APPLICATION OF LIE GROUP METHODS TO CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

by

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Declaration

I ISAIAH E. MHLANGA student number 22832947, declare that this dissertation for the degree of Master. of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Mr ISAIAH E. MHLANGA

Date: ....

This dissertation has been submitted with my approval as a university supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: ....

Dedication

To my wife and children.

Acknowledgements

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Abstract

In the first part of this work, two nonlinear partial differential equations, namely, a modified Camassa-Holm-Degasperis-Procesi equation and the generalized Korteweg-de Vries equation with two power law nonlinearities are studied. The Lie symmetry method along with the simplest equation method is used to construct exact solutions for these two equations. The second part looks at two systems of partial differential equations, namely, the generalized Boussinesq-Burgers equations and the (2+1)-dimensional Davey-Stewartson equations. The Lie symmetry method and the travelling wave hypothesis approach are utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach is used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations.

List of Acronyms

PDE:	Partial differential equation
NLPDE:	Nonlinear partial differential equation
CH:	Camassa-Holm
CH-DP:	Camassa-Holm-Degasperis-Procesi
KdV:	Korteweg-de Vries
mKdV:	modified Korteweg-de Vries
BB:	Boussinesq-Burgers
DS:	Davey-Stewartson

Introduction

The nonlinear partial differential equations (NLPDEs) are widely used as models to describe physical phenomena in different fields of applied sciences, such as fluid mechanics, solid state physics, plasma physics and plasma waves. A basic mathematical problem for such models is to obtain exact solutions to these NLPDEs for the understanding of most nonlinear physical phenomena [1–12]. Unfortunately, it is almost impossible to find all the solutions of a NLPDE. Finding solutions of such an equation is an enormous task and only in certain special cases one can write down the solutions explicitly.

However, in the last few decades important progress has been made and many effective methods for obtaining exact solutions of NLPDEs have been proposed and used to find different kinds of solutions of such physical models [2–35]. Some of the most important methods include homogeneous balance method [14], the ansatz method [15, 16], variable separation approach [17], inverse scattering transform method [18], Bäcklund transformation [19], Darboux transformation [20], Hirota's bilinear method [21], the (G'/G)-expansion method [22], the reduction mKdV equation method [23], the tri-function method [24, 25], the projective Riccati equation method [26], the sine-cosine method [27], the Jacobi elliptic function expansion method [28, 29], the F-expansion method [30], the exp-function expansion method [31] and the Lie symmetry method [2, 32–35]. The inverse scattering transform is a classical method of integration of NLPDEs whereas the other techniques of integration are some of the modern methods of integration that have flooded many NLPDEs with various kinds of solutions, including soliton solutions. Travelling wave solutions that do not change their form during propagation are called solitary waves. Solitary waves that retain their shape upon collision are called solitons [36]. Solitary-waves and solitons are results of a delicate balance between dispersion and nonlinearity [37].

In the first part of this project, two NLPDEs, namely, the modified Camassa-Holm-Degasperis-Procesi (CH-DP) equation and the generalized Korteweg-de Vries (KdV) equation with two power law nonlinearities are studied. Lie symmetry method along with the simplest equation method are used to construct exact solutions for these two equations. In the second part of the project, two systems of NLPDEs, namely, the generalized Boussinesq-Burgers equations and the (2+1)-dimensional Davey-Stewartson equations are studied. Lie symmetry method and the travelling wave hypothesis approach are utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach is used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations.

The four equations in this project have been chosen on the basis of their significance as models that describe nonlinear physical phenomena in fluid mechanics. All the four equations are nonlinear evolution partial differential equations. In this project only exact solutions to the equations are sought for. Hence boundary value problems are not considered for these equations.

Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Finding all the symmetries of a differential equation is an unattainable task. However, Sophus Lie (1842-1899), in the middle of the nineteenth century, realized that if we restrict ourselves to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), we can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. Lie's continuous symmetry groups have applications in many fields such as invariant theory, control theory, classical mechanics and relativity. In the past few decades a considerable amount of development has been made in symmetry methods for differential equations. This is evident by the number of research papers, books and many new symbolic softwares (see for example [38–42]) devoted to the subject.

The simplest equation method was introduced by Kudryashov [43] in 2005 to look for exact solutions of nonlinear differential equations. This method contains two basic ideas. The first idea was to use the simple nonlinear ordinary differential equations (Bernoulli and Riccati equations) with known general solutions for finding new special solutions. The second idea was to take all possible singularities of solutions of equations into consideration. The exact solutions of nonlinear differential equations are obtained in the form of solitary and periodic waves. The method of Kudryashov was later modified by Vitanov [44] in 2010. Recently, it has been used by several other authors (see for example [2]).

The travelling wave variable approach converts the NLPDEs into nonlinear ordinary differential equations and is often useful in obtaining exact solutions of the partial differential equations.

The modified Camassa-Holm-Degasperis-Procesi equation

$$u_t - c_0 u_x + (b_0 + 1) u u_x - \alpha^2 (u_{xxt} + u_{xxx} + b_0 u_x u_{xx}) + \gamma u_{xxx} = 0,$$

is a variation of the Camassa and Holm equation [45] and was introduced by Degasperis and Procesi in 1999 [46–48].

The generalized Korteweg-de Vries equation with two power law nonlinearities [49], given by

$$u_t + \left(\alpha u^n - \beta u^{2n}\right) u_x + u_{xxx} = 0$$

is a modification of the Korteweg-de Vries equation [50]. Lie symmetry analysis along with the simplest equation method are used to find exact solutions of these two equations.

The generalized Boussinesq-Burgers equations [51] given by

$$u_t + a u u_x + b v_x = 0,$$
$$v_t + c(uv)_x + d u_{xxx} = 0$$

arise in the study of fluid flow and describe the propagation of shallow water waves [51].

The (2+1)-dimensional Davey-Stewartson equations

$$\begin{aligned} iu_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv &= 0, \\ v_{xx} + v_{yy} + 2(|u|^2)_{xx} &= 0 \end{aligned}$$

were first introduced by Davey and Stewartson in 1974 [52] and are used to describe the long-time evolution of a two-dimensional wave packet [53–55].

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations are presented.

In Chapter two, Lie symmetry method along with the simplest equation method is employed to obtain exact solutions of the modified Camassa-Holm-Degasperis-Procesi equation and the generalized Korteweg-de Vries equation with two power law nonlinearities. The results of this chapter have been submitted for publication [56].

In Chapter three, Lie symmetry method and the travelling wave hypothesis approach is utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach is used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations. This work has also been submitted for publication [57].

In Chapter four, a summary of the results of the dissertation is presented and future work is discussed.

Bibliography is given at the end of this dissertation.

Chapter 1

Lie symmetry methods for differential equations

In this chapter, some basic methods of Lie symmetry analysis of differential equations including the algorithm to determine the Lie point symmetries of PDEs are given.

1.1 Introduction

In the late nineteenth century an outstanding mathematician Sophus Lie (1842-1899) developed a new method, known as Lie group analysis, for solving differential equations and showed that majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Recently, many good books have appeared in the literature in this field. We mention a few here, Bluman and Kumei [13], Olver [33], Stephani [58], Ibragimov [59], Cantwell [60]. Definitions and results given in this Chapter are taken from the books mentioned above.

1.2 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variable with coordinates x^i and $q = (q^1, .., q^m)$ be the dependent variable with coordinates q^α (n and m finite). Consider a change of the variables x and q involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, q, a),\ \bar{q^\alpha} = \phi^\alpha(x, q, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and q if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{q}, b) = f^i(x, q, \phi(a, b)), \\ \bar{\bar{q}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{q}, b) = \phi^\alpha(x, q, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law.* A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of q with respect to x are defined as

$$q_i^\alpha = D_i(q^\alpha), \quad q_{ij}^\alpha = D_j D_i(q_i), \cdots , \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + q_i^\alpha \frac{\partial}{\partial q^\alpha} + q_{ij}^\alpha \frac{\partial}{\partial q_j^\alpha} + \cdots , \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives q_i^α is denoted by $q_{(1)}$, i.e.,

$$q_{(1)} = \{q_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$q_{(2)} = \{q_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $q_{(3)} = \{q_{ijk}^\alpha\}$ and likewise $q_{(4)}$ etc. Since $q_{ij}^\alpha = q_{ji}^\alpha$, $q_{(2)}$ contains only q_{ij}^α for $i \leq j$. In the same manner $q_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $q_{(4)}$,$q_{(5)}$ $\cdots$.

In group analysis all variables $x, q, q_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the q_s^α are called differential variables.

We now consider a pth-order PDE, namely

$$E(x, q, q_{(1)}, ..., q_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, q)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, q, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{q^\alpha} = \phi^\alpha(x, q, a), \quad \phi^\alpha|_{a=0} = q^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, q), \quad \bar{q^\alpha} \approx q^\alpha + a\,\eta^\alpha(x, q) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = q^\alpha.$$

Thus, we have

$$\xi^i(x, q) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x, q) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{q^\alpha} \approx (1 + a\,X)q,$$

where

$$X = \xi^i(x, q)\frac{\partial}{\partial x^i} + \eta^\alpha(x, q)\frac{\partial}{\partial q^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{q}_i^\alpha = \bar{D}_j(q^\alpha), \qquad \bar{q}_{ij}^\alpha = \bar{D}_j(\bar{q}_i^\alpha) = \bar{D}_i(\bar{q}_j^\alpha), \cdots.$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{q}^\alpha) \\ &= D_i(f^j)\bar{q}_j^\alpha. \end{aligned} \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + q_i^\beta \frac{\partial f^j}{\partial q^\beta}\right)\bar{q}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + q_i^\beta \frac{\partial \phi^\alpha}{\partial q^\beta}. \tag{1.12}$$

The quantities $\bar{q}_j^\alpha$ can be represented as functions of $x, q, q_{(i)}$, a for small a, ie., (1.12) is locally invertible:

$$\bar{q}_i^\alpha = \psi_i^\alpha(x, q, q_{(1)}, a), \quad \psi^\alpha|_{a=0} = q_i^\alpha. \tag{1.13}$$

The transformations in $x, q, q_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{q}_i^\alpha \approx q_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{q}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= D_i(q^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= q_i^\alpha + aD_i\eta^\alpha \\ q_i^\alpha + a\zeta_i^\alpha + aq_j^\alpha D_i\xi^j &= q_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - q_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^{\alpha} = D_j(\eta_i^{\alpha}) - q_{ik}^{\alpha} D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^{\alpha} = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^{\alpha}) - q_{i_1,i_2,\ldots,i_{p-1}\,j}^{\alpha} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$X^{[1]} = X + \zeta_i^{\alpha} \frac{\partial}{\partial q_i^{\alpha}} \quad \text{(sum on } i, \alpha),$$

$$\vdots$$

$$X^{[p]} = X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^{\alpha} \frac{\partial}{\partial q_{i_1,\ldots,i_p}^{\alpha}} \quad p \geq 1, \tag{1.18}$$

where

$$X = \xi^i(x,q)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,q)\frac{\partial}{\partial q^{\alpha}}. \tag{1.19}$$

