

The subprime mortgage crisis: asset securitization and interbank lending

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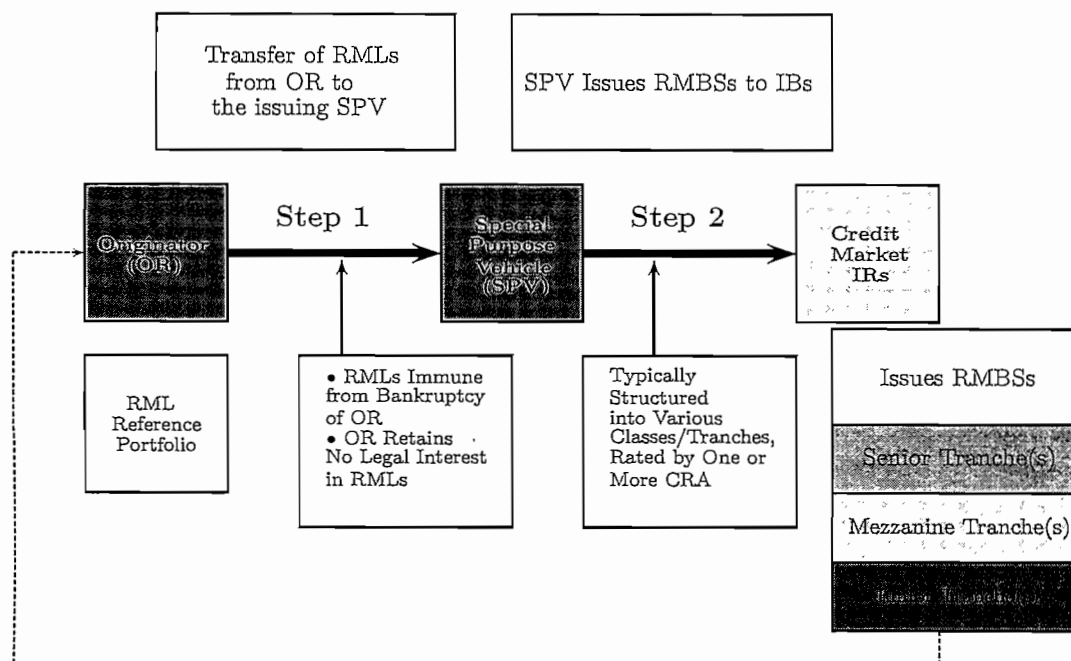


Figure 1: Diagrammatic Overview of RML Securitization Process

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Preface

One of the contributions made by the NWU-PC to the activities of the stochastic analysis community has been the establishment of an active research group FMORG that has an interest in institutional finance. In particular, FMORG has made contributions about modeling, optimization, regulation and risk management in insurance and banking. Students who have participated in projects in this programme under Prof. Petersen's supervision are listed below.

Level	Student	Graduation	Title
MSc	T Bosch	May 2003	Controllability of HJMM Interest Rate Models
MSc	CH Fouche	May 2006	Continuous-Time Stochastic Modelling of Capital Adequacy Ratios for Banks
MSc	MP Mulaudzi	May 2008	A Decision Making Problem in the Banking Industry
PhD	CH Fouche	May 2008	Dynamic Modeling of Banking Activities
PhD	F Gideon	Sept. 2008	Optimal Provisioning for Deposit Withdrawals and Loan Losses in the Banking Industry
MSc	MC Senosi	May 2009	Discrete Dynamics of Bank Credit and Capital and their Cyclicity
PhD	T Bosch	May 2009	Management and Auditing of Bank Assets and Capital
PhD	BA Tau	May 2009	Bank Loan Pricing and Profitability and Their Connections with Basel II and the Subprime Mortgage Crisis
PhD	MP Mulaudzi	May 2010	The Subprime Mortgage Crisis: Asset Securitization & Interbank Lending
MSc	B De Waal	Current	Subprime Reference Processes and Loan-to-Value Ratios
PhD	MC Senosi	Current	Discrete-Time Subprime Banking Models
PhD	S Thomas	Current	The Subprime Mortgage Crisis: Loan Securitization and Its Risks
Postdoc	J Mukuddem-Petersen	2006-9	Financial Economics

Declaration

I declare that, apart from the assistance acknowledged, the research presented in this thesis is my own unaided work. It is being submitted in partial fulfilment of the requirements for the degree Philosophiae Doctor in Applied Mathematics at the Potchefstroom Campus of the North West University. It has not been submitted before for any degree or examination to any other University.

Nobody, including Prof. Mark A. Petersen, Dr. Ilse M. Schoeman and Dr. Janine Mukuddem-Petersen, but myself is responsible for the final version of this thesis.

Signature.....

Date.....

Executive Summary

Subprime residential mortgage loan securitization and its associated risks have been a major topic of discussion since the onset of the subprime mortgage crisis (SMC) in 2007. In this regard, the thesis addresses the issues of subprime residential mortgage loan (RML) securitization in discrete-, continuous- and discontinuous-time and their connections with the SMC. In this regard, the main issues to be addressed are discussed in Chapters 2, 3 and 4.

In Chapter 2, we investigate the risk allocation choices of an investing bank (IB) that has to decide between risky securitized subprime RMLs and riskless Treasuries. This issue is discussed in a discrete-time framework with IB being considered to be regret- and risk-averse before and during the SMC, respectively. We conclude that if IB takes regret into account it will be exposed to higher risk when the difference between the expected returns on securitized subprime RMLs and Treasuries is small. However, there is low risk exposure when this difference is high. Furthermore, we assess how regret can influence IB's view – as a swap protection buyer – of the rate of return on credit default swaps (CDSs), as measured by the premium based on default swap spreads. We find that before the SMC, regret increases IB's willingness to pay lower premiums for CDSs when its securitized RML portfolio is considered to be safe. On the other hand, both risk- and regret-averse IBs pay the same CDS premium when their securitized RML portfolio is considered to be risky.

Chapter 3 solves a stochastic optimal credit default insurance problem in continuous-time that has the cash outflow rate for satisfying depositor obligations, the investment in securitized loans and credit default insurance as controls. As far as the latter is concerned, we compute the credit default swap premium and accrued premium by considering the credit rating of the securitized mortgage loans.

In Chapter 4, we consider a problem of IB investment in subprime residential mortgage-backed securities (RMBSs) and Treasuries in discontinuous-time. In order to accomplish this, we develop a Lévy process-based model of jump diffusion-type for IB's investment in subprime RMBSs and Treasuries. This model incorporates subprime RMBS losses which can be associated with credit risk. Furthermore, we use variance to measure such risk, and assume that the risk is bounded by a certain constraint. We are now able to set-up a mean-variance optimization problem for IB's investment which determines the optimal proportion of funds that needs to be invested in subprime RMBSs and Treasuries subject to credit risk measured by the variance of IB's investment. In the sequel, we also consider a mean swaps-at-risk (SaR) optimization problem for IB's investment which determines the optimal portfolio which consists of subprime RMBSs and Treasuries subject to the protection by CDSs required against the possible losses. In this regard, we define SaR as indicative to IB on how much protection from swap protection seller it must have in order to cover the losses that might occur from credit events. Moreover, SaR is expressed in terms

of Value-at-Risk (VaR).

Finally, Chapter 5 provides an analysis of discrete-, continuous- and discontinuous-time models for subprime RML securitization discussed in the aforementioned chapters and their connections with the SMC.

The work presented in this thesis is based on 7 peer-reviewed international journal articles (see [25], [44], [45], [46], [47], [48] and [55]), 4 peer-reviewed chapters in books (see [42], [50], [51] and [52]) and 2 peer-reviewed conference proceedings papers (see [11] and [12]). Moreover, the article [49] is currently being prepared for submission to an ISI accredited journal.

Key Words: Residential Mortgage Loan (RML); Residential Mortgage-Backed Security (RMBS); Treasuries; Investing Bank (IB); Special Purpose Vehicle (SPV); Credit Risk; Credit Default Swap (CDS); Tranching Risk; Counterparty Risk; Liquidity Risk; Regret; Variance; Value-at-Risk; Capital-at-Risk; Stochastic Optimization; Discrete-Time; Continuous-Time; Discontinuous-Time; Subprime Mortgage Crisis.

Samevatting

Sekuriteitslewering vir sub-prima huisverbande en die risiko daaraan verbonde is 'n onderwerp wat druk bespreek is sedert die begin van die sub-prima-verbandkrisis (SVK) in 2007. Met die oog hierop word in hierdie verhandeling aandag gegee aan die kwessies van sekuriteitslewering vir sub-prima huisverbande (SHV) in diskrete-, kontinue- en diskontinue tye en hoe dit die SVK raak. Die vernaamste aspekte van hierdie onderwerp waarop ingegaan word, word behandel in Hoofstukke 2, 3 en 4.

In Hoofstuk 2, word 'n beleggingsbank (BB) se keuses van risiko-toekenning beskou, waar besluit moet word tussen riskante verbandlenings (RVLs) met sekuriteite of risikovrye tesourie-obligasies. Hierdie aspek word bespreek in 'n raamwerk van diskrete tyd, met die aanname dat die BB risiko en berou wou vermy respektiewelik voor en gedurende die SVK. Ons maak die gevolgtrekking dat as die BB berou in ag neem, dit aan groter risiko blootgestel sal wees wanneer die verskil tussen die verwagte opbrengs op sub-prima RVLs met sekuriteite en Tesourie-obligasies klein is. Die risiko-blootstelling is egter klein wanneer hierdie verskil groot is. Ons het verder ook bepaal hoe berou die BB se oordeel kan beïnvloed – as 'n koper van swap-dekking – oor die persentasie opbrengs op kredietgebrek-swaps (KGSs), soos gemeet deur jaarlikse premie op die verstekbeskerming. Ons bevinding was dat voor die SVK, berou die bereidheid van BB's om laer premies te betaal vir KGSs wanneer gereken is dat hul portfolio van RVLs met sekuriteite veilig is, beïnvloed het. Aan die ander kant het BB's met beide risiko- en berou-vermyding dieselfde KGS -premies betaal wanneer hul portfolio van RVLs met sekuriteite as riskant beskou is.

Hoofstuk 3 bied 'n oplossing vir 'n probleem om 'n stochastiese optimale kredietverstek te verseker in kontinue tyd waar die kontant-uitvloeiempo voldoende is vir die verpligtinge teenoor die deponente met die belegging in sekuriteitslenings en versekering vir kredietverstek as kontrole. Ten opsigte van laasgenoemde bereken ons die premium vir kredietverstek swap en die opgehoopte premie, deur die kredietgradering van die verbandlenings met sekuriteite in ag te neem.

In Hoofstuk 4, beskou ons 'n probleem van BB belegging in sekuriteite wat gedek is deur sub-prima-huisverbande (RVLSe) en in tesourie-obligasies in diskontinue tyd. Vir hierdie werk het ons vir BBs se belegging in sub-prima RVLSe en in tesourie-obligasies 'n Lévy-prosesgebaseerde model van die sprong-diffusietipe ontwikkel. Hierdie model neem ook sub-prima RVLS-verliese, wat assosieer kan word met kredietrisiko, in ag. Ons maak verder gebruik van variansie om sodanige risiko te meet en ons neem aan dat die risiko begrens is deur 'n sekere beperking. Ons is nou in staat om 'n gemiddelde variansie-optimaliseringsprobleem op te stel vir beleggings deur BBs, wat die optimale gedeelte van fondse wat belê moet word in sub-prima RVLSe en tesourie-obligasies, bepaal, onderhewig aan die krediet-risiko soos gemeet deur die variansie van BBs se belegging. In die daaropvolgende werk beskou ons ook 'n gemiddelde swaps-teen-risiko (StR)-optimalisasieprobleem vir BB-

belegging wat die optimale portfolio bepaal, bestaande uit sub-prima RVLSe en tesourie-obligasies onderhewig aan die beskerming deur die KGSe wat vereis word teen moontlike verliese. In die opsig definieer ons StR as 'n aanduiding vir BBs oor hoeveel beskerming teen swap-beskermingverkoper hulle nodig om die verliese te dek wat mag voortspruit uit krediet-gevalle. Dan ook word StR uitgedruk in terme van die Waarde-onderhewig-aan-Risiko (WoaR).

Laastens verskaf Hoofstuk 5 'n analise van diskrete-, kontinue- en diskontinue-tydmodelle vir sekuriteitsaanbieding van sub-prima RVLs wat in bogenoemde hoofstukke bespreek is en hoe hulle betrekking het op die SVK.

Die werk wat in hierdie verhandeling aangebied word is gebaseer op 7 ge-evalueerde artikels in internasionale tydskrifte. (Sien [25], [44], [45], [46], [47], [48] en [55]), 4 ge-evalueerde hoofstukke in boeke (sien [42], [50], [51] en [52]) en 2 ge-evalueerde konferensie-bydraes. (sien [11] en [12]). Dan ook word die artikel [49] tans voorberei vir voorlegging aan 'n ISI-ge-akkrediteerde tydskrif.

Sleutelwoorde: Huis-verbandlening (RVL); Residensiële Verbandgedekte Sekuriteit (RVLS); Tesourie-obligasies; Beleggingsbank (BB); Spesiale doeldraer (SDD); Kredietrisiko; Kredietgebrek-swap (KGS); Oordragrisiko; Teenparty-risiko; Likwiditeitsrisiko; Berou; Variansie; Waarde-onderhewig-aan-risiko; Kapitaal-onderhewig-aan-risiko; Stochastiese optimalisering; Diskrete tyd; Kontinue tyd; Diskontinue tyd; Sub-prima Verbandlenigskrisis.

Glossary

An *adjustable-rate mortgage* (ARM) is a mortgage whose rate is adjustable throughout its term.

An *interest-only ARM* allows the homeowner to pay just the interest (not principal) during an initial period.

Borrowers borrow from lenders while *lenders* lend to borrowers.

Central banks are primarily concerned with managing the rate of inflation and avoiding recessions.

Credit crunch is a term used to describe a sudden reduction in the general availability of loans (or credit) or sudden increase in the cost of obtaining loans from banks (usually via raising interest rates).

Foreclosure is the legal proceeding in which a mortgagee, or other loanholder¹, usually a lender, obtains a court ordered termination of a mortgagor's equitable right of redemption. Usually a lender obtains a security interest from a borrower who mortgages or pledges an asset like a house to secure the loan. If the borrower defaults and the lender tries to repossess the property, courts of equity can grant the owner the right of redemption if the borrower repays the debt. When this equitable right exists, the lender cannot be sure that it can successfully repossess the property, thus the lender seeks to foreclose the equitable right of redemption. Other loanholders can and do use foreclosure, such as for overdue taxes, unpaid contractors' bills or overdue homeowners' association (HOA) dues or assessments. The foreclosure process as applied to residential mortgage loans is a bank or other secured creditor selling or repossessing a parcel of real property (immovable property) after the owner has failed to comply with an agreement between the lender and borrower called a "mortgage" or "deed of trust". Commonly, the violation of the mortgage is a default in payment of a promissory note, secured by a lien on the property. When the process is complete, the lender can sell the property and keep the proceeds to pay off its mortgage and any legal costs, and it is typically said that "the lender has foreclosed its mortgage or lien". If the promissory note was made with a recourse clause then if the sale does not bring enough to pay the existing balance of principal and fees the mortgagee can file a claim for a deficiency judgment.

Subprime lending is the practice of making loans to borrowers who do not qualify for market interest rates owing to various risk factors, such as income level, size of the down payment made, credit history and employment status.

Securitization is a structured finance process, which involves pooling and repackaging of cash-flow producing financial assets into securities that are then sold to investors. In other

¹In law, a lien is a form of security interest granted over an item of property to secure the payment of a debt or performance of some other obligation. The owner of the property, who grants the lien, is referred to as the loanor and the person who has the benefit of the lien is referred to as the loanee.

words, securitization is a structured finance process in which assets, receivables or financial instruments are acquired, classified into pools, and offered for sale to third-party investment. The name “securitization” is derived from the fact that the form of financial instruments used to obtain funds from investors are securities.

The *delinquency rate* includes loans that are at least one payment past due but does not include loans somewhere in the process of foreclosure.

The *leverage* of a financial institution refers to its debt-to-capital reserve ratio. An institution is *highly leveraged* if this ratio is high.

Abbreviations

ABS - Asset-Backed Security;
AFC - Available Funds Cap;
AIG - American International Group;
ARM - Adjustable-Rate Mortgage;
BVP - Boundary Value Problem;
CDO - Collateralized Debt Obligation;
CDOs - Collateralized Debt Obligations;
CDS - Credit Default Swap;
CE - Credit Enhancement;
CLO - Collateralized Loan Obligation;
CRA - Credit Rating Agency;
CTD - Cheapest to Deliver;
FDIC - Federal Deposit Insurance Corporation;
GIG - Generalized Inverse Gaussian;
HJBE - Hamilton-Jacobi-Bellman Equation;
IB - Investing Bank;
IO - Interest-Only;
IR - Investor;
LIBOR - London Interbank Offered Rate;
MBS - Mortgage-Backed Security;
MBSs - Mortgage-Backed Securities;
MR - Mortgagor;
NIMS - Nett Interest Margin Security;
OAD - Originate-and-Distribute;
OC - Overcollateralized;
ODE - Ordinary Differential Equation;
OR - Originator;
PD - Probability of Default;
RMBS - Residential Mortgage-Backed Security;
RML - Residential Mortgage Loan;
SaR - Swaps at Risk;
SDE - Stochastic Differential Equation;
SMC - Subprime Mortgage Crisis;
SPV - Special Purpose Vehicle;
SR - Servicer;
VaR - Value at Risk;
WAC - Weighted Average Coupon;
XS - Excess Spread.

Basic Notations

A - The Maximum Protection from Swap Protection Seller;
 B - IB's Total Investment;
 B' - IB's Nett Investment;
 B^π - IB's Total Investment Under Strategy π ;
 B'^π - IB's Nett Investment Under Strategy π ;
 C^e - Claim by Equity Tranche Holder;
 C^m - Claim by Mezzanine Tranche Holder;
 C^s - Claim by Senior Tranche Holder;
 f^s - Servicing Fee;
 f - Actual Final Fund Level;
 $f^{\max}$ - Value of the Ex-post Optimal Final Level of Funds;
 $F(S)$ - Distribution Function of Losses in Continuous-Time;
 $\mathbf{F}$ - σ -Algebra;
 $\mathbb{F}$ - Filtration;
 $\tilde{H}$ - Distribution Function of Stochastic Rate of Return, r_t^I ;
 I_u - Securitized Subprime RML in Continuous-Time;
 I^f - Face Value of the RMBS Bonds;
 L_t - Lévy Process;
 M^f - Face Value of RML;
 M^o - Overcollateralization;
 M^r - RML Reference Portfolio;
 M_t - RML Extended to Mortgagors;
 m_t - Subprime RMBS Price with Jumps;
 $\mathbf{P}$ - Probability Measure;
 N - Poisson Process;
 P_p - CDSs Premium for Regret-Averse IB in Discrete-Time;
 P_0 - CDSs Premium for Risk-Averse IB in Discrete-Time;
 p^I - Subprime RMBS Price without Jumps;
 p^A - Low Default Probability;
 p^B - High Default Probability;
 R^c - CDSs Payout in Discrete-Time;
 R_v - Recovery Rate of RML;
 r_t^M - Rate of Return on Subprime RMLs;
 r_t^I - Rate of Return on Subprime RMBS;
 r^D - Rate of Return on Deposits;
 r^T - Treasuries Rate;
 $\hat{r}$ - Rate of Return on Safe Asset;
 r^R - Recovery Rate of Subprime RMBS;
 r_t^e - Initial Teaser Rate;

r_t^i - Index Rate;
 r^x - Excess Spread;
 $\bar{r}^M$ - WAC Paid Out to RML Holder;
 $\bar{r}^I$ - WAC Paid Out to RMBS Bond Holders;
 $\bar{r}^{Mn}$ - Nett WAC;
 r^a - Available Funds Cap;
 r^ψ - Step-up Rate;
 r^{Is} - Quoted Default Swap Spread;
 R - Level of Risk;
 S - IB's Subprime RMBSs Losses;
 $s(I, m)$ - Survival Probability;
 S_v - OR's RML Losses;
 S_v^c - The Loss on the Senior Tranche of RMBS CLO;
 S_v^s - The Loss on the Senior Tranche of RMBS;
 T - Treasuries;
 U - Utility Function;
 V_t - Discounted Flow of Profits;
 $W, \tilde{W}$ - GIG Diffusion Processes;
 Z - Brownian Motion;
 Ω - Sample Space;
 ϱ_t^ξ - Risk Margin;
 σ - RMBS Price Volatility;
 Π - IB's Payout;
 $\tilde{\Pi}_t$ - Expected Profits;
 Π_v - Payout to OR;
 Π_v^c - Payout to the RMBS CLO Holder on the Senior Tranche;
 Π_v^s - Payout to the RMBS Bond Holder on the Senior Tranche;
 $\bar{\pi}$ - Face Value of Amount Protected by Swap Protection Seller;
 ρ - Weight of Regret;
 Γ - Credit Rating;
 $\Theta(C(S))$ - CDSs Premium Paid by IB in Continuous-Time;
 ϕ - Deterministic Frequency of Poisson Process, $\tilde{N}$;
 $E[B_T^\pi]$ - Expected Value of Total IB's Investment, B_T^π ;
 $E[C(S)]$ - Expected Payout of CDSs in Continuous-Time;
 $E(M_v)$ - Value of New RML;
 $E[W]$ - Expected Value of GIG Diffusion Process, W ;
 E_v - The Value at which the First RMBS Tranche Detaches;
 E_v^c - The Value at which the first RMBS CLO Tranche Detaches;
 μ^I - The Rate of Cash Inflow from Depositors to IB for Investment in Securitized Subprime RMLs;
 $\tilde{\mu}^I$ - Expected Return on Subprime RMBS in Discontinuous-Time;

k_u - The Stochastic Rate of Cash Outflow for Fulfilling Depositor Obligations;

δ^r - Discount Rate;

Φ - Accrued Premium;

α_i - Jump Size of the Lévy Process, L_t ;

ν - Lévy Measure;

λ - Intensity.

List of Figures

- Figure 1.1: Diagrammatic Overview of RML Securitization Process;
Figure 1.2: Diagrammatic Overview of Borrowing Under Securitization Strategies;
Figure 1.3: Diagrammatic Overview of RMBSs Being Protected by CDSs;
Figure 1.4: Diagrammatic Overview of Risk and Return for IBs;
Figure 1.5: Diagrammatic Overview of a Subprime RMBS Structure;
Figure 1.6: Senior/Sub 6-Pack Structure vs. XS/OC Structure;
Figure 1.7: Sample Subprime RMBS Payments;
Figure 1.8: Subprime RMBS Interest Waterfall;
Figure 1.9: Allocation of Interest;
Figure 2.1: RML Securitization Risk;
Figure 2.2: Optimal Risk and Regret in Banking;
Figure 2.3: The Certainty Equivalent;
Figure 5.1: Bank Credit Default Swaps Rates 1 January 2007 to 30 June 2009;
Figure 5.2: The Liquidity Effect;

List of Tables

- Table 5.1: Variables for the Numerical Example;
Table 5.2: Parameter Choices;
Table 6.1: Effect of Regret on Risk Allocation and Liquidity;
Table 8.1: Computational Results;
Table 8.2: Differences in Economic Conditions Before and During the SMC.

Contents

1	INTRODUCTION	1
1.1	REVIEW OF THE LITERATURE	6
1.1.1	Brief Literature Review of the Subprime Mortgage Crisis	6
1.1.2	Brief Literature Review of Subprime RML Securitization	7
1.1.3	Brief Literature Review of Credit Default Swaps	8
1.1.4	Brief Literature Review of Lévy Processes	8
1.2	PRELIMINARIES	9
1.2.1	Preliminaries About Subprime RMLs	9
1.2.2	Preliminaries About Subprime RML Securitization	10
1.2.3	Preliminaries About Credit Default Swaps	12
1.2.4	Preliminaries About Subprime Risks	13
1.2.5	Preliminaries About Subprime RMBS Deals	15
1.2.6	Preliminaries About RMBS Principal and Interest Waterfalls	17
1.2.7	Preliminaries About Poisson, Jump-Diffusions and Lévy Processes	20
1.3	MAIN PROBLEMS AND OUTLINE OF THE THESIS	22
1.3.1	Main Problems	22
1.3.2	Outline of the Thesis	24
1.3.2.1	Outline of Chapter 2	24
1.3.2.2	Outline of Chapter 3	25
1.3.2.3	Outline of Chapter 4	25
1.3.2.4	Outline of Chapter 5	26
1.3.2.5	Outline of Chapter 6	27
1.3.2.6	Outline of Chapter 7	27
1.3.2.7	Outline of Chapter 8	27
2	DISCRETE-TIME MODELS FOR SUBPRIME RML SECURITIZATION	28

2.1	SUBPRIME RMBSs AND THEIR EXPECTED RATE OF RETURN . . .	29
2.2	SUBPRIME RML SECURITIZATION RISK IN DISCRETE-TIME	30
2.2.1	IB's Risk from RMLs, RMBSs and CLOs	31
2.2.2	Formalizing IB's Securitization Risk	32
2.3	REGRET IN BANKING	33
2.3.1	Subprime RMBSs and Treasuries with Regret	33
2.3.2	Risk Allocation Spread	33
2.3.3	Utility Functions	34
2.4	CREDIT DEFAULT SWAPS IN DISCRETE-TIME	35
2.5	IB'S OPTIMIZATION PROBLEMS	37
2.5.1	IB's Optimization Problem with Risk	37
2.5.2	IB's Optimization Problem with Risk and Regret	39
2.5.2.1	IB Hedging Against Securitization Risk	46
2.5.2.2	The Main Credit Default Swaps Result	49
3	CONTINUOUS-TIME MODELS FOR SUBPRIME RML SECURITI- ZATION	54
3.1	SUBPRIME RMBS PRICE PROCESS WITHOUT JUMPS	55
3.2	IB'S SUBPRIME RMBS LOSSES IN CONTINUOUS-TIME	55
3.3	CREDIT RATINGS	56
3.4	CREDIT DEFAULT SWAPS IN CONTINUOUS-TIME	56
3.5	IB'S PAYOUT UNDER SUBPRIME RML SECURITIZATION	57
3.6	STOCHASTIC OPTIMAL CREDIT DEFAULT INSURANCE PROBLEM	57
3.7	STATEMENT OF THE OPTIMAL CREDIT DEFAULT INSURANCE PROB- LEM	58
3.8	SOLUTION TO THE OPTIMAL CREDIT DEFAULT INSURANCE PROB- LEM	58
3.8.1	General Solution to the Optimal Credit Default Insurance Problem .	59
3.8.2	Optimal Credit Default Swap Contracts in Continuous-Time	62
3.8.3	Boundary Value Problem	64
3.8.4	Stochastic Optimal Credit Default Insurance with Exponential Utility	65
3.8.5	Stochastic Optimal Credit Default Insurance with Power Utility . .	67
3.8.6	Stochastic Optimal Credit Default Insurance with Logarithmic Utility	70
4	DISCONTINUOUS-TIME MODELS FOR SUBPRIME RML SECURI- TIZATION	73

4.1	SUBPRIME RMBS PRICE PROCESS WITH JUMPS	74
4.2	TREASURIES	76
4.3	IB'S SUBPRIME RMBS LOSSES IN DISCONTINUOUS-TIME	76
4.4	STOCHASTIC DYNAMICS OF IB'S INVESTMENT IN SUBPRIME RMBSs AND TREASURIES	77
4.5	IB'S OPTIMAL INVESTMENT IN SUBPRIME RMBSs AND TREASURIES	80
4.5.1	IB's Optimal Investment Problem with Variance	80
4.5.1.1	Statement of IB's Optimal Investment Problem with Variance	80
4.5.1.2	Solution of IB's Optimal Investment Problem with Variance	81
4.5.2	IB's Optimal Investment Problem with Swaps at Risk	83
4.5.2.1	Statement of IB's Optimal Investment Problem with Swaps at Risk	83
4.5.2.2	Solution of IB's Optimal Investment Problem with Swaps at Risk	84
4.5.3	Numerical Procedure	84
4.5.3.1	Gaussian Diffusion Model for Subprime RML Securitization	84
4.5.3.2	Numerical Algorithm for Problem 4.5.4	90
5	ANALYSIS OF SUBPRIME RML SECURITIZATION MODELS	93
5.1	ANALYSIS OF CHAPTER 1	95
5.1.1	General Discussion on Subprime RML Securitization	95
5.1.2	Subprime RMLs	96
5.2	ANALYSIS OF CHAPTER 2	96
5.2.1	Subprime Risk in Discrete-Time and the SMC	96
5.2.2	IB's Optimization Problems and the SMC	96
5.2.2.1	IB's Optimization Problem with Risk and the SMC	97
5.2.2.2	IB's Optimization Problem with Risk and Regret and the SMC	97
5.2.2.3	Credit Default Swaps in Discrete-Time and the SMC	97
5.2.3	Tranching, Counterparty and Liquidity Risks	99
5.2.3.1	Tranching Risk	99
5.2.3.2	Counterparty Risk	99
5.2.3.3	Liquidity Risk	100
5.3	ANALYSIS OF CHAPTER 3	101
5.3.1	Subprime Risks in Continuous-Time and the SMC	101

5.3.2	Credit Default Insurance and the SMC	102
5.3.3	IB's Payout Under Subprime RML Securitization and the SMC . . .	103
5.3.4	Numerical Example	103
5.3.5	Stochastic Optimal Credit Default Insurance and its Connections with the SMC	104
5.3.5.1	Statement and Proof of the Credit Default Insurance Problem	104
5.3.5.2	Optimal Credit Default Swap Contract and the SMC . . .	105
5.3.5.3	Optimal Credit Default Insurance with Exponential Utility and the SMC	105
5.3.5.4	Optimal Credit Default Insurance with Power Utility and the SMC	106
5.3.5.5	Optimal Credit Default Insurance with Logarithmic Utility and the SMC	107
5.4	ANALYSIS OF CHAPTER 4	107
5.4.1	Stochastic Dynamic of IB's Investment and the SMC	107
5.4.2	Optimization Problems for IB's Investment in Subprime RMBSs & Treasuries and the SMC	108
5.4.2.1	Statement of IB's Optimal Investment Problem with Vari- ance and the SMC	108
5.4.2.2	Solution of IB's Optimal Investment Problem with Variance and the SMC	108
5.4.2.3	Statement of IB's Optimal Investment Problem with SaR and the SMC	109
5.4.2.4	Solution of IB's Optimal Investment Problem with SaR and the SMC	109
6	CONCLUDING REMARKS AND FUTURE INVESTIGATIONS	110
6.1	CONCLUDING REMARKS	111
6.1.1	Concluding Remarks About Chapter 1	111
6.1.2	Concluding Remarks About Chapter 2	111
6.1.3	Concluding Remarks About Chapter 3	112
6.1.4	Concluding Remarks About Chapter 4	113
6.1.5	Concluding Remarks About Chapter 5	113
6.1.6	Concluding Remarks About Chapter 7	113
6.1.7	Concluding Remarks About Chapter 8	113
6.2	FUTURE INVESTIGATIONS	113

<i>CONTENTS</i>	0
7 BIBLIOGRAPHY	115
8 APPENDICES	121
8.1 APPENDIX A: ITÔ'S FORMULA FOR JUMP-DIFFUSIONS PROCESSES	121
8.2 APPENDIX B: COMPUTATIONAL EXAMPLE	122
8.3 APPENDIX C: ECONOMIC CONDITIONS BEFORE AND DURING THE SMC	123
8.4 APPENDIX D: COMPARISON WITH PRIME AND ALT-A DEALS . . .	123

Chapter 1

INTRODUCTION

1.1 REVIEW OF THE LITERATURE

- 1.1.1 Brief Literature Review of the Subprime Mortgage Crisis
- 1.1.2 Brief Literature Review of Subprime RML Securitization
- 1.1.3 Brief Literature Review of Credit Default Swaps
- 1.1.4 Brief Literature Review of Lévy Processes

1.2 PRELIMINARIES

- 1.2.1 Preliminaries About Subprime RMLs
- 1.2.2 Preliminaries About Subprime RML Securitization
- 1.2.3 Preliminaries About Credit Default Swaps
- 1.2.4 Preliminaries About Subprime Risks
- 1.2.5 Preliminaries About Subprime RMBS Deals
- 1.2.6 Preliminaries About RMBS Principal and Interest Waterfalls
- 1.2.7 Preliminaries About Poisson, Jump-Diffusions and Lévy Processes

1.3 MAIN PROBLEMS AND OUTLINE OF THE THESIS

- 1.3.1 Main Problems
- 1.3.2 Outline of the Thesis
 - 1.3.2.1 Outline of Chapter 2
 - 1.3.2.2 Outline of Chapter 3
 - 1.3.2.3 Outline of Chapter 4
 - 1.3.2.4 Outline of Chapter 5

1.3.2.5 Outline of Chapter 6

1.3.2.6 Outline of Chapter 7

1.3.2.7 Outline of Chapter 8

"On the face of it, the recent economic turmoil had something to do with foolish borrowers and foolish investors who were persuaded by clever intermediaries to borrow what they could not afford and invest in what they did not understand. Without the benefit of oversight bodies with the necessary sophistication, a significant disruption hit the nerve centre of the financial system in mid-2007 which triggered the problems."

– Ian Mann (Sunday Times), 2009.

"It's now conventional wisdom that a housing bubble has burst. In fact, there were two bubbles, a housing bubble and a financing bubble. Each fueled the other, but they didn't follow the same course."

– Wall Street Journal, 2007.

"Certainly the underwriting standards for a large proportion of the U.S. home mortgages originated in 2005 and 2006 would give most people a pause. The no-downpayment, no-documents and no-stated income-or-assets loans were unprecedented in the history of mortgage finance and clearly ripe for abuse."

– Prof. Linus Wilson (Louisiana at Lafayette), 2008.

"If a guy has a good investment opportunity and he can't get funding, he won't do it. And that's when the economy collapses."

– Prof. Frederic Mishkin (Stanford), 2008.

"Although there are only a few studies, the evidence to date is consistent with the experience of a quarter century of securitization working very well. The assertions of the *originate-to-distribute* view simply are not consistent with what we know. The idea that there is a moral hazard due to the alleged ability of originators to sell loans without fear of recourse, and with no residual risk, also assumes that the buyers of these loans are irrational. That may be but the irrationality, it turns out, had to do with the belief that house prices would not fall."

– Prof. Gary Gorton (Yale and NBER), 2008.

This main body of this thesis contains discussions about subprime RML securitization in discrete-, continuous- and discontinuous-time in Chapter 2, Chapter 3 and Chapter 4, respectively. The 2007-2009 subprime mortgage crisis (SMC) was preceded by a period of favorable macroeconomic conditions with strong growth and low inflation combining with low default rates, high profitability, strong capital ratios and strong innovation involving structured financial products in the banking sector. These conditions contributed to the SMC in that they led to overconfidence and increased *regret aversion* among investors such as investing banks (IBs). In the search for yield, the growth in structured financial products would have been high impossible without IBs' strong demand for high-margin, higher risk assets such as securities backed by subprime RMLs. Such securitization involves the pooling of RMLs that are subsequently repackaged into interest-bearing securities. The interest and principal payments from RMLs are passed through to credit market investors. The risks associated with RML securitization are transferred from originators (ORs) to special purpose vehicles (SPVs) – entities set up by financial institutions – and securitized RML bond holders such as IBs. RML securitization thus represents an alternative and diversified source of housing finance based on the transfer of credit risk (and possibly also tranching and counterparty risk). In this process, some agents assumed risks beyond their capabilities and capital base and found themselves in an unsustainable position once IBs became *risk averse*. A diagrammatic overview of the securitization of RMLs is given below.

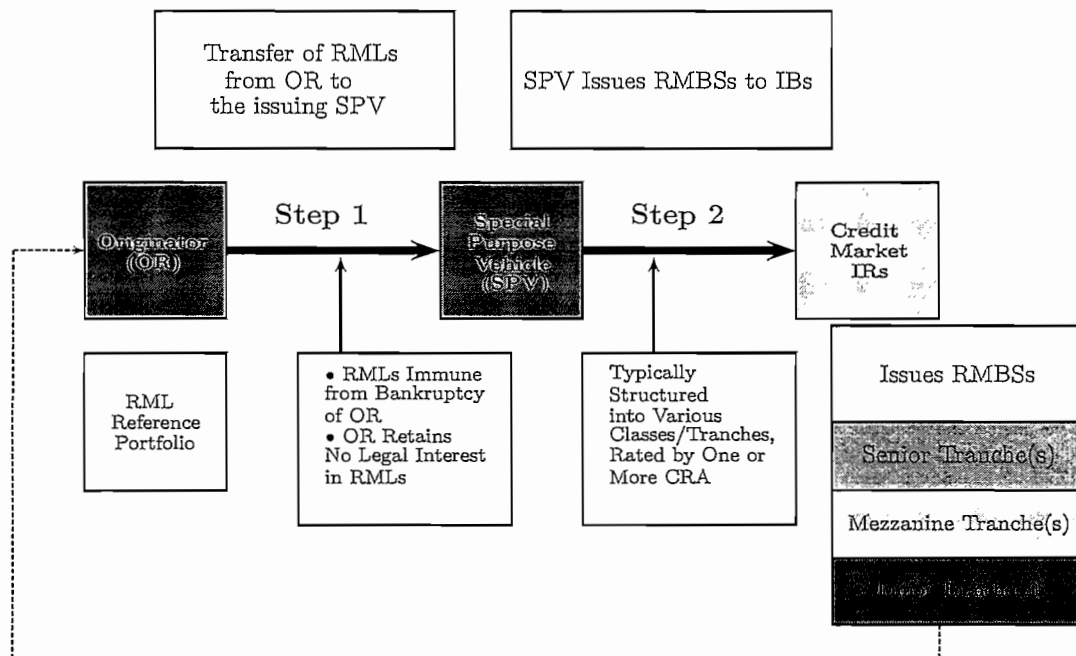


Figure 1.1: Diagrammatic Overview of the RML Securitization Process

In this thesis, we specifically investigate the securitization of subprime RMLs as illustrated in Figure 1.1. The first step in the process involves an OR that extends such loans that are subsequently removed from its balance sheet and pooled into RML reference portfolios. OR then sells these portfolios to SPV specifically to purchase RMLs and realize their off-balance-sheet treatment for legal and accounting purposes. Next, the SPV finances the acquisition of subprime RML portfolios by issuing tradable, interest-bearing securities that are sold to IBs. They receive fixed or floating rate coupons from the SPV account funded by cash flows generated by RML reference portfolios. In addition, servicers service the RML portfolios, collect payments from the original mortgagors, and pass them on – less a servicing fee – directly to SPV. Moreover, subprime RML securitization mainly refers to the securitization of such RMLs into residential mortgage-backed securities (RMBSs). For this reason we use the terms “securitized RML” and “RMBS” interchangeably. However, most of our arguments also apply to the securitization of RMBSs into RMBS collateralized loan obligations (CLOs) as well as RMBS CLOs into RMBS CLO²s. Unfortunately, the analysis in the latter cases is much more complicated and will not be attempted. The RMBSs themselves are structured into *tranches*. As in Figure 1.1, this thesis involves three such tranches: the *senior* (usually AAA rated and abbreviated as sen), *mezzanine* (usually AA, A, BBB rated and abbreviated as mezz) and *junior (equity) tranches* (usually BB, B rated and unrated and abbreviate as jun) in order of contractually specified claim priority. In the sequel, we denote the return on the RML reference portfolio by M^r , while C^s , C^m and C^e denote the claims by the senior (sen), mezzanine (mezz) and junior (jun) tranches, respectively. The following statements summarize the main interactions between these tranches.

