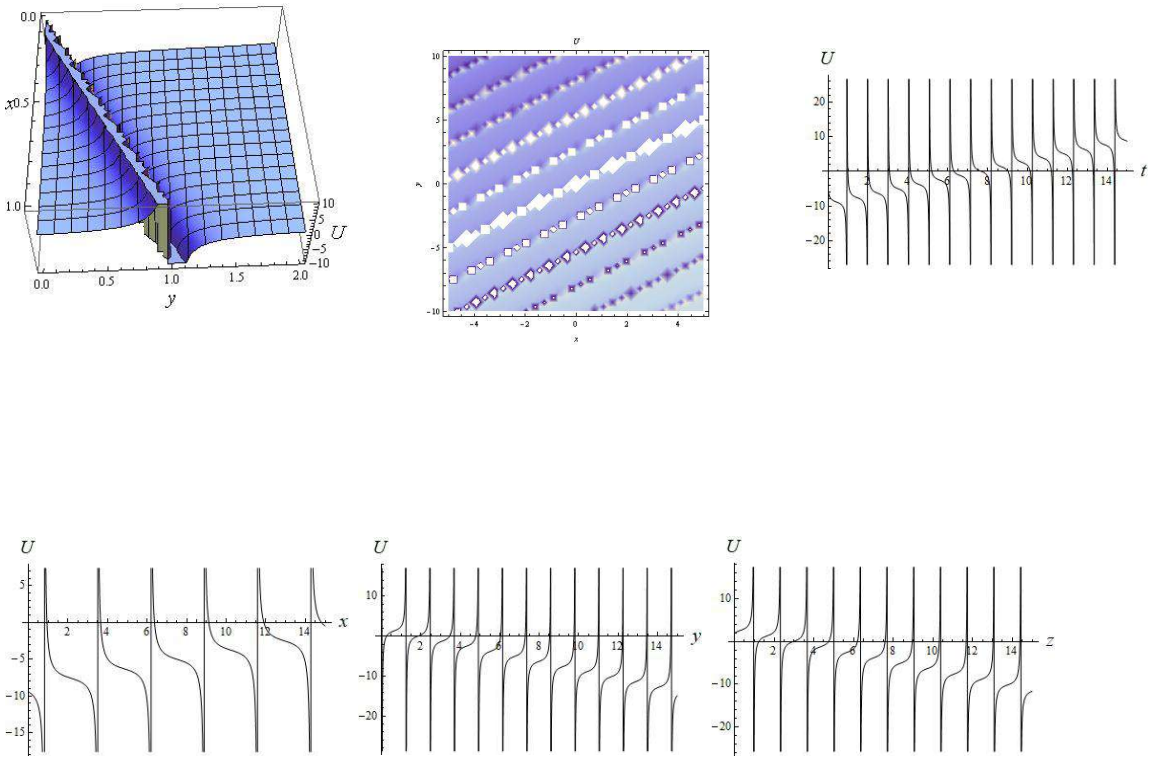


Figure 2.2: The 3D, Density plot and 2D illustration of velocity solution (2.45)

Analysis of figure 2.2.

According to Figure 2.2, the wave velocity decreases when space along x increases as well as when time evolves. The velocity, on the other note, increases when the space along y and normal space z increase. Since the increase in velocity enhances momentum and energy, this implies that momentum and energy on the surface ($x-$ y plane) decrease along the x direction and increase along the y direction. The momentum and energy as the height increase along the z direction (away from the surface) increases. The velocity decreases from maximum velocity to minimum as time evolves.

Case 2:

We reduce our original equation (2.1) using R_1 to an ODE. Working out the associated Lagrange system for R, three invariants are generated:

$$g = y, \quad f = x, \quad h = z, \quad \theta(g, f, h) = u. \tag{2.46}$$

Using the above four invariants, equation (2.1) becomes

$$\theta_{ffff} + a\theta_f\theta_{ff} - b\theta_f^2\theta_{ff} + c\theta_{gg} - d\theta_g\theta_{ff} + e\theta_{hh} = 0, \tag{2.47}$$

making it a nonlinear PDE with three independent variables named above. Equation (2.47) posses six Lie symmetries

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial f},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = \frac{\partial}{\partial h},\ \Gamma_4 = h\frac{\partial}{\partial \theta},\ \Gamma_5 = \frac{\partial}{\partial \theta},\\
&\Gamma_6 = 4\,b\,d\,h\frac{\partial}{\partial h} + 2\,b\,d\,f\frac{\partial}{\partial f} + 4\,b\,d\,g\frac{\partial}{\partial g} + (a^2 g + adf)\frac{\partial}{\partial \theta},
\end{aligned} \tag{2.48}$$

which leave the equation (2.47) unchanged. Using Γ_3 to further reduce equation (2.47), Γ_3 yields three invariants :

$$r = f, \quad w = g, \quad \theta = \phi(r, w). \tag{2.49}$$

Using the above three invariants, equation (2.47) transforms to

$$\phi_{rrrr} + a\phi_r\phi_{rr} - b\phi_r^2\phi_{rr} + c\phi_{ww} - d\phi_{rr}\phi_w = 0, \tag{2.50}$$

making it a NPDE having two independent variables. The symmetries that leave the above equation (2.50) unchanged are as follows

$$\begin{aligned}
&\Sigma_1 = \frac{\partial}{\partial w},\ \Sigma_2 = \frac{\partial}{\partial r},\ \Sigma_3 = \frac{\partial}{\partial \phi},\\
&\Sigma_4 = 4bdw\frac{\partial}{\partial w} + 2bdr\frac{\partial}{\partial r} + (a^2 w + adr)\frac{\partial}{\partial \phi}.
\end{aligned} \tag{2.51}$$

The symmetry Σ_1, gives two invariants given below

$$k = r, \quad \phi = \psi(k).$$

Utilising these two invariants, equation (2.50) takes the following form of fourth-order NODE as

$$\psi'''' - b(\psi')^2\psi'' + a\psi'\psi'' = 0. \tag{2.52}$$

Integrating (2.52) and taking the constant of integration as zero leads to the following:

$$\psi''' - \frac{b}{3}(\psi')^3 + \frac{a}{2}(\psi')^2 = 0. \tag{2.53}$$

By letting $F(k) = \psi'(k)$, equation (2.53) becomes

$$F'' - \frac{b}{3}F^3 + \frac{a}{2}F^2 = 0, \tag{2.54}$$

which gives the following solution

$$F(k) = \frac{4\,Q}{Q^2\,T^2 - 4\,R}. \tag{2.55}$$

Integrating $F(k)$ and changing back to the initial variables, the above yields the solution of (2.1) as

$$u(x, y, t, z) = -\frac{2}{\sqrt{R}} \tanh^{-1}\left(\frac{T\beta}{2\sqrt{R}}\right), \tag{2.56}$$

where $T = x$, $\beta = -\frac{a}{3}$ and $R = \frac{b}{6}$. The graphical representations of the solution (2.56) when parameter values $a = 4$, $b = 11$, $t = 0$, $z = 0$ are shown in figure 2.3

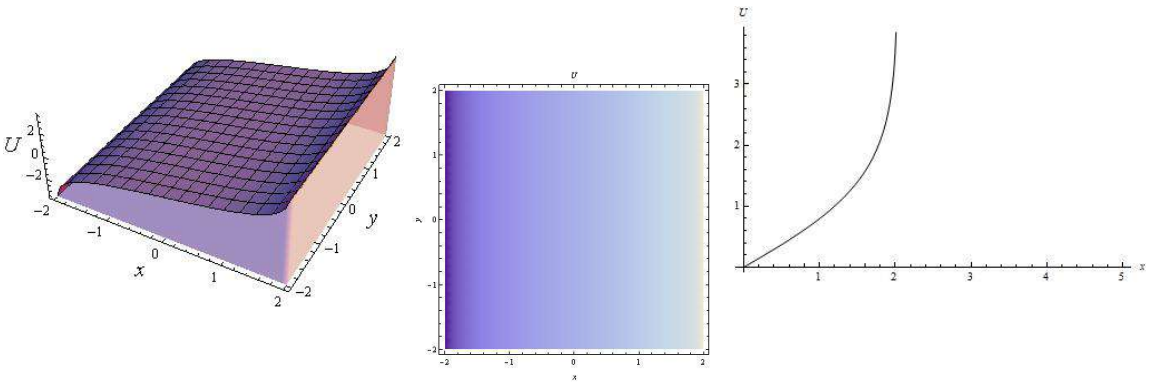


Figure 2.3: The 3D, Density and 2D illustration of solution (2.56)

Analysis of figure 2.3.

The velocity variation, according to Figure 2.3 shows that the velocity during steady

state increases with the increase in the space x such that momentum and energy become an increasing function of space variable x. It is clear from solution (2.56) that the dynamics change with respect to x since the solution is a function of x only.

Case 3:

We now utilise the symmetry R_2, to reduce (2.1) to a new PDE. Working out the Lagrange system for R, the invariants below are generated:

$$h = z, \quad f = y, \quad g = t, \quad \theta(g, f, h) = u. \tag{2.57}$$

Using the four invariants above transforms (2.1) to

$$c\theta_{ff} + e\theta_{hh} = 0. \tag{2.58}$$

Solving (2.58), yields the solution

$$\theta(f, h) = c_1 \left(\frac{f\sqrt{-ce}}{c} + h \right) + c_2 \left(h - \frac{f\sqrt{-ce}}{c} \right), \tag{2.59}$$

which, in terms of the original variables, equation (2.59) takes the following form

$$u(t, x, y, z) = c_1 \left(\frac{\sqrt{ce}\, i\, y}{c} + z \right) + c_2 \left(z - \frac{\sqrt{ce}\, i\, y}{c} \right), \tag{2.60}$$

where c_1 and c_2 are arbitrary constants. Setting parameters $c = -1$, $e = 0.1$, $x = 0$, $c_1 = 2$, $c_2 = 0$ and plotting the profile of solution (2.60), yields the following graphs

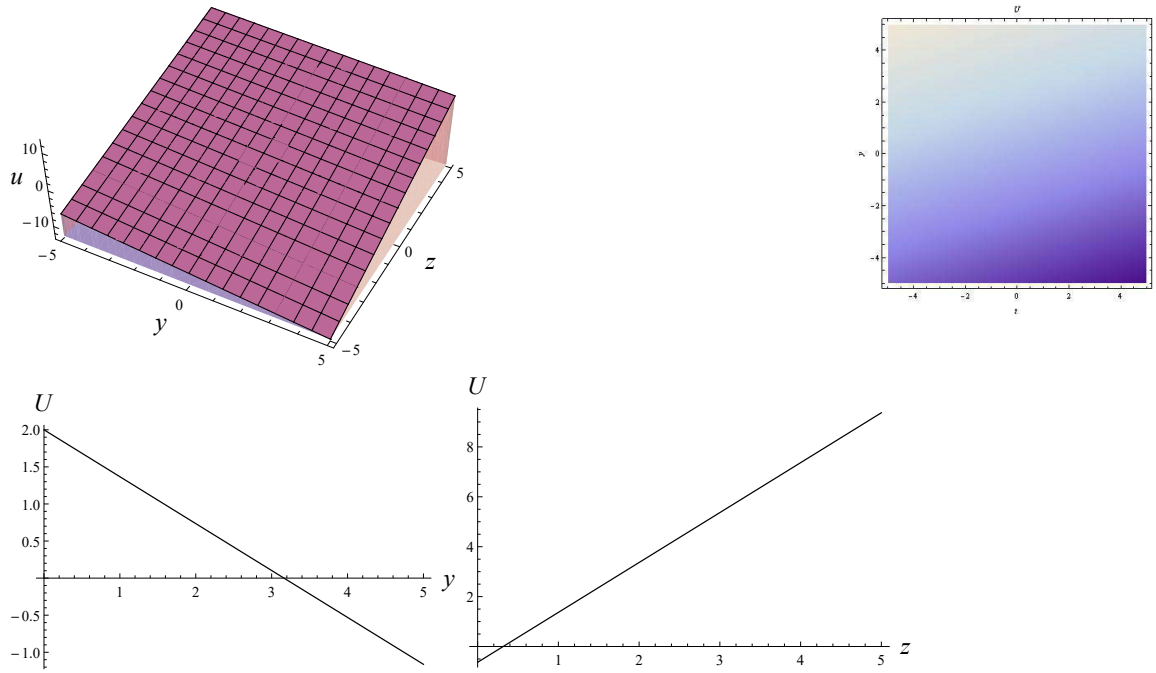


Figure 2.4: The 3D, Density and 2D illustration of velocity solution (2.60)

Analysis of figure 2.4.

According to Figure 2.4 the steady state wave velocity on the ($y-z$ plane) decreases when spatial y increases and increases when the spatial z increases. Here momentum and energy decrease along the y and increase along the z, meaning the wave moves more upwards while decreases speed along the axial direction y.