For purposes of this project (we use u instead of q), the infinitesimal generator X of the group G will be given by

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{1.20}$$

and the third prolongation of X will be given by

$$X^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{221}\frac{\partial}{\partial u_{xxt}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}, \tag{1.21}$$

where

$$\zeta_1 = \eta_t + u_t\eta_u - u_t\xi^1_t - u_t^2\xi^1_u - u_x\xi^2_t - u_tu_x\xi^2_u, \tag{1.22}$$

$$\zeta_2 = \eta_x + u_x\eta_u - u_t\xi^1_x - u_tu_x\xi^1_u - u_x\xi^2_x - u_x^2\xi^2_u, \tag{1.23}$$

$$\begin{aligned}\zeta_{22} = {} & \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi^2_x - u_x\xi^2_{xx} - 2u_x^2\xi^2_{xu} \\ & -3u_xu_{xx}\xi^2_u - u_x^3\xi^2_{uu} - 2u_{tx}\xi^1_x - u_t\xi^1_{xx} - 2u_tu_x\xi^1_{xu} - u_tu_{xx}\xi^1_u \\ & -2u_xu_{tx}\xi^1_u - u_tu_x^2\xi^1_{uu},\end{aligned} \tag{1.24}$$

$$\begin{aligned}\zeta_{221} = {} & \eta_{txx} + 2u_x\eta_{txu} + u_t\eta_{xxu} + 2u_{xt}\eta_{xu} + 2u_xu_t\eta_{xuu} - 2u_{tx}\xi^1_{tx} - u_{tx}\xi^2_{xx} \\ & -u_t\xi^1_{txx} - u_{tt}\xi^1_{xx} - 2u_tu_x\xi^1_{txu} - u_tu_x\xi^2_{xxu} - u_t^2\xi^1_{xxu} - 4u_tu_{tx}\xi^1_{xu} \\ & -2u_xu_{tt}\xi^2_{xu} - 2u_t^2u_x\xi^1_{xuu} - u_x\xi^2_{txx} - 2u_{xx}\xi^2_{tx} - 2u_x^2\xi^2_{txu} - 3u_xu_{tx}\xi^2_{xu} \\ & -2u_tu_{xxy}\xi^2_{xu} - 2u_x^2u_t\xi^2_{xuu} + u_x^2\eta_{tuu} + 2u_xu_{xt}\eta_{uu} + u_x^2u_t\eta_{uuu} - u_xu_{tx}\xi^1_{tu} \\ & -u_x^2u_t\xi^1_{tuu} - 4u_xu_tu_{tx}\xi^1_{uu} - u_x^2u_{tt}\xi^1_{uu} - u_x^2u_t^2\xi^1_{uuu} - 3u_xu_{xx}\xi^2_{tu} - u_x^3\xi^2_{tuu} \\ & -3u_x^2u_{tx}\xi^2_{uu} - 3u_xu_tu_{xx}\xi^2_{uu} - u_x^3u_t\xi^2_{uuu} + u_{xx}\eta_{tu} + u_{xx}u_t\eta_{uu} - u_{xx}u_t\xi^1_{tu} \\ & -u_{xx}u_{tt}\xi^1_u - u_{xx}u_t^2\xi^1_{uu} - 3u_{xx}u_{tx}\xi^2_u - u_{tx}u_x\xi^1_{tu} - u_{tx}u_x\xi^2_{xu} - 2u_{tx}^2\xi^1_u \\ & -u_{xxx}\xi^2_t - u_tu_{xxx}\xi^2_u + u_{txx}\eta_u - u_{txx}\xi^1_t - u_{txx}\xi^2_x - 2u_{txx}u_x\xi^2_u - u_{xxt}\xi^1_x \\ & -u_{xxt}u_x\xi^1_u - u_{xxx}\xi^2_x - u_xu_{xxx}\xi^2_u,\end{aligned} \tag{1.25}$$

$$\begin{aligned}\zeta_{222} = {} & \eta_{xxx} + 3u_x\eta_{xxu} + 3u_x^2\eta_{xuu} + 3u_{xx}\eta_{xu} + 3u_xu_{xx}\eta_{uu} + u_{xxx}\eta_u \\ & +u_x^3\eta_{uuu} - 3u_{xx}\xi^2_{xx} - 9u_xu_{xx}\xi^2_{xu} - 3u_{xxx}\xi^2_x - u_x\xi^2_{xxx} - 3u_x^2\xi^2_{xxu} \\ & -3u_{xx}^2\xi^2_u - 6u_x^2u_{xx}\xi^2_{uu} - 4u_xu_{xxx}\xi^2_u - 3u_x^3\xi^2_{xuu} - u_x^4\xi^2_{uuu} - 3u_{tx}\xi^1_{xx} \\ & -3u_tu_{xx}\xi^1_{xu} - 3u_tu_x\xi^1_{xxu} - 3u_{xxt}\xi^1_x - u_t\xi^1_{xxx} - 3u_tu_x^2\xi^1_{xu} - 3u_{tx}u_x^2\xi^1_{uu} \\ & -3u_tu_xu_{xx}\xi^1_{uu} - 3u_{tx}u_{xx}\xi^1_u - 6u_xu_{tx}\xi^1_{xu} - 3u_xu_{xxt}\xi^1_u - u_tu_x^3\xi^1_{uuu} \\ & -u_tu_{xxx}\xi^1_u.\end{aligned} \tag{1.26}$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,q)\frac{\partial}{\partial x^i} + \eta^\alpha(x,q)\frac{\partial}{\partial q^\alpha}, \tag{1.27}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E) = 0 \tag{1.28}$$

whenever $E = 0$. This can also be written as

$$X^{[p]}\, E\big|_{E=0} = 0, \tag{1.29}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.28) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{q}, \bar{q_{(1)}}, \cdots, \bar{q_{(p)}}) = 0, \tag{1.30}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x,q)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{q}) \equiv F(f^i(x,q,a), \phi^\alpha(x,q,a)) = F(x,q), \tag{1.31}$$

identically in x, q and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, q)$ to be an invariant is that

$$X\, F \equiv \xi^i(x, q)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, q)\frac{\partial F}{\partial q^\alpha} = 0\,. \tag{1.32}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has n functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, q) = c_1, \cdots, J_n(x, q) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,q)} = \cdots = \frac{dx^n}{\xi^n(x,q)} = \frac{dq^1}{\eta^1(x,q)} = \cdots = \frac{dq^n}{\eta^n(x,q)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x}, \bar{q}), \quad \frac{d\bar{q^\alpha}}{da} = \eta^\alpha(\bar{x}, \bar{q}) \tag{1.33}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{q^\alpha}\big|_{a=0} = q\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x, q)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x, q)\frac{\partial}{\partial q^\alpha}$$

and

$$X_2 = \xi_2^i(x, q)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x, q)\frac{\partial}{\partial q^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,q)\dfrac{\partial}{\partial x^i} + \eta^\alpha(x,q)\dfrac{\partial}{\partial q}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,q)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,q)\frac{\partial}{\partial q}, \quad X_2 = \xi_2^i(x,q)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,q)\frac{\partial}{\partial q}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \ [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conclusion

In this chapter, a brief introduction to the Lie group analysis of PDEs has been presented and some results which will be used throughout this project have been given. Also the algorithm to determine the Lie point symmetries of PDEs has been given.

Chapter 2

Solutions of two nonlinear PDEs using Lie symmetry and simplest equation methods

2.1 Introduction

In this chapter two nonlinear partial differential equations, namely, a modified Camassa-Holm-Degasperis-Procesi equation and the generalized Korteweg-de Vries equation with two power law nonlinearities are studied. The Lie symmetry method along with the simplest equation method are used to construct exact solutions for these two equations. This work has been submitted for publication [56].

2.2 Camassa-Holm-Degasperis-Procesi equation

The Camassa-Holm (CH) equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{2.1}$$

was proposed by Camassa and Holm [45] as a model equation for the propagation of the unidirectional gravitational waves in a shallow water approximation, with u

representing the free surface of water over a flat bed. This equation has attracted a lot of attention over the past two decades due to its interesting mathematical properties, for instance, it is completely integrable and has infinitely many conservation laws [45]. For $k = 0$, Camassa and Holm [45] showed that (2.1) has peaked solitary wave solutions. In [61] and [62], Li introduced a different approach associated with the Darboux transformation to construct the explicit expressions for multi-soliton solutions for (2.1).

In 1999, a new variant of (2.1) was introduced by Degasperis and Procesi, namely

$$m_t + um_x + b_0 u_x m = c_0 u_x - \gamma u_{xxx},$$

which is called Camassa-Holm-Degasperis-Procesi(CH-DP) equation, where $m = u - \alpha^2 u_{xx}$ is a momentum variable, α, c_0, b_0, γ are constants and $\alpha \neq 0$ [46–48]. Evidently, Eq. (2.2) can be writen as

$$u_t - c_0 u_x + (b_0 + 1)uu_x - \alpha^2(u_{xxt} + uu_{xxx} + b_0 u_x u_{xx}) + \gamma u_{xxx} = 0. \quad (2.2)$$

This equation is not only of mathematical interest but has been proved to be an approximate model of shallow water wave propagation with small amplitude and long wave length [46]. In recent years, the CH-DP equation has been widely studied by many authors. Li and Zhang [48] first transformed the CH-DP equation (2.2) to a two-dimensional system by considering $u(t, x) = \phi(x - ct)$, where c is the wave speed, and then used the method of dynamical systems to study the bifurcation behaviour of travelling wave solutions of the system. Xie and Wang [47] obtained exact explicit parameter expressions of compactons and implicit expressions of generalized kink wave solutions of the CH-DP equation for $b_0 = 3$.

2.2.1 Lie point symmetries of the CH-DP equation and symmetry reduction

First the Lie point symmetries of the CH-DP equation (2.2) are found using the Lie algorithm [33]. These Lie point symmetries are then used to transform (2.2) into an ordinary differential equation. The simplest equation method is then used to construct exact solutions of the ordinary differential equation, which leads to the exact solutions of the CH-DP equation.

