

Parameter uniform numerical methods for some singularly perturbed problems

N. A. Mbroh



orcid.org/0000-0002-2287-8258

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Promoter: Prof SC Oukouomi Noutchie

Co-promoter: Dr RY M'pika Massoukou

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Preface

The work described in this thesis was carried out in the School of Mathematical Sciences, North-West University, Mafikeng Campus, under the supervision of Professor Suares Clovis Oukouomi Noutchie and the co-supervision of Doctor Rodrigue Yves M’pika Massoukou.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Abstract

In this thesis, efficient numerical techniques have been proposed to solve different types of singularly perturbed problems. That is problems whose solutions are characterized with rapid oscillation usually known as layers. The presence of these layers makes the use of classical numerical methods unfit to solve such problems since they are unable to mimic the behaviour of the exact solution in the part of the domain where the layers are present. This thesis seeks to address this issue by modifying the classical numerical methods such that they will be able to capture the behaviour of the exact solution in the part of the domain where the layers are present. Thus we design, analyse, implement and enhance the accuracy of efficient numerical methods to solve various forms of singularly perturbed problems. The scheme is based on the Non-standard finite difference schemes and used with other time integrators to solve parabolic problems either via the full discretisation schemes or the step by step discretisation schemes (Rothe's method or the method of lines). For stationary integro-differential problems, the scheme is employed along with the Simpson's rule to provide the numerical solution. The mathematical software Matlab has been used for all simulations in this thesis.

Keywords: Singularly perturbed problems; Volterra integro-differential equation; Convection-diffusion problems; Delay parabolic problems; Degenerate problems; Robin boundary conditions; Burgers equation; Fitted operator finite difference methods.

Declaration 1 - Plagiarism

I, Nana Adjoah Mbroh, declare that

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Declaration 2 - Publication

Published papers

1. Mbroh, N. A., Oukouomi Noutchie, S. C. and M'pika Massoukou, R. Y. (2020). A second order finite difference scheme for singularly perturbed Volterra integro-differential equation. *Alexandria Engineering Journal*, 59(4):2441-2447.
2. Mbroh, N. A., Oukouomi Noutchie, S. C. and M'pika Massoukou, R. Y. (2020). A robust method of lines solution for singularly perturbed delay parabolic problem. *Alexandria Engineering Journal*, 59(4):2543-2554.
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4. M'pika Massoukou, R. Y., Oukouomi Noutchie, S. C. and Mbroh, N. A. (2020). Global dynamics for a model of amyloid fibril formation in pancreatic islet beta cells subjected to a therapy. *Journal of Biological Dynamics*, 14(1):162-186.

Submitted papers

1. A Second Order Fitted Operator Finite Difference Scheme for Singularly Perturbed Degenerate Parabolic Problem.
2. A Second Order Fitted Operator Finite Difference Scheme for Modified Burgers Equation.

Dedication

To my parents, Isaac Kingsley Mbroh & Comfort Hagan.

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Chapter 1

Introduction

Singularly Perturbed Problems (SPP) came into existence at the Third International Congress on Mathematics in Heidelberg in 1904 where Prandtl presented his short report which was later published in the proceedings (Prandtl, 1905). He introduced the boundary layer theory in his work, which has now become the basis of modern fluid dynamics. However, Friedrichs and Wasow (Friedrichs et al., 1946) were the first to introduce the term “singular perturbation”. Later, the PhD thesis of Wasow (1981), gave much more generality on the term boundary layer.

In general, singularly perturbed problems are differential equations whose highest derivatives have a coefficient with small magnitude and this small coefficient is known as the perturbation parameter ε . The solution of these problems approaches a discontinuous limit as ε approaches zero, Roos et al. (2008). Literally, solutions of singularly perturbed problems do not correspond to the solution of the reduced version, that is when the perturbation parameter $\varepsilon \rightarrow 0$. This causes the analytical solution of SPP to be equipped with some abrupt changes (layers) in some parts of the domain. Thus the best approach to gain some insight on the analytical solution as $\varepsilon \rightarrow 0$ is through the use of asymptotic expansions. However, both the straight forward asymptotic expansions and the one which uses a combination of the outer and the inner expansions have some form of limitations. Interested readers can refer to the books Bender and Orszag (2013); Bush (2018); Miller (2006); O’malley (1991); Van Dyke (1964) for more knowledge concerning asymptotic expansions of SPPs.

The other alternative is the use of numerical methods to tackle such problems. However, as the perturbation parameter tends to zero, classical numerical methods can not yield satisfactory results unless an extremely larger number of mesh points are employed. This also renders the scheme to be computationally expensive, in addition to the ill-conditioning of the system matrix. To elude this difficulty, various modification of these classical numerical methods have been made. In the context of finite differences, two main procedures which include the fitted mesh and the fitted operator finite difference schemes are known. Fitted mesh is basically the use of the classical finite difference scheme on a layer adapted mesh, and thus it is carried on a non-uniform mesh whilst fitted operator introduces a new

coefficient function into the classical finite difference scheme and it is carried on a uniform mesh. This thesis employs the fitted operator finite difference schemes to solve various forms of singularly perturbed problems and hence a brief account on its construction will be explained in subsequent sections.

1.1 Literature Review

The problems faced by numerical analysts is the fact that classical numerical methods fail to capture the behaviour of the exact solution when the perturbation parameter tends to zero. In an attempt to resolve this issue, these classical numerical methods have been modified in several ways, (fitted numerical methods) see for example the books Farrell et al. (2000); Miller et al. (2012); Morton (1996); Roos et al. (2008) and the survey articles Kadalbajoo and Reddy (1989); Kadalbajoo and Patidar (2002, 2003) for the various modifications of these classical numerical methods and forms of SPPs which were solved by these fitted numerical methods. In this thesis, we give an account of some of the recent works which have been done and we wish to state that, the works presented are the problems of interest in this work.

1.1.1 Parabolic problems with Robin boundary conditions

Hemker et al. (2002) considered a singularly perturbed problem of the form

$$-u_t + \varepsilon u_{xx} + b(x, t)u_x - c(x, t)u = f(x, t), \quad (x, t) \in Q = \Omega \times (0, T], \quad \Omega = (0, 1), \quad (1.1)$$

along with the initial and the boundary conditions

$$\begin{aligned} u(x, 0) &= \varphi(x), & 0 \leq x \leq 1, \\ -\varepsilon \alpha(0, t)u_x + \beta(0, t)u &= \psi_0(t), & t \in (0, T], \\ \varepsilon \alpha(1, t)u_x + \beta(1, t)u &= \psi_1(t), & t \in (0, T]. \end{aligned} \quad (1.2)$$

The authors designed an upwind finite difference scheme on a piecewise uniform Shishkin mesh for the spatial discretisation and the backward Euler for the time discretisation. They analysed the scheme for convergence and obtained an almost first order accuracy in space and first order accuracy in time. Using the defect correction method, they improved the accuracy of the time variable to second and third orders. Notice that the boundary layer was in the neighbourhood of $x = 0$.

Chandru and Shanthi (2015) proposed a numerical scheme to solve a stationary version of problem (1.1)–(1.2) with a discontinuous source term. They employed a simple upwind scheme on a piecewise uniform Shishkin mesh and analysed it for convergence. Their analysis yielded a second order accuracy except for a logarithmic factor.

Selvi and Ramanujam (2017) considered problem (1.1)–(1.2) with a time delay. They employed a classical finite difference scheme on a rectangular piecewise uniform fitted

mesh and analysed it for convergence. Their analysis resulted in an almost second order accuracy in space and first order in time.

Zheng et al. (2018) also studied the stationary version of problem (1.1)–(1.2) with two difference operators namely: a midpoint upwind scheme and a hybrid difference scheme. The simple upwind scheme was designed on both Shishkin and Bakhvalov-Shishkin meshes and analysed for convergence, whilst the hybrid scheme was designed on only Bakhvalov-Shishkin mesh. For the simple upwind scheme, they obtained the following accuracies $\mathcal{O}(N^{-1} \ln N)$, $\mathcal{O}(N^{-2})$, for the Shishkin mesh and $\mathcal{O}(N^{-1})$, $\mathcal{O}(N^{-2})$ for the Bakhvalov-Shishkin mesh in the layer and the non-layer regions respectively. Whilst the analysis of the hybrid scheme yielded the accuracies $\mathcal{O}(N^{-2} \ln^2 N)$ and $\mathcal{O}(N^{-1})$, in the layer and non-layer regions respectively. Clearly the hybrid scheme was more accurate than the simple upwind scheme and the Bakhvalov-Shishkin mesh was more accurate than the Shishkin mesh.

Jayalakshmi and Tamilselvan (2020a) performed a comparative study with two difference schemes on the stationary version of problem (1.1)–(1.2). The schemes were a simple upwind difference scheme and a hybrid difference scheme on a piecewise uniform Shishkin mesh. Their hybrid scheme employed the central difference scheme in the layer region and the midpoint scheme in the non-layer region. Their analysis revealed that the hybrid scheme gives a better accuracy than the simple upwind difference scheme since the accuracies were of almost second and first orders, respectively.

Jayalakshmi and Tamilselvan (2020b) proposed a numerical scheme to solve a problem similar to problem (1.1)–(1.2) which has a boundary turning point. The layer was around the right boundary in their case. The authors employed the classical upwind finite difference scheme on a rectangular piecewise uniform mesh for the discretisation. They analysed their scheme for convergence and obtained a first order accuracy in both time and space, except for a logarithmic factor in space.

Evidently, the aforementioned authors solved problem (1.1)–(1.2) on a non-uniform mesh and their accuracies had a logarithmic factor. Thus to obtain an accuracy which does not depend on a logarithmic factor, we propose a scheme which is carried on a uniform mesh and also yields a full first order accuracy to solve problem (1.1)–(1.2). The scheme is a combination of the classical finite difference scheme on a positive denominator function which reflects the qualitative properties of the problem under study and the backward Euler finite difference scheme. A first order accuracy for both the time and the space variables is established and later improved to a second order by the application of Richardson extrapolation. The scheme is then tested on an example to validate the theoretical findings.

1.1.2 Parabolic problems with delay

Consider the delay parabolic problem

$$\mathcal{L}_\varepsilon u(x, t) = f(x, t) - c(x, t)u(x, t - \tau), \quad (x, t) \in Q, \quad Q = \Omega \times (0, T], \quad \Omega = (0, 1), \quad (1.3)$$

where the above differential operator $\mathcal{L}_\varepsilon$ is expressed by

$$\mathcal{L}_\varepsilon = \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} + a(x) \frac{\partial}{\partial x} + b(x, t),$$

subject to the boundary and initial conditions

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t), \quad t \in [0, T], \quad u(x, t) = \varphi(x, t), \quad (x, t) \in [0, 1] \times [-\tau, 0]. \quad (1.4)$$

Ansari et al. (2007) designed a fitted mesh finite difference method along with an implicit Euler to solve the reaction diffusion version of problem (1.3)–(1.4). They established a temporal first order accuracy and an almost second order spatial accuracy in their analysis.

Bashier and Patidar (2011a) also considered the reaction diffusion version of problem (1.3)–(1.4) by discretising both variables at the same time. They used the Crank-Nicolson finite difference method for the temporal discretisation and the fitted operator finite difference method for the space discretisation. A second order accuracy in both time and space variables were obtained from their analysis.

In the same year, the backward Euler finite difference scheme along with a fitted operator finite difference scheme were used by Bashier and Patidar (2011b) to solve the problem (1.3)–(1.4) without the convective. They obtained an accuracy of order $\mathcal{O}(h^2 + k)$, where k and h are the step sizes in time and space variables, respectively.

Kaushik and Sharma (2012) solved problem (1.3)–(1.4) via the Rothe's method. They employed the implicit Euler method for the time discretisation and the classical finite difference scheme for the spatial discretisation. Using the discrete invariant imbedding algorithm, the discrete systems were solved and analysed for convergence. However, the result obtained was dependent on the perturbation parameter ε and thus the authors noted that the method can be extended to the case of adaptive meshes to improve the solution.

To obtain an ε -uniform result for the same problem, Das and Natesan (2015) proposed a hybrid numerical scheme to discretise the space derivatives on a piecewise homogeneous mesh and employed the backward Euler for the time discretisation. The convergence of the scheme was analysed and a first order accuracy in time and nearly second order accuracy in space were obtained. Note that their hybrid scheme was a combination of the upwind midpoint method for the non-layer region and the central difference approximation for the layer region.

Gowrisankar and Natesan (2017) studied the same problem and established a first order accuracy in time and nearly first order accuracy in space. They used the backward Euler

finite difference scheme for the time discretisation and the upwind finite difference method on a piecewise homogeneous mesh for the space discretisation.

Subsequently, Salama and Al-Amery (2017) made use of the Crank-Nicolson finite difference scheme to discretise the temporal derivative and a fourth-order compact difference scheme to discretise the spatial derivatives via the Rothe's method. They analysed their methods for convergence and obtained a second order accuracy in time, a second order accuracy in space for the non-layer region and fourth order accuracy in space for the layer region.

Das and Natesan (2018) designed an almost second order method via the Rothe's method. The authors made use of the upwind difference scheme on a Shishkin mesh to discretise the spatial variable and the backward Euler scheme to discretise the temporal variable. They analysed the methods for convergence and obtained an accuracy of $\mathcal{O}(N^{-1} \ln N + \Delta t)$. The variable N is the number of sub-intervals in the spatial direction and Δt is the step size in the temporal direction. Using Richardson extrapolation technique, they enhanced the accuracy to an almost second order in space and a second order in time.

Except for Bashier and Patidar (2011a) and Bashier and Patidar (2011b) who employed a fitted operator finite difference scheme to solve the spatial derivative of a reaction diffusion problem, all the authors in the above literature focused on only non-uniform mesh (Shishkin mesh) schemes and thus their accuracies will automatically have some logarithmic factor, thereby reducing the accuracy of the schemes. To obtain a full first or second accuracy, we solve the problem (1.3)–(1.4) using a combination of a fitted operator finite difference scheme and the Crank Nicolson finite difference schemes in the framework of the methods of line. The schemes are analysed for convergence and improved in the spatial accuracy by Richardson extrapolation.

1.1.3 Volterra-integro differential equations

Undoubtedly, the work done on singularly perturbed ordinary and partial differential equations is very huge, however, that is not the case for singular perturbed Volterra integro-differential equations of the form

$$\varepsilon u'(t) + a(t)u(t) + \int_0^t K(t,s)u(s)ds = f(t), \quad t \in I = [0, T], \quad (1.5)$$

$$u(0) = \mu, \quad (1.6)$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter. The coefficient functions and the source term are smooth functions. Moreover, the coefficient functions satisfy $a(t) \geq \alpha$ and $K(t,s) \geq \bar{K}$, $(t,s) \in (I \times I)$ and μ is a given constant. Below we give a brief account of some of the methods which have been used to solve problem (1.5)–(1.6).

Amiraliyev and Şevgin (2006) constructed an exponentially fitted finite difference scheme to solve problem(1.5)–(1.6). They analysed the scheme for convergence and obtained a first order accuracy in space, which does not depend on the perturbation parameter.

Using the midpoint difference operator with trapezoidal integration, Zhongdi and Lifeng (2006) studied problem (1.5)–(1.6) and analysed it for convergence. Their analysis resulted in an almost second order accuracy.

Şevgin (2014) proposed a uniformly convergent numerical method on a graded mesh to solve the non-linear version of problem (1.5)–(1.6). He analysed the scheme for convergence and obtained a first order accuracy in the maximum norm.

Kudu et al. (2016) designed and analysed a numerical method which is independent of the perturbation parameter to solve a delayed version of problem (1.5)–(1.6). The scheme employed the implicit finite difference scheme on a piecewise uniform mesh of Shishkin-type to solve the differential part and composite quadrature rules were used for the integral part. Their analysis was almost first order accurate in the maximum norm.

Iragi and Munyakazi (2018) proposed two difference operators: fitted mesh and a fitted operator finite difference schemes to solve problem (1.5)–(1.6). Their fitted mesh was a combination of a classical finite difference scheme and some quadrature rules on Shishkin mesh, whilst the fitted operator introduced a fitting factor through the use of integral identity and exponential basis functions along with interpolating quadrature rules. They established a first order accuracy for both schemes.

Yapman et al. (2019) constructed a uniformly convergent finite difference scheme based on the method of integral identities with exponential basis function and interpolating quadrature rules to solve a Volterra delay-integro-differential equation. They established a first order accuracy in the maximum norm for their analysis.

Yapman and Amiraliev (2020) proposed an almost second order scheme via fitted finite difference scheme on a non-uniform mesh to solve problem (1.5)–(1.6).

All the above mentioned papers made use of fitted mesh finite difference scheme and the exponentially fitted finite difference method of the fitted operator finite difference scheme. Again none of these authors have proposed a fully second order scheme.

Thus a second order finite difference scheme based on non-standard finite difference schemes, introduced by Lubuma and Patidar (2006, 2007), to solve singularly perturbed problems is proposed to solve the differential part of problem (1.5)–(1.6), whilst the composite Simpson's rule is used to solve the integral part. The scheme is analysed for convergence and enhanced in accuracy by Richardson extrapolation.

1.1.4 Problems with degenerate coefficient

Consider the singularly perturbed degenerate parabolic convection diffusion problem

$$\mathcal{L}_\varepsilon u(x, t) = f(x, t), \quad (x, t) \in Q, \quad (1.7)$$

subject to the boundary and the initial condition

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t), \quad t \in [0, T], \quad u(x, t) = \varphi(x), \quad x \in \bar{\Omega}. \quad (1.8)$$

The differential operator $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon = -\frac{\partial}{\partial t} + \varepsilon \frac{\partial^2}{\partial x^2} + a(x) \frac{\partial}{\partial x} - b(x, t).$$

We assume the source term $f(x, t)$ and the coefficient functions $a(x)$, $b(x, t)$ are smooth and bounded. Also, the coefficient functions satisfy

$$a(x) \geq a_0(x)x^p > \alpha > 0, \quad p \geq 1, \quad b(x, t) \geq \beta \geq 0.$$

The perturbation parameter ε is such that $0 < \varepsilon \ll 1$. Problems of types (1.7)–(1.8) are known as singularly perturbed turning point problems and here we give a brief account on some of the articles which studied such problems.

Dunne et al. (2000) employed the classical finite difference operator on a piecewise uniform mesh and the backward Euler to discretise the space and time variables respectively to solve a similar problem to problem (1.7)–(1.8) with non-rectangular domain. Their analysis lead to an almost first order uniform accuracy in space and first order in time.

Dunne et al. (2003) solved problem (1.7)–(1.8) by the classical finite difference operator on a Shishkin mesh and the backward Euler finite difference scheme. These authors established an almost first order accuracy in space and first order in time.

Viscor and Stynes (2010) studied the reaction diffusion version of problem (1.7)–(1.8) and obtained an almost second order accuracy in space and first order in time.

Clavero et al. (2015) investigated a singularly perturbed convection diffusion problem with degenerate coefficient having a discontinuous source term and obtained a first order accuracy in time and almost second order accuracy in space. They discretised the space and the time variables at the same time via the simple upwind scheme on a non-uniform mesh and the backward Euler schemes respectively.

Majumdar and Natesan (2017a) studied a two-dimensional version of problem (1.7)–(1.8) using an alternating direction scheme for the time derivative and an upwind finite difference operator on Shishkin mesh for the space derivative. These authors established an almost first order accuracy for their scheme.

Majumdar and Natesan (2017b) proposed a numerical scheme which employed the backward Euler and the classical finite difference operator on Shishkin mesh to solve the time and the space variables respectively. They established that their scheme was first order accurate in both variables except for a logarithmic factor in space. Using Richardson extrapolation technique, they enhanced the accuracy to almost second order in space and second order in time.

Majumdar and Natesan (2019) proposed an ε -uniform scheme of first order accuracy in time and almost second order in space to solve problem (1.7)–(1.8) via the Rothe's method.

The scheme employed a combination of the classical central difference and the midpoint upwind scheme on a Shishkin mesh to solve the spatial derivatives and the backward Euler was used for the time derivative.

Gupta et al. (2019) designed a hybrid scheme to solve a stationary singularly perturbed problem with an interior turning point. They analysed the scheme for convergence and obtained a second order accuracy except for a logarithmic factor.

Rai and Yadav (2020) discretised the time and the space variables of a singularly perturbed delayed parabolic convection diffusion problem with degenerate coefficients via the backward Euler and a hybrid scheme on a non-uniform mesh, respectively. Their analysis was almost second order in space and first order in time. Using Richardson extrapolation on the time variable they improved the time accuracy to second order.

As a contribution to literature and noticing that all the above papers focused on non-uniform mesh, we propose a scheme which uses a uniform mesh to solve problem (1.7)–(1.8). The scheme uses the Crank Nicolson finite difference scheme to solve the time derivatives and a fitted operator finite difference scheme based on a midpoint downwind scheme for the spatial derivatives. The error associated with each discretisation is presented.

1.1.5 Non-linear problems with regular layers

Consider the modified Burgers equation

$$u_t(x, t) - \varepsilon u_{xx}(x, t) + u^2 u_x(x, t) = 0, \quad (x, t) \in Q \equiv \Omega \times (0, T], \quad (1.9)$$

subject to the initial and the boundary conditions

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T]. \quad (1.10)$$

Kadalbajoo and Awasthi (2008) designed a numerical scheme which was of almost first order accuracy in space and first order in time to solve problem (1.9)–(1.10). Their scheme employed the upwind finite difference scheme on a piecewise uniform Shishkin mesh to solve the spatial derivatives and the backward Euler finite difference scheme was used for the time derivatives.

Gupta and Kadalbajoo (2010) constructed a numerical scheme to solve problem (1.9)–(1.10) for different Reynolds numbers. Their scheme was a combination of the implicit Euler and a hybrid finite difference scheme on a piecewise uniform Shishkin mesh for the time and spatial discretisations, respectively. These authors established the asymptotic bounds of the solution by using singular perturbation analysis and analysed their scheme for convergence. Their analysis led to first and second order accuracies in time and space, respectively, except for a logarithmic factor in space. Notice that their hybrid scheme employed the central difference scheme in the layer region and the midpoint scheme in the non-layer region.

Erdogan and Sakar (2017) presented a quasilinearization technique to solve a singularly perturbed delay differential equation. Their scheme employed the implicit finite difference scheme on piecewise-uniform S-meshes. Their analysis resulted in a first order uniformly convergent scheme with respect to the perturbation parameter.

Kanth and Murali (2018) considered a stationary non-linear reaction diffusion problem with delay. The authors first converted the non-linear problems into sequence of linear problems and then designed an exponentially fitted spline method to solve it. They analysed their method for convergence and obtained an almost second order accuracy.

Liu et al. (2020) considered a first order non-linear singularly perturbed problem with integral boundary condition. The proposed scheme uses the backward Euler on an equidistributing monitor function based on arc-length. They analysed their scheme for convergence and obtained a first order accuracy independent of the perturbation parameter in the maximum norm.

In literature, the work done on the general numerical solution of Burgers equation and modified Burgers equation is very extensive, however, the same cannot be said from looking at the modified Burgers equation in the context of singular perturbation. Also, almost all the aboved authors designed their schemes on non-uniform mesh. Thus to contribute to literature and fill this gap, we propose a difference scheme based on a fitted operator finite difference scheme to solve problem (1.9)–(1.10).

1.2 Preliminaries

In this section we present some basic definitions and notations which will be employed through-out the thesis.

To begin with, we denote Ω the open interval $(0, 1)$ of $\mathbb{R}$. In a time-dependent case, Q is also an open interval in $\mathbb{R}$ and takes the form $Q = \Omega \times (0, T]$. The notations $\partial\Omega$, ∂Q and $\bar{\Omega}$, $\bar{Q}$ are their boundaries and closures, respectively. For a non-negative integer, i , $\mathcal{C}^i(\Omega)$ denotes the space of all functions whose derivatives are continuous up to the orders i . In the case of time-dependent problems, $\mathcal{C}^{i,k}(Q)$ represents the space of all continuous derivatives up to the orders i in space and k in time, where k is also a positive integer.

In the analysis, the L^∞ norm, denoted by

$$\|z\|_\infty = \max_{\xi \in \Omega} |z(\xi)|,$$

for a function $z(\xi)$ on some domain Ω , will be employed. In time-dependent case, we have

$$\|z\|_\infty = \max_{(\xi_x, \xi_t) \in \bar{Q}} |z(\xi_x, \xi_t)|,$$

for a function $z(\xi_x, \xi_t)$ on some domain Q . For notational simplicity we omit the ∞ and write $\|\cdot\|_\infty$ as $\|\cdot\|$. The following finite difference operators will be employed as well. Let

Ω^n the discrete version of Ω be an arbitrary uniform mesh

$$\Omega^n \equiv \{x_0 < x_1 < \dots < x_{n-1} < x_n = 1\},$$

where the step size h is given by $h = x_i - x_{i-1}$, $i = 1, 2, \dots, n$. Subsequently, $(0, T]^m$ is the discrete version of the time domain whilst $Q^{n,m} = \Omega^n \times (0, T]^m$ is the discrete version of Q . For a discrete one-dimensional function Φ_i , we define the classical second order finite difference scheme on a uniform mesh by

$$\delta^2 \Phi_i = \frac{\Phi_{i+1} - \Phi_i + \Phi_{i-1}}{h^2}$$

and the first order backward and forward difference schemes by

$$D^- \Phi_i = \frac{\Phi_i - \Phi_{i-1}}{h} \quad \text{and} \quad D^+ \Phi_i = \frac{\Phi_{i+1} - \Phi_i}{h},$$

respectively. For a time-dependent function, we define the implicit finite difference operator by

$$D_t^- \Phi_i^{k+1} = \frac{\Phi_i^{k+1} - \Phi_i^k}{\Delta t}$$

and their corresponding spatial derivative operators are of the form

$$\delta^2 \Phi_i^{k+1} = \frac{\Phi_{i+1}^{k+1} - \Phi_i^{k+1} + \Phi_{i-1}^{k+1}}{h^2},$$

$$D^- \Phi_i^{k+1} = \frac{\Phi_i^{k+1} - \Phi_{i-1}^{k+1}}{h} \quad \text{and} \quad D^+ \Phi_i^{k+1} = \frac{\Phi_{i+1}^{k+1} - \Phi_i^{k+1}}{h}.$$

For the discrete one-dimensional function Φ_i its maximum norm takes the form

$$\|\Phi_i\| = \max_{x_i \in \Omega^n} |\Phi_i|$$

and the corresponding time-dependent case is also given by

$$\|\Phi_i^k\| = \max_{(x_i, t_k) \in Q^{n,m}} |\Phi_i^k|.$$

Definition 1.2.1 (Roos et al. (2008)). A matrix $\mathcal{A} = (a_{ij}) \in \mathbf{R}^{n \times n}$ is known as an M -matrix if $\mathcal{A}$ is positive definite, that is $\mathcal{A}^{-1} \geq 0$, $a_{ij} \leq 0$, for all $i \neq j$, $1 \leq i, j \leq n$.

Definition 1.2.2 (Miller et al. (2012)). Consider a family of mathematical problems parametrised by a singular perturbation parameter ε , where ε lies in the semi-open interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by u_ε is approximated by a sequence of numerical solutions $\{U_\varepsilon, \bar{\Omega}^n\}_{n=1}^\infty$, where U_ε is defined on the mesh $\bar{\Omega}$ and n is the discretisation parameter. Then the numerical solutions U_ε are said to converge ε -uniformly to the exact solution u_ε , if there exist a positive integer n_0 and positive numbers C and p , where n_0, p and C are all independent of n and ε , such that, for all $n \geq n_0$,

$$\sup_{0 < \varepsilon \leq 1} \|U_\varepsilon - u_\varepsilon\|_{\bar{\Omega}_T^n} \leq C n^{-p}.$$

Here p is called the ε -uniform rate of convergence and C is called the ε -uniform error constant.

1.2.1 Fitted operator finite difference methods

The focus of this thesis is the design, analysis and the implementation of fitted operator finite difference schemes to solve various forms of singularly perturbed problems.

As indicated earlier fitted operator finite difference schemes introduces a new coefficient function into the classical finite difference schemes. These methods can be grouped into the exponentially fitted and the non-standard finite difference schemes. We do not discuss exponentially fitted finite difference but only the non-standard finite difference schemes in this thesis, specifically, the scheme which uses the modelling rules of Mickens (1994). According to Mickens there is no straight forward path to construct a non-standard finite difference scheme but when constructed, they should be void of numerical instabilities and must capture the qualitative properties of the problem under study. In general singularly perturbed problems can be classified into convection diffusion problems and reaction diffusion problems and below we shall see how a fitted operator finite difference scheme is designed for a typical reaction and convection diffusion problems, respectively.

Stationary reaction-diffusion problem

Consider the reaction-diffusion problem

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in \Omega \quad (1.11)$$

$$u(0) = u(1) = \mu. \quad (1.12)$$

To construct a non-standard finite difference scheme for (1.11) we let n be a positive integer and $U(x_i) \equiv U_i$ be the approximation of $u(x_i) \equiv u_i$. Now we assume (1.11) is homogeneous with constant coefficients and use finite difference schemes to transform it into the discrete equation

$$-\varepsilon \frac{U_i - 2U_i + U_{i-1}}{\phi} + b_i U_i = 0. \quad (1.13)$$

We calculate the denominator function ϕ in (1.13) as follows: the exact solution of Equation (1.11) is the linear combination of the terms $\exp\left(-\sqrt{b/\varepsilon}h\right)$ and $\exp\left(\sqrt{b/\varepsilon}h\right)$. Now we follow Mickens (1994) to construct a second order difference equation as follows

$$\begin{vmatrix} U_i & U_{1;i} & U_{2;i} \\ U_{i+1} & U_{1;i+1} & U_{2;i+1} \\ U_{i+2} & U_{1;i+2} & U_{2;i+2} \end{vmatrix} = \begin{vmatrix} U_i & \exp\left(-\sqrt{\frac{b}{\varepsilon}}h_i\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}h_i\right) \\ U_{i+1} & \exp\left(-\sqrt{\frac{b}{\varepsilon}}h_{i+1}\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}h_{i+1}\right) \\ U_{i+2} & \exp\left(-\sqrt{\frac{b}{\varepsilon}}h_{i+2}\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}h_{i+2}\right) \end{vmatrix} = 0. \quad (1.14)$$

Simplifying the determinant (1.14) and lowering the index i by one, gives the difference scheme

$$U_{i+1} - 2 \cosh\left(\sqrt{\frac{b}{\varepsilon}}h_i\right) U_i + U_{i-1} = 0, \quad (1.15)$$

but $\sinh(h/2) = \pm\sqrt{(\cosh(h) - 1)/2}$, thus from (1.13) and (1.15) we obtain the non-standard difference scheme for Equation (1.11) as

$$-\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{(4\varepsilon/b) \sinh^2(\sqrt{b/\varepsilon} h_i/2)} + bU_i = 0,$$

with $\phi = (4\varepsilon/b) \sinh^2(\sqrt{b/\varepsilon} h_i/2)$ as the denominator function.

Stationary convection-diffusion problem

For convection diffusion problems with boundary layers the layer can either be at the right or left depending on the coefficient of the convective term $a(x)$. Here we construct the denominator function for both cases by following the ideas in Mickens (1994).

Case 1: Problem with right ($x = 1$) boundary layer

Consider the convection-diffusion problem

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega \quad (1.16)$$

$$u(0) = u(1) = \mu. \quad (1.17)$$

Assuming that Equation (1.16) is homogeneous, we rewrite it into the sub-equations

$$-\varepsilon \frac{d^2u}{dx^2} + a(x) \frac{du}{dx} = 0, \quad -\varepsilon \frac{d^2u}{dx^2} + b(x)u = 0, \quad a(x) \frac{du}{dx} + b(x)u = 0. \quad (1.18)$$

Using the same notations in the Subsection 1.2.1, the finite difference discretisation of the sub-equations in (1.18) takes the form

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} = a_i \frac{U_i - U_{i-1}}{h}, \quad \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi^2} = b_i U_i, \quad a_i \frac{U_i - U_{i-1}}{h} = b_i U_i. \quad (1.19)$$

We calculate ϕ from the scheme

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} = a_i \frac{U_i - U_{i-1}}{h},$$

as follows: we write the sub-equation

$$-\varepsilon \frac{d^2u}{dx^2} + a(x) \frac{du}{dx} = 0,$$

as a system of two first order coupled differential equations as

$$\frac{du}{dx} = y, \quad \frac{dy}{dx} = \frac{a(x)}{\varepsilon} y.$$

Now to obtain the discrete difference scheme for y we use the first order forward difference scheme

$$y_i = \frac{U_{i+1} - U_i}{h}.$$

The discrete form of dy/dx is given by

$$\frac{a_i}{\varepsilon} y_i = \frac{y_{i+1} - y_i}{h}.$$

This implies that

$$y_{i+1} = \frac{a_i h}{\varepsilon} y_i + y_i,$$

and thus

$$y_{i+1} = y_i \left(\frac{a_i h}{\varepsilon} + 1 \right). \quad (1.20)$$

Clearly, $(a_i h \varepsilon^{-1} + 1)$ is a truncated Taylor series expansion of the exponential function

$$\exp \left(\frac{a_i h}{\varepsilon} \right). \quad (1.21)$$

Thus from the coefficient of y_i in Equation (1.20) and (1.21), h takes the form

$$h = \varepsilon a_i^{-1} \exp(a_i h / \varepsilon - 1). \quad (1.22)$$

Combining the two systems of equations and replacing the y gives

$$\phi = \frac{\varepsilon h}{a_i} \left(\exp \left(\frac{a_i h}{\varepsilon} \right) - 1 \right), \quad (1.23)$$

as the value of the denominator function ϕ . Putting the sub-schemes together, we obtain the difference scheme

$$-\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} + a_i \left(\frac{U_i - U_{i-1}}{h} \right) + b_i U_i = f_i,$$

where ϕ is given by Equation (1.23).