Case 4:

The symmetry R_7 has the following characteristic equations

$$\frac{dt}{6bt} = \frac{dx}{2bx + a^2 t} = \frac{dy}{4by} = \frac{dz}{4bz} = \frac{du}{ax}. \tag{2.61}$$

Solving (2.61) gives the following invariants

$$J_1 = -\frac{a^2\, t - 4\, b\, x}{4\, b\, t^{\frac{1}{3}}},\ J_2 = \frac{y}{t^{2/3}},\ J_3 = \frac{z}{t^{2/3}},\ J_4 = \frac{a^3\, t}{12\, b^2} - \frac{ax}{2b} + u. \tag{2.62}$$

Thus the group-invariant (2.62), yields

$$u(t, x, y, z) = F\left(-\frac{a^2\, t - 4\, b\, x}{4\, b\, t^{\frac{1}{3}}}, \frac{y}{t^{2/3}}, \frac{z}{t^{2/3}}\right) - \frac{a^3\, t}{12\, b^2} + \frac{ax}{2b}. \tag{2.63}$$

Inserting the invariant (2.63) into equation (2.1), gives the reduced PDE as

$$F_{ffff} - bF_f^2F_{ff} + cF_{gg} - dF_gF_{ff} + eF_{hh} - \frac{2}{3}hF_{fh} - \frac{2}{3}gF_{fg} - \frac{1}{3}F_f - \frac{1}{3}fF_{ff} = 0, \tag{2.64}$$

where

$$f = -\frac{a^2\,t - 4\,b\,x}{4\,b\,t^{\frac{1}{3}}},\ g = \frac{y}{t^{2/3}},\ h = \frac{z}{t^{2/3}}.$$

The above equation (2.64) admits five Lie point symmetries given by:

$$\Gamma_1 = \frac{\partial}{\partial F},\ \Gamma_2 = h\frac{\partial}{\partial F},\ \Gamma_3 = 3d\frac{\partial}{\partial f} - g\frac{\partial}{\partial F},\ \Gamma_4 = -3\,h\,d\frac{\partial}{\partial f} + 9\,e\,d\frac{\partial}{\partial h} - h\,g\frac{\partial}{\partial F},$$
$$\Gamma_5 = -\frac{g}{3c}\frac{\partial}{\partial f} + \frac{\partial}{\partial g} + \left(\frac{h^2\,c - g^2e}{18\,d\,e\,c} - \frac{d\,(6\,fe + h^2)}{36\,b\,c\,e}\right)\frac{\partial}{\partial F}.$$

Using the symmetry Γ_3, we get the invariant solution as follows

$$F(f, g, h) = G(g, h) - 3dfg \tag{2.65}$$

which reduces equation (2.64) to

$$3dg + cG_{gg} + eG_{hh} = 0. \tag{2.66}$$

The above PDE (2.66) has the following solution

$$G\,(g, h) = -\frac{3\,d\,g\,(ch^2 + e\,g^2)}{8\,ce} + F_2\left(\frac{\sqrt{-ce}\,g + h\,c}{c}\right) + F_1\left(\frac{\sqrt{-ce}\,g - h\,c}{2\sqrt{-ce}}\right).$$

Reverting back to equation (2.65) by substituting $G\,(g, h)$, we get

$$F(f, g, h) = -\frac{3\,d\,g\,(ch^2 + e\,g^2)}{8\,ce} + F_2\left(\frac{\sqrt{-ce}\,g + h\,c}{c}\right) + F_1\left(\frac{\sqrt{-ce}\,g - h\,c}{2\sqrt{-ce}}\right) - 3\,d\,f\,g. \tag{2.67}$$

Substituting

$$f = -\frac{a^2\,t - 4\,b\,x}{4\,b\,t^{\frac{1}{3}}},\ g = \frac{y}{t^{2/3}},\ h = \frac{z}{t^{2/3}},$$

into (2.67) and going to the initial solution (2.63), as well as considering $i = \sqrt{-1}$, we get the following solution

$$u(t, x, y, z) = F_2\left(\frac{\sqrt{ce}\,i\,y + z\,c}{c\,t^{\frac{2}{3}}}\right) + F_1\left(\frac{\sqrt{ce}\,i\,y - z\,c}{2\,t^{\frac{2}{3}}\,i\sqrt{ce}}\right) - \frac{3\,d\,y\,(c\,z^2 + e\,y^2)}{8t^2\,e\,c} + \frac{3d\,(a^2t - 4\,bx)\,y}{4bt} - \frac{a\,(a^2t - 6\,bx)}{12b^2}. \tag{2.68}$$

The graphical representations of solution (2.68) when parameter values $b = 2$, $a = 0.2$, $d = 2$, $x = 10$, $y = 4$, $z = 3$, $c = -4$, $e = 5$, $F_1 = 6$ and $F_2 = 7$ are shown in Figure 2.5

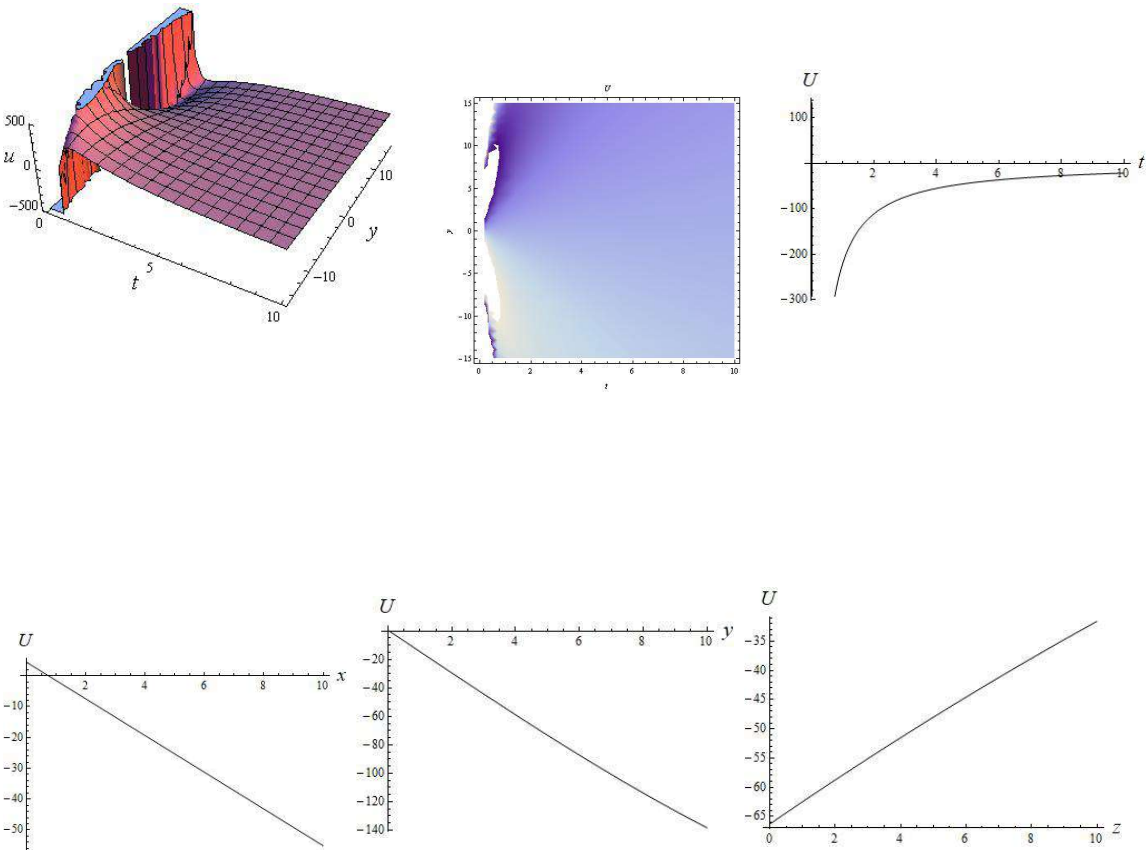


Figure 2.5: The 3D, Density plot and 2D illustration of velocity solution (2.68)

Analysis of figure 2.5.

According to Figure 2.5 the wave velocity decreases with the increase in spatial axial directions x and y, while the wave velocity increases as time evolves and when the normal direction z increases. This implies that the wave velocity along the surface decreases when the is an increase away from the surface (as z increases). Thus

momentum and energy increase as z increases and decreases along the $(x-y)$ plane.

2.2 Construction of conservation laws for (2.1)

Conservation laws of the first (3+1)-dimensional Gardner-type equation (2.1) using the Ibragimov's theorem are obtained in this section.

Recall that the seven Lie point symmetry generators below are admitted by equation (2.1):

$$\left.\begin{aligned}
R_1 &= \frac{\partial}{\partial t},\\
R_2 &= \frac{\partial}{\partial x},\\
R_3 &= \frac{\partial}{\partial y},\\
R_4 &= \frac{\partial}{\partial z},\\
R_5 &= \frac{\partial}{\partial u},\\
R_6 &= z\frac{\partial}{\partial u},\\
R_7 &= 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x + a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} + a\,x\,\frac{\partial}{\partial u}.
\end{aligned}\right\} \tag{2.69}$$

This section finds the seven conserved vectors, which correspond to these seven Lie symmetries. Adjoint equation of (2.1), when using (1.34), is as follows:

$$\begin{aligned}
E^* &= \frac{\delta}{\delta u}\Big[v(u_{xt} + au_xu_{xx} + cu_{yy} + eu_{zz} - bu_x^2u_{xx}\\
&\qquad -du_yu_{xx} + u_{xxxx})\Big],
\end{aligned}$$

for this v is equivalent $v(t,x,y,z)$ and its an dependent variable and it gives us

$$\begin{aligned}
&au_xv_{xx} + v_{xxxx} + au_{xx}v_x - bu_x^2v_{xx} - 2bu_{xx}u_xv_x + cv_{yy} - 2dv_xu_{xy}\\
&+du_{xx}v_y - du_yv_{xx} + ev_{zz} + v_{tx} = 0.
\end{aligned} \tag{2.70}$$

According to the equation (2.70), the Gardner equation (2.1) is not self-adjoint. Utilizing (1.39), we obtain fourth order Lagrangian for the system of equations in

the symmetric form:

$$L = v\left(\frac{1}{2}u_{xt} + \frac{1}{2}u_{tx} + au_xu_{xx} + u_{xxxx} + cu_{yy} + eu_{zz} - bu_x^2u_{xx} - du_yu_{xx}\right). \quad (2.71)$$

The conservation laws of equations (2.1) and (2.70) are generated by using equation (1.40) and Lie point symmetries given by (2.69).

(i) Lie point symmetry $R_1 = \frac{\partial}{\partial t}$, gives the Lie characteristic function $W = -u_t$. Y_1 and R_1 have the same operator and can be verified from equation (1.36). Invoking (1.40) and Lie symmetry generator R_1 gives components of the conserved vector below:

$$\begin{aligned}
T^1 =& au_{xx}u_xv - bu_{xx}{u_x}^2v + cu_{yy}v - du_{xx}u_yv + eu_{zz}v + \frac{1}{2}u_{tx}v + \frac{1}{2}u_tv_x + u_{xx}v_{xx},\\
T^2 =& -au_xu_{tx}v + bu_x^2u_{tx}v - du_tu_{xy}v + du_yu_{tx}v - \frac{1}{2}u_{tt}v + au_tu_xv_x - bu_t{u_x}^2v_x\\
&- du_tu_yv_x + u_tv_{xxx} + v_tu_{xxx} - v_{xx}u_{tx} - u_{xx}v_{tx} + \frac{1}{2}u_tv_t,\\
T^3 =& -cu_{ty}v + du_tu_{xx}v + cu_tv_y,\\
T^4 =& eu_tv_z - eu_{tz}v.
\end{aligned}$$

The above gives the energy conservation law.

(ii) Using the Lie point symmetry $R_2 = \frac{\partial}{\partial x}$ and the fact that equation (1.36) gives $Y_2 = R_2$, together with the corresponding characteristic function $W = -u_x$, generates the following components.

$$\begin{aligned}
T^1 =& \frac{1}{2}u_xv_x - \frac{1}{2}u_{xx}v,\\
T^2 =& cu_{yy}v - du_xu_{xy}v + eu_{zz}v + \frac{1}{2}u_{tx}v + au_x^2v_x - bu_x^3v_x - du_xu_yv_x + \frac{1}{2}v_tu_x + u_xv_{xxx}\\
&- u_{xx}v_{xx} + u_{xxx}v_x,\\
T^3 =& -cu_{xy}v + du_{xx}u_xv + cu_xv_y,\\
T^4 =& eu_xv_z - eu_{xz}v.
\end{aligned}$$

The above yields the linear momentum conservation law.