The symmetry group of the CH-DP equation (2.2) is generated by the vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.3}$$

if and only if

$$X^{[3]}(u_t - c_0 u_x + (b_0+1)uu_x - \alpha^2(u_{xxt} + uu_{xxx} + b_0 u_x u_{xx}) + \gamma u_{xxx})\big|_{(2.2)} = 0. \tag{2.4}$$

Here $X^{[3]}$ is the third prolongation of X and is given by (see (1.21))

$$X^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{221}\frac{\partial}{\partial u_{xxt}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}} + \dots,$$

where ζ_1, ζ_2, ζ_{22}, ζ_{221} and ζ_{222} are as given in chapter 1.
Now expanding (2.4) gives

$$\begin{aligned}&\Big[\xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}}\\ &+\zeta_{221}\frac{\partial}{\partial u_{xxt}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}\Big]\Big(u_t - c_0u_x + (b_0+1)uu_x - \alpha^2(u_{xxt} + u_{xxx} + b_0u_xu_{xx})\\ &+\gamma u_{xxx}\Big)\Big|_{u_t = c_0u_x - (b_0+1)uu_x + \alpha^2(u_{xxt} + uu_{xxx} + b_0u_xu_{xx}) - \gamma u_{xxx}} = 0.\end{aligned}$$

Further simplification yields

$$\begin{aligned}&\zeta_1 - c_0\zeta_2 + bu\zeta_2 + b\eta u_x + \eta u_x + u\zeta_2 - \alpha^2\zeta_{221} - \alpha^2 u_{xxx}\eta - \alpha^2 u\zeta_{222}\\ &-\alpha^2 bu_{xx}\zeta_2 - \alpha^2 bu_x\zeta_{22} + \gamma\zeta_{222}\Big|_{u_t = c_0u_x - (b_0+1)uu_x + \alpha^2(u_{xxt} + uu_{xxx} + b_0u_xu_{xx}) - \gamma u_{xxx}} = 0.\end{aligned} \tag{2.5}$$

Substituting the values of ζ_1, ζ_2, ζ_{22}, ζ_{221} and ζ_{222} in (2.5) and replacing u_t by $c_0u_x - (b_0+1)uu_x + \alpha^2(u_{xxt} + uu_{xxx} + b_0u_xu_{xx}) - \gamma u_{xxx}$, and splitting on the derivatives of u, the following overdetermined system of linear partial differential equations is obtained:

$$\xi^1_u = 0, \tag{2.6}$$
$$\xi^2_u = 0, \tag{2.7}$$
$$\eta_{uu} = 0, \tag{2.8}$$
$$\xi^1_x = 0, \tag{2.9}$$
$$\begin{aligned}&-b_0\eta_{xx}\alpha^2 - 2\eta_{txu}\alpha^2 - 2u\eta_{xxu}\alpha^2 + ub_0\eta_{xxu}\alpha^2 - c_0\eta_{xxu}\alpha^2 + \xi^2_{txx}\alpha^2\\&+u\xi^2_{xxx}\alpha^2 + \eta + \eta b_0 + u\xi^1_t + ub_0\xi^1_t - c_0\xi^1_t - \xi^2_t - u\xi^2_x - ub_0\xi^2_x\\&+c_0\xi^2_x + 3\gamma\eta_{xxu} - \gamma\xi^2_{xxx} = 0,\end{aligned} \tag{2.10}$$
$$-\eta_{txx}\alpha^2 - u\eta_{xxx}\alpha^2 + \eta_t + u\eta_x + ub_0\eta_x - c_0\eta_x + \gamma\eta_{xxx} = 0, \tag{2.11}$$
$$-b_0\eta_x\alpha^2 - \eta_{tu}\alpha^2 - 3u\eta_{xu}\alpha^2 + 2\xi^2_{tx}\alpha^2 + 3u\xi^2_{xx}\alpha^2 + 3\gamma\eta_{xu} - 3\gamma\xi^2_{xx} = 0, \tag{2.12}$$
$$\begin{aligned}&-u\eta_{xxu}\alpha^4 - \eta\alpha^2 - u\xi^1_t\alpha^2 + \xi^2_t\alpha^2 + 3u\xi^2_x\alpha^2 + \gamma\eta_{xxu}\alpha^2 + \gamma\xi^1_t\\&-3\gamma\xi^2_x = 0,\end{aligned} \tag{2.13}$$
$$\eta_{xxu}\alpha^2 + \eta_u + \xi^1_t - 3\xi^2_x = 0, \tag{2.14}$$
$$\alpha^2\eta_{xxu} - 2\xi^2_x = 0, \tag{2.15}$$
$$2\eta_{xu} - \xi^2_{xx} = 0. \tag{2.16}$$
$$\tag{2.17}$$

The above system is now solved for ξ^1, ξ^2 and η.
Eqs. (2.6) and (2.9) imply that

$$\xi^1 = a(t), \tag{2.18}$$

where $a(t)$ is an arbitrary function of t. Eq. (2.7) implies that

$$\xi^2 = b(t, x), \tag{2.19}$$

where $b(t, x)$ is an arbitrary function of t and x. Integrating Eq. (2.8) twice with respect to u gives

$$\eta = c(t, x)u + d(t, x), \tag{2.20}$$

where $c(t,x)$ and $d(t,x)$ are arbitrary functions of t and x. Substituting the values of ξ^2 and η into (2.16) gives

$$2c_x - b_{xx} = 0.$$

Integrating the above equation with respect to x, one obtains

$$c(t,x) = \frac{1}{2}(b_x + e(t)), \tag{2.21}$$

where $e(t)$ is an arbitrary function of t. Substituting the values of ξ^2 and η into (2.15) and integrating with respect to x gives

$$b(t,x) = \frac{1}{2}(\alpha^2 c_x + f(t)), \tag{2.22}$$

where $f(t)$ is an arbitrary function of t. Thus

$$\begin{aligned}
\xi^1 &= a(t), \\
\xi^2 &= \frac{1}{2}(\alpha^2 c_x + f(t)), \\
\eta &= c(t,x)u + d(t,x).
\end{aligned}$$

Now substituting these values of ξ^1, ξ^2 and η into (2.14) and simplifying gives

$$\frac{1}{2}\alpha^2 c_{xx} + c(t,x) = -a'(t).$$

The solution of the above equation is given by

$$c(t,x) = A(t)\cos\left(\frac{\sqrt{2}}{\alpha}x\right) + B(t)\sin\left(\frac{\sqrt{2}}{\alpha}x\right) - a'(t), \tag{2.23}$$

where $A(t)$ and $B(t)$ are arbitrary functions of t. Substituting this value of c into (2.21) and integrating with respect to x gives

$$b(x,t) = 2A\frac{\alpha}{\sqrt{2}}\sin\left(\frac{\sqrt{2}}{\alpha}x\right) - 2B\frac{\alpha}{\sqrt{2}}\cos\left(\frac{\sqrt{2}}{\alpha}x\right) - 2a'(t)x - e(t)x + g(t), \tag{2.24}$$

where $g(t)$ is an arbitrary function of t. Substituting this value of b into (2.22) and making use of Eq.(2.14) gives

$$\begin{aligned}
&\sqrt{2}A\alpha\sin\left(\frac{\sqrt{2}}{\alpha}x\right) - \sqrt{2}B\alpha\cos\left(\frac{\sqrt{2}}{\alpha}x\right) - 2a'(t)x - e(t)x + g(t) + \frac{1}{2}f(t) \\
&-\frac{1}{4}\alpha^2\left[2A\frac{\alpha}{\sqrt{2}}\sin\left(\frac{\sqrt{2}}{\alpha}x\right) - 2B\frac{\alpha}{\sqrt{2}}\cos\left(\frac{\sqrt{2}}{\alpha}x\right)\right] = 0.
\end{aligned} \tag{2.25}$$

This gives the following four equations:

$$\begin{aligned}
\sin\left(\frac{\sqrt{2}}{\alpha}x\right) &: \sqrt{2}A\alpha + A\frac{\sqrt{2}}{4}\alpha^3 = 0,\\
\cos\left(\frac{\sqrt{2}}{\alpha}x\right) &: -\sqrt{2}B\alpha - B\frac{\sqrt{2}}{4}\alpha^3 = 0,\\
x &: -2a'(t) - e(t) = 0,\\
1 &: g(t) + \frac{1}{2}f(t) = 0.
\end{aligned}$$

From the above four equations it implies, respectively, that

$$A = 0, \quad B = 0, \quad e(t) = -2a'(t) \text{ and } g(t) = -\frac{1}{2}f(t). \tag{2.26}$$

As a result, Eq.(2.23) becomes

$$c(t,x) = -a'(t),$$

and using the value of $a'(t)$ from (2.26) gives

$$c(t,x) = \frac{e(t)}{2}. \tag{2.27}$$

Also from Eq.(2.24) and making use of the value of $a'(t)$ from (2.26) gives

$$b(t,x) = -\frac{1}{2}f(t). \tag{2.28}$$

Thus

$$\begin{aligned}
\xi^1 &= a(t),\\
\xi^2 &= -\frac{1}{2}f(t)),\\
\eta &= -\frac{e(t)}{2}u.
\end{aligned}$$

Now substituting the value of η into (2.11) one obtains

$$\begin{aligned}
&-\alpha^2 uc_{txx} - \alpha^2 d_{txx} - \alpha^2 u^2 c_{xxx} - \alpha^2 ud_{xxx} + c_t u + d_t + u^2 c_x + ud_x\\
&+u^2 b_0 c_x + ub_0 d_x - c_0 c_x u - c_0 d_x + \gamma c_{xxx} u + \gamma d_{xxx} = 0,
\end{aligned} \tag{2.29}$$

which on splitting on u, yeilds the following set of equations:

$$u^2 : -\alpha^2 c_{xxx} + c_x + b_0 c_x = 0, \tag{2.30a}$$

$$u^1 : -\alpha^2 c_{txx} - \alpha^2 d_{xxx} + c_t + d_x + b_0 d_x - c_0 c_x + \gamma c_{xxx} = 0, \tag{2.30b}$$

$$u^0 : -\alpha^2 d_{txx} + d_t - c_0 d_x + \gamma d_{xxx} = 0. \tag{2.30c}$$

Eq.(2.30a) is satisfied since the function c depends only on t, (see Eq.(2.27)). From Eq.(2.30b),

$$-\alpha^2 dxxx - \frac{e'(t)}{2} + (1+b_0)d_x = 0,$$

which on integrating with respect to x gives

$$d(t,x) = D(t)e^{-mx} + E(t)e^{mx} + \frac{2h(t) + xe'(t)}{2(1+b_0)}, \tag{2.31}$$

where $m = \dfrac{\sqrt{1+b_0}}{\alpha}$, and $D(t)$, $E(t)$ and $h(t)$ are arbitrary functions of t. Using the above value of $d(t,x)$ in Eq.(2.30c) we obtain

$$-\alpha^2(m^2e^{-mx}D'(t) + m^2e^{mx}E'(t)) + D'(t)e^{-mx} + E'(t)e^{mx}) + \frac{2h'(t) + xe''(t)}{2(1+b_0)}$$
$$-c_0\left(- e^{-mx}mD + e^{mx}mE + \frac{e'(t)}{2(1+b_0)}\right) + \gamma(-e^{-mx}m^3D + e^{mx}m^3E) = 0,$$

which on splitting on the exponential function gives

$$e^{-mx} \; : \; -\alpha^2m^2D'(t) + D'(t) + c_0mD(t) - \gamma m^3D(t) = 0, \tag{2.32a}$$

$$e^{mx} \; : \; -\alpha^2m^2E'(t) + E'(t) - c_0mE(t) + \gamma m^3E(t) = 0, \tag{2.32b}$$

$$e^{0} \; : \; \frac{2h'(t) + xe''(t)}{2(1+b_0)} - \frac{c_0e'(t)}{2(1+b_0)} = 0. \tag{2.32c}$$