Case 2: Problem with left ($x = 0$) boundary layer

Consider the convection-diffusion problem

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega \quad (1.24)$$

$$u(0) = u(1) = \mu \quad (1.25)$$

Notice that the sub-equations for Equation (1.24) will be the same as that of Equation (1.16) with a positive ε . Following the procedures in Subsection 1.2.1, we obtain

$$-\frac{a_i}{\varepsilon} y_i = \frac{y_{i+1} - y_i}{h},$$

which implies that

$$y_{i+1} = -\frac{a_i h}{\varepsilon} y_i + y_i,$$

and hence

$$y_{i+1} = y_i \left(1 - \frac{a_i h}{\varepsilon} \right). \quad (1.26)$$

Evidently, $(1 - a_i h \varepsilon^{-1})$ is a truncated Taylor series expansion of the exponential function

$$\exp\left(-\frac{a_i h}{\varepsilon}\right). \quad (1.27)$$

Thus from (1.27) and the coefficient of y_i in Equation (1.26), h takes the form

$$h = \frac{\varepsilon}{a_i} \left(1 - \exp\left(-\frac{a_i h}{\varepsilon}\right)\right). \quad (1.28)$$

Combining the two systems of equations and replacing the y gives

$$\frac{\varepsilon h}{a_i} \left(1 - \exp\left(-\frac{a_i h}{\varepsilon}\right)\right),$$

as the value of the denominator function, ϕ . Putting the sub-schemes together, we obtain the difference scheme

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} + a_i \left(\frac{U_i - U_{i-1}}{h}\right) + b_i U_i = f_i,$$

with ϕ given by Equation (1.28)

1.2.2 Richardson extrapolation

Richardson extrapolation is a numerical method that is used to accelerate the convergence of a numerical scheme. Basically, it combines two numerical solutions to obtain a higher approximation. Thus if h is the step size and z_n is the solution for some finite difference discretisation and $0.5h$ is another step size with w_n as the solution for the same finite difference approximation, Richardson extrapolation suggests that if y_n is an approximation then it can be calculated by the formula

$$y_n = \frac{2^s w_n - z_n}{2^s - 1}.$$

Here s is the accuracy of the finite difference scheme and n is the iteration index.

1.3 Outline of the Thesis

In this thesis various forms of singularly perturbed problems are investigated. Schemes are designed for parabolic problems with Robin boundary conditions, parabolic problems with time delays, Volterra-integro differential problems, parabolic problems with degenerate coefficient and non-linear problems. Apart from the designing of the schemes, a rigorous convergence analysis is presented. Richardson extrapolation and its error estimates are presented to improve the accuracies of the proposed scheme. Extensive numerical experiments are conducted to validate the theoretical results before and after extrapolation.

The chapter-wise summary of this thesis is as follows: Chapter 2 deals with the proposal of a second order ε -uniform numerical scheme to solve a parabolic convection diffusion

problem with Robin boundary conditions. The chapter proceeds by first giving a priori estimate of the solution of the problem under study. This is followed by the construction of the proposed scheme. It is worth mentioning that the proposed scheme is a full discretisation method and it is a combination of the backward Euler and the classical finite difference operator on appropriate denominator functions. The stability of the scheme is established and a first order accuracy in both variables are estimated. Richardson extrapolation technique is then applied to accelerate the convergence of the scheme to a second order.

In Chapter 3, a delay parabolic convection diffusion problem is considered. Again an estimate of the analytical solution is presented and then a fitted operator finite difference scheme is designed along with the Crank Nicolson finite difference scheme to solve the delay problem in the frame work of method of lines. A numerical method in which the spatial discretisation is done followed by the time discretisation. Thus the delayed parabolic problem is transformed into systems of initial value problems and then integrated in time. First and second order accuracies are obtained in space and in time, respectively, from the convergence analysis. The first order spatial accuracy is then improved to a second order by the application of Richardson extrapolation.

Chapter 4 focuses on the numerical solution of a singularly perturbed Volterra-integro differential equation. A scheme which combines a fitted operator finite difference scheme and the Simpson's rule is proposed to solve the first order Volterra-integro differential equation. The stability of the scheme is investigated and a first order accuracy is established from the convergence analysis. This first order accuracy is then improved to a second order by Richardson extrapolation.

In Chapter 5, a parabolic problem with degenerate coefficient is investigated by a midpoint finite difference scheme on appropriate denominator function and the Crank Nicolson finite difference scheme via the Rothe's method. That is a numerical method where the time discretisation is done followed by the spatial discretisation. The stability and the error estimate associated with each discretisation are studied.

The procedures in Chapter 5 are employed to study a modified Burgers equation after quasilinearisation in Chapter 6. However, the implicit Euler finite difference scheme is used for the time discretisation instead of the Crank Nicolson finite difference scheme.

In Chapter 7, a contributed paper with the title, Global dynamics for a model of amyloid fibril formation in pancreatic islet beta cells subjected to a therapy is presented.

In all chapters, extensive numerical experiments will be conducted to support the theoretical findings. Suitable graphs and tables are provided in each chapter to attest to the theoretical estimates.

In the last Chapter, 8, we give brief summary of the thesis whereby we highlight our contribution and future directions of this research.

Chapter 2

A Uniformly Convergent Finite Difference Method for Singularly Perturbed Parabolic Convection Diffusion Problem with Robin Boundary Condition

2.1 Introduction

In the previous chapter, we discussed singularly perturbed problems and the various ways of solving such problems. We noted that because of the multi-scale character associated with the solutions of singularly perturbed problems, classical numerical methods are unfit to solve these problems adequately. To resolve the issue of inadequate results, two approaches which include fitted mesh and fitted operator methods are known. In this chapter, we propose a second order numerical scheme based on the latter to solve a singularly perturbed parabolic convection diffusion problem with Robin boundary conditions. The proposed scheme uses a fitted operator finite difference scheme to discretise the spatial derivatives whilst the backwards Euler finite difference scheme is used to discretise the temporal derivative. It has been proved by Hemker et al. (2002) that classical finite difference methods on Shishkin mesh yield an almost first order accuracy in space and first order accuracy in time, where-as the proposed scheme involving the fitted operator finite difference scheme results in a full first order accuracy independent of the perturbation parameter ε .

Consider the singularly perturbed parabolic convection diffusion problem

$$\mathcal{L}_\varepsilon u(x, t) \equiv -u_t + \varepsilon u_{xx} + b(x, t)u_x - c(x, t)u = f(x, t), \quad (x, t) \in Q, \quad (2.1)$$

along with the initial and the boundary conditions

$$\begin{aligned} u(x, 0) &= \varphi(x), & x \in \Omega, \\ -\varepsilon\alpha(0, t)u_x + \beta(0, t)u &= \psi_0(t), & t \in (0, T], \\ \varepsilon\alpha(1, t)u_x + \beta(1, t)u &= \psi_1(t), & t \in (0, T], \end{aligned} \quad (2.2)$$

respectively. The coefficient functions and the source terms f , $\psi_0(t)$, $\psi_1(t)$ are smooth and bounded. Moreover, we assume that the coefficient functions satisfy

$$0 < b_0 \leq b(x, t), \quad 0 < c_0 \leq c(x, t), \quad (x, t) \in \bar{Q},$$

$$\alpha(x, t), \beta(x, t) \geq 0, \quad \alpha(x, t) + \beta(x, t) \geq \alpha_0 > 0, \quad \text{for } x = 0, \text{ or } x = 1, \quad 0 < t \leq T.$$

Also, the initial condition $\varphi(x)$, satisfy, $\varphi^{(k)}(x) = 0$, $\forall k \geq 1$. The perturbation parameter ε lies in the interval $\varepsilon \in (0, 1]$. Taking into account the smoothness and the boundedness of the coefficient functions and the source term f , problem (2.1)–(2.2) admits a unique solution. Notice that this solution possesses a boundary layer in the neighborhood of $x = 0$ and $0 < t \leq T$ and that the presence of ε in the boundary conditions increases the significance of the boundary layer Ansari and Hegarty (2003).

The chapter unfolds as follows: Section 2.2 focuses on some properties on the continuity of the problem. These properties include a minimum principle, stability estimate as well as bounds of the solution and its partial derivatives. In Section 2.3, a fitted operator finite difference scheme along with the backward Euler finite difference method is designed in order to solve the Robin type parabolic convection diffusion problem. Some results which describe the stability of the numerical methods are stated in Section 2.4. In Section 2.5, these results are used to analyse the convergence of the scheme. Richardson extrapolation as well as its analysis is performed in Section 2.6 to accelerate the convergence of the scheme. In Section 2.7, the method is tested on an example and in the last section, a summary of the major findings is given.

2.2 Bounds on the Solution and its Derivatives

Below the existence and the uniqueness of the solution to the problem (2.1)–(2.2) are presented. The discrete versions of these properties will be provided later, in the next section, to aid in the analysis of the numerical method.

Lemma 2.2.1. *Continuous minimum principle*

Let $\xi(x, t)$ be a smooth function which satisfies $\xi(x, 0) \geq 0$, on $x \in \bar{\Omega}$, $-\varepsilon\alpha(0, t)\xi_x(0, t) + \beta(0, t)\xi(0, t) \geq 0$, $\varepsilon\alpha(1, t)\xi_x(1, t) + \beta(1, t)\xi(1, t) \geq 0$, on $t \in [0, T]$ and $\mathcal{L}_\varepsilon \xi(x, t) < 0$, $\forall (x, t) \in Q$. Then $\xi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$.

Proof. Let (x^*, t^*) be a point in Q such that, $\xi(x^*, t^*) < 0$ and

$$\xi(x^*, t^*) = \min_{(x, t) \in \bar{Q}} \xi(x, t).$$

It is clear that $(x^*, t^*) \notin \partial Q$. If $(x^*, t^*) \in Q$, then we have

$$\mathcal{L}_\varepsilon \xi(x^*, t^*) = -\xi_t(x^*, t^*) + \varepsilon \xi_{xx}(x^*, t^*) + b(x^*, t^*) \xi_x(x^*, t^*) - c(x^*, t^*) \xi(x^*, t^*).$$

Since $\xi_{xx}(x^*, t^*) \geq 0$ and $\xi_x(x^*, t^*) = \xi_t(x^*, t^*) = 0$, then $\mathcal{L}_\varepsilon \xi(x^*, t^*) \geq 0$, which is a contradiction.

Also $(x^*, t^*) \in Q_l = \{(0, t) : t \in (0, T]\}$, yields,

$$-\varepsilon \alpha(x^*, t^*) \xi_x(x^*, t^*) + \beta(x^*, t^*) \xi(x^*, t^*) < 0,$$

and at the boundary $Q_r = \{(1, t) : t \in (0, T]\}$, we have

$$\varepsilon \alpha(x^*, t^*) \xi_x(x^*, t^*) + \beta(x^*, t^*) \xi(x^*, t^*) < 0,$$

which contradict the assumptions. It follows that $\xi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$. $\square$

The uniqueness of the solution to the problem follows immediately from the continuous minimum principle and it is stated in the next lemma.

Lemma 2.2.2. (*Stability estimate*)

The solution $u(x, t)$ of the problem (2.1)–(2.2) satisfies

$$|u(x, t)| \leq [c_0^{-1} \|f\| + \max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)].$$

Proof. We define two smooth functions $\Psi^\pm(x, t)$, by

$$\Psi^\pm(x, t) = [c_0^{-1} \|f\| + \max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)] \pm u(x, t).$$

Now at the initial stage, we have

$$\begin{aligned} \Psi^\pm(x, 0) &= [c_0^{-1} \|f\| + \max(|\varphi(x)|, |\psi_0(0)|, |\psi_1(0)|)] \pm u(x, 0) \\ &= [c_0^{-1} \|f\| + \max(|\varphi(x)|, |\psi_0(0)|, |\psi_1(0)|)] \pm \varphi(x) \geq 0. \end{aligned}$$

At $x = 0$, we obtain

$$\Psi^\pm(0, t) = [c_0^{-1} \|f\| + \max(|\varphi(0)|, |\psi_0(t)|, |\psi_1(t)|)] \pm u(0, t). \quad (2.3)$$

From (2.3), $u(x, 0)$ and $u_x(x, 0)$ are given by

$$u(0, t) = \pm \Psi^\pm(0, t) \mp [c_0^{-1} \|f\| + \max(|\varphi(0)|, |\psi_0(t)|, |\psi_1(t)|)],$$

and

$$u_x(0, t) = \pm \Psi_x^\pm(0, t) \mp |\varphi'(0)|,$$

respectively. Substituting these results into the boundary condition at $x = 0$ results in

$$-\varepsilon \alpha(0, t) [\Psi_x^\pm(0, t) \mp |\varphi'(0)|] + \beta(0, t) (\Psi^\pm(0, t) \mp [c_0^{-1} \|f\| + \max\{|\varphi(0)|, |\psi_0(t)|, |\psi_1(t)|\}]) = \psi_0(t).$$

Simplifying further yields

$$\begin{aligned} -\varepsilon\alpha(0, t)\Psi_x^\pm(0, t) + \beta(0, t)\Psi^\pm(0, t) &= \beta(0, t)[c_0^{-1}\|f\| + \max\{|\varphi(0)|, |\psi_0(t)|, |\psi_1(t)|\}] \\ &\quad \pm \psi_0(t) - \varepsilon\alpha(0, t)|\varphi'(0)| \geq 0, \end{aligned}$$

provided that $\varphi'(0) = 0$ and $\beta(0, t)c_0^{-1}\|f\| + (\beta(0, t) \pm 1)\psi_0(t) \geq 0$.

At $x = 1$, we have

$$\Psi^\pm(1, t) = [c_0^{-1}\|f\| + \max(|\varphi(1)|, |\psi_0(t)|, |\psi_1(t)|) \pm u(1, t)]. \quad (2.4)$$

From (2.4), $u(x, 1)$ and $u_x(x, 1)$ are given by

$$u(1, t) = \pm\Psi^\pm(1, t) \mp [c_0^{-1}\|f\| + \max(|\varphi(1)|, |\psi_0(t)|, |\psi_1(t)|)],$$

and

$$u_x(1, t) = \pm\Psi_x^\pm(1, t) \mp |\varphi'(1)|,$$

respectively. Substituting these results in the boundary condition at $x = 1$ results in

$$\varepsilon\alpha(1, t)[\Psi_x^\pm \mp |\varphi'(1)|] + \beta(1, t)(\Psi^\pm(1, t) \mp [c_0^{-1}\|f\| + \max\{|\varphi(1)|, |\psi_0(t)|, |\psi_1(t)|\}]) = \psi_1(t).$$

Further simplification yields

$$\begin{aligned} \varepsilon\alpha(1, t)\Psi_x^\pm(1, t) + \beta(1, t)\Psi^\pm(1, t) &= \pm\beta(1, t)[c_0^{-1}\|f\| + \max\{|\varphi(1)|, |\psi_0(t)|, |\psi_1(t)|\}] \\ &\quad \pm \psi_1(t) + \varepsilon\alpha(1, t)|\varphi'(1)| \geq 0, \end{aligned}$$

provided that $\varphi(1) = 0$ and $\pm\psi_1(t) \pm \beta(1, t)[c_0^{-1}\|f\| + \max\{|\psi_0(t)|, |\psi_1(t)|\}] \geq 0$.

Further on the the domain Q , we have

$$\begin{aligned} \mathcal{L}_\varepsilon\Psi^\pm(x, t) &= -\Psi_t^\pm(x, t) + \varepsilon\Psi_{xx}^\pm(x, t) + b(x, t)\Psi_x^\pm(x, t) - c(x, t)\Psi^\pm(x, t) \\ &= -\max(|\psi_0'(t)|, |\psi_1'(t)|) + \varepsilon|\varphi''(x)| + b(x, t)|\varphi'(x)| - c(x, t) \\ &\quad \times (c_0^{-1}\|f\| + \max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)) \pm \mathcal{L}_\varepsilon u(x, t) \\ &= -\max(|\psi_0'(t)|, |\psi_1'(t)|) + \varepsilon|\varphi''(x)| + b(x, t)|\varphi'(x)| - c(x, t) \\ &\quad \times (c_0^{-1}\|f\| + \max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)) \pm f(x, t) \\ &= \pm f(x, t) - \frac{c(x, t)}{c_0}\|f\| - c(x, t)[\max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)] \\ &\quad - \max(|\psi_0'(t)|, |\psi_1'(t)|) + \varepsilon|\varphi''(x)| + b(x, t)|\varphi'(x)| \\ &= \pm f(x, t) - \frac{c(x, t)}{c_0}\|f\| - c(x, t)[\max(|\varphi(x)|, |\psi_0(t)|, |\psi_1(t)|)] \\ &\quad - \max(|\psi_0'(t)|, |\psi_1'(t)|) \leq 0, \end{aligned}$$

since $\varphi^{(k)}(x) = 0$, $\forall k \geq 1$. From Lemma 2.2.1, it follows that $\Psi^\pm(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$, and this yields the result. $\square$

The analysis of the numerical method we shall see in later sections requires some information about the behaviour of the solution $u(x, t)$ and its partial derivatives. These results, established from Kellogg and Tsan (1978) and Ng-Stynes et al. (1988) are provided in the subsequent lemmas.

Lemma 2.2.3. *Let $u(x, t)$ be the solution of problem (2.1)–(2.2), then it satisfies $|u(x, t)| \leq C$, $(x, t) \in \bar{Q}$.*

Proof. This results follows from the Lemma 2.2.2. □

We make the assertion that both the initial and the boundary conditions are zero.

Lemma 2.2.4. *The solution $u(x, t)$ of problem (2.1)–(2.2) is such that $|u_t(x, t)| \leq C$, $(x, t) \in \bar{Q}$.*

Proof. At $x = 0$ and $x = 1$ of $\bar{Q}$ we have $u = 0$ and thus $u_t = 0$. At $t = 0$, we have $u = 0$, $u_x = 0$, and $u_{xx} = 0$, based on the earlier assumption made on the initial and boundary condition. Taking the limit as t approaches zero in equation (2.1), gives

$$\lim_{t \rightarrow 0^+} (-u_t(x, t) + \varepsilon u_{xx}(x, t) + b(x, t)u_x(x, t) - c(x, t)u(x, t)) = \lim_{t \rightarrow 0^+} f(x, t).$$

Notice that $u = 0$, $u_x = 0$, and $u_{xx} = 0$, so we obtain

$$u_t(x, 0) = -f(x, 0), \quad 0 \leq x \leq 1. \quad (2.5)$$

Now taking the norm on both sides of (2.5) results in

$$|u_t(x, t)| \leq C_1, \quad \text{on } \partial Q,$$

when C_1 is chosen to be large. Applying the differential operator in equation (2.1), on $u_t(x, t)$ gives

$$\begin{aligned} \mathcal{L}_\varepsilon u_t(x, t) &= -u_{tt} + \varepsilon u_{txx} + b(x, t)u_{tx} - c(x, t)u_t \\ &= (-u_t + \varepsilon u_{xx} + bu_x - cu)_t + c_t u, \quad \text{where } (b(x, t) \equiv b(x)), \\ &= f_t + c_t u. \end{aligned}$$

From Lemma 2.2.3 and the fact that f and c are bounded, $|\mathcal{L}_\varepsilon u_t(x, t)| \leq C_2$, follows.

Now let $\Phi(x, t) = C_1 + C_3$, where $C_3 > 0$, for all $(x, t) \in \bar{Q}$. At $(x, t) \in \partial Q$, we have

$$\Phi(x, t) = C_1 + C_3 \geq C_1 \geq |u_t(x, t)|,$$

which implies that $|u_t(x, t)| \leq \Phi(x, t)$, $\forall (x, t) \in \partial Q$. Applying the differential operator $\mathcal{L}_\varepsilon$ on $\Phi(x, t)$ on yields

$$\mathcal{L}_\varepsilon \Phi(x, t) = -(C_1 + C_3)c(x, t) \leq -C_1 c(x, t) \leq -c(x, t)|u_t(x, t)| \leq -c_0|u_t(x, t)|. \quad (2.6)$$

The operator $\mathcal{L}_\varepsilon$ is linear, thus $|\mathcal{L}_\varepsilon(u_t)| \leq K|u_t|$, $0 < K \in \mathbb{R}$, $\forall u_t \in \mathbb{R}$. Choosing $K = c_0$, we have

$$-c_0|u_t(x, t)| \leq -|\mathcal{L}_\varepsilon u_t(x, t)|,$$

and hence

$$\mathcal{L}_\varepsilon \Phi(x, t) \leq -|\mathcal{L}_\varepsilon u_t(x, t)|$$

holds. By adding $-\mathcal{L}_\varepsilon|u_t(x, t)|$ on both sides of the latest inequality, we get

$$\mathcal{L}_\varepsilon(\Phi(x, t) - |u_t(x, t)|) \leq -(\mathcal{L}_\varepsilon|u_t(x, t)| + |\mathcal{L}_\varepsilon u_t(x, t)|) < 0, \quad \forall (x, t) \in \bar{Q}.$$

From the minimum principle, $\Phi(x, t) - |u_t(x, t)| \geq 0$. That is, $|u_t(x, t)| \leq \Phi(x, t)$ and hence $|u_t(x, t)| \leq C_1 + C_3 = C$. Thus, $|u_t(x, t)| \leq C$, on $\bar{Q}$. $\square$

Lemma 2.2.5. *The solution $u(x, t)$ of problem (2.1)–(2.2) is such that $|u_x(x, t)| \leq C(1 + \varepsilon^{-1} \exp(-b_0 x/\varepsilon))$, $(x, t) \in \bar{Q}$.*

Proof. Equation (2.1) can be written as

$$h_1(x, t) = \varepsilon u_{xx} + b(x, t)u_x, \quad (2.7)$$

where $h_1(x, t) = f(x, t) + u_t + c(x, t)u(x, t)$. Fixing $t \in [0, T]$ and multiply both sides of equation (2.7) by the integration factor $\exp(\varepsilon^{-1}B(x, t))$, $B(x, t) = \int_0^x b(s, t)ds$, yields

$$\begin{aligned} \varepsilon^{-1} \exp(\varepsilon^{-1}B(x, t))h_1(x, t) &= \exp(\varepsilon^{-1}B(x, t))u_{xx} + \varepsilon^{-1}b(x, t) \exp(\varepsilon^{-1}B(x, t))u_x, \\ \frac{d}{dx} \left(\exp(\varepsilon^{-1}B(x, t)) \frac{du}{dx} \right) &= \varepsilon^{-1} \exp(\varepsilon^{-1}B(x, t))h_1(x, t). \end{aligned} \quad (2.8)$$

Integrating (2.8) from 0 to x gives

$$\exp(\varepsilon^{-1}B(0, t))u_x(0, t) - \exp(\varepsilon^{-1}B(x, t))u_x(x, t) = \varepsilon^{-1} \int_0^x h_1(s, t) \exp(\varepsilon^{-1}B(s, t))ds.$$

This simplifies to

$$u_x(x, t) = u_x(0, t) \exp(\varepsilon^{-1}(B(0, t) - B(x, t))) + z_1(x, t), \quad (2.9)$$

where $z_1(x, t)$ is given by

$$z_1(x, t) = \varepsilon^{-1} \int_0^x h_1(s, t) \exp(\varepsilon^{-1}(B(s, t) - B(x, t)))ds.$$

Now we integrate equation (2.9) from x to 1, to derive the bound of $u_x(0, t)$

$$u(x, t) = u(1, t) + u_x(0, t) \int_x^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t)))ds + \int_x^1 z_1(s, t)ds. \quad (2.10)$$

Evaluating (2.10) at $x = 0$ gives

$$u_x(0, t) = -\frac{\int_0^1 z_1(s, t)ds}{\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t)))ds}, \quad (2.11)$$

since $u(0, t) = u(1, t) = 0$. Substituting (2.11) into equation (2.9) yields

$$u_x(x, t) = -\frac{\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))}{\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t)))ds} \int_0^1 z_1(s, t)ds + z_1(x, t). \quad (2.12)$$

Now taking the norm on both sides of (2.12) gives

$$\begin{aligned}
|u_x(x, t)| &\leq \frac{|\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))|}{|\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t)))ds|} \left| \int_0^1 z_1(s, t)ds \right| + |z_1(x, t)| \\
&\leq \frac{|\exp(\varepsilon^{-1}B(0, t))| |\exp(-\varepsilon^{-1}B(x, t))|}{|\exp(\varepsilon^{-1}(B(0, t)))| \int_0^1 \exp(-\varepsilon^{-1}B(s, t))ds|} \left| \int_0^1 z_1(s, t)ds \right| + |z_1(x, t)| \\
&= \frac{|\exp(-\varepsilon^{-1}B(x, t))|}{|\int_0^x \exp(-\varepsilon^{-1}B(s, t))ds|} \left| \int_0^1 z_1(s, t)ds \right| + |z_1(x, t)| \\
&\leq \check{C} \left(1 + \frac{\exp(-\varepsilon^{-1}b_0x)}{\int_0^1 \exp(-\varepsilon^{-1}\tilde{b}_0)ds} \right), \tag{2.13}
\end{aligned}$$

where $\tilde{b}_0$ and $\check{C}$ are the upper bounds of $B(x, t)$ and $|z_1(x, t)|$, respectively, over $\bar{Q}$. Further simplification of (2.13) results in

$$|u_x(x, t)| \leq \check{C} \left(1 + \frac{\varepsilon^{-1}\tilde{b}_0 \exp(-b_0x\varepsilon^{-1})}{(1 - \exp(-\tilde{b}_0\varepsilon^{-1}))} \right). \tag{2.14}$$

Now let $f(\tilde{b}_0) = \exp(-\tilde{b}_0\varepsilon^{-1})$ be a real valued function defined over $\bar{\Omega}$ and differentiable on Ω . Then from Maclaurin series we have

$$\tilde{b}_0 \leq (1 - \exp(-\tilde{b}_0\varepsilon^{-1})), \tag{2.15}$$

so that the estimate (2.14) reduces to

$$|u_x(x, t)| \leq C(1 + \varepsilon^{-1} \exp(-b_0x/\varepsilon)). \tag{2.16}$$

□

Lemma 2.2.6. *The solution $u(x, t)$ of problem (2.1)–(2.2) satisfies the estimate $|u_{tt}(x, t)| \leq C$, $(x, t) \in \bar{Q}$.*

Proof. The proof follows the same procedure as in the proof of Lemma 2.2.4. At the spatial boundary of Q , $u \equiv 0$ and so $u_{tt} = 0$. From equation (2.5) we have

$$u_{tx}(x, 0) = -f_x(x, 0), \quad u_{ttx}(x, 0) = -f_{xx}(x, 0), \quad \text{for } x \in \bar{\Omega}. \tag{2.17}$$

From (2.17) we have $|u_{tx}| \leq C$ and $|u_{ttx}| \leq C_1$. Differentiating equation (2.1), with respect to t gives

$$-u_{tt} + \varepsilon u_{ttx} + bu_{tx} - c_t u - cu_t = f_t, \quad b(x, t) \equiv b(x). \tag{2.18}$$

Taking the norm on both sides of equation (2.18) results in

$$|u_{tt}| \leq |f_t| + \varepsilon |u_{ttx}| + |b| |u_{tx}| + |c_t| |u| + c |u_t|,$$

which implies that

$$|u_{tt}| \leq C_2, \quad \text{for } (x, t) \in \partial Q.$$

With the differential operator acting on u_{tt} , we obtain

$$\begin{aligned}\mathcal{L}u_{tt} &= -u_{ttt} + \varepsilon u_{ttxx} + bu_{ttx} - cu_{tt} \\ &= (-u_t + \varepsilon u_{xx} + bu_x - cu)_{tt} + c_{tt}u + 2c_t u_t \quad \text{where } (b(x, t) \equiv b(x)), \\ &= f_{tt} + c_{tt}u + 2c_t u_t.\end{aligned}$$

From the bounds of u and u_t we obtain $|\mathcal{L}u_{tt}| \leq C$. Let $\Phi(x, t) = C_2 + C_3$, where $C_3 > 0$. Now for all $(x, t) \in \bar{Q}$,

$$\begin{aligned}\mathcal{L}_\varepsilon \Phi(x, t) &= -(C_2 + C_3)c(x, t) \leq -C_2 c(x, t) \\ &\leq |u_{tt}(x, t)|[-c(x, t)] \\ &\leq -c_0 |u_{tt}(x, t)| \\ &\leq -|\mathcal{L}_\varepsilon u_{tt}(x, t)|.\end{aligned}$$

Since $|\mathcal{L}_\varepsilon u_{tt}| \leq L|u_{tt}|$, $L > 0$, $\forall u \in \mathbb{R}$ with $L = c_0$. By adding $-\mathcal{L}_\varepsilon |u_{tt}(x, t)|$ on both sides of the latest equation, we obtain

$$\mathcal{L}_\varepsilon (\Phi(x, t) - |u_{tt}(x, t)|) \leq -(\mathcal{L}_\varepsilon |u_{tt}(x, t)| + |\mathcal{L}_\varepsilon u_{tt}(x, t)|) < 0, \quad \forall (x, t) \in \bar{Q}.$$

From the minimum principle, $\Phi(x, t) - |u_{tt}(x, t)| \geq 0$. That is, $|u_{tt}(x, t)| \leq \Phi(x, t)$ and hence $|u_{tt}(x, t)| \leq C_2 + C_3 = C$. Thus, $|u_{tt}(x, t)| \leq C$, $\forall (x, t) \in \bar{Q}$. $\square$

Lemma 2.2.7. *The solution $u(x, t)$ of problem (2.1)–(2.2) is such that $|u_{xt}| \leq C(1 + \varepsilon^{-1} \exp(-b_0 x/\varepsilon))$, $(x, t) \in \bar{Q}$.*

Proof. We differentiate equation (2.1) with respect to t to obtain

$$-u_{tt} + \varepsilon(u_{xt})_x + bu_{xt} - c_t u - cu_t = f_t \quad \text{on } \bar{Q}. \quad (2.19)$$

Now we rewrite equation (2.19) in the form

$$\varepsilon(u_{xt})_x + bu_{xt} = h_2(x, t), \quad (2.20)$$

where $h_2(x, t) = f_t + c_t u + cu_t + u_{tt}$. Fixing $t \in [0, T]$ and multiplying both sides of equation (2.20) by the integration factor $B(x, t) = \int_0^x b(\eta, t) d\eta$,

$$\begin{aligned}\varepsilon^{-1} \exp(-\varepsilon^{-1} B(x, t)) h_2(x, t) &= \exp(\varepsilon^{-1} B(x, t)) u_{xx} + \varepsilon^{-1} b(x, t) \exp(-\varepsilon^{-1} B(x, t)) u_x, \\ \frac{d}{dx} \left(\exp(\varepsilon^{-1} B(x, t)) \frac{du_t}{dx} \right) &= \varepsilon^{-1} \exp(\varepsilon^{-1} B(x, t)) h_2(x, t).\end{aligned}$$

Integrating from 0 to x gives

$$\exp(\varepsilon^{-1} B(0, t)) u_{tx}(0, t) - \exp(\varepsilon^{-1} B(x, t)) u_{xt}(x, t) = \varepsilon^{-1} \int_0^x h_2(s, t) \exp(\varepsilon^{-1} B(s, t)) ds,$$

which leads to

$$u_{tx}(x, t) = u_{xt}(0, t) \exp(\varepsilon^{-1} (B(0, t) - B(x, t))) + z_2(x, t), \quad (2.21)$$

where $z_2(x, t)$ is given by

$$z_2(x, t) = \varepsilon^{-1} \int_0^x h_2(s, t) \exp(\varepsilon^{-1}(B(s, t) - B(x, t))) ds.$$

Now we integrate equation (2.21) from x to 1, to derive the bound of $u_{tx}(0, t)$

$$u_t(x, t) = u_t(1, t) + u_{tx}(0, t) \int_x^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds + \int_x^1 z_2(s, t) ds.$$

Evaluating this equation at $x = 0$ gives

$$u_{xt}(0, t) = - \frac{\int_0^1 z_2(s, t) ds}{\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds}. \quad (2.22)$$

Substituting (2.22) into equation (2.21) yields

$$u_{xt}(x, t) = - \frac{\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))}{\int_0^1 \exp(B(0, t) - B(s, t)) ds} \int_0^1 z_2(s, t) ds + z_2(x, t). \quad (2.23)$$

Now taking the norm on both sides of (2.23) gives

$$\begin{aligned} |u_{xt}(x, t)| &\leq \frac{|\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))|}{|\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds|} \left| \int_0^1 z_1(s, t) ds \right| + |z_1(x, t)| \\ &\leq \frac{|\exp(\varepsilon^{-1}B(0, t))| |\exp(-\varepsilon^{-1}B(x, t))|}{|\exp(\varepsilon^{-1}(B(0, t)))| \int_0^1 \exp(-\varepsilon^{-1}B(s, t)) ds|} \left| \int_0^1 z_1(s, t) ds \right| + |z_1(x, t)| \\ &= \frac{|\exp(-\varepsilon^{-1}B(x, t))|}{|\int_0^x \exp(-\varepsilon^{-1}B(s, t)) ds|} \left| \int_0^1 z_1(s, t) ds \right| + |z_1(x, t)| \\ &\leq \check{C} \left(1 + \frac{\exp(-\varepsilon^{-1}b_0x)}{\int_0^1 \exp(-\varepsilon^{-1}\tilde{b}_0) ds} \right), \end{aligned} \quad (2.24)$$

where $\tilde{b}_0$ and $\check{C}$ are the upper bounds of $B(x, t)$ and $|z_1(x, t)|$, respectively, over $\bar{Q}$. Further simplification of (2.13) results in

$$|u_{xt}(x, t)| \leq \check{C} \left(1 + \frac{\varepsilon^{-1}\tilde{b}_0 \exp(-b_0x\varepsilon^{-1})}{(1 - \exp(-\tilde{b}_0\varepsilon^{-1}))} \right). \quad (2.25)$$

Applying the argument used in the proof of Lemma 2.2.5 yields the estimate

$$|u_{tx}(x, t)| \leq C(1 + \varepsilon^{-1} \exp(-b_0x/\varepsilon)). \quad (2.26)$$

□

Lemma 2.2.8. *The solution $u(x, t)$ of problem (2.1)–(2.2) satisfies the estimate $|u_{xx}| \leq C(1 + \varepsilon^{-2} \exp(-b_0x/\varepsilon))$, $(x, t) \in \bar{Q}$.*

Proof. Here we differentiate equation (2.1) with respect to the spatial variable x and rearrange to obtain

$$\varepsilon u_{xxx} + b(x, t)u_{xx} = h_3(x, t),$$

where $h_3(x, t) = f_x + c_x u + c u_x - b_x u_x + u_{tx}$, and satisfies the bound

$$|h_3(x, t)| \leq C\varepsilon^{-1},$$

from the Lemmas 2.2.3, 2.2.5 and 2.2.7. Employing the integration factor technique gives

$$u_{xx}(x, t) = u_{xx}(0, t) \exp(\varepsilon^{-1}(B(0, t) - B(x, t))) + z_3(x, t), \quad (2.27)$$

where $z_3(x, t)$ is given by

$$z_3(x, t) = \varepsilon^{-1} \int_x^1 h_3(\gamma, t) \exp(\varepsilon^{-1}B(\gamma, t) - B(x, t)) d\gamma.$$

Now we integrate (2.27) from x to 1, to obtain

$$u_x(1, t) - u_x(x, t) = u_{xx}(0, t) \int_x^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds + \int_x^1 z_3(s, t) ds.$$

Evaluating this at $x = 0$ yields

$$u_{xx}(1, t) = \frac{-\int_0^1 z_3(s, t) ds}{\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds}. \quad (2.28)$$

Substituting (2.28) into equation (2.27) gives

$$u_{xx}(x, t) = -\frac{\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))}{\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds} \int_0^1 z_3(s, t) ds + z_3(x, t). \quad (2.29)$$

Taking the norm on both sides of (2.29) gives

$$|u_{xx}(x, t)| \leq \frac{|\exp(\varepsilon^{-1}(B(0, t) - B(x, t)))|}{|\int_0^1 \exp(\varepsilon^{-1}(B(0, t) - B(s, t))) ds|} \int_0^1 |z_3(s, t)| ds + |z_3(x, t)|. \quad (2.30)$$

The coefficient of $\int_0^1 |z_3(s, t)| ds$ in (2.30) is equivalent to that of $\int_0^1 |z_2(s, t)| ds$ in (2.24) thus the inequality 2.30 reduces to

$$|u_{xx}(x, t)| \leq \varepsilon^{-1} \exp(-b_0 x / \varepsilon) \int_0^1 |z_3(x, t)| ds + |z_3(x, t)|, \quad (2.31)$$

by similar argument used in the Proof of Lemma 2.2.5. Moreover, notice that $|z_3(x, t)|$ depends on $|h_3(x, t)| \leq C_8 \varepsilon^{-1}$, thus we have that

$$|u_{xx}(x, t)| \leq C_8 \varepsilon^{-2} \exp(-b_0 x / \varepsilon) + C_8 \varepsilon^{-1},$$

which reduces to

$$|u_{xx}(x, t)| \leq C_8 (1 + \varepsilon^{-2} \exp(-b_0 x / \varepsilon)),$$

since $0 < \varepsilon < 1$. □

Theorem 2.2.1. *For all non-negative integers such that $0 \leq i + k \leq 4$, the solution of problem (2.1)-(2.2) and its partial derivatives satisfy*

$$\left| \frac{\partial^{i+k} u(x, t)}{\partial x^i \partial t^k} \right| \leq C(1 + \varepsilon^{-i} e^{-b_0 x / \varepsilon}), \quad (x, t) \in \bar{Q}. \quad (2.32)$$

Proof. The Lemmas 2.2.3–2.2.8 prove Theorem 2.2.1 for the integers $0 \leq i + k \leq 2$, and analogously the higher order derivatives up to the fourth order can be obtained. □

In the next section we solve the continuous problem numerically.