If $M^r < C^s$, then $C^s = M^r$ and $C^m = C^e = 0$. If $M^r > C^s$, then C^s is paid out.

If $C^s < M^r \leq C^s + C^m$, then $C^m = M^r - C^s$. If $M^r > C^s + C^m$, then C^s and C^m is paid out.

If $M^r < C^s + C^m$, then $C^e = 0$. If $M^r \geq C^s + C^m$, then $C^e = M^r - C^s - C^m$.

At this stage, the location and extent of subprime risk cannot be clearly described. This is due to the chain of interacting securities that cause the risk characteristics to be opaque. Another contributing factor are the derivatives that resulted in negative basis trades moving CLO risk and credit derivatives that created additional long exposure to subprime RMLs. Determining the extent of the risk is also difficult because the effects on expected RML losses depend on house prices as the first order risk factor. Simulating the effects of this through the chain of interacting securities is very difficult.

By way of motivating our study and illustrating the aforementioned risk issues and their cascading effects, we consider payouts from an interacting subprime RML, a sen/sub tranche RMBS securitization of this single RML and a sen/sub tranche RMBS CLO, which has purchased the sen tranche of the RMBS. In our example, all payouts take place at time v .

The RML has a face value of M^f . At time v , the RML experiences a step-up rate, r^ψ , and will either be refinanced or not. If it is not refinanced, then it defaults, in which case OR will recover R_v . Therefore, OR will suffer a loss of S_v which is given by $S_v = M_v^f - R_v$, where S_v and R_v are the RML losses and recovery, respectively. In the case where no default occurs, the new RML is expected to be worth $E(M_v)$. If we assume no dependence of R_v and $E(M_v)$ on house prices, the payout to OR is given by $\Pi_v = \max[M_v, R_v]$, where M_v is the value of the new RML after refinancing. If $M_v < R_v$ then OR does not refinance and the mortgagor defaults. OR finances RML extensions via securitization, where the RML is sold at par of M^f . The subprime RMBS transaction has two tranches: the first tranche attaches at 0 and detaches at E_v , the second tranche attaches at E_v and detaches at the end value M^f . The face value of the sen tranche is the difference between the face value of RML and the first loss to be absorbed by the equity tranche, i.e., $M^f - E_v$. It then follows that the losses that may occur on a sen tranche is given by

$$S_v^s = \max[S_v - E_v, 0] \quad (1.1)$$

where E_v is the value at which the first RMBS tranche detaches. Here, the payout to the RMBS bond holder on the sen tranche has the form

$$\Pi_v^s = \min \begin{cases} \max[M_v^f - E_v, 0]; \\ M_v^f - E_v - S_v^s. \end{cases}$$

In this case, if $\max[M_v^f - E_v, 0] = M_v^f - E_v$, then

$$S_v^s \leq M_v^f - E_v.$$

This implies that

$$\Pi_v^s = M_v^f - E_v - S_v^s,$$

which, in turn, implies that

$$\Pi_v^s = \min[M_v^f - E_v, M_v^f - S_v].$$

Next, we consider a situation in which the sen tranche of the subprime RMBS is sold to a CLO, which has two tranches: the first tranche attaches at 0 and detaches at E_v^c ; the second tranche attaches at E_v^c and detaches at the end value $M^f - E_v$. We note that the size of the CLO is $M^f - E_v$ since it only purchases the sen tranche of the subprime RMBS. Moreover, the amount E_v^c will be less than E_v because the CLO portfolio is smaller; the sub tranche of the CLO could be large in percentage terms though. In this case, we have that the loss on the sen tranche is

$$S_v^c = \max \left[\min[S_v^s, M_v^f - E_v] - E_v^c, 0 \right]. \quad (1.2)$$

Furthermore, the payout to the RMBS CLO holder on this tranche is given by

$$\Pi_v^c = \min \begin{cases} \max[M_v^f - E_v - E_v^c, 0]; \\ M_v^f - E_v - E_v^c - S_v^c. \end{cases} \quad (1.3)$$

If we substitute (1.2) into (1.3), then Π_v^c takes the form

$$\Pi_v^c = \min \begin{cases} \max[M_v^f - E_v - E_v^c, 0]; \\ E_v - E_v^c - \max \left[\min[S_v^s, M_v^f - E_v] - E_v^c, 0 \right]. \end{cases} \quad (1.4)$$

Finally, substituting (1.1), we obtain

$$\Pi_v^c = \min \begin{cases} \max[M_v^f - E_v - E_v^c, 0]; \\ E_v - E_v^c - \max \left[\min[\max[\min[S_v^s, M_v^f - E_v] - E_v^c, 0], \right. \\ \left. M_v^f - E_v] - E_v^c, 0 \right]. \end{cases}$$

1.1 REVIEW OF THE LITERATURE

In this subsection, we present strands of literature related to the SMC, RML securitization, credit default swaps and Lévy process

1.1.1 Brief Literature Review of the Subprime Mortgage Crisis

The SMC began with the bursting of the U.S. housing bubble (see, for instance, [9] and [34]¹) and high default rates on subprime and ARMs. The working paper [16] provides evidence that the rise and fall of the subprime mortgage market follows a classic lending boom-bust scenario, in which unsustainable growth leads to the collapse of the market. Loan incentives, such as easy initial terms, in conjunction with an acceleration in rising housing prices encouraged borrowers to assume difficult mortgages on the belief they would be able to quickly refinance at more favorable terms. However, once housing prices started to

¹a quote from this article states that “It’s now conventional wisdom that a housing bubble has burst. In fact, there were two bubbles, a housing bubble and a financing bubble. Each fueled the other, but they didn’t follow the same course.”

drop moderately in 2006-2007 in many parts of the U.S., refinancing became more difficult. Defaults and foreclosure activity increased dramatically, as easy initial terms expired, home prices failed to go up as anticipated and ARM interest rates reset higher.

A model that has become important during this crisis is the Diamond-Dybvig model (see, for instance, [17] and [18]). Despite the fact that these contributions consider a simpler model than ours, they are able to explain important features of bank liquidity that reflect reality. The quarterly reports [21] and [22] of the Federal Deposit Insurance Corporation (FDIC) intimate that profits decreased from \$ 35.6 billion to \$ 19.3 billion during the first quarter of 2008 versus the previous year, a decline of 46 %. Foreclosures accelerated in the U.S. in late 2006 and triggered a global financial crisis through 2007 and 2008. During 2007, nearly 1.3 million U.S. housing properties were subject to foreclosure activity, up 79 % from 2006 (see [3] for more details). The mortgage lenders that retained credit risk were the first to be affected, as borrowers became unable or unwilling to make payments. Corporate, individual and institutional investors holding MBSs or CDOs faced significant losses, as the value of the underlying mortgage assets declined. Stock markets in many countries declined significantly.

1.1.2 Brief Literature Review of Subprime RML Securitization

Asset securitization began with the creation of private mortgage pools in the 1970s (see, for instance, [2]). In 1995, the Community Reinvestment Act was revised to allow mortgages to be securitized. In 1997, Bear Sterns was the first to take advantage of this law (see [5]). Under the guidelines of this act, the OR receives credit for originating subprime mortgages or buying mortgages on a whole loan basis but not holding subprime mortgage loans. This rewarded ORs for originating subprime mortgages, then selling them to others who would securitize them. Thus any credit risk from subprime mortgages was passed from OR to others, including financial institutions, SPV and investors globally. In this regard, the *originate-to-distribute* (OTD) model of lending as for RMBSs, where the OR sells its RMLs to various third party investors, has become a popular vehicle for credit and liquidity risk management. This method of lending was very popular in the RML market till the freeze began in June-July 2007.

As far as we know, the study of securitization problems from a mathematically rigorous point of view is virtually non-existent. Notwithstanding this, we note the contribution in [1] that elucidates connections between the SMC and securitization from a non-technical point of view. In the case of subprime banking models, we refer to our accepted book manuscript [53] and papers ([24], [55] and [44]) that deals with modeling aspects of subprime mortgage credit, subprime mortgage securitization, subprime bank bailouts (see, also, [54]) as well as interbank lending and credit crunches. In the aforementioned book and papers, we specifically employ discrete- and continuous-time modeling techniques to, amongst many other things, explore the connections between optimal securitization and the SMC. To the

best of our knowledge, Lévy process-driven models that deal with securitization and its relationship with the SMC were first introduced in [44]. However, the latter paper does not deal with optimization aspects of securitization. There are several references to support the adoption of stochastic models for subprime RMBS prices and IB's investment as well as subprime RMBS losses in this thesis. For our study, the most relevant of these are [15] that discusses bank asset prices such as subprime RMBSs prices that are driven by Brownian motion and, of course, [35] that is one of the standard references involving the stochastic dynamics of (bank) asset price processes.

1.1.3 Brief Literature Review of Credit Default Swaps

Credit default swaps (CDSs) are financial instruments that are used as a hedge and protection for debtholders, in particular subprime RMBS investors, from the risk of default (see, for instance, [30]). Like all swaps and other credit derivatives, CDSs may either be used to hedge risks (specifically, to insure IBs against default) or to profit from speculation. In the SMC, as the nett payout to IBs decreased because of subprime RMBS losses, the probability increased that protection sellers would have to compensate their counterparties (see [64] for further discussion). This created uncertainty across the system, as IBs wondered which agents would be required to pay to cover RML defaults. Our work has a strong connection with this issue via IB's payout model under RML securitization that incorporates CDS dynamics and the rate of cash outflow to fulfill depositor obligations. CDSs are largely not regulated. As of 2008, there was no central clearinghouse to honor CDSs in the event a party to a CDS proved unable to perform its obligations under the CDS contract. Required disclosure of CDS-related obligations has been criticized as inadequate (compare with [20] and [30]).

1.1.4 Brief Literature Review of Lévy Processes

Although not explicitly discussed, some of the contributions that are pertinent to Lévy processes are [13], [57] and [59]. Our contribution is comparable with the above contributions since we consider general Lévy process-driven subprime banking models that accommodate jumps in IB's investment in RMBSs and Treasuries. For instance, one of the main novelties of [10] is the solution of an optimal control problem involving bank reserves and a rate of depository consumption that is of importance during a (random) audit of the reserve requirements. Here, the specific choice of a power utility function is made in order to obtain an analytic solution in a Lévy process setting. In addition, [10] (see, also, [40]) solves an optimal auditing time problem for the Basel II capital adequacy requirement by making use of Lévy process-based models.

1.2 PRELIMINARIES

In this section, we provide preliminaries about subprime RMLs and their securitization, CDSs and risks.

1.2.1 Preliminaries About Subprime RMLs

Adjustable-rate mortgages (ARMs) are complex financial instruments with payoff features similar to those of interest rate derivatives. By contrast to a fixed-rate mortgage (FRM), MRs holding ARMs retain most of the interest rate risk, subject to a collar (floor and cap). Note that most mortgagors are not in a position to easily hedge away this interest rate risk. RMLs are usually hybrid loans since they incorporate the features of both FRMs and ARMs.

In the sequel, the monthly RML repayment is initially based on a *teaser* interest rate, r^ϵ , that is fixed for the first two (for 2/28 RMLs) or three (for 3/27 RMLs) years, and is lower than what MR would pay for a 30-year FRM. At the beginning of period t , OR extends subprime RMLs, M_0 , at the *subprime RML rate*, r_t^M , that may coincide with an initial teaser rate, r^ϵ , given by

$$r_t^\epsilon = r_t^i + \varrho_t^\epsilon, \quad (1.5)$$

where r^i is the index rate (i.e., 6-month LIBOR) and ϱ^ϵ is the margin or risk premium for r^ϵ . By 2006, a fifth of all new RMLs were subprime. The interest rates on many of these were adjustable, unlike those on most U.S. mortgages. Low r_t^ϵ were charged for a while before higher, market-based rates kicked in. An estimated one-third of subprime RMLs originated between 2004 and 2006 had $r_t^\epsilon < 4\%$, which then increased significantly after some initial period, as much as doubling the monthly payment. Teaser rate RML products artificially inflated the U.S. home-ownership market.

During the SMC, many new mortgagors eventually had trouble making their monthly repayments when house prices started to decrease and the teaser rate, r_t^ϵ , increased to the step-up rate, r^ψ , in period $t + 1$. However, after this initial period, during period $t + 1$, the monthly payment may be based on a higher (step-up) interest rate, r^ψ , equal to the value of r_{t+1}^i plus a margin, ϱ^ψ , that is fixed for the remaining life (in our case, 28 years for 2/28 RMLs and 27 years for 3/27 RMLs) of the RML. Symbolically, this means that

$$r_{t+1}^\psi = r_{t+1}^i + \varrho_{t+1}^\psi, \quad (1.6)$$

where ϱ^ψ is the margin or risk premium for r^ψ . This interest rate is updated every six months for the life of the RML, and is subject to limits called *adjustment caps* on the

amount that it can increase: the cap on the first adjustment is called the *initial cap*; the cap on each subsequent adjustment is called the *period cap*; the cap on the interest rate over the life of the loan is called the *lifetime cap*; and the floor on the interest rate is called the *floor*.

Before and during the SMC, interest rate cuts were made in order to lower subprime RML rates and stimulate the economy, respectively. Before the SMC, the average difference between prime and subprime RML interest rates (the *subprime markup*) declined quite dramatically. In other words, the risk premium, ϱ , required by OR to offer a subprime RML declined. This continued to occur during the SMC even though the level of macroeconomic activity of subprime MRs and the quality of subprime RMLs, both declined.

1.2.2 Preliminaries About Subprime RML Securitization

A diagrammatic overview of borrowing under RML securitization strategies may be represented as follows.

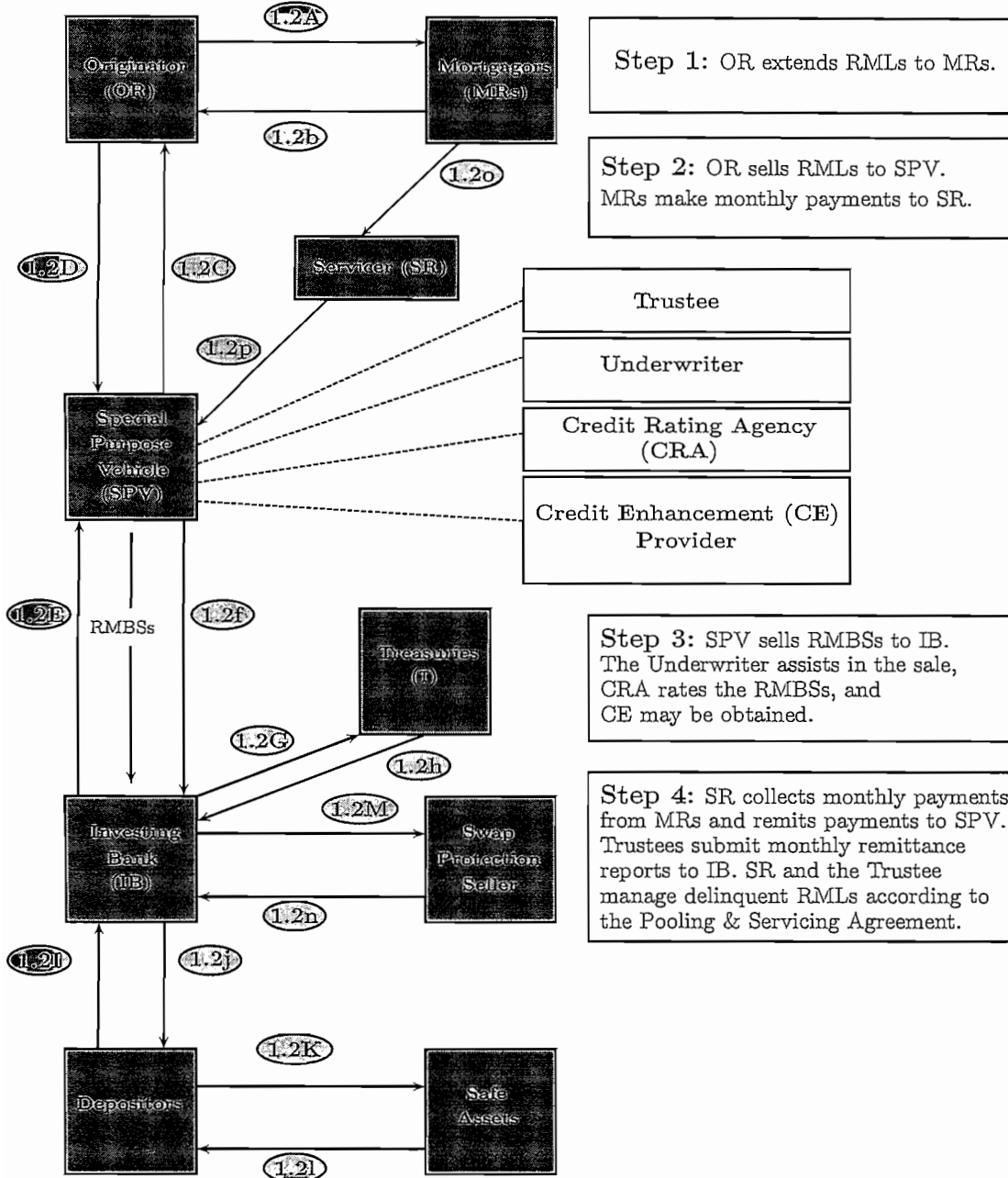


Figure 1.2: Diagrammatic Overview of Borrowing Under RML Securitization Strategies

From Figure 1.2, at the outset, OR extends RMLs, M_t , to mortgagors (see 1.2A). By agreement, mortgagors will be required to pay an interest rate, r_t^M , on their RMLs (compare

with 1.2b). Next, OR pools its RMLs, and sells them to SPV (1.2D). The SPV pays OR an amount which is slightly greater than the value of the pool of RMLs as in 1.2C. In addition, the SPV divides this pool into sen, mezz and jun tranches which are exposed to differing levels of credit risk. Moreover, the SPV sells these tranches as securities backed by subprime RMLs to IB (see 1.2E). IB is paid out at an interest rate, r^I , that is determined by the RML default rate, prepayment and foreclosure (see 1.2f). On the other hand, IB has an option of investing in Treasuries with a return of r^T (see 1.2G and 1.2h). The deposits attracted by IB are paid a stochastic rate, r^D , (see 1.2I and 1.2j). Depositors may also invest funds in safe assets, where their realized rate of returns, $\hat{r} \geq 1$, are known in advance (see 1.2K and 1.2l). IB may attract depositors because of deposit insurance and the possibility that it pays higher rates than those for riskless assets, i.e., $r^D > \hat{r}$. In anticipation of losses arising from investment in securitized subprime RMLs, IB purchases credit protection from a swap protection seller (see 1.2M). For the recovery rate, r^R , 1.2n represents payment, $1 - r^R$, made by the swap protection seller after a *credit event*². More is said about CDSs in Subsection 1.2.3. Furthermore, 1.2o is the interest rate, r^M , paid by mortgagors to the servicer of the RMLs. For the servicing fee, f^s , 1.2p is the interest rate, $r^M - f^s$, passed by a servicer of the RMLs to the SPV.

1.2.3 Preliminaries About Credit Default Swaps

In this subsection, a diagrammatic overview of RMBSs being protected by CDSs is provided. Our dynamic model allows for protection against securitized RML losses via CDS contracts. The CDS counterparty, IB, who is the protection buyer makes a regular stream of payments, known as the *premium leg* (see 1.3A) to the RMBS SPV. This SPV, in turn, makes regular coupon payments to the protection seller (refer to 1.3B). These payments are made until a credit event occurs or until maturity, whichever happens first. The size of premium payments is dependent on the quoted *default swap spread* which is paid on the face value of the protection and is directly related to credit ratings. If there is no credit event, the seller of protection receives the periodic fee from the buyer, and profits if the RML reference portfolio remains fully functional through the life of the contract and no payout takes place. However, the protection seller is taking the risk of big losses if a credit event occurs. Depending on the terms agreed upon at the onset of the contract, when such an event takes place, the protection seller may deliver either the current cash value of the referenced bonds or the actual bonds to the protection buyer via the RMBS SPV (refer to 1.3C and 1.3D). This payment to the protection buyer, is known as the *protection leg* (see 1.3D). It equals the difference between par and the price of the cheapest to deliver (CTD) asset associated with the RML portfolio on the face value of the protection and compensates the protection buyer for the RML loss. The value of a CDS contract fluctuates based on the increasing or decreasing probability that a RML reference portfolio will have a credit event (compare

²A credit event is a legally defined event that typically includes bankruptcy, failure-to-pay and restructuring.

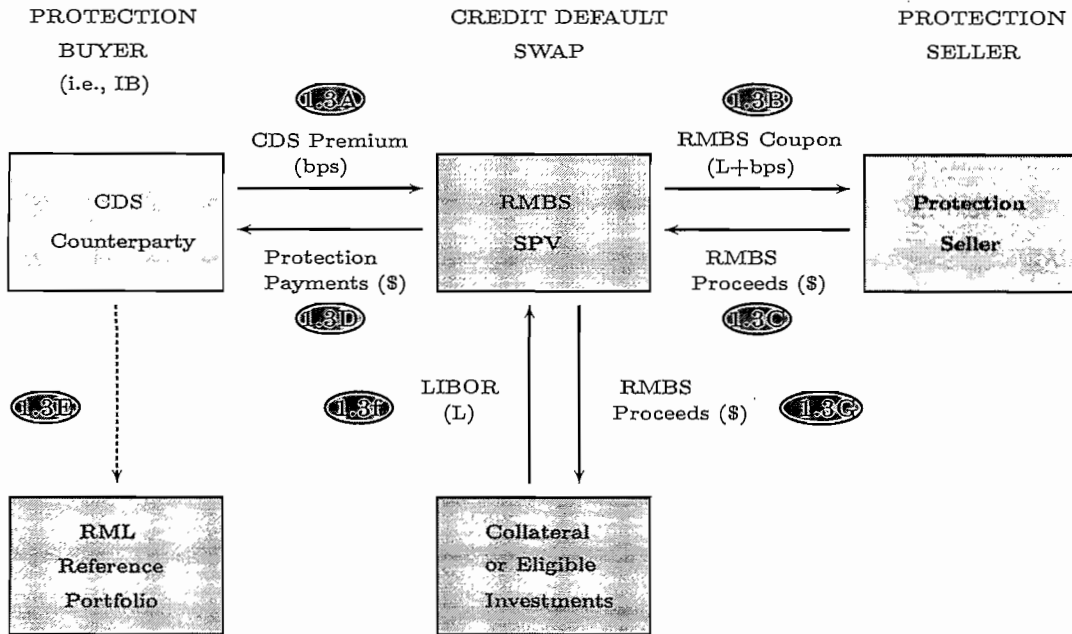


Figure 1.3: Diagrammatic Overview of RMBSs Being Protected by CDSs

with 1.3E). Increased probability of such an event would make the contract worth more for the buyer of protection, and worth less for the seller. The opposite occurs if the probability of a credit event decreases. Collateral or eligible investments are highly rated, highly liquid financial instruments purchased from the sale proceeds of the initial RMBS (represented by 1.3G). These investments contribute to the index portion (see 1.3f) of the RMBS coupon and provides protection payments or the return of principal to RMBS bond holders.

1.2.4 Preliminaries About Subprime Risks

The main subprime RML risks that can be identified from the discussions above are credit (including, prepayment), tranching, counterparty, liquidity, price, interest rate, systemic and maturity mismatch risks. *Credit risk* emanates from the inability of subprime mortgagors to make regular repayments on the underlying RML portfolio under any interest rate regime. This risk category generally includes both default and delinquency risk. *Prepayment risk* results from the ability of the subprime mortgagor to repay his/her RML after an interest rate change – usually from teaser to step-up rate – has been implemented by OR. *Counterparty risk* refers to the ability of economic agents – such as ORs, mortgagors, servicers, investors, SPVs, trustees, underwriters and depositors – to fulfill their obligations towards each other. *Liquidity risk* arises from situations in which SPV as a holder

of RMBSs cannot trade because no economic agent in the credit market is willing to do so. We consider *price risk* to be the risk that RML securitizations will depreciate in value, resulting in financial losses, markdowns and possibly margin calls. Subcategories of price risk are valuation risk (resulting from the valuation of long-term RML investments) and re-investment risk (resulting from the valuation of short-term RML investments). *Interest rate risk* arises from the adjustable and unpredictable nature of subprime RMBS interest rates, as was observed before. The aggregate effect of these and other risks has recently been called *systemic risk*, which refers to when formerly uncorrelated risks shift and become highly correlated, damaging the entire banking system. A risk that is also of issue for RMBSs is *maturity mismatch risk* that results from the discrepancy between the economic lifetimes of RMBSs and the investment horizons of IBs. In this thesis, our main interests are in credit, tranching, counterparty and liquidity risks. For sake of argument, risks falling in these categories are cumulatively known as *subprime risks* – just called *risks* hereafter.

In Figure 1.4 below, we provide a diagrammatic overview of the aforementioned risk and its relationship with returns for IB.

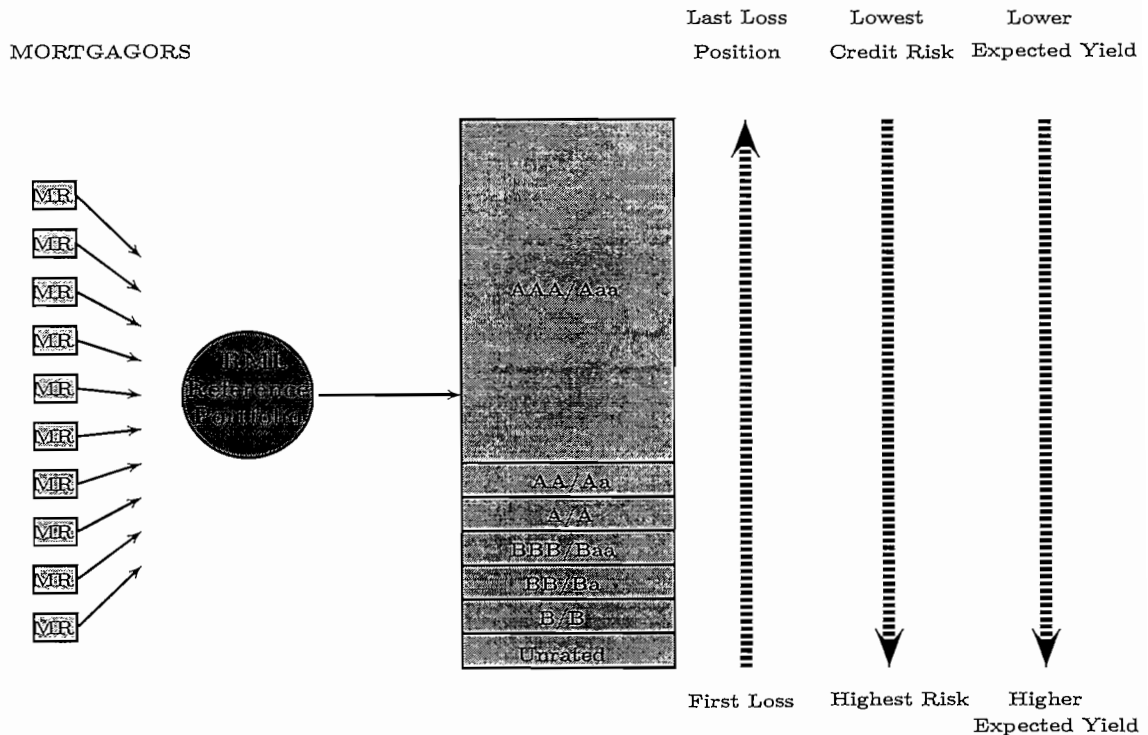


Figure 1.4: Diagrammatic Overview of Risk and Return for IBs

1.2.5 Preliminaries About Subprime RMBS Deals

The RMBS structure can be explained with the help of diagrams due to [33] (see, also, [28]). Figure 1.5 below provides a diagrammatic overview of the structure of a subprime RMBS deal.

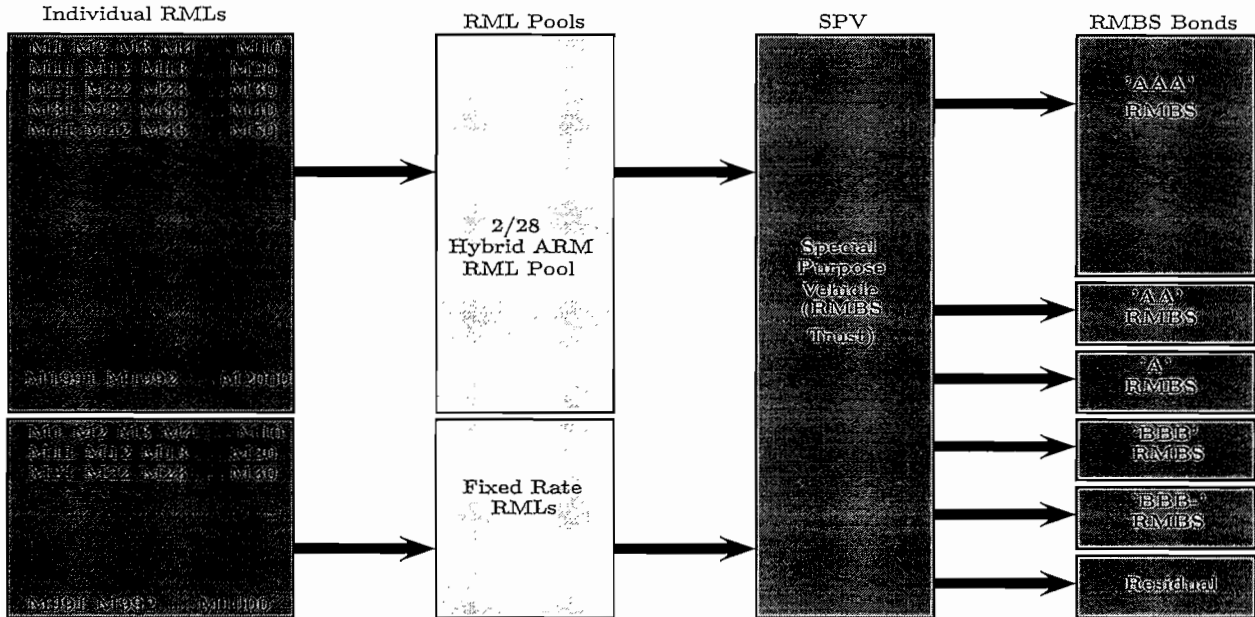


Figure 1.5: Diagrammatic Overview of a Subprime RMBS Structure; Source: [33].

RMBSs mainly use one or both of the sen/sub shifting of interest structure, sometimes called the *6-pack* structure (with 3 mezz and 3 sub RMBS bonds junior to the AAA bonds), or an XS/OC structure (see, for instance, [6]). Here, XS and OC denote *excess spread* and *overcollateralization*, respectively. Like sen/sub deals, XS is used to increase OC, by accelerating principal payments on sen RMBS bonds via sequential amortization; a process known as *turboing*. An OC target is a fraction of the original RML balance, and is designed to be in the second loss position against collateral losses with the interest-only strip (IO) being first. Typically, the initial OC amount is less than 100 % of the OC target, and it is then increased over time via the XS until the target is reached. When this happens, the OC is said to be *fully funded* and Nett Interest Margin Securities (NIMSS) can begin to receive cash flows from the deal. Once the OC target has been reached, and subject to certain performance tests, XS can be released for other purposes, including payment to the residual holder. OC implies that initial deal assets exceed liabilities so that

$$M^f \geq I^f, \quad M^o = M^f - I^f, \quad (1.7)$$

where M^f is the face value of the subprime RML reference portfolio (collateral), I^f is the face value of the RMBS bonds (liabilities) and M^o is the OC that can be created in either of two ways. It can be accumulated over time using XS or it is part of the deal from the beginning when (1.7) holds. Since credit risk mitigation is critical, in addition to a sen/sub structure (as in prime RMBSs), subprime RMBS bonds have an extra layer of support that arises from XS. In this regard, we have that

$$r^x = \bar{r}^M - \bar{r}^I, \quad (1.8)$$

where r^x is the XS, $\bar{r}^M$ is the weighted average coupon (WAC) paid into the deal from the subprime RML reference portfolio and $\bar{r}^I$ is the WAC paid out to RMBS bond holders. At the conclusion of the deal, the remaining OC described in (1.7) reverts to an equity claim.

The lock-out and step-down provisions are common structural features of RMBS deals. The *lock-out* provision locks out the mezz and sub bonds from receiving principal payments and prepayments for a period of time after the deal is initiated. This means that during the lock-out period, amortization is sequential³. The lock-out period, and other details, differ depending on the type of deal collateral. During the deal, the XS/OC feature of RMBSs leads to an accumulation of credit enhancement (CE) from the RML reference portfolio itself. Prior to the step-down date, the sen bonds receive 100 % of the principal payments. When the sen bonds are completely amortized away, prepaid principal continues to sequentially amortize, with the next class being the outstanding mezz bonds. After the lock-out period, deals are allowed to *step-down*, i.e., principal payments can be distributed to the sub bonds provided that CE limits are twice the original levels and the deal passes other performance criteria, measured by triggers. The step-down date in an XS/OC deal is the later of a specified month (e.g., 36 months) and the date at which the sen CE reaches a specified level (e.g., 52 %).

The allocation of CE over time depends on triggers that reflect the credit quality of the subprime RML reference portfolios. Under certain circumstances, triggers will cause a reallocation of principal to protect or increase subordination levels. Generally speaking, the two types of triggers are *delinquency* and *loss* triggers. A trigger is said to *pass* if the collateral does not breach the specified constraints, and to *fail* if those conditions are hit or breached. If a trigger fails, principal payments to mezz and sub bonds are delayed or stopped, preventing a reduction of CE for the sen bonds. Loss triggers are target levels of cumulative losses as of specific dates after the RMBS deal was initiated. XS builds up

³Sequential amortization means that there is a sequential elimination of RMBS bond liabilities in regular payments over a specified period of time.

throughout the deal with a CE threshold (target) level eventually being attained. Once this threshold is breached, XS can be paid to the residual holder, and becomes unavailable to cover RML losses. The aforementioned triggers have many complicated features⁴ (see, e.g., [36], [37] and [38]). Moreover, there may be cross-collateralization, where some deals contain multiple RML groups. After interest payments are made on RMBS bonds in one group, funds that remain can be used to pay interest to bonds in another group.

Figure 1.6 below displays the two types of deal structures, viz., sen/sub and OC structures.

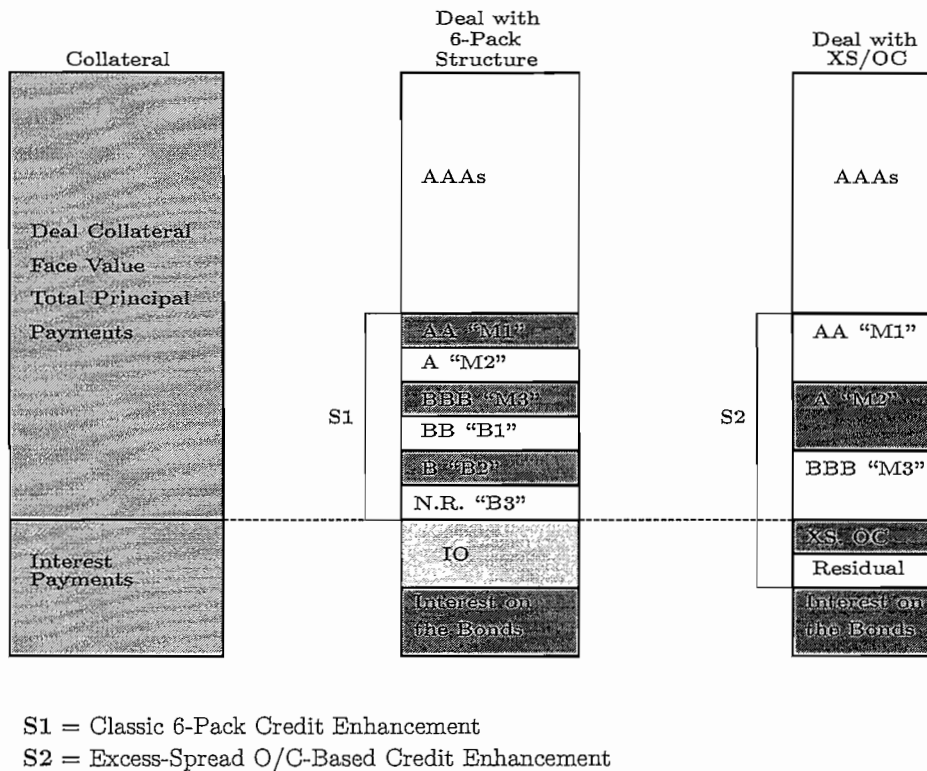


Figure 1.6: Senior/Sub 6-Pack Structure vs. XS/OC Structure; Source: UBS.