Integrating Eqs. (2.32a) and (2.32b) with respect to t gives respectively

$$D = C_1e^{M_1t} \quad \text{and} \quad E = C_2e^{M_2t},$$

where $M_1 = \dfrac{-m^3\gamma + mc_0}{-1+m^2\alpha^2}$ and $M_2 = \dfrac{m^3\gamma - mc_0}{-1+m^2\alpha^2}$. C_1 and C_2 are arbitrary constants of integration. Now substituting the values of D and E into Eq.(2.31) we obtain

$$d = C_1e^{M_1t-mx} + C_2e^{M_2t+mx} + \frac{2h(t) + xe'(t)}{2(1+b_0}. \tag{2.33}$$

Thus

$$\xi^1 = a(t),$$
$$\xi^2 = -\frac{1}{2}f(t)),$$
$$\eta = -\frac{e(t)}{2}u + C_1e^{M_1t-mx} + C_2e^{M_2t+mx} + \frac{2h(t) + xe'(t)}{2(1+b_0)}.$$

Substituting the values of ξ^2 and η into (2.16) gives

$$-b_0\alpha^2\Big(-mC_1e^{M_1t-mx}+mC_2e^{M_2t+mx}+\frac{e'(t)}{2(1+b_0)}\Big)+\alpha^2\frac{e'(t)}{2}=0,$$

and splitting this equation on the exponential function yields

$$\begin{aligned}
e^{M_1t-mx} &: \ b_0\alpha^2mC_1=0,\\
e^{M_1t+mx} &: \ -b_0\alpha^2mC_2=0,\\
e^0 &: \ \frac{-b_0\alpha^2e'(t)}{2(1+b_0)}+\frac{\alpha^2e'(t)}{2}=0.
\end{aligned}$$

The above three equations imply that

$$C_1=0,\ C_2=0,\ \text{and } e(t)=C_3, \tag{2.34}$$

where C_3 is an arbitrary constant of integration. Substituting the value of $e(t)$ from (2.34) into (2.26) and integrating with respect to t we obtain

$$a(t)=-\frac{1}{2}C_3t+C_4,$$

where C_4 is an arbitrary constant of integration. Thus the new values of ξ^1, ξ^2 and η are

$$\xi^1=-\frac{1}{2}C_3t+C_4, \tag{2.35a}$$

$$\xi^2=-\frac{1}{2}f(t)), \tag{2.35b}$$

$$\eta=\frac{C_3}{2}u+\frac{h(t)}{1+b_0}. \tag{2.35c}$$

Now substituting the values of ξ^1, ξ^2 and η into Eq.(2.13) and simplifying gives

$$h(t)=-\frac{1+b_0}{2}f'-\frac{C_3(1+b_0)\gamma}{2\alpha^2}.$$

Substituting this value of $h(t)$ into (2.35c) and simplifying gives the new value of η as

$$\eta=\frac{C_3}{2}u-\frac{1}{2}f'-\frac{C_3\gamma}{2\alpha^2}. \tag{2.36}$$

Now substituting the new values of ξ^1, ξ^2 and η into Eq.(2.10) and simplifying gives

$$-\frac{1}{2}(1+b_0)f'(t)-\frac{C_3(1+b_0)\gamma}{2\alpha^2}+\frac{c_0C_3}{2}+\frac{1}{2}f'(t)=0.$$

Further simplification and integrating with respect to t yields

$$f(t) = C_3\left(\frac{-(1+b_0)\gamma + c_0\alpha^2}{b_0\alpha^2}\right)t + C_5,$$

where C_5 is an arbitrary constant of integration. Thus finally the following values for ξ^1, ξ^2 and η are obtained as

$$\begin{aligned}
\xi^1 &= -\frac{1}{2}C_3 t + C_4, \\
\xi^2 &= -C_3\left(\frac{-(1+b_0)\gamma + c_0\alpha^2}{2b_0\alpha^2}\right)t - \frac{C_5}{2}, \\
\eta &= \frac{C_3}{2}u - C_3\left(\frac{-(1+b_0)\gamma + c_0\alpha^2}{2b_0\alpha^2}\right) - \frac{C_3\gamma}{2\alpha^2}.
\end{aligned}$$

Therefore it can be deduced that the symmetry group of the CH-DP equation (2.2) is spanned by the three vector fields

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= -\alpha^2 b_0 t\frac{\partial}{\partial t} + (\gamma b_0 - \alpha^2 c_0 + \gamma)t\frac{\partial}{\partial x} + (\alpha^2 b_0 u - \alpha^2 c_0 + \gamma)\frac{\partial}{\partial u}.
\end{aligned}$$

Now utilizing the linear combination of the translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$, namely, the symmetry $X = X_1 + \nu X_2$, where ν is a constant, leads to the associated Lagrange equations

$$\frac{dt}{1} = \frac{dx}{\nu},$$

which after solving yields the two invariants

$$z = x - \nu t, \quad F = u. \tag{2.37}$$

Treating F as the new dependent variable and z as the new independent variable, gives

$$\begin{aligned}
u_t &= -\nu F', \\
u_x &= F', \\
u_{xx} &= F'', \\
u_{xxx} &= F''', \\
u_{xxt} &= -\nu F''',
\end{aligned}$$

which when we substitute into the CH-DP equation (2.2) it is then transformed to a third-order nonlinear ordinary differential equation

$$-(\nu + c_0)F' + (b_0 + 1)FF' - \alpha^2(FF''' + b_0F'F'') + (\alpha^2\nu + \gamma)F''' = 0. \quad (2.38)$$

2.2.2 Exact solutions of the CH-DP equation using simplest equation method

In this subsection the simplest equation method, which was introduced by Kudryashov [43] and modified by Vitanov [44], is used to solve the ODE (2.38) and as a result the exact solutions of the CH-DP equation (2.2) are obtained.

The simplest equations that will be used are the Bernoulli and Riccati equations. We now briefly recall the simplest equation method here.

The solutions of the ODE (2.38) are considered to be in the form

$$F(z) = \sum_{i=0}^{M} \mathcal{A}_i(H(z))^i, \quad (2.39)$$

where $H(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [43,44] and $\mathcal{A}_i, (i = 0, 1, \cdots, M)$, are parameters to be determined. The solutions of the Bernoulli and Riccati equations can be expressed in terms of the elementary functions.

The Bernoulli equation which we use here, is given by

$$H'(z) = aH(z) + bH^2(z), \quad (2.40)$$

where a and b are constants. The solution of the Bernoulli equation is

$$H(z) = a\left\{\frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b\cosh[a(z + C)] - b\sinh[a(z + C)]}\right\}, \quad (2.41)$$

where C is a constant of integration. For the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \quad (2.42)$$

where a, b and c are constants, the solutions to be used are

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z + C)\right] \quad (2.43)$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}, \tag{2.44}$$

with $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

Solutions of (2.2) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (2.38) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H + \mathcal{A}_2 H^2. \tag{2.45}$$

It follows from (2.45) that

$$F'(z) = \mathcal{A}_1 H'(z) + 2\mathcal{A}_2 H H', \tag{2.46a}$$

$$F''(z) = \mathcal{A}_1 H'' + 2\mathcal{A}_2[H'^2 + HH''], \tag{2.46b}$$

$$F'''(z) = \mathcal{A}_1 H''' + 2\mathcal{A}_2[3H'H'' + HH''']. \tag{2.46c}$$

Now repeatedly making use of the Bernoulli equation

$$H'(z) = aH(z) + bH^2(z) \tag{2.47}$$

one obtains that

$$H''(z) = a^2 H + 3abH^2 + 2b^2 H^3 \tag{2.48}$$

and

$$H'''(z) = a^3 H + (a^2 b + 6ab)H^2 + (6b^2 + 6ab^2)H^3 + 6b^3 H^4. \tag{2.49}$$

Substituting (2.47), (2.48) and (2.49) into (2.46) gives respectively

$$
\begin{aligned}
F'(z) &= a\mathcal{A}_1 H + b\mathcal{A}_1 H^2 + 2a\mathcal{A}_2 H^2 + 2b\mathcal{A}_2 H^2, && (2.50\text{a})\\
F''(z) &= a^2\mathcal{A}_1 H + 3b\mathcal{A}_1 H^2 + 2b^2\mathcal{A}_1 H^3 + 4a^2\mathcal{A}_2 H^2 + 4ab\mathcal{A}_2 H^3\\
&\quad +6b^2\mathcal{A}_2 H^4 + 6b\mathcal{A}_2 H^3, && (2.50\text{b})\\
F'''(z) &= a^3\mathcal{A}_1 H + a^2 b\mathcal{A}_1 H^2 + 6ab\mathcal{A}_1 H^2 + 6b^2\mathcal{A}_1 H^3 + 6ab^2\mathcal{A}_1 H^3\\
&\quad +6b^3\mathcal{A}_1 H^4 + 8a^3\mathcal{A}_2 H^2 + 30ab\mathcal{A}_2 H^3 + 24ab^2\mathcal{A}_2 H^4 + 8a^2 b\mathcal{A}_2 H^3\\
&\quad +30b^2\mathcal{A}_2 H^4 + 24b^3\mathcal{A}_2 H^5. && (2.50\text{c})
\end{aligned}
$$