2.3 Numerical Scheme

In this section, a fitted operator finite difference scheme is designed along with backward Euler finite difference scheme to discretise the continuous problem in space and in time respectively.

We employ the notation $U(x_i, t_{k+1}) \equiv U_i^{k+1}$ to be the approximation of $u(x_i, t_{k+1}) \equiv u_i^{k+1}$. Notice that both discretisations are performed on a uniform mesh. From the theory of classical and non-standard finite difference schemes Mickens (1994) we discretise problem (2.1)–(2.2) in the space and the time variables concurrently to obtain the discrete problem

$$\begin{aligned} \mathcal{L}_\varepsilon^{n,m} U_i^{k+1} &\equiv - \left(\frac{U_i^{k+1} - U_i^k}{\Delta t} \right) + \varepsilon \left(\frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{\phi_i^2} \right) + b_i^{k+1} \left(\frac{U_{i+1}^{k+1} - U_i^{k+1}}{h} \right) \\ &\quad - c_i^{k+1} U_i^{k+1} = f_i^{k+1}, \text{ for } i = 1, 2, \dots, n-1, \text{ and } k = 0, 1, \dots, m, \end{aligned} \quad (2.33)$$

along with the discrete initial and boundary conditions

$$U_i^0 = \varphi_i, \quad (2.34)$$

$$-\varepsilon \alpha_0^{k+1} \left(\frac{U_1^{k+1} - U_0^{k+1}}{\phi_0^*} \right) + \beta_0^{k+1} U_0^{k+1} = \psi_0^{k+1}, \quad (2.35)$$

$$\varepsilon \alpha_n^{k+1} \left(\frac{U_n^{k+1} - U_{n-1}^{k+1}}{\phi_n^*} \right) + \beta_n^{k+1} U_n^{k+1} = \psi_n^{k+1}, \quad (2.36)$$

respectively. The denominator functions ϕ_0^* , ϕ_i^2 and ϕ_n^* are given by

$$\phi_0^* = \frac{\alpha_0^{k+1} \varepsilon}{\beta_0^{k+1}} \left(\exp \left(\frac{\beta_0^{k+1} h}{\alpha_0^{k+1} \varepsilon} \right) - 1 \right), \quad (2.37)$$

$$\phi_i^2 = \frac{h \varepsilon}{b_i^{k+1}} \left(1 - \exp \left(-\frac{b_i^{k+1} h}{\varepsilon} \right) \right), \quad i = 1, 2, 3, \dots, n-1, \quad (2.38)$$

$$\phi_n^* = \frac{\alpha_n^{k+1} \varepsilon}{\beta_n^{k+1}} \left(1 - \exp \left(\frac{-\beta_n^{k+1} h}{\alpha_n^{k+1} \varepsilon} \right) \right), \quad (2.39)$$

respectively. The expansion of (2.38) in Taylor series yields

$$\phi_i^2 = h^2 + \mathcal{O}(h^3).$$

The fitted operator finite difference scheme (2.33)–(2.36) can be represented in the matrix notation

$$AU = F,$$

where A is a square matrix of size $((0, 1, \dots, n) \times (0, 1, \dots, n))$ and F is a vector of size $(0, 1, \dots, n)$ with their respective entries

$$\begin{aligned} A_{0,0} &= \frac{\varepsilon \alpha_0^{k+1}}{\phi_0^*} + \beta_0^{k+1}, & A_{0,1} &= -\frac{\varepsilon \Delta t}{\phi_0}, \\ A_{n,n-1} &= -\frac{\varepsilon \alpha_n^{k+1}}{\phi_n^*}, & A_{n,n} &= \frac{\varepsilon \alpha_n^{k+1}}{\phi_n^*} + \beta_n^{k+1}, \end{aligned}$$

$$\begin{aligned}
A_{ii} &= -1 - \frac{2\varepsilon\Delta t}{\phi_i^2} - \frac{b_i^{k+1}\Delta t}{h} - c_i^{k+1}\Delta t, & i = 1, 2, \dots, n-1, \\
A_{i,i+1} &= \frac{\varepsilon\Delta t}{\phi_i^2} + \frac{b_i^{k+1}\Delta t}{h}, & i = 1, 2, \dots, n-1, \\
A_{i,i-1} &= \frac{\varepsilon\Delta t}{\phi_i^2}, & i = 1, 2, \dots, n-1, \\
F_0 &= \psi_0^{k+1} - U_0^k, \\
F_i &= \Delta t f_i^{k+1} - U_i^k, & i = 1, 2, 3, \dots, n-1, \\
F_n &= \psi_n^{k+1} - U_n^k,
\end{aligned}$$

with $k = 0, 1, \dots, m$. Next we prove some useful attributes of the scheme. These attributes show that the discrete operator $\mathcal{L}_\varepsilon^{n,m}$ in the scheme (2.33)–(2.36) satisfies a discrete minimum principle which is then used to prove the uniform stability of the solution.

2.4 Stability of the Scheme

Some properties of the discrete operator which will be useful in the convergence analysis of the scheme (2.33)–(2.36) are given in this section.

Lemma 2.4.1. *Discrete minimum principle*

The discrete operator $\mathcal{L}_\varepsilon^{m,n}$ defined by the scheme (2.33)–(2.36) satisfies a discrete minimum principle. That is, if Ψ_i^{k+1} is a mesh function which satisfies

$$\begin{aligned}
\Psi_i^0 &\geq 0, & \psi_0^{k+1} &\equiv -\varepsilon\alpha_0^{k+1} \left(\frac{\Psi_1^{k+1} - \Psi_0^{k+1}}{\phi_0^*} \right) + \beta_0^{k+1}\Psi_0^{k+1} \geq 0, \\
\psi_n^{k+1} &\equiv \varepsilon\alpha_n^{k+1} \left(\frac{\Psi_n^{k+1} - \Psi_{n-1}^{k+1}}{\phi_n^*} \right) + \beta_n^{k+1}\Psi_n^{k+1} \geq 0,
\end{aligned}$$

and $\mathcal{L}_\varepsilon^{m,n}\Psi(x_i, t_k) \leq 0$ for all $(x_i, t_k) \in Q^{n,m}$, then $\Psi_i^{k+1} \geq 0$, for all $(x_i, t_k) \in \bar{Q}^{n,m}$.

Proof. Suppose j and s are indices which satisfy

$$\Psi_j^{s+1} = \min_{(i,k) \in \bar{Q}^{n,m}} \Psi_i^{k+1} \text{ and } \Psi_j^{s+1} < 0.$$

Then $(j, s) \in [1, \dots, n-1] \times [1, 2, \dots, m]$, $\Psi_{j+1}^{s+1} - \Psi_j^{s+1} > 0$, $\Psi_j^{s+1} - \Psi_{j-1}^{s+1} > 0$ and $\Psi_j^{s+1} - \Psi_j^s > 0$. On the domain $Q^{n,m}$, we obtain

$$\begin{aligned}
\mathcal{L}_\varepsilon^{n,m}\Psi_j^{s+1} &\equiv -\frac{1}{\Delta t}(\Psi_j^{s+1} - \Psi_j^s) + \frac{\varepsilon}{\phi_i^2}(\Psi_{j+1}^{s+1} - 2\Psi_j^{s+1} + \Psi_{j-1}^{s+1}) + \frac{b_j^{s+1}}{h}(\Psi_{j+1}^{s+1} - \Psi_j^{s+1}) \\
&\quad - c_j^{s+1}\Psi_j^{s+1} \\
&= -\frac{1}{\Delta t}(\Psi_j^{s+1} - \Psi_j^s) + \frac{\varepsilon}{\phi_i^2}[(\Psi_{j+1}^{s+1} - \Psi_j^{s+1}) - (\Psi_j^{s+1} - \Psi_{j-1}^{s+1})] \\
&\quad + \frac{b_j^{s+1}}{h}(\Psi_{j+1}^{s+1} - \Psi_j^{s+1}) - c_j^{s+1}\Psi_j^{s+1} \geq 0,
\end{aligned}$$

and at x_0 and x_n , we have

$$-\varepsilon \frac{\alpha_1^{s+1}}{\phi_0^*} (\Psi_1^{s+1} - \Psi_0^{s+1}) + \beta_0^{s+1} \Psi_0^{s+1} < 0,$$

and

$$\varepsilon \frac{\alpha_n^{s+1}}{\phi_n^*} (\Psi_n^{s+1} - \Psi_{n-1}^{s+1}) + \beta_n^{k+1} \Psi_n^{s+1} \leq 0,$$

respectively. This is a contradiction, and therefore $\Psi_j^{s+1} \geq 0$ and hence $\Psi_i^{k+1} \geq 0$, $\forall i = 0, 1, \dots, n$ and $k = 0, 1, \dots, m$. $\square$

Now we show that the solution of the discrete problem (2.33)–(2.36) is bounded.

Lemma 2.4.2. (*Stability estimate*)

Let u_i^{k+1} be the solution of the discrete problem, then we have

$$\|u_i^{k+1}\| \leq c_0^{-1} \max_{(i,k) \in \bar{Q}^{n,m}} \|\mathcal{L}_\varepsilon^{n,m} u_i^{k+1}\| + \max_{(i,k) \in \bar{Q}^{n,m}} (|\varphi_i|, |\psi_0^{k+1}|, |\psi_n^{k+1}|).$$

Proof. We define the barrier functions $(\Psi^\pm)_i^{k+1}$ given by

$$(\Psi^\pm)_i^{k+1} = M \pm u_i^{k+1},$$

where the definition

$$M = c_0^{-1} \max_{(i,k) \in \bar{Q}^{n,m}} \|\mathcal{L}_\varepsilon^{n,m} u_i^{k+1}\| + \max_{(i,k) \in \bar{Q}^{n,m}} (|\varphi_i|, |\psi_0^{k+1}|, |\psi_n^{k+1}|),$$

has been used. At the initial stage we obtain

$$(\Psi^\pm)_i^0 = M \pm u_i^0 = M \pm \varphi_i \geq 0,$$

and at the boundaries we get

$$\begin{aligned} (\Psi^\pm)_0^{k+1} &= M \pm u_0^{k+1} = M \pm \psi_0^{k+1} \geq 0, \\ (\Psi^\pm)_n^{k+1} &= M \pm u_n^{k+1} = M \pm \psi_n^{k+1} \geq 0. \end{aligned}$$

Further on the domain $Q^{n,m}$ we have

$$\begin{aligned} \mathcal{L}_\varepsilon^{n,m} \Psi^\pm(x_i, t_{k+1}) &= - \left(\frac{(M \pm u_i^{k+1}) - (M \pm u_i^k)}{\Delta t} \right) + b_i^{k+1} \left(\frac{M \pm u_{i+1}^{k+1} - M \pm u_i^{k+1}}{h} \right) \\ &\quad + \varepsilon \left(\frac{M \pm u_{i+1}^{k+1} - 2(M \pm u_i^{k+1}) + M \pm u_{i-1}^{k+1}}{\phi_i^2} \right) - c_i^{k+1} (M \pm u_i^{k+1}) \\ &= -c_i^{k+1} M \pm \mathcal{L}_\varepsilon^{n,m} u_i^{k+1} \\ &= - \left(\max_{(x_i, t_k) \in \bar{Q}^{n,m}} \|\mathcal{L}_\varepsilon^{n,m} u_i^{k+1}\| + \max(|\varphi_i|, |\psi_0^{k+1}|, |\psi_n^{k+1}|) \right) \pm f_i^{k+1} \\ &\quad - \frac{c_i^{k+1}}{c_0} \|f_i^{k+1}\| \\ &\leq 0. \end{aligned}$$

From Lemma 2.4.1 it follows that $(\Psi^\pm)_i^{k+1} \geq 0$, $\forall (x_i, t_k) \in \bar{Q}^{n,m}$. $\square$

The Lemma below was proved in Patidar (2005) and it is also needed in the error analysis of the numerical method.

Lemma 2.4.3. *For all $k \in \mathbb{I}^+$ on a fixed mesh, we have that*

$$\lim_{\varepsilon \rightarrow 0} \max_{1 < i < n-1} \frac{\exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \max_{1 < i < n-1} \frac{\exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0,$$

where $x_i = ih$, $h = 1/n$, $\forall i = 1, 2, \dots, n-1$.

Proof. The domain Ω , is transformed into the discrete domain

$$[0 = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = 1].$$

It is clear that the interior grid points satisfy the inequalities

$$\max_{1 < i < n-1} \frac{\exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{j/2}} \leq \frac{\exp(-Cx_1/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = \frac{\exp(-Ch/\sqrt{\varepsilon})}{\varepsilon^{j/2}},$$

and

$$\max_{1 \leq i \leq n-1} \frac{\exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{j/2}} \leq \frac{\exp(-C(1-x_{n-1})/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = \frac{\exp(-Ch/\sqrt{\varepsilon})}{\varepsilon^{j/2}}.$$

Since $x_1 = h$, $1 - x_{n-1} = 1 - (n-1)h = 1 - nh + h = h$. Applying L'Hospital's rule results in

$$\lim_{\varepsilon \rightarrow 0} \frac{\exp(-Ch/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = \lim_{(p=1/\sqrt{\varepsilon}) \rightarrow \infty} \frac{p^j}{\exp(Chp)} \equiv \lim_{p \rightarrow \infty} \frac{j!}{(Ch)^j \exp(Chp)} = 0.$$

□

In the next section we analyse the numerical method for convergence.

2.5 Convergence Analysis

In this section, the scheme (2.33)–(2.36) is analysed for convergence and the result below estimates the error associated with the discretisation process.

Lemma 2.5.1. *The error associated with the time and the space discretisation satisfies*

$$\sup_{0 < \varepsilon \leq 1} \max_{0 \leq i \leq n; 0 \leq k \leq m} \|u_i^{k+1} - U_i^{k+1}\| \leq C(h + \Delta t),$$

where u_i^{k+1} , U_i^{k+1} are the analytical and the numerical solutions, respectively and C is a constant independent of the perturbation parameter ε and the mesh widths h and Δt .

Proof. The truncation error of the scheme (2.33)–(2.36) is given by

$$\begin{aligned}\mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - U_i^{k+1}) &= \mathcal{L}_\varepsilon^{n,m}u_i^{k+1} - \mathcal{L}_\varepsilon^{n,m}U_i^{k+1} = \mathcal{L}_\varepsilon^{n,m}u_i^{k+1} - f_i^{k+1} = \mathcal{L}_\varepsilon^{n,m}u_i^{k+1} - \mathcal{L}_\varepsilon u_i^{k+1} \\ &= \left[-\left(\frac{u_i^{k+1} - u_i^k}{\Delta t} \right) + \varepsilon \left(\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{\phi_i^2} \right) + b_i^{k+1} \left(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{h} \right) \right] \\ &\quad - \left[-(u_t)_i^{k+1} + \varepsilon(u_{xx})_i^{k+1} + b_i^{k+1}(u_x)_i^{k+1} \right].\end{aligned}$$

For any (i, k) such that $i = 1, 2, \dots, n-1$ and $k = 0, 1, 2, \dots, m$. A truncated Taylor series expansion of the terms u_i^{k+1} , u_{i+1}^{k+1} and u_{i-1}^{k+1} reduces the error into

$$\begin{aligned}\mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - U_i^{k+1}) &= -\frac{\Delta t}{2}(u_{tt})_i^k + \dots - \varepsilon(u_{xx})_i^{k+1} + \frac{\varepsilon}{\phi_i^2} \left(h^2(u_{xx})_i^{k+1} + \frac{h^4}{12}(u_{xxxx})_i^{k+1} + \dots \right) \\ &\quad + b_i^{k+1} \frac{h}{2}(u_{xx})_i^{k+1} + \dots\end{aligned}$$

Further expansion of the denominator function and simplification yields

$$\begin{aligned}\mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - U_i^{k+1}) &= -\frac{\Delta t}{2}(u_{tt})_i^k + \dots + \frac{\varepsilon}{12}h^2(u_{xxxx})_i^{k+1} + b_i^{k+1}h(u_{xx})_i^{k+1} \\ &\quad + \frac{b_i^{k+1}}{24}h^3(u_{xxxx})_i^{k+1}.\end{aligned}$$

From the bounds of the coefficient function, Theorem 2.2.1 and Lemma 2.4.3, we obtain

$$\left\| \mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - U_i^{k+1}) \right\| \leq C(h + \Delta t).$$

The error at the left boundary is estimated as follows:

$$\begin{aligned}u_0^{k+1} - U_0^{k+1} &= -\varepsilon\alpha_0^{k+1}u_x(0) + \beta_0^{k+1}u(0) - \psi_0^{k+1} \\ &\quad - \varepsilon\alpha_0^{k+1}u_x(0) + \left(\varepsilon\alpha_0^{k+1} \left(\frac{u_1^{k+1} - u_0^{k+1}}{\phi_0^*} \right) \right).\end{aligned}$$

Appropriate Taylor series expansions yields

$$u_0^{k+1} - U_0^{k+1} = -\varepsilon\alpha_0^{k+1}(u_x^{k+1})_0 + \left(\frac{\varepsilon\alpha_0^{k+1}}{h} - \frac{\beta_0^{k+1}}{2} \right) \left(h(u_x^{k+1})_0 + \frac{h^2}{2}(u_{xx}^{k+1})_0 + \dots \right).$$

Further simplification and the use of the bounds 2.2.1 results in

$$\left\| u_0^{k+1} - U_0^{k+1} \right\| = \left\| \frac{\varepsilon\alpha_0^{k+1}}{2}(u_{xx}^{k+1})_0 h - \frac{\beta_0^{k+1}}{2}(u_x^{k+1})_0 h - \frac{\beta_0^{k+1}}{4}(u_{xx}^{k+1})_0 h^2 \right\| \leq Ch.$$

At the right boundary, we estimate the error as follows:

$$\begin{aligned}u_n^{k+1} - U_n^{k+1} &= \varepsilon\alpha_n^{k+1}u_x(1) + \beta_n^{k+1}u(1) - \psi_n^{k+1} \\ &= \varepsilon\alpha_n^{k+1}u_x(1) - \left(\varepsilon\alpha_n^{k+1} \left(\frac{u_n^{k+1} - u_{n-1}^{k+1}}{\phi_n^*} \right) \right).\end{aligned}$$

Appropriate Taylor series expansions yields

$$u_n^{k+1} - U_n^{k+1} = \varepsilon\alpha_n^{k+1}(u_x)_n - \left(\frac{\varepsilon\alpha_n^{k+1}}{h} + \frac{\beta_n^{k+1}}{2} \right) \left(h(u_x^{k+1})_n + \frac{h^2}{2}(u_{xx}^{k+1})_n + \dots \right).$$

Simplifying further and the use of the bounds 2.2.1 yields

$$\left\| u_n^{k+1} - U_n^{k+1} \right\| = \left\| -\frac{\varepsilon \alpha_n^{k+1}}{2} (u_{xx}^{k+1})_n h - \frac{\beta_n^{k+1}}{2} (u_x^{k+1})_n h - \frac{\beta_n^{k+1}}{4} (u_{xx}^{k+1})_n h^2 \right\| \leq Ch.$$

Now let $\mathcal{R}_i^{k+1}$ be a mesh function defined by

$$\mathcal{R}_i^{k+1} = C(h + \Delta t), \quad 0 \leq i \leq n, \quad 0 \leq k \leq m.$$

Notice that $\mathcal{R}_0^{k+1} = \mathcal{R}_n^{k+1} = C(h + \Delta t)$ and $\mathcal{R}_i^0 = C(h + \Delta t)$ at the boundary and initial of $\bar{Q}^{n,m}$ respectively. Further on the domain $Q^{n,m}$ we have

$$\begin{aligned} \mathcal{L}_\varepsilon^{n,m} \mathcal{R}_i^{k+1} &\equiv -\frac{\mathcal{R}_i^{k+1} - \mathcal{R}_i^k}{\Delta t} + \frac{\varepsilon}{\phi_i^2} (\mathcal{R}_{i+1}^{k+1} - 2\mathcal{R}_i^{k+1} + \mathcal{R}_{i-1}^{k+1}) + \frac{b_i^{k+1}}{h} (\mathcal{R}_i^{k+1} - \mathcal{R}_{i-1}^{k+1}) - c_i^{k+1} \mathcal{R}_i^{k+1} \\ &= -c_i^{k+1} C(h + \Delta t). \end{aligned}$$

Suppose $\mathcal{H}_i^{k+1}$ is also another function which is given by

$$\mathcal{H}_i^{k+1} = \mathcal{R}_i^{k+1} - \|u_i^{k+1} - U_i^{k+1}\|.$$

At $\partial Q^{m,n}$, we have

$$\begin{aligned} \mathcal{H}_i^0 &= \mathcal{R}_i^0 - \|u_i^0 - U_i^0\| = C(h + \Delta t) > 0, \\ \mathcal{H}_0^{k+1} &= \mathcal{R}_0^{k+1} - \|u_0^{k+1} - U_0^{k+1}\| = C(h + \Delta t) - Ch > 0, \\ \mathcal{H}_n^{k+1} &= \mathcal{R}_n^{k+1} - \|u_n^{k+1} - U_n^{k+1}\| = C(h + \Delta t) - Ch > 0. \end{aligned}$$

Further on $Q^{m,n}$, we have

$$\begin{aligned} \mathcal{L}_\varepsilon^{n,m} \mathcal{H}_i^{k+1} &= \mathcal{L}_\varepsilon^{n,m} \mathcal{R}_i^{k+1} - \mathcal{L}_\varepsilon^{n,m} \|u_i^{k+1} - U_i^{k+1}\| \\ &= -c_i^{k+1} C(h + \Delta t) - \mathcal{L}_\varepsilon^{n,m} \|u_i^{k+1} - U_i^{k+1}\| \\ &= -(c_i^{k+1} C(h + \Delta t) + \mathcal{L}_\varepsilon^{n,m} \|u_i^{k+1} - U_i^{k+1}\|) < 0. \end{aligned}$$

From Lemma 2.4.1, we obtain

$$\mathcal{H}_i^{k+1} = \mathcal{R}_i^{k+1} - \|u_i^{k+1} - U_i^{k+1}\| \geq 0,$$

and hence

$$\|u_i^{k+1} - U_i^{k+1}\| \leq \mathcal{R}_i^{k+1},$$

which leads to

$$\|u_i^{k+1} - U_i^{k+1}\| \leq C(h + \Delta t).$$

□

In the next section we use Richardson extrapolation to improve the order of the fitted operator and the backward Euler finite difference schemes.

2.6 Richardson Extrapolation

Let $Q^{n,m} \subset Q^{2n,2m}$ where $Q^{n,m}$ is as defined earlier and $Q^{2n,2m}$ is the mesh obtained from bisecting the mesh widths h and Δt . We denote the numerical solution obtained with the mesh $Q^{2n,2m}$ by $\tilde{U}_i^{k+1}$. From 2.5.1 we have

$$u_i^{k+1} - U_i^{k+1} \leq C(h + \Delta t) + R^{n,m}, \quad (x_i, t_{k+1}) \in Q^{n,m}, \quad (2.40)$$

$$u_i^{k+1} - \tilde{U}_i^{k+1} \leq C\left(\frac{(h + \Delta t)}{2}\right) + R^{2n,2m}, \quad (x_i, t) \in Q^{2n,2m}, \quad (2.41)$$

where R^μ is the remainder term of the error. Now we subtract the inequality 2.41 from 2.40 to obtain the extrapolation formula

$$u_i^{k+1} - U_i^{k+1} - 2(u_i^{k+1} - \tilde{U}_i^{k+1}) = R_i^n - 2R_i^{2n}, \quad (2.42)$$

$$\mathcal{U}_i^{k+1} = 2\tilde{U}_i^{k+1} - U_i^{k+1}. \quad (2.43)$$

Similarly, we use the extrapolation formula

$$\mathcal{U}_0^{k+1} = 2\tilde{U}_0^{k+1} - U_0^{k+1}, \quad (2.44)$$

to calculate the numerical solution at the boundary point x_0 .

Lemma 2.6.1. *Let u_i^{k+1} be the exact solution of the continuous problem (2.1)-(2.2) and $\mathcal{U}_i^{k+1}$ be the numerical solution obtained via 2.42. Then the error is of the form*

$$\sup_{0 < \varepsilon \leq 1} \max_{0 \leq i \leq 2n; 0 \leq k \leq 2m} \|u_i^{k+1} - \mathcal{U}_i^{k+1}\| \leq C(h^2 + (\Delta t)^2).$$

Proof. The error at the point x_0 takes the form

$$\begin{aligned} u_0^{k+1} - \mathcal{U}_0^{k+1} &= 2 \left[-\varepsilon \alpha_0^{k+1} u_x(0) + \left(\varepsilon \alpha_0^{k+1} \left(\frac{u_{0+1}^{k+1} - u_0^{k+1}}{\phi_0} \right) \right) \right] - (u_0^{k+1} - U_0^{k+1}) \\ &= 2 \left[-\varepsilon \alpha_0^{k+1} (u_x)_0 + \left(\frac{2\varepsilon \alpha_0^{k+1}}{h} - \frac{\beta_0^{k+1}}{2} \right) \left(\frac{h}{2} (u_x)_0 + \frac{h^2}{8} (u_{xx})_0 + \dots \right) \right] \\ &\quad - Ch. \end{aligned}$$

Simplifying further results in

$$\left\| u_0^{k+1} - \mathcal{U}_0^{k+1} \right\| \leq Ch^2. \quad (2.45)$$

Both the extrapolation formula and the error at the mesh points x_n can be obtained analogously. To estimate the error at the interior mesh points after extrapolation, we define it by

$$\mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - \mathcal{U}_i^{k+1}) = 2\mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - \tilde{U}_i^{k+1}) - \mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - U_i^{k+1}).$$

Using appropriate Taylor series expansions reduces the error into

$$\begin{aligned} \mathcal{L}_\varepsilon^{n,m}(u_i^{k+1} - \mathcal{U}_i^{k+1}) = 2 \left[- \left(\frac{\Delta t}{4} (u_{tt})_i^k - \frac{(\Delta t)^2}{24} (u_{ttt})_i^k + \dots \right) + \frac{\varepsilon}{48} (u_{xxx})_i^{k+1} h^2 + \frac{b_i^{k+1}}{2} h (u_{xx})_i^{k+1} \right. \\ \left. + \frac{b_i^{k+1}}{192} h^3 (u_{xxx})_i^{k+1} \right] \\ - \left(- \frac{\Delta t}{2} (u_{ttt})_i^k + \dots + \frac{\varepsilon}{12} h^2 (u_{xxx})_i^{k+1} + b_i^{k+1} h (u_{xx})_i^{k+1} \right. \\ \left. + \frac{b_i^{k+1}}{24} h^3 (u_{xxx})_i^{k+1} \right). \end{aligned}$$

Application of the bound and Lemma 2.4.1, leads to the result. $\square$

Theorem 2.6.1. *Let $u(x, t_k)$ the exact solution of the continuous problem and $U(x, t_k)$ be the numerical solution. The error of the scheme (2.33)–(2.34) satisfies the estimate*

$$\max_{(i,k) \in Q^{n,m}} |u(x_i, t_k) - U(x_i, t_k)| \leq C(h + \Delta t),$$

before extrapolation and

$$\max_{(i,k) \in Q^{n,m}} |u(x_i, t_k) - \mathcal{U}(x_i, t_k)| \leq C(h^2 + \Delta t^2),$$

after extrapolation.

Proof. The proof of this theorem follows from the Lemmas 2.5.1 and 2.6.1. $\square$

Now we demonstrate the theory in practice through numerical experiment.

2.7 Numerical Results

In this section we test the proposed method with an example. We compute the maximum pointwise error and the numerical rate of convergence before and after extrapolation. These results are then displayed in tables formats. In the example below the exact solution of the problem is not known, thus to calculate the errors $E_{n,\Delta t}^1$ and $E_{n,\Delta t}^{\text{ex}}$, before and after extrapolation we use the double mesh principle (a numerical solution obtained by doubling the number of sub-intervals serves as the exact solution) given by

$$E_{n,\Delta t}^1 = \max_{0 \leq i \leq n; 0 \leq k \leq m} \left\| U_{i,n}^{k;\Delta t} - U_{i,2n}^{k;\frac{\Delta t}{2}} \right\|, \quad (2.46)$$

before extrapolation, and after extrapolation we use the formula Here $U_{i,n}^{k;\Delta t}$ is the numerical solution obtained before extrapolation with (n, m) number of sub-intervals and $U_{i,2n}^{k;\frac{\Delta t}{2}}$ is also a numerical solution but on the finer mesh $(2n, 2m)$. After extrapolation $\mathcal{U}_{i,n}^{k;\Delta t}$ is obtained by the mesh $Q^{n,m}$ and $\mathcal{U}_{i,2n}^{k;\frac{\Delta t}{2}}$ is by the mesh $Q^{2n,2m}$. Also, we compute the ε -uniform pointwise error given by

$$E_{n,m}^1 = \max_{0 < \varepsilon \leq 1} E_{n,\Delta t}^1$$

and

$$E_{n,m}^{\text{ex}} = \max_{0 < \varepsilon \leq 1} E_{n,\Delta t}^{\text{ex}}.$$

Furthermore, we compute the numerical rate of convergence with the formulae

$$r_1 = \log_2 \left(\frac{E_{n,m}^1}{E_{2n,2m}^1} \right) \quad \text{and} \quad r_2 = \log_2 \left(\frac{E_{n,m}^{\text{ex}}}{E_{2n,2m}^{\text{ex}}} \right)$$

and the ε -uniform rate of convergence with

$$R^1 = \max_{0 < \varepsilon \leq 1} r_1, \quad R^{\text{ex}} = \max_{0 < \varepsilon \leq 1} r_2,$$

respectively.

Example 2.7.1. *Hemker et al. (2002)* We consider the problem

$$\begin{aligned} -u_t + \varepsilon u_{xx} + u_x &= -4t^3, \quad (x, t) \in (0, 1) \times (0, 1] \\ u(x, 0) &= 0, \quad x \in \bar{\Omega}, \\ -\varepsilon u_x(0, t) = t^2 \quad u(1, t) &= 0, \end{aligned}$$

Table 2.1: Maximum pointwise error and rate of convergence for Example 2.7.1 before extrapolation.

ε	n=m=16	32	64	128	256	512	1024
10^{-4}	8.66E-02	5.28E-02	2.97E-02	1.58E-02	8.16E-03	4.15E-03	2.09E-03
	0.7122	0.8320	0.9093	0.9530	0.9761	0.9905	
10^{-5}	8.66E-02	5.28E-02	2.97E-02	1.58E-02	8.16E-03	4.15E-03	2.09E-03
	0.7122	0.8320	0.9093	0.9530	0.9761	0.9879	
10^{-6}	8.66E-02	5.28E-02	2.97E-02	1.58E-02	8.16E-03	4.15E-03	2.09E-03
	0.7122	0.8320	0.9093	0.9530	0.9761	0.9879	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	8.66E-02	5.28E-02	2.97E-02	1.58E-02	8.16E-03	4.15E-03	2.09E-03
	0.7122	0.8320	0.9093	0.9530	0.9761	0.9879	
$E_{n,m}^1$	8.66E-02	5.28E-02	2.97E-02	1.58E-02	8.16E-03	4.15E-03	2.09E-03
R^1	0.7122	0.8320	0.9093	0.9530	0.9761	0.9879	

Table 2.2: Maximum pointwise error and rate of convergence for Example 2.7.1 after extrapolation.