(iii) From the operator $R_3 = \frac{\partial}{\partial y}$ and the fact that equation (1.36) gives $Y_3 = R_3$ together with the corresponding characteristic function $W = -u_y$, the following

components are obtained

$$
\begin{aligned}
T^1 =& \frac{1}{2}u_y v_x - \frac{1}{2}u_{xy}v,\\
T^2 =& -au_x u_{xy}v + bu_x^2 u_{xy}v - \frac{1}{2}u_{ty}v + au_x u_y v_x - bu_x^2 u_y v_x + du_y^2 v_x + \frac{1}{2}v_t u_y + u_y v_{xxx}\\
& - u_{xx}v_{xy} - v_{xx}u_{xy} + u_{xxx}v_y,\\
T^3 =& au_{xx}u_x v - bu_{xx}{u_x}^2 v + eu_{zz}v + u_{tx}v + cu_y v_y + u_{xx}v_{xx},\\
T^4 =& eu_y v_z - eu_{yz}v,
\end{aligned}
$$

which provides us with the linear momentum conservation law.

(iv) Since $Y_4 = R_4$ according to equation (1.36), the translation symmetry $R_4 = \frac{\partial}{\partial z}$ and its Lie characteristic function $W = -u_z$, yield the following conserved quantity

$$
\begin{aligned}
T^1 =& \frac{1}{2}u_z v_x - \frac{1}{2}u_{xz}v,\\
T^2 =& -au_x u_{xz}v + bu_x^2 u_{xz}v + du_y u_{xz}v - du_z u_{xy}v - \frac{1}{2}u_{tz}v + au_x u_z v_x - bu_x^2 u_z v_x - du_y u_z v_x\\
& + \frac{1}{2}v_t u_z - u_{xx}v_{xz} - v_{xx}u_{xz} + u_{xxx}v_z + u_z v_{xxx},\\
T^3 =& -cu_{yz}v + du_{xx}u_z v + cu_z v_y,\\
T^4 =& au_{xx}u_x v - bu_{xx}u_x^2 v + cu_{yy}v - du_{xx}u_y v + u_{tx}v + eu_z v_z + u_{xx}v_{xx},
\end{aligned}
$$

which also leads to law of conservation of linear momentum.

(v) The symmetry generator $R_5 = \frac{\partial}{\partial u}$ and the fact that $Y_5 = R_5$ according to the equation (1.36), gives Lie characteristic function $W = 1$, such that the resulting conserved quantities are as follows

$$
\begin{aligned}
T^1 =& -\frac{v_x}{2},\\
T^2 =& du_{xy}v - au_x v_x + bu_x^2 v_x + du_y v_x - \frac{v_t}{2} - v_{xxx},\\
T^3 =& -du_{xx}v - cv_y,\\
T^4 =& -ev_z.
\end{aligned}
$$

(vi) $R_6 = z\frac{\partial}{\partial u}$, similarly yields $Y_6 = R_6$ and $W = z$. Thus the resulting conserved

vectors are given by

$$\begin{aligned}
T^1 &= -\frac{1}{2}zv_x,\\
T^2 &= dzu_{xy}v - azu_xv_x + bzu_x^2v_x + dzu_yv_x - \frac{zv_t}{2} - zv_{xxx},\\
T^3 &= -dzu_{xx}v - czv_y,\\
T^4 &= ev - ezv_z,
\end{aligned}$$

(viii) Finally, the symmetry generator

$$R_7 = 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x + a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} + a\,x\,\frac{\partial}{\partial u}, \tag{2.72}$$

have the value of Y_7 which is different from R_7 given by

$$Y_7 = 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x + a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} + a\,x\,\frac{\partial}{\partial u} - 8bv\frac{\partial}{\partial v}. \tag{2.73}$$

Using the above together with the Lie characteristic function $W = a\,x - 6\,b\,t\,u_t - (2\,b\,x + a^2\,t)u_x - 4\,b\,y\,u_y - 4\,b\,z\,u_z$, the following conserved vector is obtained

$$\begin{aligned}
T^1 = &-\frac{1}{2}a^2tu_{xx}v + 6abtu_xu_{xx}v - 6b^2tu_x{}^2u_{xx}v + 6bctu_{yy}v - 6bdtu_{xx}u_yv + 6betu_{zz}v\\
&+ 3btu_{tx}v - 2byu_{xy}v - 2bzu_{xz}v - bu_xv - bxu_{xx}v + \frac{1}{2}av + \frac{1}{2}a^2tu_xv_x - \frac{1}{2}axv_x\\
&+ 3btu_tv_x + 6btu_{xx}v_{xx} + bxu_xv_x + 2byu_yv_x + 2bzu_zv_x,
\end{aligned}$$

$$
\begin{aligned}
T^2 = {} & tu_x{}^2v_xa^3 + etvu_{zz}a^2 + ctvu_{yy}a^2 + \frac{1}{2}vu_xa^2 - btu_x{}^3v_xa^2 - xu_xv_xa^2 - dtu_yu_xv_xa^2 \\
& - dtvu_xu_{xy}a^2 - tu_{xx}v_{xx}a^2 + tv_xu_{xxx}a^2 + tu_xv_{xxx}a^2 + \frac{1}{2}tu_xv_ta^2 + \frac{1}{2}tvu_{tx}a^2 \\
& - 3bvu_x{}^2a - dvu_ya + 3bxu_x{}^2v_xa + dxu_yv_xa + 4bzu_zu_xv_xa + 4byu_yu_xv_xa \\
& - 4bzvu_xu_{xz}a + dxvu_{xy}a - 4byvu_xu_{xy}a + v_{xx}a - xv_{xxx}a + 6btu_xv_xu_ta - \frac{1}{2}xv_ta \\
& - 6btvu_xu_{tx}a + 2b^2vu_x{}^3 + 2bexvu_{zz} + 2bcxvu_{yy} + 2bdvu_yu_x - 2b^2xu_x{}^3v_x \\
& - 4bdyu_y{}^2v_x - 4b^2zu_zu_x{}^2v_x - 4b^2yu_yu_x{}^2v_x - 4bdzu_zu_yv_x - 2bdxu_yu_xv_x \\
& + 4b^2zvu_x{}^2u_{xz} + 4bdzvu_yu_{xz} + 4b^2yvu_x{}^2u_{xy} - 4bdzvu_zu_{xy} - 2bdxvu_xu_{xy} - 2bv_xu_{xx} \\
& - 4bzv_{xz}u_{xx} - 4byv_{xy}u_{xx} - 2bu_xv_{xx} - 4bzu_{xz}v_{xx} - 4byu_{xy}v_{xx} - 2bxu_{xx}v_{xx} + 4bzv_zu_{xxx} \\
& + 4byv_yu_{xxx} + 2bxv_xu_{xxx} + 4bzu_zv_{xxx} + 4byu_yv_{xxx} + 2bxu_xv_{xxx} - 3bvu_t \\
& - 6b^2tu_x{}^2v_xu_t - 6bdtu_yv_xu_t - 6bdtvu_{xy}u_t + 6btv_{xxx}u_t + 2bzu_zv_t + 2byu_yv_t + bxu_xv_t \\
& + 6btu_{xxx}v_t + 3btu_tv_t - 2bzvu_{tz} - 2byvu_{ty} + 6b^2tvu_x{}^2u_{tx} + bxvu_{tx} + 6bdtvu_yu_{tx} \\
& - 6btv_{xx}u_{tx} - 6btu_{xx}v_{tx} - 3btvu_{tt}, \\
T^3 = {} & - a^2ctu_{xy} + a^2dtu_xu_{xx} + 4abyu_xu_{xx} - adxu_{xx} - 4b^2yu_x{}^2u_{xx} - 6bctu_{ty} - 2bcxu_{xy} \\
& - 4bczu_{yz} - 4bcu_y + 6bdtu_tu_{xx} + 4bdzu_{xx}u_z + 2bdxu_xu_{xx} + 4beyu_{zz} + 4byu_{tx} \\
& + a^2ctu_xv_y - acxv_y + 6bctu_tv_y + 2bcxu_xv_y + 4bcyu_yv_y + 4bczu_zv_y + 4byu_{xx}v_{xx}, \\
T^4 = {} & - a^2etu_{xz} + 4abzu_xu_{xx} - 4b^2zu_x{}^2u_{xx} + 4bczu_{yy} - 4bdzu_{xx}u_y - 6betu_{tz} - 2bexu_{xz} \\
& - 4beyu_{yz} - 4beu_z + 4bzu_{tx} + a^2etu_xv_z - aexv_z + 6betu_tv_z + 2bexu_xv_z \\
& + 4beyu_yv_z + 4bezu_zv_z + 4bzu_{xx}v_{xx}.
\end{aligned}
$$

Remark: It should be noted that the above conservation laws include the energy conservation law, which corresponds to the time translation and three momentum conservation laws, which correspond to the three space translations.

2.3 Conservation laws using multiplier method

This section invokes the multiplier technique to generate the conservation laws of (2.1). Considering the zeroth-order multiplier $\Lambda = \Lambda(t, x, y, z, u)$ of the NPDE men-

tioned in (2.1), these multipliers are determined by using the determining equation

$$\frac{\delta}{\delta u}\left\{\Lambda\left(u_{tx}+au_xu_{xx}+u_{xxxx}-bu_x^2u_{xx}+cu_{yy}-du_{xx}u_y+eu_{zz}\right)\right\}=0, \tag{2.74}$$

having $\frac{\delta}{\delta u}$ as the Euler operator given by

$$\begin{aligned}\frac{\delta}{\delta u}=&\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}\\&+D_z^2\frac{\partial}{\partial u_{zz}}+D_x^4\frac{\partial}{\partial u_{xxxx}},\end{aligned}$$

where D_t, D_x, D_y and D_z are total derivative operators. Expanding (2.74) and splitting on the derivatives of u gives the determining equations:

$$\left.\begin{aligned}\Lambda_{zz}&=0,\\\Lambda_x&=0,\\\Lambda_y&=0,\\\Lambda_u&=0.\end{aligned}\right\} \tag{2.75}$$

Working out the PDEs (2.75) gives,

$$\Lambda=G(t)+F(t)z,$$

where F and G are functions of t. The conserved quantities of equation (2.1) are derived by applying the divergence identity

$$\begin{aligned}&D_tT^t+D_xT^x+D_yT^y+D_zT^z=\\&\Lambda\left(u_{tx}+au_xu_{xx}+u_{xxxx}-bu_x^2u_{xx}+cu_{yy}-du_{xx}u_y+eu_{zz}\right)\end{aligned} \tag{2.76}$$

with T^t being conserved density and T^x, T^y and T^z are spatial fluxes. Thus, using (2.76) we obtain the following conserved vectors corresponding to the multiplier:

Case 1. For $\Lambda_1 = F(t)z$, we have the corresponding conserved vector as

$$\begin{aligned}
T^t &= \frac{1}{2}zF(t)u_x,\\
T^x &= azF(t)u_x - \frac{1}{3}zbF(t)u_x^3 + \frac{1}{2}azu_x^2 + \frac{1}{2}zdF(t)uu_{xy} + \frac{1}{2}zdF(t)u_xu_y + \frac{1}{2}zF'(t)u\\
&\quad + zF(t)u_{xxx} - \frac{1}{2}zF'(t)u + \frac{1}{2}zF(t)u_t,\\
T^y &= zcF(t)u_y - \frac{1}{2}zF(t)uu_{xx},\\
T^z &= dzeF(t)u_z - deF(t)u;
\end{aligned}$$

Case 2. For $\Lambda_2 = G(t)$, we have the conserved vector given by

$$\begin{aligned}
T^t &= \frac{1}{2}G(t)u_x,\\
T^x &= G(t)u_{xxx} - \frac{1}{2}G'(t)u + \frac{1}{2}u_t + \frac{1}{2}dG(t)uu_{xy} - \frac{1}{2}dG(t)u_xu_y - \frac{1}{3}bG(t)u_x^3 + \frac{1}{2}aG(t)u_x^2,\\
T^y &= cG(t)u_y - \frac{1}{2}dG(t)uu_{xx},\\
T^z &= eG(t)u_z.
\end{aligned}$$

NB. Our conservation laws have arbitrary functions, therefore we can have infinitely many conservation laws.

2.4 Concluding remarks

This chapter used the theory of Lie symmetry analysis to investigate the first (3+1)-dimensional Gardner-type equation (2.1). Lie symmetries were attained and utilised to accomplish symmetry reductions that leave the model under investigation invariant. As a result of symmetry reduction performed, various solvable representations of the model of interest in this chapter were obtained from different linear combinations of the symmetries admitted by (3+1)-dimensional Gardner-type equation (2.1). Transformed solvable representations were solved using different integration techniques. The solutions obtained were further used to study momentum and energy variation graphically. The increase in sound projection along a certain direction and plane leads to a decrease in the direction perpendicular to either the plane or the direction that the sound projection increases along. Lastly, conservation laws representing mass, energy and momentum, among others, were derived using the help of Ibragimov theorem and the multiplier method, respectively, to construct the conserved vectors of the equation under investigation.