Substituting (2.45) and (2.50) into (2.38), simplifying and equating all coefficients of the functions H^i to zero, yields the following algebraic system of seven equations in terms of $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$:

$$
\begin{aligned}
&\alpha^2 a^3 \mathcal{A}_1 \nu - \alpha^2 a^3 \mathcal{A}_0 \mathcal{A}_1 + a^3 \mathcal{A}_1 \gamma - a\mathcal{A}_1 \nu + a\mathcal{A}_0\mathcal{A}_1 + a\mathcal{A}_0\mathcal{A}_1 b_0 - a\mathcal{A}_1 c_0 = 0,\\
&8\alpha^2 a^3 \mathcal{A}_2 \nu - \alpha^2 a^3 \mathcal{A}_1^2 - 8\alpha^2 a^3 \mathcal{A}_0\mathcal{A}_2 + 8a^3\mathcal{A}_2\gamma - \alpha^2 a^3 \mathcal{A}_1^2 b_0 + 7\alpha^2 a^2 \mathcal{A}_1 b\nu\\
&-7\alpha^2 a^2 \mathcal{A}_0\mathcal{A}_1 b + 7a^2\mathcal{A}_1 b\gamma - 2a\mathcal{A}_2\nu + a\mathcal{A}_1^2 + 2a\mathcal{A}_0\mathcal{A}_2 + a\mathcal{A}_1^2 b_0 + 2a\mathcal{A}_0\mathcal{A}_2 b_0\\
&-2a\mathcal{A}_2 c_0 - \mathcal{A}_1 b\nu + \mathcal{A}_0\mathcal{A}_1 b + \mathcal{A}_0\mathcal{A}_1 b b_0 - \mathcal{A}_1 b c_0 = 0,\\
&-9\alpha^2 a^3\mathcal{A}_1\mathcal{A}_2 - 6\alpha^2 a^3 \mathcal{A}_1\mathcal{A}_2 b_0 + 38\alpha^2 a^2 \mathcal{A}_2 b\nu - 7\alpha^2 a^2\mathcal{A}_1^2 b - 4\alpha^2 a^2 \mathcal{A}_1^2 b b_0\\
&-38\alpha^2 a^2\mathcal{A}_0\mathcal{A}_2 b + 38a^2\mathcal{A}_2 b\gamma + 3a\mathcal{A}_1\mathcal{A}_2 + 12\alpha^2 a\mathcal{A}_1 b^2\nu - 12\alpha^2 a\mathcal{A}_0\mathcal{A}_1 b^2\\
&+12a\mathcal{A}_1 b^2\gamma + 3a\mathcal{A}_1\mathcal{A}_2 b_0 - 2\mathcal{A}_2 b\nu + \mathcal{A}_1^2 b + \mathcal{A}_1^2 b b_0 + 2\mathcal{A}_0\mathcal{A}_2 b + 2\mathcal{A}_0\mathcal{A}_2 b b_0\\
&-2\mathcal{A}_2 b c_0 = 0,\\
&-8\alpha^2 a^3\mathcal{A}_2^2 - 8\alpha^2 a^3\mathcal{A}_2^2 b_0 - 45\alpha^2 a^2\mathcal{A}_1\mathcal{A}_2 b - 22\alpha^2 a^2\mathcal{A}_1\mathcal{A}_2 b b_0 + 2a\mathcal{A}_2^2\\
&+54\alpha^2 a\mathcal{A}_2 b^2\nu - 12\alpha^2 a\mathcal{A}_1^2 b^2 - 5\alpha^2 a\mathcal{A}_1^2 b^2 b_0 - 54\alpha^2 a\mathcal{A}_0\mathcal{A}_2 b^2 + 54a\mathcal{A}_2 b^2\gamma\\
&+2a\mathcal{A}_2^2 b_0 + 6\alpha^2\mathcal{A}_1 b^3\nu - 6\alpha^2\mathcal{A}_0\mathcal{A}_1 b^3 + 6\mathcal{A}_1 b^3\gamma + 3\mathcal{A}_1\mathcal{A}_2 b + 3\mathcal{A}_1\mathcal{A}_2 b b_0 = 0,\\
&-38a^2\alpha^2\mathcal{A}_2^2 b - 28a^2\alpha^2\mathcal{A}_2^2 b_0 b - 66a\alpha^2\mathcal{A}_1\mathcal{A}_2 b^2 - 26a\alpha^2\mathcal{A}_1\mathcal{A}_2 b_0 b^2 + 24\alpha^2\mathcal{A}_2 b^3\nu\\
&-6\alpha^2\mathcal{A}_1^2 b^3 - 2\alpha^2\mathcal{A}_1^2 b_0 b^3 - 24\alpha^2\mathcal{A}_0\mathcal{A}_2 b^3 + 24\mathcal{A}_2 b^3\gamma + 2\mathcal{A}_2^2 b_0 b + 2\mathcal{A}_2^2 b = 0,\\
&-54a\alpha^2\mathcal{A}_2^2 b^2 - 32a\alpha^2\mathcal{A}_2^2 b_0 b^2 - 30\alpha^2\mathcal{A}_1\mathcal{A}_2 b^3 - 10\alpha^2\mathcal{A}_1\mathcal{A}_2 b_0 b^3 = 0,\\
&-24\alpha^2\mathcal{A}_2^2 b^3 - 12\alpha^2\mathcal{A}_2^2 b_0 b^3 = 0.
\end{aligned} \tag{2.51}
$$

The solution of the above equations, with the aid of Mathematica, is

$$
\begin{aligned}
c_0 &= \frac{-2\alpha^2\nu - \gamma}{\alpha^2},\\
b_0 &= -2,\\
a &= \frac{1}{\alpha},\\
\mathcal{A}_0 &= -c_0 - \nu,\\
\mathcal{A}_1 &= \mathcal{A}_1,\\
\mathcal{A}_2 &= a\alpha^2\mathcal{A}_1 b.
\end{aligned}
$$

As a result, a solution of the CH-DP equation (2.2) is obtained as

$$
\begin{aligned}
u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\} \\
+ \mathcal{A}_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}^2, \qquad (2.52)
\end{aligned}
$$

where $z = x - \nu t$.

Solutions of (2.2) using Riccati equation as the simplest equation

The same procedure is followed to obtain solutions of the CH-DP equation(2.2) using Riccati equation (2.42) as the simplest equation. The balancing procedure yields $M = 2$, so the solutions of the third-order nonlinear ordinary differential equation (2.38) are of the form

$$
F(z) = \mathcal{A}_0 + \mathcal{A}_1 H + \mathcal{A}_2 H^2. \qquad (2.53)
$$

Substituting (2.53) into (2.38) and making use of the Riccati equation (2.42) and equating all coefficients of the functions H^i to zero, an algebraic system is obtained in terms of $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$. Solving the resultant algebraic equations gives the following solution:

$$
\begin{aligned}
c_0 &= -\frac{1}{\alpha^2}(2\alpha^2\nu + \gamma), \\
b_0 &= -2, \\
b &= \frac{i}{\sqrt{3}\alpha}, \\
a &= -\frac{b^2}{2c}, \\
\mathcal{A}_0 &= -c_0 - \nu, \\
\mathcal{A}_1 &= \mathcal{A}_1, \\
\mathcal{A}_2 &= -3a\alpha^2 \mathcal{A}_1 b.
\end{aligned}
$$

Hence the two solutions of the CH-DP equation (2.2) are given by

$$
\begin{aligned}
u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left[\frac{1}{2}\theta(z+C)\right] \right\} \\
+ \mathcal{A}_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left[\frac{1}{2}\theta(z+C)\right] \right\}^2 \qquad (2.54)
\end{aligned}
$$

and

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \right\}$$
$$+\mathcal{A}_2\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \right\}^2, \qquad (2.55)$$

where $z = x - \nu t$.

A profile of the solution (2.54) is given in Figure 2.1.

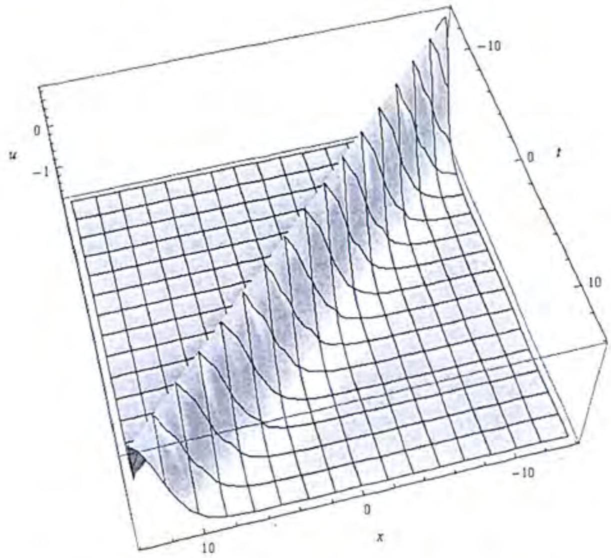

Figure 2.1: Profile of the solution (2.54)

2.3 Generalized Korteweg-de Vries equation with two power law nonlinearities

The generalized Korteweg-de Vries (KdV) equation with two power law nonlinearities [49], given by

$$u_t + \left(\alpha u^n - \beta u^{2n}\right) u_x + u_{xxx} = 0 \qquad (2.56)$$

describes the propagation of nonlinear long acoustic-type waves [63]. It should be noted that when $n = 1$ one obtains the well-known Gardner equation which is also called the combined KdV-mKdV equation. The generalized Korteweg-de Vries equation is a modification of the Korteweg-de Vries equation [50]

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.57}$$

which describes the evolution of weakly nonlinear and weakly dispersive waves used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory [64, 65].

2.3.1 Lie point symmetries of the KdV equation and symmetry reduction

The symmetries of the KdV equation (2.56) are generated by the vector field of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.58}$$

if and only if

$$X^{[3]}\left(u_t + \left(\alpha u^n - \beta u^{2n}\right)u_x + u_{xxx}\right)\Big|_{(2.56)} = 0, \tag{2.59}$$

where $X^{[3]}$ is the third prolongation of X and is given by (1.21).
Now expanding (2.59) gives

$$\begin{aligned}&\left[\xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}}\right.\\ &\left.+\zeta_{221}\frac{\partial}{\partial u_{xxt}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}\right]\left(u_t + \left(\alpha u^n - \beta u^{2n}\right)u_x + u_{xxx}\right)\Big|_{u_t=-(\alpha u^n-\beta u^{2n})u_x-u_{xxx}} = 0,\end{aligned}$$

where ζ_1, ζ_2, ζ_{22}, ζ_{221} and ζ_{222} are as given by in chapter 1.
Further simplification yields

$$\alpha n\eta u^{n-1}u_x - 2n\beta\eta u^{2n-1}u_x + \zeta_1 + \alpha u^n\zeta_2 + \zeta_{222}\Big|_{u_t=-(\alpha u^n-\beta u^{2n})u_x-u_{xxx}} = 0. \tag{2.60}$$