ε	n=m=16	32	64	128	256	512	1024
10^{-4}	1.52E-02	6.14E-03	1.98E-03	5.60E-04	1.56E-04	4.22E-05	1.09E-5
	1.3052	1.6328	1.8211	1.8402	1.8862	1.9529	
10^{-5}	1.52E-02	6.14E-03	1.98E-03	5.60E-04	1.56E-04	4.22E-05	1.09E-5
	1.3052	1.6328	1.8206	1.8466	1.8862	1.9529	
10^{-6}	1.52E-02	6.14E-03	1.98E-03	5.60E-04	1.56E-04	4.22E-05	1.09E-5
	1.3052	1.6328	1.8206	1.8466	1.8862	1.9529	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.52E-02	6.14E-03	1.98E-03	5.60E-04	1.56E-04	4.22E-05	1.09E-5
	1.3052	1.6328	1.8206	1.8466	1.8862	1.9529	
$E_{n,m}^{\text{ex}}$	1.52E-02	6.14E-03	1.98E-03	5.60E-04	1.56E-04	4.22E-05	1.09E-5
R^{ex}	1.3052	1.6328	1.8206	1.8466	1.8862	1.9529	

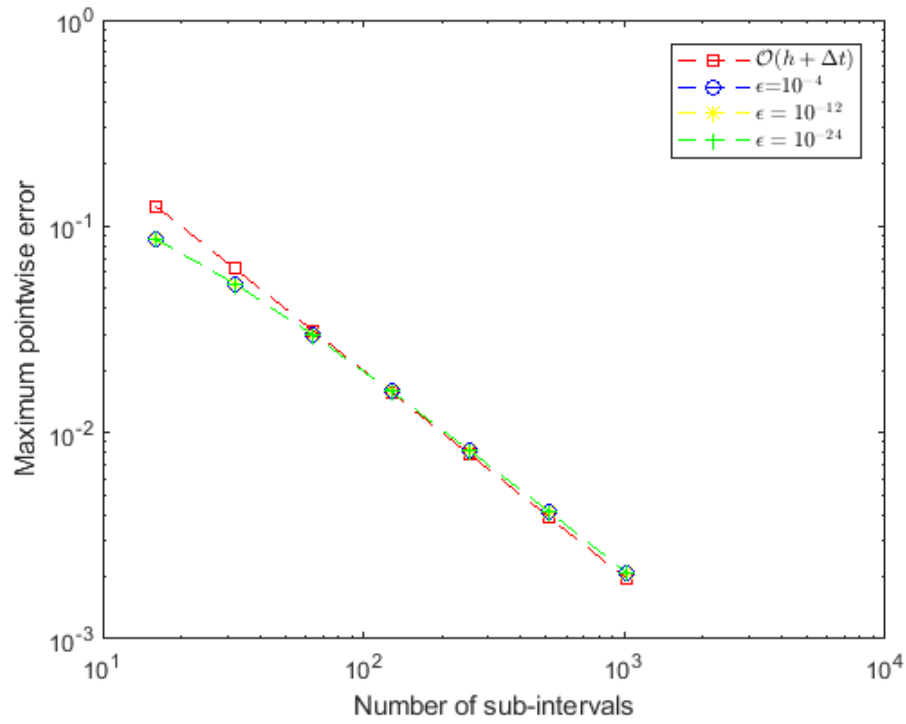


Figure 2.1: Loglog plot of the maximum pointwise errors of Example 2.7.1 before extrapolation.

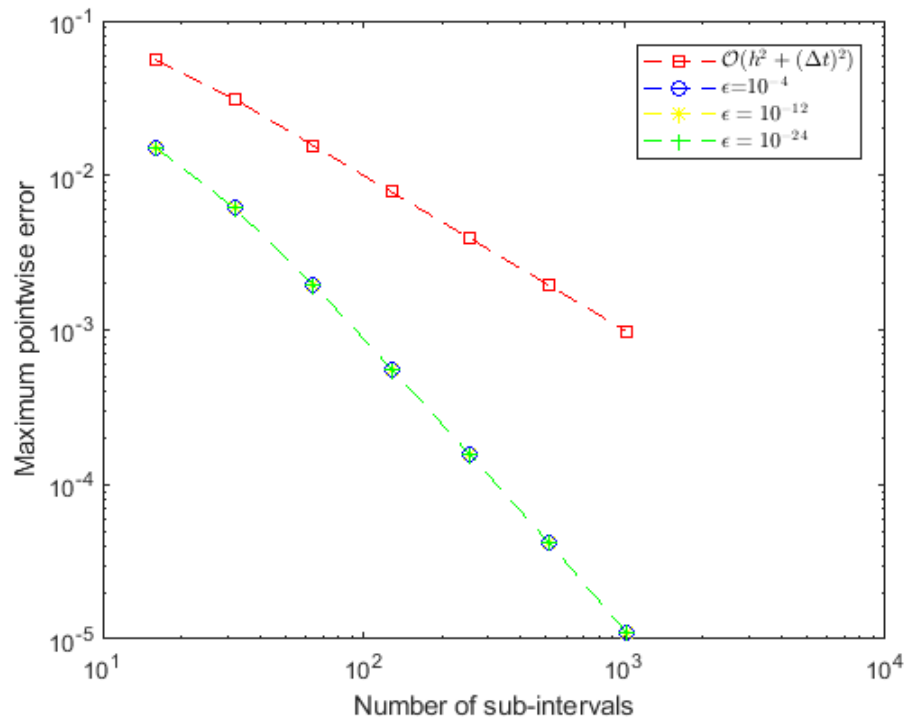


Figure 2.2: Loglog plot of the maximum pointwise errors of Example 2.7.1 after extrapolation.

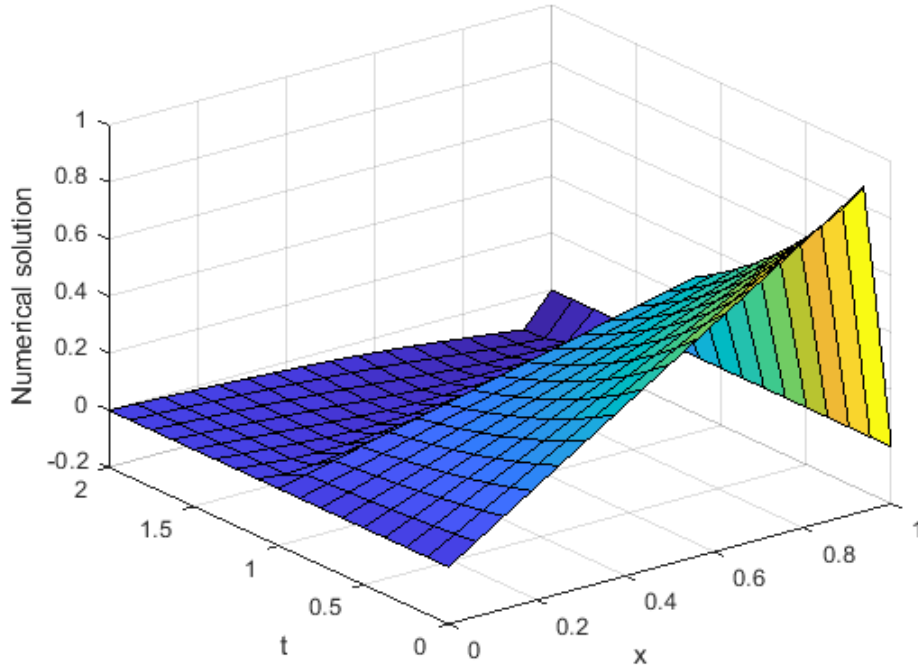


Figure 2.3: Surface plot of the numerical solution of Example 2.7.1 with $n = 16$ and $\varepsilon = 10^{-24}$.

2.8 Conclusion

A uniformly convergent finite difference method was proposed to solve a time-dependent convection diffusion problem with Robin boundary conditions in this chapter. The spatial derivatives at both the boundaries and the interior point of the domain were solved using a fitted operator finite difference method and the time derivative was solved by a backward Euler method. Some useful properties of the discrete problem were given and then used to analyse the numerical methods for convergence. The convergence analysis resulted in a first order accuracy in both the space and the time variables. These results were then improved to a second order accuracy after the use of Richardson extrapolation. To validate the proposed method, we tested the method on an example and the results obtained before and after extrapolation were displayed in Tables 2.1 and 2.2 respectively. In each table, we have both the maximum pointwise error and the numerical rate of convergence for different values of ε, n and m . The rates of convergence in the Tables 2.1 and 2.2 adhered to the Theorems 2.5.1 and 2.6.1 respectively. The Figures 2.1 and 2.2 also displayed the log-log plot of the maximum pointwise errors and the number of sub-intervals. The surface plot of the numerical solution was shown in Figure 2.3. From the Figures 2.1 and 2.2 it is evident that indeed the maximum pointwise error and the rate of convergence do not depend on ε and also the maximum pointwise error decreases with an increase in the number of sub-intervals. These results were in adherence to the theoretical findings.

In the next chapter, a fitted operator finite difference scheme which is similar to the one introduced in this chapter is designed along with the Crank Nicolson finite difference scheme to solve a parabolic convection diffusion problem with time delay.

Chapter 3

A Robust Method of Lines Solution for Singularly Perturbed Delay Parabolic Problem

3.1 Introduction

In Chapter 1, we gave an overview of fitted operator finite difference schemes. From Section 1.1.2 of Chapter 1, we noticed that singularly perturbed delay parabolic problems have been solved mainly by the fitted mesh finite difference schemes. Thus to fill the void in that part of literature we propose a fitted operator finite difference scheme in the context of the non-standard finite difference scheme to solve the spatial variables and the Crank Nicolson finite difference scheme is used to solve the temporal derivatives in the framework of method of lines. We analyse the scheme for convergence and improve the spatial accuracy by Richardson extrapolation.

Singularly perturbed problems (SPPs) are applicable in various fields of studies. They are differential equations which depend on a small parameter known as the perturbation parameter and their solutions possess steep gradient usually referred to as layers. These layers can either be at the boundaries or even in the interior of the domain. SPPs exist in various forms, as shown by Amiraliyev and Şevgin (2006); Kadalbajoo and Patidar (2003); Kadalbajoo and Gupta (2010); Kauthen (1997); Munyakazi (2015); Patidar (2007).

In this chapter, we are interested in the initial boundary value problem

$$\mathcal{L}_\varepsilon u(x, t) = f(x, t) - c(x, t)u(x, t - \tau), \quad (x, t) \in Q, \quad (3.1)$$

where the above differential operator $\mathcal{L}_\varepsilon$ is expressed by

$$\mathcal{L}_\varepsilon = \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} + a(x) \frac{\partial}{\partial x} + b(x, t),$$

subject to the boundary and initial conditions

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t), \quad t \in [0, T], \quad u(x, t) = \varphi(x, t), \quad (x, t) \in [0, 1] \times [-\tau, 0]. \quad (3.2)$$

The perturbation parameter ε is such that $0 < \varepsilon \ll 1$. The time delay τ is strictly positive and is chosen such that ($\tau p = T$, for some integer $p > 1$). The coefficient functions $a(x)$, $b(x, t)$ and $c(x, t)$ are smooth, bounded and satisfy

$$0 < \alpha \leq a(x), \quad 0 < \beta \leq b(x, t), \quad c(x, t) \geq 0.$$

Also, the source term, the initial and the boundary conditions are smooth and bounded. The compatibility conditions

$$\varphi(0, 0) = \eta_0(0), \quad (3.3)$$

$$\varphi(1, 0) = \eta_1(0), \quad (3.4)$$

$$\frac{d\eta_0(0)}{dt} - \varepsilon \frac{\partial \varphi(0, 0)}{\partial x^2} + a(0) \frac{\partial \varphi(0, 0)}{\partial x} + b(0, 0) \varphi(0, 0) = -c(0, 0) \varphi(0, -\tau) + f(0, 0) \quad (3.5)$$

$$\frac{d\eta_1(0)}{dt} - \varepsilon \frac{\partial \varphi(1, 0)}{\partial x^2} + a(1) \frac{\partial \varphi(1, 0)}{\partial x} + b(1, 0) \varphi(1, 0) = -c(1, 0) \varphi(1, -\tau) + f(1, 0), \quad (3.6)$$

are imposed. The SPP (3.1)–(3.2) has a unique solution exhibiting a boundary layer along $x = 1$, of width $\mathcal{O}(\varepsilon)$, Das and Natesan (2015). In this chapter, our main objective is to design, analyse and implement a uniformly convergent finite difference method to solve problem (3.1)–(3.2) in the frame work of method of lines. With this in mind, first the continuous problem is spatially discretised by means of a fitted operator finite difference method followed by the time discretisation using the Crank Nicolson finite difference method. The methods are then analysed for convergence and then improved in the spatial accuracy by Richardson extrapolation.

The rest of this chapter is as follows: In Section 3.2, some properties of the continuous problem, whose replication in the pseudo-discrete form will be employed in the analysis of the numerical scheme, are given. Section 3.3 is devoted to the spatial semi-discretisation of the continuous problem by the fitted operator finite difference scheme. This is then followed by error estimate of this semi-discretisation process in Section 3.5. Richardson extrapolation on the spatial discretisation and its analysis are presented in Section 3.6. Section 3.7 focuses on the time discretisation of the resulting initial value problems and its theoretical estimate. Numerical simulations are carried out in Section 3.8 to support the theoretical estimates. In the last section we conclude this chapter with a summary of the main results.

3.2 Bounds on the Solution and its Derivatives

The purpose of this section is to derive the bounds of the solution and its partial derivatives. To achieve this, some useful lemmas will be introduced.

Lemma 3.2.1. *Continuous maximum principle*

Suppose $\Psi(x, t)$ is such that, $\Psi(x, t) \geq 0$, $\forall (x, t) \in \partial\Omega \times [0, T]$ and $\Psi(x, t) \geq 0$, $\forall (x, t) \in \bar{\Omega} \times [-\tau, 0]$. If $\mathcal{L}_\varepsilon \Psi(x, t) \geq 0$, $\forall (x, t) \in Q$, then $\Psi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$.

Proof. Assume the function $\Psi(x, t)$ takes its minimum value at the point (x^*, t^*) and that its minimum is negative, that is

$$\Psi(x^*, t^*) = \min_{(x, t) \in \bar{Q}} \Psi(x, t) \text{ and } \Psi(x^*, t^*) < 0.$$

Clearly $(x^*, t^*) \in Q$. Now for the operator $\mathcal{L}_\varepsilon$ at the point (x^*, t^*) we have

$$\mathcal{L}_\varepsilon \Psi(x^*, t^*) = \Psi_t(x^*, t^*) - \varepsilon \Psi_{xx}(x^*, t^*) + a(x^*) \Psi_x(x^*, t^*) + b(x^*, t^*) \Psi(x^*, t^*).$$

Since $0 < \Psi_{xx}(x^*, t^*)$ and $0 = \Psi_t(x^*, t^*) = \Psi_x(x^*, t^*)$, we obtain $0 < \mathcal{L}_\varepsilon \Psi(x^*, t^*)$, which is a contradiction. Therefore, $\Psi(x, t) \geq 0$ for all $(x, t) \in \bar{Q}$. $\square$

Lemma 3.2.2. *The solution $u(x, t)$ of the problem ((3.1)–(3.2)) is such that*

$$|u(x, t) - \varphi(x, 0)| \leq Ct, \quad (x, t) \in \bar{Q},$$

where C is a constant which does not depend of ε .

Proof. Let $q(x, t) = \begin{cases} u(x, t) - \varphi(x, 0), & 0 \leq t \leq T, \\ u(x, t) - \varphi(x, t), & -\tau \leq t \leq 0, \end{cases}$

so that we have $u(x, t) = q(x, t) + \varphi(x, t)$.

$$\begin{aligned} \mathcal{L}_\varepsilon(q(x, t) + \varphi(x, 0)) &= -(c(x, t)q(x, t - \tau) + c(x, t)\varphi(x, -\tau)) + f(x, t) \\ \mathcal{L}_\varepsilon q(x, t) &= -c(x, t)q(x, t - \tau) - c(x, t)\varphi(x, -\tau) + f(x, t) - \mathcal{L}_\varepsilon \varphi(x, 0). \end{aligned}$$

Since $q(x, t) = u(x, t) - \varphi(x, t) = \varphi(x, t) - \varphi(x, t) = 0$, $-\tau \leq t \leq 0$, we obtain

$$\mathcal{L}_\varepsilon q(x, t) = -c(x, t)\varphi(x, -\tau) + f(x, t) - \mathcal{L}_\varepsilon \varphi(x, 0), \quad (x, t) \in Q.$$

At the boundaries of Q , we have

$$\begin{aligned} q(x, t) &= u(x, t) - \varphi(x, t) = 0, \\ q(0, t) &= \eta_0(t) - \varphi(0, 0), \quad q(1, t) = \eta_1(t) - \varphi(1, 0), \text{ for all } t \in [0, T]. \end{aligned}$$

From the compatibility conditions, we have

$$|q(0, t)| = |\eta_0(t) - \eta_0(0)| \leq C_1 t, \quad |q(1, t)| = |\eta_1(t) - \eta_1(0)| \leq C_2 t, \text{ for all } t \in [0, T],$$

for some constants $C_1, C_2 > 0$. Let $r(x, t)$ be a function defined by

$$r(x, t) = \begin{cases} Ct, & 0 \leq t \leq T, \\ 0, & -\tau \leq t \leq 0, \end{cases}$$

where C has been chosen to satisfy $C = \max\{C_1, C_2\}$. Applying the differential operator $\mathcal{L}_\varepsilon$, on $r(x, t)$ yields

$$\mathcal{L}_\varepsilon r(x, t) = (1 + tb(x, t))C \geq (1 + \beta t)C,$$

and at the boundaries of Q we have

$$r(x, t) = 0, \quad r(0, t) = r(1, t) = Ct.$$

Now we define another function $h(x, t)$ given by

$$h(x, t) = r(x, t) - |q(x, t)|.$$

At the boundaries of Q , we obtain

$$\begin{aligned} h(x, 0) &= r(x, 0) - |q(x, 0)| = 0, \\ h(0, t) &= r(0, t) - |q(0, t)| = Ct - |\eta_0(t) - \eta_0(0)| = Ct - C_1 t = (C - C_1)t \geq 0 \end{aligned}$$

and

$$h(1, t) = r(1, t) - |q(1, t)| = Ct - |\eta_1(t) - \eta_1(0)| = Ct - C_2 t = (C - C_2)t \geq 0,$$

respectively, since $C = \max\{C_1, C_2\}$. Now applying the operator in equation (3.1) on $h(x, t)$ gives

$$\mathcal{L}_\varepsilon h(x, t) = \mathcal{L}_\varepsilon r(x, t) - \mathcal{L}_\varepsilon |q(x, t)| \geq C^* - \sup_{(x, t) \in Q} \mathcal{L}_\varepsilon |q(x, t)| \geq 0,$$

in view of the fact that $C^* = (1 + \beta t)C$ satisfies

$$C^* \geq \sup_{(x, t) \in Q} \mathcal{L}_\varepsilon |q(x, t)|.$$

Making use of Lemma 3.2.1, we obtain

$$h(x, t) = r(x, t) - |q(x, t)| \geq 0.$$

It follows that

$$|q(x, t)| \leq r(x, t),$$

and hence we get

$$|q(x, t)| = |u(x, t) - \varphi(x, 0)| \leq Ct.$$

□

Lemma 3.2.3. *Let $u(x, t)$ be the solution of problem (3.1)–(3.2), then it satisfies $|u(x, t)| \leq C_3$, for any $(x, t) \in \bar{Q}$.*

Proof. By Lemma 3.2.2, we get $|u(x, t) - \varphi(x, 0)| \leq Ct \leq CT$. It follows that

$$\begin{aligned} |u(x, t)| &= |u(x, t) - \varphi(x, 0) + \varphi(x, 0)| \\ &\leq |u(x, t) - \varphi(x, 0)| + |\varphi(x, 0)| \\ &\leq CT + \sup_{x \in [0, 1]} |\varphi(x, 0)| = C_3, \end{aligned}$$

that is

$$|u(x, t)| \leq C_3.$$

□

The following lemmas provides the bounds of the partial derivatives of the solution $u(x, t)$ and their proofs follow from in Das and Natesan (2015) and Ng-Stynes et al. (1988). We consider the assumptions

$$u(x, t) = 0, \quad (x, t) \in \partial Q, \quad (3.7)$$

$$u(x, t) = u_t(x, t) = 0, \quad x \in \bar{\Omega}, \quad t \in [-\tau, 0]. \quad (3.8)$$

Lemma 3.2.4. *The solution $u(x, t)$ of problem (3.1)–(3.2) satisfies the estimate $|u_t(x, t)| \leq C_4$, for any $(x, t) \in \bar{Q}$,*

Proof. At $x = 0$ and $x = 1$ of $\bar{Q}$, $u = 0$, and thus $u_t(0, 1) = u_t(1, t) = 0$, from the assumption made on the initial and the boundary conditions. At $t = 0$, we have $u(x, 0) = 0$, $u_x(x, 0) = 0$, and $u_{xx}(x, 0) = 0$. Now taking the limit as t approaches zero in equation (3.1) yields

$$\lim_{t \rightarrow 0^+} u_t(x, t) - \varepsilon u_{xx}(x, t) + a(x)u_x(x, t) + b(x, t)u(x, t) = \lim_{t \rightarrow 0^+} (f(x, t) - c(x, t)u(x, t - \tau)).$$

Since $u(x, 0) = u_x(x, 0) = u_{xx}(x, 0) = 0$, and $u(x, -\tau) = 0$, we get

$$u_t(x, 0) = f(x, 0), \quad 0 \leq x \leq 1. \quad (3.9)$$

Now taking the norm on both sides of (3.9), we obtain

$$|u_t(x, t)| \leq \bar{C} \quad \text{on } \partial Q,$$

when $\bar{C}$ is chosen to be large. Applying the differential operator in equation (3.1) on $u_t(x, t)$ gives

$$\begin{aligned} \mathcal{L}_\varepsilon u_t(x, t) &= b(x, t)u_t + a(x)u_{tx} + u_{tt} - \varepsilon u_{txx} \\ &= (b(x, t)u + a(x)u_x + u_t - \varepsilon u_{xx})_t - b_t u \\ &= (-c(x, t)u(x, t - \tau) + f(x, t))_t - b_t u. \end{aligned} \quad (3.10)$$

From (3.10), when the delay term is a given function, that is when $t \in [0, \tau]$, we have

$$\mathcal{L}_\varepsilon u_t(x, t) = f_t(x, t) - b_t u(x, t).$$

From Lemma 3.2.3 and the fact that f and b are bounded, it follows that $|\mathcal{L}_\varepsilon u_t(x, t)| \leq C_6$, holds. Let $\Phi_1(x, t) = \bar{C} + \tilde{C}$, where $\tilde{C} > 0$. For all $(x, t) \in \partial\Omega \times [-\tau, 0]$,

$$\Phi_1(x, t) = \bar{C} + \tilde{C} \geq \bar{C} \geq |u_t(x, t)|.$$

This implies that $|u_t(x, t)| \leq \Phi_1(x, t)$, $\forall (x, t) \in \partial\Omega \times [0, \tau]$. Applying the operator $\mathcal{L}_\varepsilon$ on $\Phi_1(x, t)$, results in

$$\mathcal{L}_\varepsilon \Phi_1(x, t) = (\bar{C} + \tilde{C})b(x, t) \geq \bar{C}\beta \geq \beta|u_t(x, t)|. \quad (3.11)$$

Using the linearity of the operator $\mathcal{L}_\varepsilon$, we have

$$|\mathcal{L}_\varepsilon u_t(x, t)| \leq c|u_t(x, t)|, \quad c > 0, \quad \text{for any } (x, t) \in Q. \quad (3.12)$$

It follows that

$$|\mathcal{L}_\varepsilon(-u_t(x, t))| = |-\mathcal{L}_\varepsilon u_t(x, t)| = |\mathcal{L}_\varepsilon u_t(x, t)| \leq c|u_t(x, t)| \leq c|-u_t(x, t)|. \quad (3.13)$$

From the estimates (3.12) and (3.13), we have

$$|\mathcal{L}_\varepsilon|u_t|| \leq c|u_t|. \quad (3.14)$$

Moreover since $\mathcal{L}_\varepsilon|u_t(x, t)| \leq |\mathcal{L}_\varepsilon|u_t(x, t)||$, using (3.14), we get

$$\mathcal{L}_\varepsilon|u_t(x, t)| \leq c|u_t(x, t)|. \quad (3.15)$$

Therefore, choosing $c = \beta$ in (3.15) and using (3.11) results in

$$\mathcal{L}_\varepsilon\Phi_1(x, t) \geq \beta|u_t(x, t)| \geq \mathcal{L}_\varepsilon|u_t(x, t)|. \quad (3.16)$$

Subtracting $\mathcal{L}_\varepsilon|u_t(x, t)|$ from both side of (3.16) results in

$$\begin{aligned} \mathcal{L}_\varepsilon\Phi_1(x, t) - \mathcal{L}_\varepsilon|u_t(x, t)| &\geq 0 \\ \mathcal{L}_\varepsilon(\Phi_1(x, t) - |u_t(x, t)|) &\geq 0. \end{aligned}$$

From Lemma 3.2.1, we obtain $\Phi_1(x, t) - |u_t(x, t)| \geq 0$, thus $\Phi_1(x, t) \geq |u_t(x, t)|$, and hence $|u_t(x, t)| \leq \bar{C} + \hat{C} = C_4$, that is $|u_t(x, t)| \leq C_4$, $(x, t) \in \bar{\Omega} \times [0, \tau]$.

Now the domain $(x, t) \in (0, 1) \times [\tau, 2\tau]$, is considered. For $t \in [\tau, 2\tau]$, (3.10) yields

$$\mathcal{L}_\varepsilon u_t(x, t) = f_t(x, t) - c_t(x, t)u(x, t - \tau) - c(x, t)u_t(x, t - \tau) - b_t(x, t)u(x, t).$$

Since each term on the right hand side of this equation is bounded, we obtain the following estimate $|\mathcal{L}_\varepsilon u_t(x, t)| \leq \hat{C}$. Defining the function $\Phi_2(x, t) = \bar{C} + \hat{C}$, $\hat{C} > 0$ and using the same procedures we employed in the domain $t \in [0, \tau]$, the result follows. Analogously, the bounds on the entire time domain can be obtained. $\square$

Lemma 3.2.5. *The solution $u(x, t)$ of problem (3.1)–(3.2) satisfies the estimate $|u_x(x, t)| \leq C_5(1 + \varepsilon^{-1} \exp(-\bar{\alpha}(1 - x)/\varepsilon))$, for any $(x, t) \in \bar{Q}$.*

Proof. Equation (3.1) is written as

$$a(x)u_x - \varepsilon u_{xx} = h_1(x, t), \quad (3.17)$$

where $h_1(x, t) = f(x, t) - u_t(x, t) - c(x, t)u(x, t - \tau) - b(x, t)u(x, t)$. Now we fix $t \in [0, T]$, and multiply through by the integration factor $A(x) = \int_0^x a(\gamma)d\gamma$, to obtain

$$\begin{aligned} \exp(-\varepsilon^{-1}A(x))u_{xx} - \varepsilon^{-1}a(x)\exp(-\varepsilon A(x))u_x &= -\varepsilon^{-1}\exp(-\varepsilon^{-1}A(x))h_1(x, t) \\ \frac{d}{dx} \left(\exp(-\varepsilon^{-1}A(x)) \frac{du}{dx} \right) &= -\varepsilon^{-1}\exp(-\varepsilon^{-1}A(x))h_1(x, t). \end{aligned}$$

Integrating from x to 1 leads to

$$\begin{aligned} \exp(-\varepsilon^{-1}A(1))u_x(1, t) - \exp(-\varepsilon^{-1}A(x))u_x(x, t) &= -\varepsilon^{-1} \int_x^1 h_1(\gamma, t) \exp(-\varepsilon^{-1}A(\gamma))d\gamma, \\ u_x(x, t) &= u_x(1, t) \exp(-\varepsilon^{-1}(A(1) - A(x))) + z_1(x, t), \end{aligned} \quad (3.18)$$

where $z_1(x, t)$ is given by

$$z_1(x, t) = \varepsilon^{-1} \int_x^1 h_1(\gamma, t) \exp(-\varepsilon^{-1}A(\gamma) - A(x)) d\gamma.$$

To derive the bound of $u_x(1, t)$ in (3.18), we integrate (3.18) from x to 1, to obtain

$$u(1, t) - u(x, t) = u_x(1, t) \int_x^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds + \int_x^1 z_1(s, t) ds. \quad (3.19)$$

Evaluating (3.19) at $x = 0$ yields

$$u_x(1, t) = -\frac{\int_0^1 z_1(s, t) ds}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds}, \quad (3.20)$$

since $0 = u(0, t) = u(1, t)$. Substituting (3.20) into equation (3.18) gives

$$u_x(x, t) = -\frac{\exp(-\varepsilon^{-1}(A(1) - A(x)))}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds} \int_0^1 z_1(s, t) ds + z_1(x, t). \quad (3.21)$$

Now we take the norm on both sides of (3.21) to obtain

$$\begin{aligned} |u_x(x, t)| &\leq \frac{|\exp(-\varepsilon^{-1}(A(1) - A(x)))|}{|\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds|} \left| \int_0^1 z_1(s, t) ds \right| + |z_1(x, t)| \\ &\leq \frac{|\exp(-\varepsilon^{-1}A(1))| |\exp(\varepsilon^{-1}A(x))|}{|\exp(-\varepsilon^{-1}A(1))| |\int_0^1 \exp(\varepsilon^{-1}A(s)) ds|} \int_0^1 |z_1(s, t)| ds + |z_1(x, t)| \\ &= \frac{\exp(\varepsilon^{-1}A(x))}{\int_0^1 \exp(\varepsilon^{-1}A(s)) ds} \int_0^1 |z_1(s, t)| ds + |z_1(x, t)| \\ &\leq \check{C} \left(1 + \frac{\exp(\varepsilon^{-1}\bar{\alpha}x)}{\int_0^1 \exp(\varepsilon^{-1}\alpha) ds} \right), \end{aligned} \quad (3.22)$$

where $\bar{\alpha}$ is an upper bound of $a(x)$ over $[0, 1]$ and $\check{C}$ is also an upper bound of $|z_1(x, t)|$ over $\bar{Q}$. Further simplification of (3.22) results in

$$\begin{aligned} |u_x(x, t)| &\leq \check{C} \left(1 + \frac{\exp(\bar{\alpha}x\varepsilon^{-1})}{\varepsilon\alpha^{-1}(\exp(\alpha\varepsilon^{-1}) - 1)} \right) \\ &= \check{C} \left(1 + \frac{\alpha\varepsilon^{-1} \exp(\bar{\alpha}\varepsilon^{-1}) \exp(-\bar{\alpha}(1-x)\varepsilon^{-1})}{\exp(\alpha\varepsilon^{-1}) - 1} \right). \end{aligned} \quad (3.23)$$

Since $0 < \varepsilon < 1$, we can find ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and hence $\exp(\bar{\alpha}\varepsilon^{-1}) < \exp(\bar{\alpha}\varepsilon_0^{-1})$. Thus the estimate (3.23) can be written as

$$|u_x(x, t)| \leq \check{C} \exp(\bar{\alpha}\varepsilon_0^{-1}) \left(1 + \frac{\alpha\varepsilon^{-1} \exp(-\bar{\alpha}(1-x)\varepsilon^{-1})}{\exp(\alpha\varepsilon^{-1}) - 1} \right). \quad (3.24)$$

Let f be a real valued function defined by $f(\alpha) = \exp(\alpha\varepsilon^{-1})$ over the interval $[0, 1]$. Now expanding $f(\alpha)$ in Maclaurin series yields $\alpha \leq (\exp(\alpha\varepsilon^{-1}) - 1)$ and then the estimate (3.24) reduces to

$$|u_x(x, t)| \leq C_5(1 + \varepsilon^{-1} \exp(-\bar{\alpha}(1-x)/\varepsilon)), \quad (3.25)$$

where $C_5 = \check{C} \exp(\bar{\alpha}\varepsilon_0^{-1})$. $\square$

Lemma 3.2.6. *The solution $u(x, t)$ of problem (3.1)–(3.2) satisfies the estimate $|u_{tt}(x, t)| \leq C_6$, for all $(x, t) \in \bar{Q}$.*

Proof. To prove the Lemma we follow the procedures in the proof of Lemma 3.2.4. At ∂Q , $u = 0$ and so $u_{tt} = 0$. From Equation (3.8) and (3.9), we have

$$u_t(x, t) = \begin{cases} 0 & , \text{ for all } t \in [-\tau, 0), \\ f(x, 0) & , \text{ for } t = 0, \end{cases}$$

$$u_{tx}(x, t) = \begin{cases} 0 & , \text{ for all } t \in [-\tau, 0), \\ f_x(x, 0) & , \text{ for } t = 0 \end{cases}$$

and

$$u_{txx}(x, t) = \begin{cases} 0 & , \text{ for all } t \in [-\tau, 0), \\ f_{xx}(x, 0) & , \text{ for } t = 0. \end{cases}$$

Now taking the norm on both sides of the equal sign yields $C^+ \geq |u_{tx}|$ and $C^+ \geq |u_{txx}|$ on ∂Q . The differentiation of equation (3.1) with respect to the time variable t and the rearrangement of terms results in

$$u_{tt} = \varepsilon u_{txx} - a(x)u_{tx} - b_t u - b u_t - c_t u(x, t - \tau) - c u_t(x, t - \tau) + f_t. \quad (3.26)$$

Now taking the norm on both sides of (3.26) yields

$$\begin{aligned} |u_{tt}| &\leq |f_t| + |c_t||u(x, t - \tau)| + |c||u_t(x, t - \tau)| + \varepsilon|u_{txx}| + |a(x)||u_{tx}| + |b_t||u| - |b||u_t| \\ &\leq C_{10}, \text{ for } (x, t) \in \partial Q, \end{aligned}$$

because each term on the right side is bounded. Applying the operator in equation (3.1) on u_{tt} gives

$$\begin{aligned} \mathcal{L}u_{tt} &= u_{ttt} - \varepsilon u_{ttxx} + a u_{ttx} + b(x, t)u_{tt} \\ &= (b u + u_t + a(x)u_x - \varepsilon u_{xx})_{tt} - b_{tt}u - 2b_t u_t. \end{aligned} \quad (3.27)$$