1.2.6 Preliminaries About RMBS Principal and Interest Waterfalls

As is shown in Figure 1.7, principal waterfalls are usually paid sequentially for the first 36 months. This means that all scheduled principal and prepayments are utilized to repay the sen bond holders first, until they are paid in full. Then, these payments go to the next senior RMBS bond holder, until they are fully paid. This process repeats itself until eventually

⁴For example, the loss trigger in months 1-36 might be 3.75 %, rise to 6.75 % in months 37-50, 6.85 % in months 51-72, and stay flat at 7.25 % thereafter.

all bond holders are fully compensated.

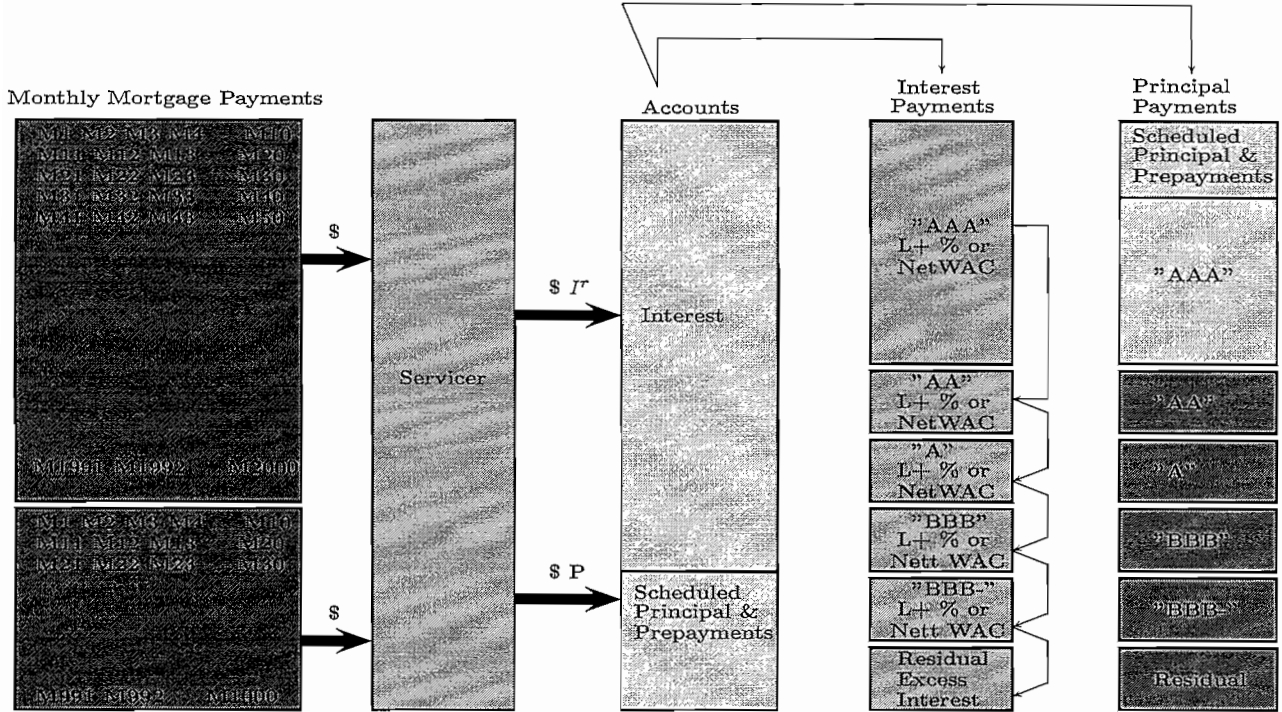


Figure 1.7: Sample Subprime RMBS Payments; Source: [33].

As discussed before, after the first 36 months (Scenario 1 below), CE steps down, if certain performance tests have been met (Scenario 2 below). For example, if OC targets have been met, the CE steps down by repaying sub bonds holders. OC targets are set to double the original subordination.

Interest waterfalls involve sequential interest pay outs to RMBS bond holders, r^I , capped at the weighted average RML coupon nett expenses (Nett WAC), $\bar{r}^{Mn}$, or available funds cap (AFC), r^a , so that

$$r^I = \max[\bar{r}^{Mn}, r^a].$$

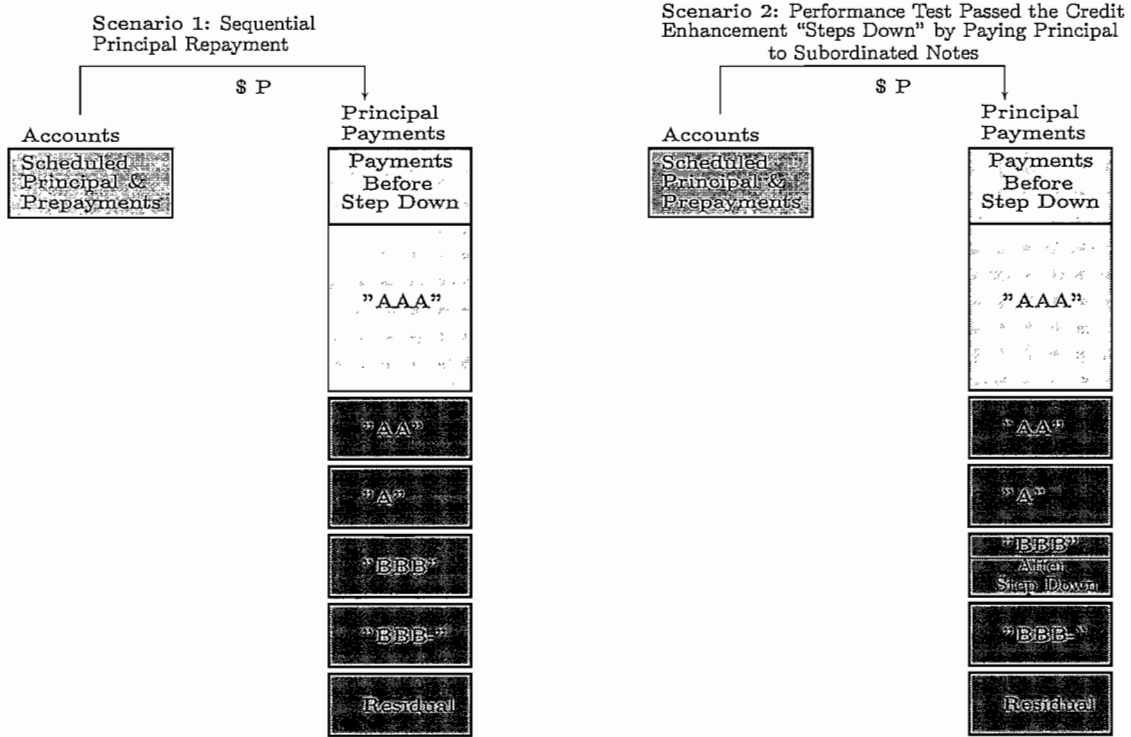


Figure 1.8: Subprime RMBS Interest Waterfall; Source: [33].

As we have seen before, XS is the remaining interest, r^x , which contributes to the interest collection account, after paying RMBS bond holders regular interest. The first use of XS is to cover realized RML reference portfolio (collateral) losses. Secondly, r^x is used to cover any interest shortfalls in the situation where

$$\bar{r}^{Mn} \leq r^I,$$

with $\bar{r}^{Mn}$ and r^I being the nett WAC and RMBS bond coupon, respectively. Lastly, the remaining XS accrues to the holder of the residual bond which is usually the subprime RML originator.

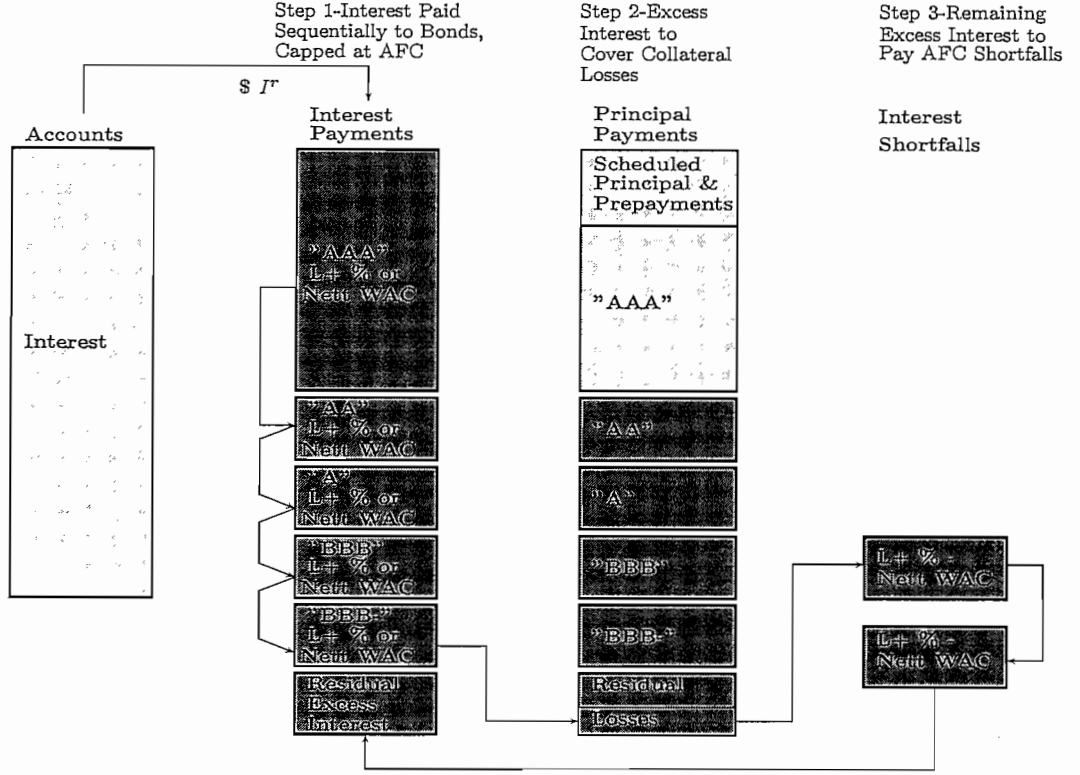


Figure 1.9: Allocation of Interest; Source: [33].

1.2.7 Preliminaries About Poisson, Jump-Diffusions and Lévy Processes

The preliminaries presented in this subsection are important for a number of reasons. Firstly, a Poisson process is a stochastic process with a discontinuous path and is used as a building block for constructing more complex jump processes, e.g., Lévy processes. Moreover, jump diffusions are solutions of SDEs that are driven by Lévy processes. Since a Lévy process can be expressed as the linear sum of t , a Brownian motion, Z , and a pure jump process, jump diffusions represent a natural and useful generalization of Itô diffusions.

Throughout our contribution, we suppose for the sample space Ω , σ -algebra $\mathbb{F}$, filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and real probability measure $\mathbb{P}$, that $(\Omega, \mathbb{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space. In this subsection, for sake of completeness, we firstly provide general descriptions of different types of Poisson processes and define a Lévy process and its measure as well as Lévy-Itô decomposition.

Definition 1.2.1 (Poisson Process): Let $(t_j)_{j \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_k = \sum_{j=1}^k t_j$. The process $(N_t)_{t \geq 0}$ which is given by

$$N_t = \sum_{k=1}^{\infty} 1_{t \geq T_k}$$

is called a Poisson process with intensity λ .

In line with the definition above, a Poisson process can be considered to be a counting process which counts the number of random times, T_k , which occur between 0 and t , when $(T_k - T_{k-1})_{k \geq 1}$ is an independent, identical distributed (iid) sequence of exponential variables.

Definition 1.2.2 (Compound Poisson Process): A compound Poisson process with intensity, $\lambda > 0$, and jump size distribution function, $\tilde{F}$, is a process, N_t^c , which is given by

$$N_t^c = \sum_{j=1}^{N_t} \tilde{J}_j$$

where $\tilde{J}_j$ represents the jump sizes which are iid with distribution function $\tilde{F}$ and N_t^c is a Poisson process with intensity λ , which does not depend on $(\tilde{J}_j)_{j \geq 1}$.

Next, we assume that $\phi(\xi)$ is the characteristic function of a distribution. If for every positive integer n , $\phi(\xi)$ is also the n -th power of a characteristic function, we say that the distribution is *infinitely divisible*. For each infinitely divisible distribution, a stochastic process $L = (L_t)_{0 \leq t}$ called a *Lévy process* exists.

Definition 1.2.3 (Lévy Process): A Lévy process *initiates at zero, has independent and stationary increments and has $(\phi(u))^t$ as a characteristic function for the distribution of an increment over $[s, s+t]$, $0 \leq s, t$, such that $L_{t+s} - L_s$.*

Next, we provide more important definitions and a useful result.

Definition 1.2.4 (Càdlàg Stochastic Process): A stochastic process X is said to be càdlàg if it almost surely (a.s.) has sample paths which are right continuous with left-hand limits.

The jump process $\Delta L = (\Delta L_t)_{t \geq 0}$ associated with a Lévy process, L , is denoted by $\Delta L_t = L_t - L_{t-}$, for each $t \geq 0$, where $L_{t-} = \lim_{s \uparrow t} L_s$ is the left limit at t . Let $L = (L_t)_{0 \leq t \leq \tau}$

with $L_0 = 0$ a.s. be the càdlàg version of a Lévy process. Also, we assume that the Lévy measure, ν , satisfies the following integrals

$$\int_{|y|<1} |y|^2 \nu(dy) < \infty, \quad \int_{|y|\geq 1} \nu(dy) < \infty. \quad (1.9)$$

In addition, the following definition of the Lévy-Itô decomposition is important in our analysis.

Definition 1.2.5 (Lévy-Itô Decomposition): *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ and ν be its Lévy measure which satisfies the integrals given by (1.9). Then there exist a constant, $a \in \mathbb{R}$, and Brownian motion, Z_t , such that the Lévy-Itô decomposition is given by*

$$L_t = at + bZ_t + N_t^{cf} + \lim_{\varepsilon \downarrow 0} N_t^{c\varepsilon},$$

where N_t^{cf} is a compound Poisson process with a finite number of terms and $N_t^{c\varepsilon}$ is also a compound Poisson process. However, there can be infinitely many small jumps.

An implication of the Lévy-Itô decomposition is that every Lévy process is a combination of a Brownian motion and a sum of independent compound Poisson processes. In line with this, every Lévy process can be approximated as a jump diffusion process, viz., by the sum of a Brownian motion with drift and a compound Poisson process. In our contribution, we shall consider the Lévy-Itô decomposition of the form

$$dL_t = \sigma dZ_t + \sum_{i=1}^n \left(\alpha_i dN_i(t) - \alpha_i \lambda_i dt \right), \quad t \geq 0, \quad L_0 = 0, \quad (1.10)$$

where $\sigma > 0$ is a volatility, $n \in \mathbb{N}$, and for $i = 1, \dots, n$ the process N_i is a homogeneous Poisson process with intensity λ_i . It counts the number of jumps of size α_i of L_t .

1.3 MAIN PROBLEMS AND OUTLINE OF THE THESIS

In this subsection, we state the main problems and provide an outline of the thesis.

1.3.1 Main Problems

Our general objective is to investigate aspects of subprime RML securitization as well as their connections with the SMC. In this regard, specific research objectives are listed below.

Problem 1.3.1 (Utility Function of IB Funds Under Regret): *Can we choose a utility function that incorporates IB's risk allocation preferences in a regret framework ? (Subsection 2.3.3 in Chapter 2).*

Problem 1.3.2 (IB's Optimization Problem with Risk): *Can we solve IB's optimization problem that determines the optimal leverage profit subject to unanticipated deposit withdrawals and risk ? (Subsection 2.5.1 in Chapter 2).*

Problem 1.3.3 (IB's Optimization Problem with Risk and Regret): *Can we solve IB's optimization problem that determines the optimal allocation of funds by IB between subprime RMBSs and Treasuries under risk and regret ? (Theorem 2.5.3 in Subsection 2.5.2 of Chapter 2).*

Problem 1.3.4 (Credit Default Swaps): *How can regret influence IB's view of the rate of return on CDSs, as measured by a premium that risk- and regret-averse IBs are willing to pay for protection ? (Theorem 2.5.6 in Subsection 2.5.2.2 of Chapter 2).*

Problem 1.3.5 (Discrete-Time Securitization Models and the SMC): *How does IB's optimization problems and risk- and regret-aversions relate to the SMC ? (Subsection 5.2 in Chapter 5).*

Problem 1.3.6 (IB's Payout From Investment in Securitized Subprime RMLs): *Can we construct a stochastic dynamic model to describe IB's payout from subprime RML securitization in continuous-time ? (Subsection 3.5 in Chapter 3).*

Problem 1.3.7 (Stochastic Optimal Credit Default Insurance Problem): *Which decisions about the rate of cash outflow for fulfilling depositor obligations, value of IB's investment in securitized subprime RMLs and credit default insurance must be made in order to attain an optimal payout for IB ? (Theorem 3.8.1 in Subsection 3.8 of Chapter 3).*

Problem 1.3.8 (Continuous-Time Securitization Models and the SMC): *How does the stochastic optimal credit default insurance problem solved in Subsection 3.8 relate to the SMC ? (Subsection 5.3 in Chapter 5).*

Problem 1.3.9 (Dynamic Model of IB's Investment in Subprime RMBSs and Treasuries): *Can we construct a stochastic dynamic model of IB's investment in subprime RMBSs and Treasuries which involves the proportion of funds invested as well as the nett subprime RMBS losses in a Lévy process setting ? (Subsection 4.4 in Chapter 4).*

Problem 1.3.10 (IB's Optimal Investment Problem with Variance): *Can we solve IB's optimization problem that determines the optimal proportion of funds invested in subprime RMBSs and Treasuries for credit risk with variance ? (Theorem 4.5.2 in Subsection 4.5.1 of Chapter 4).*

Problem 1.3.11 (IB's Optimal Investment Problem with Swaps at Risk): *Can we solve IB's optimization problem that determines the optimal proportion of funds invested in subprime RMBSs and Treasuries with swaps-at-risk ? (Subsection 4.5.2 in Chapter 4).*

Problem 1.3.12 (Discontinuous-Time Securitization Models and the SMC): *How does IB's optimal investment problems solved in Subsections 4.5.1 and 4.5.2 relate to the SMC ? (Subsection 5.4 in Chapter 5).*

1.3.2 Outline of the Thesis

The current chapter is introductory in nature. The remaining chapters of the thesis are structured as follows.

1.3.2.1 Outline of Chapter 2

In Chapter 2, we present discrete-time models for subprime RML securitization. The work in this chapter is based on the research contributions [42], [48], [55] and [52]. Pertinent facts about subprime RMBSs issued by the SPV and their rate of return are presented in Subsection 2.1 in discrete-time. In Subsection 2.2, we determine the optimal level of IB's risk (see Proposition 2.2.1). The impact of regret on allocation of funds by IB is outlined in more detail in Subsection 2.3. In this regard, Subsection 2.3.1 analyzed the possible scenario between the subprime RMBSs rate of return, r^I , and Treasuries rate, r^T . In particular, it gives the mathematical formulation of the ex-post optimal final level of funds, i.e., the fund level that IB could have attained if it had made the optimal choice with respect to the realized state of the economy. Subsection 2.3.2 illustrates a situation where the risk allocation spread is low and high. In Subsection 2.3.3, we provide a construction of the utility function of IB funds which incorporate both risk and regret. The cash flow in CDS contracts in discrete-time is discussed in Subsection 2.4.

Subsection 2.5.1 offers the mathematical formulation and solution of IB's optimization problem with risk. The results of this problem are summarized in Lemma 2.5.1. In Subsection 2.5.2, Theorem 2.5.3 proves that a regret-averse IB will always allocate away from $\pi_p^* = 0$ and $\pi_p^* = 1$, where π_p^* denote the optimal fraction of available IB funds invested in subprime RMBSs. The next important result shows the existence of a Treasuries rate at which regret has no impact on IB's optimal proportion invested in subprime RMBSs (see Corollary

2.5.4). Also, Proposition 2.5.5 in Subsection 2.5.2.1 proposes that higher regret amplifies the effect of IB hedging its bets. In Subsection 2.5.2.2, we suggest a way of mitigating risk and regret via CDSs. In particular, Theorem 2.5.6 in Subsection 2.5.2.2 shows that when the fraction of available funds invested in the subprime RMBSs is low, a regret-averse IB values the CDSs contract less than the risk-averse IB. On the other hand, both risk- and regret-averse IBs pay the same CDS premium when their RMBS portfolio is considered to be risky.

1.3.2.2 Outline of Chapter 3

The contents of Chapter 3 which discusses a problem of subprime RML securitization in continuous-time, emanate from [44], [45], [46], [47] and [51]. We start by presenting the dynamics of the RMBS price process (see Subsection 3.1), subprime RMBS losses (see Subsection 3.2) and credit ratings (see Subsection 3.3). Furthermore, Subsection 3.4 describes the type of CDS contract that we consider and includes the mathematical formulation of the associated premium in (3.3). The above discussions about subprime mortgage credit enable us to develop a stochastic model for IB's payout under RML securitization in (3.4). The stochastic differential equation (3.5) allows us to state and prove a stochastic optimal credit default insurance problem in Subsections 3.7 and 3.8, respectively. More specifically, Theorem 3.8.1 and Proposition 3.8.2 give the general solution to Problem 3.7.1. Subsequently, we determine explicit solutions for Problem 3.7.1 when exponential, power and logarithmic utility functions are chosen. For instance, Proposition 3.8.3 provides an explicit solution to the stochastic optimal credit default insurance problem with the choice of an exponential utility function. In this case, IB's optimal investment in securitized subprime RMLs, rate of cash outflow for fulfilling depositor obligations and accrued premium⁵ for CDSs, denoted by I^* , k^* and Φ^* , respectively, are not random variables, since they are not dependent on IB's payout, Π . On the other hand, the choice of a power utility function in Proposition 3.8.4 yields an explicit solution where the optimal control processes are expressed as linear functions of IB's optimal payout, Π^* . Moreover, Proposition 3.8.5 provides an explicit solution to Problem 3.7.1 with the choice of a logarithmic utility function – here the optimal controls are found to be comparable with those in Proposition 3.8.4.

1.3.2.3 Outline of Chapter 4

In Chapter 4, we discuss subprime RML securitization in a jump-diffusion framework. In this regard, the research in this chapter originated from [11], [12], [25], [50] and [49]. In Subsection 4.1, we provide a stochastic model of subprime RMBS prices driven by a jump process. Moreover, Subsection 4.2 gives a short description of Treasuries and their dynamics. Also, we incorporate the possibility that subprime RMBS losses may occur. In

⁵Accrued premium is the amount owing to the swap protection seller for IB's credit default protection for the period between the previous premium payment and the negative credit event.

particular, Subsection 4.3 briefly explains the losses that IB could suffer from its investment in subprime RMBSs. In Subsection 4.4, we assume that IB has initial funds, B_0 , that can be invested in subprime RMBSs and Treasuries with price processes given by (4.1) and (4.7), respectively. From the analysis above, we are able to present a stochastic dynamic model of IB's investment in subprime RMBSs and Treasuries which appears in Subsection 4.4. In particular, we introduce the IB's investment strategy, π , and the total and nett investment under π which we denote by B^π and B'^π , respectively. B^π is expressible in terms of the expected rate of return, $\tilde{\mu}^I$, a Treasuries rate, r^T , jump size α_i of the Lévy process L_t , homogeneous Poisson process, N_i , intensity, λ_i , and the volatility of the subprime RMBS price process, σ .

In Section 4.5, we provide a statement and proof of the optimization problem for IB's investment in subprime RMBSs and Treasuries with both variance and swaps-at-risk (SaR). More specifically, we investigate an IB's optimal investment problem that consists of maximizing expected terminal investment value under a constraint of an upper bound for the risk. Here, we measure risk by the variance. In the second problem, we solve IB's optimization problem subject to SaR. Furthermore, in the first problem, we find that the optimal investment strategy, π^* , and the maximal expected IB's investment process at the end of the planning term, $E[B_T^{\pi^*}]$, can be expressed in terms of $\tilde{\mu}^I$, r^T and α_i . A solution of our second problem shows that the optimal investment strategy, π^* , is the largest $\pi \in [0, 1]$ subject to SaR-constraint in (4.25) in Subsection 4.5.2.1 and conditions (4.13) in Subsection 4.4 of Chapter 4.

1.3.2.4 Outline of Chapter 5

Chapter 5 provides the analysis of securitization models and their connections with the SMC. A discussion on subprime RML securitization models in discrete-time and the SMC is presented in Subsection 5.2.1. The analysis of the main problems in Subsections 2.5.1 and 2.5.2 solved in Chapter 2 is provided in Subsection 5.2.2. CDSs contract and its function of mitigating risk and regret is discussed in Subsection 5.2.2.3. Subsection 5.2.3.3, relates the liquidity effect and the SMC. In particular, we consider the impact of risk allocation away from subprime RMBSs towards Treasuries to the economy within the context of the SMC.

In Subsection 5.3, we provide an analysis of issues related to IB's payout model under RML securitization and stochastic optimal credit default insurance problem as well as their connections with the SMC. Furthermore, we furnish a numerical example to illustrate the influence of some of the key parameters.

Finally, Subsection 5.4 contains the analysis of IB's optimal investment problem in a Lévy process setting and its connection with SMC.

1.3.2.5 Outline of Chapter 6

Chapter 6 presents a few concluding remarks and highlights some possible topics for future research.

1.3.2.6 Outline of Chapter 7

In Chapter 7, we list the articles, books and other sources that is cited in the thesis.

1.3.2.7 Outline of Chapter 8

Chapter 8 provides appendices containing information about Itô's formula for jump-diffusion processes, a computational example, a comparison between economic conditions before and during the SMC as well as features of prime and Alt-A mortgage loan deals (see Subsections 8.1, 8.2, 8.3 and 8.4, respectively).

Chapter 2

DISCRETE-TIME MODELS FOR SUBPRIME RML SECURITIZATION

2.1 SUBPRIME RMBSs AND THEIR EXPECTED RATE OF RETURN

2.2 SUBPRIME RML SECURITIZATION RISK IN DISCRETE-TIME

2.2.1 IB's Risk from RMLs, RMBSs and CLOs

2.2.2 Formalizing IB's Securitization Risks

2.3 REGRET IN BANKING

2.3.1 Subprime RMBSs and Treasuries with Regret

2.3.2 Risk Allocation Spread

2.3.3 Utility Functions

2.4 CREDIT DEFAULT SWAPS IN DISCRETE-TIME

2.5 IB'S OPTIMIZATION PROBLEMS

2.5.1 IB's Optimization Problem with Risks

2.5.2 IB's Optimization Problem with Risk and Regret

2.5.2.1 IB Hedging Against Securitization Risk

2.5.2.2 The Main Credit Default Swaps Result

The work in this chapter on discrete-time modeling of subprime RML securitization originated from [42], [48], [55] and [52]. In line with these contributions, we consider subprime RML securitization in discrete-time, risk and regret as well as credit default swaps. The main agents in our model are subprime mortgagors, ORs, SPVs and IBs (subprime RMBS investor and swap protection buyer). Each participant except IB – allowed to be risk-averse – is risk neutral. Other agents that are mentioned on occasion are swap protection sellers, depositors and CRAs. All events are scheduled to take place in period t that begins at time instant 0 and ends at time 1. At certain junctures in the discussion, also consider period $t+1$. Here, we drop the time subscripts when the financial variable has recursive behaviour.

2.1 SUBPRIME RMBSs AND THEIR EXPECTED RATE OF RETURN

In period t , IB invests a proportion of its funds in a subprime RMBS portfolio with stochastic returns, r_t^I . On the other hand, IB has the option of investing in Treasuries at the deterministic rate, $r^T(t) \leq r_t^I$. IB's investment in subprime RMBS portfolios enables OR to expand its subprime lending activities.

In making its risk allocation choice, IB takes into account that it may regret its choice if the investment proves to be suboptimal after the expiry of the RMBS contract. An important assumption throughout our discussion is that IB avoids deleterious consequences of a result that is worse than the best that could be achieved had knowledge of investment losses been known ex-ante. For example, if IB invests heavily in subprime RMBSs and then incurs a large loss, IB would experience some additional disutility of not having invested less in such RMBSs. Of course, the subprime RMBS rate, r^I , earned by IB is a function of the subprime RML rate, r^M , paid by mortgagors. The following assumption is an important one.

Assumption 2.1.1 (IB's Normalized Fund Supply): *We assume that the aggregate supply of funds by IB to SPV in exchange for interest and principal payments from securitized subprime RMLs is fixed and normalized to unity.*

Furthermore, IB can affect the risk-return profile of its subprime RMBS portfolio, I . More precisely, we assume that a subprime RMBS rate, r^I , is distributed according to the two-point distribution

$$r^I = \begin{cases} I & \text{with probability } q(I, \mathbf{m}); \\ 0 & \text{with probability } 1 - q(I, \mathbf{m}), \end{cases}$$

where $\mathbf{m} \in [0, 1]$ is a stochastic i.i.d. variable representing a random variable related to the level of macroeconomic activity, distributed over the interval $[0, 1]$ with a continuous

density function $f(\mathfrak{m})$, and a cumulative distribution function, $F(\mathfrak{m})$, $F(1) = 1$. For sake of simplicity, we assume that the functional form for the probability of success is given by

$$q(I, \mathfrak{m}) = \mathfrak{m}q(I),$$

so that expected returns can be written as

$$\mathbb{E}[r^I] = \xi q(I)I, \tag{2.1}$$

with $q(I) \in C^2$ and

$$\xi \equiv \int_0^1 \mathfrak{m} \tilde{\varphi}(\mathfrak{m}) d\mathfrak{m} < 1,$$

where $\tilde{\varphi}$ is chosen such that the higher the realization of $\mathfrak{m}$, the higher the expected returns, $\mathbb{E}[r^I]$, for any given choice I . Also, the higher the realization of $\mathfrak{m}$, the higher the probability of success, $q(I)$. We assume that a higher I is associated with a lower probability of success q . This means that $q'(I) < 0$. In addition, to avoid corner solutions with infinite risk, we assume that $q''(I) \leq 0$, so that (2.1) is strictly concave in the control variable I , and that there exists a $\bar{I} < \infty$, such that $q(\bar{I}) = 0$. Furthermore, we also assume that

$$I \geq \underline{I} = r^T / \xi$$

with $q(\underline{I}) = 1$ and $q'(\underline{I}) > -1/\underline{I}$. From the above, the following result is immediate.

Proposition 2.1.2 (Expected Returns from Treasuries versus RMBS Portfolios):
For IB, the riskless Treasuries is dominated in expected returns by (at least) some risky RMBS portfolio.

2.2 SUBPRIME RML SECURITIZATION RISK IN DISCRETE-TIME

In this subsection, we discuss subprime RMLs and RMBSs as well as their expected rate of return.

2.2.1 IB's Risk from RMLs, RMBSs and CLOs

A cash collateralized loan obligation (CLO) is a SPV, which buys a portfolio of fixed income loans, and finances the purchase of the portfolio via issuing different tranches of risk in the capital markets. Of particular interest are ABS CLOs, CLOs which have the underlying portfolios consisting of ABSs, including RMBSs. CLO portfolios typically include tranches of subprime and Alt-A deals, sometimes quite significant amounts. The interlinking of RMLs, RMBSs and CLOs is portrayed in the Figure 2.1 below. A representation of the creation of a RMBS deal is given on the left-hand side of the figure. Some of the bonds issued in this deal go into ABS CLOs. In particular, as shown on the right-hand side of the figure, RMBS bonds rated AAA, AA, and A form part of a *high grade* CLO portfolio, so called because the portfolio bonds have these ratings. The BBB bonds from the RMBSs deal go into a *mezz CLO*, so named because its portfolio consists entirely, or almost entirely, of BBB rated ABS and RMBS tranches. If bonds issued by mezz CLOs are put into another CLO portfolio, then new CLO – now holding mezz CLO tranches – is called a *CLO squared* or *CLO²*.

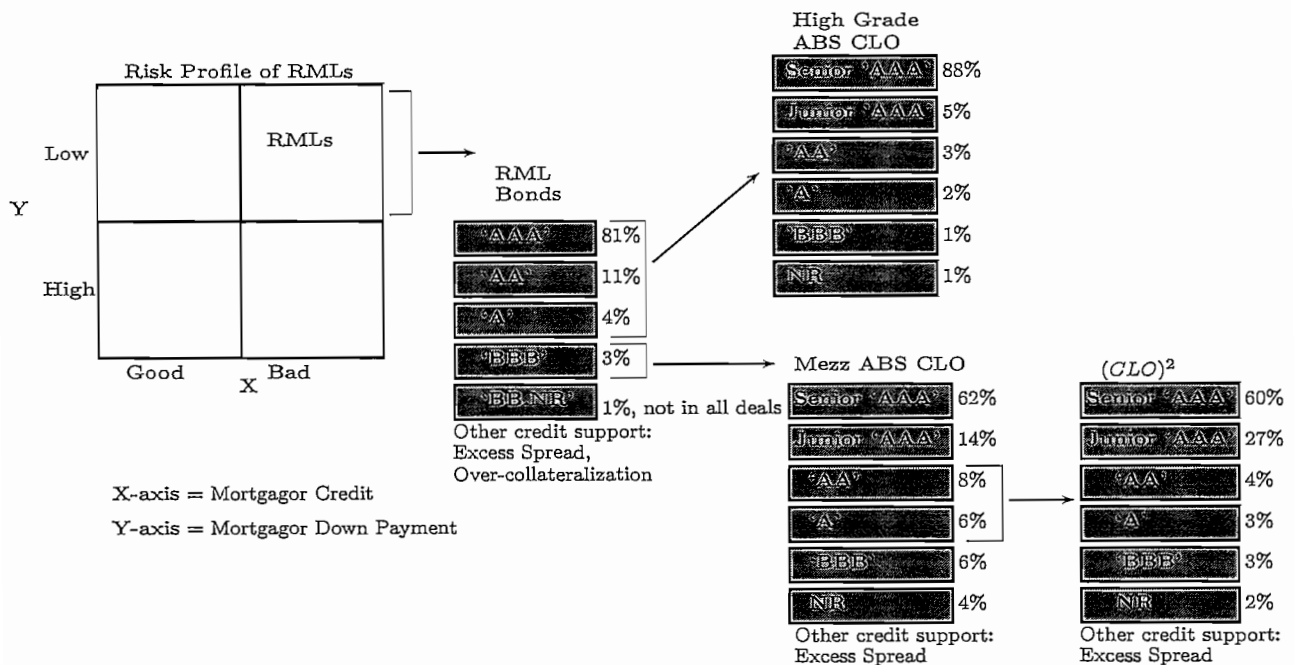


Figure 2.1: RML Securitization Risk; Source: [33].

2.2.2 Formalizing IB's Securitization Risk

As far as IB's risk is concerned, we are interested in the expression

$$\max_m \xi q(I)I.$$

In the above expression, we have an unconstrained optimization problem which maximizes the expected return of IB by choosing an appropriate portfolio of subprime RMBSs. From a necessary and sufficient first-order condition, we have that the optimal choice of I (denoted by I^*) is the unique solution to

$$q'(I)I + q(I) = 0.$$

We can rewrite this expression as

$$\alpha(I) \equiv \frac{q'(I)I}{q(I)} = -1. \quad (2.2)$$

The equality (2.2) means that, at the first best solution, the elasticity of the probability of success with respect to the RMBS portfolio return in case of success equals minus one. Furthermore, (2.2) implies that, since

$$\frac{\partial \alpha(I)}{\partial I} = \frac{q''(I)I + q'(I) - [q'(I)]^2 I}{[q(I)]^2} < 0, \quad (2.3)$$

excessive risk (i.e., a super-optimal risk level) is associated with $\alpha(I) < -1$. Furthermore, we have that

$$\alpha(\underline{I}) = \frac{q'(\underline{I})\underline{I}}{q(\underline{I})} > -1. \quad (2.4)$$

The discussion above leads to the following result.

Proposition 2.2.1 (Optimal Level of IB's Risk): *The optimal level of risk is strictly positive, i.e., $I^* > \underline{I}$.*

2.3 REGRET IN BANKING

In this subsection, we discuss subprime RMBSs and Treasuries in a regret framework as well as the associated risk allocation spread $\xi q(I)I - r^T$. Finally, we consider appropriate utility functions.

2.3.1 Subprime RMBSs and Treasuries with Regret

In the sequel, we make a distinction between the cases where the interest rate earned by IB on the subprime RMBSs, r^I , exceeds the Treasuries rate, $r^I \geq r^T$. For some RMBS portfolios, this possibility is guaranteed by Proposition 2.1.2. However, the opposite may also be true, i.e., $r^I < r^T$. In the first instance, for optimal returns, the regret-averse IB would have wanted to invest all available funds in the subprime RMBSs. On the other hand, in the second case, it would have been optimal to invest all funds in the Treasuries. Symbolically, we can express this as

$$f^{\max} = \begin{cases} f_0(1 + r^I), & \text{if } r^I \geq r^T; \\ f_0(1 + r^T), & \text{if } r^I < r^T, \end{cases}$$

where $f^{\max}$ is the value of the ex-post optimal final level of funds, i.e., the fund level that IB could have attained if it had made the optimal choice with respect to the realized state of the economy. Also, we have that

$$f = f_0 \left(1 + \pi r^I + (1 - \pi) r^T \right)$$

is the actual final fund level. In reality, the ex-post optimal final level of funds will always be greater than the actual final fund level.