Now substituting the values of ζ_1, ζ_2 and ζ_{222} in (2.60) and replacing u_t by $-\left(\alpha u^n - \beta u^{2n}\right)u_x - u_{xxx}$, one can separate the resultant equation on the derivatives of u to obtain the following overdetermined system of linear partial differential equations:

$$\xi^2_u = 0, \tag{2.61}$$

$$\xi^1_x = 0, \tag{2.62}$$

$$\xi^1_u = 0, \tag{2.63}$$

$$\eta_{uu} = 0, \tag{2.64}$$

$$\xi^2_{xx} - \eta_{xu} = 0, \tag{2.65}$$

$$-\eta_t + \eta_x \beta\, u^{2\,n} - \eta_x \alpha\, u^n - \eta_{xxx} = 0, \tag{2.66}$$

$$2\,u^{2\,n}\beta\, n\eta - u^n \eta n\alpha + u\xi^2_t + 2\,\xi^2_x u^{1+2\,n}\beta - 2\,\xi^2_x u^{n+1}\alpha + u\xi^2_{xxx}$$

$$-3\,u\eta_{xxu} = 0, \tag{2.67}$$

$$2\,u^{2\,n}\beta\, n\eta - u^n \eta n\alpha + \xi^1_t u^{1+2\,n}\beta - \xi^1_t u^{n+1}\alpha + u\xi^2_t - \xi^2_x u^{1+2\,n}\beta$$

$$+\xi^2_x u^{n+1}\alpha + u\xi^2_{xxx} - 3\,u\eta_{xxu} = 0. \tag{2.68}$$

Now we solve this system of linear partial differential equations for ξ^1, ξ^2 and η. From Eq.(2.61),

$$\xi^2 \equiv \xi^2(t) = B(t,x), \tag{2.69}$$

where $B(t,x)$ is an arbitrary function of t and x. Eqs. (2.62) and (2.63) imply that

$$\xi^1 = A(t), \tag{2.70}$$

where $A(t)$ is an arbitrary function of t. Integrating Eq. (2.64) twice with respect to u gives

$$\eta = C(t,x)u + D(t,x), \tag{2.71}$$

where $C(t,x)$ and $D(t,x)$ are arbitrary functions of t and x. Substituting the value of η into equation (2.66) yields

$$-uC_t - D_t + \beta u^{2n+1} C_x + \beta u^{2n} D_x - \alpha u^{n+1} C_x - \alpha u^n D_x - uC_{xxx} - D_{xxx} = 0. \tag{2.72}$$

Now splitting (2.72) with respect to powers of u yields

$$u^{n+1}: \quad C_x = 0, \tag{2.73}$$

$$u^5: \quad D_x = 0, \tag{2.74}$$

$$u: \quad C_t + C_{xxx} = 0, \tag{2.75}$$

$$1: \quad D_t + D_{xxx} = 0. \tag{2.76}$$

Integrating Eqs. (2.73) and (2.74) gives

$$C = E(t) \tag{2.77}$$

and

$$D = F(t) \tag{2.78}$$

respectively, where $E(t)$ and $F(t)$ are arbitrary functions of t. Substituting the value of C into (2.75) gives

$$E'(t) = 0 \;\Rightarrow\; E(t) = C_1, \tag{2.79}$$

where C_1 is a constant of integration. Also, substituting the value of D into (2.76) gives

$$F'(t) = 0 \;\Rightarrow\; F(t) = C_2, \tag{2.80}$$

where C_2 is a constant of integration. Thus

$$\begin{aligned} \xi^1 &= A(t), \\ \xi^2 &= B(t,x), \\ \eta &= C_1 u + C_2. \end{aligned} \tag{2.81}$$

Now substituting values of ξ^2 and η into (2.65) and integrating twice with respect to x gives

$$B = xG(t) + H(t), \tag{2.82}$$

where $G(t)$ and $H(t)$ are arbitrary functions of t. Also substituting the values of ξ^2 and η into (2.67) gives

$$2u^{2n+1}\beta nC_1 + 2u^{2n}\beta nC_2 - n\alpha u^{n+1}C_1 - n\alpha u^n C_2 + xuG'(t) + uH'(t)$$
$$+2u^{2n+1}\beta G(t) - 2\alpha u^{n+1}G(t) = 0. \tag{2.83}$$

Now splitting (2.83) with respect to powers of u one obtains

$$u^{2n+1}: \quad 2\beta nC_1 + 2\beta G(t) = 0, \tag{2.84}$$
$$u^{2n}: \quad 2\beta nC_2 = 0 \Rightarrow C_2 = 0, \tag{2.85}$$
$$u^{n+1}: \quad -n\alpha C_1 - 2\alpha G(t) = 0, \tag{2.86}$$
$$u: \quad xG'(t) + H'(t) = 0. \tag{2.87}$$

Eq. (2.84) implies that

$$G(t) = -nC_1. \tag{2.88}$$

Substituting Eq.(2.88) into (2.86) gives

$$-n\alpha C_1 + 2n\alpha C_1 = 0 \quad \text{implying that} \quad C_1 = 0. \tag{2.89}$$

Thus $G(t) = 0$, and from (2.87) it implies $H'(t) = 0$. Therefore

$$H(t) = C_3, \tag{2.90}$$

where C_3 is an arbitrary constant of integration. Substituting C_1, C_2 and C_3 into (2.81) gives

$$\xi^1 = A(t), \quad \xi^2 = C_3 \quad \text{and} \quad \eta = 0. \tag{2.91}$$

Now substituting these values of ξ^1, ξ^2 and η into Eq.(2.68) gives

$$\beta u^{2n+1}A'(t) - \alpha u^{n+1}A'(t) = 0.$$

Integrating the above equation with respect to t gives

$$A(t) = C_4,$$

where C_4 is an arbitrary constant of integration. Therefore

$$\xi^1 = C_4,$$
$$\xi^2 = C_3,$$
$$\eta = 0.$$

So now from (2.58) it can be deduced that (2.56) has two translation symmetries, namely

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x}.$$

Considering the linear combination of the two translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$, namely $X_1 + \nu X_2$, where ν is a constant and solving the corresponding Lagrange system for the symmetry $X_1 + \nu X_2$, one obtains an invariant $z = x - \nu t$, and so the group-invariant solution of (2.56) is of the form

$$u = F(z),$$

where the function F satisfies the nonlinear ordinary differential equation

$$F'''(z) + \alpha F(z)^n F'(z) - \beta F(z)^{2n} F'(z) - \nu F'(z) = 0. \tag{2.92}$$

By introducing the transformation

$$F(z) = G(z)^{\frac{1}{n}}, \tag{2.93}$$

equation (2.92) transforms to the nonlinear ordinary differential equation for G, viz.,

$$\begin{aligned} &n^2 G'''(z) G(z)^2 + \alpha n^2 G(z)^3 G'(z) - n^2 \beta G(z)^4 G'(z) - \nu n^2 G(z)^2 G'(z) + 2n^2 G'(z)^3 \\ &-3nG'(z)^3 + G'(z)^3 - 3n^2 G(z) G'(z) G''(z) + 3nG(z) G'(z) G''(z) = 0. \end{aligned} \tag{2.94}$$

2.3.2 Exact solutions of the KdV equation using simplest equation method

The solutions of the ODE (2.94) are considered in the form

$$G(z) = \sum_{i=0}^{M} A_i (H(z))^i, \tag{2.95}$$

where $H(z)$ satisfies the Bernoulli and Riccati equations.

Solutions of (2.56) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 1$, and so the solution of (2.94) is of the form

$$G(z) = A_0 + A_1 H. \tag{2.96}$$

Substituting (2.96) into (2.94) and making use of the Bernoulli equation (2.40) and then equating the coefficients of the functions H^i to zero, one obtains an algebraic system of six equations in terms of A_i, $(i = 0, 1, 2)$.

$$
\begin{aligned}
&-n^2 {A_1}^5 \beta\, b + 2\, n^2 {A_1}^3 b^3 + {A_1}^3 b^3 + 3\, n {A_1}^3 b^3 = 0,\\
&-n^2 A_1 \nu\, a {A_0}^2 + n^2 A_1 a^3 {A_0}^2 + n^2 A_1 \alpha\, a {A_0}^3 - n^2 A_1 \beta\, a {A_0}^4 = 0,\\
&6\, n {A_1}^2 b^3 A_0 + 3\, n^2 {A_1}^3 a b^2 + 3\, {A_1}^3 a b^2 + 6\, n {A_1}^3 a b^2 - 4\, n^2 {A_1}^4 \beta\, b A_0\\
&-n^2 {A_1}^5 \beta\, a + 6\, n^2 {A_1}^2 b^3 A_0 + n^2 {A_1}^4 \alpha\, b = 0,\\
&7\, n^2 A_1 a^2 b {A_0}^2 - 2\, n^2 {A_1}^2 \nu\, a A_0 - n^2 A_1 \nu\, b {A_0}^2 + n^2 A_1 \alpha\, b {A_0}^3 - n^2 {A_1}^2 a^3 A_0\\
&+3\, n^2 {A_1}^2 \alpha\, a {A_0}^2 + 3\, n {A_1}^2 a^3 A_0 - 4\, n^2 {A_1}^2 \beta\, a {A_0}^3 - n^2 A_1 \beta\, b {A_0}^4 = 0,\\
&3\, n^2 {A_1}^2 \alpha\, b {A_0}^2 - 6\, n^2 {A_1}^3 \beta\, a {A_0}^2 + 12\, n^2 A_1 a b^2 {A_0}^2 - 4\, n^2 {A_1}^2 \beta\, b {A_0}^3\\
&+{A_1}^3 a^3 - n^2 {A_1}^3 \nu\, a - 2\, n^2 {A_1}^2 \nu\, b A_0 + 3\, n^2 {A_1}^3 \alpha\, a A_0 + 2\, n^2 {A_1}^2 a^2 b A_0\\
&+12\, n {A_1}^2 a^2 A_0 b = 0,\\
&-6\, n^2 {A_1}^3 \beta\, b {A_0}^2 - n^2 {A_1}^3 \nu\, b + 3\, n {A_1}^3 a^2 b + n^2 {A_1}^3 a^2 b + 3\, n^2 {A_1}^3 \alpha\, b A_0\\
&+9\, n^2 {A_1}^2 a b^2 A_0 + n^2 {A_1}^4 \alpha\, a + 3\, {A_1}^3 a^2 b - 4\, n^2 {A_1}^4 \beta\, a A_0 + 6\, n^2 A_1 b^3 {A_0}^2\\
&+15\, n {A_1}^2 a A_0 b^2 = 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Mathematica, one possible set of values of A_i is

$$
\begin{aligned}
a &= 1, \\
b &= 2, \\
\alpha &= \frac{2(2+n^2+3\,n)}{n^2 A_1}, \\
\beta &= \frac{4(3\,n+1+2\,n^2)}{n^2 {A_1}^2}, \\
\nu &= n^{-2}, \\
A_0 &= \frac{1}{2} A_1.
\end{aligned}
$$

As a result, a solution of the generalized Korteweg-de Vries equation with two power law nonlinearities (2.56) is

$$
u(t,x) = \left[A_0 + A_1 a \left\{ \frac{\cosh[a(x-\nu t+C)] + \sinh[a(x-\nu t+C)]}{1 - b\cosh[a(x-\nu t+C)] - b\sinh[a(x-\nu t+C)]} \right\} \right]^{1/n} \tag{2.97}
$$