For all $t \in [0, \tau]$, we have

$$\mathcal{L}_\varepsilon u_{tt} = f_{tt} - b_{tt}u - 2b_t u_t, \quad (3.28)$$

when the assumption 3.8 is employed. Each term on the right hand side of (3.28) is bounded, thus $|\mathcal{L}_\varepsilon u_{tt}(x, t)| \leq C_{11}$ follows. Defining the function $\xi_1(x, t) = C_{10} + C_{12}$, where $C_{12} > 0$. For all $(x, t) \in (0, 1) \times [0, \tau]$, we have

$$\xi_1(x, t) = C_{10} + C_{12} \geq C_{10} \geq |u_{tt}(x, t)|.$$

This results in $|u_{tt}(x, t)| \leq \xi_1(x, t) = C_{10} + C_{12} = C$, $\forall (x, t) \in (0, 1) \times [0, \tau]$. Applying the differential operator on $\xi_1(x, t)$ gives

$$\begin{aligned} \mathcal{L}_\varepsilon \xi_1(x, t) &= (C_{10} + C_{12})b(x, t) \geq C_{10}\beta \\ &\geq \beta|u_{tt}(x, t)|. \end{aligned} \quad (3.29)$$

The operator $\mathcal{L}_\varepsilon$ is linear, thus we have

$$|\mathcal{L}_\varepsilon u_{tt}(x, t)| \leq c_0 |u_{tt}(x, t)|, \quad c_0 > 0, \quad \text{for any } (x, t) \in Q \quad (3.30)$$

and

$$|\mathcal{L}_\varepsilon(-u_{tt}(x, t))| = |-\mathcal{L}_\varepsilon u_{tt}(x, t)| = |\mathcal{L}_\varepsilon u_{tt}(x, t)| \leq c_0 |u_{tt}(x, t)|. \quad (3.31)$$

From the estimates (3.30) and (3.31), we get

$$|\mathcal{L}_\varepsilon |u_{tt}|| \leq c_0 |u_{tt}|. \quad (3.32)$$

Since $\mathcal{L}_\varepsilon |u_{tt}(x, t)| \leq |\mathcal{L}_\varepsilon |u_{tt}(x, t)||$, using (3.32), we obtain

$$\mathcal{L}_\varepsilon |u_{tt}(x, t)| \leq c_0 |u_{tt}(x, t)|. \quad (3.33)$$

Considering the estimates (3.33) and choosing $c_0 = \beta$, we obtain

$$\begin{aligned} \mathcal{L}_\varepsilon \xi_1(x, t) &\geq \beta |u_{tt}| \geq \mathcal{L}_\varepsilon |u_{tt}(x, t)|, \\ \mathcal{L}_\varepsilon \xi_1(x, t) - \mathcal{L}_\varepsilon |u_{tt}(x, t)| &\geq 0, \\ \mathcal{L}_\varepsilon (\xi_1(x, t) - |u_{tt}(x, t)|) &\geq 0. \end{aligned}$$

From Lemma 3.2.1, we have $\xi_1(x, t) - |u_{tt}(x, t)| \geq 0$, thus $\xi_1(x, t) \geq |u_{tt}(x, t)|$, and hence $|u_{tt}(x, t)| \leq C_{10} + C_{12} = C_6$. Therefore $|u_{tt}(x, t)| \leq C_6$, $(x, t) \in (0, 1) \times [0, \tau]$,

From (3.27), when $t \in [\tau, 2\tau]$, we obtain

$$\begin{aligned} \mathcal{L}_\varepsilon u_{tt}(x, t) &= f_{tt}(x, t) - c_{tt}(x, t)u(x, t - \tau) - 2c_t(x, t)u_t(x, t - \tau) \\ &\quad - c_t(x, t)u_{tt}(x, t - \tau) - b_t(x, t)u(x, t). \end{aligned} \quad (3.34)$$

Notice that each term on the right hand side of (3.34) is bounded, thus $|\mathcal{L}_\varepsilon u_t(x, t)| \leq C_{13}$ follows. Defining the function $\xi_2(x, t) = C_9 + C_6$, for $C_6 > 0$. Performing similar computations on the domain $t \in [0, \tau]$, yields the result. Further, we use the same argument to derive the bounds on the entire time domain. $\square$

Lemma 3.2.7. *The solution $u(x, t)$ of problem (3.1)–(3.2) satisfies the estimate*

$$|u_{xt}| \leq C_7(1 + \varepsilon^{-1} \exp(-\bar{\alpha}(1 - x)/\varepsilon)) \text{ for all } (x, t) \in \bar{Q}.$$

Proof. We differentiate equation (3.1) with respect to t to obtain

$$u_{tt} - \varepsilon(u_{xt})_x + au_{xt} + b_t u + bu_t = f_t - c_t u(x, t - \tau) - c(x, t)u_t(x, t - \tau) \quad \text{on } \bar{Q}.$$

Now we rewrite it in the form

$$a(x)u_{xt} - \varepsilon(u_{xt})_x = h_2(x, t), \quad (3.35)$$

where $h_2(x, t) = -c_t u(x, t - \tau) - c(x, t)u_t(x, t - \tau) - u_{tt} - b_t u - bu_t + f_t$. Fixing $t \in [0, T]$ and multiplying both sides of (3.35) by the integration factor $A(x) = \int_0^x a(\theta)d\theta$ results in

$$\begin{aligned} \exp(-\varepsilon^{-1}A(x))(u_{xt})_x - \varepsilon^{-1}a(x)\exp(-\varepsilon A(x))u_{xt} &= -\varepsilon^{-1}\exp(-\varepsilon^{-1}A(x))h_2(x, t), \\ \frac{d}{dx}(\exp(-\varepsilon^{-1}A(x))u_{xt}) &= -\varepsilon^{-1}\exp(-\varepsilon^{-1}A(x))h_2(x, t). \end{aligned}$$

Now we integrate this equation from x to 1 to obtain

$$\begin{aligned} \exp(-\varepsilon^{-1}A(1))u_{xt}(1, t) - \exp(-\varepsilon^{-1}A(x))u_{xt}(x, t) &= - \int_x^1 \varepsilon^{-1}h_2(s, t) \exp(-\varepsilon^{-1}A(s))ds \\ u_{xt}(x, t) &= u_{xt}(1, t) \exp(-\varepsilon^{-1}(-A(x) + A(1))) + z_2(x, t). \end{aligned} \quad (3.36)$$

Here $z_2(x, t)$ is of the form

$$z_2(x, t) = \varepsilon^{-1} \int_x^1 h_2(s, t) \exp(-\varepsilon^{-1}A(s) - A(x))ds.$$

To bound $u_{xt}(1, t)$, (3.36) is integrated from x to 1, to obtain

$$u_t(x, t) - u_t(1, t) = -u_{xt}(1, t) \int_x^1 \exp(-\varepsilon^{-1}(A(1) - A(s)))ds - \int_x^1 z_2(s, t)ds.$$

Evaluating this equation at $x = 0$ yields

$$u_{xt}(1, t) = \frac{u_t(1, t) - u_t(0, t) - \int_0^1 z_2(s, t)ds}{\int_0^1 \exp(\varepsilon^{-1}(A(1) - A(s)))ds}. \quad (3.37)$$

Substituting (3.37) into equation (3.36) gives

$$u_{xt}(x, t) = - \frac{\exp(-\varepsilon^{-1}(A(1) - A(x)))}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s)))ds} \int_0^1 z_2(s, t)ds + z_2(x, t). \quad (3.38)$$

Taking the norm on both sides of (3.38) yields

$$\begin{aligned} |u_{xt}(x, t)| &\leq \frac{|\exp(-\varepsilon^{-1}(A(1) - A(x)))|}{|\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s)))ds|} \left| \int_0^1 z_2(s, t)ds \right| + |z_2(x, t)| \\ &\leq \frac{|\exp(-\varepsilon^{-1}A(1))||\exp(\varepsilon^{-1}A(x))|}{|\exp(-\varepsilon^{-1}A(1))||\int_0^1 \exp(\varepsilon^{-1}A(s))ds|} \int_0^1 |z_2(s, t)|ds + |z_2(x, t)| \\ &= \frac{\exp(\varepsilon^{-1}A(x))}{\int_0^1 \exp(\varepsilon^{-1}A(s))ds} \int_0^1 |z_2(s, t)|ds + |z_2(x, t)| \\ &\leq C \left(1 + \frac{\exp(\varepsilon^{-1}\bar{\alpha}x)}{\int_0^1 \exp(\varepsilon^{-1}\alpha s)ds} \right) \end{aligned} \quad (3.39)$$

$$\begin{aligned} &= C \left(1 + \frac{\exp(\varepsilon^{-1}\bar{\alpha}x)}{\varepsilon\alpha^{-1}(\exp(\varepsilon^{-1}\alpha) - 1)} \right) \\ &= C \left(1 + \frac{\alpha\varepsilon^{-1}\exp(\varepsilon^{-1}\bar{\alpha})\exp(-\bar{\alpha}(1-x)\varepsilon^{-1})}{\exp(\varepsilon^{-1}\alpha) - 1} \right). \end{aligned} \quad (3.40)$$

Using the similar argument as in the proof of Lemma 3.2.5, the latter estimates reduces to

$$|u_{xt}(x, t)| \leq C_7(\varepsilon^{-1}\exp(-\bar{\alpha}(1-x)/\varepsilon + 1), \quad (3.41)$$

which completes the proof. $\square$

Lemma 3.2.8. *The solution $u(x, t)$ of problem (3.1)–(3.2) satisfies the estimate $|u_{xx}| \leq C_8(1 + \varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon})$, for all $(x, t) \in \bar{Q}$.*

Proof. Here we differentiate the equation (3.1) with respect to the space variable x and rearrange to obtain

$$-\varepsilon u_{xxx} + au_{xx} = h_3(x, t),$$

where $h_3(x, t) = f_x - bu_x - b_x u - a_x u_x - u_{tx} - c_x u(x, t - \tau) - c(x, t)u_x(x, t - \tau)$, and satisfies the bound

$$|h_3(x, t)| \leq C\varepsilon^{-1},$$

from the Lemmas 3.2.3, 3.2.5 and 3.2.7. Employing the integration factor technique gives

$$u_{xx}(x, t) = u_{xx}(1, t) \exp(-\varepsilon^{-1}(A(1) - A(x))) + z_3(x, t), \quad (3.42)$$

where $z_3(x, t)$ is given by

$$z_3(x, t) = \varepsilon^{-1} \int_x^1 h_3(\gamma, t) \exp(-\varepsilon^{-1}A(\gamma) - A(x)) d\gamma.$$

Now we integrate (3.42) from x to 1, to obtain

$$u_x(1, t) - u_x(x, t) = u_{xx}(1, t) \int_x^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds + \int_x^1 z_3(s, t) ds. \quad (3.43)$$

Evaluating (3.43) at $x = 0$ yields

$$u_{xx}(1, t) = \frac{u_x(1, t) - u_x(0, t) - \int_0^1 z_3(s, t) ds}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds}. \quad (3.44)$$

Substituting (3.44) into equation (3.42) gives

$$u_{xx}(x, t) = -\frac{\exp(-\varepsilon^{-1}(A(1) - A(x)))}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds} \int_0^1 z_3(s, t) ds + z_3(x, t).$$

Taking the norm on both sides of this equation leads to

$$|u_{xx}(x, t)| \leq \frac{|\exp(-\varepsilon^{-1}(A(1) - A(x)))|}{|\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds|} \int_0^1 |z_3(s, t)| ds + |z_3(x, t)|. \quad (3.45)$$

The coefficient of $\int_0^1 |z_3(s, t)| ds$ in (3.45) is equivalent to that of $\int_0^1 |z_2(s, t)| ds$ in (3.39), thus the inequality (3.45) reduces to

$$|u_{xx}(x, t)| \leq \varepsilon^{-1} \exp(-\bar{\alpha}(1 - x)/\varepsilon) \int_0^1 |z_3(x, t)| ds + |z_3(x, t)|, \quad (3.46)$$

by similar argument used in the proof of Lemma 3.2.5. Moreover, notice that $|z_3(x, t)|$ depends on $|h_3(x, t)| \leq C_8 \varepsilon^{-1}$, thus we have that

$$|u_{xx}(x, t)| \leq C_8 \varepsilon^{-2} \exp(-\bar{\alpha}(1 - x)/\varepsilon) + C_8 \varepsilon^{-1},$$

which reduces to

$$|u_{xx}(x, t)| \leq C_8(1 + \varepsilon^{-2} \exp(-\bar{\alpha}(1 - x)/\varepsilon)),$$

since $0 < \varepsilon < 1$. □

Theorem 3.2.1. *The solution and its partial derivatives are such that,*

$$\left| \frac{\partial^{i+j} u(x, t)}{\partial x^i \partial t^j} \right| \leq C(1 + \varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon)), \quad (x, t) \in Q,$$

where i and j are non-negative integers which satisfy $0 \leq i + j \leq 4$.

Proof. The proof of this Theorem for $0 \leq i + j \leq 2$, follows from the Lemmas 3.2.3, 3.2.4, 3.2.5, 3.2.6, 3.2.7, 3.2.8. The proof of the higher order derivatives, that is $2 \leq i + j \leq 4$, can be obtained analogously. $\square$

In the next section, we integrate the continuous problem in space.

3.3 Spatial Discretisation

Below we design a fitted operator finite difference scheme to discretise the problem (3.1)–(3.2) in space. We let $U_i(t) \equiv U(x_i, t)$ be the approximation of $u(x_i, t) \equiv u(x_i, t)$ and perform the spatial discretisation as follows:

$$\begin{aligned} \mathcal{L}_\varepsilon^n \equiv \frac{dU_i(t)}{dt} - \varepsilon \left[\frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\phi_i^2(h, \varepsilon)} \right] + a_i \left[\frac{U_i(t) - U_{i-1}(t)}{h} \right] \\ + b_i(t)U_i(t) = f_i(t) - c_i(t)U_i(t - \tau), \end{aligned} \quad (3.47)$$

along with the semi-discrete boundary and the initial conditions

$$U_0(t) = \eta_0(t), \quad U_n(t) = \eta_1(t), \quad t \in [0, T], \quad U_i(t) = \varphi_i(t), \quad (x_i, t) \in \bar{\Omega}^n \times [-\tau, 0]. \quad (3.48)$$

The denominator function $\phi_i^2(h, \varepsilon)$ takes the form

$$\phi_i^2(h, \varepsilon) = \frac{\varepsilon h}{a_i} \left(\exp \left(\frac{a_i h}{\varepsilon} \right) - 1 \right). \quad (3.49)$$

The fitted operator finite difference scheme (3.47)–(3.49) is equivalent to the initial value problem

$$U'(t) + A(t)U(t) = F(t), \quad (3.50)$$

where $U(t)$ and $F(t)$ are vectors of size $(n-1)$ and $A(t)$ is a tridiagonal matrix of size $(n-1) \times (n-1)$. The entries of the coefficient matrix $A(t)$ are of the form

$$A_{ij}(t) = \begin{cases} r_i^- &= -\varepsilon \phi_i^{-2}(h, \varepsilon) - a_i h^{-1}, & i \in \{2, 3, \dots, n-1\}, j = i-1 \\ r_i^c &= 2\varepsilon \phi_i^{-2}(h, \varepsilon) + a_i h^{-1} + b_i(t), & i \in \{1, 2, \dots, n-1\}, i = j \\ r_i^+ &= -\varepsilon \phi_i^{-2}(h, \varepsilon), & i \in \{1, 2, \dots, n-2\}, j = i+1. \end{cases}$$

The column vectors $F(t)$ and $U(t)$ have the entries

$$\begin{aligned} F_1(t) &= g_1(t) - r_0^- \eta_0(t), \\ F_i(t) &= g_i(t), \quad i = 2, 3, \dots, n-1, \\ F_{n-1}(t) &= g_{n-1}(t) - r_{n-1}^+ \eta_1(t), \end{aligned}$$

where $g_i(t)$ is given by

$$g_i(t) = f_i(t) - c_i(t)U_i(t - \tau), \quad (3.51)$$

and

$$U(t) = [U_1(t), U_2(t), \dots, U_{n-1}(t)].$$

Notice that $U'(t)$ is the first derivative of the vector $U(t)$ in time.

The entries of the coefficient matrix suggest that, it is a strictly diagonally dominant matrix which makes it invertible in the maximum norm. Therefore, the semi-discrete Initial Value Problem (IVP) (3.50) has a unique solution and hence satisfies a semi-discrete maximum principle which is stated in the Lemma 3.4.1 below. This result is established from Clavero and Jorge (2015) and it is needed in the analysis of the scheme (3.47)–(3.49).

3.4 Stability of the Scheme

Here we establish the discrete stability result for the semi-discrete scheme.

Lemma 3.4.1. *Semi-discrete maximum principle*

The operator $\mathbf{L}_\varepsilon^n$ defined by the scheme

$$\begin{aligned} \mathbf{L}_\varepsilon^n &\equiv \frac{dU_i(t)}{dt} - \varepsilon \left[\frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\phi_i^2(h, \varepsilon)} \right] + a_i \left[\frac{U_i(t) - U_{i-1}(t)}{h} \right] \\ &\quad + b_i(t)U_i(t) + c_i(t)U_i(t - \tau), = f_i(t), \quad 1 \leq i \leq n-1, \\ U_0(t) &= \eta_0(t), \quad U_n(t) = \eta_1(t), \quad t \in [0, T], \quad U_i(t) = \varphi_i(t), \quad (x_i, t) \in \bar{\Omega}^n \times [-\tau, 0], \end{aligned}$$

satisfies a semi-discrete maximum principle. That is, if $\xi(x_i, t)$ is a mesh function defined on $\bar{\Omega}^n \times [0, T]$ and satisfies $\xi(x_i, t) \geq 0$, $\forall (x_i, t) \in \partial\Omega^n \times [-\tau, 0]$ and $\mathcal{L}_\varepsilon^n \xi(x_i, t) \geq 0$, $\forall (x_i, t) \in \Omega^n \times (0, T]$. Then $\xi(x_i, t) \geq 0$, $\forall (x_i, t) \in \bar{\Omega}^n \times [0, T]$.

Proof. First the above mentioned scheme is represented in the matrix form

$$\tilde{U}'(t) + \tilde{A}(t)\tilde{U}(t) = \tilde{F}(t). \quad (3.52)$$

The entries of $\tilde{U}'(t) + \tilde{A}(t)\tilde{U}(t) = \tilde{F}(t)$ are given by

$$[\tilde{A}_{i,j}(t)] = \begin{cases} r_i^- &= -\varepsilon\phi_i^{-2}(h, \varepsilon) - a_i h^{-1}, & i = 2, 3, \dots, n-1, \\ & j = i-1 \\ r_i^c &= 2\varepsilon\phi_i^{-2}(h, \varepsilon) + a_i h^{-1} + b_i(t) + c_i(t) & i = 1, 2, \dots, n-1, \\ & i = j \\ r_i^+ &= -\varepsilon\phi_i^{-2}(h, \varepsilon), & i = 1, 2, \dots, n-2, \\ & j = i+1, \end{cases}$$

$$\tilde{U}(t) = [U_0(t), U_1(t), \dots, U_{n-1}(t), U_n(t)]^T$$

and

$$\tilde{F}(t) = [\eta_0(t), \mathcal{L}_\varepsilon^n U_1(t), \dots, \mathcal{L}_\varepsilon^n U_{n-1}(t), \eta_1(t)]^T.$$

From the entries of $\tilde{A}(t)$ it is evident that

$$r_i^- < 0, \quad r_i^+ < 0 \quad \text{and} \quad r^- + r^c + r^+ > 0,$$

thus $\tilde{A}(t)$ is an M-matrix and therefore its inverse exit and it is positive. Therefore the spatial discretisation process is stable. $\square$

Lemma 3.4.2. *The solution of the semi-discrete problem is such that*

$$|u_i(t)| \leq \beta^{-1} \max_{(x_i, t) \in \Omega \times [0, T]} |\mathbf{L}_\varepsilon u_i(t)| + \max_{(x_i, t) \in \bar{\Omega} \times [0, T]} \max(|\varphi_i(t)|, |\eta_0(t)|, |\eta_1(t)|).$$

Proof. Let

$$z = \beta^{-1} \max_{(x_i, t) \in \bar{\Omega} \times [0, T]} |\mathbf{L}_\varepsilon u_i(t)| + \max_{(x_i, t) \in \bar{\Omega} \times [0, T]} \max(|\varphi_i(t)|, |\eta_0(t)|, |\eta_1(t)|),$$

and consider the mesh function $\Psi_i^\pm(t)$ defined by

$$\Psi_i^\pm(t) = z \pm u_i(t).$$

At $x_i \in \partial\Omega^n$ and at $t \in [-\tau, 0]$, we have

$$\begin{aligned} \Psi_0^\pm(t) &= z \pm u_i(t) = z \pm \eta_0(t) \geq 0, \\ \Psi_n^\pm(t) &= z \pm u_i(t) = z \pm \eta_n(t) \geq 0, \\ \Psi_i^\pm(t) &= z \pm u_i(t) = z \pm \varphi_i(t) \geq 0. \end{aligned}$$

On the domain $(x_i, t) \in \Omega^n \times (0, T]$,

$$\begin{aligned} \mathbf{L}_\varepsilon^n \Psi_i^\pm(t) &= (z + u_i(t))' - \varepsilon \left(\frac{z + u_{i+1}(t) - 2(z + u_i(t)) + z \pm u_{i-1}(t)}{\phi_i^2(\varepsilon, h)} \right) + b_i(t)(z \pm u_i(t)) \\ &\quad + a_i \left(\frac{z \pm u_i(t) - z \pm u_{i-1}(t)}{h} \right) + c_i(t)(z \pm u_i(t - \tau)) \\ &= z' + b_i(t)z + c_i(t)z \pm \mathbf{L}_\varepsilon^n u_i(t) \geq 0, \end{aligned}$$

provided $z' > 0$ and $c_i(t)z > 0$. From Lemma 3.4.1, $\Psi_i^\pm(t) \geq 0$, $\forall (x_i, t) \in \bar{\Omega}^n \times [0, T]$, as required. $\square$

3.5 Error Estimate

This section is concerned with the error analysis of the spatial variable.

Lemma 3.5.1. *The error for the spatial semi-discretisation process satisfies the bound*

$$\max_{i \in [0, n], t \in [0, T]} |u(x_i, t) - U(x_i, t)| \leq Ch,$$

where $U_i(t)$ and $u_i(t)$ are the numerical and the exact solutions of the initial value problems (3.50) and problem ((3.1)–(3.2)) respectively.

Proof. The local truncation error of the spatial discretisation is of the form

$$\begin{aligned} \mathbf{L}_\varepsilon^n(u_i(t) - U_i(t)) &= \mathbf{L}_\varepsilon^n u_i(t) - \mathbf{L}_\varepsilon^n U_i(t) = \mathbf{L}_\varepsilon^n u_i(t) - \mathbf{L}_\varepsilon u_i(t) \\ &= -\varepsilon \left(\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\phi_i^2(\varepsilon, h)} \right) + a_i \left(\frac{u_i(t) - u_{i-1}(t)}{h} \right) \\ &\quad + \varepsilon(u_{xx}(t))_i - a_i(u_x(t))_i \\ &= \varepsilon(u_{xx}(t))_i - \left(\frac{\varepsilon}{h^2} - \frac{a_i}{2h} \right) \left(h^2(u_{xx}(t))_i + \frac{h^4}{12}(u_{xxxx}(t))_i \xi_i \right) \\ &\quad - \frac{a_i h}{2}(u_{xx}(t))_i, \quad \xi_i \in (u_{i+1}, u_{i-1}). \end{aligned}$$

Using appropriate Taylor series expansions yields

$$\begin{aligned} \mathbf{L}_\varepsilon^n(u_i(t) - U_i(t)) &= \varepsilon \frac{h^2}{12}(u_{xxxx}(t))_i \xi_i + \frac{a_i' h^2}{2}(u_{xx}(t))_i + \frac{a_i' h^4}{24}(u_{xxxx}(t))_i \xi_i \\ &\quad - \frac{a_i h}{2}(u_{xx}(t))_i. \end{aligned}$$

Application of Theorem 3.2.1 and noticing that the exponential terms vanish as $\varepsilon \rightarrow 0$, see Lemma 2.4.3 for proof, results in

$$|\mathbf{L}_\varepsilon^n(u_i(t) - U_i(t))| \leq \varepsilon \frac{h^2}{12} \xi_i + \frac{a_i' h^2}{2} + \frac{a_i' h^4}{24} \xi_i - \frac{a_i h}{2}.$$

From the relation $h > h^2 > \dots, h^n$, the bounds on the coefficients,

$$|\mathbf{L}_\varepsilon^n(u_i(t) - U_i(t))| \leq Ch,$$

follows. Now let $\mathcal{R}_i(t)$ be a mesh function defined by

$$\mathcal{R}_i(t) = \begin{cases} Ch, & 0 \leq i \leq n, \quad 0 \leq t \leq T, \\ 0, & 0 \leq i \leq n, \quad -\tau \leq t \leq 0, \end{cases}$$

Notice that $\mathcal{R}_0(t) = \mathcal{R}_n(t) = Ch$, and

$$\begin{aligned} \mathbf{L}_\varepsilon^n \mathcal{R}_i(t) &\equiv -\frac{\varepsilon}{\phi_i^2}(\mathcal{R}_{i+1}(t) - 2\mathcal{R}_i(t) + \mathcal{R}_{i-1}(t)) + \frac{a_i}{h}(\mathcal{R}_i(t) - \mathcal{R}_{i-1}(t)) + b_i(t)\mathcal{R}_i(t) \\ &\quad + c_i(t)\mathcal{R}_i(t - \tau) \\ &= b_i(t)\mathcal{R}_i(t) + c_i(t)\mathcal{R}_i(t - \tau). \end{aligned}$$

Suppose $\mathcal{H}_i(t)$ is also another function which given is by

$$\mathcal{H}_i(t) = \mathcal{R}_i(t) - |u_i(t) - U_i(t)|$$

At $i = 0$ and $i = n$, we have

$$\mathcal{H}_0(t) = \mathcal{R}_0(t) - |u_0(t) - U_0(t)| = Ch > 0,$$

and

$$\mathcal{H}_n(t) = \mathcal{R}_n(t) - |u_n(t) - U_n(t)| = Ch > 0,$$

respectively. Further on $1 \leq i \leq n-1$, we have

$$\begin{aligned} \mathbf{L}_\varepsilon^n \mathcal{H}_i(t) &= \mathbf{L}_\varepsilon^n \mathcal{R}_i(t) - \mathbf{L}_\varepsilon^n |u_i(t) - U_i(t)| \\ &= b_i(t) \mathcal{R}_i(t) + c_i(t) \mathcal{R}_i(t - \tau) - \mathbf{L}_\varepsilon^n |u_i(t) - U_i(t)| > 0. \end{aligned}$$

From Lemma 3.4.1, we obtain

$$\mathcal{H}_i(t) = \mathcal{R}_i(t) - |u_i(t) - U_i(t)| \geq 0,$$

and hence

$$|u_i(t) - U_i(t)| \leq \mathcal{R}_i(t),$$

which leads to

$$|u_i(t) - U_i(t)| \leq Ch.$$

□

Now Richardson extrapolation is used to accelerate the convergence of the scheme (3.47)–(3.49).

3.6 Richardson Extrapolation

Let $\Omega^n \subset \Omega^{2n}$ where Ω^n is as defined earlier and Ω^{2n} is the mesh obtained from bisecting the stepsize h . Now we denote the numerical solution obtained from Ω^{2n} by $\tilde{U}_i(t)$. From Lemma 3.5.1, we have

$$u_i(t) - U_i(t) \leq Ch + R_i^n(t), \quad (x_i, t) \in \Omega^n \times (0, T] \quad (3.53)$$

$$u_i(t) - \tilde{U}_i(t) \leq C \left(\frac{h}{2} \right) + R_i^{2n}(t), \quad (x_i, t) \in \Omega^{2n} \times (0, T], \quad (3.54)$$

where $R_i^\mu(t)$ is the remainder term of the error. To derive the extrapolation formula, we subtract the inequality 3.54 from 3.53 to obtain formula

$$u_i(t) - U_i(t) - 2(u_i(t) - \tilde{U}_i(t)) = R_i^n(t) - 2R_i^{2n}(t), \quad (3.55)$$

$$\mathcal{U}_i(t) = 2\tilde{U}_i(t) - U_i(t). \quad (3.56)$$

The next lemma provides the error associated with the application of Richardson extrapolation.

Lemma 3.6.1. *The error associated with the Richardson extrapolation process is of the form*

$$\max_{i \in [0, n], t \in [0, T]} |u_i(t) - \mathcal{U}_i(t)| \leq Ch^2,$$

where $\mathcal{U}_i(t)$ and $u_i(t)$ are the numerical and the exact solutions of the IVPs 3.50.

Proof. To estimate the error after extrapolation, we define it by

$$\mathbf{L}_\varepsilon^n(u_i(t) - \mathcal{U}_i(t)) = 2\mathbf{L}(u_i(t) - \tilde{U}_i(t)) - \mathbf{L}_\varepsilon(u_i(t) - U_i(t)) = e_i^{ext}.$$

Using appropriate Taylor series expansions leads to

$$\begin{aligned} e_i^{ext} &= 2 \left[\varepsilon(u_{xx}(t))_i - \left(\frac{4\varepsilon}{h^2} - \frac{a_i}{h} \right) \left(\frac{h^2}{4}(u_{xx}(t))_i + \frac{h^4}{16 \times 12}(u_{xxxx}(t))_i \kappa_i \right) - \frac{a_i}{4}h(u_{xx}(t))_i \right] \\ &\quad - \mathbf{L}_\varepsilon(u_i(t) - U_i(t)), \quad \kappa_i \in (u_{i+1}, u_{i-1}) \\ &= \frac{\varepsilon h^2}{24}(u_{xxxx}(t))_i + \frac{a'_i h^2}{4}(u_{xx}(t))_i + \frac{a'_i h^2}{192}(u_{xxxx}(t))_i - \frac{a_i h}{4}(u_{xx}(t))_i \\ &\quad - \mathbf{L}_\varepsilon(u_i(t) - U_i(t)), \quad \kappa_i \in (u_{i+1}, u_{i-1}) \\ &\leq Ch^2, \end{aligned}$$

where the relation $h^2 > h^4$, the bounds on the derivatives been used. Following how the maximum principle was employed in Section 3.5, the result follows. $\square$

Next, we integrate the Initial Value Problems (IVPs) (3.50) in time.

3.7 Time Discretisation

In this section we discretise the IVPs (3.50) by the Crank Nicolson finite difference scheme. We sub-divide the domain $(0, T]$ into the uniform sub-intervals

$$t_0 = 0 \quad t_k = t_0 + k\Delta t, \quad k = 1(1)K-1, \quad \Delta t = t_k - t_{k-1}, \quad t_K = T.$$

The variable K is the number of sub-intervals and it is chosen such that the step-size Δt meets the condition $\tau = s\Delta t$, where s is a nonnegative integer. Also, we denote the approximation of $u_i(t_k) := u_i^k$ by U_i^k , however for notational simplicity, we drop the subscript index. Now the fully discrete scheme is written as

$$\begin{aligned} \frac{U^{k+1} - U^k}{\Delta t} + A(t_{k+\frac{1}{2}}) \left(\frac{U^{k+1} + U^k}{2} \right) &= F(t_{k+\frac{1}{2}}), \quad \text{for } k = 1, \dots, K, \\ U^0 &= \varphi(t_k), \quad -\tau \leq t_k \leq 0. \end{aligned} \quad (3.57)$$

Rearranging (3.57) yields

$$\left[I + \frac{\Delta t}{2} A(t_{k+\frac{1}{2}}) \right] U^{k+1} = G^{k+1} \quad (3.58)$$

Here G^{k+1} is of the form

$$\Delta t \left[F(t_{k+\frac{1}{2}}) \right] + \left[I - \frac{\Delta t}{2} A(t_{k+\frac{1}{2}}) \right] U^k,$$

and $F(t_{k+\frac{1}{2}})$ is given by

$$F(t_{k+\frac{1}{2}}) = \frac{g^{k+1} + g^k}{2},$$

with the definition of g given by equation (3.51). The delay term $U_i(t_k - \tau)$ is approximated by the function

$$H_i^k = \begin{cases} \varphi(x_i, t_k - \tau), & t_k < \tau, \\ U(x_i, t_{k-s}), & t_k > \tau. \end{cases}$$

Lemma 3.7.1. *The local truncation error corresponding to the integration, with respect to the time variable, satisfies*

$$\|e^k\| \leq C\Delta t^3,$$

where the constant C does not depend on ε and K .

Proof. We define the local truncation error as

$$e^{k+1} = u(t_{k+1}) - U^{k+1}, \quad (3.59)$$

where $u(t_{k+1})$ and U^{k+1} denote the exact solution and the numerical solution of (3.47)-(3.48), respectively.

Truncated Taylor series representations, of order 3, for $u(t_{k+1})$ and $u(t_k)$ are given by

$$u(t_{k+1}) = u(t_{k+1/2}) + \frac{\Delta t}{2}u_t(t_{k+1/2}) + \frac{\Delta t^2}{8}u_{tt}(t_{k+1/2}) + \frac{\Delta t^3}{48}u_{ttt}(t_{k+1/2}) + \mathcal{O}\Delta t^4, \quad (3.60)$$

$$u(t_k) = u(t_{k+1/2}) - \frac{\Delta t}{2}u_t(t_{k+1/2}) + \frac{\Delta t^2}{8}u_{tt}(t_{k+1/2}) - \frac{\Delta t^3}{48}u_{ttt}(t_{k+1/2}) + \mathcal{O}\Delta t^4. \quad (3.61)$$

From (3.60) and (3.61) we have

$$\frac{u(t_{k+1}) - u(t_k)}{\Delta t} = u_t(t_{k+1/2}) + \mathcal{O}(\Delta t^2). \quad (3.62)$$

From the semi-discrete problems, equation (3.62) can be written as

$$\frac{u(t_{k+1}) - u(t_k)}{\Delta t} = [F(t_{k+1/2}) - A(t_{k+1/2})u(t_{k+1/2})] + \mathcal{O}\Delta t^2,$$

Further simplification yields

$$\left[I + \frac{\Delta t}{2}A(t_{k+1/2}) \right] u(t_{k+1}) = G(t_{k+1}) + \mathcal{O}\Delta t^3, \quad (3.63)$$

where $G(t_{k+1})$ is given by

$$\Delta t[(F(t_{k+1/2}))] + \left[I - \frac{\Delta t}{2}A(t_{k+1/2}) \right] u(t_k).$$

From Equations (3.58) and (3.63) the local error e^{k+1} which is given by Equation (3.59) satisfies

$$\left[I + \frac{\Delta t}{2}A(t_{k+1}) \right] e^{k+1} = \mathcal{O}(\Delta t)^3. \quad (3.64)$$

Since $A(t)$ is a non-singular matrix, we obtain

$$\|e^{k+1}\| \leq C(\Delta t)^3.$$

□

From Lemma 3.7.1 we obtain the global error $\mathcal{E}^K$ by

$$\|\mathcal{E}^K\| \leq \left\| \sum_{k=1}^K e^k \right\| \leq CK\Delta t^3 = C(\Delta t)^2. \quad (3.65)$$

Lemma 3.7.2. *The global error $\mathcal{E}^K$ of the time discretisation, satisfies*

$$\|\mathcal{E}^K\| \leq C(\Delta t)^2.$$

The following theorem summarizes the main result in this chapter.