2.3.2 Risk Allocation Spread

The investment decisions between the subprime RMBSs and Treasuries will partly be based on their allocation spread

$$\xi q(I)I - r^T$$

whose realized value is not known in advance (compare with (2.1)). Moreover, we will show a particular interest in the situations where

$$q(I) = \frac{r^T}{\xi I} \quad (2.5)$$

and

$$q(I) = \frac{r^T \mathbb{E} \left[U'(f_0(1 + r^I)) \right] + \text{cov} \left[-r^I, U'(f_0(1 + r^I)) \right]}{\xi I \mathbb{E} \left[U'(f_0(1 + r^I)) \right]}. \quad (2.6)$$

In this chapter, (2.5) represents the case where the risk allocation spread $\xi q(I)I - r^T$ is zero, whereas (2.6) corresponds to the case where the spread is high. A motivation for considering a special form for the right hand side of (2.6) is given as follows. If the risk allocation spread is nonnegative, i.e., $\xi q(I)I - r^T > 0$, so that $q(I) > \frac{r^T}{\xi I}$, then $f^{\max} = f_0(1 + r^I)$. In this regard, for $U' > 0$, with $q(I)$ given by (2.6), we are guaranteed that the risk allocation spread will be high. Note that we will sometimes use the notation $q(I) \gg \frac{r^T}{\xi I}$ when referring to (2.6). Because the subprime RMBSs are riskier than Treasuries, the risk allocation spread should generally be nonnegative which makes scenario (2.6) more realistic than (2.5).

2.3.3 Utility Functions

Expected utility theory is a major paradigm in investment theory (see, for instance, [7]). In our contribution, we choose a regret theoretic expected utility of the form

$$\int \left[U(f_{\mathbf{m}}) - \rho \cdot g \left(U(f_{\mathbf{m}}^{\max}) - U(f_{\mathbf{m}}) \right) \right] dF(\mathbf{m}),$$

where $F(\mathbf{m})$ is a cumulative distribution function that incorporates institutional views about macroeconomic states, $\mathbf{m}$, where $f_{\mathbf{m}}$ is the result in state, $\mathbf{m}$, of action f being taken. With this in mind, we investigate the impact of regret on IB's ex-ante risk allocation by representing its preferences as a two-component Bernoulli utility function, $U_{\rho} : \mathbb{R}^+ \rightarrow \mathbb{R}$, given by

$$U_{\rho}(f) = U(f) - \rho \cdot g \left(U(f^{\max}) - U(f) \right), \quad (2.7)$$

where $U : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the traditional Bernoulli utility (value) function over funding positions.

In the above, regret aversion corresponds to the convexity of g , and IB's preference is

assumed to be representable by maximization subject to U . The second term in (2.7) is concerned with the prospect of IB regret. The function $g(\cdot)$ measures the amount of regret that IB experiences, which depends on the difference between the value it assigns to the ex-post optimal fund level, $f^{\max}$, that it could have achieved, and the value that it assigns to its actual final level of funds, f . The parameter $\rho \geq 0$ measures the weight of the regret attribute with respect to the first attribute that is indicative of risk aversion. The ex-post optimal funds level should be greater than the actual final level of funds, i.e. $f^{\max} > f$. The first term in (2.7) relates to risk aversion and involves IB's utility function $U(\cdot)$ with $U'(\cdot) > 0$ and $U''(\cdot) < 0$. Therefore, the utility function of ex-post optimal funds level is greater than the utility function of actual final level of funds, i.e. $U(f^{\max}) > U(f)$ because $U(\cdot)$ is an increasing function. In the sequel, for $\rho > 0$, it is necessary that $U(f^{\max}) > U(f)$. In this case, the utility function of IB includes some compensation for regret and we call IB *regret-averse*. Throughout the thesis, $g(\cdot)$ is increasing and strictly convex, i.e. $g'(\cdot) > 0$ and $g''(\cdot) > 0$, which also implies regret-aversion. For $\rho = 0$, IB's utility function does not include regret and we call IB *risk-averse*. In particular, IB would be a maximizer of risk-averse expected utility, which means that $U_0(\cdot) = U(\cdot)$. The mathematical conditions which imply risk-aversion are $U'(\cdot) > 0$ and $U''(\cdot) < 0$.

2.4 CREDIT DEFAULT SWAPS IN DISCRETE-TIME

Let $r^{Is} \geq 0$ be a quoted default swap spread which is paid on the face value of the protection, $\bar{\pi}$. In the situation where no protection against risk related to subprime RMBS portfolios is bought, the default swap spread should be zero, i.e. $r^{Is} = 0$. If a credit event occurs, the swap protection seller will pay $1 - r^R$. In the case where no such event takes place, the swap protection seller pays nothing. In this case, IB (the swap protection buyer) will receive the normal rate of return on subprime RMBSs, r^I , throughout the term of the RML. In essence, the payout on CDSs is given by

$$R^c = \max(r^I, 1 - r^R). \quad (2.8)$$

In the sequel, the CDS contract does not alter the ex-post optimal level of funds, $f^{\max}$. Therefore, the ex-post optimal preference is for IB to invest all its available funds in subprime RMBSs, in the event that the realized return, r^I , is above the Treasuries rate, r^T , and all of it in the Treasuries otherwise. Mathematically, this may be expressed as

$$f^{\max} = f_0 \left(1 + \max(r^I, r^T) \right).$$

Furthermore, suppose that $P_\rho(r^{Is}, \bar{\pi})$ is the maximum premium that IB with regret para-

meter $\rho \geq 0$, is willing to pay for credit protection for $\bar{\pi}$. The premium size is dependent on the default swap spread, r^{Is} . For a very risky investment, r^{Is} is likely to be very high, which will force IB to pay a high CDS premium. In this case, IB's premium is governed by the indifference equation

$$\begin{aligned} & \mathbb{E} \left[U_\rho \left(f_0 \left(1 + \bar{\pi} r^I + (1 - \bar{\pi}) r^T \right) \right) \right] \\ &= \mathbb{E} \left[U_\rho \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \left(1 + \bar{\pi} R^c + (1 - \bar{\pi}) r^T \right) \right) \right]. \end{aligned} \quad (2.9)$$

The right-hand side of (2.9) describes the situation where no credit protection is bought while the left-hand side incorporates the cash flow on the CDS contract purchased by IB. In the case where no credit protection is bought, i.e. $r^{Is} = 0$, IB's premium for CDSs is zero. This means that

$$P_\rho(0, \bar{\pi}) = 0, \quad \forall \quad 0 \leq \bar{\pi} \leq 1.$$

If we rewrite (2.9), then the result below follows immediately.

Proposition 2.4.1 (Hedging Against IB's Securitization Risk Via Credit Default Swaps): *For R^c given by (2.8), if we put*

$$\aleph(R^c, \bar{\pi}) = 1 + \bar{\pi} R^c + (1 - \bar{\pi}) r^T,$$

then

$$\mathbb{E} \left[U_\rho \left(f_0 \aleph(r^I, \bar{\pi}) \right) \right] = \mathbb{E} \left[U_\rho \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right]. \quad (2.10)$$

Of course, if all IB's funds were allocated to Treasuries, its premium for CDSs should be zero, so that

$$P_\rho(r^{Is}, 0) = 0, \quad \forall \quad 0 \leq r^{Is} \leq r^T.$$

2.5 IB'S OPTIMIZATION PROBLEMS

In this subsection, we consider two optimization problems involving IB. In the first problem, we consider the optimization of IB's profit by considering risk only. On the other hand, the second involves an optimal RML securitization problem incorporating both risk and regret.

2.5.1 IB's Optimization Problem with Risk

IB invests in subprime RMBSs issued by the SPV that cannot be liquidated quickly at a high price. On the other hand, IB issues demand deposits that allow depositors to withdraw at any time. This mismatch of liquidity, in which IB's liabilities are more liquid than its investment in subprime RMBSs, has caused the failure of financial institutions due to bank runs. In our case, IB is subject to unanticipated deposit withdrawals and risk on their RMBS investment. In particular, IB has the risk-averse utility function, i.e. $U_0(f) = U(f)$. Therefore, its objective would be a maximization of risk-averse expected utility, $\mathbb{E}[U_0(f)]$, such that it attains an optimal profit at the end. This problem can also be interpreted as profit maximization. In this regard, IB's problem, which consists in maximizing the discounted flow of profits V_t , valued in period t , is thus given by

$$V_t = \tilde{\Pi}_t + \delta s_t(\cdot) \tilde{\Pi}_{t+1} + \delta^2 s_t(\cdot) s_{t+1}(\cdot) \tilde{\Pi}_{t+2} + \dots,$$

where $s(\cdot)$ is the probability of survival, $\delta < 1$ is a discount factor, $s_t(I_t, \mathbf{m}, \cdot)$ is the IB's probability of survival from period t to period $t+1$, and $\tilde{\Pi}_t$ denotes current expected profits, that is

$$\tilde{\Pi}_t = \xi q(I_t)(I_t - r_t^D), \quad (2.11)$$

with $r_t^D = \hat{r}$. Furthermore, to rule out supergames, by restricting our attention to Markov strategies. This in turn implies that the problem that IB faces is a recursive (able to drop time subscript) one and IB's value can be rewritten as

$$V = \max_I \frac{\tilde{\Pi}(I)}{1 - \delta s(\cdot)}. \quad (2.12)$$

If, in the event of zero returns, IB's shareholders choose not to recapitalize, IB's survival probability is given by

$$s(I, \mathbf{m}) = \xi q(I),$$

and (2.12) can be rewritten as

$$V^a = \max_I \frac{\xi q(I)(I - r^D)}{1 - \delta \xi q(I)}, \quad (2.13)$$

In this case, the necessary and sufficient first-order condition for the maximization in (2.12) can be expressed as

$$\alpha(I) = - \left[1 - \frac{q(I)}{\theta^a} \right] \frac{I}{I - r^D}, \quad \theta^a \equiv \frac{1}{\delta \xi}, \quad (2.14)$$

where the superscript a stands for absence of a bailout policy. Using (2.2), (2.4), and (2.14), and denoting by I^a the (unique) solution to (2.14), one can immediately verify that

$$\frac{1}{\delta} \geq \frac{\xi q(I^a)(I^a)}{r^D} \Rightarrow I^a \geq I^*. \quad (2.15)$$

Accordingly, for a given discount rate, IB is more prone to engage in excessive risk-taking the lower the returns of the profit maximizing RMBS portfolio relative to those of riskless Treasuries are. Alternatively, for given rates of return, IB with a stronger preference for the present (a higher discount rate $1/\delta$), tends to invest in riskier RMBS portfolios.

Note that (2.15) suggests that IB may in some cases choose too little risk ($I^a < I^*$). However, it can be shown that in those cases, it is in the interest of IB shareholders to recapitalize IB and avoid default. More precisely, if at the end of period t , the realized RMBS portfolio returns are zero, shareholders still have the option to raise the necessary funds in the capital market, thereby avoiding default. This "non-default" option will be exercised whenever the value of the charter continues to be positive, that is, when the discounted value of future expected profits under the assumption of recapitalization given by

$$V^a = \max_I \frac{\xi q(I)I - r^D}{1 - \delta}, \quad (2.16)$$

exceeds the stock of outstanding deposit liabilities, so that, in equilibrium we have

$$r_t^D \leq \delta V^a = \delta \frac{\xi q(I^a)I^a - r^D}{1 - \delta}.$$

This condition can be rewritten as

$$\frac{\xi q(I^a)I^a}{r^D} \geq \frac{1}{\delta}. \quad (2.17)$$

But, if condition (2.17) holds and shareholders anticipate that IB never defaults, the optimal strategy is then necessarily that of maximizing expected returns in each time period, in which case IB's problem coincides with the central planner's. In summary, we have the following optimization result.

Lemma 2.5.1 (Suboptimal Equilibrium Risk): *Equilibrium risk is never suboptimal. In this case, we have that*

1. *If*

$$\frac{\xi q(I^a)I^a}{r^D} < \frac{1}{\delta},$$

IB selects a RMBS portfolio I^a strictly riskier than I , and defaults whenever portfolio returns are below deposit liabilities;

2. *If*

$$\frac{\xi q(I^a)I^a}{r^D} \geq \frac{1}{\delta},$$

IB chooses the optimal RMBS portfolio I^* and never defaults.

2.5.2 IB's Optimization Problem with Risk and Regret

In this subsection, we consider how IB's optimal risk allocation is influenced by regret theoretic issues in a stylized framework. Let π_ρ denote the fraction of available IB funds invested in the subprime RMBSs with regret parameter $\rho \geq 0$. For the case where π_ρ is optimal (denoted by π_ρ^*), we have that π_0^* denotes the optimal fraction invested in the subprime RMBSs by the risk-averse IB. For the two-attribute Bernoulli utility function (2.7), the objective function is given by

$$J(\pi) = \mathbf{E}[U_\rho(f(\pi))]. \quad (2.18)$$

In order to determine the optimal risk allocation, π_ρ^* , we consider the *set of admissible controls* given by

$$\mathcal{A} = \left\{ \pi_\rho : 0 \leq \pi_\rho \leq 1 \text{ and (2.18) has a finite value} \right\}. \quad (2.19)$$

Also, if π is the proportion of available IB funds invested in subprime RMBSs the *value function* is given by

$$\begin{aligned} V &= \max_{\pi \in \mathcal{A}} \mathbf{E}[U_\rho(f(\pi))] \\ &= \max_{\pi \in \mathcal{A}} \mathbf{E} \left[U(f(\pi)) - \rho \cdot g \left(U(f^{\max}) - U(f(\pi)) \right) \right]. \end{aligned} \quad (2.20)$$

The optimal risk allocation problem with regret may be formally stated as follows.

Problem 2.5.2 (Optimal Investment in Subprime RMBSs and Treasuries): *Suppose that the Bernoulli utility function, U_ρ , objective function, J , and admissible class of control laws, $\mathcal{A} \neq \emptyset$, are described by (2.7), (2.18) and (2.19), respectively. In this case, characterize V in (2.20) and the optimal control law, π^* , if it exists.*

The ensuing optimization result demonstrates that a regret-averse IB (before the SMC) will always allocate away from $\pi_\rho^* = 0$ and $\pi_\rho^* = 1$. In other words, by comparison with risk-averse IBs (during the SMC), regret-averse IBs will commit to a riskier allocation if the difference $\xi q(I)I - r^T$ is low, and a less risky allocation if $\xi q(I)I - r^T$ is high. In the years leading up to the SMC, RMBS investment by the majority of IBs – considered to be regret-averse – was driven by high spreads. Spread size was an indication that risk was perceived to be low. This encouraged many IBs to invest more in RMBS portfolios. However, during the SMC, when mortgagors failed to make repayments on their RMLs, the value of RMBSs as well as the spread declined. In this period, risk was considered to have increased, with many IBs becoming risk-averse and preferring investment in safer assets such as Treasuries.

Theorem 2.5.3 (Optimal Investment in Subprime RMBSs and Treasuries): *Suppose that Assumption 2.1.1 holds and that U_ρ is the two-attribute Bernoulli utility function defined by (2.7). Regret-averse IBs always invest funds in subprime RMBSs even if the risk allocation spread is zero as in (2.5). However, risk-averse IBs would hold only Treasuries in its portfolio in that case. Moreover, for a sufficiently large risk allocation spread as in (2.6), regret-averse IBs always invest a positive amount in Treasuries, whereas risk-averse IBs hold subprime RMBSs in their portfolio.*

Proof. The statement of Theorem 2.5.3 is equivalent to

1. If (2.5) holds then $\pi_\rho^* > 0$ for all $\rho > 0$, with $\pi_0^* = 0$;
2. If (2.6) holds then $\pi_\rho^* < 1$ for all $\rho > 0$, with $\pi_0^* = 1$.

In this regard, we use standard maximization arguments to prove the above results. In particular, we must show that the first derivative of (2.20) with respect to π , at $\pi_\rho^* = 0$ and $\pi_\rho^* = 1$ does not vanish. In this regard, we have that

$$f(\pi) = f_0 \left(1 + \pi r^I + (1 - \pi) r^T \right) \text{ and } f^{\max} = f_0 \left(1 + \max(r^I, r^T) \right)$$

denote IB final fund level and ex-post optimal fund level, respectively. The first- and second-order conditions for (2.20) are

$$\frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} = 0 \quad (2.21)$$

and

$$\frac{d^2\mathbf{E}[U_\rho(f(\pi))]}{d\pi^2} < 0, \quad (2.22)$$

respectively. But

$$\frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} = \mathbf{E} \left[f_0(r^I - r^T) U'(f(\pi)) \left(1 + \rho g' \left(U(f^{\max}) - U(f(\pi)) \right) \right) \right]$$

and

$$\begin{aligned} \frac{d^2\mathbf{E}[U_\rho(f(\pi))]}{d\pi^2} &= \mathbf{E} \left[f_0^2 (r^I - r^T)^2 U''(f(\pi)) \left(1 + \rho g' \left(U(f^{\max}) - U(f(\pi)) \right) \right) \right] \\ &\quad - \mathbf{E} \left[f_0^2 (r^I - r^T)^2 \rho U'^2(f(\pi)) g'' \left(U(f^{\max}) - U(f(\pi)) \right) \right]. \end{aligned}$$

It then follows that (2.21) and (2.22) take the forms

$$\begin{aligned} \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} &= \mathbf{E}\left[f_0(r^I - r^T)U'(f(\pi))\left(1 + \rho \cdot g' \left(U(f^{\max}) - U(f(\pi))\right)\right)\right] \\ &= 0 \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} \frac{d^2\mathbf{E}[U_\rho(f(\pi))]}{d\pi^2} &= \mathbf{E}\left[f_0^2(r^I - r^T)^2 U''(f(\pi))\left(1 + \rho \cdot g' \left(U(f^{\max}) - U(f(\pi))\right)\right)\right] \\ &\quad - \mathbf{E}\left[f_0^2(r^I - r^T)^2 \rho U'^2(f(\pi))g'' \left(U(f^{\max}) - U(f(\pi))\right)\right] < 0, \end{aligned} \quad (2.24)$$

respectively. This implies that $\mathbf{E}[U_\rho(f(\pi))]$ is strictly concave in π , so that any solution of the first order condition (2.23) uniquely fixes the global maximum. Furthermore, in this case, a decomposition of (2.23) may be given by

$$\begin{aligned} \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} &= \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} + \int_{-1}^{r^T} \rho f_0(r^I - r^T)U'(f(\pi))g' \left(U(f(0)) - U(f(\pi))\right) dF(r^I) \\ &\quad + \int_{r^T}^{\infty} \rho f_0(r^I - r^T)U'(f(\pi))g' \left(U(f(1)) - U(f(\pi))\right) dF(r^I). \end{aligned}$$

If we evaluate this first derivative at $\pi = 0$ and $\pi = 1$, then we obtain

$$\begin{aligned} \left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=0} &= \left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=0} + \rho f_0 U'(f(0))g'(0) \int_{-1}^{r^T} (r^I - r^T) dF(r^I) \\ &\quad + \rho f_0 U'(f(0)) \int_{r^T}^{\infty} (r^I - r^T)g' \left(U(f(1)) - U(f(0))\right) dF(r^I) \\ &> \left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=0} + \rho f_0 U'(f(0))g'(0) \int_{-1}^{r^T} (r^I - r^T) dF(r^I) \\ &= f_0 U'(f(0)) \left(\mathbf{E}[r^I] - r^T \right) (1 + \rho g'(0)) \\ &= f_0 U'(f(0)) \left(\xi q(I)I - r^T \right) (1 + \rho g'(0)) \end{aligned}$$

and

$$\begin{aligned}
 \left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=1} &= \left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=1} \\
 &+ \int_{-1}^{r^T} \rho f_0(r^I - r^T) U'(f(1)) g'(U(f(0)) - U(f(1))) dF(r^I) \\
 &+ \int_{r^T}^{\infty} \rho f_0(r^I - r^T) U'(f(1)) g'(0) dF(r^I) \\
 &< \left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=1} + \rho f_0 g'(0) \int_{r^T}^{\infty} (r^I - r^T) U'(f(1)) dF(r^I) \\
 &= f_0 \mathbf{E} \left[(r^I - r^T) U'(f_0(1 + r^I)) \right] (1 + \rho g'(0)) \\
 &= \left\{ \mathbf{E} \left[r^I U'(f_0(1 + r^I)) \right] - r^T \mathbf{E} \left[U'(f_0(1 + r^I)) \right] \right\} f_0 (1 + \rho g'(0)),
 \end{aligned}$$

respectively. As a result of this, if (2.5) holds, then

$$\left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=0} > 0$$

for all $\rho > 0$. On the other hand, if (2.6) holds, i.e.,

$$q(I) = \frac{r^T \mathbf{E} \left[U'(f_0(1 + r^I)) \right] + \text{cov} \left[-r^I, U'(f_0(1 + r^I)) \right]}{\xi I \mathbf{E} \left[U'(f_0(1 + r^I)) \right]}.$$

and taking into account that

$$r^T = \frac{\mathbf{E}[r^I U'(f_0(1 + r^I))]}{\mathbf{E}[U'(f_0(1 + r^I))]}$$

then

$$\left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=1} < 0$$

for all $\rho > 0$. This implies, in the former instance, that $\pi_\rho^* > 0$ for all $\rho > 0$ and $\pi_\rho^* < 1$ for all $\rho > 0$ in the second situation. $\square$

Theorem 2.5.3 can be illustrated as follows.

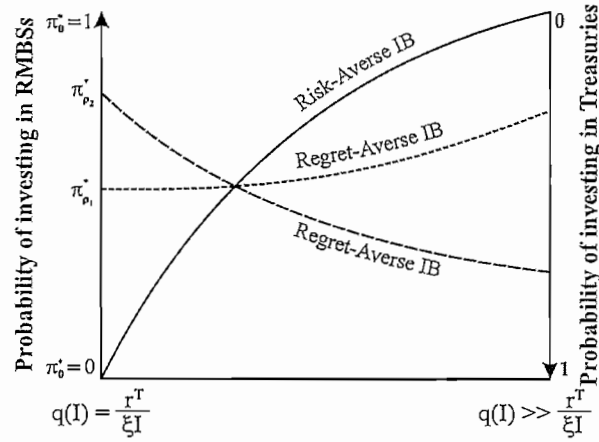


Figure 2.2: Optimal Risk and Regret in Banking

We use Theorem 2.5.3 to show that the next corollary holds.

Corollary 2.5.4 (Risk Allocation of Risk- and Regret-Averse IBs): *There exists a Treasuries rate, $\tilde{r}^T$, and therefore a level $\xi q(I)I - \tilde{r}^T$, for which regret does not affect the optimal proportion invested in subprime RMBSs, π^* . At this specific $\xi q(I)I - \tilde{r}^T$, the risk allocation for a regret-averse IB will correspond to that of a risk-averse IB.*

Proof. In Corollary 2.5.4, we must show that there exists a Treasuries rate, $\tilde{r}^T$, such that

$$0 < \xi q(I)I - \tilde{r}^T < \frac{\text{cov}\left(-r^I, U'(f_0(1+r^I))\right)}{\mathbf{E}\left(U'(f_0(1+r^I))\right)}$$

and, for all $\rho > 0$, we have that $\pi_\rho^* = \pi_0^*$.

We have proved in Theorem 2.5.3, for any fixed $\rho > 0$, that

$$\begin{cases} \pi_\rho^* > 0 \text{ and } \pi_0^* = 0, & \text{if (2.5) holds;} \\ \pi_\rho^* < 1 \text{ and } \pi_0^* = 1, & \text{if (2.6) holds.} \end{cases}$$

Furthermore, the Intermediate Value Theorem suggests the existence of a Treasuries rate, $\tilde{r}^T$, with the property that

$$\xi g(I)I > \tilde{r}^T > \frac{\mathbf{E}[r^I U'(f_0(1 + r^I))]}{\mathbf{E}[U'(f_0(1 + r^I))]}$$

and $\pi_\rho^* = \pi_0^*$. The first-order derivative conditions

$$\left. \frac{d\mathbf{E}[U(f(\pi))]}{d\pi} \right|_{\pi=\pi_0^*} = \mathbf{E} \left[f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) \right] = 0$$

and (2.23) at $\pi = \pi_\rho^*$, i.e.,

$$\begin{aligned} & \left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^*} \\ &= \mathbf{E} \left[f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) \left(1 + \rho g' \left(U(f^{\max}) - U(f(\pi_0^*)) \right) \right) \right] = 0. \end{aligned}$$

It then follows that

$$\mathbf{E} \left[f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) \right] = \mathbf{E} \left[f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) \left(1 + g' \left(U(f^{\max}) - U(f(\pi_0^*)) \right) \right) \right].$$

Assuming a continuous-time environment, we write

$$\begin{aligned} & \int_{-1}^{\infty} f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) dF(r^I) \\ &= \int_{-1}^{\infty} f_0(r^I - \tilde{r}^T(\rho)) U'(f(\pi_0^*)) \left(1 + \rho g' \left(U(f^{\max}) - U(f(\pi_0^*)) \right) \right) dF(r^I). \end{aligned}$$

The above expression holds if and only if

$$f_0(r^I - \tilde{r}^T(\rho))U'(f(\pi_0^*)) = f_0(r^I - \tilde{r}^T(\rho))U'(f(\pi_0^*)) \left(1 + \rho g' \left(U(f^{max}) - U(f(\pi_0^*)) \right) \right),$$

which simply means that

$$f_0(r^I - \tilde{r}^T(\rho))U'(f(\pi_0^*))\rho g' \left(U(f^{max}) - U(f(\pi_0^*)) \right) = 0.$$

Therefore, we have that

$$r^I - \tilde{r}^T(\rho) = 0,$$

since $f_0 > 0$, $\rho > 0$, $U'(\cdot) > 0$ and $g'(\cdot) > 0$. Thus, we conclude that for all $\rho > 0$, we have $\tilde{r}^T(\rho) = r^I$. □

The results of Corollary 2.5.4 can be represented graphically as shown below.

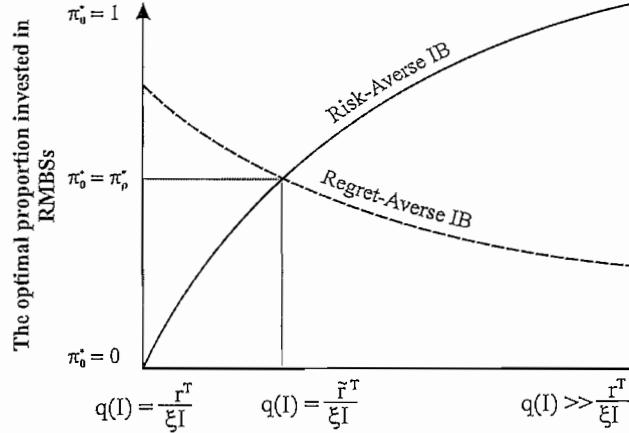


Figure 2.3: The Certainty Equivalent

2.5.2.1 IB Hedging Against Securitization Risk

In the following proposition, we show that higher regret exacerbates the effect of IB hedging its bets.

Proposition 2.5.5 (Hedging Against Risk): *Suppose that IB is more regret- than risk-averse (as measured by ρ). Then, under (2.5), it invests more in subprime RMBSs, whereas*

under (2.6) it invests less in subprime RMBSs. In particular, the more regret-averse the IB, the more likely it will be to hold subprime RMBSs in its portfolio as long as the risk allocation spread is zero. Conversely, it will hold less subprime RMBSs when the risk allocation spread is high.

Proof. In Proposition 2.5.5, we are required to show that

$$\frac{\partial \pi_\rho^*}{\partial \rho} \begin{cases} > 0, & \text{if (2.5) holds;} \\ < 0, & \text{if (2.6) holds.} \end{cases}$$

Taking the total differential of the first-order condition (2.23) with respect to π and ρ yields

$$d \left[\frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right] \Big|_{\pi=\pi_\rho^*} = \frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi^2} \Big|_{\pi=\pi_\rho^*} \cdot d\pi + \frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \Big|_{\pi=\pi_\rho^*} \cdot d\rho = 0.$$

In this case, we therefore have that

$$\frac{\partial \pi_\rho^*}{\partial \rho} = - \frac{\frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \Big|_{\pi=\pi_\rho^*}}{\frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi^2} \Big|_{\pi=\pi_\rho^*}}.$$

Since it is true that

$$\frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi^2} \Big|_{\pi=\pi_\rho^*} < 0$$

we may conclude that

$$\text{sign} \left(\frac{\partial \pi_\rho^*}{\partial \rho} \right) = \text{sign} \left(\frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \Big|_{\pi=\pi_\rho^*} \right).$$

We observe that the mixed partial derivative yields

$$\frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \Big|_{\pi=\pi_\rho^*} = \mathbf{E} \left[f_0(r^I - r^T) U'(f(\pi_\rho^*)) g' \left(U(f^{\max}) - U(f(\pi_\rho^*)) \right) \right].$$

Furthermore, from the first-order condition (2.23), it follows that

$$\left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^*} = \left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^*} + \rho \cdot \left. \frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \right|_{\pi=\pi_\rho^*}.$$

Since we have that

$$\left. \frac{d\mathbf{E}[U_\rho(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^*} = 0$$

we can deduce that

$$\text{sign}\left(\frac{\partial \pi_\rho^*}{\partial \rho}\right) = \text{sign}\left(\left. \frac{\partial^2 \mathbf{E}[U_\rho(f(\pi))]}{\partial \pi \partial \rho} \right|_{\pi=\pi_\rho^*}\right) = -\text{sign}\left(\left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^*}\right). \quad (2.25)$$

Our conclusion is that if (2.5) holds then $\pi_\rho^* > 0$ for all $\rho > 0$, and $\pi_0^* = 0$ according to Theorem 2.5.3. This implies that

$$\left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^* > 0} < 0$$

and, as a consequence, we have

$$\frac{\partial \pi_\rho^*}{\partial \rho} > 0$$

as suggested by (2.25).

If, on the other hand, (2.6) holds then $\pi_\rho^* < 1$ for all $\rho > 0$, and $\pi_0^* = 1$ according to Theorem 2.5.3. By the method used in the above, this implies that

$$\left. \frac{d\mathbf{E}[U_0(f(\pi))]}{d\pi} \right|_{\pi=\pi_\rho^* < 1} > 0$$

and thus

$$\frac{\partial \pi_\rho^*}{\partial \rho} < 0$$

by (2.25). □

2.5.2.2 The Main Credit Default Swaps Result

In the following theorem, if the proportion of available funds invested in subprime RMBSs is low, then we have that a risk-averse IB values the CDSs more than a regret-averse IB. On the other hand, both risk- and regret-averse IBs would like to pay the same premium when the proportion of available funds invested in subprime RMBSs is high.

Theorem 2.5.6 (Risk Mitigation via Credit Default Swaps): *We have that*

$$P_\rho(r^{Is}, \bar{\pi}) < P_0(r^{Is}, \bar{\pi}) \quad (2.26)$$

for low levels of $\bar{\pi}$ and all r^{Is} . On the other hand, it is true that

$$P_\rho(r^{Is}, \bar{\pi}) = P_0(r^{Is}, \bar{\pi}) \quad (2.27)$$

for high levels of $\bar{\pi}$ and low levels of r^{Is} .

Proof. IB's premium for CDSs is implicitly defined through the conditional indifference equation (2.10). Regret-averse IB is willing to pay a smaller premium for the CDSs contract than the risk-averse IB, i.e., (2.26) holds for all r^{Is} , if and only if

$$\begin{aligned} \mathbb{E} \left[U \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right] &> \mathbb{E} \left[U \left(\left(f_0 - P_0(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right] \\ &= \mathbb{E} \left[U \left(f_0 \aleph(r^I, \bar{\pi}) \right) \right] \end{aligned}$$

for all r^{Is} . Define the function $h : [0, 1] \rightarrow \mathbb{R}$ as

$$h(\bar{\pi}) = \mathbb{E} \left[U \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right] - \mathbb{E} \left[U \left(f_0 \aleph(r^I, \bar{\pi}) \right) \right]. \quad (2.28)$$

A first observation is that for $\bar{\pi} = 0$, we have $h(0) = 0$. In order to prove that (2.26) holds for small $\bar{\pi}$ and all r^{Is} , we thus have to show that $h'(0) > 0$. Finding the derivative of h with respect to $\bar{\pi}$ yields

$$\begin{aligned}
 h'(\bar{\pi}) &= -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial \bar{\pi}} \mathbf{E} \left[U'((f_0 - P_\rho(r^{Is}, \bar{\pi}))\aleph(R^c, \bar{\pi}))\aleph(R^c, \bar{\pi}) \right] \\
 &+ \mathbf{E} \left[(f_0 - P_\rho(r^{Is}, \bar{\pi}))(R^c - r^T) U'((f_0 - P_\rho(r^{Is}, \bar{\pi}))\aleph(R^c, \bar{\pi})) \right] \\
 &- \mathbf{E} \left[f_0(r^I - r^T) U'(f_0\aleph(r^I, \bar{\pi})) \right].
 \end{aligned}$$

and thus

$$h'(0) = U' \left(f_0(1 + r^T) \right) \left[-\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial \bar{\pi}} \Big|_{\bar{\pi}=0} (1 + r^T) + f_0 \mathbf{E}[R^c - r^I] \right]. \quad (2.29)$$

If we differentiate (2.10) with respect to $\bar{\pi}$, we obtain

$$\begin{aligned}
 &\mathbf{E} \left[f_0(r^I - r^T) U' \left(f_0\aleph(r^I, \bar{\pi}) \right) \left(1 + \rho g' \left(U(f^{\max}) - U \left(f_0\aleph(r^I, \bar{\pi}) \right) \right) \right) \right] \\
 &= -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial \bar{\pi}} \mathbf{E} \left[\aleph(r^I, \bar{\pi}) U' \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(r^I, \bar{\pi}) \right) \right. \\
 &\quad \times \left. \left(1 + \rho g' \left(U(f^{\max}) - U \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right) \right) \right] \\
 &+ \mathbf{E} \left[\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) (R^c - r^T) U' \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right. \\
 &\quad \times \left. \left(1 + \rho g' \left(U(f^{\max}) - U \left(\left(f_0 - P_\rho(r^{Is}, \bar{\pi}) \right) \aleph(R^c, \bar{\pi}) \right) \right) \right) \right].
 \end{aligned}$$

If we set $\bar{\pi} = 0$, it follows that

$$\begin{aligned}
 &f_0 U' \left(f_0(1 + r^T) \right) \mathbf{E} \left[(r^I - r^T) \left(1 + \rho g' \left(U(f^{\max}) - U \left(f_0(1 + r^T) \right) \right) \right) \right] \\
 &= -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial \bar{\pi}} \Big|_{\bar{\pi}=0} (1 + r^T) U' \left(f_0(1 + r^T) \right) \mathbf{E} \left[\left(1 + \rho g' \left(U(f^{\max}) - U \left(f_0(1 + r^T) \right) \right) \right) \right] \\
 &+ f_0 U' \left(f_0(1 + r^T) \right) \mathbf{E} \left[(R^c - r^T) \left(1 + \rho g' \left(U(f^{\max}) - U \left(f_0(1 + r^T) \right) \right) \right) \right]
 \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} & \left. \frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial \bar{\pi}} \right|_{\bar{\pi}=0} \\ &= \frac{f_0 \mathbf{E} \left[(R^c - r^I) \left(1 + \rho g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right) \right]}{(1 + r^T) \mathbf{E} \left[1 + \rho g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right]}. \end{aligned} \quad (2.30)$$

If we substitute (2.30) into (2.29), we may conclude that

$$\begin{aligned} & h'(0) = f_0 U' \left(f_0(1 + r^T) \right) \\ & \times \left(- \frac{(1 + r^T) \mathbf{E} \left[(R^c - r^I) \left(1 + \rho g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right) \right]}{(1 + r^T) \mathbf{E} \left[1 + \rho g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right]} + \mathbf{E}[R^c - r^I] \right) \\ &= - \frac{f_0 U' (f_0(1 + r^T)) \rho}{\mathbf{E} \left[1 + \rho g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right]} \cdot \text{cov} \left(R^c - r^I, g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right). \end{aligned}$$

Also, we observe that

$$\begin{aligned} & \text{cov} \left(R^c - r^I, g' \left(U(f^{\max}) - U(f_0(1 + r^T)) \right) \right) \\ &= \text{cov} \left(R^c - r^I, g' \left(U(f_0(1 + \max(r^I, r^T))) - U(f_0(1 + r^T)) \right) \right) < 0. \end{aligned}$$

In this case, we may conclude that $h'(0) > 0$, which implies $h(\bar{\pi}) > 0$ for small $\bar{\pi}$ since $h(0) = 0$. From this we can deduce that (2.26) holds for low levels of $\bar{\pi}$ and all r^{Is} .