Chapter 3

Symmetry analysis and conservation laws of the second (3+1)-dimensional Gardner-type equation

3.1 Exact solutions for (6)

In this section, similarity transformation based on Lie's theory is used to transform equation (6) representing the case study under investigation into a solvable form. The obtained solvable forms will be integrated to find exact solutions of the PDE (6). Similarly by letting $a = 6\,\beta$, $b = -\frac{3}{2}\,\alpha^2$, $c = 3\,\sigma^2$, $d = -3\,\alpha\,\sigma$, $e = 3\,\sigma^2$ and $f = -3\,\alpha\,\sigma$ for simplicity, equation (6) takes the following form

$$u_{xt} + au_xu_{xx} + u_{xxxx} + bu_x^2u_{xx} + cu_{yy} + du_{xx}u_y + eu_{zz} + fu_{xx}u_z = 0, \tag{3.1}$$

3.1.1 Lie Symmetries

In this section, Lie symmetry operators admitted by equation (3.1) are generated. Using one-parameter group of transformations with an infinitesimal generator defined

by (1.7) whenever

$$R^{[4]}(u_{xt}+au_xu_{xx}+u_{xxxx}+bu_x^2u_{xx}+cu_{yy}+du_{xx}u_y+eu_{zz}+fu_{xx}u_z)\Big|_{(3.1)}=0, \quad (3.2)$$

where by $R^{[4]}$ is the fourth prolongation of R given in (1.13), yields

$$\begin{aligned}&\big(\zeta_{12}+\zeta_{2222}+du_{xx}\zeta_3+fu_{xx}\zeta_4+e\zeta_{44}+c\zeta_{33}+\zeta_2(au_{xx}+2bu_x)+\\&\zeta_{22}(au_x+bu_x^2+du_y+fu_z)\big)\Big|_{(3.1)}=0. \quad (3.3)\end{aligned}$$

Expanding (3.3) and separating the resulting equation on derivatives of u, yields the following system of PDEs:

$$\begin{aligned}\xi^1_x &= 0, && (3.4)\\ \xi^1_y &= 0, && (3.5)\\ \xi^1_u &= 0, && (3.6)\\ \xi^2_y &= -\frac{\xi^3_t}{2\,c}, && (3.7)\\ \xi^2_u &= 0, && (3.8)\\ \xi^3_x &= 0, && (3.9)\\ \xi^3_u &= 0, && (3.10)\\ \xi^4_x &= 0, && (3.11)\\ \xi^4_z &= \frac{2\xi^1_t}{3}, && (3.12)\\ \xi^4_y &= 0, && (3.13)\\ \xi^4_u &= 0, && (3.14)\\ \eta_u &= 0, && (3.15)\\ \xi^1_t &= \frac{3\,\xi^4_z}{2}, && (3.16)\\ \xi^1_z &= 0, && (3.17)\end{aligned}$$

$$\xi^2_x = \frac{\xi^1_t}{3}, \tag{3.18}$$

$$\xi^2_z = -\frac{\xi^4_t}{2\,e}, \tag{3.19}$$

$$\xi^3_y = \frac{2\xi^1_t}{3}, \tag{3.20}$$

$$\xi^3_z = 0, \tag{3.21}$$

$$\xi^1_{tt} = 0, \tag{3.22}$$

$$\xi^4_{zz} = 0, \tag{3.23}$$

$$\eta_x = -\frac{a\,\xi^1_t}{6\,b} - \frac{f\,\xi^4_t}{4\,b\,e} - \frac{d\,\xi^3_t}{4\,b\,c}, \tag{3.24}$$

$$\eta_y = \frac{a^2\xi^1_t}{6\,d\,b} + \frac{a\,\xi^3_t}{4\,b\,c} + \frac{\xi^2_t}{d} - \frac{f\,\eta_z}{d} + \frac{a\,f\,\eta^4_t}{4\,e\,d\,b}, \tag{3.25}$$

$$\eta_{zz} = -\frac{f\,(2\,b\,c - d^2)\,\xi^4_{tt}}{4\,b\,e\,(c\,f^2 + e\,d^2)} + \frac{d\,(2\,b\,c + d^2)\,\xi^3_{tt}}{4\,b\,e\,(c\,f^2 + e\,d^2)}. \tag{3.26}$$

Solving the above system of PDEs, we obtain the following values of ξ^1, ξ^2, ξ^3, ξ^4 and η as

$$\begin{aligned}
\xi^1 &= k_1\,t + k_2,\\
\xi^2 &= \frac{1}{3}k_1\,x + d\,F_4(t) + f\,F_5(t) - \frac{a^2\,k_1\,t}{6\,b} - \frac{d\,a\,F_1(t)}{4\,b\,c} - \frac{f\,a\,F_1(t)}{4\,b\,e} + k_3,\\
\xi^3 &= \frac{2}{3}y\,k_1 + F_3(t),\\
\xi^4 &= \frac{2}{3}z\,k_1 + F_2(t),\\
\eta &= -\frac{a\,x\,k_1}{6\,b} + F_6(t).
\end{aligned} \tag{3.27}$$

where k_1, k_2, k_3 are constants and $F_1(t)$ to $F_6(t)$ are arbitrary functions with respect to t. Similarly, we set functions of t to be 1, to obtain physically interesting solutions that yields travelling wave solutions. Thus, one obtains the following six Lie

symmetries of (3.1) as follows:

$$\left.\begin{aligned} R_1 &= \frac{\partial}{\partial t}, \\ R_2 &= \frac{\partial}{\partial x}, \\ R_3 &= \frac{\partial}{\partial y}, \\ R_4 &= \frac{\partial}{\partial z}, \\ R_5 &= \frac{\partial}{\partial u}, \\ R_6 &= 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u}. \end{aligned}\right\} \tag{3.28}$$

3.1.2 Commutator table for (3.1)

In this subsection, we obtain the commutation relations based on the symmetries (3.28) and represent them in a tabular form. Executing the first commutation by making us of $[R_3, R_6]$

$$\begin{aligned} [R_3, R_6] &= R_3\,R_6 - R_6\,R_3 \\ &= \frac{\partial}{\partial y}\left(6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u}\right) \\ &\quad -\left(6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u}\right)\frac{\partial}{\partial y} \\ &= 4\,b\,\frac{\partial}{\partial y} \\ &= 4\,b\,R_3. \end{aligned}$$

This approach applies to all the remaining commutation relations, thus representing all the remaining commutation relations in a table below

	R_1	R_2	R_3	R_4	R_5	R_6
R_1	0	0	0	0	0	$6\,b\,R_1 - a^2\,R_2$
R_2	0	0	0	0	0	$2\,b\,R_2 - a\,R_5$
R_3	0	0	0	0	0	$4\,b\,R_3$
R_4	0	0	0	0	0	$4\,b\,R_4$
R_5	0	0	0	0	0	0
R_6	$-6\,b\,R_1 + a^2\,R_2$	$-2\,b\,R_2 + a\,R_5$	$-4\,b\,R_3$	$-4\,b\,R_4$	0	0

Table 2. Commutator table of the equation (3.1)

3.1.3 Symmetry reductions

In this subsection, we carry out symmetry reductions by utilizing the Lie point symmetries admitted by equation (3.1). Thereafter use the obtained symmetries to transform the PDE under investigation in this chapter to NODEs. The NODEs representations of the case study will then be solved to obtain solutions of (3.1), which we further represent graphically to display the dynamics of the model under investigation.

Case 1: $R = R_1 + R_2 + R_3 + \alpha R_4$

We start by considering the linear combination of four translation symmetries R_1, R_2, R_3 and R_4, taking α as a constant. Solving the resulting Lagrange system gives the following similarity transformations

$$j = t - y, \quad k = t - x, \quad l = \alpha t - z, \quad u = \theta(k, j, l). \tag{3.29}$$

The above similarity transformations reduce the second (3+1)-dimensional Gardner-type equation (3.1) to the following nonlinear PDE

$$\begin{aligned} &\theta_{kkkk} - \theta_{kk} - \theta_{kj} - \alpha\theta_{kl} - a\theta_k\theta_{kk} + b\theta_k^2\theta_{kk} + c\theta_{jj} - d\theta_{kk}\theta_j \\ &+e\theta_{ll} - f\theta_{kk}\theta_l = 0. \end{aligned} \tag{3.30}$$

The above transformed equation (3.30) admits the following six Lie point symmetries:

$$\Gamma_1 = \frac{\partial}{\partial k},\ \Gamma_2 = \frac{\partial}{\partial j},\ \Gamma_3 = \frac{\partial}{\partial l},\ \Gamma_4 = (d\,l - f\,j)\frac{\partial}{\partial \theta},\ \Gamma_5 = \frac{\partial}{\partial \theta},$$

$$\begin{aligned} \Gamma_6 = &\left(2\,a^2cel + a\alpha\,cfl - 2\,acefk + 2\,\alpha^2bcl - \alpha\,cf^2k + adel + 8\,bcel - defk + 2\,bel\right)\frac{\partial}{\partial \theta} \\ &- 8\,bcefj\frac{\partial}{\partial j} + \left(2\,\alpha\,bcfl - 4\,bcefk + 2\,fbej\right)\frac{\partial}{\partial k} - 8\,bcefl\frac{\partial}{\partial l}. \end{aligned}$$

Using this symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \beta\,\Gamma_3$, where β is a constant, this symmetry yields the following three invariants:

$$r = j - k, \quad w = \beta j - l, \quad \theta = \phi(r, w). \tag{3.31}$$

The above invariants (3.31) transform equation (3.30) as follows

$$\phi_{rrrr} + (\beta - \alpha + 2\,c\,\beta)\phi_{rw} + (a - d)\phi_r\phi_{rr} + b\phi_r^2\phi_{rr} + c\phi_{rr} + (c\,\beta^2 + e)\phi_{ww} + (f - d\,\beta)\,\beta\,\phi_{rr}\phi_w = 0. \tag{3.32}$$

Equation (3.32) above admits the following operators:

$$\Sigma_1 = \frac{\partial}{\partial w},\ \Sigma_2 = \frac{\partial}{\partial r},\ \Sigma_3 = \frac{\partial}{\partial \phi},$$

$$\begin{aligned}\Sigma_4 = {} & (8\,b\,\beta^2\,c\,f\,w - 8\,b\,\beta^3\,c\,d\,w - 8\,b\,\beta\,d\,e\,w + 8\,b\,e\,f\,w)\frac{\partial}{\partial w} + (4\,b\,\beta^3\,c\,d\,r - 4\,b\,\beta^2\,c\,d\,w \\ & + 4\,b\,\beta^2\,c\,f\,r + 2\alpha\,b\,\beta\,d\,w - 2\,b\,\beta^2\,d\,w + 4\,b\,\beta\,c\,f\,w - 4\,b\,\beta\,d\,e\,r - 2\,\alpha\,b\,f\,w + 2\,b\,\beta\,f\,w \\ & + 4\,b\,e\,f\,r)\frac{\partial}{\partial r} + (2\,a\,\beta^3\,c\,d\,r + 2\,a^2\beta^2\,c\,w - 2\,a\,\beta^2\,c\,d\,w - 2\,a\,\beta^2\,c\,f\,r - \alpha\,\beta^2\,d^2\,r \\ & + \beta^3\,d^2\,r - 2\,\beta^2\,c\,d\,f\,r - a\,\alpha\,\beta\,d\,w + a\,\beta^2\,d\,w - 2\,a\,\beta\,c\,f\,w + 2\,a\,\beta\,d\,e\,r - 8\,\alpha\,b\,\beta\,c\,w \\ & + \alpha\,\beta\,d^2\,w + 2\,\alpha\,\beta\,d\,f\,r + 8\,b\,\beta^2\,c\,w - \beta^2\,d^2\,w - 2\,\beta^2\,d\,f\,r + 2\,\beta\,c\,d\,f\,w + 2\,\beta\,c\,f^2\,r \\ & - 2\,\beta\,d^2\,e\,r + 2\,a^2\,e\,w + a\,\alpha\,f\,w - a\,\beta\,f\,w - 4\,a\,d\,e\,w - 2\,a\,e\,f\,r + 2\,\alpha^2\,b\,w - 4\,\alpha\,b\,\beta\,w \\ & - \alpha\,d\,f\,w - \alpha\,f^2\,r + 2\,b\,\beta^2\,w - 8\,b\,c\,e\,w + \beta\,d\,f\,w + \beta\,f^2\,r + 2\,d^2\,e\,w + 2\,d\,e\,f\,r)\frac{\partial}{\partial \phi}.\end{aligned}$$

Using the combination of Σ_1 and Σ_2 such that the symmetry $\Sigma = \Sigma_1 + \delta\Sigma_2$, taking δ as a constant, gives two invariants

$$k = \delta w - r, \quad \phi = \psi(k),$$

which maps equation (3.32) to an ODE given as follows

$$\begin{aligned}&\psi'''' + b(\psi')^2\psi'' + [-\delta(\beta - \alpha + 2c\beta) + c + \delta^2(e + c\beta^2)]\psi'' \\ &+[\delta(f - d\beta) - a + d]\psi'\psi'' = 0.\end{aligned} \tag{3.33}$$