Theorem 3.7.1. *(Uniform Convergence)*

Let $u \in C^{4,2}(\bar{Q})$ be the exact solution of the evolution problem (3.1)-(3.2) and let U be the approximate solution. We have the error of the numerical methods; fitted operator finite difference method and the Crank Nicolson finite difference scheme as

$$\max_{(i,k) \in Q^{n,K}} |u(x_i, t_k) - U(x_i, t_k)| \leq C(h + \Delta t^2),$$

before extrapolation and

$$\max_{(i,k) \in Q^{n,K}} |u(x_i, t_k) - \mathcal{U}(x_i, t_k)| \leq C(h^2 + \Delta t^2),$$

after extrapolation.

Proof. The proof of this theorem follows from the Lemmas 3.5.1, 3.6.1 and 3.7.2. $\square$

3.8 Numerical Results

Here the efficiency of the numerical methods is validated by the test problem below. In example, the largest pointwise error as well as the numerical rate of convergence are calculated. From these computations, we obtain the ε -uniform error as well as the ε -uniform numerical rate of convergence. Results obtained from all computations will be displayed in the form of tables.

Example 3.8.1. *Consider the parabolic equation*

$$u_t - \varepsilon u_{xx} + (2 - x^2)u_x + xu = u(x, t - 1) + 10t^2 \exp(-t)x(1 - x), \quad (x, t) \in (0, 1) \times (0, 2],$$

with initial condition

$$u(x, t) = 0, \quad (x, t) \in [0, 1] \times [-1, 0]$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \in [0, 2].$$

The above problem, whose exact solution is unknown, was considered in Das and Natesan (2015) using a hybrid scheme on Shishkin mesh. We are going to use a different approach and obtain similar result before extrapolation. The maximum pointwise error $E_{n,\Delta t}^1$ and $E_{n,\Delta t}^2$, before and after extrapolation will be computed using the double mesh principle given by

$$E_{n,\Delta t}^1 = \max_{i \in [0,n], k \in [0,K]} |U_{i,k}^{\varepsilon,n,K} - U_{i,k}^{\varepsilon,2n,K}|,$$

$$E_{n,\Delta t}^2 = \max_{i \in [0,n], k \in [0,K]} |\mathcal{U}_{i,k}^{\varepsilon,2n,K} - \mathcal{U}_{i,k}^{\varepsilon,2n,2K}|,$$

respectively, where $U_{i,k}^{\varepsilon,n,K}$ and $\mathcal{U}^{\varepsilon,n,K}$ are numerical solutions on the mesh $\mu(n, K)$ and $U_{i,k}^{\varepsilon,2n,K}$ is obtained using the mesh $\mu(2n, K)$. Whilst $\mathcal{U}^{\varepsilon,2n,2K}$ is on the mesh $\mu(2n, 2K)$. The $2n$ and K were used in the computations before extrapolation to be able to balance the error. This is because the numerical methods considered here is of first order in space and second order in time. For the calculation after the extrapolation, $2n$ and $2K$ were used because both methods converge at a second order rate.

Also the numerical rate of convergence uses the formulae

$$r_1 = \log_2 \left(\frac{E_{n,\Delta t}^1}{E_{2n,\Delta t}^1} \right) \quad \text{and} \quad r_2 = \log_2 \left(\frac{E_{n,\Delta t}^2}{E_{2n,\Delta t/2}^2} \right).$$

Further, the ε -uniform error before and after extrapolation is obtained via the formulae

$$E^1 = \max_{0 < \varepsilon \leq 1} E_{n,\Delta t}^1, \quad E^2 = \max_{0 < \varepsilon \leq 1} E_{n,\Delta t}^2,$$

and the ε -uniform rate of convergence by

$$R_1 = \max_{0 < \varepsilon \leq 1} r_1, \quad R_2 = \max_{0 < \varepsilon \leq 1} r_2.$$

Table 3.1: Maximum pointwise error and rate of convergence for Example 3.8.1 before extrapolation.

ε	n=16	32	64	128	256	512	1024
	m=16	16	16	16	16	16	16
10^{-3}	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.16E-04	3.97E-04	1.55E-04
	0.9195	0.9582	0.9799	1.0248	1.2069	1.3534	
10^{-4}	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.38E-04	4.71E-04	2.36E-04
	0.9195	0.9582	0.9794	0.9898	0.9949	0.9975	
10^{-5}	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.38E-04	4.71E-04	2.36E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-9}	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.38E-04	4.71E-04	2.36E-04
	0.9195	0.9582	0.9794	0.9898	0.9949	0.9974	
10^{-10}	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.38E-04	4.71E-04	2.36E-04
	0.9195	0.9582	0.9794	0.9898	0.9949	0.9974	
E_1	1.35E-02	7.14E-03	3.67E-03	1.86E-03	9.38E-04	4.71E-04	2.36E-04
R_1	0.9195	0.9582	0.9794	0.9898	0.9949	0.9974	

Table 3.2: Maximum pointwise error and rate of convergence for Example 3.8.1 after extrapolation.

ε	n=m=16	32	64	128	256	512	1024
10^{-3}	7.02E-03	1.97E-03	5.22E-04	2.49E-04	2.55E-04	1.10E-04	3.26E-05
	1.8376	1.9115	1.0700	-0.0369	1.2084	1.7614	
10^{-4}	7.02E-03	1.97E-03	5.32E-04	1.41E-04	3.65E-05	9.01E-06	2.03E-05
	1.8374	1.8858	1.9171	1.9468	2.0188	-1.1692	
10^{-5}	7.02E-03	1.97E-03	5.32E-04	1.41E-04	3.65E-05	9.33E-06	2.36E-06
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-9}	7.02E-03	1.97E-03	5.32E-04	1.41E-04	3.65E-05	9.33E-06	2.36E-06
	1.8374	1.8858	1.9171	1.9467	1.9694	1.9835	
10^{-10}	7.02E-03	1.97E-03	5.32E-04	1.41E-04	3.65E-05	9.33E-06	2.36E-06
	1.8374	1.8858	1.9171	1.9467	1.9694	1.9835	
E_2	7.02E-03	2.30E-03	7.52E-04	2.04E-04	3.65E-05	1.27E-05	2.030E-05
R_2	1.8374	1.8858	1.9171	1.9468	2.0188	2.0105	

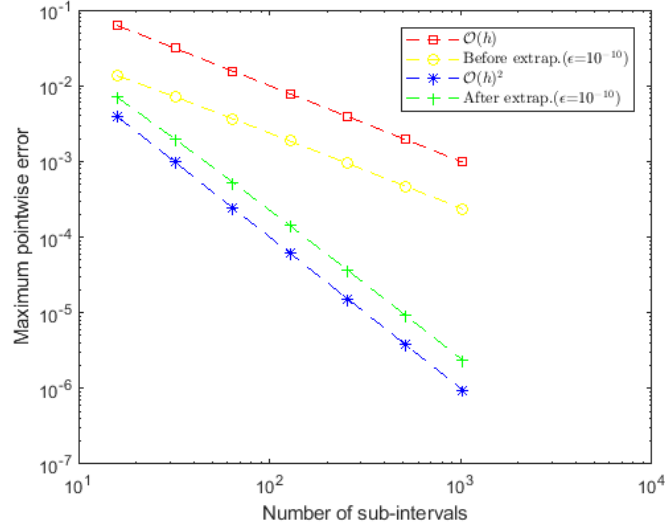


Figure 3.1: Loglog plot of the maximum pointwise errors of Example 3.8.1 before extrapolation.

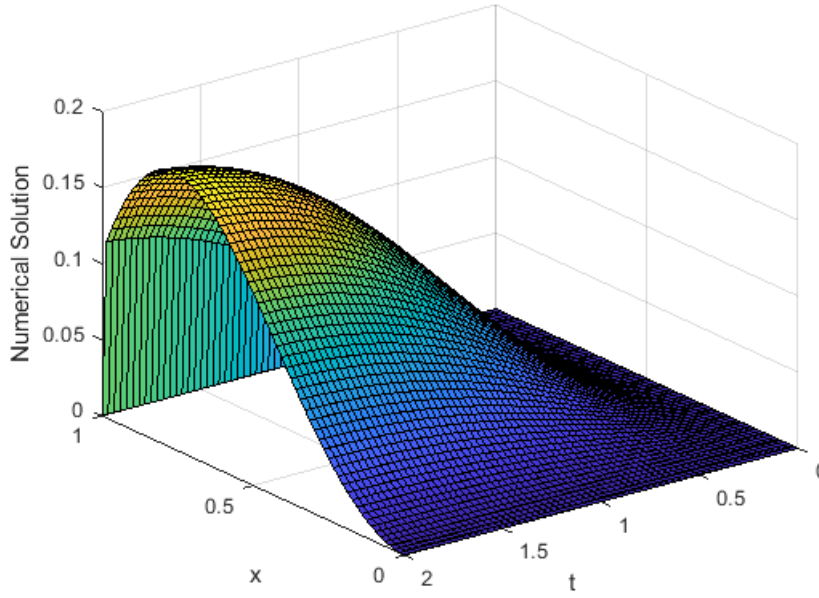


Figure 3.2: Surface plot of the numerical solution of Example 3.8.1 with $n = 16$ and $\varepsilon = 10^{-24}$.

3.9 Conclusion

A numerical method for solving a time-dependent convection diffusion problem with time delay was proposed in this chapter. This method employed the fitted operator finite difference scheme to discretise the spatial variable and the Crank Nicholson finite difference scheme for the time variable. The analysis of the proposed numerical resulted in a first order accuracy in space and a second order in time. Using Richardson extrapolation technique, the accuracy in the spatial direction was improved into second order. To demonstrate the reliability of the method, it was tested on a problem and the results were displayed in Tables 3.1–3.2. Table 3.1 shows the results obtained before extrapolation whilst Table 3.2 indicates the results after extrapolation. Both tables displayed the maximum pointwise error as well as the numerical rate of convergence for several values of ε , m and n . From the Tables, the uniform rate of convergence as well as the first and second order rate of convergence can be observed. Figures 3.1 and 3.2 also displayed a log-log plot of the maximum pointwise error against n and the surface plot of the numerical solution. Clearly the errors approach zero as the stepsizes approach zero in 3.1, whilst the right boundary layer in the solution can be observed from the surface plot of the numerical solution 3.2. Therefore we conclude that the numerical results are in accordance with Theorem 3.7.1.

In the next chapter, we design a similar scheme to solve a singularly perturbed Volterra integro-differential equation.

Chapter 4

A Second Order Finite Difference Scheme for Singularly Perturbed Volterra Integro-Differential Equation

4.1 Introduction

Fitted operator finite difference scheme has been designed along with either the backward Euler finite difference scheme or the Crank Nicolson finite difference scheme to solve singularly perturbed problems in the previous chapters. In this chapter, the scheme is designed along with the Simpson's composite rule to solve a singularly perturbed Volterra integro-differential equation.

Consider the following Singularly Perturbed Volterra Integro-differential Equation (SP-VIDE)

$$\mathcal{L}_\varepsilon u(t) = \int_0^t K(t, s)u(s)ds + \varepsilon u'(t) + a(t)u(t) = f(t), \quad t \in \Omega = (0, T], \quad (4.1)$$

$$u(0) = \mu, \quad (4.2)$$

where ε ($\varepsilon \in (0, 1]$) is the perturbation parameter. The source term f and the coefficient functions a , K are smooth. Moreover, the coefficient functions satisfy $a(t) \geq \alpha$ and $K(t, s) \geq \bar{K}$, $(t, s) \in (\Omega \times \Omega)$ and μ is a given constant. Under these assumptions, problem (4.1)–(4.2) admits a unique solution which exhibits a boundary layer in the neighbourhood of $t = 0$. As indicated earlier, we design, analyse and implement a uniformly convergent finite difference scheme to solve (4.1)–(4.2). The scheme uses a non-standard finite difference scheme to solve the differential part of the Volterra integro-differential equation whilst the Simpson's rule is used to for the integral part.

The rest of the content of this chapter is organised as follows: Some properties of the

solution are studied in Section 4.2. In Section 4.3, the numerical method is presented, followed by the stability and the convergence of the scheme in Sections 4.4 and 4.5 respectively. Richardson extrapolation and its analysis are presented in Section 4.6 to improve the accuracy of the method. Numerical simulations are carried out in Section 4.7, while conclusion and discussions of the main results are provided in Section 4.8.

4.2 Bounds on the Solution and its Derivatives

Below we present an estimate of the analytical solution to SPVIDE (4.1)–(4.2). The derivatives of this solution are also provided. The ideas of Amiraliyev and Şevgin (2006) are borrowed to establish the proofs of these bounds.

Theorem 4.2.1. *Let $u(t)$, be the solution of the continuous problem (4.1)–(4.2). The following estimate holds*

$$\|u\|_{\infty, \Omega} \leq C, \quad (4.3)$$

$$|u^k(t)| \leq C \left(1 + \varepsilon^{-k} \exp \left(\frac{-\alpha t}{\varepsilon} \right) \right), \quad t \in \Omega, \quad k = 1, 2, 3, 4. \quad (4.4)$$

Proof. To prove the inequality 4.3, equation (4.1) is written in the standard form as

$$\mathcal{P}(t) = \varepsilon^{-1} a(t) u(t) + u'(t), \quad (4.5)$$

where

$$\mathcal{P}(t) = \varepsilon^{-1} f(t) - \varepsilon^{-1} \int_0^t K(t, s) u(s) ds. \quad (4.6)$$

Multiplying both sides of equation (4.5) by the integration factor $\exp(\varepsilon^{-1} A(t))$, and simplifying yields

$$\frac{d}{dt} (u(t) \exp(A(t)\varepsilon^{-1})) = \exp(A(t)\varepsilon^{-1}) \mathcal{P}(t), \quad (4.7)$$

where $A(t) = \int_0^t a(\kappa) d\kappa$. Integrating equation (4.7), from 0 to t , gives

$$\begin{aligned} \exp(A(t)\varepsilon^{-1}) u(t) - \exp(A(0)\varepsilon^{-1}) u(0) &= \int_0^t \exp(A(\kappa)\varepsilon^{-1}) \mathcal{P}(\kappa) d\kappa, \\ u(t) &= \int_0^t \mathcal{P}(\kappa) \exp(\varepsilon^{-1} [A(\kappa) - A(t)]) d\kappa + u(0) \exp(-\varepsilon^{-1} A(t)). \end{aligned} \quad (4.8)$$

Now taking the norm on both sides of equation (4.8), while noticing that $K(s, t) \geq \bar{K}$, $a(t) \geq \alpha$ and $\mathcal{P}(t)$ is given by equation (4.6), yields

$$\begin{aligned} |u(t)| &\leq |u(0)| \exp(-\varepsilon^{-1} \alpha t) + \varepsilon^{-1} \int_0^t |f(\kappa)| \exp(\varepsilon^{-1} \alpha(\kappa - t)) d\kappa \\ &\quad - \varepsilon^{-1} \bar{K} \int_0^t \left[\int_0^\kappa |u(s)| ds \right] \exp(\varepsilon^{-1} \alpha(\kappa - t)) d\kappa. \end{aligned} \quad (4.9)$$

Simplifying equation (4.9), gives

$$\begin{aligned}
|u(t)| &\leq |u(0)| \exp(-\varepsilon^{-1}\alpha t) + \frac{1}{\varepsilon} \int_0^t |f(\kappa)| \exp(\alpha(\kappa - t)\varepsilon^{-1}) d\kappa \\
&\quad - \frac{1}{\varepsilon} \bar{K} \exp(-\alpha t \varepsilon^{-1}) \int_0^t |u(s)| \left(\int_\kappa^t \exp(\varepsilon^{-1}\alpha \kappa) d\kappa \right) ds \\
|u(t)| &\leq |u(0)| \exp(-\varepsilon^{-1}\alpha t) + \varepsilon^{-1} \int_0^t |f(\kappa)| \exp(\varepsilon^{-1}\alpha(\kappa - t)) d\kappa \\
&\quad - \varepsilon^{-1} \bar{K} \exp(-\varepsilon^{-1}\alpha t) \int_0^t |u(s)| \left(\int_0^t \exp(\varepsilon^{-1}\alpha \kappa) d\kappa \right) ds.
\end{aligned}$$

Further simplification results in

$$\begin{aligned}
|u(t)| &\leq |\mu| \exp(-\varepsilon^{-1}\alpha t) + \alpha^{-1} \|f\| (1 - \exp(-\varepsilon^{-1}\alpha t) + \bar{K} \alpha^{-1} (1 - \exp(-\varepsilon^{-1}\alpha t))) \int_0^s |u(s)| ds, \\
&\leq |\mu| + \alpha^{-1} \|f\| + \alpha^{-1} \bar{K} \int_0^t |u(s)| ds, \text{ since } \exp(-\varepsilon^{-1}\alpha t) > 1.
\end{aligned} \tag{4.10}$$

Employing Gronwall's inequality on equation (4.10), gives

$$|u(t)| \leq \exp\left(\frac{1}{\alpha} \bar{K} t\right) \left(|\mu| + \frac{1}{\alpha} \|f\|_\infty\right),$$

and this completes the proof of (4.3).

Now to prove (4.4) for $k = 1$, we differentiate equation (4.1) to obtain

$$\mathcal{Q}(t) = \varepsilon \mathcal{V}'(t) + a(t) \mathcal{V}(t), \tag{4.11}$$

where the functions $\mathcal{V}(t)$ and $\mathcal{Q}(t)$ satisfy

$$\begin{aligned}
\mathcal{V}(t) &= u'(t) \text{ and} \\
\mathcal{Q}(t) &= -a'(t)u(t) - \int_0^t \frac{\partial K(t, \sigma)}{\partial t} u(\sigma) d\sigma - K(t, t)u(t) + f'(t),
\end{aligned}$$

after making use of Leibniz's rule for differentiation under the integral sign, with $\mathcal{V}(0) = u'(0)$. We show that

$$|u'(0)| \leq \frac{|f(0) - a(0)u(0)|}{\varepsilon} = \frac{|a(0)\mu + |f(0)|}{\varepsilon} \leq \frac{C}{\varepsilon}, \tag{4.12}$$

provided u and its derivatives are continuous at $t = 0$. Using the integration factor technique over equation (4.11), gives

$$\mathcal{V}(t) = \mathcal{V}(0) \exp(A(t) - \varepsilon^{-1}) + \varepsilon^{-1} \int_0^t \mathcal{Q}(\nu) \exp(\varepsilon^{-1}(A(\nu) - A(t))) d\nu. \tag{4.13}$$

Under the smoothness condition of the source term and the coefficient functions, we have

$$\varepsilon^{-1} \left| \int_0^t \mathcal{Q}(\nu) \exp(\varepsilon^{-1}(A(\nu) - A(t))) d\nu \right| \leq C. \tag{4.14}$$

From (4.12), (4.13) and (4.14), we obtain

$$|u'(t)| \leq \frac{C}{\varepsilon} \exp\left(\frac{-\alpha t}{\varepsilon}\right) + C,$$

for $k = 1$. Now for $k = 2$, we differentiate equation (4.11) to obtain

$$\mathcal{G}(t) = a(t)\mathcal{V}'(t) + \varepsilon\mathcal{V}''(t). \quad (4.15)$$

Here

$$\mathcal{V}'(t) = u''(t) \quad \text{and} \quad \mathcal{G}(t) = \mathcal{Q}'(t) - a'(t)\mathcal{V}(t).$$

From (4.11) and (4.12), we have

$$|u''(0)| \leq \frac{|\mathcal{Q}(0) - a(0)u'(0)|}{\varepsilon} \leq \frac{|\mathcal{Q}(0)| + |a(0)|u'(0)}{\varepsilon} \leq \frac{C}{\varepsilon^2}. \quad (4.16)$$

From (4.15), we have

$$\mathcal{V}'(t) = \mathcal{V}'(0) \exp(-A(t)\varepsilon^{-1}) + \varepsilon^{-1} \int_0^t \mathcal{G}(\xi) \exp(-(A(\xi) - A(t))\varepsilon^{-1}) d\xi.$$

Since $\mathcal{V}'(t) = u''(t)$, we obtain

$$u''(t) = C\varepsilon^{-2} \exp(-A(t)\varepsilon^{-1}) + \frac{1}{\varepsilon} \int_0^t \mathcal{G}(\xi) \exp(-(A(\xi) - A(t))\varepsilon^{-1}) d\xi.$$

Since

$$\varepsilon^{-1} \left| \int_0^t \mathcal{G}(\xi) \exp(-(A(\xi) - A(t))\varepsilon^{-1}) d\xi \right| \leq C,$$

then we have

$$|u''(t)| \leq C\varepsilon^{-2} \exp(-\varepsilon^{-1}\alpha t) + C, \quad (4.17)$$

which completes the proof for $k = 2$. A similar argument can be used to obtain the estimate for $k = 3, 4$. $\square$

In the next section we construct the numerical scheme to integrate problem (4.1)–(4.2) numerically.

4.3 Numerical Scheme

In this section, the theory on non-standard finite difference schemes is employed to discretise the differential part and the composite Simpson's rule is applied on the integral part. We adopt $U_i \equiv U(t_i)$ as the numerical solution of $u(t_i) \equiv u_i$. Consider the partitioning of $\bar{\Omega}$, into the discrete form $\bar{\Omega}^n$,

$$t_0 = 0, \quad t_n = T, \quad t_i = t_0 + ih, \quad i = 1(1)n-1 \quad h = t_i - t_{i-1}. \quad (4.18)$$

The parameters n and h are the total number of sub-divisions and the step size respectively. problem (4.1)–(4.2) is now transformed into the discrete problem

$$\begin{aligned} \mathcal{L}_\varepsilon^n U_i &\equiv \varepsilon \frac{U_i - U_{i-1}}{\phi_i} + a_i U_i + H(t_{i-1}; U_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1})U_{i-1} + K(t_i, t_i)U_i] \\ &= f_i, \quad i = 1, 2, \dots, n, \end{aligned} \quad (4.19)$$

$$U_0 = \mu. \quad (4.20)$$

The term $H(t_{i-1}; U_{i-1})$ and the denominator function ϕ_i , are given by

$$H(t_{i-1}; U_{i-1}) = \begin{cases} 0, & i = 1, \\ \frac{4}{3}hK(t_i, t_{i-1})U_{i-1}, & i = 2, \\ \sum_{j=1}^{\lfloor i/2 \rfloor} \frac{4}{3}hK(t_i, t_{2j-1})U_{2j-1} + \sum_{j=1}^{\lfloor (i-1)/2 \rfloor} \frac{2}{3}hK(t_i, t_{2j})U_{2j}, & i > 2 \end{cases}$$

and

$$\phi_i = \frac{\varepsilon}{a_i} \left(\exp \left(\frac{a_i h}{\varepsilon} \right) - 1 \right),$$

respectively. In matrix notation, the scheme (4.19)–(4.20) takes the form

$$AU = F, \quad (4.21)$$

with A as a square lower triangular matrix of size n^2 and U, F are column vectors of size n .

The system matrix A and F , have the following entries

$$\begin{aligned} A_{i,i} &= \frac{\varepsilon}{\phi_i} + a_i + \frac{h}{3}K_{i,i}, & i = 1, 2, \dots, n, \\ A_{i,i-1} &= -\frac{\varepsilon}{\phi_i} + \frac{h}{3}K_{i,i-1} + \frac{4h}{3}K_{i,i-1}, & i = 2, 4, \dots, n, \\ A_{i,i-1} &= -\frac{\varepsilon}{\phi_i} + \frac{h}{3}K_{i,i-1} + \frac{2h}{3}K_{i,i-1}, & i = 3, 5, \dots, n, \\ A_{i,j} &= \frac{4h}{3}K_{i,j}, \\ A_{i,j} &= \frac{2h}{3}K_{i,j}, \end{aligned}$$

and

$$\begin{aligned} F_1 &= f_1 - \left(-\frac{\varepsilon}{\phi_1} + \frac{h}{3}K_{1,0} \right) \mu \\ F_i &= f_i, & i = 2, 3, \dots, n, \end{aligned}$$

respectively. Next we prove some properties of the discrete problem (4.19)–(4.20) which ensure the existence of solution of the discrete problem.

4.4 Stability of the Scheme

Some properties of the discrete problem which implies the stability of the scheme in the form of lemmas are presented in this section.

Lemma 4.4.1. *Discrete maximum principle*

From the scheme (4.19)–(4.20), $\mathcal{L}_\varepsilon^n$ satisfies a discrete maximum principle. That is, if $\Psi_0 \geq 0$ and $\mathcal{L}_\varepsilon^n \Psi_i \geq 0$, $\forall i \in [1, n]$, then $\Psi_i \geq 0$, $\forall 0 \leq i \leq n$.

Proof. Suppose Ψ_s takes its minimum at the index s , $0 \leq s \leq n$, $\Psi_s = \min_{i \in \bar{\Omega}^n} \Psi_i$ and that $\Psi_s < 0$. Then $s \neq 0$ and $\Psi_s - \Psi_{s-1} < 0$. Rewriting the scheme (4.19) as

$$\mathcal{L}_\varepsilon^n U_i \equiv \mathcal{A}U_i - \mathcal{B}U_{i-1}, \quad 1 \leq i \leq n,$$

where the $\mathcal{A} > 0$ and $\mathcal{B} > 0$ and are given by,

$$\mathcal{A} = \frac{\varepsilon}{\phi_i} + a_i + \frac{h}{3} [K(t_i, t_i)], \quad \mathcal{B} = -\frac{\varepsilon}{\phi_i} + \frac{h}{3} [K(t_i, t_{i-1})] + H(t_{i-1}; U_{i-1}).$$

Now on the domain ($i = 1, \dots, n$), we have

$$\begin{aligned} \mathcal{L}_\varepsilon^n \Psi_s &= \mathcal{A}\Psi_s - \mathcal{B}\Psi_{s-1} \\ &= \mathcal{A}\Psi_s - \mathcal{B}\Psi_s + \mathcal{B}\Psi_s - \mathcal{B}\Psi_{s-1} \\ &= (\mathcal{A} - \mathcal{B})\Psi_s + \mathcal{B}(\Psi_s - \Psi_{s-1}) < 0, \text{ provided } \mathcal{A} - \mathcal{B} > 0. \end{aligned}$$

This is a contradiction and therefore $\Psi_i \geq 0$ for all $0 \leq i \leq n$. $\square$

The stability of the scheme (4.19)–(4.20) follows from Lemma 4.4.1.

Lemma 4.4.2. *Let u_i be the solution of the discrete problem (4.19)–(4.20). Then it satisfies the estimate*

$$|u_i| \leq |\mu| + \alpha^{-1} \max_{0 \leq i \leq n} |\mathcal{L}_\varepsilon^n u_i|.$$

Proof. Consider the smooth functions $\Psi_i^\pm$, given by

$$\Psi_i^\pm = z \pm u_i,$$

where z takes the form

$$|\mu| + \frac{1}{\alpha} \max_{0 \leq i \leq n} |\mathcal{L}_\varepsilon^n u_i|.$$

For $i = 0$, we have

$$\Psi_0^\pm = |\mu| + \frac{1}{\alpha} \max_{0 \leq i \leq n} |\mathcal{L}_\varepsilon^n u_0| \pm u_0 \geq 0,$$

and that for $i \geq 1$, we have

$$\begin{aligned} \mathcal{L}_\varepsilon^n \Psi_i^\pm &= \mathcal{A}(z \pm u_i) - \mathcal{B}(z \pm u_{i-1}) \\ &= (\mathcal{A} - \mathcal{B})z \pm \mathcal{A}u_i - \mathcal{B}u_{i-1} \\ &= (\mathcal{A} - \mathcal{B})z \pm \mathcal{L}_\varepsilon^n u_i \\ &= (\mathcal{A} - \mathcal{B}) \left[|\mu| + \frac{1}{\alpha} \max_{1 \leq i \leq n} |\mathcal{L}_\varepsilon^n u_i| \right] \pm f_i \\ &\geq 0. \end{aligned}$$

From Lemma 4.4.1, $\Psi_i^\pm \geq 0$, $\forall 0 \leq i \leq n$, as required. $\square$

4.5 Error Estimate

Here, the scheme is analysed for convergence.

Theorem 4.5.1. *Let u_i be the analytical solution and U_i be the numerical solution. Then the error is of the form*

$$|u_i - U_i| \leq Ch,$$

where C is a constant which is not constrained by ε and h .

Proof. Below the error associated with the discretisation process is estimated as follows:

$$\begin{aligned} \mathcal{L}_\varepsilon^n(u_i - U_i) &= \mathcal{L}_\varepsilon^n u_i - \mathcal{L}_\varepsilon^n U_i = \mathcal{L}_\varepsilon^n u_i - f_i, \\ &= \left(\varepsilon \frac{u_i - u_{i-1}}{\phi_i} + a_i u_i + \frac{h}{3} [K(t_i, t_{i-1})u_{i-1} + K(t_i, t_i)u_i] + H(t_{i-1}; U_{i-1}) \right) \\ &\quad - \left(\varepsilon u'_i + a_i u_i + \int_0^t K(t, s)u(s)ds \right), \\ &= \varepsilon \left[\left(\frac{u_i - u_{i-1}}{\phi_i} \right) - u'_i \right] + \left[H(t_{i-1}; U_{i-1}) - \int_0^{t_{i-1}} K(t_i, s)u(s)ds \right] \\ &\quad + \left[\frac{h}{3} [K(t_i, t_{i-1})u_{i-1} + K(t_i, t_i)u_i] - \int_{t_{i-1}}^{t_i} K(t_i, s)u(s)ds \right], \\ &\leq \left| \varepsilon \left[\left(\frac{1}{h} - \frac{a_i}{2\varepsilon} + \dots \right) \left(hu'_i - \frac{h^2}{2}u''_i + \dots \right) - u'_i \right] \right. \\ &\quad \left. + C \left[h[u_{i-1} + u_i] - \int_{t_{i-1}}^{t_i} u(s)ds \right] + C \left[h \sum_{j=1}^{i-1} [u_j + u_{j-1}] - \int_0^{t_{i-1}} u(s)ds \right] \right|. \end{aligned}$$

Notice that we have used Taylor series of the terms ϕ_i and u_{i-1} . Simplifying gives

$$|\mathcal{L}_\varepsilon^n(u_i - U_i)| \leq \left| \left(-\frac{\varepsilon}{2}u'' - \frac{a_i}{2}u' \right) h + \frac{a_i}{4}u''h^2 \right| + Chu_i - Ch^2u'_i + \dots + Chu_0 - Ch^2u'_i + \dots$$

From Theorem 4.2.1 and noticing that the exponential terms go to zero yields

$$|\mathcal{L}_\varepsilon^n(u_i - U_i)| \leq Ch.$$

Now let $\mathcal{R}_i$ be a mesh function defined by

$$\mathcal{R}_i = Ch, \quad 0 \leq i \leq n.$$

Notice that $\mathcal{R}_0 = Ch$ and

$$\begin{aligned} \mathcal{L}_\varepsilon^n \mathcal{R}_i &\equiv \varepsilon \frac{\mathcal{R}_i - \mathcal{R}_{i-1}}{\phi_i} + a_i \mathcal{R}_i + H(t_{i-1}; \mathcal{R}_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1})\mathcal{R}_{i-1} + K(t_i, t_i)\mathcal{R}_i] \\ &= a_i \mathcal{R}_i + H(t_{i-1}; \mathcal{R}_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1})\mathcal{R}_{i-1} + K(t_i, t_i)\mathcal{R}_i]. \end{aligned}$$

Suppose $\mathcal{H}_i$ is also another function which takes the form

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i|.$$

At $t = 0$, we have

$$\mathcal{H}_0 = \mathcal{R}_0 - |u_0 - U_0| = Ch > 0,$$

Further on $(0, T]$ we have

$$\begin{aligned} \mathcal{L}_\varepsilon^n \mathcal{H}_i &= \mathcal{L}_\varepsilon^n \mathcal{R}_i - \mathcal{L}_\varepsilon^n |u_i - U_i| \\ &= a_i \mathcal{R}_i + H(t_{i-1}; \mathcal{R}_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1}) \mathcal{R}_{i-1} + K(t_i, t_i) \mathcal{R}_i] - \mathcal{L}_\varepsilon^n |u_i - U_i|. \end{aligned}$$

Since $\mathcal{L}_\varepsilon^n |\gamma| \leq |\mathcal{L}_\varepsilon^n \gamma|$, we get

$$\begin{aligned} \mathcal{L}_\varepsilon^n \mathcal{H}_i &= a_i \mathcal{R}_i + H(t_{i-1}; \mathcal{R}_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1}) \mathcal{R}_{i-1} + K(t_i, t_i) \mathcal{R}_i] - |\mathcal{L}_\varepsilon^n |u_i - U_i|| \\ &= a_i \mathcal{R}_i + H(t_{i-1}; \mathcal{R}_{i-1}) + \frac{h}{3} [K(t_i, t_{i-1}) \mathcal{R}_{i-1} + K(t_i, t_i) \mathcal{R}_i] - Ch > 0. \end{aligned}$$

From Lemma 4.4.1,

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i| \geq 0,$$

and hence

$$|u_i - U_i| \leq \mathcal{R}_i,$$

which leads to

$$|u_i - U_i| \leq Ch.$$

□

4.6 Richardson Extrapolation

In this section, we employ Richardson extrapolation to improve the accuracy and accelerate the convergence of the numerical method. Suppose ω_n is the mesh function in 4.18 and $\tilde{U}_i$ is the numerical solution obtained via the mesh function ω_{2n} by halving h . Now from Theorem 4.5.1 we have

$$|u_i - U_i| \leq h + R_n, \quad (4.22)$$

$$|u_i - \tilde{U}_i| \leq \frac{h}{2} + R_{2n}. \quad (4.23)$$

Here R_n and R_{2n} are the errors from the mesh functions ω_n and ω_{2n} respectively. A simple subtraction of the inequalities (4.23) from (4.22) and further simplification yields the extrapolation formula

$$\mathcal{U}_i = 2\tilde{U}_i - U_i. \quad (4.24)$$

Theorem 4.6.1. *The error associated with the Richardson extrapolation satisfies the estimate*

$$|u_i - \mathcal{U}_i| \leq Ch^2,$$

where C is not constrained by ε .