Next, we would like to show that (2.27) holds for high levels of $\bar{\pi}$ and small r^{Is} . This inequality holds if and only if $h(\bar{\pi}) = 0$ for $\bar{\pi}$ and small r^{Is} (see (2.28) for the definition of $h(\cdot)$). A first observation is that

$$h(1) = \mathbf{E} \left[U \left(\left(f_0 - P_\rho(r^{Is}, 1) \right) (1 + R^c) \right) \right] - \mathbf{E} \left[U(f_0(1 + r^I)) \right].$$

A further observation is that at $r^{Is} = 0$, we have that $h(1)|_{r^{Is}=0} = 0$. If we differentiate $h(1)$ with respect to r^{Is} , we obtain

$$\frac{\partial h(1)}{\partial r^{Is}} = -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1} \mathbf{E} \left[(1 + R^c) U' \left((f_0 - P_\rho(r^{Is}, 1))(1 + R^c) \right) \right].$$

Determining the value at $r^{Is} = 0$ yields

$$\frac{\partial h(1)}{\partial r^{Is}} = -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1, r^{Is}=0} \mathbf{E} \left[(1 + r^I) U' (f_0(1 + r^I)) \right].$$

Furthermore, if we differentiate (2.10) with respect to r^{Is} , it follows that

$$\begin{aligned} & -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \mathbf{E} \left[\aleph(R^c, \bar{\pi}) U' \left((f_0 - P_\rho(r^{Is}, \bar{\pi})) \aleph(R^c, \bar{\pi}) \right) \right. \\ & \times \left. \left(1 + \rho g' \left(U(f^{\max}) - U \left((f_0 - P_\rho(r^{Is}, \bar{\pi})) \aleph(R^c, \bar{\pi}) \right) \right) \right) \right] = 0. \end{aligned} \quad (2.31)$$

For $\bar{\pi} = 1$, we obtain

$$\begin{aligned} & -\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1} \mathbf{E} \left[(1 + R^c) u' \left((f_0 - P_\rho(r^{Is}, 1))(1 + R^c) \right) \right. \\ & \times \left. \left(1 + \rho g' \left(u(f^{\max}) - U \left((f_0 - P_\rho(r^{Is}, 1))(1 + R^c) \right) \right) \right) \right] = 0. \end{aligned} \quad (2.32)$$

If we evaluate at $r^{Is} = 0$, then it follows that

$$\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1, r^{Is}=0} = 0$$

and as a consequence

$$\frac{\partial h(1)}{\partial r^{Is}} \Big|_{r^{Is}=0} = 0.$$

If we differentiate again we obtain

$$\begin{aligned} \frac{\partial^2 h(1)}{\partial (r^{Is})^2} &= -\frac{\partial^2 P_\rho(r^{Is}, \bar{\pi})}{\partial (r^{Is})^2} \Big|_{\bar{\pi}=1} \mathbb{E} \left[(1 + R^c) U' \left((f_0 - P_\rho(r^{Is}, 1))(1 + R^c) \right) \right] \\ &\quad + \left(\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1} \right)^2 \mathbb{E} \left[(1 + R^c)^2 U'' \left((f_0 - P_\rho(r^{Is}, 1))(1 + R^c) \right) \right]. \end{aligned}$$

Recall that when $r^{Is} = 0$, we have

$$\frac{\partial P_\rho(r^{Is}, \bar{\pi})}{\partial r^{Is}} \Big|_{\bar{\pi}=1} = 0$$

and thus

$$\frac{\partial^2 h(1)}{\partial (r^{Is})^2} \Big|_{r^{Is}=0} = -\frac{\partial^2 P_\rho(r^{Is}, \bar{\pi})}{\partial (r^{Is})^2} \Big|_{\bar{\pi}=1, r^{Is}=0} \mathbb{E} \left[(1 + r^I) U' \left(f_0(1 + r^I) \right) \right].$$

If we differentiate (2.32) with respect to r^{Is} and determine a value at $r^{Is} = 0$, then it follows that

$$-\frac{\partial^2 P_\rho(r^{Is}, \bar{\pi})}{\partial (r^{Is})^2} \Big|_{\bar{\pi}=1, r^{Is}=0} = 0.$$

From this it follows that

$$\frac{\partial^2 h(1)}{\partial (r^{Is})^2} \Big|_{r^{Is}=0} = 0.$$

Since we have

$$\frac{\partial^2 h(1)}{\partial (r^{Is})^2} \Big|_{r^{Is}=0} = 0 \text{ and } \frac{\partial h(1)}{\partial r^{Is}} \Big|_{r^{Is}=0} = 0,$$

it follows that $h(1) = 0$ for small quoted default swap spread. This, in turn, confirms that (2.27) holds for large $\bar{\pi}$ and small r^{Is} .

□

Chapter 3

CONTINUOUS-TIME MODELS FOR SUBPRIME RML SECURITIZATION

- 3.1 SUBPRIME RMBS PRICE PROCESS WITHOUT JUMPS
- 3.2 IB'S SUBPRIME RMBS LOSSES IN CONTINUOUS-TIME
- 3.3 CREDIT RATINGS
- 3.4 CREDIT DEFAULT SWAPS IN CONTINUOUS-TIME
- 3.5 IB'S PAYOUT UNDER SUBPRIME RML SECURITIZATION
- 3.6 STOCHASTIC OPTIMAL CREDIT DEFAULT INSURANCE
PROBLEM
- 3.7 STATEMENT OF THE OPTIMAL CREDIT DEFAULT INSURANCE
PROBLEM
- 3.8 SOLUTION TO THE OPTIMAL CREDIT DEFAULT INSURANCE
PROBLEM
 - 3.8.1 General Solution to the Optimal Credit Default Insurance Problem
 - 3.8.2 Optimal Credit Default Swap Contracts in Continuous-Time
 - 3.8.3 Boundary Value Problem
 - 3.8.4 Stochastic Optimal Credit Default Insurance with Exponential Utility
 - 3.8.5 Stochastic Optimal Credit Default Insurance with Power Utility
 - 3.8.6 Stochastic Optimal Credit Default Insurance with Logarithmic Utility

The contents of this chapter is based on [44], [45], [46], [47] and [51]. In this regard, we discuss subprime RML securitization and construct a stochastic model of IB's payout under such securitization in continuous-time. In particular, we consider the modeling of subprime RMBSs, RMBS losses, credit ratings and credit default insurance.

3.1 SUBPRIME RMBS PRICE PROCESS WITHOUT JUMPS

Excess spread/Overcollateralization (XS/OC) transactions are prevalent in subprime RML securitization. They are more complex than straight sen/sub 6-pack deals for typical prime and Alt-A structures. For subprime RML securitization, further complications for IB's payout arise from the *available funds cap risk* (AFC). Generally, RMBS bonds (liabilities) in XS/OC deals pay a floating coupon, r^I , while subprime RML reference portfolios (collateral) typically pay a fixed rate, r^M , until the reset date on hybrid adjustable rate mortgages (ARMs). In this case, the risk that interest paid into the deal from the RML reference portfolio, r^M , is not sufficient to make coupon payments, r^I , to RMBS bond holders may arise. To mitigate this situation, the deal may be subject to an AFC. Here, IBs receive interest as the minimum of the sum of the index rate, r^i , (i.e., 6-month LIBOR) and margin, ϱ , or the weighted average AFC, r^a . Symbolically, this means that

$$r^I = \min[r^i + \varrho, r^a]. \quad (3.1)$$

Given (3.1), the stochastic dynamics of the securitized subprime RML price process, P^I , may be represented via geometric Brownian motion as

$$dP_t^I = P_t^I \left[r_t^I dt + \sigma_t^I dZ_t^I \right], \quad (3.2)$$

where σ^I and Z^I are RMBS price volatility and standard Brownian motion, respectively.

3.2 IB'S SUBPRIME RMBS LOSSES IN CONTINUOUS-TIME

We suppose that losses suffered by IB from RML reference portfolio defaults is a random variable, S , with the distribution function, $F(S)$. In the sequel, we define this loss as

$$S : \Omega \rightarrow \mathbb{R}^+ = [0, \infty),$$

where Ω takes on nonnegative real values that may not necessarily be measurable. Moreover, let $\theta \geq 0$ be a nonnegative real number which is an upper bound of $S(\eta)$, for all $\eta \in \Omega$, where η is defined as IB's payout. Therefore, $\{\eta \in \Omega : S(\eta) > \theta\}$ is empty. This enables us

to define the smallest essential upper bound for the aggregate securitization losses, S , as

$$\text{ess sup } S(\eta) = \inf\{\theta \in R^+ : P(\{\eta : S(\eta) > \theta\}) = 0\}.$$

Furthermore, we assume that S is modeled as a compound Poisson process, for which $\tilde{N}$ is a Poisson process with a deterministic frequency parameter, $\phi(t)$. In this case, $\tilde{N}$ is stochastically independent of the Brownian motion, Z^I , given in (3.2).

3.3 CREDIT RATINGS

Concerns about credit ratings have resurfaced during the SMC, where banks have been allowed to use ratings to determine the risk attached to their subprime RML securitizations. Based on private and public information about RMBS quality and a published credit rating, IBs have to decide whether or not to continue investing in securitized RMLs.

We suppose that at time t , a continuum of RMBSs are eligible to be rated. To simplify notation, this mass of RMBSs is normalized to 1. There are two types of RMBSs, viz., A and B. If no shock occurs, type-A RMBSs have a low default probability, p^A , and type-B RMBSs have a high default probability, p^B , where $0 < p^A < p^B < 1$. Let $\Gamma(t)$ denote the mass of type-A RMBSs at time t , where $0 \leq \Gamma(t) \leq 1$. In principle, the value of Γ rises when perceived credit risk (or probability of default) is low and decreases when such risk is high. In general, there is substantial evidence to suggest that credit rating changes exhibit procyclical behavior.

At each t , RMBSs are uniformly located along the unit interval according to their type. The CRA chooses a fee and offers a rating to each RMBS. There are two rating categories, viz., $\mathfrak{A}$ and $\mathfrak{B}$. Rating category $\mathfrak{A}$ indicates that an RMBS is of type A (for instance, AAA, AA and Aaa) and rating category $\mathfrak{B}$ indicates that an RMBS is of type B (for instance, BBB, BB and B). The CRA chooses fee $f_t \in \mathfrak{R}_0^+$ and rating threshold $a_t \in [0, 1]$. The CRA offers RMBSs, who are located on or to the right of a_t on the unit interval, an $\mathfrak{A}$ rating, and RMBSs, who are located to the left of a_t on the unit interval, a $\mathfrak{B}$ rating. If $a_t = \Gamma(t)$, the CRA gives all type-A RMBSs an $\mathfrak{A}$ rating and all type-B RMBSs a $\mathfrak{B}$ rating. If $a_t > \Gamma(t)$, the CRA extends inflated ratings to the RMBS portfolio. If $a_t < \Gamma(t)$, the CRA offers type-B RMBSs an $\mathfrak{A}$ rating.

3.4 CREDIT DEFAULT SWAPS IN CONTINUOUS-TIME

IB's investment in securitized RMLs may yield substantial returns but may also result in losses as suggested in Subsection 3.2. In particular, our dynamic IB payout model allows for protection against such losses via CDS contracts (see Subsection 1.2.3 for more details). Our contribution uses CDSs to hedge risk rather than for speculation. In this process, we assume that the CDS premium paid by IB takes the form of a continuous contribution that

is expressed as

$$\Theta(C(S)) = [1 - \Gamma(u)]\phi(u)\mathbf{E}[C_u(S)], \quad u \geq t, \quad (3.3)$$

where Γ is given as in Subsection 3.3, S is IB's aggregate losses from RML securitization investments (see Subsection 3.2) and C is the payment to the swap protection buyer for such losses as in the premium leg. This means that if the losses are $S = l$ at time u , the payment, $C_u(l)$, equals the difference between par and the price of the CTD of the RML on the face value of protection.

3.5 IB'S PAYOUT UNDER SUBPRIME RML SECURITIZATION

In this subsection, we provide a stochastic model for IB's payout under subprime RML securitization. In the sequel, we denote IB's rate of return on investments in securitized RMLs by r^I , the rate of cash inflow from depositors to IB for investments in securitized subprime RMLs by μ^I , and the stochastic rate of cash outflow for fulfilling depositor obligations by k_u . In this case, the stochastic model of IB's payout under subprime RML securitization, is given by

$$\begin{aligned} d\Pi_u = & \left[r_u^I I_u + \mu_u^I - k_u - [1 - \Gamma(u)]\phi(u)\mathbf{E}[C_u(S)] \right] du + \sigma^I I_u dZ_u^I \\ & - \left(S(\Pi_u, u) - C_u(S(\Pi_u, u)) \right) d\tilde{N}_u, \quad u \geq t, \quad \Pi_t = \eta. \end{aligned} \quad (3.4)$$

If we assume that IB receives fixed coupons from its investment in RMBSs, i.e., $r_u^I = r^I$. In addition, let $\sigma^I = \sigma$ be a constant. Then the stochastic model (3.4) may be rewritten as

$$\begin{aligned} d\Pi_u = & \left[r^I I_u + \mu_u^I - k_u - [1 - \Gamma(u)]\phi(u)\mathbf{E}[C_u(S)] \right] du + \sigma I_u dZ_u^I \\ & - \left(S(\Pi_u, u) - C_u(S(\Pi_u, u)) \right) d\tilde{N}_u, \quad u \geq t, \quad \Pi_t = \eta. \end{aligned} \quad (3.5)$$

3.6 STOCHASTIC OPTIMAL CREDIT DEFAULT INSURANCE PROBLEM

In this subsection, we solve IB's stochastic optimal credit default insurance problem using the model of IB's payout under RML securitization described in equation (3.5) from Subsection 3.5.

3.7 STATEMENT OF THE OPTIMAL CREDIT DEFAULT INSURANCE PROBLEM

Let a set of control laws, $\mathcal{B}$, which is adapted to IB's payout under subprime RML securitization, Π , be given by

$$\mathcal{B} = \{(k_t, I_t, C_t) : \text{measurable w.r.t. filtration } \mathcal{F}_t; (3.5) \text{ has unique solution}\}. \quad (3.6)$$

The objective function of the stochastic optimal credit default insurance problem is given by

$$J(\eta, t) = \sup_{\mathcal{B}} \mathbf{E}^{t, \eta} \left[\int_t^T \exp\{-\delta^r(u-t)\} U^{(1)}(k_u) du + \exp\{-\delta^r(T-t)\} U^{(2)}(\Pi_t) \right], \quad (3.7)$$

where $DU^{(1)}(.) > 0$, $D^2U^{(1)}(.) < 0$, $DU^{(2)}(.) > 0$ and $D^2U^{(2)}(.) < 0$. Here, D and D^2 are the first- and second-order differential operators. Also, $U^{(1)}$ and $U^{(2)}$ are increasing, concave utility functions and $\delta^r > 0$ is the rate at which the utility functions of IB's rate of cash outflow for fulfilling depositor obligations, k , and terminal payout, Π_T are discounted. We considered several choices of utility functions such as power, logarithmic and exponential utility functions.

We are now in a position to state the stochastic optimal credit default insurance problem for a fixed adjustment period, $[t, T]$.

Problem 3.7.1 (Optimal Rate of Depositor Cash Outflow and Payout): *Suppose that the admissible class of control laws, $\mathcal{B} \neq \emptyset$, is given by (3.6). Moreover, let the controlled stochastic differential equation for the Π -dynamics be given by (3.5) and the objective function, $J : \mathcal{B} \rightarrow \mathbb{R}^+$, by (3.7). In this case, we solve*

$$\sup_{\mathcal{B}} J(\Pi_t; k_t, I_t, C_t),$$

and the optimal control law (k_t^*, I_t^*, C_t^*) , if it exists,

$$(k_t^*, I_t^*, C_t^*) = \arg \sup_{\mathcal{B}} J(\Pi_t; k_t, I_t, C_t) \in \mathcal{B}.$$

3.8 SOLUTION TO THE OPTIMAL CREDIT DEFAULT INSURANCE PROBLEM

In this subsection, we determine a solution to Problem 3.7.1 by using the dynamic programming method involving a Hamilton-Jacobi-Bellman equation (HJBE).

3.8.1 General Solution to the Optimal Credit Default Insurance Problem

In the sequel, we assume that the optimal control laws exist, with the objective function, J , given by (3.7) being continuous twice-differentiable. Next, we assume that $F(u, k_u) = \exp[-\delta^r(u - t)]U^{(1)}(k_u)$ and

$$J(\Pi_T, T) = \exp[-\delta^r(T - t)]U^{(2)}(\Pi_T).$$

Then (3.7) takes the form

$$J(\eta, t) = \sup_{\mathcal{B}} \mathbf{E}^{t, \eta} \left\{ \int_t^T F(u, k_u) du + J(\Pi_T, T) \right\} \quad (3.8)$$

and from (3.8), we see that

$$J(\eta, t) \geq \mathbf{E}^{t, \eta} \left[\int_t^T F(u, k_u) du + J(\Pi_T, T) \right] = \mathbf{E}^{t, \eta} \left[\int_t^T F(u, k_u) du \right] + \mathbf{E}^{t, \eta} \left[J(\Pi_T, T) \right].$$

Applying a form of Itô's formula that is appropriate to our problem, we have

$$\begin{aligned} dJ(\Pi_u, u) &= \frac{\partial J(\Pi_u, u)}{\partial t} du + \frac{\partial J(\Pi_u, u)}{\partial \eta} d\Pi_u + \frac{1}{2} \frac{\partial^2 J(\Pi_u, u)}{\partial \eta^2} (d\Pi_u)^2 \\ &= \frac{\partial J(\Pi_u, u)}{\partial \eta} \left[\left(r^I I + \mu_u^I - k_u - [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] \right) du \right] \\ &\quad + \frac{\partial J(\Pi_u, u)}{\partial \eta} \sigma I dZ_u^I + \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] d\tilde{N}_u \\ &\quad + \frac{1}{2} \frac{\partial^2 J(\Pi_u, u)}{\partial \eta^2} \sigma^2 I^2 du + \frac{\partial J(\Pi_u, u)}{\partial t} du. \end{aligned}$$

The above expression can be rewritten as

$$\begin{aligned} dJ(\Pi_u, u) &= \left[r^I I J_\eta(\Pi_u, u) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_u, u) \right. \\ &\quad + \mu_u^I J_\eta(\Pi_u, u) - k_u J_\eta(\Pi_u, u) - [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] J_\eta(\Pi_u, u) + J_t(\Pi_u, u) \left. \right] du \\ &\quad + J_\eta(\Pi_u, u) \sigma I dZ_u^I + \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] d\tilde{N}_u. \end{aligned}$$

Furthermore, if we integrate from t to T we obtain

$$\begin{aligned} J(\Pi_T, T) &= J(\Pi_t, t) + \int_t^T \left[r^I I J_\eta(\Pi_u, u) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_u, u) \right. \\ &\quad + \left. \mu_u^I J_\eta(\Pi_u, u) - k_u J_\eta(\Pi_u, u) - [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] J_\eta(\Pi_u, u) + J_t(\Pi_u, u) \right] du \\ &\quad + \int_t^T J_\eta(\Pi_u, u) \sigma I dZ_u^I + \int_t^T \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] d\tilde{N}_u. \end{aligned}$$

Taking the mathematical expectation of the above expression, we get

$$\begin{aligned} \mathbf{E} \left[J(\Pi_T, T) \right] &\geq \mathbf{E} \left[J(\Pi_T, T) \right] + \mathbf{E} \left[\int_t^T F(u, k_u) du \right] + \mathbf{E} \left[\int_t^T \left(r^I I J_\eta(\Pi_u, u) \right. \right. \\ &\quad + \left. \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_u, u) + \mu_u^I J_\eta(\Pi_u, u) - k_u J_\eta(\Pi_u, u) \right. \\ &\quad - \left. [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] J_\eta(\Pi_u, u) + J_t(\Pi_u, u) \right) du \Big] + \mathbf{E} \left[\int_t^T J_\eta(\Pi_u, u) \sigma I dZ_u^I \right] \\ &\quad + \mathbf{E} \left[\int_t^T \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] d\tilde{N}_u \right]. \end{aligned}$$

Assuming integrability as in [35], we have

$$\mathbf{E} \left[\int_t^T J_\eta(\Pi_u, u) \sigma I dZ_u^I \right] = 0.$$

It then follows that

$$\begin{aligned} \mathbf{E} \left[\int_t^T F(u, k_u) du \right] &+ \mathbf{E} \left[\int_t^T \left(r^I I J_\eta(\Pi_u, u) \right. \right. \\ &\quad + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_u, u) + \mu_u^I J_\eta(\Pi_u, u) - k_u J_\eta(\Pi_u, u) \\ &\quad - \left. [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] J_\eta(\Pi_u, u) + J_t(\Pi_u, u) \right) du \Big] \\ &\quad + \int_t^T \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] d\tilde{N}_u \leq 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{E} \left[\int_t^T \left(F(u, k_u) \right. \right. &+ \left. r^I I J_\eta(\Pi_u, u) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_u, u) \right. \\ &\quad + \left. \mu_u^I J_\eta(\Pi_u, u) - k_u J_\eta(\Pi_u, u) - [1 - \Gamma(u)] \phi(u) \mathbf{E}[C(S)] J_\eta(\Pi_u, u) + J_t(\Pi_u, u) \right. \\ &\quad + \left. \left. \mathbf{E} \left[J(\Pi_u - (S - C(S)), u) - J(\Pi_u, u) \right] \phi(u) \right) du \right] \leq 0. \end{aligned}$$

If we assume regularity as in [35], the above expression can be reduced to

$$\begin{aligned}
 F(t, k_t) &+ r^I I J_\eta(\Pi_t, t) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\Pi_t, t) \\
 &+ \mu_t^I J_\eta(\Pi_t, t) - k_t J_\eta(\Pi_t, t) - [1 - \Gamma(t)] \phi(t) \mathbf{E}[C(S)] J_\eta(\Pi_t, t) + J_t(\Pi_t, t) \\
 &+ \mathbf{E} \left[J(\Pi_t - (S - C(S)), t) - J(\Pi_t, t) \right] \phi(t) \leq 0.
 \end{aligned}$$

In the sequel, we recall that $\Pi_t = \eta$, and equality is attained at the maximum. Thus

$$\begin{aligned}
 J_t(\eta, t) &+ \max_{\mathcal{B}} \left\{ F(t, k_t) + r^I I J_\eta(\eta, t) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\eta, t) \right. \\
 &+ \mu_t^I J_\eta(\eta, t) - k_t J_\eta(\eta, t) - [1 - \Gamma(t)] \phi(t) \mathbf{E}[C(S)] J_\eta(\eta, t) \\
 &+ \left. \phi(t) \left[\mathbf{E}[J(\eta - (S - C(S)), t)] - J(\eta, t) \right] \right\} = 0.
 \end{aligned}$$

Again, since $F(t, k_t) = U^{(1)}(k)$ and $\mathcal{B}$ is given by (3.6), it follows that

$$\begin{aligned}
 J_t(\eta, t) &+ \max_k \left[U^{(1)}(k_t) - k_t J_\eta(\eta, t) \right] + \max_I \left[r^I I J_\eta(\eta, t) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\eta, t) \right] \\
 &+ \max_C \left[\phi(t) \left[\mathbf{E}[J(\eta - (S - C(S)), t)] - J(\eta, t) \right] - [1 - \Gamma(t)] \phi(t) \mathbf{E}[C(S)] J_\eta(\eta, t) \right] \\
 &+ \mu_t^I J_\eta(\eta, t) = 0.
 \end{aligned}$$

Also, the boundary condition from (3.8) implies that $J(\eta, T) = U^{(2)}(\eta)$. Then J satisfies the HJBE given by

$$\begin{cases}
 0 = \max_k \left[U^{(1)}(k) - k J_\eta(\eta, t) \right] + \max_I \left[r^I I J_\eta(\eta, t) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\eta, t) \right] \\
 + \max_C \left[\phi(t) \left(\mathbf{E}[J(\eta - (S - C(S)), t)] - J(\eta, t) \right) - [1 - \Gamma(t)] \phi(t) \mathbf{E}[C(S)] J_\eta(\eta, t) \right] \\
 + \mu_t^I J_\eta(\eta, t) + J_t(\eta, t) \\
 J(\eta, T) = U^{(2)}(\eta).
 \end{cases} \quad (3.9)$$

Note that

$$J_t = \frac{\partial J}{\partial t}, \quad J_\eta = \frac{\partial J}{\partial \eta} \quad \text{and} \quad J_{\eta\eta} = \frac{\partial^2 J}{\partial \eta^2}.$$

The objective function, J , is increasing and concave with respect to IB's payout, η , because the utility functions $U^{(1)}$ and $U^{(2)}$ are increasing and concave. It is important to note that we can use verification theorems to show that if the objective function, J , has a smooth solution as well as the related HJBE, $\widehat{J}$, then under the regularity conditions in this thesis, we have $J = \widehat{J}$.

Theorem 3.8.1 (Optimal Rate of Depositor Cash Outflow and Payout): *Suppose that the objective function, $J(\eta, t)$, solves the HJBE (3.9). In this case, a solution to IB's stochastic optimal credit default insurance problem is*

$$I_t^* = -\frac{r^I}{\sigma^2} \frac{J_\eta(\eta, t)}{J_{\eta\eta}(\eta, t)}, \quad (3.10)$$

where $\Pi_t^* = \eta$ is IB's optimally controlled payout under subprime RML securitization. Also, the optimal cash outflow for satisfying depositor obligations, $\{k_t^*\}_{t \geq 0}$, solves the equation

$$D_k U^{(1)}(k_t^*) = J_\eta(\eta, t), \quad (3.11)$$

where D_k represents the ordinary derivative with respect to k .

Proof. In our proof, we consider a static optimization problem given by

$$\max_k \left[U^{(1)}(k) - k J_\eta(\eta, t) \right] + \max_I \left[r^I I J_\eta(\eta, t) + \frac{1}{2} \sigma^2 I^2 J_{\eta\eta}(\eta, t) \right]. \quad (3.12)$$

In order to verify (3.10) and (3.11), we differentiate the expression inside the square brackets of (3.12) with respect to I and k . Then we set the resulting partial derivatives to zero so that

$$r^I J_\eta(\eta, t) + \sigma^2 I^* J_{\eta\eta}(\eta, t) = 0 \quad (3.13)$$

$$D_k U^{(1)}(k_t^*) - J_\eta(\eta, t) = 0 \quad (3.14)$$

From (3.13) and (3.14), we have that

$$I_t^* = -\frac{r^I}{\sigma^2} \frac{J_\eta(\eta, t)}{J_{\eta\eta}(\eta, t)}$$

and

$$D_k U^{(1)}(k_t^*) = J_\eta(\eta, t),$$

respectively. An alternative method of proof of Theorem 3.8.1 is by using the martingale approach that is expounded in [14] and [32]. $\square$

3.8.2 Optimal Credit Default Swap Contracts in Continuous-Time

In this subsection, we determine the optimal CDS contract, C_t^* , when the premium is given by the functional, $\Theta(C)$, in (3.3). In this chapter, we consider a CDS market which can trade the type of CDS contract described earlier. In addition, we assume that $0 \leq C \leq S$. Taking our lead from insurance theory and the assumption that $\Theta(C)$ is proportional to

the nett CDS premium for a portfolio with mass of type-A RMBSs, Γ , the optimal CDS contract takes the form

$$C(S) = \begin{cases} 0, & \text{if } S \leq \Phi; \\ S - \Phi, & \text{if } S > \Phi. \end{cases} \quad (3.15)$$

Some features of the aforementioned CDS contract are as follows. If $S \leq \Phi$, then it would be optimal for IB not to buy CDS protection. If $S > \Phi$, then it would be optimal to buy CDS protection. In the sequel, the maximization of the CDS contract purchased by IB is now reduced to the problem of determining the optimal accrued premium, Φ .

Proposition 3.8.2 (Optimal Credit Default Insurance): *The optimal CDS contract is either no swap protection or per-loss accrued premium CDSs, in which the accrued premium, Φ , varies with time. In particular, at a specified time, the optimal accrued premium, Φ_t^* , solves*

$$J'(\Pi_t^* - \Phi_t, t) = [1 - \Gamma(t)]J_\eta(\Pi_t^*, t). \quad (3.16)$$

No CDSs contract is optimal at time t if and only if

$$J'(\Pi_t^* - \text{ess sup} S(\Pi_t^*, t), t) \leq [1 - \Gamma(t)]J_\eta(\Pi_t^*, t). \quad (3.17)$$

Proof. We consider

$$\max_{\mathcal{C}} \left[\phi(t) \left(\mathbb{E}[J(\eta - (S - C(S)), t)] - J(\eta, t) \right) - [1 - \Gamma(t)]\phi(t)\mathbb{E}[C(S)]J_\eta(\eta, t) \right]. \quad (3.18)$$

Let H be the expression inside the square brackets of (3.18) so that

$$H = \phi(t) \left\{ \mathbb{E}[J(\eta - (S - C(S)), t)] - J(\eta, t) \right\} - [1 - \Gamma(t)]\phi(t)\mathbb{E}[C(S)]J_\eta(\eta, t) \quad (3.19)$$

If we utilize (3.15), then (3.19) becomes

$$\begin{aligned} H = \phi(t) \left\{ \int_0^\Phi J(\eta - S, t) dF(S) + \int_\Phi^\infty J(\eta - \Phi, t) dF(S) - J(\eta, t) \right\} \\ - [1 - \Gamma(t)]\phi(t)J_\eta(\eta, t) \int_\Phi^\infty (S - \Phi) dF(S). \end{aligned} \quad (3.20)$$

Differentiating (3.20) with respect to Φ , we obtain

$$\begin{aligned} \frac{dH}{d\Phi} &= \phi(t) \left\{ J'(\eta - \Phi, t) \int_{\Phi}^{\infty} dF(S) \right\} + [1 - \Gamma(t)] \phi(t) J_{\eta}(\eta, t) \int_{\Phi}^{\infty} dF(S) \quad (3.21) \\ &= -\phi(t) J'(\eta - \Phi, t) [1 - F(\Phi)] + [1 - \Gamma(t)] \phi(t) J_{\eta}(\eta, t) [1 - F(\Phi)]. \end{aligned}$$

To determine the optimal accrued premium we set (3.21) to zero, so that

$$-\phi(t) J'(\eta - \Phi, t) [1 - F(\Phi)] + [1 - \Gamma(t)] \phi(t) J_{\eta}(\eta, t) [1 - F(\Phi)] = 0. \quad (3.22)$$

From the above equation we can produce (3.16) so that

$$J'(\eta - \Phi, t) = [1 - \Gamma(t)] J_{\eta}(\eta, t). \quad (3.23)$$

The expression (3.17) follows immediately from (3.23). $\square$

In order to determine an exact (closed form) solution for the stochastic optimization problem in Theorem 3.8.1, we are required to make a specific choice for the utility functions $U^{(1)}$ and $U^{(2)}$. Essentially these functions can be almost any function involving k and η , respectively. However, in order to obtain smooth analytic solutions to the stochastic optimal credit default insurance problem, in the ensuing discussion, we choose power, logarithmic and exponential utility functions and analyze the effect of the different choices.

3.8.3 Boundary Value Problem

From what we have done so far, we see that by substituting the optimal control processes, I_t^* and k_t^* , our problem can be further reduced to a boundary value problem (BVP) that consists of a partial differential equation involving the unknown objective function, J , and one boundary condition. Our argument leads to a BVP that may be expressed as

$$\begin{cases} J_t(\eta, t) + U^{(1)}(k^*) - k^* J_{\eta}(\eta, t) + r^I I^* J_{\eta}(\eta, t) + \frac{1}{2} \sigma^2 I^{*2} J_{\eta\eta}(\eta, t) \\ + \phi(t) (\mathbf{E}[J(\eta - \Phi^*, t)] - J(\eta, t)) - [1 - \Gamma(t)] \phi(t) \mathbf{E}[S - \Phi^*] J_{\eta}(\eta, t) + \mu_t^I J_{\eta}(\eta, t) = 0 \\ J(\eta, T) = U^{(2)}(\eta). \end{cases} \quad (3.24)$$

In the sequel, we determine the objective function $J(\eta, t)$ which solves (3.24).

3.8.4 Stochastic Optimal Credit Default Insurance with Exponential Utility

Assume that

$$U^{(1)}(k) = 0 \text{ and } U^{(2)}(\eta) = -\frac{1}{\varrho} \exp\{-\varrho\eta\}, \quad \varrho > 0. \quad (3.25)$$

The following proposition provides a closed form solution to Problem 3.7.1 in the case of an exponential utility function.

Proposition 3.8.3 (Stochastic Optimal Credit Default Insurance with Exponential Utility): *Let the exponential utility functions be given by (3.25) and assume that IB's RML securitization losses, S , are independent of its payout, with the probability distribution of S being a deterministic function of time. Under the exponential utility, the objective function is given by*

$$J(\eta, t) = -\frac{1}{\varrho} \exp\left(-\varrho\eta - \frac{r^I 2}{2\sigma^2}(T-t)\right) \vartheta(t), \quad (3.26)$$

with $\vartheta(t)$ being given by

$$\vartheta(t) = \exp\left\{-\int_t^T Y(r)dr\right\}, \quad (3.27)$$

where Y has the form

$$Y(r) = \phi(r)\mathbb{E}[\exp(\varrho\Phi^*)] - \phi(r) + [1 - \Gamma(r)]\phi(r)\varrho\mathbb{E}[S - \Phi^*] - \mu_r^I \varrho.$$

Then IB's optimal investment in securitized subprime RMLs is

$$I_t^* = \frac{r^I}{\sigma^2 \varrho}. \quad (3.28)$$

Furthermore, $U^{(1)}(k_t) = 0$ leads to the optimal rate of cash outflow for satisfying depositor obligations being equal to zero, so that

$$k_t^* = 0. \quad (3.29)$$

The optimal accrued premium is given by

$$\Phi_t^* = \min \left\{ \frac{1}{\varrho} \ln(1 - \Gamma(t)), \text{ess sup } S(t) \right\}. \quad (3.30)$$

Proof. For the objective function

$$J(\eta, t) = -\frac{1}{\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t)$$

we have that

$$\begin{aligned} J_t(\eta, t) &= -\frac{r^{I2}}{2\sigma^2\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t) - \frac{1}{\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta'(t), \\ J_\eta(\eta, t) &= \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t), \quad J_{\eta\eta}(\eta, t) = -\varrho \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t). \end{aligned}$$

If we consider the above partial derivatives as well as Theorem 3.8.1 and Proposition 3.8.2, we obtain the optimal control laws (3.28), (3.29) and (3.30). If we substitute the control laws, i.e., B_t^* , k_t^* and Φ_t^* into (3.9), then we obtain a solution for the BVP in (3.24) so that

$$\begin{aligned} & -\frac{1}{\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta'(t) - \phi(t) \frac{1}{\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t) \times \mathbf{E}[\exp(\varrho\Phi^*)] \\ & + \phi(t) \frac{1}{\varrho} \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t) - [1 - \Gamma(t)] \phi(t) \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t) \mathbf{E}[S - \Phi^*] \\ & + \mu_t^I \exp \left(-\varrho\eta - \frac{r^{I2}}{2\sigma^2}(T-t) \right) \vartheta(t) = 0, \\ & \vartheta'(t) + \phi(t) \vartheta(t) \mathbf{E}[\exp(\varrho\Phi^*)] - \phi(t) \vartheta(t) + [1 - \Gamma(t)] \phi(t) \varrho \vartheta(t) \mathbf{E}[S - \Phi^*] - \mu_t^I \varrho \vartheta(t) = 0 \end{aligned}$$

and $\vartheta'(t) + \vartheta(t) \left[\phi(t) \mathbf{E}[\exp(\varrho\Phi^*)] - \phi(t) + [1 - \Gamma(t)] \phi(t) \varrho \mathbf{E}[S - \Phi^*] - \mu_t^I \varrho \right] = 0$. The boundary condition which corresponds to the ordinary differential equation above is given by

$$J(\eta, T) = -\frac{1}{\varrho} \exp(-\varrho\eta) \vartheta(T) = -\frac{1}{\varrho} \exp(-\varrho\eta), \quad \vartheta(T) = 1.$$

If we set $Y(t) = \phi(t) \mathbf{E}[\exp(\varrho\Phi^*)] - \phi(t) + [1 - \Gamma(t)] \phi(t) \varrho \mathbf{E}[S - \Phi^*] - \mu_t^I \varrho$ then, $\vartheta(t)$ is a solution to the BVP given by

$$\begin{cases} \vartheta'(t) + Y(t) \vartheta(t) = 0 \\ \vartheta(T) = 1. \end{cases}$$

This verifies that $\vartheta(t)$ is indeed of the form (3.27). The objective function (3.26) together with the optimal control processes (3.28) and (3.29) solve the BVP (3.24). From the verification theorems, we can also conclude that (3.26) is the optimal value function. In addition,

the uniqueness of the solution to (3.5) under exponential utility can be established via [57, Chapter V, Section 3]. $\square$

3.8.5 Stochastic Optimal Credit Default Insurance with Power Utility

For a choice of power utility we have that

$$\overline{U}^{(1)}(k) = \frac{k^\varrho}{\varrho} \text{ and } \overline{U}^{(2)}(\eta) = \tilde{b} \frac{\eta^\varrho}{\varrho}, \quad (3.31)$$

for some $\varrho < 1$, $\varrho \neq 0$, and $\tilde{b} \geq 0$. The parameter $\tilde{b}$ represents the weight that IB gives to terminal payout versus the rate of cash outflow for fulfilling depositor obligations and can be viewed as a measure of IB's propensity to retain earnings. This leads to the following result.