The equation above can be written as

$$\psi'''' + b\psi'^2\psi'' + A\psi'' + B\psi'\psi'' = 0, \tag{3.34}$$

by taking $A = -\delta(\beta - \alpha + 2c\beta) + c + \delta^2(e + c\beta^2)$ and $B = \delta(f - d\beta) - a + d$.
If we integrate equation (3.34) once and taking constant of integration to be zero gives

$$\psi''' + \frac{b}{3}\psi'^3 + \frac{1}{2}B\psi'^2 + A\psi' = 0. \tag{3.35}$$

Integrating equation (3.35) gives

$$\frac{1}{2}\psi''^2 + \frac{b}{12}\psi'^4 + \frac{1}{6}B\psi'^3 + \frac{1}{2}A\psi'^2 = 0. \tag{3.36}$$

Setting $\psi'(k) = G(k)$, transforms equation (3.36) as follows:

$$\frac{1}{2}G'^2 + \frac{b}{12}G^4 + \frac{1}{6}BG^3 + \frac{1}{2}AG^2 = 0, \tag{3.37}$$

which yields the following separable equation

$$\int \left(\frac{1}{G\sqrt{-\frac{b}{6}G^2 - \frac{B}{3}G - A}} \right) \mathrm{d}G = \int dk.$$

Integrating the above separable equation gives

$$\ln\left[\frac{-72\,A - 12\,B\,G + 12\,\sqrt{-A}\sqrt{-6\,b\,G^2 - 12\,B\,G - 36\,A}}{G} \right] = \sqrt{-A}\,(c_1 - k),$$

which can be simplified as follows

$$-\frac{72\,A}{G} + \frac{12\,\sqrt{-A}}{G}\sqrt{-6\,b\,G^2 - 12\,B\,G - 36\,A} = \mathbf{e}^{\sqrt{-A}(c_1-k)} + 12\,B.$$

Isolating the value of G, we obtain the following solution of $G(k)$:

$$G(k) = \frac{144\,A e^{\sqrt{-A}\,(c_1+k)}}{864\,A\,b\,e^{2\sqrt{-A}\,k} - 144\,B^2\,e^{2\sqrt{-A}\,k} - 24\,B\,e^{\sqrt{-A}\,(c_1+k)} - e^{2\sqrt{-A}c_1}}. \tag{3.38}$$

Using the fact that $\psi'(k) = G(k)$ and integrating the resulting equation gives

$$\psi(k) = 144\,A\,\mathbf{e}^{c_1\sqrt{-A}} \int \left(\frac{\mathbf{e}^{k\sqrt{-A}}}{(864\,A\,b - 144\,B)\mathbf{e}^{2\,k\sqrt{-A}} - \mathbf{e}^{2\,c_1\sqrt{-A}} - 24\,B\,\mathbf{e}^{\sqrt{-A}\,(c_1+k)}} \right) dk. \tag{3.39}$$

Since $z = \mathbf{e}^{k\sqrt{-A}}$, $dk = \frac{\mathbf{e}^{-k\sqrt{-A}}}{\sqrt{-A}}\,dz$, thus equation (3.39) becomes

$$\psi(k) = \frac{144\,A\,\mathbf{e}^{c_1\sqrt{-A}}}{\sqrt{-A}} \int \left(\frac{1}{24[6(6\,A\,b - B^2)\,z^2 - B\,\mathbf{e}^{c_1\sqrt{-A}}\,z] - \mathbf{e}^{2\,c_1\sqrt{-A}}} \right) dz. \tag{3.40}$$

Using completing the square, equation (3.40) gives

$$\psi(k) = \frac{144\,A\,\mathbf{e}^{c_1\sqrt{-A}}}{\sqrt{-A}} \int \left(\frac{1}{\left[12\sqrt{6\,A\,b - B^2}\,z - \frac{B\,\mathbf{e}^{c_1\sqrt{-A}}}{\sqrt{6\,A\,b - B^2}}\right]^2 - \frac{B^2\,\mathbf{e}^{2\,c_1\sqrt{-A}}}{6\,A\,b - B^2} - \mathbf{e}^{2\,c_1\sqrt{-A}}} \right) dz. \tag{3.41}$$

By taking $p = \frac{12\,(6\,A\,b-B^2)z-B\,\mathbf{e}^{c_1\,\sqrt{-A}}}{\sqrt{6\,A\,b-B^2}\,\sqrt{-\frac{B^2\,\mathbf{e}^{2\,c_1\,\sqrt{-A}}}{6\,A\,b-B^2}-\mathbf{e}^{2\,c_1\,\sqrt{-A}}}}$, such that $dz = \frac{\sqrt{-\frac{B^2\,\mathbf{e}^{2\,c_1\,\sqrt{-A}}}{6\,A\,b-B^2}-\mathbf{e}^{2\,c_1\,\sqrt{-A}}}}{12\,\sqrt{6\,A\,b-B^2}}\,dp$, thus equation (3.41) becomes

$$\psi(k) = \frac{144\,A\,\mathbf{e}^{c_1\,\sqrt{-A}}}{\sqrt{-A}}\,\frac{\sqrt{6\,A\,b-B^2}}{(72\,A\,b-12\,B^2)\,\sqrt{-\frac{B^2\,\mathbf{e}^{2\,c_1\,\sqrt{-A}}}{6\,A\,b-B^2}}-\mathbf{e}^{2\,c_1\,\sqrt{-A}}}\int\frac{1}{p^2+1}\,dp. \quad (3.42)$$

Using the identity

$$\int\frac{1}{p^2+1}\,dp = \ tan^{-1}(p),$$

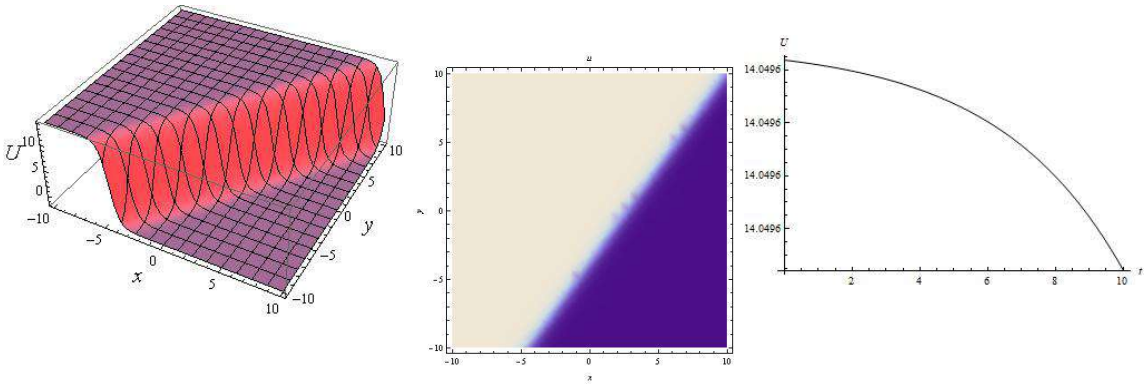
solution (3.42) in terms of the original variables becomes

$$u(t,x,y,z) = -\frac{2\,\sqrt{6}\,\sqrt{-A}\,\mathbf{e}^{\sqrt{-A}\,c_1}}{\sqrt{-A\,b\,\mathbf{e}^{2\sqrt{-A}\,c_1}}}\,\tan^{-1}\left[\frac{72\,A\,b\,\mathbf{e}^{k\,\sqrt{-A}}-12B^2\mathbf{e}^{\sqrt{-A}k}-B\mathbf{e}^{c_1\sqrt{-A}}}{12\sqrt{6}\sqrt{-A\,b\,\mathbf{e}^{2\sqrt{-A}\,c_1}}}\right], \quad (3.43)$$

where

$$A = c(\beta\delta-1)^2+\delta(\alpha-\beta+\delta e),$$
$$B = -a-\beta d\delta+d+\delta f,$$
$$k = t(\beta\delta-\alpha\delta)-x+y(1-\beta\delta)+\delta z.$$

Representing the solution (3.43) graphically when $\delta = 0.4$, $B = -0.6$, $b = 0.3$, $\alpha = 1$, $A = -5$, $t = 5$, $\beta = 0.7$, $c_1 = 0.7$, $z = 5.628$, $x = 0.4$, yields the following plots



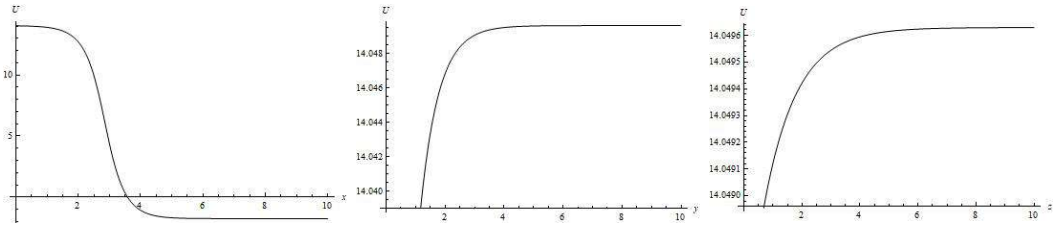


Figure 3.1: The 3D, Density plot and 2D illustration of velocitysolution (3.43)

Analysis of figure 3.1.

As seen from Figure 3.1 the wave velocity decreases as x increases and increases along the y direction as well as along the normal direction z. Since the increase in velocity amplifies momentum and energy, these plots show that there is an increase in both momentum and energy along the surface ($y-z$ plane) and a decrease in momentum along the x direction. Also, it should be noted that over time the velocity decreases.

Case 2 : $R = R_2 + \alpha R_5$

Lie combination $R_2 + \alpha R_5$, yields the following four Lie similarity transformations

$$g = y, \quad j = x, \quad h = z, \quad \theta(j,h,g) = u - \alpha t. \tag{3.44}$$

Using the above transformation (3.44) in equation (3.1) yields

$$a\theta_j\theta_{jj} + \theta_{jjjj} + b\theta_{jj}\theta_j^2 + c\theta_{gg} + d\theta_{jj}\theta_g + e\theta_{hh} + f\theta_{jj}\theta_h = 0. \tag{3.45}$$

Similarly equation (3.45) admits the following six operators

$$\Gamma_1 = \frac{\partial}{\partial j},\ \Gamma_2 = \frac{\partial}{\partial h},\ \Gamma_3 = \frac{\partial}{\partial g},\ \Gamma_4 = (d\,h - f\,g)\frac{\partial}{\partial \theta},\ \Gamma_5 = \frac{\partial}{\partial \theta},$$
$$\Gamma_6 = 4\,h\,f\,b\frac{\partial}{\partial h} + 2\,f\,j\,b\frac{\partial}{\partial j} + 4\,g\,f\,b\frac{\partial}{\partial g} + (a^2\,h - a\,f\,j)\frac{\partial}{\partial \theta}.$$

Utilising operator Γ_1 to reduce (3.45), we get the following invariants,

$$r = g, \quad s = h, \quad \phi = \theta,$$

which further transform equation (3.45) to the following simpler PDE

$$c\,\phi_{rr} + e\,\phi_{ss} = 0. \tag{3.46}$$

The above equation (3.46) admits the following four Lie symmetries

$$\Sigma_1 = \frac{\partial}{\partial s},\ \Sigma_2 = \frac{\partial}{\partial r},\ \Sigma_3 = \frac{\partial}{\partial \phi},\ \Sigma_4 = \phi\frac{\partial}{\partial \phi}.$$

Using the symmetry resulting from combining $\Sigma_1+\Sigma_2+\beta\Sigma_4$ above, gives the following invariants

$$k = r - s, \quad F = \phi\, e^{-\beta s},$$

which further reduce equation (3.46) into an ODE given by

$$e\,\beta^2\,F - 2\,\beta\,e\,F' + c\,F'' + e\,F'' = 0. \tag{3.47}$$

The general solution of the ODE above is given by

$$F(k) = c_1\,\mathbf{e}^{\frac{e+\sqrt{-ce}\,\beta\,k}{c+e}} + c_2\,\mathbf{e}^{-\frac{-e+\sqrt{-ce}\,\beta\,k}{c+e}}. \tag{3.48}$$

where c_1, c_2 are arbitrary constants and $k = y - z$. Consequentially the solution becomes

$$u(t,x,y,z) = \mathbf{e}^{z\,\beta}[\alpha\,t\,\mathbf{e}^{-z\,\beta} + c_1\,\mathbf{e}^{\frac{e+i\sqrt{ce}\,\beta\,(y-z)}{c+e}} + c_2\,\mathbf{e}^{-\frac{-e+i\sqrt{ce}\,\beta\,(y-z)}{c+e}}]. \tag{3.49}$$

Using the solution (3.49) the plots when $e = 10$, $z = 1$, $\alpha = 2$, $\beta = 0.8$, $c_1 = 10$, $\delta = 0.9$, $c_2 = 0.8$, $c = -0.8$, are shown in figure 3.2