Proof. Now the error after extrapolation is estimated as follows:

$$\begin{aligned} \mathcal{L}_\varepsilon^n(u_i - \mathcal{U}_i) = 2 \left[\left(\varepsilon \frac{\tilde{u}_i - \tilde{u}_{i-1}}{\phi_i} + a_i \tilde{u}_i + \frac{h}{6} [K(t_i, t_{i-1}) \tilde{u}_{i-1} + K(t_i, t_i) \tilde{u}_i] + H(t_{i-1}, \tilde{u}_{i-1}) \right) \right. \\ \left. - \left(\varepsilon \tilde{u}'_i + a_i \tilde{u}_i + \int_0^t K(t, s) u(s) ds \right) \right] - \mathcal{L}_\varepsilon^n(u_i - U_i). \end{aligned}$$

Appropriate Taylor series expansions and further simplifications results into all the terms with the coefficient h cancelling out. Thus we have

$$\mathcal{L}_\varepsilon^n(u_i - \mathcal{U}_i) = \frac{a_i}{4} u''_i h^2 + \dots - C u'_i h^2 + \dots - C u'_i h^2 + \dots$$

Application of Theorem 4.2.1, yields

$$|\mathcal{L}_\varepsilon^n(u_i - \mathcal{U}_i)| \leq C h^2.$$

From Lemma 4.4.1 we obtain

$$|u_i - \mathcal{U}_i| \leq C h^2.$$

□

To illustrate the above scheme, we apply the method on test examples in the next section.

4.7 Numerical Results

Here, we perform simulations with two test problems to attest to the theoretical results. For each test problem, the maximum error and the rate of convergence will be computed and presented by tables. Each table will comprise of both the numerical rate of convergence and the errors. The first column of the table will have the values of the perturbation parameter changing whilst the varying of the number of sub-intervals will be seen in the first row. The formulae

$$E_{\varepsilon,n} = \max_{0 \leq i \leq 1} |u_i - U_i| \quad \text{and} \quad Ex_{\varepsilon,n} = \max_{0 \leq i \leq 1} |u_i - \mathcal{U}_i|,$$

will be used to calculate the maximum point wise errors before and after extrapolation respectively. Also, we employ

$$r = \log_2 \left(\frac{E_{\varepsilon,n}}{E_{\varepsilon,2n}} \right), \quad \text{and} \quad rx = \log_2 \left(\frac{Ex_{\varepsilon,n}}{Ex_{\varepsilon,2n}} \right) \quad (4.25)$$

to find the rates of convergence. Further, to evaluate the ε -uniform maximum errors we employ

$$E = \max_{0 < \varepsilon \leq 1} |E_{\varepsilon,n}| \quad \text{and} \quad Ex = \max_{0 < \varepsilon \leq 1} |Ex_{\varepsilon,n}| \quad (4.26)$$

with the corresponding ε -uniform rates of convergence obtained using

$$R = \max_{0 < \varepsilon \leq 1} |r|, \quad \text{and} \quad Rx = \max_{0 < \varepsilon \leq 1} |rx|. \quad (4.27)$$

Example 4.7.1. Amiraliyev and Şevgin (2006) *Consider the SPVIDE*

$$\begin{aligned} f(t) &= \varepsilon u'(t) + u(t) \int_0^t su(s)ds, \quad t \in (0, 1], \\ f(t) &= 10\varepsilon^2 + \exp\left(\frac{-t}{\varepsilon}\right) (2 + 9\varepsilon + \varepsilon t + 11t + t^2) \exp(-t) - \exp\left(\frac{-t}{\varepsilon}\right) 10(\varepsilon t + \varepsilon^2) \\ &\quad + 5t^2 - 2, \\ u(0) &= 10, \end{aligned}$$

along with the analytical solution $u(t) = 10 - (10 + t) \exp(-t) + 10 \exp\left(\frac{-t}{\varepsilon}\right)$.

Example 4.7.2. Zhongdi and Lifeng (2006) *Consider the problem*

$$\begin{aligned} \varepsilon u'(t) + (t+1)u(t) \int_0^t (t+s)u(s)ds &= f(t), \quad t \in (0, 1], \\ f(t) &= \exp(-t/\varepsilon)(t - 2t\varepsilon + \varepsilon^2) + \varepsilon \cos t - \varepsilon^2 + t \sin t + 2 \sin t + t - 2t \cos t + \varepsilon t, \\ u(0) &= 1. \end{aligned}$$

The analytical solution takes the form

$$u(t) = \sin t + \exp(-t/\varepsilon).$$

Table 4.1: Maximum point wise error and rate of convergence for Example 4.7.2 before extrapolation.

ε	n=32	64	128	256	512	1024	2048
10^{-3}	7.43E-03	3.62E-03	1.68E-03	9.24E-04	6.90E-04	4.21E-04	2.30E-04
	1.0366	1.1080	0.8633	0.4206	0.7137	0.8707	
10^{-4}	7.69E-03	3.88E-03	1.94E-03	9.58E-04	4.65E-04	2.18E-04	9.65E-05
	0.9864	1.0016	1.0171	1.0417	1.0908	1.1796	
10^{-5}	7.72E-03	3.91E-03	1.96E-03	9.84E-04	4.91E-04	2.44E-04	1.21E-04
	0.9816	0.9920	0.9977	1.0020	1.0074	1.0166	
10^{-6}	7.72E-03	3.91E-03	1.97E-03	9.87E-04	4.94E-04	2.47E-04	1.23E-04
	0.9811	0.9911	0.9958	0.9982	0.9998	1.0011	
10^{-8}	7.72E-03	3.91E-03	1.97E-03	9.87E-04	4.94E-04	2.47E-04	1.23E-04
	0.9811	0.9911	0.9958	0.9982	0.9998	1.0011	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-10}	7.72E-03	3.91E-03	1.97E-03	9.87E-04	4.94E-04	2.47E-04	1.24E-04
	0.9811	0.9910	0.9956	0.9978	0.9989	0.9995	
E	7.69E-03	3.91E-03	1.97E-03	9.87E-04	4.94E-04	2.47E-04	2.30E-04
R	1.0366	1.1080	1.0171	1.0890	1.2854	1.7924	

Table 4.2: Maximum point wise error and rate of convergence for Example 4.7.2 after extrapolation.

ε	n=32	64	128	256	512	1024	2048
10^{-2}	3.40E-05	8.45E-06	2.10E-06	5.20E-07	1.25E-07	2.70E-08	9.99E-09
	2.0079	2.0069	2.0145	2.0521	2.2153	1.4356	
10^{-4}	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.31E-07	3.28E-08	8.16E-09
	2.0072	2.0038	2.0020	2.0014	2.0025	2.0083	
10^{-6}	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.32E-07	3.29E-08	8.22E-09
	2.0071	2.0037	2.0019	2.0009	2.0005	2.0003	
10^{-8}	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.32E-07	3.29E-08	8.22E-09
	2.0071	2.0037	2.0019	2.0009	2.0005	2.0002	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.32E-07	3.29E-08	8.22E-09
	2.0071	2.0037	2.0019	2.0009	2.0005	2.0002	
10^{-20}	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.32E-07	3.29E-08	8.22E-09
	2.0071	2.0037	2.0019	2.0009	2.0005	2.0002	
Ex	3.40E-05	8.46E-06	2.11E-06	5.26E-07	1.32E-07	3.29E-08	8.22E-09
Rx	2.0079	2.0069	2.0145	2.0521	2.2153	2.0083	

Table 4.3: Maximum point wise error and rate of convergence for Example 4.7.1 before extrapolation.

ε	n=32	64	128	256	512	1024	2048
10^{-3}	5.32E-02	2.63E-02	1.24E-02	8.22E-03	6.05E-03	3.66E-03	2.00E-03
	1.0153	1.0821	0.5963	0.4422	0.7237	0.8700	
10^{-4}	5.47E-02	2.79E-02	1.40E-02	6.95E-03	3.40E-03	1.61E-03	8.66E-04
	0.9736	0.9938	1.0104	1.0327	1.0739	0.8982	
10^{-6}	5.49E-02	2.80E-02	1.42E-02	7.12E-03	3.57E-03	1.79E-03	8.93E-04
	0.9692	0.9850	0.9927	0.9966	0.9989	1.0005	
10^{-8}	5.49E-02	2.80E-02	1.42E-02	7.12E-03	3.57E-03	1.79E-03	8.94E-04
	0.9692	0.9849	0.9925	0.9963	0.9982	0.9991	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0.9692	0.9849	0.9925	0.9963	0.9981	0.9991	
10^{-18}	5.49E-02	2.80E-02	1.42E-02	7.12E-03	3.57E-03	1.79E-03	8.94E-04
	0.9692	0.9849	0.9925	0.9963	0.9981	0.9991	
10^{-20}	5.49E-02	2.80E-02	1.42E-02	7.12E-03	3.57E-03	1.79E-03	8.94E-04
	0.9692	0.9849	0.9925	0.9963	0.9981	0.9991	
E	5.49E-02	2.80E-02	1.42E-02	8.22E-03	6.05E-03	3.66E-03	2.00E-03
R	0.9692	1.0821	0.9927	1.0327	1.0739	1.0005	

Table 4.4: Maximum point wise error and rate of convergence for Example 4.7.1 after extrapolation.

ε	n=32	64	128	256	512	1024	2048
10^{-2}	2.99E-04	7.47E-05	1.86E-05	4.64E-06	1.14E-06	2.70E-07	8.99E-08
	2.0011	2.0016	2.0057	2.0222	2.0835	1.5846	
10^{-4}	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.91E-07	7.27E-08
	2.0008	2.0002	2.0001	2.0002	2.0009	2.0036	
10^{-6}	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	7.29E-08
	2.0008	2.0002	2.0001	2.0000	2.0000	2.0000	
10^{-8}	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	7.29E-08
	2.0008	2.0002	2.0001	2.0000	2.0000	2.0000	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	7.29E-08
10^{-18}	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	7.29E-08
	2.0008	2.0002	2.0001	2.0000	2.0000	2.0000	
10^{-20}	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	7.29E-08
	2.0008	2.0002	2.0001	2.0000	2.0000	2.0000	
Ex	2.99E-04	7.47E-05	1.87E-05	4.67E-06	1.17E-06	2.92E-07	8.99E-08
Rx	2.0011	2.0016	2.0057	2.0222	2.0835	2.0036	

Table 4.5: Comparison of numerical results of Example 4.7.1 by the proposed scheme and the scheme in Amiraliyev and Şevgin (2006) before extrapolation.

	Scheme	n=16	32	64	128
Maximum errors	Proposed Scheme	1.05E-01	5.49E-02	2.80E-02	1.42E-02
	Amiraliyev and Şevgin (2006)	2.61E-01	1.29E-01	6.06E-02	2.60E-02
Orders of Convergence	Proposed scheme	0.94	0.97	0.99	
	Amiraliyev and Şevgin (2006)	1.06	1.11	1.23	

Table 4.6: Comparison of numerical results of Example 4.7.1 by the proposed scheme and the scheme in Amiraliyev and Şevgin (2006) after extrapolation.

	Scheme	n=16	32	64	128
Maximum errors	Proposed Scheme	1.20E-03	2.99E-04	7.47E-05	1.87E-05
	Amiraliyev and Şevgin (2006)	2.61E-01	1.29E-01	6.06E-02	2.60E-02
Orders of Convergence	Proposed scheme	2.0	2.0	2.0	
	Amiraliyev and Şevgin (2006)	1.06	1.11	1.23	

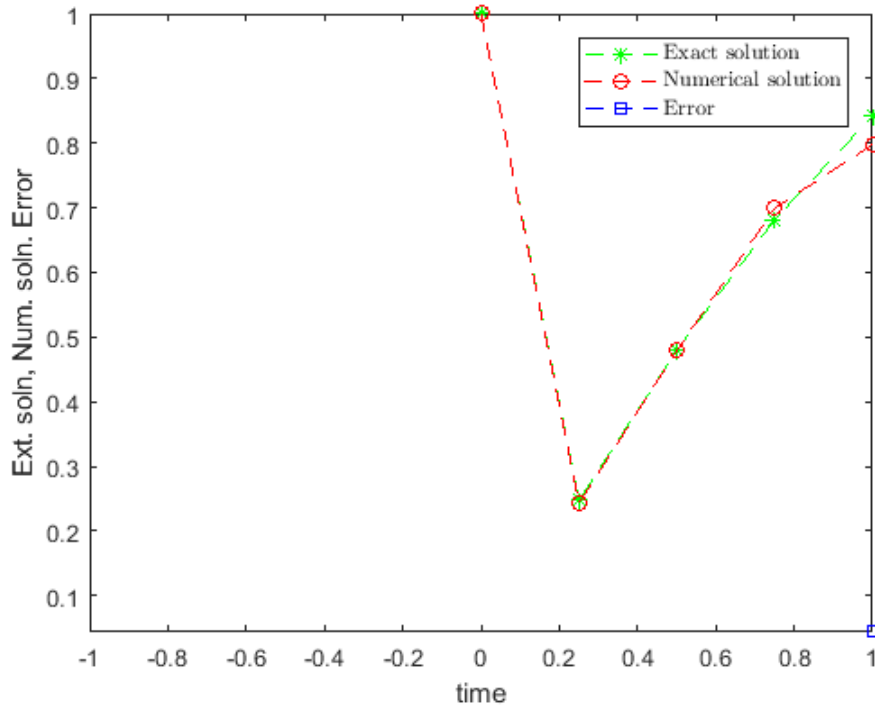


Figure 4.1: Plot of the exact, numerical solution and the error of Example 4.7.2 for $n = 4$ and $\varepsilon = 10^{-24}$.

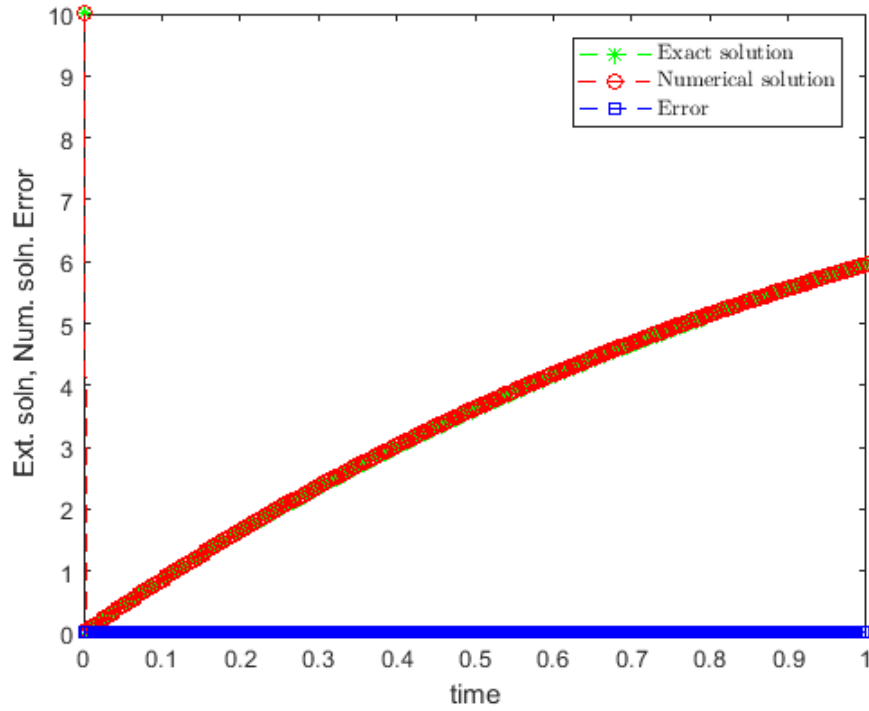


Figure 4.2: Plot of the exact, numerical solution and the error of Example 4.7.1 for $n = 128$ and $\varepsilon = 10^{-24}$.

4.8 Conclusion

A numerical method was proposed to solve a singularly perturbed Volterra integro-differential equation of first order. The method employed the non-standard finite difference scheme and the composite Simpson's rule to convert the continuous singularly perturbed Volterra integro-differential equation into systems of algebraic equations. Stability of the scheme was studied and a first order accuracy was obtained from the convergence analysis. Using Richardson extrapolation technique, the accuracy of the method was improved to a second order. Numerical simulations which were carried out validated the theoretical findings before and after extrapolation. Moreover, our numerical results were also compared to those in Amiraliyev and Şevgin (2006) and the comparison revealed that their scheme converges faster than the proposed scheme before extrapolation while vice versa occurred after extrapolation. Figures 4.1-4.2 displayed the numerical, exact and the error of Examples 4.7.2 and 4.7.1 respectively. As shown from the graph, the errors between the numerical solutions and the exact solutions are practically zero which indicates that there is no difference between the numerical solutions and the exact solutions. In the next chapter, the ideas used to solve the differential part of problem (4.1)–(4.2) is employed along with the Crank Nicolson finite difference scheme to solve a parabolic convection diffusion problem with degenerate coefficients by the Rothe's method.

Chapter 5

A Second Order Fitted Operator Finite Difference Scheme for Singularly Perturbed Degenerate Parabolic Problem

5.1 Introduction

This chapter presents the study of a singularly perturbed parabolic problem with degenerate coefficient. The solution of this problem is characterised by boundary layer in the neighbourhood of $x = 0$. Thus to solve this problem, we construct a numerical method based on the Crank Nicolson finite difference scheme and a fitted operator finite difference scheme via the Rothe's method. Notice that the fitted operator scheme we propose here is based on a midpoint scheme. A second order accuracy independent of the perturbation parameter is established for both the time and the space variables.

Consider the singularly perturbed degenerate parabolic convection diffusion problem

$$\mathcal{L}_\varepsilon u(x, t) = f(x, t), \quad (x, t) \in Q, \quad (5.1)$$

subject to the boundary and the initial condition

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t), \quad t \in [0, T], \quad u(x, t) = \varphi(x), \quad x \in \bar{\Omega}. \quad (5.2)$$

The differential operator $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon = -\frac{\partial}{\partial t} + \varepsilon \frac{\partial^2}{\partial x^2} + a(x) \frac{\partial}{\partial x} - b(x, t).$$

We assume the source term $f(x, t)$ and the coefficient functions $a(x)$, $b(x, t)$ are smooth and bounded. Also, the coefficient functions satisfy

$$a(x) = a_0(x)x^p, \quad a_0(x) > \alpha > 0, \quad p \geq 1, \quad b(x, t) \geq \beta > 0.$$

The perturbation parameter ε is such that $0 < \varepsilon \ll 1$. Problems of type (5.1)–(5.2) are known as singularly perturbed turning point problem since the coefficient of the convective term vanishes at $x = 0$ and subsequently, the point $x = 0$ is the turning point. Depending on the value of p we can have either a simple turning point ($p = 1$) or a multiple point ($p > 1$). Setting $\varepsilon = 0$ in problem (5.1)–(5.2) yields the reduced problem

$$-(u_0)_t(x, t) + a(x)(u_0)_x(x, t) - b(x, t)(u_0)(x, t) = f(x, t), \quad (5.3)$$

which is first order hyperbolic problem along with the initial and boundary conditions

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t), \quad t \in [0, T], \quad u(x, 0) = \varphi(x), \quad x \in \bar{\Omega}. \quad (5.4)$$

Evidently, solving the reduced problem (5.3)–(5.4) will require only one boundary condition and thus, one of the boundary conditions will be underutilized leading to a boundary layer in that neighborhood. Notice that the convective term is positive throughout the domain Ω therefore the boundary layer will occur in the neighborhood of $x = 0$. The width of that layer takes the form $\mathcal{O}(\sqrt{\varepsilon})$, which is similar to that of a reaction diffusion problem.

The content of the rest of the chapter is as follows: In Section 5.2, some theoretical results are presented. The discretisation of the continuous problem in time by the Crank Nicolson finite difference scheme and its convergence are presented in Section 5.3. Section 5.4 focuses on the spatial discretisation which uses a fitted operator finite difference scheme based on the midpoint rule. In Section 5.5, the stability of the scheme is presented and then analysed for convergence in Section 5.6. In Section 5.7 numerical experiments are conducted to validate the theoretical findings whilst a summary of the main result is presented in Section 5.8 .

5.2 Bounds on the Solution and its Derivatives

In this section, we estimate the solution of problem (5.1)–(5.2) by providing the bounds of the solution and its partial derivatives. To prove these bounds, first some lemmas which will be needed in this proof will be provided. The proofs of the lemmas in this section follow the lines of the proofs in Majumdar and Natesan (2019); Ng-Stynes et al. (1988).

Lemma 5.2.1. *Continuous minimum principle*

Suppose $\Psi(x, t)$ is a smooth functions which satisfies $\Psi(x, 0) \geq 0$, $\Psi(0, t) \geq 0$, $\Psi(1, t) \geq 0$ and $\mathcal{L}_\varepsilon \Psi(x, t) \leq 0$, $\forall (x, t) \in Q$. Then $\Psi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$.

Proof. Let $(x^*, t^*) \in \bar{Q}$, such that

$$\Psi(x^*, t^*) = \min_{(x, t) \in \bar{Q}} \Psi(x, t) \text{ and } \Psi(x^*, t^*) < 0.$$

It is clear that $(x^*, t^*) \in Q$. At the point (x^*, t^*) the operator $\mathcal{L}_\varepsilon$ on $\Psi(x^*, t^*)$ becomes

$$\mathcal{L}_\varepsilon \Psi(x^*, t^*) = -\Psi_t(x^*, t^*) + \varepsilon \Psi_{xx}(x^*, t^*) + a(x^*) \Psi_x(x^*, t^*) - b(x^*, t^*) \Psi(x^*, t^*).$$

Since $\Psi_{xx}(x^*, t^*) > 0$ and $\Psi_x(x^*, t^*) = \Psi_t(x^*, t^*) = 0$, we obtain $\mathcal{L}_\varepsilon \Psi(x^*, t^*) \geq 0$, which is a contradiction. Therefore, $\Psi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$, which completes the proof. $\square$

Lemma 5.2.2. (*Stability estimate*)

Suppose $u(x, t)$ is the solution of problem (5.1)–(5.2). Then the following estimate holds

$$|u(x, t)| \leq [\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)].$$

Proof. Consider the smooth functions $\Psi^\pm$, by

$$\Psi^\pm(x, t) = [\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)] \pm u(x, t).$$

At $t = 0$,

$$\begin{aligned} \Psi^\pm(x, 0) &= [\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(0)|, |\eta_1(0)|)] \pm u(x, 0), \\ &= [\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(0)|, |\eta_1(0)|)] \pm \varphi(x) \geq 0. \end{aligned}$$

At $x \in \partial\Omega$,

$$\begin{aligned} \Psi^\pm(0, t) &= [\beta^{-1} \|f\| + \max(|\varphi(0)|, |\eta_0(t)|, |\eta_1(t)|)] \pm u(0, t), \\ \Psi^\pm(0, t) &= [\beta^{-1} \|f\| + \max(|\varphi(0)|, |\eta_0(t)|, |\eta_1(t)|)] \pm \eta_0(t) \geq 0, \\ \Psi^\pm(1, t) &= [\beta^{-1} \|f\| + \max(|\varphi(1)|, |\eta_0(t)|, |\eta_1(t)|)] \pm u(1, t), \\ \Psi^\pm(1, t) &= [\beta^{-1} \|f\| + \max(|\varphi(1)|, |\eta_0(t)|, |\eta_1(t)|)] \pm \eta_1(t) \geq 0. \end{aligned}$$

Further on the the domain Q , we have

$$\begin{aligned} \mathcal{L}_\varepsilon \Psi^\pm(x, t) &= -\Psi_t^\pm(x, t) + \varepsilon \Psi_{xx}^\pm(x, t) + a(x) \Psi_x^\pm(x, t) - b(x, t) \Psi^\pm(x, t) \\ &= -\max(|\eta_0'(t)|, |\eta_1'(t)|) + \varepsilon |\varphi''(x)| + a(x) |\varphi'(x)| - b(x, t) \\ &\quad \times (\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)) \pm \mathcal{L}_\varepsilon u(x, t), \\ &= -\max(|\eta_0'(t)|, |\eta_1'(t)|) + \varepsilon |\varphi''(x)| + a(x) |\varphi'(x)| - b(x, t) \\ &\quad \times (\beta^{-1} \|f\| + \max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)) \pm f(x, t), \\ &= \pm f(x, t) - \frac{b(x, t)}{\beta} \|f\| - b(x, t) [\max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)] \\ &\quad - \max(|\eta_0'(t)|, |\eta_1'(t)|) + \varepsilon |\varphi''(x)| + a(x) |\varphi'(x)|, \\ &= \pm f(x, t) - \frac{b(x, t)}{\beta} \|f\| - b(x, t) [\max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)] \\ &\quad - \max(|\eta_0'(t)|, |\eta_1'(t)|) \leq 0, \end{aligned}$$

provided $\varphi^k(x) = 0$, $k \geq 1$. From Lemma 5.2.1, it follows that $\Psi^\pm(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$ and this yields the result. $\square$

To estimate the bounds of the partial derivatives we follow the ideas of Kellogg and Tsan (1978); Vulanović and Farrell (1993). Now we fix t and let $g(x, t)$ be a function such that $|g^i(x, t)| \leq C(1 + \varepsilon^{i/2} \exp(-\mu_* x))$, $1 \leq i \leq 6$, $\mu_* = \sqrt{q_*/\varepsilon}$ and satisfies the problem

$$\mathcal{L}_\varepsilon u(x, t) = g(x, t), \quad (x, t) \in Q, \quad (5.5)$$

$$u(0, t) = \eta_0(t), \quad u(1, t) = \eta_1(t). \quad (5.6)$$

Now for a fixed $t \in [0, T]$, let $q_i(x, t)$ be given by

$$q_i(x, t) = b(x, t) - ia'(x), \quad x \in \Omega, \quad 1 \leq i \leq 6,$$

with $b(0, t) > a_0(0)$. Then $q_i(0, t) > 0$, follows for a fixed $t \in [0, T]$ and there exist a point $\theta \in \Omega$ which does not dependent on ε , such that

$$q_i(x, t) > q_* > 0, \quad 1 \leq i \leq 6, \quad x \in [0, \theta_0]. \quad (5.7)$$

Lemma 5.2.3. *The solution $u(x, t)$ of problem (5.1)–(5.2) satisfies the estimate $|u_t(x, t)| \leq C_1^*$, for all $(x, t) \in \bar{Q}$.*

Proof. The proof of this Lemma is similar to that of Lemma 2.2.4 in Chapter 2. $\square$

Lemma 5.2.4. *Let $u(x, t)$ be the solution of problem (5.5)–(5.6), then it satisfies $|u_x(x, t)| \leq C(1 + \varepsilon^{-1/2} \exp(-\mu_* x))$, for all $(x, t) \in \bar{Q}$.*

Proof. To prove this lemma, the domain $\bar{\Omega}$ is divided into the sub-domain $[0, \theta_0]$ and $[\theta_0, 1]$ so that the bounds in each domain can be estimated separately.

Let $\mathcal{L}_1$ be an operator given by

$$\begin{aligned} \mathcal{L}_1 u &= \mathcal{L}_x u + 1a'(x)u = \varepsilon u'' + a(x)u' - b(x, t)u + 1a'(x)u \\ &= \varepsilon u'' + a(x)u' - q_1(x, t)u, \end{aligned}$$

where $\mathcal{L}_x = \mathcal{L}_\varepsilon - \frac{\partial}{\partial t}$ and $q_1(x, t) = b(x, t) - 1a'(x)$. Now let $\rho_i(x, t)$ be a barrier function given by

$$\rho_i(x, t) = C_1(1 + \varepsilon^{-i/2} \exp(-\mu_* x)), \quad x \in [0, \theta_0], \quad 1 \leq i \leq 6,$$

where $\mu_* = \sqrt{q_*/\varepsilon}$. Thus at $i = 1$, we have

$$\rho_1(x, t) = C_1(1 + \varepsilon^{-1/2} \exp(-\mu_* x)).$$

At $x = 0$ and $x = \theta_1 \in [0, \theta_0]$, we have

$$\rho_1(0, t) = C_1(1 + \varepsilon^{-1/2}), \quad \rho_1(\theta_1, t) = C_1(1 + \varepsilon^{-1/2} \exp(-\mu_* \theta_1)), \quad (5.8)$$

respectively. Further on the domain $(0, \theta_1)$, we obtain

$$\begin{aligned} \mathcal{L}_1 \rho_1(x, t) &= \varepsilon C_1(-\mu_*)^2 \varepsilon^{-1/2} \exp(-\mu_* x) + a(x)C_1(-\mu_*) \varepsilon^{-1/2} \exp(-\mu_* x) \\ &\quad - q_1(x, t)C_1(1 + \varepsilon^{-1/2} \exp(-\mu_* x)) \\ &= C_1 \left[-q_1(x, t) + \varepsilon^{-1/2}(-q_1(x, t) + q_* - a(x)(q_* \varepsilon)^{-1/2}) \exp(-\mu_* x) \right] \\ &\leq -C_1 \left[q_1(x, t) + \varepsilon^{-1/2}(q_1(x, t) - q_*) \exp(-\mu_* x) \right]. \end{aligned} \quad (5.9)$$

Consider the first derivative of the solution of problem (5.5) in space, that is $u_x(x, t)$, in the domain $[0, \theta_0]$. At $x = 0$, we rewrite equation (5.5) in the form

$$\varepsilon u_{xx}(x, t) + (a(x)u(x, t))_x - (a_x(x) + b(x, t))u(x, t) = g(x, t) + u_t(x, t),$$

and integrate from 0 to the point x^* , which satisfies

$$u_x(x^*, t) = \frac{u(\sqrt{\varepsilon}, t) - u(0, t)}{\sqrt{\varepsilon}}, \quad x^* \in (0, \sqrt{\varepsilon}), \quad (5.10)$$

to obtain

$$\varepsilon u_x(0, t) = \varepsilon u_x(x^*, t) + a(x^*)u(x^*, t) - \int_0^{x^*} g_1(x, t) dx, \quad (5.11)$$

where $g_1(x, t) = g(x, t) + u_t(x, t) + (a_x(x) + b(x, t))u(x, t)$ and satisfies the bound $|g_1(x, t)| \leq C_2\varepsilon^{-1/2}$. Further simplification of equation (5.11) yields

$$\begin{aligned} u_x(0, t) &= \varepsilon^{-1/2} + C_3\varepsilon^{-1} - C_4\varepsilon^{-1}\varepsilon^{-1/2} \leq C_5\varepsilon^{-1/2} \\ |u_x(0, t)| &\leq C_5\varepsilon^{-1/2}. \end{aligned} \quad (5.12)$$

At $x = \theta_1 \in (0, \theta_0)$, from the mean value theorem we have

$$u_x(\theta_1, t) = \frac{u(\theta_0, t) - u(0, t)}{\theta_0}.$$

It follows that

$$|u_x(\theta_1, t)| \leq \theta_0^{-1}[|u(\theta_0, t)| + |u(0, t)|] = C_6. \quad (5.13)$$

Now the operator $\mathcal{L}_1$ on $u_x(x, t)$ yields

$$\begin{aligned} \mathcal{L}_1 u_x(x, t) &= \mathcal{L}_x u_x + 1a'(x)u_x = \varepsilon u_{xxx} + a(x)u_{xx} - q_1(x, t)u_x \\ &= \varepsilon u_{xxx} + a(x)u_{xx} - b(x, t)u_x + a'(x)u_x \\ &= (\varepsilon u_{xx} + a(x)u_x - b(x, t)u)_x - a'(x)u_x + b_x(x, t)u + a'(x)u_x \\ &= g_x(x, t) + b_x(x, t)u \\ |\mathcal{L}_1 u_x(x, t)| &\leq C_*(1 + \varepsilon^{-1/2} \exp(-\mu_* x)). \end{aligned} \quad (5.14)$$

Suppose $\mathcal{H}(x, t)$ is a function which satisfies $\mathcal{H}(x, t) = \rho(x, t) - |u_x(x, t)|$, so that we have

$$\begin{aligned} \mathcal{H}(0, t) &= \rho(0, t) - |u_x(0, t)| \\ &= C_1(1 + \varepsilon^{-1/2}) - C_5\varepsilon^{-1/2} \geq 0 \\ \mathcal{H}(\theta_1, t) &= \rho(\theta_1, t) - |u_x(\theta_1, t)| \\ &= C_1(1 + \varepsilon^{-1/2} \exp(-\mu_* \theta_1)) - C_6 \geq 0, \end{aligned}$$

at $x = 0$ and $x = \theta_1$, respectively. The operator $\mathcal{L}_1$ on $\mathcal{H}(x, t)$ yields

$$\begin{aligned} \mathcal{L}_1 \mathcal{H}(x, t) &= \mathcal{L}_1 \rho(x, t) - \mathcal{L}_1 |u_x(x, t)| \\ &= -C_1 \left[q_1(x, t) + \varepsilon^{1/2}(q_1(x, t) - q_*) \exp(-\mu_* x) \right] - C_*(1 + \varepsilon^{-1/2} \exp(-\mu_* x)) \\ &\leq 0, \end{aligned} \quad (5.15)$$

where the estimate $\mathcal{L}_\varepsilon |\kappa| \leq |\mathcal{L}_\varepsilon \kappa|$, has been used. From the minimum principle,

$$\rho_1(x, t) - |u_x(x, t)| \geq 0 \quad \forall x \in [0, \theta_1] \text{ and a fixed } t \in [0, T].$$

Therefore $|u_x(x, t)| \leq C(1 + \varepsilon^{-1/2} \exp(-\mu_* x))$. To estimate the solution in the domain $[\theta_0, 1]$, we rewrite equation (5.1) as

$$\varepsilon u_{xx}(x, t) + a(x)u_x(x, t) = h_1(x, t),$$

where $h_1(x, t)$ satisfies $h_1(x, t) = f(x, t) + b(x, t)u(x, t) + u_t(x, t)$. Let $A(x) = \int_{\theta_0}^x a(t)dt$ be the integration factor so that we have

$$\begin{aligned} \exp(\varepsilon^{-1}A(x))u_{xx} + \varepsilon^{-1}a(x)\exp(\varepsilon^{-1}A(x))u_x &= \varepsilon^{-1}\exp(\varepsilon^{-1}A(x))h_1(x, t), \\ \frac{d}{dx} \left(\exp(\varepsilon^{-1}A(x)) \frac{du}{dx} \right) &= \varepsilon^{-1}\exp(\varepsilon^{-1}A(x))h_1(x, t). \end{aligned} \quad (5.16)$$