Proposition 3.8.4 (Stochastic Optimal Credit Default Insurance with Power Utility): *Suppose that the power utility functions are given as in (3.31) and assume that the RML securitization losses, S , are proportional to IB's payout under subprime RML securitization so that*

$$S(\eta, t) = \varphi(t)\eta,$$

for some deterministic S and severity function, $\varphi(t)$, where $0 \leq \varphi(t) \leq 1$. Under power utility, the objective function may be represented by

$$\overline{J}(\eta, t) = \frac{\eta^\varrho}{\varrho} \zeta(t), \quad (3.32)$$

where $\zeta(t)$ is given by

$$\zeta(t) = \left[\tilde{b} \exp \left(- \int_t^T \frac{Q(s)}{1-\varrho} ds \right) + \frac{\varrho}{1-\varrho} \exp \left(- \int_t^T \frac{Q(s)}{1-\varrho} ds \right) \int_t^T \exp \left(\int_t^s \frac{Q(u)}{1-\varrho} du \right) ds \right]^{1-\varrho} \quad (3.33)$$

with Q having the form

$$\begin{aligned} Q(t) &= \phi(t) \left(1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right)^\varrho - 1 \Big] + \Delta \\ &- [1 - \Gamma(t)] \phi(t) \varrho \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] + \mu_t^I \varrho \eta^{-1} \end{aligned}$$

and

$$\Delta = \frac{r^{I2}}{2\sigma^2(1-\varrho)} \varrho.$$

In this case, IB's optimal rate of cash outflow for fulfilling depositor obligations is given by

$$k_t^* = \zeta(t)^{-\frac{1}{1-\varrho}} \eta, \quad (3.34)$$

and IB's optimal investment in securitized RMLs is

$$I_t^* = \frac{r^I}{\sigma^2(1-\varrho)} \eta. \quad (3.35)$$

Furthermore, under power utility, the optimal accrued premium is given by

$$\Phi_t^* = \min \left\{ \left[1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}} \right], \varphi(t) \right\} \eta. \quad (3.36)$$

Proof. The proof of Proposition 3.8.4 is analogous to that of Proposition 3.8.3. In this regard, for the objective function

$$\bar{J}(\eta, t) = \frac{\eta^\varrho}{\varrho} \zeta(t)$$

we have that

$$\bar{J}_t(\eta, t) = \frac{\eta^\varrho}{\varrho} \zeta'(t), \quad \bar{J}_\eta(\eta, t) = \eta^{\varrho-1} \zeta(t) \quad \text{and} \quad \bar{J}_{\eta\eta}(\eta, t) = (\varrho - 1) \eta^{\varrho-2} \zeta(t).$$

Using the partial derivatives above and Theorem 3.8.1 as well as Proposition 3.8.2, we can verify that the optimal control laws are given by (3.34), (3.35) and (3.36). If we substitute these control laws into (3.24), we obtain

$$\begin{aligned} & \frac{\eta^\varrho}{\varrho} \zeta'(t) + \frac{\eta^\varrho}{\varrho} \zeta(t)^{-\frac{\varrho}{1-\varrho}} - \eta^\varrho \zeta(t)^{-\frac{\varrho}{1-\varrho}} + \frac{1}{2} \frac{r^{I2}}{\sigma^2(1-\varrho)} \eta^\varrho \zeta(t) \\ & + \phi(t) \left\{ \frac{\eta^\varrho}{\varrho} \zeta(t) \left[1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right]^\varrho - \frac{\eta^\varrho}{\varrho} \zeta(t) \right\} \\ & - [1 - \Gamma(t)] \phi(t) \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] \eta^\varrho \zeta(t) + \mu_t^I \eta^{\varrho-1} \zeta(t) = 0 \end{aligned}$$

so that

$$\begin{aligned} & \zeta'(t) + \zeta(t)^{-\frac{\varrho}{1-\varrho}} - \varrho \zeta(t)^{-\frac{\varrho}{1-\varrho}} + \frac{1}{2} \frac{r^{I2}}{\sigma^2(1-\varrho)} \varrho \zeta(t) + \phi(t) \zeta(t) \left[1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right]^\varrho \\ & - \phi(t) \zeta(t) - [1 - \Gamma(t)] \phi(t) \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] \varrho \zeta(t) + \mu_t^I \varrho \eta^{-1} \zeta(t) = 0. \end{aligned}$$

In turn, the above expression implies that

$$\begin{aligned} & \zeta'(t) + \zeta(t) \left\{ \phi(t) \zeta(t) \left(1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right)^{\varrho} - 1 \right\} + \frac{1}{2} \frac{r^{I2}}{\sigma^2(1-\varrho)} \varrho \\ & - [1 - \Gamma(t)] \phi(t) \varrho \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] + \mu_t^I \varrho \eta^{-1} \Big\} = [\varrho - 1] \zeta(t)^{-\frac{\varrho}{1-\varrho}}. \end{aligned}$$

This ordinary differential equation is of Bernoulli-type with a boundary condition given by $\zeta(T) = \tilde{b} \geq 0$. Next, we set $\Delta = \frac{1}{2} \frac{r^{I2}}{\sigma^2(1-\varrho)} \varrho$, so that

$$\begin{aligned} & \zeta'(t) + \zeta(t) \left\{ \phi(t) \left(1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right)^{\varrho} - 1 \right\} + \Delta \\ & - [1 - \Gamma(t)] \phi(t) \varrho \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] + \mu_t^I \varrho \eta^{-1} \Big\} = [\varrho - 1] \zeta(t)^{-\frac{\varrho}{1-\varrho}}. \end{aligned}$$

Furthermore, let

$$\begin{aligned} Q(t) &= \phi(t) \left(1 - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right)^{\varrho} - 1 \Big] + \Delta \\ &- [1 - \Gamma(t)] \phi(t) \varrho \left[\varphi(t) - \min \left(1 - (1 - \Gamma(t))^{-\frac{1}{1-\varrho}}, \varphi(t) \right) \right] + \mu_t^I \varrho \eta^{-1}, \\ \varrho - 1 &= \tilde{\varrho} \quad \text{and} \quad \bar{\varrho} = \frac{\varrho}{1-\varrho}. \end{aligned}$$

We restate our problem as

$$\begin{cases} \zeta'(t) + Q(t) \zeta(t) = \tilde{\varrho} \zeta(t)^{-\bar{\varrho}} \\ \zeta(T) = \tilde{b} \geq 0. \end{cases} \quad (3.37)$$

In order to solve the above ordinary differential equation, we let $z = \zeta^{1+\bar{\varrho}}$ and divide (3.37) by $\zeta^{-\bar{\varrho}}$, so that

$$\zeta^{\bar{\varrho}} \frac{d\zeta}{dt} + Q(t) \zeta^{1+\bar{\varrho}} = \tilde{\varrho}.$$

Since $z = \zeta^{1+\bar{\varrho}}$, we have that

$$\zeta^{\bar{\varrho}} \frac{d\zeta}{dt} = \frac{1}{1+\bar{\varrho}} \frac{dz}{dt}$$

and it follows that

$$\frac{dz}{dt} + (1 + \bar{\varrho})Q(t)z = \bar{\varrho}(1 + \bar{\varrho}).$$

From the above equation, we see that the integrating factor is given by

$$\exp \left(\int (1 + \bar{\varrho})Q(t)dt \right).$$

The general solution of the above ODE is given by

$$z(t) = \exp \left(- \int_t^T (1 + \bar{\varrho})Q(s)ds \right) \left[\bar{A} + \int_t^T \bar{\varrho} \exp \left(\int_t^s (1 + \bar{\varrho})Q(u)du \right) ds \right], \quad z(T) = \bar{A} = \bar{b} \geq 0.$$

This yields the expression

$$\begin{aligned} z(t) &= \bar{b} \exp \left(- \int_t^T (1 + \bar{\varrho})Q(s)ds \right) + \bar{\varrho} \exp \left(- \int_t^T (1 + \bar{\varrho})Q(s)ds \right) \int_t^T \exp \left(\int_t^s (1 + \bar{\varrho})Q(u)du \right) ds \\ &= \bar{b} \exp \left(- \int_t^T \frac{Q(s)}{1 - \varrho} ds \right) + \frac{\varrho}{1 - \varrho} \exp \left(- \int_t^T \frac{Q(s)}{1 - \varrho} ds \right) \int_t^T \exp \left(\int_t^s \frac{Q(u)}{1 - \varrho} du \right) ds. \end{aligned}$$

Again from $z = \zeta^{1+\bar{\varrho}}$, we have $\zeta(t) = z^{\frac{1}{1+\bar{\varrho}}}$, which implies that

$$\zeta(t) = \left[\bar{b} \exp \left(- \int_t^T \frac{Q(s)}{1 - \varrho} ds \right) + \frac{\varrho}{1 - \varrho} \exp \left(- \int_t^T \frac{Q(s)}{1 - \varrho} ds \right) \int_t^T \exp \left(\int_t^s \frac{Q(u)}{1 - \varrho} du \right) ds \right]^{1-\varrho}.$$

As before, (3.32) and the corresponding optimal control laws (3.34) and (3.35) satisfy the BVP (3.24) with (3.32) being an optimal value function. A consideration of [57, Chapter V, Section 3] yields a unique solution to (3.5) under power utility. $\square$

3.8.6 Stochastic Optimal Credit Default Insurance with Logarithmic Utility

Suppose we let $\varrho \rightarrow 0$ in Proposition 3.8.4, so that

$$\tilde{U}^{(1)}(k) = \ln k \text{ and } \tilde{U}^{(2)}(\eta) = \bar{b} \ln \eta. \quad (3.38)$$

In this case, it is clear that the following special extension to Proposition 3.8.4 holds.

Proposition 3.8.5 (Stochastic Optimal Credit Default Insurance with Logarithmic Utility): *Suppose the logarithmic utility functions are given as in (3.38). In this case, we have that the objective function has the form*

$$\tilde{J}(\eta, t) = \tilde{b} \ln \eta + P(t), \quad (3.39)$$

where $P(t)$ is defined by

$$P(t) = - \int_t^T \tilde{C}(s) ds \quad (3.40)$$

and

$$\begin{aligned} \tilde{C}(t) &= \ln \left[\frac{\eta}{\tilde{b}} \right] + \frac{1}{2} \frac{r^{I2}}{\sigma^2} \tilde{b} + \phi(t) \tilde{b} \ln \left[1 - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] \\ &- [1 - \Gamma(t)] \phi(t) \tilde{b} \left[\varphi(t) - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] + \mu_t^I \tilde{b} \eta^{-1} - 1. \end{aligned}$$

In this case, IB's optimal rate of cash outflow for fulfilling depositor obligations is given by

$$k_t^* = \frac{\eta}{\tilde{b}} \quad (3.41)$$

and its optimal investment in securitized RMLs is

$$I_t^* = \eta \frac{r^I}{\sigma^2}. \quad (3.42)$$

Also, in this case, IB's optimal accrued premium is given by

$$\Phi_t^* = \min \left\{ -\frac{\Gamma(t)}{[1 - \Gamma(t)]}, \varphi(t) \right\} \eta. \quad (3.43)$$

Proof. As before, Theorem 3.8.1 is the main tool for verifying (3.41) and (3.42) as optimal control laws. In this regard, we note that the objective function

$$\tilde{J}(\eta, t) = \tilde{b} \ln \eta + P(t)$$

has the partial derivatives

$$\tilde{J}_t = P'(t), \quad \tilde{J}_\eta = \frac{\tilde{b}}{\eta} \quad \text{and} \quad \tilde{J}_{\eta\eta} = -\frac{\tilde{b}}{\eta^2}.$$

Substituting (3.41), (3.42) and (3.43) into (3.24), we obtain

$$\begin{aligned} P'(t) &+ \ln \left[\frac{\eta}{\tilde{b}} \right] - 1 + \frac{1}{2} \frac{r^{I2}}{\sigma^2} \tilde{b} + \phi(t) \tilde{b} \ln \left[1 - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] \\ &- [1 - \Gamma(t)] \phi(t) \tilde{b} \left[\varphi(t) - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] + \mu_t^I \tilde{b} \eta^{-1} = 0. \end{aligned}$$

If we let

$$\begin{aligned} \tilde{C}(t) &= \ln \left[\frac{\eta}{\tilde{b}} \right] + \frac{1}{2} \frac{r^{I2}}{\sigma^2} \tilde{b} + \phi(t) \tilde{b} \ln \left[1 - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] \\ &- [1 - \Gamma(t)] \phi(t) \tilde{b} \left[\varphi(t) - \min \left(-\frac{\Gamma(t)}{1 - \Gamma(t)}, \varphi(t) \right) \right] + \mu_t^I \tilde{b} \eta^{-1} - 1, \end{aligned}$$

then it follows that $P'(t) + \tilde{C}(t) = 0$. Since $P(T) = 0$, the solution of this ODE is given by (3.40). From the above, we conclude that substituting the optimal objective function (3.39) as well as (3.41) and (3.42) into the right-hand side of (3.24) leads to the desired result. Finally, [57, Chapter V, Section 3] yields the unique solution to (3.5) under logarithmic utility. $\square$

Chapter 4

DISCONTINUOUS-TIME MODELS FOR SUBPRIME RML SECURITIZATION

4.1 SUBPRIME RMBS PRICE PROCESS WITH JUMPS

4.2 TREASURIES

4.3 IB'S SUBPRIME RMBS LOSSES IN DISCONTINUOUS-TIME

4.4 STOCHASTIC DYNAMICS OF IB'S INVESTMENT IN SUBPRIME RMBSs AND TREASURIES

4.5 IB'S OPTIMAL INVESTMENT IN SUBPRIME RMBSs AND TREASURIES

4.5.1 IB's Optimal Investment Problem with Variance

4.5.2 IB's Optimal Investment Problem with Swaps at Risk

4.5.2.1 Statement of IB's Optimal Investment Problem with Swaps-at-Risk

4.5.2.2 Solution of IB's Optimal Investment Problem with Swaps-at-Risk

4.5.3 Numerical Procedure

4.5.3.1 Gaussian Diffusion Model for Subprime RML Securitization

4.5.3.2 Numerical Algorithm for Problem 4.5.4

The contents of this chapter is based on research presented in [11], [12], [25], [50] and [49]. In this regard, we consider the subprime mortgage credit as well as IB's subprime RMBS losses. Also, we construct a stochastic dynamic model for IB's investment in subprime RMBSs and Treasuries.

4.1 SUBPRIME RMBS PRICE PROCESS WITH JUMPS

In the sequel, IB invests a proportion of its funds in subprime RMBSs issued by SPV with stochastic rate of return, r_t^I . This rate is distributed according to some cumulative distribution function, $\tilde{H}$. IB's investment in subprime RMBS portfolios enables OR to expand its subprime lending activities. Of course, the subprime RMBS rate, r_t^I , earned by IB is a function of the subprime RML rate, r_t^M , paid by MR, so that

$$r_t^I = y(r_t^M).$$

In our thesis, the mathematical expectation of the rate of return on subprime RMBSs is denoted by $\tilde{\mu}^I = \mathbb{E}[r_t^I]$. We assume that the dynamics of the subprime RMBS price process is given by

$$dM_t = M_{t-} \left\{ \left(\tilde{\mu}^I - \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \sigma dZ_t + \sum_{i=1}^n \alpha_i dN_i(t) \right\}, \quad t \geq 0, \quad M_0 = \iota, \quad (4.1)$$

where $\tilde{\mu}^I \in \mathbb{R}$, $\sigma > 0$ is a constant volatility of M_t , $n \in \mathbb{N}$, and for $i = 1, \dots, n$ the process N_i is a homogeneous Poisson process with intensity λ_i . In particular, N_i is the number of jumps of size α_i of the Lévy process, L_t . In order to avoid negative subprime RMBS prices, we make the assumption that

$$-1 < \alpha_1 < \dots < \alpha_n < \infty. \quad (4.2)$$

Lemma 4.1.1 (Solution of Stochastic Dynamics (4.1)): *Suppose that the dynamic of subprime RMBS prices, M_t , is represented by (4.1) and condition (4.2) holds. Then M_t can explicitly expressed as*

$$M_t = \iota \exp \left\{ \left(\tilde{\mu}^I - \frac{1}{2} \sigma^2 - \sum_{i=1}^n \alpha_i \lambda_i \right) t + \sigma Z_t + \sum_{i=1}^n (N_i(t)) \ln(1 + \alpha_i) \right\}, \quad t \geq 0, \quad M_0 = \iota. \quad (4.3)$$

Proof. Let $G(t) = G(M_t) = \ln M_t$. If we apply Itô's formula from Lemma 8.1.1 to our problem, we have that

$$\begin{aligned} dG &= \frac{dG(M_t)}{dM_t} \left(\tilde{\mu}^I - \sum_{i=1}^n \alpha_i \lambda_i \right) M_t dt + \frac{dG(M_t)}{dM_t} M_t \sigma dZ_t + \frac{1}{2} \frac{d^2 G(M_t)}{dM_t^2} M_t^2 \sigma^2 dt \\ &+ \sum_{i=1}^n \left\{ \ln(M_t + \alpha_i M_t) - \ln M_t \right\} dN_i(t). \end{aligned} \quad (4.4)$$

In this case, we observe that

$$\frac{dG}{dM_t} = \frac{1}{M_t}, \quad \frac{d^2 G}{dM_t^2} = -\frac{1}{M_t^2}. \quad (4.5)$$

Substituting (4.5) into (4.4), we have that

$$\begin{aligned} d \ln M_t &= \frac{1}{M_t} \left(\tilde{\mu}^I - \sum_{i=1}^n \alpha_i \lambda_i \right) M_t dt + \frac{1}{M_t} M_t \sigma dZ_t - \frac{1}{2 M_t^2} M_t^2 \sigma^2 dt + \sum_{i=1}^n \ln(1 + \alpha_i) dN_i(t) \\ &= \left(\tilde{\mu}^I - \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \sigma dZ_t - \frac{1}{2} \sigma^2 dt + \sum_{i=1}^n \ln(1 + \alpha_i) dN_i(t). \end{aligned}$$

Integrating the above equation from 0 to t , we obtain

$$\ln \frac{M_t}{M_0} = \left(\tilde{\mu}^I - \sum_{i=1}^n \alpha_i \lambda_i \right) t + \sigma Z_t - \frac{1}{2} \sigma^2 t + \sum_{i=1}^n (N_i(t)) \ln(1 + \alpha_i).$$

Therefore,

$$M_t = M_0 \exp \left\{ \left(\tilde{\mu}^I - \frac{1}{2} \sigma^2 - \sum_{i=1}^n \alpha_i \lambda_i \right) t + \sigma Z_t + \sum_{i=1}^n (N_i(t)) \ln(1 + \alpha_i) \right\}, \quad (4.6)$$

$t \geq 0, \quad M_0 = M.$

□

4.2 TREASURIES

Treasuries are bonds issued by national Treasuries and are the debt financing instruments of the federal government. There are four types of Treasuries: treasury bills, treasury notes, treasury bonds and savings bonds. All of the treasuries besides savings bonds are very liquid and are heavily traded on the secondary market. During the SMC, IBs hold their investments in safe assets such as Treasuries. However, such investments contribute to the prolonging of the crisis. In the sequel, we denote the *interest rate on Treasuries* or *Treasuries rate* by r^T . Suppose that the value process of the *Treasuries*, T , follows

$$T(t) = \exp\{r^T t\}, \quad t \geq 0.$$

By using Lemma 8.1.1, the corresponding SDE for the above value process becomes an ODE which is given by

$$dT(t) = r^T T(t) dt, \quad t > 0, \quad T(0) = 1. \quad (4.7)$$

This form is indicative of the fact that the value process for the Treasuries is riskless because it does not contain a volatility term.

4.3 IB'S SUBPRIME RMBS LOSSES IN DISCONTINUOUS-TIME

We suppose that the nett RMBS losses that IB experiences is a random variable, S , with the distribution function, $F(S)$. In the sequel, we define this loss $S : \Omega \rightarrow \mathbb{R}^+ = [0, \infty)$ as

$$S_t = \tilde{S}_t - C_t, \quad (4.8)$$

where $\tilde{S}$ and C are the total RMBS losses and swap protection seller payout, respectively. S is a random variable defined on Ω with nonnegative real values, which are not necessarily measurable. Moreover, we make an assumption that IB's nett RMBS losses, S , is modeled as a compound Poisson process, in which N is a Poisson process with a deterministic frequency parameter, $\phi(t)$. Here, N is stochastically independent of the Brownian motion, Z_t , given in (1.10).

4.4 STOCHASTIC DYNAMICS OF IB'S INVESTMENT IN SUBPRIME RMBSs AND TREASURIES

In this subsection, we construct a stochastic dynamics model of IB's investment in subprime RMBSs and Treasuries. IB starts with initial funds, B_0 , and the expected rate of return on subprime RMBSs, $\tilde{\mu}^I \in \mathbb{R}$. The investment in subprime RMBSs may yield substantial return but may also result in losses. The subprime RMBS losses, S , are modeled by a compound Poisson process. Moreover, for the planning term $[0, T]$, we consider a characterization of the IB's *nett investment in subprime RMBSs and Treasuries*, B' , of the form

$$\text{IB's Nett Investment in RMBSs \& Treasuries } (B') = \quad (4.9)$$

$$\text{IB's Total Investment in RMBSs \& Treasuries } (B) - \text{IB's Nett RMBS Losses } (S),$$

where the *IB's total investment in subprime RMBSs and Treasuries*, B , is the stochastic process $(B_t)_{t \geq 0}$, defined on the probability space, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$. Let $\pi \in [0, 1]$ be the proportion of IB's funds invested in subprime RMBSs, and $1 - \pi$ is the proportion of funds invested in Treasuries. For $t > 0$ and S , M_t , L_t and $T(t)$ given by (4.8), (4.6), (1.10) and (4.7), respectively, we use (4.9) to represent the dynamics of IB's total and nett investment in subprime RMBSs & Treasuries by

$$dB_t^\pi = B_{t-}^\pi \left\{ \left((1 - \pi)r^\pi + \pi\tilde{\mu}^I - \pi \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \pi \sigma dZ_t + \pi \sum_{i=1}^n \alpha_i dN_i(t) \right\} \quad (4.10)$$

and

$$dB_t'^\pi = B_{t-}^\pi \left\{ \left((1 - \pi)r^\pi + \pi\tilde{\mu}^I - \pi \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \pi \sigma dZ_t + \pi \sum_{i=1}^n \alpha_i dN_i(t) \right\} - dS_t, \quad (4.11)$$

$$t \geq, \quad B_0^\pi = \ell,$$

respectively. The solution of (4.10) is obtained via the same Itô's formula in Lemma 8.1.1 and, for $B_0^\pi = \ell \in \mathbb{R}^+$, $t = T$ is found to be

$$\begin{aligned} B_T^\pi &= \ell \exp \left\{ \left(r^\pi + (\tilde{\mu}^I - r^\pi)\pi - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T \right. \\ &\quad \left. + \pi \sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \right\}. \end{aligned} \quad (4.12)$$

In the sequel, to avoid the possibility IB's total investment value process, B^π , being negative due to an *unpleasant* jump in the subprime RMBS price (4.6), we restrict the investment strategy as follows

$$\pi \in \begin{cases} \left[-\frac{1}{\alpha_n}, -\frac{1}{\alpha_1} \right) & \text{if } \alpha_n > 0 > \alpha_1; \\ \left(-\infty, -\frac{1}{\alpha_1} \right] & \text{if } \alpha_n < \infty; \\ \left[-\frac{1}{\alpha_n}, \infty \right) & \text{if } \alpha_1 > 0. \end{cases} \quad (4.13)$$

The next result provides explicit formulae for the expectation and variance of B_T^π . In this regard, we assume that the moments of B_T^π exist.

Proposition 4.4.1 (Explicit Formulae for $\mathbb{E}[B_T^\pi]$ and $\text{var}(B_T^\pi)$): *Suppose that (4.12) holds, and the moment generating function of homogeneous Poisson process $N_i(T)$ exists, that is,*

$$\mathbb{E}\left(\exp\{uN_i(T)\}\right) = \varphi_{N_i(T)}(u) = \exp\left\{\lambda_i T(e^u - 1)\right\}, \quad u > 0.$$

Then we have the expectation and variance of B_T^π as below

1. An explicit formula for $\mathbb{E}[B_T^\pi]$ is given by

$$\mathbb{E}[B_T^\pi] = \ell \exp\left\{(r^T + (\tilde{\mu}^I - r^T)\pi)T\right\}. \quad (4.14)$$

2. An explicit formula for $\text{var}(B_T^\pi)$ is given by

$$\text{var}\left(B_T^\pi\right) = \ell^2 \exp\left\{2(r^T + (\tilde{\mu}^I - r^T)\pi)T\right\} \left[\exp\left\{\left(\pi^2 \sigma^2 + \pi^2 \sum_{i=1}^n \alpha_i^2 \lambda_i\right)T\right\} - 1 \right].$$

Proof. If we take the mathematical expectation of (4.12), we obtain

$$\begin{aligned} \mathbb{E}[B_T^\pi] &= \ell \exp \left\{ \left(r^T + (\tilde{\mu}^I - r^T)\pi - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T \right\} \\ &\quad \times \mathbb{E} \left\{ \exp[\pi \sigma Z_T] \exp \left(\sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \right) \right\}. \end{aligned}$$

Since the moment generating function of homogeneous Poisson process, $N_i(T)$, exists, and also Z_T and $N_i(T)$ are two independent stochastic process. In addition, let $\tilde{h}_i = \ln(1 + \pi \alpha_i)$. Then we write

$$\begin{aligned} &\mathbb{E} \left\{ \exp[\pi \sigma Z_T] \exp \left(\sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \right) \right\} \\ &= \mathbb{E} \left\{ \exp[\pi \sigma Z_T] \right\} \mathbb{E} \left\{ \exp \left(\sum_{i=1}^n \tilde{h}_i N_i(T) \right) \right\} \\ &= \exp \left[\frac{1}{2} \pi^2 \sigma^2 T \right] \exp \left[\pi \sum_{i=1}^n \alpha_i \lambda_i T \right] \\ &= \exp \left[\left(\frac{1}{2} \pi^2 \sigma^2 + \pi \sum_{i=1}^n \alpha_i \lambda_i \right) T \right]. \end{aligned}$$

It then follows that

$$\mathbb{E}[B_T^\pi] = \ell \exp \left\{ (r^T + (\tilde{\mu}^I - r^T)\pi) T \right\}.$$

To compute the variance, we proceed as follows

$$\text{var} \left(B_T^\pi \right) = \mathbb{E}[(B_T^\pi)^2] - [\mathbb{E}[B_T^\pi]]^2. \quad (4.15)$$

But,

$$[\mathbb{E}[B_T^\pi]]^2 = \ell^2 \exp \left\{ 2(r^T + (\tilde{\mu}^I - r^T)\pi) T \right\}. \quad (4.16)$$

Moreover, we find that the second moment of B_T^π is given by

$$\begin{aligned}
 \mathbb{E}[(B_T^\pi)^2] &= \ell^2 \exp \left\{ \left(2(r^T + (\tilde{\mu}^I - r^T)\pi) - 2\pi \sum_{i=1}^n \alpha_i \lambda_i - \pi^2 \sigma^2 \right) T \right\} \\
 &\quad \times \mathbb{E} \left\{ \exp[2\pi \sigma Z_T] \right\} \mathbb{E} \left\{ \exp \left(2 \sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \right) \right\}. \\
 &= \ell^2 \exp \left\{ \left(2(r^T + (\tilde{\mu}^I - r^T)\pi) + \pi^2 \sigma^2 + \pi^2 \sum_{i=1}^n \alpha_i^2 \lambda_i \right) T \right\}
 \end{aligned} \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.15), we obtain

$$\text{var} \left(B_T^\pi \right) = \ell^2 \exp \left\{ 2(r^T + (\tilde{\mu}^I - r^T)\pi) T \right\} \left[\exp \left\{ \left(\pi^2 \sigma^2 + \pi^2 \sum_{i=1}^n \alpha_i^2 \lambda_i \right) T \right\} - 1 \right].$$

□

4.5 IB'S OPTIMAL INVESTMENT IN SUBPRIME RMBSs AND TREASURIES

In this subsection, we state and prove a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries for credit risk with both variance and swaps at risk.

4.5.1 IB's Optimal Investment Problem with Variance

In this subsection, we state and prove a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries for credit risk with variance.

4.5.1.1 Statement of IB's Optimal Investment Problem with Variance

In this subsection, we consider a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries, in which we use the variance as a measure for risk that is bounded by an appropriate constraint. The problem can be formally stated as follows.

Problem 4.5.1 (Statement of IB's Optimal Investment Problem with Variance):

Suppose that L_t is a Lévy process given by (1.10) and that B_T^π is given by (4.12). Then the mathematical formulation of constrained optimization problem for IB's investment is given as follows

$$\max_{\pi \in [0,1]} \mathbb{E}[B_T^\pi] \text{ subject to (4.12) and } \text{var}\left(B_T^\pi\right) \leq R, \quad (4.18)$$

where T is some given planning horizon and R is a given level of risk.

4.5.1.2 Solution of IB's Optimal Investment Problem with Variance

In this subsection, we solve a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries as outlined in Problem 4.5.1.

Theorem 4.5.2 (Solution of IB's Optimal Investment Problem with Variance):

The solution of the constrained optimization problem for IB's investment in subprime RMBSs and Treasuries, stated in Problem 4.5.1, is given by

$$\pi^* = \wp^* \sigma^{-1}, \quad \pi^* \in [0, 1] \text{ and also satisfies condition (4.13)}. \quad (4.19)$$

In addition, $\wp^*$ is the unique positive solution of

$$(\tilde{\mu}^I - r^T)\pi_{\wp^*} T = F(\wp^*),$$

$$\text{where } F(\wp) = \frac{1}{2} \ln \left[\frac{R}{\ell^2 \{\exp((\wp^2 + \wp^2 \sigma^{-2} \sum_{i=1}^n \alpha_i^2 \lambda_i) T) - 1\}} \right] - r^T T, \quad \wp > 0.$$

The maximal expected terminal IB's total investment in subprime RMBSs and Treasuries under the variance constraint (4.18) corresponding to (4.19) equals

$$\mathbb{E}[B_T^{\pi^*}] = \ell \exp \left\{ (r^T + (\tilde{\mu}^I - r^T)\wp^* \sigma^{-1}) T \right\}. \quad (4.20)$$

Proof. Using the explicit formula for $\text{var}(B_T^\pi)$ in Proposition 4.4.1, we can rewrite the variance constraint in Problem 4.5.1 as

$$\ell^2 \exp \left\{ 2(r^T + (\tilde{\mu}^I - r^T)\pi) T \right\} \left[\exp \left\{ \left(\pi^2 \sigma^2 + \pi^2 \sum_{i=1}^n \alpha_i^2 \lambda_i \right) T \right\} - 1 \right] \leq R. \quad (4.21)$$

Assume that $\pi \sigma = \wp$, for all $\wp > 0$. Therefore, (4.21) becomes

$$(\tilde{\mu}^I - r^T)\pi T \leq \frac{1}{2} \ln \left[\frac{R}{\ell^2 \{\exp((\wp^2 + \wp^2 \sigma^{-2} \sum_{i=1}^n \alpha_i^2 \lambda_i)T) - 1\}} \right] - r^T T. \quad (4.22)$$

In the sequel, let the right-hand side of (4.22) be denoted by $F(\wp)$. In this case, it follows that

$$(\tilde{\mu}^I - r^T)\pi T \leq F(\wp). \quad (4.23)$$

Furthermore, if $\pi \in \mathbb{R}$ solves the constraints in (4.23) for $\wp > 0$, then it must also solves the variance constraint in (4.18) and vice versa. In this proof, we note that $F(\wp)$ is strictly decreasing in $\wp > 0$ with

$$\lim_{\wp \downarrow 0} F(\wp) = \infty, \quad \lim_{\wp \rightarrow \infty} F(\wp) = -\infty.$$

In this regard, the left-hand side of (4.23) should be smaller than the right-hand one for some $\wp > 0$ if we plug in $\pi = \pi_\wp = \wp \sigma^{-1}$. If we have equality in (4.23), that is

$$(\tilde{\mu}^I - r^T)\pi_{\wp^*} T = F(\wp^*) \quad (4.24)$$

for the first time with increasing $\wp > 0$ then this determines the optimal $\wp^* > 0$. To see this, note that we have

$$\mathbb{E}[B_T^\pi] \leq \mathbb{E}[B_T^{\pi_{\wp^*}}], \quad \forall \pi, \text{ such that } \pi \sigma \leq \wp^*,$$

and for all admissible π with $\wp = \pi \sigma > \wp^*$ we obtain

$$(\tilde{\mu}^I - r^T)\pi T \leq F(\wp) < F(\wp^*) = (\tilde{\mu}^I - r^T)\pi_{\wp^*} T.$$

By solving the nonlinear equation (4.24) for $\wp^*$, we have determined the solution of the constrained optimization problem for IB's investment in subprime RMBSs and Treasuries stated in Problem 4.5.1. $\square$

4.5.2 IB's Optimal Investment Problem with Swaps at Risk

In this subsection, we state and prove a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries with swaps at risk. In this regard, we give a definition of swaps at risk (SaR) which is appropriate to our analysis.

Definition 4.5.3 (Mathematical Definition of SaR): *Suppose that ℓ is the initial IB's funds and $[0, T]$ is a fixed term. Moreover, let y_γ be the γ -quantile of the distribution of $\pi\sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi\alpha_i)$ for the proportion of funds $\pi \in [0, 1]$, invested in subprime RMBSs, and B_T^π the corresponding terminal value of IB's investment. That is, y_γ is a real number such that*

$$\begin{aligned} & \mathbf{P} \left(\pi\sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi\alpha_i) \leq y_\gamma \right) \\ &= \sum_{n_1, \dots, n_n=0}^{\infty} \left\{ \chi \left(\frac{1}{|\pi\sigma|\sqrt{T}} \left(y_\gamma - \sum_{i=1}^n (n_i \ln(1 + \pi\alpha_i)) \right) \right) \exp \left(-T \sum_{i=1}^n \lambda_i \right) \prod_{i=1}^n \frac{(T\lambda_i)^{n_i}}{n_i} \right\} = \gamma \end{aligned}$$

Therefore, we define the value at risk (VaR) of IB's investment as

$$\text{VaR}(\ell, \pi, T) = \ell \exp \left\{ \left(r^T + (\tilde{\mu}^I - r^T)\pi - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T + y_\gamma \right\}.$$

Finally, we give a definition

$$\begin{aligned} \text{SaR}(\ell, \pi, T) &= \ell \exp(r^T T) - \text{VaR}(\ell, \pi, T) \\ &= \ell \exp(r^T T) \left\{ 1 - \exp \left[\left(\pi(\tilde{\mu}^I - r^T) - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T + y_\gamma \right] \right\} \end{aligned}$$

of the SaR of the proportion of IB funds invested in RMBS, π , with initial IB's funds, ℓ , and planning term $[0, T]$.

4.5.2.1 Statement of IB's Optimal Investment Problem with Swaps at Risk

In this subsection, we formulate a constrained optimization problem for IB's investment in subprime RMBSs and Treasuries subject to SaR, where SaR is defined as a measure of the protection required against possible subprime RMBS losses, and it is bounded by a constant, A . The mathematical formulation of this problem is given below.

Problem 4.5.4 (Statement of IB's Optimal Investment Problem with Swaps at Risk):

Assume that L_t is a Lévy process which is represented by (1.10) and that B_T^π is given by (4.12). For $t \geq 0$, we give a mathematical statement of IB's investment problem with SaR as follows

$$\max_{\pi \in [0,1]} \mathbf{E}[B_T^\pi] \quad \text{subject to (4.12) and } \text{SaR}(\ell, \pi, T) \leq A, \quad (4.25)$$

where $[0, T]$ is a fixed term and A is the maximum protection that can be acquired.

4.5.2.2 Solution of IB's Optimal Investment Problem with Swaps at Risk

The problem we face is that y_γ cannot be represented explicitly. This brings us to the conclusion that the analytical solution of Problem 4.5.4 is not possible. However, using (4.14) we can be able to tell the behavior of the solution. In this regard, we note that

$$\frac{\partial \mathbf{E}[B_T^\pi]}{\partial \pi} > 0$$

provided $\tilde{\mu}^I > r^T$, i.e. $\mathbf{E}[B_T^\pi]$ is increasing function over the interval $0 \leq \pi \leq 1$. It then follows that the optimal solution of Problem 4.5.4 is the largest IB's investment strategy $\pi \in [0, 1]$ that satisfies the SaR constraint and condition (4.13). This strategy can only be found through numerical methods for optimization problems of this kind. Although the scope of the thesis does not cover the numerical solution of Problem 4.5.4, we shall present a numerical algorithm that can be used to solve this problem.

4.5.3 Numerical Procedure

In this subsection, we use the contribution of Gaussian in stochastic process (see [13]) to develop a numerical algorithm that can be used to approximate the solution of Problem 4.5.4.

4.5.3.1 Gaussian Diffusion Model for Subprime RML Securitization

In this subsection, we replace the aforementioned model in Subsection 4.1 with the generalized inverse Gaussian (GIG) diffusion model. In particular, we provide the stochastic analysis behind Gaussian with application to subprime RML securitization Problem 4.5.4. Suppose that the dynamic of Treasuries, T , is given by (4.7) and the subprime RMBS price process in (4.1) is given by

$$dM_t = M_t \{ \tilde{\mu}^I dt + dL_t \}, t \geq 0, M_0 = \iota, \quad (4.26)$$

$$\text{where } L_t = W(t) - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s)ds - w, \quad L_0 = 0.$$

Using (4.7) and (4.26), we write the value process of IB's total investment as below.

$$B_t^\pi = \ell \exp \left\{ ((1-\pi)r^T + \pi \tilde{\mu}^I)t + \pi \left(W(t) - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s)ds - w \right) \right\}, \quad B_0^\pi = \ell. \quad (4.27)$$

Moreover, $W(t)$ is defined as the GIG diffusion process which satisfy the following SDE

$$dW(t) = \frac{1}{4}\sigma^2 W^{2\beta-2}(t) \left(\varpi + 2(2\beta + \lambda - 1)W(t) - vW^2(t) \right) dt + \sigma W^\beta(t) dZ_t, \quad (4.28)$$

$$W(0) = w,$$

where $\sigma > 0$, $\beta \geq \frac{1}{2}$, $v, \varpi \geq 0$, $\max(v, \varpi) > 0$, and

$$\begin{aligned} \lambda &\in \mathbb{R} && \text{if } v, \varpi > 0; \\ \lambda &\leq \min(0, 2(1-\beta)) && \text{if } v = 0, \varpi > 0; \\ \lambda &\geq \min(0, 2(1-\beta)) && \text{if } v > 0, \varpi = 0. \end{aligned} \quad (4.29)$$

The Gaussian diffusion model is a generalization of the Black-Scholes model, which correspond to $\beta = \varpi = 0$, $\lambda = 1$, $v = -2$.