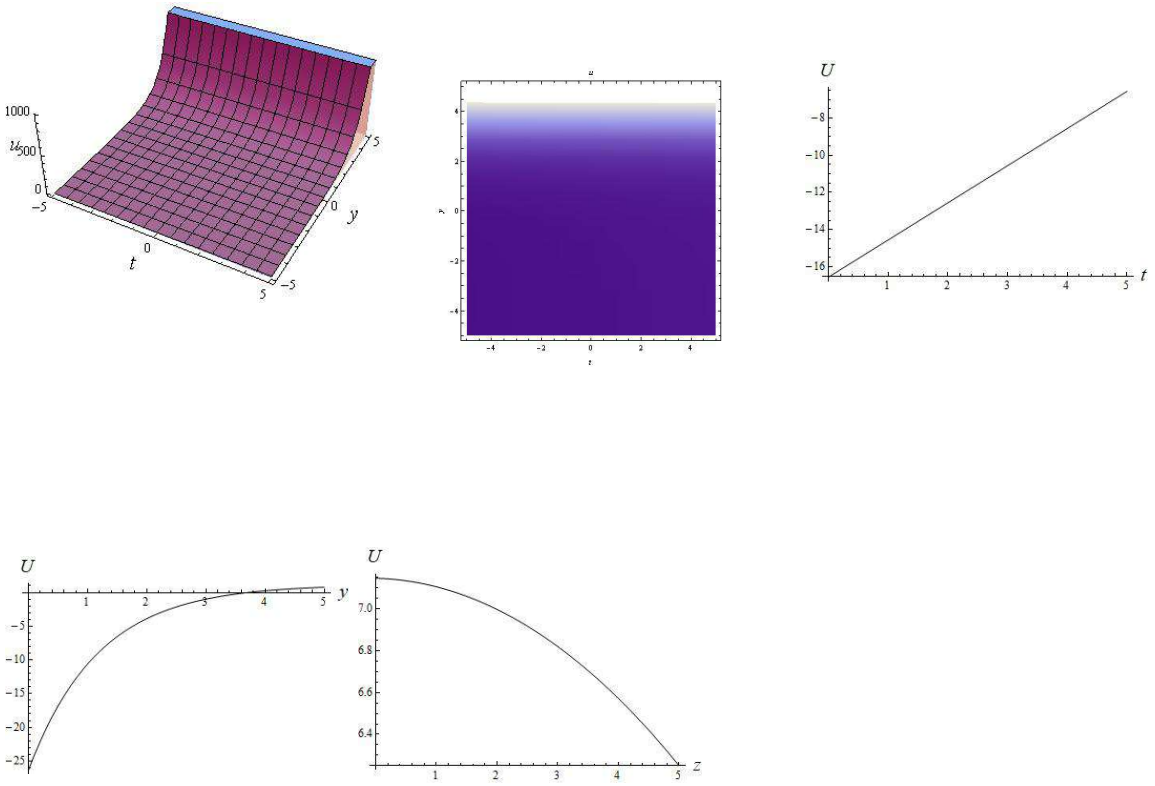


Figure 3.2: The 3D, Density plot and 2D illustration of solution (3.49)

Analysis of figure 3.2.

As seen from Figure 3.2 the wave velocity increases with the increase in axial space y while the increase in normal space z decreases velocity. This implies that momentum and energy increase along the y direction and decrease when normal space z becomes more (away from the surface). Time enhances the velocity according to the plots as seen in the above figure.

Case 3 : $R = R_4 + \alpha R_2$

This case uses a combination of two translation symmetries R_4, R_2 with α as a

constant, giving the invariants

$$g = t, \quad j = y, \quad h = x - \alpha\, z, \quad \theta(j, g, h) = u. \tag{3.50}$$

The use of invariants (3.50) reduces (3.1) to

$$a\theta_h\theta_{hh} + \theta_{hhhh} + b\theta_{hh}\theta_h^2 + c\theta_{jj} + \theta_{hg} + d\theta_{hh}\theta_j + e\alpha^2\,\theta_{hh} + f\,\alpha\,\theta_{hh}\theta_h = 0. \tag{3.51}$$

Equation (3.51) admits the following Lie operators

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial j},\ \Gamma_2 = \frac{\partial}{\partial h},\ \Gamma_3 = \frac{\partial}{\partial g},\ \Gamma_4 = \frac{\partial}{\partial \theta},\\
&\Gamma_5 = -2\,j\,b\,\frac{\partial}{\partial h} + 4\,g\,b\,c\,\frac{\partial}{\partial j} + (a\,j - \alpha\,f\,j - d\,h)\frac{\partial}{\partial \theta},\\
&\Gamma_6 = 2\,h\,b\,d\,\frac{\partial}{\partial h} + 4\,j\,b\,d\,\frac{\partial}{\partial j} + 6\,g\,b\,d\,\frac{\partial}{\partial g} + (\alpha^2\,f^2\,j - 4\,\alpha^2\,b\,e\,j - 2\,f\,\alpha\,a\,j +\\
&\quad \alpha\,d\,f\,h + j\,a^2 - d\,a, h)\frac{\partial}{\partial \theta}.
\end{aligned}$$

Utilizing Γ_6, yields three invariants given by

$$r = \frac{h}{g^{\frac{1}{3}}}, \quad s = \frac{h}{g^{\frac{1}{3}}}, \quad \phi = -\frac{j\,(a - \alpha\,f)^2}{4\,b\,d} + \frac{h(a - \alpha\,f)}{2\,b} + \frac{\alpha^2\,e\,j}{d} + \theta.$$

Substituting the above invariants into (3.51) transforms the equation (3.51) into

$$-3\,b\,\phi_{rr}\,\phi_r^2 - 3\,d\,\phi_{rr}\,\phi_s + r\,\phi_{rr} + 2\,s\,\phi_{rs} - 3\,c\,\phi_{ss} - 3\,\phi_{rrrr} + \phi_r = 0. \tag{3.52}$$

Which remains invariant under the following two symmetries

$$\Sigma_1 = \frac{\partial}{\partial \phi},\ \Sigma_2 = 3\,d\,\frac{\partial}{\partial r} + s\,\frac{\partial}{\partial \phi}.$$

Using Σ_2 to further reduce (3.52) gives the following invariants

$$k = s, \quad F = \frac{3\,d\,\phi - r\,s}{3\,d}.$$

Utilizing the above invariants in equation (3.52), gives the following ODE.

$$F''(k) = \frac{k}{3\,c\,d} \tag{3.53}$$

Solving (3.53) yields

$$F(k) = \frac{k^3}{18\,c\,d} + c_1\,k + c_2, \tag{3.54}$$

where c_1 and c_2 are constants. Thus, the solution of (3.1) according to (3.54) becomes

$$u(t,x,y,z) = \frac{y(a-\alpha f)^2}{4\,b\,d} + \frac{(a-\alpha f)(\alpha z - x)}{2\,b} + \frac{y^3}{18\,c\,d\,t^2} + \frac{c_1 y}{t^{\frac{2}{3}}} + c_2 - \frac{y\,(3\alpha^2 e t - x + \alpha z)}{3\,d\,t}. \tag{3.55}$$

The graphical representations of the solution (3.55) are as follows when $d = -1$, $c = 0.2$, $b = 0.2$, $a = 0.2$, $f = 0.2$, $\alpha = 0.2$, $z = 0.2$, $x = 0.2$, $e = 0.2$, $c_1 = 0.2$, $c_2 = 0.2$, $y = 0.2$

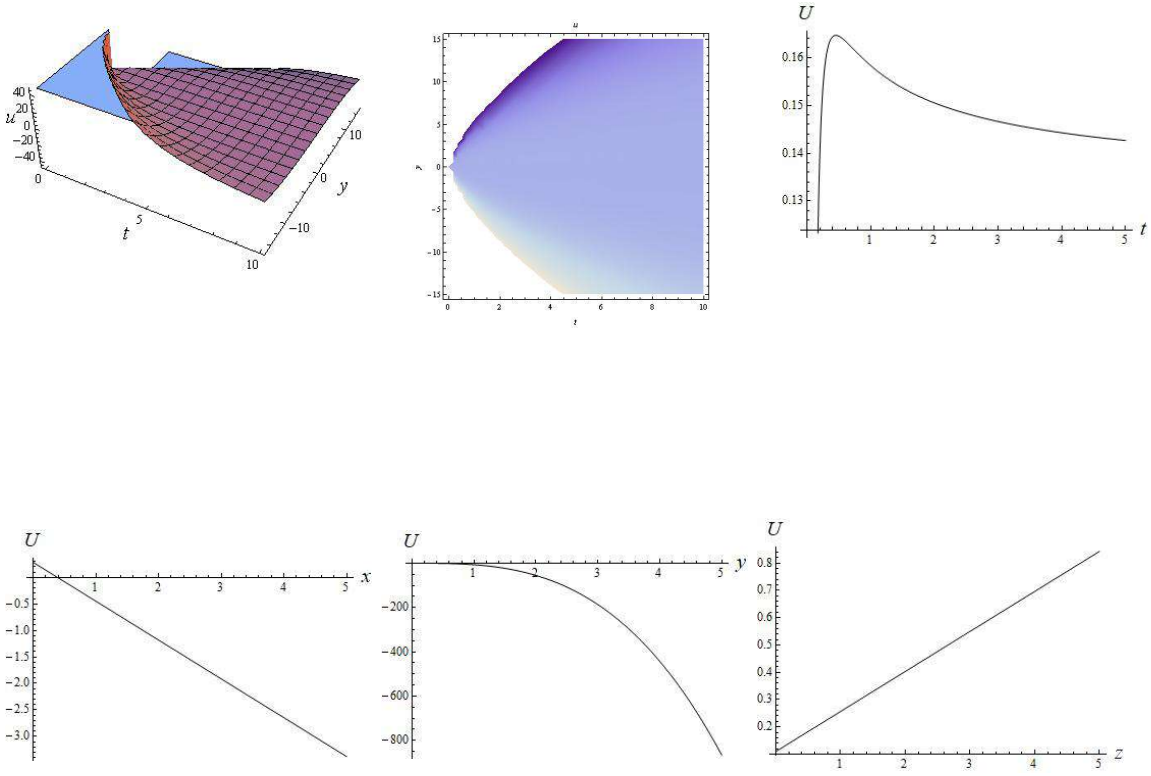


Figure 3.3: The 3D, Density plot and 2D illustration of solution (3.55)

Analysis of figure 3.3.

As seen from Figure 3.3, the wave velocity decreases when space along x and y increases while the velocity increases along the normal direction z. Since the increase in velocity enhances momentum and energy, these implies that momentum and energy decrease along the surface ($x-y$ plane) and there is an increase in momentum and energy as the height increase along the z direction (away from the surface). Initially, the velocity quickly reaches maximum velocity, thereafter, it decreases as time evolves.

Case 4 : $R = R_1 + \alpha R_4$

Using $R_1 + \alpha R_4$, we obtain invariants below, here α, is a constant

$$k = x, \quad j = y, \quad l = -\alpha\, t + z, \quad \theta(j,k,l) = u. \tag{3.56}$$

The above invariants reduce (3.1) as follows

$$-\alpha\, \theta_{l,k} + a\, \theta_k\, \theta_{kk} + \theta_{kkkk} + b\, \theta_k^2\, \theta_{kk} + c\, \theta_{jj} + d\, \theta_{kk}\, \theta_j + e\, \theta_{ll} + f\, \theta_{kk}\, \theta_l = 0. \tag{3.57}$$

Equation (3.57) yields the following symmetries

$$\Gamma_1 = \frac{\partial}{\partial j},\ \Gamma_2 = \frac{\partial}{\partial l},\ \Gamma_3 = \frac{\partial}{\partial k},\ \Gamma_4 = (d\, l - f\, j)\frac{\partial}{\partial \theta},\ \Gamma_5 = \frac{\partial}{\partial \theta},$$
$$\Gamma_6 = (4\, b\, e\, f\, k - 2\, \alpha\, b\, f\, l)\frac{\partial}{\partial k} + 8\, e\, f\, b\, j\, \frac{\partial}{\partial j} + 8\, l\, e\, f\, b\, \frac{\partial}{\partial l} + (2\, a^2\, e\, l + a\, \alpha\, f\, l - 2\, a\, e\, f\, k$$
$$+ 2\, \alpha^2\, b\, l - \alpha\, f^2\, k)\frac{\partial}{\partial \theta},$$

which leaves the same equation (3.57) invariant. The combination Γ_3 and Γ_4 which yields the operator $\Gamma = \Gamma_3 + \beta\, \Gamma_4$, taking β as a constant, gives the invariants

$$r = j, \quad s = l, \quad \phi(r,s) = -\beta\, d\, k\, l + \beta\, f\, j\, k + \theta. \tag{3.58}$$

The use of (3.58) in (3.57) gives

$$-\beta\, d\, \alpha + c\, \phi_{rr} + e\, \phi_{ss} = 0. \tag{3.59}$$

Similarly, the above PDE (3.59) admits the following four Lie symmetries

$$\Sigma_1 = \frac{\partial}{\partial s},\ \Sigma_2 = \frac{\partial}{\partial r},\ \Sigma_3 = \frac{\partial}{\partial \phi},\ \Sigma_4 = \phi\, \frac{\partial}{\partial \phi}.$$