A profile of the solution (2.97) is given in Figure 2.2

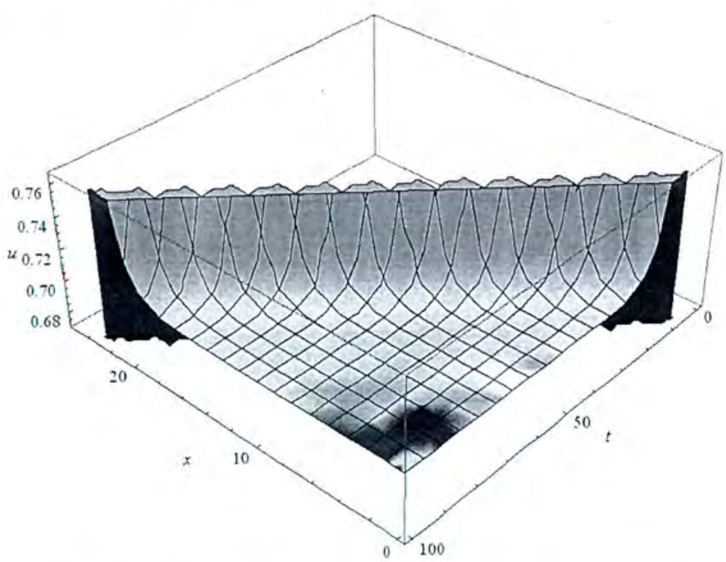

Figure 2.2: Profile of solitary wave solution (2.97)

Solutions of (2.56) using Riccati equation as the simplest equation

In this case the balancing procedure yields $M = 1$, and so the solution of (2.94) is of the form

$$G(z) = A_0 + A_1 H. \tag{2.98}$$

Substituting (2.98) into (2.94) and making use of the Riccati equation (2.42), one obtains an algebraic system of equations in terms of A_i, $(i = 0, 1)$ by equating all coefficients of the functions H^i to zero. Solving the resultant system, one possible set of values is

$$\begin{aligned}
a &= 2,\\
b &= 3,\\
c &= 1,\\
a &= \frac{A_1\,(3\,A_0 - A_1)}{{A_0}^2},\\
\alpha &= \frac{(3\,A_0 - A_1)\,(6\,A_0 + 9\,nA_0 + 3\,n^2 A_0 - 4\,A_1 - 6\,nA_1 - 2\,n^2 A_1)}{n^2 {A_0}^3},\\
\beta &= \frac{(3\,A_0 - A_1)^2\,(1 + 3\,n + 2\,n^2)}{{A_0}^4 n^2},\\
\nu &= \frac{9\,{A_0}^2 - 12\,A_1 A_0 + 4\,{A_1}^2}{n^2 {A_0}^2}.
\end{aligned}$$

Hence the two solutions of (2.56) are

$$u(t,x) = \left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z + C)\right]\right\}\right)^{1/n}$$

and

$$u(t,x) = \left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\right)^{1/n},$$

where $z = x - \nu t$.

2.4 Conclusions

In this chapter two nonlinear partial differential equations, namely, the modified Camassa-Holm-Degasperis-Procesi equation (2.2) and the generalized Korteweg-de Vries equation with two power law nonlinearities (2.56) were studied from the point of view of Lie symmetry analysis. The Lie symmetries obtained were then used to transform both the equations to nonlinear ordinary differential equations. Thereafter the simplest equation method was employed to construct exact solutions of both equations. The correctness of the solutions obtained here was checked by substituting them back into the equations.

Chapter 3

Exact solutions of two systems of NLPDEs using Lie symmetry method and the travelling wave hypothesis approach

3.1 Introduction

In this chapter two coupled systems of nonlinear partial differential equations, namely, the generalized Boussinesq-Burgers equations and the (2+1)-dimensional Davey-Stewartson equations are studied. The Lie symmetry method and the travelling wave hypothesis approach is utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach is used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations. This work has been submitted for publication [57].

3.2 Generalized Boussinesq-Burgers equations

The generalized Boussinesq-Burgers (BB) equations [3] are given by

$$u_t + auu_x + bv_x = 0, \tag{3.1a}$$

$$v_t + c(uv)_x + du_{xxx} = 0, \tag{3.1b}$$

where a, b, c and d are real nonzero constants. These equations arise in the study of fluid flow and describe the propagation of shallow water waves, where x and t represent the normalized space and time, respectively. Here $u(x,t)$ represents the horizontal velocity and at the leading order it is the depth averaged horizontal field, while $v(x,t)$ denotes the height of the water surface above the horizontal level at the bottom [3]. The Lie symmetry method and the travelling wave hypothesis approach is utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations.

3.2.1 Lie symmetry analysis of the BB equations

The symmetry group of the generalized BB equations (3.1) is generated by the vector field given by

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Applying the third prolongation $X^{[3]}$ [33] to (3.1) and solving the resultant overdetermined system of linear partial differential equations using Mathematica [38] one obtains the following three Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= -2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}. \end{aligned}$$

Taking the linear combination of the two translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$, namely the symmetry $X_1 + \nu X_2$, where ν is an arbitrary constant, gives rise to the group-invariant solution

$$u = F(z), \quad v = G(z), \tag{3.2}$$

where $z = x - \nu t$ is an invariant of the symmetry $X_1 + \nu X_2$. Substitution of (3.2) into (3.1) results in the system of ordinary differential equations

$$aF(z)F'(z) - \nu F'(z) + bG'(z) = 0, \tag{3.3a}$$

$$dF'''(z) + cG(z)F'(z) + cF(z)G'(z) - \nu G'(z) = 0. \tag{3.3b}$$

Integration of (3.3a) with respect to z yields

$$\frac{1}{2}aF(z)^2 - \nu F(z) + bG(z) = 0,$$

where the constant of integration is chosen to be zero, since a soliton solution is being sought. Solving for $G(z)$ gives

$$G(z) = \frac{2\nu F(z) - aF(z)^2}{2b}. \tag{3.4}$$

Substituting (3.4) into (3.3b) results in the third-order nonlinear ordinary differential equation

$$2bdF'''(z) - 3acF(z)^2F'(z) + 2\nu(a+2c)F(z)F'(z) - 2\nu^2F'(z) = 0,$$

which can be integrated twice to obtain

$$F'(z)^2 - \frac{ac}{4bd}F(z)^4 + \frac{\nu(a+2c)}{3bd}F(z)^3 - \frac{\nu^2}{bd}F(z)^2 = 0. \tag{3.5}$$

Here again the constants of integration are taken to be zero for the same reason as given above. Integrating (3.5) with the aid of Mathematica and reverting back to our original variables, one obtains the solitary wave solution

$$u(x,t) = F(z) = \frac{12\nu}{2(a+2c) + \alpha\cosh(\gamma z + \delta) - \beta\sinh(\gamma z + \delta)}. \tag{3.6}$$

Now substituting (3.6) into (3.4) gives the solution for the variable v as

$$v(x,t) = G(z) = \frac{12\nu^2\left[-6a + 2(a+2c) + \alpha\cosh(\gamma z + \delta) - \beta\sinh(\gamma z + \delta)\right]}{b[2(a+2c) + \alpha\cosh(\gamma z + \delta) - \beta\sinh(\gamma z + \delta)]^2}, \tag{3.7}$$

where α, β, γ and δ are constants given by

$$\alpha = a^2 - 5ac + 4c^2 + 1,$$
$$\beta = a^2 - 5ac + 4c^2 - 1,$$
$$\gamma = \frac{\nu}{\sqrt{bd}},$$
$$\delta = 2\sqrt{3}c\nu,$$
$$z = x - \nu t$$

with $bd > 0$.

Profiles of the solutions (3.6) and (3.7) are given in Figure 3.1 and Figure 3.2 respectively.

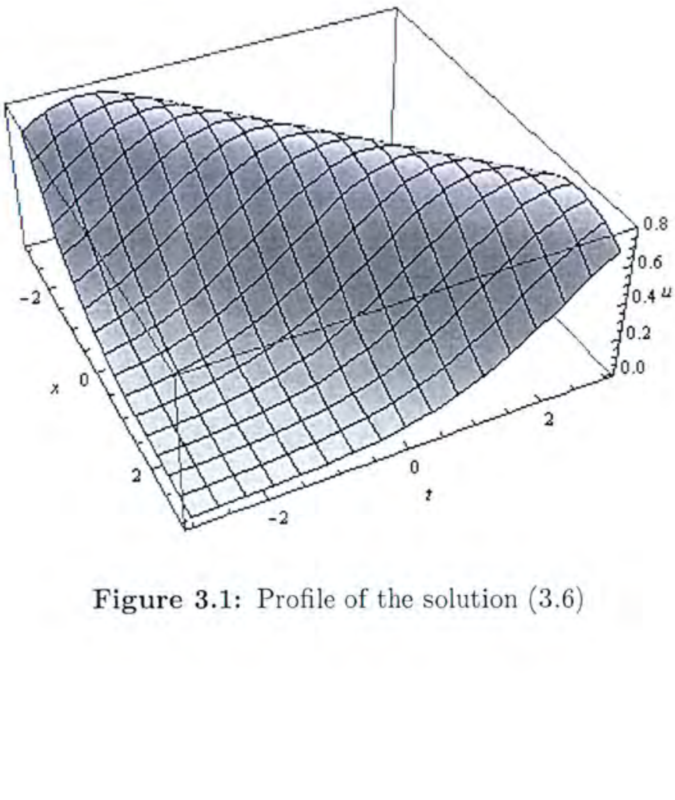

Figure 3.1: Profile of the solution (3.6)

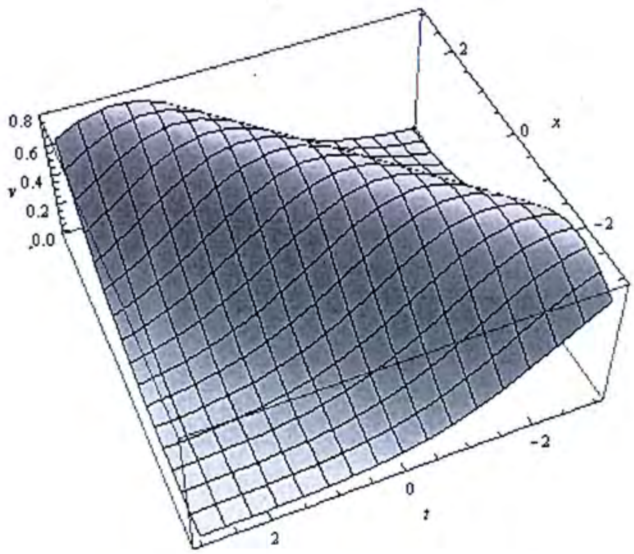

Figure 3.2: Profile of the solution (3.7)