In the following lemma we decompose the subprime RMBS price process presented in (4.26). The idea of the lemma is to reveal the advantage of our Gaussian diffusion model construction.

Lemma 4.5.5 (Decomposition of the subprime RMBS price (4.26)): *Suppose that M_t and $W(t)$ satisfy (4.26) and (4.28), respectively. Then we can decompose M_t into a drift term multiplied by a local martingale, i.e.,*

$$\begin{aligned} M_t &= \iota \exp \left\{ \tilde{\mu}^I t + \frac{1}{4}\sigma^2 \int_0^t W^{2\beta-2}(s) \left(\varpi + 2(2\beta + \lambda - 1)W(s) - vW^2(s) \right) ds \right\} \\ &\times \exp \left\{ \sigma \int_0^t W^\beta(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s) ds \right\}, \quad t \geq 0. \end{aligned}$$

Proof. From (4.26), we have

$$dL_t = dW(t) - \frac{1}{2}\sigma^2 W^{2\beta}(t)dt \quad (4.30)$$

Substituting (4.28) into (4.30), we obtain

$$\begin{aligned} dL_t = \frac{1}{4}\sigma^2 W^{2\beta-2}(t) & \left(\varpi + 2(2\beta + \lambda - 1)W(t) - vW^2(t) \right) dt + \sigma W^\beta(t) dZ_t \\ & - \frac{1}{2}\sigma^2 W^{2\beta}(t) dt. \end{aligned} \quad (4.31)$$

Integrating from 0 to t , we get

$$\begin{aligned} L_t &= \frac{1}{4}\sigma^2 \int_0^t W^{2\beta-2}(s) \left(\varpi + 2(2\beta + \lambda - 1)W(s) - vW^2(s) \right) ds \\ &+ \sigma \int_0^t W^\beta(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s) ds. \end{aligned} \quad (4.32)$$

Furthermore, solving (4.26) for M_t , we see that

$$M_t = \iota \exp\{\tilde{\mu}^I t + L_t\} \quad (4.33)$$

If we substitute (4.32) into (4.33), we get

$$\begin{aligned} M_t &= \iota \exp \left\{ \tilde{\mu}^I t + \frac{1}{4}\sigma^2 \int_0^t W^{2\beta-2}(s) \left(\varpi + 2(2\beta + \lambda - 1)W(s) - vW^2(s) \right) ds \right\} \\ &\times \exp \left\{ \sigma \int_0^t W^\beta(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s) ds \right\}, \quad t \geq 0. \end{aligned}$$

□

The next lemma contains useful results concerning (4.28), which will be required in order to describe IB's total investment value process.

Lemma 4.5.6 (Scalar Multiple of GIG Diffusion is also GIG Diffusion): *Suppose that $W(t)$ is the GIG diffusion given by (4.28) and $\pi > 0$ is the proportion of IB funds invested in subprime RMBS. Then the process $\widetilde{W}(t) = \pi W(t)$ is also a GIG diffusion with $\widetilde{W}(0) = \pi W(0)$ and parameters $\tilde{\sigma} = \sigma\pi^{1-\beta}$, $\tilde{\varpi} = \varpi\pi$, $\tilde{v} = v\pi^{-1}$. In addition, β and λ in (4.28) remained the same.*

Proof. Since $\widetilde{W}(t) = \pi W(t)$, implies that $d\widetilde{W}(t) = d(\pi W(t)) = \pi dW(t)$. Then

$$d\widetilde{W}(t) = \frac{1}{4}\sigma^2\pi W^{2\beta-2}(t)\left(\varpi + 2(2\beta + \lambda - 1)W(t) - vW^2(t)\right)dt + \sigma\pi W^\beta(t)dZ_t$$

If we set $W(t) = \pi^{-1}\widetilde{W}(t)$, we obtain

$$d\widetilde{W}(t) = \frac{1}{4}\sigma^2\pi^{2-2\beta}\widetilde{W}^{2\beta-2}(t)\left(\varpi\pi + 2(2\beta + \lambda - 1)\widetilde{W}(t) - v\pi^{-1}\widetilde{W}^2(t)\right)dt + \sigma\pi^{1-\beta}\widetilde{W}^\beta(t)dZ_t$$

The above SDE can be simplified further to obtain

$$d\widetilde{W}(t) = \frac{1}{4}\widetilde{\sigma}^2\widetilde{W}^{2\beta-2}(t)\left(\widetilde{\varpi} + 2(2\beta + \lambda - 1)\widetilde{W}(t) - \widetilde{v}\widetilde{W}^2(t)\right)dt + \widetilde{\sigma}\widetilde{W}^\beta(t)dZ_t, \quad \widetilde{W}(0) = \pi w > 0.$$

From this equation, we can conclude that $\widetilde{W}$ is a GIG diffusion with $\widetilde{\sigma} = \sigma\pi^{1-\beta}$, $\widetilde{\varpi} = \varpi\pi$, $\widetilde{v} = v\pi^{-1}$. $\square$

Remark 4.5.7 (IB's Investment Value Process): *The results of Lemma 4.5.6 together with (4.27) gives us IB's total investment value process of the form*

$$B_t^\pi = \ell \exp \left\{ ((1 - \pi)r^\pi + \bar{\mu}^I)T + \widetilde{L}_t \right\}, \quad t \geq 0, \quad (4.34)$$

where

$$\bar{\mu}^I = \pi\widetilde{\mu}^I \quad \text{and} \quad \widetilde{L}_t = \widetilde{W}(t) - \frac{1}{2}\widetilde{\sigma}^2 \int_0^t \widetilde{W}^{2\beta}(s)ds - \pi w, \quad t \geq 0. \quad (4.35)$$

In definition 4.5.3, we see that the γ -quantile of $\widetilde{L}_T$ needs to be determined. Moreover, the advantage behind $SaR(\ell, \pi, T)$ is that, it is independent of moments of B_T^π . Therefore, it can be defined for finite or infinite moments of B_T^π . Although SaR is not dependent on the moments, but to solve Problem 4.5.4, we require the existence of first moment of B_T^π and it has to be finite. However, it is not possible to decide if B_T^π has a finite mathematical expectation. In this regard, we assume that $W(T)$ or $\widetilde{W}(T)$ have the stationary distribution of the process W or $\widetilde{W}$ respectively, for time horizon T which is sufficiently large.

Proposition 4.5.8 (Finite Mean of B_T^π): *Suppose that $W(T)$ and $\widetilde{W}(T)$ are GIG diffusion processes with constants ϖ , v , λ , β and $\widetilde{\varpi}$, $\widetilde{v}$, λ , β respectively. Let $\pi > 0$ be the proportion of IB funds invested in subprime RMBS. Then B_T^π has a finite mean if $\widetilde{v} = v\pi^{-1} > 2$.*

Proof. Since $\widetilde{W}$ is always positive, we approximate IB's total investment as below

$$B_T^\pi \leq \ell \exp \left\{ ((1 - \pi)r^T + \bar{\mu}^I)T + \widetilde{W}(T) - \pi w \right\}$$

If the moment generating function of $\widetilde{W}(T)$ is finite, i.e., $\mathbb{E}[\exp(\widetilde{W}(T))] < \infty$, then the mathematical expectation of B_T^π is also finite, that is, $\mathbb{E}[B_T^\pi] < \infty$.

The explicit formula for moment generating function of $\widetilde{W}(T)$ which is GIG random variable is given

$$\mathbb{E}[\exp(\widetilde{W}(T))] = \frac{\mathcal{G}_\lambda(\sqrt{v\varpi}(1 - \frac{2}{\tilde{v}}))}{\mathcal{G}_\lambda(\sqrt{v\varpi})(1 - \frac{2}{\tilde{v}})^{\frac{\lambda}{2}}}, \quad (4.36)$$

where $\mathcal{G}_\lambda(\cdot)$ denotes the generalized Bessel function of the third kind. Note that (4.36) is only finite if $\tilde{v} > 2$. Therefore, if the parameter $v > 2$ and $\pi \in [0, 1]$, then also $\tilde{v} > 2$ and $\mathbb{E}[B_T^\pi]$ is finite. $\square$

In line with Proposition 4.5.8, the optimization Problem 4.5.4 is well-defined and can be solved, however the question remained unanswered, whether the approach can be analytical or numerically. In the next corollary we discuss the analytic approach for solving Problem 4.5.4.

Corollary 4.5.9 (Analytic Approach for Solving Problem 4.5.4) *Suppose that the dynamic of $W(t)$ is given by (4.28). Then for $\beta = 1$, $v = 0$, we have*

$$dW(t) = \left(\frac{1}{4}\sigma^2\varpi + \frac{1}{2}\sigma^2(1 + \lambda)W(t) \right) dt + \sigma W(t)dZ_t, \quad W(0) = w, \quad (4.37)$$

which is interpreted as mean-reverting model. Furthermore, the solution of (4.37) is given by

$$W(t) = \exp \left(\frac{1}{2}\sigma^2\lambda t + \sigma Z_t \right) \left\{ w + \frac{1}{4}\sigma^2\varpi \int_0^t \exp \left(-\frac{1}{2}\sigma^2\lambda s - \sigma Z_s \right) ds \right\}, \quad t \geq 0,$$

and its expected value is written as

$$\mathbb{E}[W(t)] = \begin{cases} \exp\left(\frac{1}{2}\sigma^2 t(\lambda + 1)\right) \left\{ w + \frac{1}{4\lambda}\varpi \left(1 - \exp(-\sigma^2 \lambda t)\right) \right\} & \text{if } \lambda \neq -1; \\ w - \frac{1}{4}\varpi \left(1 - \exp(\sigma^2 t)\right) & \text{if } \lambda = -1. \end{cases}$$

Also,

$$L_t = \frac{1}{4}\sigma^2 \varpi t + \frac{1}{2}(1 + \lambda)\sigma^2 \int_0^t W(s)ds + \sigma \int_0^t W(s)dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^2(s)ds. \quad (4.38)$$

We obtain the same representations for $\widetilde{W}(t)$ and $\widetilde{L}_t$ if we substitute ϖ by $\widetilde{\varpi} = \pi\varpi$.

Proof. Consider the associated homogeneous equation for (4.37)

$$dY_t = \frac{1}{2}\sigma^2(1 + \lambda)Y_t dt + \sigma Y_t dZ_t, \quad Y_0 = 1.$$

Then through Itô's formula, we obtain

$$Y_t = \exp\left\{\frac{1}{2}\sigma^2 \lambda t + \sigma Z_t\right\}.$$

The solution of (4.37) is then given by

$$\begin{aligned} W(t) &= Y_t \left\{ w + \int_0^t \frac{1}{4}\sigma^2 \varpi \left(Y_s\right)^{-1} ds \right\} \\ &= \exp\left(\frac{1}{2}\sigma^2 \lambda t + \sigma Z_t\right) \left\{ w + \frac{1}{4}\sigma^2 \varpi \int_0^t \exp\left(-\frac{1}{2}\sigma^2 \lambda s - \sigma Z_s\right) ds \right\}. \end{aligned}$$

Taking the mathematical expectation of the above expression, we have

$$\begin{aligned}
\mathbb{E}[W(t)] &= \exp\left(\frac{1}{2}\sigma^2\lambda t\right) \left\{ w\mathbb{E}[\exp(\sigma W(t))] + \frac{1}{4}\sigma^2\varpi \int_0^t \exp\left(-\frac{1}{2}\sigma^2\lambda s\right) \mathbb{E}[\exp(\sigma(Z_t - Z_s))] ds \right\} \\
&= \exp\left(\frac{1}{2}\sigma^2\lambda t\right) \left\{ w \exp\left(\frac{1}{2}\sigma^2 t\right) + \frac{1}{4}\sigma^2\varpi \int_0^t \exp\left(-\frac{1}{2}\sigma^2\lambda s\right) \exp\left(\frac{1}{2}\sigma^2(t-s)\right) ds \right\} \\
&= \exp\left(\frac{1}{2}\sigma^2\lambda t\right) \left\{ w \exp\left(\frac{1}{2}\sigma^2 t\right) + \frac{1}{4}\sigma^2\varpi \exp\left(\frac{1}{2}\sigma^2 t\right) \int_0^t \exp\left(-\sigma^2\lambda s\right) ds \right\} \\
&= \exp\left(\frac{1}{2}\sigma^2 t(\lambda + 1)\right) \left\{ w + \frac{1}{4}\sigma^2\varpi \left[-\frac{1}{\sigma^2\lambda} \left(\exp\left(-\sigma^2\lambda t\right) - 1 \right) \right] \right\} \\
&= \exp\left(\frac{1}{2}\sigma^2 t(\lambda + 1)\right) \left\{ w + \frac{1}{4\lambda}\varpi \left(1 - \exp(-\sigma^2\lambda t) \right) \right\}, \text{ for } \lambda \neq -1.
\end{aligned}$$

If $\lambda = -1$ then

$$\mathbb{E}[W(t)] = w - \frac{1}{4}\varpi \left(1 - \exp(\sigma^2 t) \right).$$

The SDE in (4.37) can be written as an integral equation of the form

$$W(t) = w + \frac{1}{4}\sigma^2\varpi t + \frac{1}{2}\sigma^2(1 + \lambda) \int_0^t W(s) ds + \sigma \int_0^t W(s) dZ_s. \quad (4.39)$$

Combining (4.39) and L_t in (4.26), we achieve the following

$$L_t = \frac{1}{4}\sigma^2\varpi t + \frac{1}{2}\sigma^2(1 + \lambda) \int_0^t W(s) ds + \sigma \int_0^t W(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^2(s) ds. \quad (4.40)$$

Finally, we get the same representations for $\widetilde{W}(t)$ and $\widetilde{L}_t$ if we replace ϖ by $\widetilde{\varpi} = \pi\varpi$. $\square$

Although this corollary gives us some analytical properties of the process that seems to be important for solving the mean-SaR optimization problem, the closed form solution cannot be determine. This brings us to the presentation of the following numerical algorithm that can be used to determine a numerical solution to Problem 4.5.4.

4.5.3.2 Numerical Algorithm for Problem 4.5.4

The rest of this subsection discusses a numerical approximation to Problem 4.5.4. In particular, we discuss an iterative method that can be implemented in order to find the optimal investment strategy for IB subject to swap at risk. The Monte-Carlo method may be helpful to achieve the solution. In the sequel, we first present the properties of Problem 4.5.4. The

properties of expectation operator together with (4.35) allow us to write equation below.

$$\begin{aligned}
 \mathbb{E}[B_T^\pi] &\geq \ell \exp \left\{ ((1 - \pi)r^T + \bar{\mu}^I)T + \mathbb{E}[\tilde{L}_T] \right\} \\
 &= \ell \exp \left\{ ((\bar{\mu}^I - r^T)\pi + r^T)T + \frac{1}{4}\sigma^2\varpi\pi T + \frac{1}{2}\pi^2\lambda\sigma^2 \int_0^T \mathbb{E}[W^2(s)]ds \right\} \\
 &= \ell \exp(r^T T) \exp(\pi\gamma + \pi^2\lambda\alpha).
 \end{aligned} \tag{4.41}$$

If the expected rate of return on subprime RMBS is greater than Treasuries rate, i.e., $\bar{\mu}^I > r^T$, the constants γ and α are both positive, whether the right-hand side of (4.41) increases depends on the intensity, λ . On the other hand, if the right-hand side of (4.41) increases in π and is large, so is the left-hand side. More precisely, for a non-negative λ the right-hand side can be made arbitrarily large by increasing the proportion of IB funds invested in subprime RMBS, π . Thus, in solving Problem 4.5.4 numerically, we replace the expected terminal IB's investment value by the right-hand side of (4.41). The consequence of this is that we only have to find the largest proportion of IB funds invested in subprime RMBS such that it comply with the protection against the possible losses from that investment, which is measured by $SaR(\ell, \pi, T)$. In the sequel, if we use the inequality

$$\widetilde{W}(t) \geq \widetilde{W}(t) - \frac{1}{2}\sigma^2 \int_0^t \widetilde{W}(s)ds$$

and consider the stationary distribution for $\widetilde{W}(T)$ as better approximation for its exact distribution then we can solve the SaR-constraint to obtain the optimal investment in subprime RMBS such that the constraint is still satisfied. Take note that in this case the stationary distribution of $\widetilde{W}(T)$ is an *inverse gamma distribution*.

A numerical iterations that can be used to solve Problem 4.5.4 are given below. For $i = 1, \dots, K$ with K being large.

Step 1: Simulate the trajectories $(Z_i^t)_{0 \leq t \leq T}$ of the Brownian motion $(Z_t)_{0 \leq t \leq T}$.

Step 2: Determine the numerical value for $W_i(t)$ and $\int_0^T W_i^2(t)dt$ of $W(T)$ and $\int_0^T W^2(t)dt$, respectively, from the path followed by $(Z_i^t)_{0 \leq t \leq T}$.

Step 3: $\forall \pi \in \mathbb{R}$ compute

$$\tilde{L}_T^{\pi^i} = \pi W_i(T) - \int_0^T W_i^2(t) dt - \pi w.$$

Step 4: Find the approximations $\tilde{\Psi}(\pi)$ for $\mathbb{E}[B_T^\pi]$ and $\tilde{\Upsilon}(\ell, \pi, T)$ for $SaR(\ell, \pi, T)$:

$$\tilde{\Psi}(\pi) = \frac{\ell}{K} \sum_{i=1}^K \exp \left((r^T + (\tilde{\mu}^I - r^T)\pi)T + \tilde{L}_T^{\pi^i} \right).$$

$$\tilde{\Upsilon}(\ell, \pi, T) = \ell \exp(r^T T) \left[1 - \exp \left(\pi(\tilde{\mu}^I - r^T)T + \tilde{y}_\gamma \right) \right],$$

where $\tilde{y}_\gamma$ is the γ -quantile of the empirical distribution of $\tilde{L}_T^{\pi^i}$.

Step 5: Finally, select a proportion of IB funds invested in subprime RMBS with the largest value of $\tilde{\Psi}(\pi)$ such that $\tilde{\Upsilon}(\ell, \pi, T)$ is below or equal to a constant A for the SaR.

Chapter 5

ANALYSIS OF SUBPRIME RML SECURITIZATION MODELS

5.1 ANALYSIS OF CHAPTER 1

5.1.1 General Discussion on Subprime RML Securitization

5.1.2 Subprime RMLs

5.2 ANALYSIS OF CHAPTER 2

5.2.1 Subprime RML Securitization Risk in Discrete-Time and the SMC

5.2.2 IB's Optimization Problems and the SMC

5.2.2.1 IB's Optimization Problem with Risk and the SMC

5.2.2.2 IB's Optimization Problem with Risk and Regret and the SMC

5.2.2.3 Credit Default Swaps in Discrete-Time and the SMC

5.2.3 Tranching, Counterparty and Liquidity Risks

5.2.3.1 Tranching Risk

5.2.3.2 Counterparty Risk

5.2.3.3 Liquidity Risk

5.3 ANALYSIS OF CHAPTER 3

5.3.1 Subprime Securitization Risks in Continuous-Time and the SMC

5.3.2 Credit Default Insurance and the SMC

5.3.3 IB's Payout Under Subprime RML Securitization and the SMC

5.3.4 Numerical Example

5.3.5 Stochastic Optimal Credit Default Insurance and its Connections with the SMC

5.3.5.1 Statement and Proof of the Credit Default Insurance Problem

5.3.5.2 Optimal Credit Default Swap Contracts and the SMC

5.3.5.3 Optimal Credit Default Insurance with Exponential Utility and the SMC

5.3.5.4 Optimal Credit Default Insurance with Power Utility and the SMC

5.3.5.5 Optimal Credit Default Insurance with Logarithmic Utility and the SMC

5.4 ANALYSIS OF CHAPTER 4

5.4.1 Stochastic Dynamic of IB's Investment and the SMC

5.4.2 Optimization Problems for IB's Investment in Subprime RMBSs & Treasuries and the SMC

5.4.2.1 Statement of IB's Optimal Investment Problem with Variance and the SMC

5.4.2.2 Solution of IB's Optimal Investment Problem with Variance and the SMC

5.4.2.3 Statement of IB's Optimal Investment Problem with SaR and the SMC

5.4.2.4 Solution of IB's Optimal Investment Problem with SaR and the SMC

"The current credit crisis will come to an end when the overhang of inventories of newly built homes is largely liquidated, and home price deflation comes to an end. That will stabilize the now-uncertain value of the home equity that acts as a buffer for all home mortgages, but most importantly for those held as collateral for residential MBSs. Very large losses will, no doubt, be taken as a consequence of the crisis. But after a period of protracted adjustment, the U.S. economy, and the world economy more generally, will be able to get back to business."

– Alan Greenspan, 2007.

"On Wednesday, 1 October 2008, the U.S. Senate approved an amended version of the plan which was ratified by the House on Friday, 3 October 2008 and immediately signed into law by President Bush. After the law was passed, the U.S. Treasury instead primarily used the first \$ 350 billion of bailout funds to buy preferred stock in banks instead of troubled mortgage assets."

– Wikipedia, 2008.

In this chapter, we provide the analysis of subprime RML securitization models presented in Chapters 1, 2, 3 and 4 as well as their relationship with the SMC. As is well-known, this ongoing crisis is characterized by shrinking liquidity in global credit markets and the opaqueness of risks associated with structured financial products. A downturn in the U.S. housing market, risky practices by IBs and mortgagors and excessive individual and corporate debt levels have affected the world economy adversely on a number of levels. As a result, the SMC has exposed pervasive weaknesses in the global banking system and regulatory framework.

5.1 ANALYSIS OF CHAPTER 1

In this subsection, we provide a general discussion on subprime RML securitization which was introduced in Chapter 1.

5.1.1 General Discussion on Subprime RML Securitization

From (1.3) in Section 1 of Chapter 1, it is clear that Π_j^c is dependent on the structure of the securitization, E_u , and on the underlying single subprime RML and its loss, S_j^s . In addition, we note that S_j^s is dependent on house price appreciation. The variable M_u does not appear in Π_j^c , because if the RML is refinanced at the end of the period, then it is paid out and there are no losses. Therefore, M_u should be under the expectation operator. Also, the relationship between the house price and Π_j^c only appears through the recovery value of the house if there is a default. In essence, the valuation of Π_j^c requires an integration of (1.3) over a distribution of house prices so that

$$\int \Pi_j^c dD(H^p),$$

where H^p is the house price and $D(H^p)$ is its distribution function. Two practical problems arise from this situation. First, the dependence on house prices creates a valuation problem in practice – even if the distribution of house prices is known. For instance, as in the computational example in Section 8.2 of Chapter 8, the subprime securitization has four portfolios, each consisting of many RMLs. The CLO has purchased 100 tranches from different securitizations, including, say, twenty sen subprime tranches from different deals. In principle, the issue is how to evaluate the sen CLO tranche (even ignoring all the OC tests and other complications of the CLO structure). Besides the fact that this is very difficult to accomplish, interlinking the three structures together in a meaningful way is virtually impossible. In principle, an IB who actually purchased a particular CLO tranche or subprime RMBS tranche would receive trustee reports and would, therefore, have some knowledge about the RML reference portfolios. However, since the computational complexity is very

high, it remains difficult for the subprime RMBS IB to look-through to the RML reference portfolios and determine the value of such tranches. The second problem involves accounting for all the structure. Despite the fact that there are vendor-provided packages that model the structure of structured products, the valuation is based on (point estimate) assumptions that are input by the user, rather than simulation of the performance of the RML reference portfolios.

5.1.2 Subprime RMLs

We recall that the rate of return on subprime RMBSs, r_t^I , is a function of the subprime RML rate, r_t^M , that is defined as the sum of the index rate, r_t^i , and risk margin, ϱ_t^e (see equation (1.6) in Subsection 1.2.1 of Chapter 1). The risk margin is an indication of perceived subprime risk. Before the SMC, the average difference between prime and subprime RML interest rates (the *subprime markup*) declined quite dramatically. In other words, the risk margin, ϱ_t^e , required by OR to offer a subprime RML declined. This continued to occur during the SMC even though the level of macroeconomic activity and the quality of subprime RMLs, both declined.

5.2 ANALYSIS OF CHAPTER 2

In this subsection, we provide an analysis of the subprime RML securitization models and IB's optimization problems in discrete-time (see Chapter 2) and their connections with SMC.

5.2.1 Subprime Risk in Discrete-Time and the SMC

Under Assumption 2.1.1 in Subsection 2.1 of Chapter 2, we have that the rate of return on subprime RMBSs takes the value $r^I = I$ with a probability of success $q(I, m)$. The value of I depends on the level of macroeconomic activity, m , where $q'(I, m) < 0$. Before the SMC, q was high because of minimal default rates on reference RML portfolios. In turn, this prompted CRAs to assign high ratings to subprime RMBSs which drove IBs to hold large quantities of such RMBSs. During the SMC, mortgagors started to default, and this increased the probability of failure, $1 - q(I, m)$, which led many IBs to charge higher interest rates. As this situation worsened, IBs started to invest their funds in riskless assets such as Treasuries. The behavior of these IBs exacerbated the financial crisis. In particular, due to the decisions taken by IBs, the global RML market froze.

5.2.2 IB's Optimization Problems and the SMC

In this subsection, we briefly discuss the solutions emanating from the optimization problems solved in Subsection 2.5 of Chapter 2 and their relationships with the SMC.

5.2.2.1 IB's Optimization Problem with Risk and the SMC

A maturity mismatch problem arises from the fact that IB takes deposits which are very liquid and invests it in illiquid subprime RMBSs. In order to alleviate this problem, IB must maximize its expected return on subprime RMBSs. Lemma 2.5.1 in Subsection 2.5.1 of Chapter 2 summarizes the related result as follows. In the absence of a bailout policy, when the expected return on subprime RMBSs-to-deposit rate ratio is less than the capitalization factor then IB selects a portfolio I^a instead of I^* . In this case, if the RMBS portfolio returns are lower than those of the deposits, IB defaults on its obligations. On the other hand, if the expected return on subprime RMBSs-to-expected returns on deposits ratio is greater than the capitalization factor, IB chooses to invest I^* . In this case, IB may be able to honor its debt obligations.

5.2.2.2 IB's Optimization Problem with Risk and Regret and the SMC

Theorem 2.5.3 in Subsection 2.5.2 of Chapter 2 shows that regret-averse IBs would prefer to hold a positive proportion of its available funds in the subprime RMBSs for (2.5) in Subsection 2.3.2 of Chapter 2 while risk-averse IB would hold all funds in Treasuries in that case. On the other hand, for (2.6) in Subsection 2.3.2 of Chapter 2, the regret-averse IB always invests in Treasuries, whereas the risk-averse IB holds all available funds in the subprime RMBSs. Intuitively, this means that $\pi_\rho^* = 0$ exposes IB to the possibility of facing extreme regret if subprime RMBSs do well. By contrast, where $\pi_\rho^* = 1$, IB will feel less regret if subprime RMBSs do well but, in return, IB will feel some regret if it does badly.

Corollary 2.5.4 in Subsection 2.5.2 of Chapter 2 claims that for some intermediate level of $\xi q(I)I - r^T$, a regret-averse IB can choose a risk allocation as if regret was not considered. As the level of regret aversion rises, i.e., the value of ρ increases, the amount of available funds invested in the subprime RMBSs increases. With a relatively large $\xi q(I)I - r^T$, the risk-averse IB allocates all of its available funds to the subprime RMBSs, while the regret-averse IB invests some money in Treasuries, T. As the level of regret aversion increases, with a high $\xi q(I)I - r^T$, the amount of available funds invested in the subprime RMBSs decreases. Hence, the certainty equivalent is the point $q(I) = \frac{\tilde{r}^T}{\xi I}$ where a regret-averse IB chooses an optimal risk allocation as if regret was not considered. In Proposition 2.5.5 in Subsection 2.5.2.1 of Chapter 2, IBs that weigh regret-aversion heavily, i.e., have large values for ρ , are more likely to hold subprime RMBSs in its portfolio under the assumption that $\xi q(I)I - r^T$ is low.

5.2.2.3 Credit Default Swaps in Discrete-Time and the SMC

In this subsection, we briefly explain some of the issues related to CDS contracts discussed in Subsection 2.4 of Chapter 2 and their relationship with the SMC. Our specific interest

is in inefficient liquidation, moral hazard and the comparison between the behavior of risk- and regret-averse IBs.

One measure of the distress that IBs are suffering during the SMC, and the risk their counterparties are facing, is the rate on CDSs for selected banks shown in Figure 5.1 below. These CDS spreads clearly show concerns about the solvency of IBs, with local peaks appearing on 16 August 2008 for Washington Mutual and Countrywide. These banks are connected to financial institutions that are heavily involved in securitizing subprime RMLs and are thus subject to delinquency and default risk being put back to them (see, for instance, [19]).

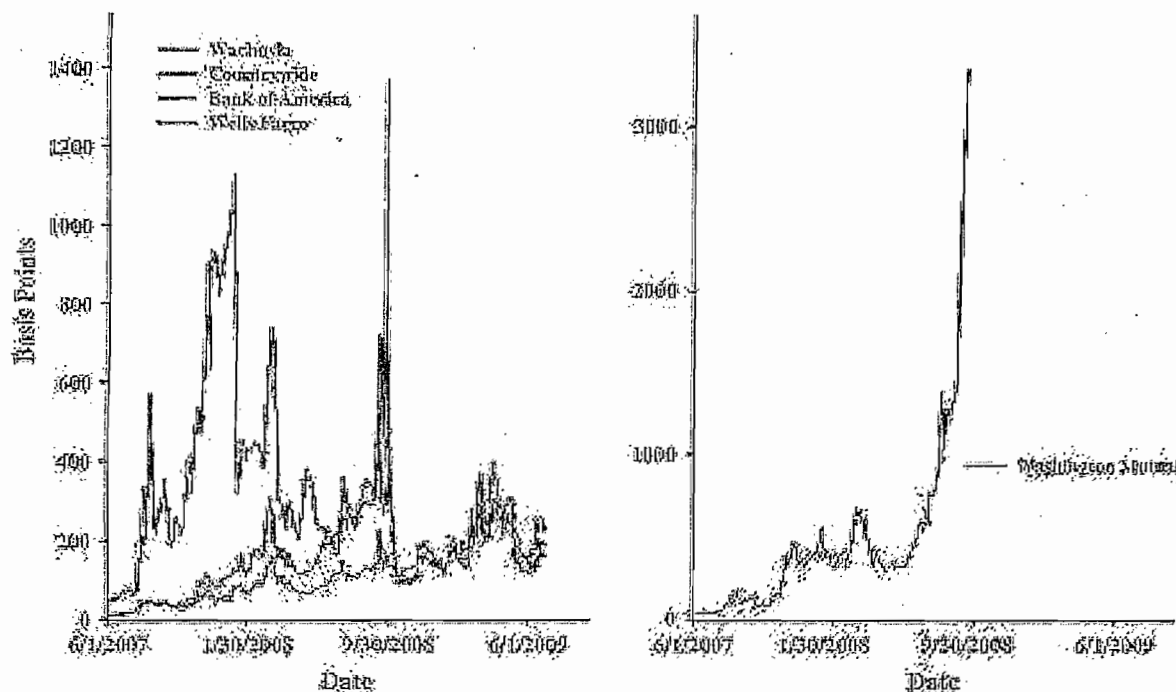


Figure 5.1: Bank Credit Default Swaps Rates 1 January 2007 to 30 June 2009; Source: [19].

It is clear that if collateral pledged during RML extension exceeds a critical value, mortgagors may be inefficiently liquidated once they become financially distressed (see, for instance, [19]). It is likely that fairly priced CDSs provided by a swap protection seller can partially alleviate this inefficient liquidation problem. The costs of providing CDS contracts depends, of course, on how the contracts are designed. Firstly, it depends on how often the seller must honor its promise. For example, it might be sufficient to structure the program so that the minimum return is evaluated only at the expiry of the CDS term, rather than annually or more frequently. Second, the CDS premiums depend on how much risk is borne

by IB. IB could make the CDSs more valuable, and hence more costly, if they have an opportunity to choose riskier subprime RMBSs subsequent to receiving swap protection. During the SMC, this moral hazard problem has been recognized and has prompted some countries to impose regulations on IB's risk allocations. For instance, under adverse economic conditions, a requirement may be that IBs hold most of their available funds in Treasuries. The securitized RML market could offer IB some protection from market fluctuations without making a portfolio with Treasuries compulsory. This can be accomplished by providing CDS contracts on the subprime RMBSs.

Theorem 2.5.6 in Subsection 2.5.2.2 of Chapter 2 demonstrates that if the proportion of available funds invested in the subprime RMBSs is low, the regret-averse IB would be willing to pay smaller premiums for the CDSs than would a risk-averse IB. In this case, the benefits from the CDSs in mitigating regret are small and the additional regret cost through the price weighs more. On the other hand, when investment in the subprime RMBSs is large and the quoted default swap spread is small, the benefits of regret mitigation would be large and would outweigh its cost. Under these conditions, both risk- and regret-averse IBs pay the same premium.

5.2.3 Tranching, Counterparty and Liquidity Risks

RML securitization largely involves credit, tranching, counterparty and liquidity risks. In this subsection, we discuss these risks.

5.2.3.1 Tranching Risk

From the above, we can conclude that RML securitization via tranching poses the following risks. As was evident from the motivating example in Section 1 of Chapter 1, tranching makes RMBS deals very complex. Besides the difficulties related to the estimation of the RML reference portfolio's loss distribution, tranching requires comprehensive, deal-specific documentation to ensure that the desired characteristics, such as claim seniority, will be carried through (see, for instance, Figure 1.4 in Subsection 1.2.4 of Chapter 1). Moreover, complexity may be further exacerbated by regret-averse asset managers and other agents, whose own incentives to act in the interest of some investor classes at the expense of others may need to be curtailed. As complexity increases, less sophisticated IBs have more difficulty understanding RMBS tranching and thus a limited capacity to make prudent investment decisions (see the role of IB in Figure 1.4). For instance, tranches from the same deal may have different risk, reward and/or maturity characteristics. Modeling the performance of tranching transactions based on historical performance may have led to the over-rating (by CRAs) and underestimation of risks (by IBs) of RMBSs with high-yield RML reference portfolios. All these factors have contributed towards the SMC.

5.2.3.2 Counterparty Risk

The protection seller assumes the credit risk that IB does not wish to bear in exchange for periodic premiums, and is obligated to pay only if a negative credit event occurs. In this regard, IB will be exposed to counterparty risk rather than credit risk. Here, counterparty risk refers to the risk that a swap protection seller will not be able to make a payment to the protection buyer if the mortgagors default. This could happen because CDSs are over-the-counter and unregulated, and the contracts often get traded so much that it is hard to know who stands at each end of a transaction. There is the possibility that the seller may not have the financial strength to abide by the contract's provisions, making it difficult to value the contracts. The leverage involved in many CDS transactions, and the possibility that a widespread downturn in the market could cause massive defaults and challenge the ability of protection sellers to pay their obligations, adds to the uncertainty. During the SMC, as the net worth of banks and other financial institutions deteriorated because of losses related to subprime RMLs, the likelihood increased that those providing swap protection would have to pay their counterparties.

5.2.3.3 Liquidity Risk

In our case, *liquidity* refers to the degree to which RMBSs can be bought or sold in the secondary market without affecting its price. Liquidity is characterized by a high level of trading in RMBSs in this market. The complexity of RMBSs conceal risk and reduce liquidity. In circumstances where RML default rates increase, RMBSs that have blended varying types of credit risk in a complex transactions network become *toxic*. Toxicity spreads across the banking sector, the wholesale markets, the retail markets, insurance companies, the asset management industry, and into the household. The situation further deteriorates when holders of RMBSs have trouble finding other IBs to buy these as the secondary RML market runs short of liquidity. Those holding RMBSs, and who took out financing to do so, may have margin calls that force them to trade, at a discount, what are illiquid underlying investments. Liquidity is further restricted because financially distressed IBs will hold on to cash as an insurance against further charge off's of irrecoverable RMLs. In the sequel, we comment on the possible liquidity problems that arise from the allocation of funds by IB under risk and regret.

Theorem 2.5.3 in Subsection 2.5.2 of Chapter 2 shows that for (2.5) in Subsection 2.3.2 of Chapter 2 the regret-averse IB invests more in subprime RMBSs, whereas the risk-averse IB would allocate its available funds in Treasuries. If (2.6) in Subsection 2.3.2 of Chapter 2 holds, the risk-averse IB makes a decision to invest all of its funds in subprime RMBSs. However, the regret-averse IB would choose to invest less in subprime RMBSs and the rest in Treasuries. In particular, when (2.6) holds, the investment strategies of both risk- and regret-averse IBs are likely to produce lower and higher liquidity in the secondary RML market, respectively. In the case where (2.5) is true, the allocating strategies of risk- and

regret-averse IBs are likely to result in higher and lower liquidity in the market, respectively. In the first case, if every IB is risk-averse, there will be a slowing of fund inflow in the RML market. Similarly, in the second instance, i.e., (2.5), if every bank is regret-averse, we expect a boom in the economy. Proposition 2.5.5 in Subsection 2.5.2.1 of Chapter 2 shows that when IB is more regret-averse, under (2.5) it will invest more in subprime RMBSs, whereas for (2.6) it chooses to invest a smaller proportion of its funds in subprime RMBSs. In line with this discussion, we recall that before the SMC, institutions were more willing to lend, which led to more investment in subprime RMBSs. This is related to the previous analysis on investment strategies for regret- and risk-averse IBs under (2.5) and (2.6). During the SMC, the same IBs switched from investing in subprime RMBSs to Treasuries. This can also be linked with (2.5) and (2.6) for risk- and regret-averse IBs, respectively. The investment away from subprime RMBSs to Treasuries was a root cause of the lack of liquidity in secondary RML markets. Below, Figure 5.2 illustrates the behavior of IB and its association with liquidity.