Making use of the combination of the above symmetries $\Sigma = \Sigma_2 + \delta\,\Sigma_3$, where δ is considered to be a constant, gives the invariants below

$$k = s, \quad F(k) = -\delta\, r + \phi. \tag{3.60}$$

Utilizing these invariants into (3.59), we obtain the following ODE:

$$-\beta\, d\, \alpha + e\, F'' = 0, \tag{3.61}$$

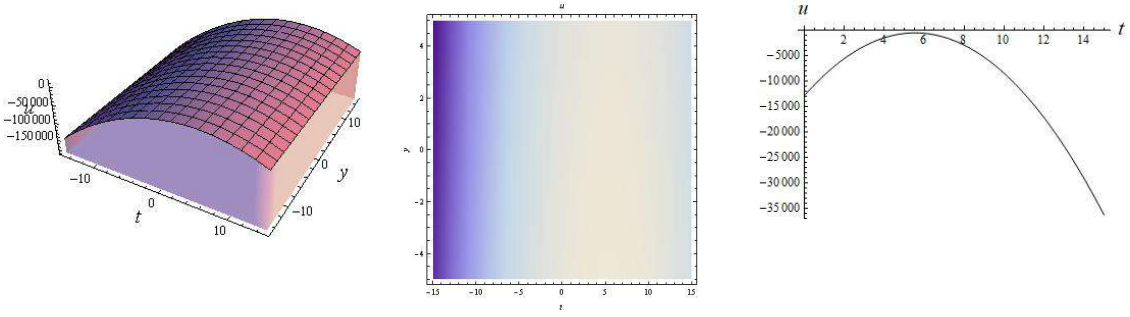
which, upon integration yields

$$F(k) = \frac{\beta\, d\, \alpha\, k^2}{2\, e} + c_1\, k + c_2, \tag{3.62}$$

where c_1 and c_2 are constants with $k = z - \alpha\, t$. Using solution $F(k)$ from (3.62) yields the solution in terms of the original variables as follows

$$u(t, x, y, z) = c_1(\alpha t + z) + c_2 + \frac{\alpha\beta d(\alpha t - z)^2}{2e} - \beta x(\alpha d t - dz + fy) + \delta y. \tag{3.63}$$

Ploting the solution (3.63) when $d = -1, e = 0.1, z = 11, \alpha = 2, \beta = 10, f = 10, c_1 = 10, \delta = 0.9, y = 10, c_2 = 0.8$, gives the plots in figure 3.4



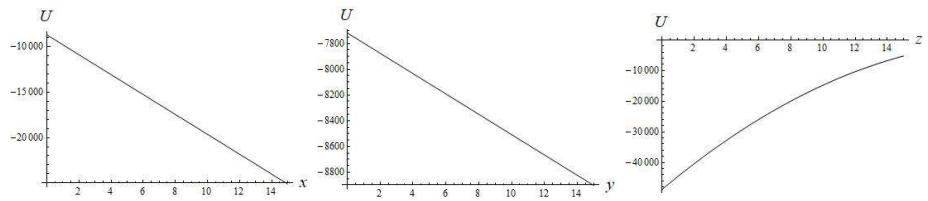


Figure 3.4: The 3D, Density plot and 2D illustration of solution (3.63)

Analysis of figure 3.4.

Figure 3.3 shows that increasing space along x and y leads to a decrease in wave velocity whereas increasing normal space z enhances velocity. Since the increase in velocity enhances momentum and energy, this implies that momentum and energy decrease along the surface ($x - y$ plane) and there is an increase in momentum and energy as the height increase along the z direction (away from the surface). The velocity increases until it reaches maximum after six seconds and decreases over time.

3.2 Construction of conservation laws for (3.1)

In this subsection, we generate conservation laws of the second (3+1)-dimensional Gardner-type equation (3.1) using the Ibragimov's theorem .

Recall that equation (3.1) admits the following generators:

$$\left.\begin{aligned} R_1 &= \frac{\partial}{\partial t}, \\ R_2 &= \frac{\partial}{\partial x}, \\ R_3 &= \frac{\partial}{\partial y}, \\ R_4 &= \frac{\partial}{\partial z}, \\ R_5 &= \frac{\partial}{\partial u}, \\ R_6 &= 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u}. \end{aligned}\right\} \tag{3.64}$$

In this section, six conserved vectors corresponding to these six Lie symmetries (3.64) are obtained. The adjoint equation of (3.1), when utilising (1.34), is as follows

$$\begin{aligned} E^* =& \frac{\delta}{\delta u}[v(u_{xt} + a u_x u_{xx} + c u_{yy} + e u_{zz} + b u_x^2 u_{xx} + d u_y u_{xx} \\ &+ f u_{xx} u_z + u_{xxxx})] \end{aligned} \tag{3.65}$$

for this we consider v to be equivalent to $v(t, x, y, z)$, thus (3.65) becomes

$$\begin{aligned} &a u_x v_{xx} + a u_{xx} v_x + b {u_x}^2 v_{xx} + 2b u_{xx} u_x v_x + c v_{yy} + 2d v_x u_{xy} - d u_{xx} v_y + d u_y v_{xx} \\ &+ e v_{zz} + 2f v_x u_{xz} - f u_{xx} v_z + f u_z v_{xx} + v_{tx} + v_{xxxx} = 0. \end{aligned} \tag{3.66}$$

According to (3.66), the Gardner-type equation (3.1) is not self-adjoint. Recall (1.39), we end up getting the fourth order Lagrangian which leads to the following

$$\begin{aligned} L \;=\; v\Big(&\frac{1}{2}u_{xt} + \frac{1}{2}u_{tx} + a u_x u_{xx} + u_{xxxx} + c u_{yy} + e u_{zz} + b u_x^2 u_{xx} \\ &+ d u_y u_{xx} + f u_z u_{xx}\Big). \end{aligned} \tag{3.67}$$

The conservation laws of equations (3.1) and (3.66) are obtained using Lie symmetries (3.64), and can be obtained by utilizing (1.40)

(i) For the first generator $R_1 = \frac{\partial}{\partial t}$, we have $W = -u_t$. Recall (1.36) which proves $Y_1 = R_1$. From (1.40), the Lie symmetry R_1 gives the following components of the

conserved vector:

$$\begin{aligned}
T^1 =& \alpha u_{xx}u_x v + b u_{xx}{u_x}^2 v + c u_{yy} v + d u_{xx} u_y v + e u_{zz} v + f u_{xx} u_z v + \frac{1}{2} u_{tx} v \\
&+ \frac{1}{2} u_t v_x + u_{xx} v_{xx}, \\
T^2 =& - a u_x u_{tx} v - b {u_x}^2 u_{tx} v + d u_t u_{xy} v - d u_y u_{tx} v + f u_t u_{xz} v - f u_z u_{tx} v \\
&- \frac{1}{2} u_{tt} v + a u_t u_x v_x + b u_t {u_x}^2 v_x + d u_t u_y v_x + f u_t u_z v_x + u_t v_{xxx} + v_t u_{xxx} \\
&- v_{xx} u_{tx} - u_{xx} v_{tx} + \frac{1}{2} u_t v_t, \\
T^3 =& - c u_{ty} v - d u_t u_{xx} v + c u_t v_y, \\
T^4 =& - e u_{tz} v - f u_t u_{xx} v + e u_t v_z.
\end{aligned}$$

The above conserved quantity gives the energy conservation law.

(ii) The second generator $R_2 = \frac{\partial}{\partial x}$, gives $W = -u_x$ with $Y_2 = R_2$ from equation (1.36). Utilizing (1.40), the components of the conserved vector are given by

$$\begin{aligned}
T^1 =& \frac{1}{2} u_x v_x - \frac{1}{2} u_{xx} v, \\
T^2 =& c u_{yy} v + d u_x u_{xy} v + e u_{zz} v + f u_x u_{xz} v + \frac{1}{2} u_{tx} v + a {u_x}^2 v_x \\
&+ b {u_x}^3 v_x + d u_x u_y v_x + f u_x u_z v_x + \frac{1}{2} v_t u_x + u_x v_{xxx} - u_{xx} v_{xx} + u_{xxx} v_x, \\
T^3 =& - c u_{xy} v - d u_{xx} u_x v + c u_x v_y, \\
T^4 =& - e u_{xz} v - f u_{xx} u_x v + e u_x v_z,
\end{aligned}$$

which leads to the linear momentum conservation law.

(iii) The third generator $R_3 = \frac{\partial}{\partial y}$, having $Y_3 = R_3$ from (1.36) with the Lie characteristic function $W = -u_y$, gives

$$\begin{aligned}
T^1 =& \frac{1}{2} u_y v_x - \frac{1}{2} u_{xy} v, \\
T^2 =& - a u_x u_{xy} v - b {u_x}^2 u_{xy} v + f u_y u_{xz} v - f u_z u_{xy} v - \frac{1}{2} u_{ty} v + a u_x u_y v_x \\
&+ b {u_x}^2 u_y v_x + d {u_y}^2 v_x + f u_y u_z v_x + \frac{1}{2} v_t u_y + u_y v_{xxx} - u_{xx} v_{xy} - v_{xx} u_{xy} + u_{xxx} v_y, \\
T^3 =& a u_{xx} u_x v + b u_{xx} {u_x}^2 v + e u_{zz} v + f u_{xx} u_z v + u_{tx} v + c u_y v_y + u_{xx} v_{xx}, \\
T^4 =& - e u_{yz} v - f u_{xx} u_y v + e u_y v_z,
\end{aligned}$$

which provides us with linear momentum conservation law.

(iv) Fourth translational Lie symmetry $R_4 = \frac{\partial}{\partial z}$, with the Lie characteristic function of $W = -u_z$ and having $R_4 = Y_4$ from (1.36), gives the following

$$\begin{aligned}
T^1 =& \frac{1}{2}u_z v_x - \frac{1}{2}u_{xz}v,\\
T^2 =& -au_x u_{xz}v - b{u_x}^2 u_{xz}v + du_z u_{xy}v - du_y u_{xz}v - \frac{1}{2}u_{tz}v + au_x u_z v_x + b{u_x}^2 u_z v_x\\
&+ du_y u_z v_x + f{u_z}^2 v_x + \frac{1}{2}v_t u_z + u_z v_{xxx} - u_{xx}v_{xz} - v_{xx}u_{xz} + u_{xxx}v_z,\\
T^3 =& -cu_{yz}v - du_{xx}u_z v + cu_z v_y,\\
T^4 =& au_{xx}u_x v + bu_{xx}{u_x}^2 v + cu_{yy}v + du_{xx}u_y v + u_{tx}v + eu_z v_z + u_{xx}v_{xx},
\end{aligned}$$

which provides us with linear momentum conservation law.

(v) For the Lie symmetry $R_5 = \frac{\partial}{\partial u}$, it gives Lie characteristic function $W = 1$, and according to (1.36), Y_5 is still the same as R_5. We end up with the following

$$\begin{aligned}
T^1 =& -\frac{v_x}{2},\\
T^2 =& -du_{xy}v - fu_{xz}v - au_x v_x - b{u_x}^2 v_x - du_y v_x - fu_z v_x - \frac{v_t}{2} - v_{xxx},\\
T^3 =& du_{xx}v - cv_y,\\
T^4 =& fu_{xx}v - ev_z.
\end{aligned}$$

(vi) Finally, for the last Lie symmetry

$$R_6 = 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u}. \tag{3.68}$$

the value of Y_6 is different to R_6 which is then given by

$$Y_6 = 6\,b\,t\frac{\partial}{\partial t} + (2\,b\,x - a^2\,t)\frac{\partial}{\partial x} + 4\,b\,y\frac{\partial}{\partial y} + 4\,b\,z\frac{\partial}{\partial z} - a\,x\,\frac{\partial}{\partial u} - 8bv\frac{\partial}{\partial v}. \tag{3.69}$$