3.2.2 Travelling wave approach

In this case solutions of generalized BB equations (3.1) are obtained using the traveling wave variables

$$u(x,t) = g(kx + lt + m), \tag{3.8a}$$

$$v(x,t) = h(kx + lt + m), \tag{3.8b}$$

where k, l and m are constants. Substituting these into (3.1) gives

$$lg' + akgg' + bkh' = 0, \tag{3.9a}$$

$$lh' + ckhg' + ckgh' + dk^3 g''' = 0, \tag{3.9b}$$

where $'$ denotes differentiation with respect to the variable $s = kx + lt + m$. On integrating (3.9a) with respect to s once, gives

$$lg + \frac{ak}{2}g^2 + bkh = 0$$

and solving for h gives

$$h = -\frac{l}{bk}g - \frac{a}{2b}g^2. \tag{3.10}$$

Here the constant of integration is taken to be zero, since the goal is to obtain a soliton solution. Integration of (3.9b) with respect to s once, and again taking the constant of integration to be zero, gives

$$lh + ckgh + dk^3 g'' = 0. \tag{3.11}$$

Substituting the value of h from (3.10) into (3.11) and simplifying one obtains a second-order nonlinear ordinary differential equation

$$g'' = \frac{l^2}{bdk^4} g + \frac{l}{dk^3}\left(\frac{a+2c}{2b}\right) g^2 + \frac{ac}{2bdk^2} g^3$$

which can be written as

$$g'' = Ag + Bg^2 + Cg^3, \tag{3.12}$$

where

$$A = \frac{l^2}{bdk^4}, \quad B = \frac{l(a+2c)}{2bdk^3} \quad \text{and} \quad C = \frac{ac}{2bdk^2}.$$

After multiplying both sides of (3.12) by g', one can integrate the equation once. Again choosing the constant of integration to be zero, gives

$$(g')^2 = Ag^2 + \frac{2B}{3} g^3 + \frac{C}{2} g^4.$$

This is a variables separable equation, which on separating the variables yields

$$\int \frac{dg}{\sqrt{Ag^2 + (2B/3)g^3 + (C/2)g^4}} = \int ds.$$

On integration this leads to

$$g(s) = \frac{12A}{4B + (\alpha - 1)\cosh(\sqrt{A}s) + (\alpha + 1)\sinh(\sqrt{A}s)},$$

or

$$u(x,t) = \frac{12A}{4B + (\alpha - 1)\cosh[\sqrt{A}(kx + lt + m)] + (\alpha + 1)\sinh[\sqrt{A}(kx + lt + m)]}, \tag{3.13}$$

where $\alpha = 18AC - 4B^2$. Substituting the above value of g into (3.10), one can now obtain the solution for the variable v.

3.3 The (2+1)-dimensional Davey-Stewartson equations

The (2+1)-dimensional Davey-Stewartson (DS) equations

$$\begin{aligned} iu_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv &= 0, \qquad (3.14a) \\ v_{xx} + v_{yy} + 2(|u|^2)_{xx} &= 0 \qquad (3.14b) \end{aligned}$$

were first introduced by Davey and Stewartson in 1974 [52]. This system of equations is completely integrable and is often used to describe the long-time evolution of a two-dimensional wave packet [53–55]. In [53, 54] the Jacobi elliptic function method and the extended Jacobi elliptic function method were used respectively to obtain a series of exact solutions while the sine-cosine method was used in [55] to obtain soliton solutions of (3.14). In this project we use the travelling wave hypothesis approach to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations (3.14).

3.3.1 Travelling wave approach

First, this system of partial differential equations (3.14) is transformed to a system of nonlinear ordinary differential equations in order to derive its exact solutions.

The transformation

$$u = e^{i\theta}u(\xi), \; v = v(\xi), \; \theta = px + qy + rt, \; \xi = kx + cy + dt, \qquad (3.15)$$

where p, q, r, k, c and d are real constants, is made. Using equations (3.15) one can easily derive that

$$\begin{aligned} u_t &= ire^{i\theta}u + de^{i\theta}u', \\ u_x &= ipe^{i\theta}u + ke^{i\theta}u', \\ u_{xx} &= i^2p^2e^{i\theta}u + 2ipke^{i\theta}u' + k^2e^{i\theta}u'', \\ u_y &= iqe^{i\theta}u + ce^{i\theta}u', \\ u_{yy} &= i^2q^2e^{i\theta}u + 2icqe^{i\theta}u' + c^2e^{i\theta}u'', \\ -2|u|^2u &= -2u^2e^{i\theta}u. \qquad (3.16) \end{aligned}$$

Also

$$\begin{aligned}
v_{xx} &= k^2 v'', \\
v_{yy} &= c^2 v'', \\
2(|u|^2)_{xx} &= 2(u^2)''.
\end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.14) the (2+1)-dimensional DS equations (3.14) transform to

$$(-r - p^2 + q^2)u + i(d + 2pk - 2cq)u' + (k^2 - c^2)u'' - 2u^3 - 2uv = 0, \tag{3.18a}$$

$$(k^2 + c^2)v'' + 2(u^2)'' = 0. \tag{3.18b}$$

Integration of (3.18b) twice and taking the constants of integration to be zero one obtains

$$(k^2 + c^2)v + 2(u^2) = 0$$

and solving for v gives

$$v = \frac{-2u^2}{k^2 + c^2}. \tag{3.19}$$

Now substituting (3.19) into (3.18a) leads to the second-order nonlinear ordinary differential equation

$$u'' = \left(\frac{r + p^2 - q^2}{k^2 - c^2}\right)u + \left(\frac{2k^2 + 2c^2 - 4}{k^4 - c^4}\right)u^3,$$

which can be written in the form

$$u'' = Au + Bu^3, \tag{3.20}$$

where
$A = \dfrac{r + p^2 - q^2}{k^2 - c^2}$ and $B = \dfrac{2k^2 + 2c^2 - 4}{k^4 - c^4}$.

Solving the equation (3.20), with the aid of Mathematica, one obtains the solution

$$u = \pm\frac{1}{P_2}\, i\,\mathrm{sn}(P_1|\omega), \tag{3.21}$$

where $\mathrm{sn}(P_1|\omega)$ is a Jacobian elliptic function of the *sine-amplitude* [66],

$$P_1 = \frac{\sqrt{\left(\sqrt{A^2 - 2Bc_1} - A\right)(z + c_2)^2}}{\sqrt{2}}, \quad P_2 = \sqrt{\frac{B}{\sqrt{A^2 - 2Bc_1} + A}}$$

and

$$\omega = \frac{-Bc_1 + A\left(\sqrt{A^2 - 2Bc_1} + A\right)}{Bc_1}$$

is the modulus of the elliptic function with $0 < \omega < 1$. Here c_1 and c_2 are constants of integration and $Bc_1 < 0$. Reverting back to the original variables, one can now write the solution of the (2+1)-dimensional DS equations as

$$u(x, y, t) = \pm\frac{1}{P_2}\, i\, \mathrm{sn}(P_1|\omega), \tag{3.22}$$

where

$$P_1 = \frac{\sqrt{\left(\sqrt{A^2 - 2Bc_1} - A\right)(kx + cy + dt + c_2)^2}}{\sqrt{2}},$$

ω and P_2 are as above.

Now substituting (3.22) into (3.19) one obtains the solution for $v(x, y, t)$ and it is given by

$$v(x, y, t) = \frac{2}{P_2^2}\mathrm{sn}^2(P_1|\omega). \tag{3.23}$$

It should be noted that these solutions are valid for $0 < \omega < 1$ and as ω approaches zero, the solution becomes the normal sine function, $\sin z$, and as ω approaches 1, the solution tends to the tanh function, $\tanh z$.

The profiles of the solutions (3.22) and (3.23) are given in the Figures 3.3 and 3.4 respectively.

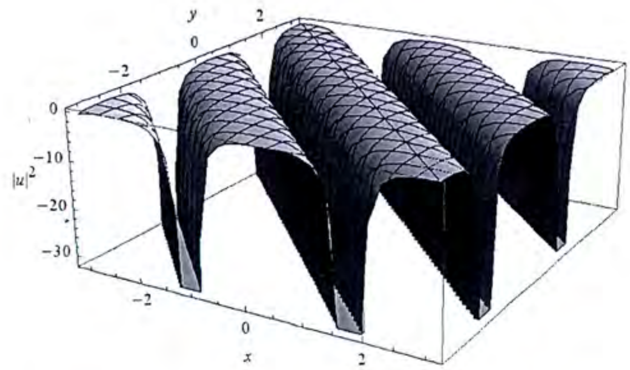

Figure 3.3: Profile of the solution (3.22)

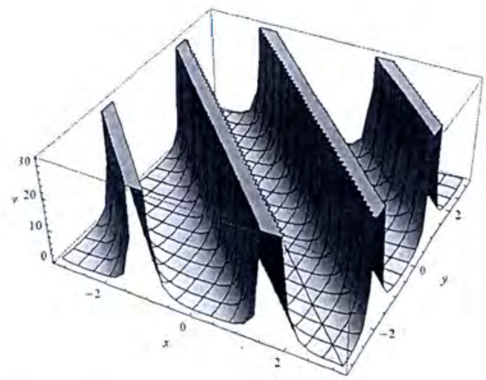

Figure 3.4: Profile of the solution (3.23)

3.4 CONCLUSION

In this chapter, two systems of nonlinear partial differential equations were studied. Firstly, exact solutions of the generalized Boussinesq-Burgers equations given by (3.1) were obtained using two different approaches; the Lie symmetry method and the travelling wave hypothesis approach. The solutions obtained were travelling wave solutions. Secondly, exact solutions of the (2+1)-dimensional Davey-Stewartson equations (3.14) were found using the travelling wave hypothesis. The Davey-Stewartson system was first transformed to a system of nonlinear ordinary differential equations, which was then solved to obtain the exact solutions.

Chapter 4

Concluding remarks

In this research project some important definitions and results from Lie group theory were first recalled and were later used in the dissertation. In Chapter 2, exact solutions of two nonlinear partial differential equations, namely, the modified Camassa-Holm-Degasperis-Procesi equation and the generalized Korteweg-de Vries equation with two power law nonlinearities were obtained. Lie symmetry method along with the simplest equation method were used to construct exact solutions for these two equations. In Chapter 3, two systems of nonlinear partial differential equations, namely, the generalized Boussinesq-Burgers equations and the (2+1)-dimensional Davey-Stewartson equations were studied. Lie symmetry method and the travelling wave hypothesis approach were utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach was used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations.

In future the conservation laws of these nonlinear partial differential equations will be studied.

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