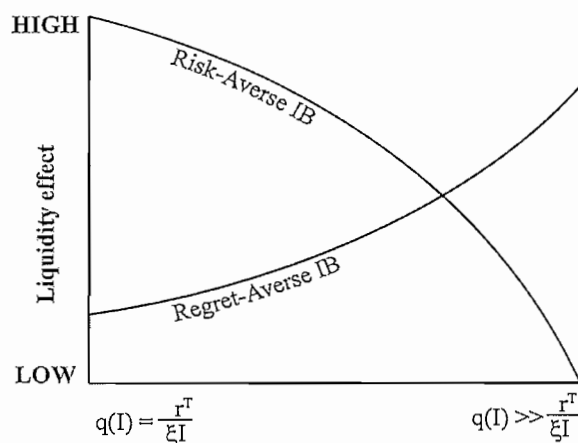


Figure 5.2: The Liquidity Effect

5.3 ANALYSIS OF CHAPTER 3

In this subsection, we briefly discuss some issues emanating from the subprime RML securitization models constructed in Chapter 3 and optimal credit default insurance problem (see Subsection 3.6 of Chapter 3) as well as their relationship with the SMC.

5.3.1 Subprime Risks in Continuous-Time and the SMC

In a CDS contract, the protection seller assumes the credit risk that IB does not wish to bear in exchange for periodic premiums, and is obligated to pay only if negative credit events

occur. As a consequence, in addition to credit risk, IB will also be exposed to counterparty risk. In this thesis, counterparty risk refers to the risk that a swap protection seller will not be able to make a payment to IB when a credit event occurs. For instance, the credit default insurance term, $C_u(S(\Pi_u, u))$, in equation (3.5) for IB's payout dynamics in Subsection 3.5 of Chapter 3 may be compromised. This could happen because CDSs are over-the-counter and unregulated. Also, the contracts often get heavily traded with the definition of the roles of all security transaction agents becoming difficult. There is the possibility that the protection seller may lack the financial muscle to honor the CDS contract's provisions, making it difficult to value them. The leverage involved in many CDS contracts, and the possibility that a widespread market downturn could cause massive RML reference portfolio defaults and challenge the ability of protection sellers to meet their obligations, contributes to the uncertainty. During the SMC, as IB charter values deteriorated on the back of subprime RML losses, the likelihood increased that swap protection sellers would have to compensate their counterparties. Also, counterparty risk is of importance in IB-depositor relationships, where IB is under pressure to fulfill obligations towards depositors (see, for instance, the role of k in equation (3.5) in Subsection 3.5 of Chapter 3).

In our case, *liquidity* refers to the degree to which RMBSs can be bought or sold in the secondary market without affecting their price. Liquidity is characterized by a high level of trading in RMBSs in this market with complexity reducing liquidity. In circumstances where RML default rates increase, RMBSs that mixed and matched credit risk types in a intricate web of transactions become "toxic." Toxicity spreads across the banking sector, the wholesale markets, the retail markets, insurance companies, the asset management industry, and into the household. In particular, as liquidity dries up in the secondary RML market, the situation deteriorates even more when RMBS bond holders experience difficulty finding other IBs to trade with. Those holding RMBSs, who may have raised deposits to do so, may have margin calls that force them to trade illiquid RMLs and their securities at a discount (see, for instance, the role of μ^I in equation (3.5) in Subsection 3.5 of Chapter 3). During the SMC, liquidity was further restricted when financially distressed IBs hoarded cash as a buffer against rampant subprime RML reference portfolio losses.

5.3.2 Credit Default Insurance and the SMC

The volume of CDSs outstanding increased 100-fold from 1998 to 2008, with estimates of the debt covered by CDS contracts, as of November 2008, ranging from \$ 33 to \$ 47 trillion. During the SMC, IBs were not sure whether swap protection sellers would be able to pay to cover subprime RML defaults (see, for instance, [64]). For instance, when investment bank Lehman Brothers went bankrupt on Monday, 15 September 2008, there was much uncertainty as to which financial firms would be required to honor the CDS contracts on its \$ 600 billion of bonds outstanding (see, for instance, [64]). Merrill Lynch's large losses in 2008 were attributed in part to the drop in value of its unhedged portfolio of collateralized debt obligations (CDOs) after the American International Group (AIG) ceased offering

CDSs on Merrill's CDOs. The loss of confidence by trading partners in Merrill Lynch's solvency and its ability to refinance its short-term debt led to its acquisition by the Bank of America.

5.3.3 IB's Payout Under Subprime RML Securitization and the SMC

In Subsection 3.5 of Chapter 3, the dynamic model of IB's payout under subprime RML securitization given by (3.4), involves the mass of type-A RMBSs, Γ (see Subsection 3.3 of Chapter 3) and a CDS premium, $\Theta(C)$ (see (3.3) in Subsection 3.4 of Chapter 3). In this regard, our model shows that when Γ is high this premium is low because high Γ means that a low probability of default (PD) is associated with the securitized subprime RMLs. In our payout model, the size of Γ is indicative of credit risk, while Π , k and μ^I are indicators of tranching, counterparty and liquidity risk, respectively. Before the SMC, low credit risk created liquidity in the credit market since IBs were more willing to indulge in RML securitization activities. This liquidity caused some of the financial institutions to more readily lend to subprime mortgagors because of competition in the credit market.

During the SMC, depreciation in house prices contributed to a decline in profits (related to IB's payout model in (3.4)) of many U.S. banks. IBs that retained credit risk were the first to be affected, as mortgagors became unable or unwilling to make payments and the value of RML reference portfolios declined. In this regard, profits at the 8 533 U.S. banks insured by the Federal Deposit Insurance Corporation (FDIC) fell from \$ 35.2 billion to \$ 646 million (effectively by 89 %) during Quarter 4 of 2007 when compared with the previous year. This was largely due to escalating RML losses and provisions for such losses. This decline in profits contributed to the worst bank and thrift quarterly performance since 1990. In 2007, these banks earned approximately \$ 100 billion, which represented a decline of 31 % from the record profit of \$ 145 billion in 2006. Profits decreased from \$ 35.6 billion to \$ 19.3 billion during the first quarter of 2008 versus the previous year, a decline of 46 %. The quarterly reports [21] and [22] of the FDIC intimate that profits decreased from \$ 35.6 billion to \$ 19.3 billion during the first quarter of 2008 versus the previous year, a decline of 46 %.

5.3.4 Numerical Example

In this subsection, for an anonymous IB, we compute the inputs of our model before (i.e. 2001 - 2006) and during the SMC (i.e., 2007 - 2009) and explore changes in optimal credit default insurance between the two periods. In order to accomplish this, data was sourced from the U.S. Federal Reserve Bank (see [63]) and the paper [31].

Date	Type-A RMBS Mass	Rate of Return	Index Rate	Margin
2001	0.86	6.19 %	3.40 %	3.51 %
2002	0.88	4.67 %	2.17 %	2.50 %
2003	0.89	4.12 %	2.11 %	2.01 %
2004	0.9	4.34 %	2.34 %	2.00 %
2005	0.91	6.19 %	4.19 %	2.00 %
2006	0.91	7.96 %	5.96 %	2.00 %
2007	0.50	8.05 %	5.86 %	2.19 %
2008	0.20	5.09 %	2.39 %	2.70 %
2009	0.15	3.25 %	0.50 %	2.75 %

Table 5.1: Numerical Example Variables; Source: U.S. Federal Reserve Bank and [31]

The following parameter choices were used for before and after the SMC.

Period	η	r^I	S	$C(S)$	ϱ	φ	$\tilde{b}$
Before SMC	35.2 bn	5.7 %	13.2 bn	5.28 bn	0.111	0	0
During SMC	19.3 bn	5.46 %	7.3 bn	2.92 bn	0.111	0	0

Table 5.2: Parameter Choices

By inputting the data above in our stochastic model (3.5), it follows that before the SMC the optimal cash outflow for fulfilling depositor obligations, k^* , is given as \$ 154.398 billion and IB's optimal investment in RMBSs, I^* , has the value \$ 983.714 million. During the SMC, k^* and I^* is given as \$ 531.148 billion and \$ 160.517 million, respectively. This means, for instance, that as was evidenced in practice, the investment in RMBSs decreased dramatically from before to during the SMC.

5.3.5 Stochastic Optimal Credit Default Insurance and its Connections with the SMC

This subsection discusses connections between the optimal credit default insurance problem in Subsection 3.6 of Chapter 3 and the SMC. Here, we include an analysis of the results from Propositions 3.8.2, 3.8.3, 3.8.4 and 3.8.5.

5.3.5.1 Statement and Proof of the Credit Default Insurance Problem

The objective function in (3.7) in Subsection 3.7 of Chapter 3 is additively separable in $U^{(1)}$ and $U^{(2)}$ which is not necessarily true for all IBs. In our problem, we have a discount rate, δ^r , which is used to discount these utility functions but is not the market discount rate. The stochastic credit default insurance problem determines the optimal rate of cash outflow for satisfying depositor obligations, k^* , IB's optimal investment in securitized RMLs, I^* , and

optimal credit default insurance, Φ^* . In this regard, Theorem 3.8.1 in Subsection 3.8.1 of Chapter 3 provides the general solution to this optimization problem (see Problem 3.7.1 in Subsection 3.7 of Chapter 3). In the sequel, connections between specific solutions of the optimal credit default insurance problem and the SMC are forged.

5.3.5.2 Optimal Credit Default Swap Contract and the SMC

From Proposition 3.8.2 in Subsection 3.8.2 of Chapter 3, we deduce that the optimal CDS contract coincides with the optimal accrued premium, Φ^* . In this regard, Φ^* is attained when the marginal cost of decreasing or increasing Φ in (3.15) is equals to the marginal benefit of the CDS contract. Moreover, if $\Gamma = 0$ in (3.16) then the optimal accrued premium should be zero, i.e., $\Phi^* = 0$. In this case, if IB holds no type-A RMBSs, Γ , – which is indicative of a high PD for reference portfolio RMLs – then it may be optimal for IB to purchase a CDS contract which protects against all such losses. However, full protection may also introduce high costs in the event that the swap protection seller fails to honor its obligations. In particular, during the SMC, many IBs that purchased CDS contracts promising to cover all losses, regretted making this decision when the swap protection sellers were unable to make payments after a credit event. Notwithstanding this, certain IBs that bought CDS contracts that only pay when the losses exceed a certain level set by the swap protection seller found swap protection beneficial. In particular, they did not experience the same volume of losses as those who purchased full protection (see, for instance, [16]).

5.3.5.3 Optimal Credit Default Insurance with Exponential Utility and the SMC

In Proposition 3.8.3 in Subsection 3.8.4 of Chapter 3, IB's optimal investment in securitized subprime RMLs, I^* , the optimal rate of cash outflow for fulfilling depositor obligations, k^* , and the optimal accrued premium, Φ^* , given by (3.28), (3.29) and (3.30), respectively, are not random variables since they are not dependent on IB's payout, Π in (3.4). Furthermore, IB's optimal investment in securitized RMLs (3.28) from Proposition 3.8.3 is fixed. Moreover, the expression (3.28) also involves the risk aversion measure, ϱ . Ceteris parabis, before the SMC, if ϱ was very high, IB was unlikely to engage in extensive RML securitization. Moreover, in the aforementioned result, $k^* = 0$. This may be indicative of the fact that a troubled IB may no longer be able to fulfill depositor obligations (compare with the formula (3.27)) thus exacerbating counterparty risk. This does not mean, however, that deposits necessarily will dry up since $\mu^I - k^* = \mu^I$, where μ^I is the rate of cash inflow from depositors to IB for investments in securitized subprime RMLs. Another factor that could deter IB from investing in lower rated RMBSs is the cost of credit default insurance. In this case, we note from (3.30) that Φ^* is dependent on ϱ , Γ and S , with Γ having a major effect on the cost of CDS contracts.

The relevance for the SMC of the analysis in the previous paragraph may be identified as

follows. It is clear that during the SMC the ability of IBs to obtain funds for subprime RML securitization investment (compare with μ^I in equation (3.5)) has been dramatically curtailed. Also, spreads have narrowed, as depositors are demanding higher returns to lend money to highly leveraged IBs (compare with k in equation (3.5)). Furthermore, there is substantial evidence to suggest that with increasing delinquencies and foreclosures during 2007-2008, the credit ratings of subprime RMBSs declined dramatically. Depositors became concerned and, in some cases, demanded refunds resulting in margin calls – immediate need to sell/liquidate these securities at fire-sale prices. Consequently, highly leveraged IBs suffered huge losses, bankruptcy or merged with other institutions. Because RMBSs were considered to be *toxic* due to uncertainty in the housing market and could not be sold readily (i.e., they are *illiquid*), their values plummeted. In addition, the market value of houses were penalized by the inability to sell the RMBSs – sometimes they were less than the value the actual cash inflow would merit.

5.3.5.4 Optimal Credit Default Insurance with Power Utility and the SMC

In Proposition 3.8.4 in Subsection 3.8.5 of Chapter 3, the optimal controls in (3.34), (3.35) and (3.36) are expressed as linear functions of IB's optimal payout under subprime RML securitization, Π^* . In this case, we see that the optimal rate of cash outflow for satisfying depositor obligations, k^* , is independent of the frequency and severity parameters ϕ and φ , of the aggregate securitized RML losses, S , respectively. These results are true because the power utility function exhibits constant relative risk aversion which means that

$$-\frac{\eta D^2 \bar{U}^{(2)}(\eta)}{D \bar{U}^{(2)}(\eta)} = 1 - \varrho.$$

Here, we see that if the relative risk aversion increases, the amount invested in RMBSs decreases which may be indicative of the fact that the mass of type-A RMBSs, Γ , is low at that time. The expression for ζ in (3.33) reveals that not only the objective function, J , is affected by the horizon T , but also the optimal rate of cash outflow for satisfying depositor obligations, k^* . Moreover, IB's optimal investment in securitized subprime RMLs, I^* , is affected by the time horizon T via the optimal rate of cash outflow, k^* , which impacts on IB's payout. In addition, the expression for ζ in (3.33) shows that k^* depends on the frequency and severity parameters, ϕ and φ , of the RMBS losses, S , respectively. Furthermore, IB's optimal investment, I^* , is affected by RML losses that indirectly involves k^* . From Proposition 3.8.4, it is clear that the amount invested in RMBSs, I , depends on the payout, Π in (3.4). RML reference portfolio defaults will cause a decrease in IB's payout under subprime RML securitization, which will later affect the rate of cash outflow for fulfilling depositor obligations, k . In particular, this may cause a liquidity problem in the secondary RML market since μ^I may decrease as a result of this effect on k .

If payouts, Π , decrease, it is natural to expect that some IBs will fail as in the SMC (see, for

instance, [41]). For instance, both the failure of the Lehman Brothers investment bank and the acquisition in September 2008 of Merrill Lynch and Bear Stearns by Bank of America and JP Morgan, respectively, was preceded by a decrease in payouts from securitization. A similar trend was discerned for the U.S. mortgage companies, Fannie Mae and Freddie Mac, who had to be bailed out by the U.S. government at the beginning of September 2008.

5.3.5.5 Optimal Credit Default Insurance with Logarithmic Utility and the SMC

As expected, the optimal controls (3.41), (3.42) and (3.43) in Proposition 3.8.5 (see Subsection 3.8.6 in Chapter 3) are consistent in the case where $\varrho \neq 0$ with those in Proposition 3.8.4 (see Subsection 3.8.5 in Chapter 3). In particular, these optimal controls may be expressed as linear functions of IB's optimal payout under subprime RML securitization, Π^* . By contrast to Proposition 3.8.4, the optimal rate of cash outflow for fulfilling depositor obligations, k^* , does not depend on the frequency and severity parameters ϕ and φ , respectively. If the inclination of IB towards retaining earnings, $\tilde{b}$, given by (3.41), is high, the optimal rate of cash outflow, k^* , decreases. Furthermore, if the RMBS price process is very volatile, IB's optimal investment in securitized RMLs, I^* , declines. Indeed, high volatility associated with the price of RMBSs is indicative of high credit risk, which will deter investment in such structured financial products.

The relevance of the discussion in the above, for the SMC, is as follows. Our contention is that if the inclination of IB, $\tilde{b}$, towards retaining earnings is very high, the rate of cash outflow for satisfying depositor obligations, k , decreases. Before the SMC, k was very high as depositors demanded higher returns. This meant that not much of the IB's payout was retained. On the other hand, during the SMC, there was a tendency for IBs to hoard cash so that the value of $\tilde{b}$ in (3.41) increased.

5.4 ANALYSIS OF CHAPTER 4

In this subsection, we briefly discuss some issues emanating from the subprime RML securitization models constructed in Chapter 4 and IB's optimal investment problems (see Section 4.5 in Chapter 4) as well as their connections with the SMC.

5.4.1 Stochastic Dynamic of IB's Investment and the SMC

In Subsection 4.4 of Chapter 4, we have seen that the dynamic model of IB's investment given by (4.11) involves subprime RMBS losses, S , Treasuries rate, r^T , expected return on subprime RMBSs, $\tilde{\mu}^I$, jump heights, α_i , proportions of IB funds invested in subprime RMBS, π , and Treasuries, $1 - \pi$. Before the SMC, we believe that $0 < \alpha_i < \infty$. This was mainly due to the house price appreciation which caused the value of subprime RMBS to

increase. In this regard, IBs investment could have performed well as a result of positive returns, $\tilde{\mu}^I > 0$, due to an increase in subprime RMBS prices as in (4.1) in Subsection 4.1 of Chapter 4. In particular, IBs experienced lower default rates because of increases in house prices. This trend attracted IB investment in the subprime RML market. However, as the SMC unravelled, house prices depreciated dramatically. Accordingly, the value of subprime RMBS declined and $-1 < \alpha_i < 0$. As a consequence, the default rate increased considerably with IBs incurring large losses from their investments. In addition, some IBs started to allocate away from risky assets towards safe assets such as Treasuries. Although this strategy was considered to be safe, it was another factor that prolonged the SMC.

5.4.2 Optimization Problems for IB's Investment in Subprime RMBSs & Treasuries and the SMC

In this subsection, we provide a brief discussion of some of the optimization aspects of the statement and solution of the main result of Chapter 4 as well as their connections with the SMC.

5.4.2.1 Statement of IB's Optimal Investment Problem with Variance and the SMC

We have seen that Problem 4.5.1 in Subsection 4.5.1 of Chapter 4 is a constrained one. This problem maximizes the expected terminal value of IB's investment in subprime RMBS and Treasuries for a given level of investment risk. This may be associated with the fluctuation of the subprime RMBS price in (4.1) in Subsection 4.1 of Chapter 4. This problem enables us to analyze some occurrences during the SMC in a mathematically rigorous way.

5.4.2.2 Solution of IB's Optimal Investment Problem with Variance and the SMC

Theorem 4.5.2 in Subsection 4.5.1.2 of Chapter 4 provides an optimal investment strategy, π^* , which appeared in (4.19). This strategy has an interesting form. In particular, it contains important financial variables that are related to IB's investment decision, namely, the expected return on subprime RMBS, $\tilde{\mu}^I$, Treasuries rate, r^T , as well as the level of IB's investment risk, R , via a variable, $\wp$. If the risk allocation spread, $\tilde{\mu}^I - r^T$, is very high, we expect IBs to invest more in subprime RMBSs and less in Treasuries because high spread could mean low risk is associated subprime RMBS investments (see Appendix C in Section 8.3 of Chapter 8). On the other hand, when the spread is low, it is natural for IBs to invest less in subprime RMBSs and more in Treasuries. Before the SMC, because the spread was high, investments in subprime RML market were very attractive propositions. However, during the SMC, low spread (see Appendix C in Section 8.3 of Chapter 8) caused IBs to allocate away from subprime RMBSs towards Treasuries.

The maximal expected terminal value of IB's total investment, $\mathbb{E}[B_T^{\pi^*}]$, given by (4.20) subject to the variance constraint in (4.18), was found to be dependent on the expected return on subprime RMBSs, $\tilde{\mu}^I$, Treasuries rate, r^T , and the level of IB's investment risk, R . This means that both the optimal investment strategy, π^* , and $\mathbb{E}[B_T^{\pi^*}]$ depend on the same parameters at the end of a planning horizon, T . In this regard, $\mathbb{E}[B_T^{\pi^*}]$ depends on the dynamics of Treasuries in (4.7) in Subsection 4.2 and the subprime RMBS price process in (4.1) in Subsection 4.1 of Chapter 4. Furthermore, if $\alpha_i \in (0, \infty)$, the realized value of IB's total investment will be high. In the case where $\alpha_i \in (-1, 0)$, the realized value of IB's investment will be low. This is exactly what was happening before and during the SMC, respectively (compare with Table 8.2 in Section 8.3).

5.4.2.3 Statement of IB's Optimal Investment Problem with SaR and the SMC

In Subsection 4.5.2.1 of Chapter 4, we consider a mean-SaR optimization problem (see Problem 4.5.4). This problem maximizes the expected terminal value of IB's total investment subject to SaR – defined as a measure of the protection required against possible losses from subprime RMBS investments. In this regard, Problem 4.5.4 complements Problem 4.5.1. Moreover, this problem enables us to analyze the features of CDSs which was another derivative that fueled the SMC.

5.4.2.4 Solution of IB's Optimal Investment Problem with SaR and the SMC

The solution of Problem 4.5.4 can only be obtained through numerical methods. However, this is beyond the scope of this thesis. By way of partially addressing this problem, Subsection 4.5.2.2 of Chapter 4 only contains an illustration of the behavior of the solution. In particular, through differential calculus, we found that the solution to our problem has to be the largest investment strategy, π , which satisfies both SaR-constraint in (4.25) and conditions in (4.13). Furthermore, Subsection 4.5.3 of Chapter 4 discussed a numerical algorithm that can be followed in order to approximate a solution. Also, Problem 4.5.4 reveals that IBs will be exposed to counterparty risk rather than credit risk. In addition, if IBs experience the losses, and the swap protection seller failed to honor its obligation, IBs will have to use its capital to absorb all theses losses. In the situation where IBs do not have enough capital, they will go bankrupt. This situation can cause systemic risk in the case where IBs engaged in interbank lending. On the other hand, if IBs investments perform well, minimal RML losses will be ably compensated by credit default insurance via CDS contracts.

Chapter 6

CONCLUDING REMARKS AND FUTURE INVESTIGATIONS

6.1 CONCLUDING REMARKS

6.1.1 Concluding Remarks About Chapter 1

6.1.2 Concluding Remarks About Chapter 2

6.1.3 Concluding Remarks About Chapter 3

6.1.5 Concluding Remarks About Chapter 4

6.1.6 Concluding Remarks About Chapter 5

6.1.8 Concluding Remarks About Chapter 7

6.1.7 Concluding Remarks About Chapter 8

6.2 FUTURE INVESTIGATIONS

In this chapter, we present a few concluding remarks and highlights some possible topics for future research.

6.1 CONCLUDING REMARKS

In this thesis, we analyzed the subprime RML securitization models in different times framework, i.e., discrete-, continuous- and discontinuous-time. In particular, through the theory of stochastic analysis we developed the securitization models that enabled us to set-up the optimization problems for IB. Finally, we briefly discussed the results emanating from the securitization models and their connection with SMC.

6.1.1 Concluding Remarks About Chapter 1

In Chapter 1, we introduced the structure of the securitization which is considered to be the main cause of 2007-2009 financial crisis. In this regard, we provided the cash flow of all participants which are involved in the structure. We reviewed literature about SMC, securitization, CDSs and jump-diffusion processes. In addition, we gave the preliminaries about subprime RML securitization, CDSs, RML securitization risks, subprime RMLs, subprime RMBS deals, RMBS principal and interest waterfalls and jump-diffusions processes. Furthermore, we list all problems that we address in the thesis. Lastly, we outlined the structure of the thesis.

6.1.2 Concluding Remarks About Chapter 2

Chapter 2 shows how regret can influence the risk allocation behavior of IB. In this regard, the outcomes show that an IB with regret-averse attributes will select risk allocations that are less extreme than those predicted by conventional expected utility. If very risky subprime RMBSs were selected by a purely risk-averse IB, its regret-averse counterpart will choose less risky subprime RMBSs. Conversely, when the purely risk-averse IB picks a riskless portfolio, the regret-averse bank would prefer a riskier portfolio. In essence, regret-averse IBs tend to hedge their bets, taking into account the possibility that their preferences may turn out to be suboptimal after the maturity of the subprime RMBS contract.

More specifically, from Proposition 2.5.5 in Subsection 2.5.2.1 of Chapter 2, we conclude that IBs that weigh regret-aversion more strongly than risk-aversion (as measured by ρ), are more likely to hold subprime RMBSs in their portfolio when $\xi q(I)I - r^T$ is low. Conversely, IB will hold less subprime RMBSs if $\xi q(I)I - r^T$ is high. Corollary 2.5.4 in Subsection 2.5.2 of Chapter 2 claims that for $q(I) = \frac{\tilde{r}^T}{\xi I}$, a regret-averse IB chooses a asset portfolio allocation as if regret was not considered. We also commented on how much a regret-averse IB is willing to pay for credit protection via CDSs, given a fixed asset portfolio allocation. Theorem 2.5.3 in Subsection 2.5.2 of Chapter 2 shows that regret allows IB decisions to

move away from $\pi_\rho^* = 0$ and $\pi_\rho^* = 1$, if no credit protection is bought. This means that IBs who take regret into account will hold more subprime RMBSs when $\xi q(I)I - r^T$ is low, but less when $\xi q(I)I - r^T$ is high. However, under credit protection, Theorem 2.5.6 in Subsection 2.5.2.2 of Chapter 2 shows that regret-averse IBs value CDS contracts less than purely risk-averse IBs, when the investment in subprime RMBSs is small. On the other hand, both risk- and regret-averse IBs would like to pay the same premium when the proportion of available funds invested in subprime RMBSs is high. Table 6.1 below gives a summary of the main results obtained here.

Cases	Behavior with Respect to:	Risk-Averse IB	Regret-Averse IB
<u>Case 1:</u> $q(I) = \frac{r^T}{\xi I}$	Probability of Investing in Subprime RMBSs	$= 0$ $\therefore \pi_0^* = 0$	> 0 $\therefore \pi_\rho^* > 0$
<u>Case 2:</u> $q(I) \gg \frac{r^T}{\xi I}$	Probability of Investing in Treasuries	$= 0$ $\therefore \pi_0^* = 1$	> 0 $\therefore \pi_\rho^* < 1$
<u>Case 1:</u> $q(I) = \frac{r^T}{\xi I}$	Liquidity Effect	High	Low
<u>Case 2:</u> $q(I) > \frac{r^T}{\xi I}$ $q(I) \gg \frac{r^T}{\xi I}$	Liquidity Effect	Low Lower	Low if $\rho \gg 0$. Higher if $\rho \gg 0$.

Table 6.1: Effect of Regret on Risk Allocation and Liquidity

The other main thrust of Chapter 2 is the discussion of risks associated with the securitization of subprime RMLs. In this regard, we mainly comment on credit, tranching, counterparty and liquidity risks and their characteristics.

6.1.3 Concluding Remarks About Chapter 3

In Chapter 3, we have built a stochastic dynamic model for IB's payout under subprime RML securitization. This model enabled us to set-up a stochastic optimal credit default insurance problem (see Problem 3.7.1 in Subsection 3.7 of Chapter 3) for a fixed term, $[t, T]$, which optimizes the rate of cash outflow for fulfilling depositor obligations, k , value of IB's investment in securitized subprime RMLs, I , and CDS contract accrued premium, Φ . Explicit closed form solutions to this problem was determined by choosing exponential, power and logarithmic utility functions. We also highlighted some of the risk issues and related key outcomes to the SMC.

6.1.4 Concluding Remarks About Chapter 4

In Chapter 4, we investigated the optimal investment strategy for IB in subprime RMBSs and Treasuries in a discontinuous-time. In this regard, we have constructed the stochastic dynamic of IB's investment in subprime RMBSs and Treasuries in a Lévy process setting. This model enabled us to state and solve an optimization problems for IB's investment with both variance and SaR. The solution in Theorem 4.5.2 in Subsection 4.5.1.2 of Chapter 4 shows that the optimal strategy, π^* , depends on the expected return on subprime RMBS, $\tilde{\mu}^I$, Treasuries rate, r^T , and the level of IB's total investment risk, R , via a variable, $\wp$. In Subsection 4.5.2 of Chapter 4, we have an optimization problem for IB's investment subject to SaR that cannot be solved by analytical approach. However, through differential calculus, we have found that the optimal solution of Problem 4.5.4 is the largest IB's investment strategy $\pi \in [0, 1]$ that satisfies the SaR constraint in (4.25) and condition (4.13). We also gave some relation between the issues discussed in Chapter 4 and the SMC.

6.1.5 Concluding Remarks About Chapter 5

In Chapter 5, we briefly discussed the results of the main issues of the thesis and their connections with the SMC. In particular, we compared the outcomes of the stochastic models established in Chapters 2, 3 and 4, with the current dramatically event in the actual financial and economic systems.

6.1.6 Concluding Remarks About Chapter 7

In Chapter 7, we gave all references that have been used to produce the thesis.

6.1.7 Concluding Remarks About Chapter 8

Chapter 8 provided the appendices of the thesis. In this regard, Appendix A gave the statement of Itô's formula for jump-diffusion processes, that have been used where it is applicable in the thesis. Subsection 8.2 contained the numerical analysis of the formulas that appeared in Chapter 1. Appendix C discussed what was happening before and during the SMC to the global economy. Lastly, Appendix D briefly discussed a comparison between the prime and Alt-A deals.

6.2 FUTURE INVESTIGATIONS

In future studies, an open problem is to solve the stochastic optimal credit default insurance problem solved in Chapter 3 in a Lévy process framework (see, for instance, Protter in [57, Chapter I, Section 4]). This will involve expected subprime securitized RML losses that are modeled via exponential Lévy processes. Such processes have an advantage over the

more traditional modeling tools such as Brownian motion in that they describe the non-continuous evolution of the value of economic and financial items more accurately. For instance, because the behavior of RMLs, payout, capital and capital adequacy ratio are characterized by jumps, the representation of the dynamics of these items by means of Lévy processes is more realistic. As a result of this, recent research has strived to replace the existing Brownian motion-based bank models (see, for instance, [44]) by systems driven by more general processes. Furthermore, problem of interest is the optimization of IB's payout from securitized and unsecuritized subprime RMLs simultaneously. In this case, we would like to ascertain which investment decision must be made about the proportion invested in securitized and unsecuritized RMLs in order to generate optimal payout (compare with [55]). Another problem of interest could be a use of different numerical approximations to Problem 4.5.4.

Chapter 7

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Chapter 8

APPENDICES

8.1 APPENDIX A: ITÔ'S FORMULA FOR JUMP-DIFFUSIONS PROCESSES

8.2 APPENDIX B: COMPUTATIONAL EXAMPLE

8.3 APPENDIX C: ECONOMIC CONDITIONS BEFORE AND DURING THE SMC

8.4 APPENDIX D: COMPARISON WITH PRIME AND ALT-A DEALS

In this chapter, we provide an appendix about Itô's formula for jump-diffusion processes, computational example, economic conditions and comparison with prime and Alt-A deals.

8.1 APPENDIX A: ITÔ'S FORMULA FOR JUMP-DIFFUSIONS PROCESSES

This subsection contains an appropriate version of Ito's formula for jump-diffusions processes (compare [57, Chapter 2] and [13, Chapter 8]).

Lemma 8.1.1 (Itô's Formula for Jump-Diffusions Processes): *Suppose that $(X_t)_{t \geq 0}$ is a stochastic process described by a stochastic differential equation, where the random fluctuations are generated by both Brownian motion, Z_t , and n Poisson counters $(N_i(t))_{1 \leq i \leq n}$. That is,*

$$dX_t = f(X_t)dt + g(X_t)dZ_t + \sum_{i=1}^n \tilde{h}_i(X_t)dN_i(t),$$

where $f(\cdot)$, $g(\cdot)$ and $\tilde{h}_i(\cdot)$ are real-valued functions. Moreover, $dN_i(t)$ is given by

$$dN_i(t) = \begin{cases} 0, & \text{no event at time } t; \\ 1, & \text{event at time } t. \end{cases}$$

Then, for $\bar{P}(t) = \bar{P}(X_t)$, we have that

$$\begin{aligned} d\bar{P}(X_t) &= \frac{d\bar{P}(X_t)}{dX_t} f(X_t) dt + \frac{d\bar{P}(X_t)}{dX_t} g(X_t) dZ_t + \frac{1}{2} \frac{d^2 \bar{P}(X_t)}{dX_t^2} [g(X_t)]^2 dt \\ &+ \sum_{i=1}^n [\bar{P}(X_t + \tilde{h}_i(X_t)) - \bar{P}(X_t)] dN_i(t). \end{aligned}$$

8.2 APPENDIX B: COMPUTATIONAL EXAMPLE

In this subsection, we provide a computational example from [28] that illustrates the issues raised on pages 4, 5 and 6 of Chapter 1. Suppose that the subprime RML face value is \$ 100; the size of RMBS sub tranche is \$ 20 and the size of the sen RMBS tranche is \$ 80. Furthermore, the subprime RMBS tranche is sold to a CLO, which only buys this tranche, so that the size of the CLO is \$ 80. Let \$ 15 be the size of CLO sub tranche, so that the sen tranche's size is \$ 65. If we keep these parameters constant and vary the recovery amount, Table 8.1 shows the loss on the sen RMBS tranche, the payout from the sen RMBS tranche, the loss on the sen CLO tranche, and the payout from the sen CLO tranche at the end of the period.

Parameters								
R_v (\$)	90	70	60	50	40	30	20	10
Results								
S_v^s	0	10	20	30	40	50	60	70
Π_v^s	80	70	60	50	40	30	20	10
Π_v^s as % of 80	100 %	87.5 %	75 %	62.5 %	50 %	37.5 %	25 %	12.5 %
S_v^c	0	0	5	15	25	35	45	55
Π_v^c	65	65	60	50	40	30	20	10
Π_v^c as % of 65	100 %	100 %	92.3 %	76.9 %	61.5 %	46.2 %	30.8 %	15.4 %

Table 8.1: Computational Results

8.3 APPENDIX C: ECONOMIC CONDITIONS BEFORE AND DURING THE SMC

Table 8.2 below compares economic conditions before and during the SMC.

Before SMC (Year < 2007)	During SMC (Year $\geq$ 2007)
High Level of Macroeconomic Activity	Lower Level of Macroeconomic Activity
Boom Conditions	Recession Conditions
Low Perceived Credit Risk	Higher Perceived Credit Risk
Low Delinquency Rate	Higher Delinquency Rate
Low Foreclosure Rate	Higher Foreclosure Rate
Regret-Averse Agents	Risk-Averse Agents
House Prices Increase	House Prices Decline
Low Counterparty Risk	Higher Counterparty Risk
High Rate of Securitization of Subprime RMLs	Lower Rate of Securitization of Subprime RMLs
Low Investment in Safe Assets such as Treasuries	Higher Investment in Safe Assets such as Treasuries
High Spreads	Low Spreads
High Market Liquidity	Low Market Liquidity
Few Credit Crunches	Many Credit Crunches
Highly Leveraged Financial Institutions	Less Highly Leveraged Financial Institutions

Table 8.2: Differences in Economic Conditions Before and During the SMC

8.4 APPENDIX D: COMPARISON WITH PRIME AND ALT-A DEALS

This subsection enables us to compare some of the structural features of subprime deals with prime jumbo and Alt-A deals. The latter type of deals use a 6-pack structure while the XS/OC structure is used by most subprime RMBSs and some Alt-A deals (see, for instance, [28]). The structure that is chosen is mainly determined by the amount of XS in the deal. In an XS/OC structure, r^x is typically between 300-400 basis points (see (1.8) for a precise formulation of r^x). There is no OC in a 6-pack structure where RML collateral is tranching into sen (AAA), mezz (AA, A, BBB) and sub (BB, B and unrated) tranches. The most junior bond, essentially equity, is unrated because it is in the *first loss* position, meaning that it will absorb the first losses from the RML reference portfolio.

In a sen/sub or 6-pack, structure, the mezz and sub tranches are of sufficient thickness to absorb RML reference portfolio (collateral) losses. This ensures that the sen bonds have a probability of loss sufficiently low to justify a AAA credit rating. This is accomplished by

reversing the order of the priority of the cash flow payments and losses in the transaction. At the beginning of the deal, prepaid principal is allocated from the top down (sequential amortization), with only the sen bonds being paid, while the mezz bonds and sub bonds are *locked out* from receiving prepaid principal. Losses are allocated from the bottom up, that is, the lowest-rated class outstanding at the time will absorb any principal losses. By using sequential amortization, the sen bonds are paid down first, and there is an increase in the percentage of the remaining RML reference portfolio collateral that is covered by the mezz and sub bonds. This continues during the lock-out period, which may be the first five years, in a fixed rate transaction, or for as long as 10 years in a prime ARM transaction.

In prime and Alt-A ARM deals there may be triggers that reduce the length of the lock-out period subject to the achievement of certain performance criteria. The two most common metrics in prime ARM sen/sub structures are (1) a *step-down test* and (2) the *double-down test*. A step-down test refers to when prepaid principal switches from sequential pay to pro-rata amortization. According to [28], prepaid principal switches from sequential pay to pro rata for all outstanding classes if: (a) the senior CE is twice the original percentage; (b) the average 60+ day delinquency percentage for the prior six months is less than 50 % of the current balance and (c) cumulative losses are under a specified percentage of the original RML balance. The doubled-down test means that prior to the initial three-year period, 50 % of prepaid principal can be allocated to the mezz and sub bonds if (a) to (c) hold.