$W = -a\,x - 6\,b\,t\,u_t - (2\,b\,x - a^2\,t)u_x - 4\,b\,y\,u_y - 4\,b\,z\,u_z$. When using (1.40), we get

$$
\begin{aligned}
T^1 =& \frac{1}{2}a^2 t u_{xx} v + 6abt u_x u_{xx} v + 6b^2 t {u_x}^2 u_{xx} v + 6bct u_{yy} v + 6bdt u_{xx} u_y v + 6bet u_{zz} v \\
&+ 6bft u_{xx} u_z v + 3bt u_{tx} v - 2by u_{xy} v - 2bz u_{xz} v - b u_x v - bx u_{xx} v - \frac{1}{2}av \\
&- \frac{1}{2}a^2 t u_x v_x + \frac{1}{2}ax v_x + 3bt u_t v_x + 6bt u_{xx} v_{xx} + bx u_x v_x + 2by u_y v_x + 2bz u_z v_x, \\
T^2 =& - t {u_x}^2 v_x a^3 - etv u_{zz} a^2 - ctv u_{yy} a^2 - \frac{1}{2} v u_x a^2 - bt {u_x}^3 v_x a^2 + x u_x v_x a^2 \\
&- ft u_z u_x v_x a^2 - dt u_y u_x v_x a^2 - ftv u_x u_{xz} a^2 - dtv u_x u_{xy} a^2 + t u_{xx} v_{xx} a^2 \\
&- t v_x u_{xxx} a^2 - t u_x v_{xxx} a^2 - \frac{1}{2} t u_x v_t a^2 - \frac{1}{2} tv u_{tx} a^2 - 3bv {u_x}^2 a - dv u_y a \\
&+ 3bx {u_x}^2 v_x a + fx u_z v_x a + dx u_y v_x a + 4bz u_z u_x v_x a + 4by u_y u_x v_x a + fxv u_{xz} a \\
&- 4bzv u_x u_{xz} a + dxv u_{xy} a - 4byv u_x u_{xy} a - v_{xx} a + x v_{xxx} a + 6bt u_x v_x u_t a \\
&+ \frac{1}{2} x v_t a - 6btv u_x u_{tx} a - 2b^2 v {u_x}^3 - afv u_z + 2bexv u_{zz} + 2bcxv u_{yy} - 2bfv u_z u_x \\
&- 2bdv u_y u_x + 2b^2 x {u_x}^3 v_x + 4bfz {u_z}^2 v_x + 4bdy {u_y}^2 v_x + 4b^2 z u_z {u_x}^2 v_x + 4b^2 y u_y {u_x}^2 v_x \\
&+ 4bfy u_z u_y v_x + 4bdz u_z u_y v_x + 2bfx u_z u_x v_x + 2bdx u_y u_x v_x - 4b^2 zv {u_x}^2 u_{xz} \\
&+ 4bfyv u_y u_{xz} - 4bdzv u_y u_{xz} + 2bfxv u_x u_{xz} - 4b^2 yv {u_x}^2 u_{xy} - 4bfyv u_z u_{xy} \\
&+ 4bdzv u_z u_{xy} + 2bdxv u_x u_{xy} - 2b v_x u_{xx} - 4bz v_{xz} u_{xx} - 4by v_{xy} u_{xx} - 2b u_x v_{xx} \\
&- 4bz u_{xz} v_{xx} - 4by u_{xy} v_{xx} - 2bx u_{xx} v_{xx} + 4bz v_z u_{xxx} + 4by v_y u_{xxx} + 2bx v_x u_{xxx} \\
&+ 4bz u_z v_{xxx} + 4by u_y v_{xxx} + 2bx u_x v_{xxx} - 3bv u_t + 6b^2 t {u_x}^2 v_x u_t + 6bft u_z v_x u_t \\
&+ 6bdt u_y v_x u_t + 6bftv u_{xz} u_t + 6bdtv u_{xy} u_t + 6bt v_{xxx} u_t + 2bz u_z v_t + 2by u_y v_t \\
&+ bx u_x v_t + 6bt u_{xxx} v_t + 3bt u_t v_t - 2bzv u_{tz} - 2byv u_{ty} - 6b^2 tv {u_x}^2 u_{tx} + bxv u_{tx} \\
&- 6bftv u_z u_{tx} - 6bdtv u_y u_{tx} - 6bt v_{xx} u_{tx} - 6bt u_{xx} v_{tx} - 3btv u_{tt},
\end{aligned}
$$

$$
\begin{aligned}
T^3 =& a^2 c t u_{xy} v + a^2 d t u_x u_{xx} v + 4 a b y u_x u_{xx} v - a d x u_{xx} v + 4 b^2 y {u_x}^2 u_{xx} v - 6 b c t u_{ty} v \\
& - 2 b c x u_{xy} v - 4 b c z u_{yz} v - 4 b c u_y v - 6 b d t u_t u_{xx} v - 4 b d z u_{xx} u_z v - 2 b d x u_x u_{xx} v \\
& + 4 b e y u_{zz} v + 4 b f y u_{xx} u_z v + 4 b y u_{tx} v - a^2 c t u_x v_y + a c x v_y + 6 b c t u_t v_y + 2 b c x u_x v_y \\
& + 4 b c y u_y v_y + 4 b c z u_z v_y + 4 b y u_{xx} v_{xx}, \\
T^4 =& a^2 e t u_{xz} v + a^2 f t u_x u_{xx} v + 4 a b z u_x u_{xx} v - a f x u_{xx} v + 4 b^2 z {u_x}^2 u_{xx} v + 4 b c z u_{yy} v \\
& + 4 b d z u_{xx} u_y v - 6 b e t u_{tz} v - 2 b e x u_{xz} v - 4 b e y u_{yz} v - 4 b e u_z v - 6 b f t u_t u_{xx} v \\
& - 4 b f y u_{xx} u_y v - 2 b f x u_x u_{xx} v + 4 b z u_{tx} v - a^2 e t u_x v_z + a e x v_z + 6 b e t u_t v_z \\
& + 2 b e x u_x v_z + 4 b e y u_y v_z + 4 b e z u_z v_z + 4 b z u_{xx} v_{xx}.
\end{aligned}
$$

Remark: It should be noted that the above conservation laws include the energy conservation law, which corresponds to the time translation and three momentum conservation laws, which correspond to the three space translations.

3.3 Conservation laws using multiplier method

Consider the following second (3+1)-dimensional Gardner-type equation (3.1). For this entire section, we will be using the multiplier technique to generate the conservation laws of the equation (3.1). We first seek zeroth-order multiplier $\Lambda = \Lambda(t, x, y, z, u)$. The multiplier are determined using the following determining equation

$$
\begin{aligned}
\frac{\delta}{\delta u}\Big[\Lambda\Big(& u_{tx} + a u_x u_{xx} + u_{xxxx} + b u_x^2 u_{xx} + c u_{yy} + d u_{xx} u_y \\
& + e u_{zz} + f u_{xx} u_z \Big)\Big] = 0,
\end{aligned} \tag{3.70}
$$

where by $\frac{\delta}{\delta u}$ is the well known Euler operator given as

$$
\begin{aligned}
\frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}} \\
& + D_z^2 \frac{\partial}{\partial u_{zz}} + D_x^4 \frac{\partial}{\partial u_{xxxx}},
\end{aligned}
$$

where D_t, D_x, D_y and D_z are total derivative operators. If we expand (3.70) and split on appropriate derivatives of u we obtain the system of four multiplier determining

equations:

$$\Lambda_{zz} = 0,$$
$$\Lambda_x = 0,$$
$$\Lambda_y = -\frac{f\,\Lambda_z}{d},$$
$$\Lambda_u = 0.$$

The solution of the above PDEs gives

$$\Lambda = F(t)\,z - \frac{f\,y\,F(t)}{d} + G(t),$$

where $F(t)$ and $G(t)$ are functions of t. The conserved quantities of equation (3.1) are then generated by applying the divergence identity (1.42)

$$D_t T^t + D_x T^x + D_y T^y + D_z T^z =$$
$$\Lambda\left(u_{tx} + a u_x u_{xx} + u_{xxxx} + b u_x^2 u_{xx} + c u_{yy} + d u_{xx} u_y + e u_{zz} + f u_{xx} u_z\right) \tag{3.71}$$

with T^t being conserved density and T^x, T^y , T^z spatial fluxes. Thus, using (3.71), we obtain the following conserved vectors corresponding to the two multipliers:

Case 1. For $\mathcal{A}_1 = \frac{F(t)}{d}\,(d\,z - f\,y)$, we have the corresponding conserved vector as

$$\begin{aligned}
T^t =& \frac{F(t)}{2\,d}\,(d\,z - f\,y)\,u_x,\\
T^x =& -\frac{1}{6\,d}\big[-3\,F(t)\,d\,f\,u\,u_{xy}\,y + 3\,F(t)\,d\,f\,u\,u_{xz}\,z + 3\,F(t)\,d\,f\,u_x\,u_y\,y\\
&- 3\,F(t)\,d\,f\,u_x\,u_z\,z - 3\,F(t)\,d\,u_t\,z - 6\,F(t)\,d\,u_{xxx}\,z + 3\,F(t)\,f\,u_t\,y\\
&+ 6\,F(t)\,f\,u_{xxx}\,y + 3\,F'(t)\,d\,u\,z - 3\,F'(t)\,f\,u\,y - 3\,F(t)\,a\,d\,u_x^2\,z\\
&+ 3\,F(t)\,a\,f\,u_x^2\,y + 3\,F(t)\,d^2\,u\,u_{xy}\,z - 3\,F(t)\,d^2\,u_x\,u_y\,z - 3\,F(t)\,f^2\,u\,u_{xz}\,y\\
&+ 3\,F(t)\,f^2\,u_x\,u_z\,y + 2\,F(t)\,b\,f\,u_x^3\,y - 2\,F(t)\,b\,d\,u_x^3\,z\big],\\
T^y =& \frac{1}{2}\,F(t)\,d\,u\,u_{xx}\,z - \frac{1}{2}\,F(t)\,f\,u\,u_{xx}\,y + F(t)\,c\,u_y\,z - \frac{F(t)}{d}\,c\,f\,u_y\,y + \frac{F(t)}{d}\,u\,f\,c,\\
T^z =& -\frac{1}{2}\,F(t)\,f\,u\,u_{xx}\,z + \frac{F(t)}{2\,d}\,f^2\,u\,u_{xx}\,y - F(t)\,e\,u_z\,z + \frac{F(t)}{d}\,e\,f\,u_z\,y + F(t)\,u\,e.
\end{aligned}$$

Case 2. For $\mathcal{A}_2 = G(t)$, we have the conserved vector given by

$$
\begin{aligned}
T^t &= \frac{1}{2}\,G(t)\,u_x,\\
T^x &= \frac{1}{3}\,G(t)\,b\,u_x^3 + \frac{1}{2}\,G(t)\,a\,u_x^2 - \frac{1}{2}\,G(t)\,d\,u\,u_{xy} + \frac{1}{2}\,G(t)\,d\,u_x\,u_y - \frac{1}{2}\,G(t)\,f\,u\,u_{xz}\\
&\quad + \frac{1}{2}\,G(t)\,f\,u_x\,u_z - \frac{1}{2}\,G'(t)\,u + \frac{1}{2}\,G(t)\,u_t + G(t)\,u_{xxx},\\
T^y &= \frac{1}{2}\,G(t)\,(d\,u\,u_{xx} + 2\,c\,u_y),\\
T^z &= \frac{1}{2}\,G(t)\,(f\,u\,u_{xx} + 2\,e\,u_z).
\end{aligned}
$$

NB. Our conservation laws have arbitrary functions, thus give infinitely many conservation laws.

3.4 Concluding remarks

This chapter further investigates the second (3+1)-dimensional Gardner-type equation (3.1) to advance the dynamics of sound waves modelled by Gardner-type equation. Lie point symmetries that leave the equation (3.1) unchanged are obtained and used to perform similarity transformations that represent the PDE model to ODEs that conserved the physics of the PDE model. Furthermore, the obtained ODEs are solved to compute invariant solutions of the PDE model of interest in this chapter. Graphically representations of solutions obtained were used to study momentum and energy variation modelled by PDE studied. The increase in sound projection along a certain direction and plane leads to a decrease in the direction perpendicular to either the plane or the direction that the sound projection increases along. Lastly, the conservation laws such as mass, energy and momentum, among others, were derived using Ibragimov theorem and the multiplier method to find yet another conserved vector.

Chapter 4

Concluding remarks

The purpose of this research project was to investigate the nonlinear partial differential equations, namely Gardner-type equations and study the dynamics of sound wave phenomena. Exact solutions obtained using Lie symmetry analysis were used to study momentum (velocity) and energy variation during sound wave movement modelled by Gardner-type equation. In the first chapter, we reviewed the main definitions and outcomes of the Lie group theory used in this project. Also, a brief introduction to the conservation laws is introduced in Chapter 1.

In Chapter 2, we generated Lie point symmetries of the first (3+1)-dimensional Gardner-type equation with the help of the Lie group theory. A commutator table was generated using the obtained Lie point symmetries. Symmetry reductions were performed using different combinations of point symmetries to obtain various NODEs from the (3+1)-dimensional Gardner-type equation without changing the dynamics of the sound wave phenomena modelled by the (3+1)-dimensional Gardner-type equation. Exact solutions obtained from those NODEs were represented graphically and analysed to investigate sound wave momentum and energy variation. Representations of sound wave conserved quantities were obtained using Ibragimov theorem and the multiplier method.

Chapter 3 advanced the study of sound wave phenomena by tackling the second (3+1)-dimensional Gardner-type equation. All symmetry operators which conserve

the physics of sound wave phenomena under investigation was obtained using Lies theory. Various combinations of those operators were used to transform the model of interest into various simpler representations (ODEs) without changing the physics modelled by the initial model, the second (3+1)-dimensional Gardner-type equation. The simpler representations were integrated to obtain various exact solutions of the second type Gardner equation which were used to study sound wave dynamics graphically. In addition to the solutions, we derived the conservation laws representing conserved vectors during wave sound flow, such as mass, moment, concentration and energy, using Ibragimov theorem and the multiplier's approach, respectively.

For future work, we plan to focus our studies on stability analysis of the momentum and energy during flow dynamics of wave sound. Also, the conserved quantities will be used to integrate the PDE model that represents the wave sound phenomena.

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