Integrating (5.16) from θ_1 to x yields

$$\exp(\varepsilon^{-1}A(x))u_x(x, t) - \exp(\varepsilon^{-1}A(\theta_1))u_x(\theta_1, t) = \varepsilon^{-1} \int_{\theta_1}^x h_1(\gamma, t) \exp(\varepsilon^{-1}A(\gamma))d\gamma.$$

Further simplification yields

$$u_x(x, t) = \exp(\varepsilon^{-1}(A(\theta_1) - A(x)))u_x(\theta_1, t) + z_1(x, t), \quad (5.17)$$

where

$$z_1(x, t) = \varepsilon^{-1} \int_{\theta_1}^x h_1(\gamma, t) \exp(\varepsilon^{-1}(A(\gamma) - A(x)))d\gamma$$

and is bounded by a constant. From (5.13), Equation (5.17) now takes the form

$$|u_x(x, t)| \leq C_6 |\exp(\varepsilon^{-1}(A(\theta_1)))| |\exp(-\varepsilon^{-1}A(x))| + |z_1(x, t)| = C.$$

A combination of these bounds in each sub-domain proves the Lemma. $\square$

Lemma 5.2.5. *The solution $u(x, t)$ of problem (5.1)–(5.2) satisfies the estimate $|u_{tt}(x, t)| \leq C_3^*$, $(x, t) \in \bar{Q}$.*

Proof. The lines in this proof follows that of the Lemma 2.2.6 in Chapter 2. $\square$

Lemma 5.2.6. *Let $u(x, t)$ be the solution of problem (5.5)–(5.6), then it satisfies*

$$|u_{xx}(x, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu_* x)), \quad (x, t) \in \bar{Q}.$$

Proof. The proof of this Lemma follows the same lines as in the proof of Lemma 5.2.4. We define the operator $\mathcal{L}_2$ by

$$\begin{aligned} \mathcal{L}_2 u &\equiv \mathcal{L}_x u + 2a'(x)u = \varepsilon u'' + a(x)u' - b(x, t)u + 2a'(x)u, \\ &= \varepsilon u'' + a(x)u' - q_2(x, t)u. \end{aligned} \quad (5.18)$$

Now let $\rho_2(x, t)$ be a barrier function given by

$$\rho_2(x, t) = C_2(1 + \varepsilon^{-1} \exp(-\mu_* x)), \quad x \in [0, \theta_0].$$

At $x = 0$ and $x = \theta_2 \in [0, \theta_0]$, we have

$$\rho_2(0, t) = C_2(1 + \varepsilon^{-1}), \quad \rho_2(\theta_2, t) = C_2(1 + \varepsilon^{-1} \exp(-\mu_* \theta_2)). \quad (5.19)$$

On the domain $(0, \theta_2)$, we get

$$\begin{aligned}
\mathcal{L}_2 \rho_2(x, t) &= \varepsilon C_2 (-\mu_*)^2 \varepsilon^{-1} \exp(-\mu_* x) + a(x) C_2 (-\mu_*) \varepsilon^{-1} \exp(-\mu_* x) \\
&\quad - q_2(x, t) C_2 (1 + \varepsilon^{-1} \exp(-\mu_* x)), \\
&= C_2 \left[-q_2(x, t) - \varepsilon^{-1} (q_2(x, t) - q_* + a(x) (q_* \varepsilon)^{-1/2}) \exp(-\mu_* x) \right] \\
&\leq -C_2 \left[q_2(x, t) + \varepsilon^{-1} (q_2(x, t) - q_*) \exp(-\mu_* x) \right].
\end{aligned} \tag{5.20}$$

Consider the second derivative of the solution of equation (5.5) in space, that is $u_{xx}(x, t)$, in the domain $[0, \theta_0]$. At $x = 0$, we obtain

$$u_{xx}(0, t) = \varepsilon^{-1} g_2(x, t), \tag{5.21}$$

where $g_2(x, t) = g^2(x, t) + u_t(x, t) + b(x, t)u(x, t)$ and satisfies the bound $|g_2(x, t)| \leq C\varepsilon^{-1}$. Further simplification of equation (5.21) yields

$$|u_{xx}(0, t)| \leq \varepsilon^{-1} (C\varepsilon^{-1}) = C_7 \varepsilon^{-2}. \tag{5.22}$$

At $x = \theta_2 \in (0, \theta_0)$, we let $v(x, t)$ be a function given by

$$v(x, t) = u\left(x + \frac{\theta_0}{2}, t\right) - u(x, t), \tag{5.23}$$

where $x \in \left[0, \frac{\theta_0}{2}\right]$, $\left[0, \frac{\theta_0}{2}\right] \subset [0, \theta_0]$ and t is fixed in $[0, T]$. Clearly,

$$v\left(\frac{\theta_0}{2}, t\right) = u(\theta_0, t) - u\left(\frac{\theta_0}{2}, t\right) \quad \text{and} \quad v(0, t) = u\left(\frac{\theta_0}{2}, t\right) - u(0, t).$$

Thus

$$v\left(\frac{\theta_0}{2}, t\right) - v(0, t) = u(\theta_0, t) - 2u\left(\frac{\theta_0}{2}, t\right) + u(0, t).$$

From the mean value theorem, there exist some point $\theta_1 \in (0, \frac{\theta_0}{2})$ such

$$\begin{aligned}
v_x(\theta_1, t) &= \frac{v\left(\frac{\theta_0}{2}, t\right) - v(0, t)}{\theta_0/2}, \\
&= \frac{u(\theta_0, t) - 2u\left(\frac{\theta_0}{2}, t\right) + u(0, t)}{\theta_0/2}
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
v\left(\frac{\theta_0}{2}\right) - v(0, t) &= \frac{b-a}{2} v_x(x, \theta_1) \\
&= \frac{b-a}{2} \left(u'\left(\theta_1 + \frac{b-a}{2}\right) - u'(\theta_1) \right).
\end{aligned} \tag{5.25}$$

From (5.23), (5.24) can be written as

$$u_x\left(\theta_1 + \frac{\theta_0}{2}, t\right) - u_x(\theta_1, t) = \frac{u(\theta_0, t) - 2u\left(\frac{\theta_0}{2}, t\right) + u(0, t)}{\theta_0/2}. \tag{5.26}$$

Again we apply the mean value theorem for u_x over $\left[\theta_1, \theta_1 + \frac{\theta_0}{2}\right] \subset [0, \theta_0]$. Thus for some $\theta_2 \in \left(\theta_1, \theta_1 + \frac{\theta_0}{2}\right)$, we have

$$u_{xx}(\theta_2, t) = \frac{u_x\left(\theta_1 + \frac{\theta_0}{2}, t\right) - u_x(\theta_1, t)}{\theta_0/2}, \quad (5.27)$$

which implies

$$u_x\left(\theta_1 + \frac{\theta_0}{2}, t\right) = \frac{\theta_0}{2} u_{xx}(\theta_2, t) + u_x(\theta_1, t). \quad (5.28)$$

Substituting (5.27) into (5.26) and further simplification result in

$$u_{xx}(\theta_2, t) = 4\theta_0^{-2} \left[u(\theta_0, t) - 2u\left(\frac{\theta_0}{2}, t\right) + u(0, t) \right],$$

which yields

$$|u_{xx}(\theta_2, t)| = 4\theta_0^{-2} \left| u(\theta_0, t) - 2u\left(\frac{\theta_0}{2}, t\right) + u(0, t) \right|.$$

It follows that

$$|u_{xx}(\theta_2, t)| \leq 4\theta_0^{-2} \left[|u(\theta_0, t)| + 2 \left| u\left(\frac{\theta_0}{2}, t\right) \right| + |u(0, t)| \right] = C_8.$$

Now the operator $\mathcal{L}_2$ on $u_{xx}(x, t)$ yields

$$\begin{aligned} \mathcal{L}_2 u_{xx}(x, t) &= \mathcal{L}_x u_{xx} + 2a'(x)u_{xx} = \varepsilon u_{xxxx} + a(x)u_{xxx} - q_2(x, t)u_{xx}, \\ &= \varepsilon u_{xxxx} + a(x)u_{xxx} - b(x, t)u_{xx} + 2a'(x)u_{xx}, \\ &= (\varepsilon u_{xx} + a(x)u_x - b(x, t)u)_{xx} - 2a'(x)u_{xx} + 2b_x(x, t)u_x - b_{xx}u + 2a'(x)u_{xx}, \\ &= g_{xx}(x, t) + 2b_x(x, t)u_x - b_{xx}u. \end{aligned}$$

Taking the norm yields

$$|\mathcal{L}_2 u_{xx}(x, t)| \leq C_*(1 + \varepsilon^{-1} \exp(-\mu_* x)). \quad (5.29)$$

Suppose $\mathcal{H}(x, t)$ is a function which satisfies $\mathcal{H}(x, t) = \rho(x, t) - |u_{xx}(x, t)|$, so that we have

$$\begin{aligned} \mathcal{H}(0, t) &= \rho(0, t) - |u_{xx}(0, t)| \\ &= C_2(1 + \varepsilon^{-1}) - C_7\varepsilon^{-2} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}(\theta_2, t) &= \rho(\theta_2, t) - |u_{xx}(\theta_2, t)| \\ &= C_2(1 + \varepsilon^{-1} \exp(-\mu_* \theta_2)) - C_8 \geq 0, \end{aligned}$$

at $x = 0$ and $x = \theta_2$, respectively, provided C_2 greater than C_7 and C_8 . The operator $\mathcal{L}_2$ on $\mathcal{H}(x, t)$ yields

$$\begin{aligned} \mathcal{L}_2 \mathcal{H}(x, t) &= \mathcal{L}_2 \rho(x, t) - \mathcal{L}_2 |u_{xx}(x, t)| \\ &= -C_2 [q_2(x, t) + \varepsilon^{-1}(q_2(x, t) - q_*) \exp(-\mu_* x)] - C_*(1 + \varepsilon^{-1} \exp(-\mu_* x)), \\ &\leq 0. \end{aligned} \quad (5.30)$$

The minimum principle gives

$$\rho_2(x, t) - |u_{xx}(x, t)| \geq 0, \quad \forall x \in [0, \theta_2] \text{ and a fixed } t \in [0, T].$$

Therefore $|u_{xx}(x, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu x))$. The estimate of the second spatial derivative in the domain $[\theta_0, 1]$ is similar to that of the first derivative. $\square$

Theorem 5.2.1. *The solution and its partial derivatives are such that,*

$$\left| \frac{\partial^{i+j} u(x, t)}{\partial x^i \partial t^j} \right| \leq C(1 + \varepsilon^{-i/2} \exp(-\mu x)), \quad (x, t) \in Q,$$

where $\mu = \sqrt{q_*/\varepsilon}$ and i and j are non-negative integers which satisfy $0 \leq i + j \leq 6$.

Proof. The proof of this Theorem for $0 \leq i + j \leq 2$, follows from the Lemmas 5.2.2, 5.2.3, 5.2.4, 5.2.5. The proof of the higher order derivatives, that is $2 \leq i + j \leq 6$, can be obtained analogously. $\square$

5.3 Time Discretisation

Here we transform problem (5.1)–(5.2) to a boundary value problem by discretising in time, whilst leaving the spatial domain continuous. The discretisation in time via the Crank Nicolson finite difference scheme on a uniform mesh yields the following semi-discrete boundary value problems

$$u(x, 0) = \varphi(x), \quad x \in \bar{\Omega}, \quad (5.31)$$

$$\begin{aligned} \mathcal{L}_\varepsilon^* u(x, t_{k+1/2}) &\equiv -\frac{u^{k+1} - u^k}{\Delta t} + \varepsilon u_{xx}(x, t_{k+1/2}) + a(x)u_x(x, t_{k+1/2}) \\ &\quad - b(x, t_{k+1/2})u(x, t_{k+1/2}) = f(x, t_{k+1/2}), \\ k &= 0, 1, 2, \dots, m, \end{aligned} \quad (5.32)$$

along with the boundary conditions

$$u(0, t_{k+1}) = \eta_0(t_{k+1}), \quad u(1, t_{k+1}) = \eta_1(t_{k+1}). \quad (5.33)$$

The notation $Z(x, t_{k+1/2})$ is given by

$$Z(x, t_{k+1/2}) = \frac{Z(x, t_{k+1}) + Z(x, t_k)}{2}.$$

The scheme (5.31)–(5.33) is rewritten as

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad (5.34)$$

$$\mathcal{L}_\varepsilon^m u^{k+1}(x) \equiv \frac{\varepsilon}{2} u_{xx}^{k+1}(x) + \frac{a(x)}{2} u_x^{k+1}(x) - d(x) u^{k+1}(x) = g(x, t_{k+1}), \quad (5.35)$$

$$u^{k+1}(0) = \eta_0(t_{k+1}), \quad u^{k+1}(1) = \eta_1(t_{k+1}), \quad (5.36)$$

where

$$g(x, t_{k+1}) = f(x, t_{k+1/2}) - \frac{\varepsilon}{2} u_{xx}^k(x) - \frac{a(x)}{2} u_x^k(x) - e(x) u^k(x),$$

$$d(x) = (1/\Delta t + b(x, t_{k+1})/2),$$

and

$$e(x) = (1/\Delta t) - b(x, t_{k+1})/2,$$

or the scheme (5.31)–(5.33) is rewritten as

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad (5.37)$$

$$\mathcal{L}_\varepsilon^{**} u^{k+1}(x) \equiv \frac{\Delta t \varepsilon}{2} u_{xx}^{k+1}(x) + \frac{\Delta t a(x)}{2} u_x^{k+1}(x) - c(x) u^{k+1}(x) = g(x, t_{k+1}), \quad (5.38)$$

$$u^{k+1}(0) = \eta_0(t_{k+1}), \quad u^{k+1}(1) = \eta_1(t_{k+1}), \quad (5.39)$$

where

$$g(x, t_{k+1}) = f(x, t_{k+1/2}) - \frac{\Delta t \varepsilon}{2} u_{xx}^k(x) - \frac{a(x) \Delta t}{2} u_x^k(x) - p(x) u^k(x),$$

$$c(x) = 1 + \Delta t (b(x, t_{k+1})/2),$$

and

$$p(x) = 1 - \Delta t b(x, t_{k+1})/2.$$

Lemma 5.3.1. *Semi-discrete minimum principle*

Let $\Psi^{k+1}(x) \in \mathcal{C}^4(\bar{\Omega})$. If $\Psi^{k+1}(x) \geq 0$, $\forall x \in \partial\Omega$ and $\mathcal{L}_\varepsilon^m \Psi^{k+1}(x) \leq 0$, for $x \in \Omega$ then, $\Psi^{k+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. Suppose there exist $x^* \in \bar{\Omega}$ such that

$$\Psi^{k+1}(x^*) = \min_{x \in \bar{\Omega}} \Psi^{k+1}(x) < 0.$$

It is evident that $x^* \neq 0$, nor 1 but $x^* \in \Omega$. Now applying the differential operator $\mathcal{L}_\varepsilon^{**}$ on Ψ^{k+1} gives

$$\mathcal{L}_\varepsilon^{**} \Psi^{k+1}(x^*) = \frac{\Delta t \varepsilon}{2} \Psi_{xx}^{k+1}(x^*) + \frac{\Delta t a(x^*)}{2} \Psi_x^{k+1}(x^*) - c(x^*) \Psi^{k+1}(x^*).$$

Notice that $\Psi_{xx}^{k+1}(x^*) \geq 0$, and $\Psi_x^{k+1}(x^*) = 0$, $\mathcal{L}_\varepsilon^{**} \Psi^{k+1}(x^*) \geq 0$, which is a contradiction. Thus $\Psi^{k+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$, as required. $\square$

5.3.1 Error Estimate

In this section, we analyse the time discretisation process for convergence.

Lemma 5.3.2. *The local truncation error of the time semi-discretisation satisfies*

$$\|e^{k+1}\| \leq C(\Delta t)^3. \quad (5.40)$$

Proof. The local truncation error of the time semi-discetization method is given by

$$e^{k+1} = u(x, t_{k+1}) - \hat{u}(x, t_{k+1}), \quad (5.41)$$

where $\hat{u}^{k+1}$ is the computed solution of

$$\hat{u}(x, 0) = \varphi(x), \quad x \in \Omega, \quad (5.42)$$

$$\mathcal{L}_\varepsilon^{**} \hat{u}^{k+1}(x) \equiv \frac{\Delta t}{2} \varepsilon \hat{u}_{xx}^{k+1}(x) + \frac{\Delta t}{2} a(x) \hat{u}_x^{k+1}(x) - c(x) \hat{u}^{k+1}(x) = g(x, t_{k+1}), \quad (5.43)$$

$$\hat{u}^{k+1}(0) = \eta_0(t_{k+1}), \quad \hat{u}^{k+1}(1) = \eta_1(t_{k+1}), \quad (5.44)$$

where

$$g(x, t_{k+1}) = \Delta t \left(f(x, t_{k+1/2}) - \frac{\varepsilon}{2} \hat{u}_{xx}^k(x) - \frac{a(x)}{2} \hat{u}_x^k(x) - p(x) \hat{u}^k(x) \right),$$

$$c(x) = (1 + b(x, t_{k+1})/2),$$

and

$$p(x) = (1 - b(x, t_{k+1})/2).$$

Truncated Taylor series expansions of $u(x, t_{k+1})$ and $u(x, t_k)$ take the form

$$u(x, t_{k+1}) = u(x, t_{k+1/2}) + \frac{\Delta t}{2} u_t(x, t_{k+1/2}) + \frac{\Delta t^2}{8} u_{tt}(x, t_{k+1/2}) + \mathcal{O}(\Delta t)^3 \quad (5.45)$$

and

$$u(x, t_k) = u(x, t_{k+1/2}) - \frac{\Delta t}{2} u_t(x, t_{k+1/2}) + \frac{\Delta t^2}{8} u_{tt}(x, t_{k+1/2}) + \mathcal{O}(\Delta t)^3, \quad (5.46)$$

respectively. From Equations (5.45) and (5.46), we have

$$\begin{aligned} \frac{u(x, t_{k+1}) - u(x, t_k)}{\Delta t} &= u_t(x, t_{k+1/2}) + \mathcal{O}((\Delta t)^2), \\ &= -\varepsilon u_{xx}(x, t_{k+1/2}) - a(x) u_x(x, t_{k+1/2}) + c(x) u(x, t_{k+1/2}) \\ &\quad + g(x, t_{k+1/2}) + \mathcal{O}(\Delta t)^2. \end{aligned} \quad (5.47)$$

Further simplification yields

$$\mathcal{L}_\varepsilon^{**} u^{k+1}(x) \equiv \frac{\Delta t}{2} \varepsilon u_{xx}^{k+1}(x) + \frac{\Delta t}{2} a(x) u_x^{k+1}(x) - c(x) u^{k+1}(x) = g(x, t_{k+1}) + \mathcal{O}(\Delta t)^3. \quad (5.48)$$

From Equation (5.43) and (5.48), the local truncation error satisfies

$$\mathcal{L}_\varepsilon^{**} e^{k+1} = \mathcal{O}((\Delta t)^3)$$

and

$$e^{k+1}(0) = e^{k+1}(1) = 0.$$

Now let $\mathcal{R}(x)$ be a function given by

$$\mathcal{R}(x) = C(\Delta t)^3, \quad 0 \leq x \leq 1. \quad (5.49)$$

At the boundaries, we have $\mathcal{R}(0) = C(\Delta t)^3$ and $\mathcal{R}(1) = C(\Delta t)^3$ and applying the differential operator $\mathcal{L}_\varepsilon^{**}$ on $\mathcal{R}(x)$ yields

$$\mathcal{L}_\varepsilon^{**}\mathcal{R}(x) = -c^{k+1}C(\Delta t)^3. \quad (5.50)$$

Let $\mathcal{H}(x) = \mathcal{R}(x) - |e^{k+1}(x)|$ be another function which satisfies

$$\mathcal{H}(0) = C(\Delta t)^3 - |e^{k+1}(0)| \geq 0, \text{ and } \mathcal{H}(1) = C(\Delta t)^3 - |e^{k+1}(1)| \geq 0, \quad (5.51)$$

at the boundary of Ω and

$$\mathcal{L}_\varepsilon^{**}\mathcal{H}(x) = \mathcal{L}_\varepsilon^{**}\mathcal{R}(x) - \mathcal{L}_\varepsilon^{**}|e^{k+1}(x)| = -c^{k+1}C(\Delta t)^3 - \mathcal{L}_\varepsilon^{**}|e^{k+1}(x)| \leq 0, \quad (5.52)$$

at the interior of Ω . From the Lemma 5.2.1, we obtain

$$\mathcal{H}(x) = \mathcal{R}(x) - |e^{k+1}(x)| \geq 0.$$

It follows that

$$|e^{k+1}(x)| \leq \mathcal{R}(x)$$

and hence

$$|e^{k+1}(x)| \leq C(\Delta t)^3,$$

which completes the proof. $\square$

The global error of the time discretisation satisfies the next result.

Lemma 5.3.3. *The global error $\mathcal{E}^k$ takes the form*

$$\|\mathcal{E}^m\| \leq C(\Delta t)^2. \quad (5.53)$$

Proof.

$$\|\mathcal{E}^m\| \leq \left| \sum_{k=1}^m e^{k+1}(x) \right| \leq Cm(\Delta t)^3 = C(\Delta t)^2. \quad (5.54)$$

$\square$

Next we design a fitted operator finite difference scheme based on the midpoint downwind scheme to discretise problem (5.34)–(5.36) in space.

5.4 Spatial Discretisation

In this section, the boundary value problem (5.34)–(5.36) is discretised via a fitted operator finite difference scheme. We employ the notation $U_i^{k+1} \equiv U^{k+1}(x_i) \equiv U(x_i, t_{k+1})$ as the

numerical solution of $u^{k+1}(x)$ and perform the discretisation on a uniform mesh as follows:

$$u(x, 0) = \varphi(x_i), \quad x \in \bar{\Omega}^n, \quad (5.55)$$

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,n} U^{k+1}(x_i) &\equiv \frac{\varepsilon}{2} \delta^2(U^{k+1}(x_i)) + \frac{a(x_{i+1/2})}{2} (D^+ U^{k+1}(x_i)) - d(x_{i+1/2}) U^{k+1}(x_i), \\ &= g(x_{i+1/2}, t_{k+1}), \end{aligned} \quad (5.56)$$

$$u^{k+1}(0) = \eta_0(t_{k+1}), \quad u^{k+1}(1) = \eta_1(t_{k+1}). \quad (5.57)$$

The source term $g(x_{i+1/2}, t_{k+1})$ takes the form

$$\begin{aligned} g(x_{i+1/2}, t_{k+1}) &= \frac{f(x_{i+1/2}, t_{k+1}) + f(x_{i+1/2}, t_k)}{2} - \frac{\varepsilon}{2} \delta^2 U^k(x_i) \\ &\quad - \frac{a(x_{i+1/2})}{2} D^+ U^k(x_i) - e(x_{i+1/2}) U^k(x_i). \end{aligned} \quad (5.58)$$

The notations D^+ , δ^2 are the first and second order finite difference schemes on appropriate denominator functions and are given by

$$D^+ z_i = \frac{z_{i+1} - z_i}{h}, \quad \delta^2 z = \frac{z_{i+1} - 2z_i + z_{i-1}}{\phi_i^2(\varepsilon, h)}, \quad (5.59)$$

respectively. The denominator function $\phi_i^2(h, \varepsilon)$ is of the form

$$\phi_i^2(h, \varepsilon) = \frac{\varepsilon h}{a_i} \left(1 - \exp \left(\frac{-a(x_{i+1/2})h}{\varepsilon} \right) \right), \quad (5.60)$$

and satisfies

$$\phi_i^2(h, \varepsilon) = h^2 + \mathcal{O}(h^3).$$

The scheme (5.55)–(5.57) can be represented in a compact form

$$U + AU = F, \quad (5.61)$$

where U and F are vectors of size $(n-1)$ and A is a tridiagonal matrix of size $(n-1) \times (n-1)$. Their entries are as follows:

$$\begin{aligned} F_1 &= g_1(x_{i+1/2}, t_{k+1}) - r_0^- \eta_0(t_{k+1}), \\ F_i &= g_i, \quad i = 2(1)n - 2, \\ F_{n-1} &= g_{n-1}(x_{i+1/2}, t_{k+1}) - r_{n-1}^+ \eta_n(t_{k+1}), \\ U &= [U_1, U_2, \dots, U_{n-1}] \end{aligned}$$

and

$$A_{ij} = \begin{cases} r_i^-, & j = i - 1, \\ r_i^c, & i = j, \\ r_i^+, & j = i + 1, \end{cases}$$

where the r_i^- , r_i^c and r_i^+ are given by

$$r_i^- = \frac{\varepsilon}{2\phi_i^{-2}(h, \varepsilon)}, \quad i = 2, 3, \dots, n-1,$$

$$r_i^c = -\frac{\varepsilon}{\phi_i^2(h, \varepsilon)} - \frac{a(x_{i+1/2})}{2h} - d(x_{i+1/2}), \quad i = 1, 2, \dots, n-1,$$

$$r_i^+ = \frac{\varepsilon}{2\phi_i^{-2}(h, \varepsilon)} + \frac{a(x_{i+1/2})}{2h}, \quad i = 1, 2, \dots, n-2,$$

respectively. The operator $\mathcal{L}_\varepsilon^{m,n}$ satisfies a discrete minimum principle which is presented in the next Lemma. For simplicity, we have dropped the superscript indices.

5.5 Stability of the Scheme

Below we show that the spatial discretisation is stable and that the scheme (5.55)–(5.56)–(5.57) has a unique solution.

Lemma 5.5.1. *Discrete minimum principle*

Let $\xi(x_i)$ be a mesh function defined on $\bar{\Omega}^n$, satisfies $\xi(x_i) \geq 0, \forall (x_i) \in \partial\Omega^n$ and $\mathcal{L}_\varepsilon^{m,n}\xi(x_i) \leq 0, \forall (x_i) \in \Omega^n$. Then $\xi(x_i) \geq 0, \forall (x_i) \in \bar{\Omega}^n$.

Proof. The matrix A is a square tridiagonal matrix which assumes the properties of an M-matrix thus the scheme is positive. $\square$

From Lemma 5.5.1 we prove further that the scheme (5.55)–(5.57) satisfies a uniform stability estimate below.

Lemma 5.5.2. *The operator $\mathcal{L}_\varepsilon^{m,n}$ is uniformly stable. That is the solution u_i of the discrete problem (5.55)–(5.57) satisfies*

$$|u_i| \leq \rho^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,n} u_i| + \max_{x_i \in \bar{\Omega}} (|\eta_0|, |\eta_1|).$$

Proof. Let

$$z = \rho^{-1} \max_{(x_i) \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,n} u_i| + \max_{(x_i) \in \bar{\Omega}} (|\eta_0|, |\eta_1|),$$

and define the mesh function $\Psi_i^\pm$ by

$$\Psi_i^\pm = z \pm u_i.$$

At $i = 0$ and $i = n$, we have

$$\begin{aligned} \Psi_0^\pm &= z \pm u_i = z \pm \eta_0 \geq 0, \\ \Psi_n^\pm &= z \pm u_i = z \pm \eta_n \geq 0. \end{aligned}$$

Further on the domain $x_i \in \Omega^n$, we obtain

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,n} \Psi_i^\pm &= \frac{\varepsilon}{2} \left(\frac{z + u_{i+1} - 2(z + u_i) + z \pm u_{i-1}}{\phi_i^2(h, \varepsilon)} \right) + \frac{a_{i+1/2}}{2} \left(\frac{z \pm u_{i+1} - (z \pm u_i)}{h} \right) - d_{i+1/2}(z \pm u_i), \\ &= -d_{i+1/2} z \pm \mathcal{L}_\varepsilon^{m,n} u_i, \\ &= -d_{i+1/2} \left(\rho^{-1} \max_{(x_i) \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,n} u_i| + \max_{(x_i) \in \bar{\Omega}} (|\eta_0|, |\eta_n|) \right) \pm g_{i+1/2} \leq 0. \end{aligned}$$

From Lemma 5.5.1, $\Psi_i^\pm \geq 0, \forall (x_i) \in \bar{\Omega}^n$, as required. $\square$

5.6 Error Estimate

Below, the error associated with the spatial discretisation is estimated as follows:

Lemma 5.6.1. *The error associated with the fitted operator finite difference scheme based on the midpoint satisfies*

$$|u_i - U_i| \leq Ch^2.$$

Proof. The spatial discretisation error is such that

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,n}(u(x_i) - U(x_i)) &= \mathcal{L}_\varepsilon^{m,n}u(x_i) - \mathcal{L}_\varepsilon^{m,n}U(x_i) = \mathcal{L}_\varepsilon^{m,n}u(x_i) - g_{i+1/2} \\ &= \mathcal{L}_\varepsilon^{m,n}u(x_i) - \mathcal{L}_\varepsilon^m u(x_{i+1/2}) \\ &= \frac{\varepsilon}{2} (\delta^2 u_i - u''(x_{i+1/2})) + \frac{a_{i+1/2}}{2} (D^+ u_i - u'(x_{i+1/2})). \end{aligned}$$

A truncated Taylor series expansion and of the terms u_{i+1} and ϕ_i^2 and simplifying further results in

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,n}(u(x_i) - U(x_i)) &= \frac{\varepsilon}{2h^2} \left(h^2 u^2 + \frac{h^4}{4!} u^{iv} + \frac{h^6}{16.5!} u^{vi} + \dots \right) - \frac{\varepsilon}{2} u'' + \frac{a_{i+1/2} h^2}{8.3!} u''' \\ &\quad + \frac{a_{i+1/2} h^4}{32.5!} u^v + \dots, \\ &= \left(\frac{\varepsilon}{2.4!} u^{iv} + \frac{a_{i+1/2}}{8.3!} u^{iii} \right) h^2 + \left(\frac{\varepsilon}{32.5!} u^{vi} + \frac{a_{i+1/2}}{32.5!} u^v \right) h^4 + \dots \end{aligned}$$

From Theorem 5.2.1 and noticing that the exponential terms vanish as $\varepsilon \rightarrow 0$ see Lemma 2.4.3 yields

$$|\mathcal{L}_\varepsilon^{m,n}(u_i - U_i)| \leq Ch^2.$$

Now let $\mathcal{R}_i$ be a mesh function defined by

$$\mathcal{R}_i = Ch^2, \quad 0 \leq i \leq n.$$

Notice that $\mathcal{R}_0 = \mathcal{R}_n = Ch^2$ and

$$\mathcal{L}_\varepsilon^{m,n} \mathcal{R}_i \equiv \frac{\varepsilon}{2\phi_i^2} (\mathcal{R}_{i+1} - 2\mathcal{R}_i + \mathcal{R}_{i-1}) + \frac{a_{i+1/2}}{2h} (\mathcal{R}_i - \mathcal{R}_{i-1}) - d_{i+1/2} \mathcal{R}_i = -d_{i+1/2} \mathcal{R}_i.$$

Suppose $\mathcal{H}_i$ is also another function given by

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i|.$$

At $i = 0$ and $i = n$, we have

$$\mathcal{H}_0 = \mathcal{R}_0 - |u_0 - U_0| = Ch^2 > 0, \quad \mathcal{H}_n = \mathcal{R}_n - |u_n - U_n| = Ch^2 > 0.$$

Further on $1 \leq i \leq n-1$, we have

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,n} \mathcal{H}_i &= \mathcal{L}_\varepsilon^{m,n} \mathcal{R}_i - \mathcal{L}_\varepsilon^{m,n} |u_i - U_i|, \\ &= -d_{i+1/2} Ch^2 - \mathcal{L}_\varepsilon^{m,n} |u_i - U_i| \leq 0. \end{aligned}$$

From Lemma 5.5.1, we obtain

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i| \geq 0,$$

and hence

$$|u_i - U_i| \leq \mathcal{R}_i,$$

which leads to

$$|u_i - U_i| \leq Ch^2.$$

□

The following theorem summarizes the main result in this chapter.

Theorem 5.6.1. (*Uniform Convergence*)

Let u be the exact solution of the continuous problem (5.1)-(5.2) and U be the approximate solution. The error associated with the proposed numerical scheme satisfies

$$\max_{0 \leq i \leq n; 0 \leq k \leq K} |u(x_i, t_k) - U(x_i, t_k)| \leq C(h^2 + \Delta t^2).$$

Proof. The proof of this theorem follows from Lemma 5.3.3 and 5.6.1. □

Next we apply the method on test problem to validate the theoretical estimate.

5.7 Numerical Results

In this section, we test the efficiency of the proposed scheme through numerical experiments. Similar to the previous examples in the early chapters the exact solution of the problem is not known, thus to compute the maximum pointwise errors we use the double mesh principle given by

$$E_{n,\Delta t}^1 = \max_{0 \leq i \leq n; 0 \leq k \leq m} |U_{i:n}^{k:m} - U_{i:2n}^{k:2m}|. \quad (5.62)$$

Notice that we have used $2n$ and $2K$ in the formula because the proposed scheme is of second order accuracy in both the time and the space variables. To compute the corresponding rate of convergence we use the formula

$$r_1 = \log_2 \left(\frac{E_{n,\Delta t}^1}{E_{2n,\Delta t}^1} \right). \quad (5.63)$$

Further ε -uniform error is obtained via the formula

$$E^1 = \max_{0 < \varepsilon \leq 1} E_{n,\Delta t}^1 \quad (5.64)$$

and the ε -uniform rate of convergence by

$$R_1 = \max_{0 < \varepsilon \leq 1} r_1 \quad (5.65)$$

Example 5.7.1 (Majumdar and Natesan (2019)). *Consider the parabolic degenerate problem*

$$\begin{aligned} -u_t(x, t) + \varepsilon u_{xx}(x, t) + x^p u_x(x, t) - (x+p)u(x, t) &= p(x^2 - 1)\exp(-t), (x, t) \in \omega \times (0, 1], \\ u(x, 0) &= (1 - x)^2, \quad x \in (0, 1) \quad u(0, t) = 0, \quad u(1, t) = 0, \quad t \in [0, 1]. \end{aligned}$$

Table 5.1: Maximum pointwise error and rate of convergence for Example 5.7.1.

ε	n=8	32	128	512	2048
	m=4	16	64	256	1024
10^{-4}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-5}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-6}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-7}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-8}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-9}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
10^{-10}	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
	0.9103	1.6813	1.9209	1.9803	
E^1	2.91E-02	1.55E-02	4.83E-03	1.28E-03	3.23E-04
R_1	0.9103	1.6813	1.9209	1.9803	

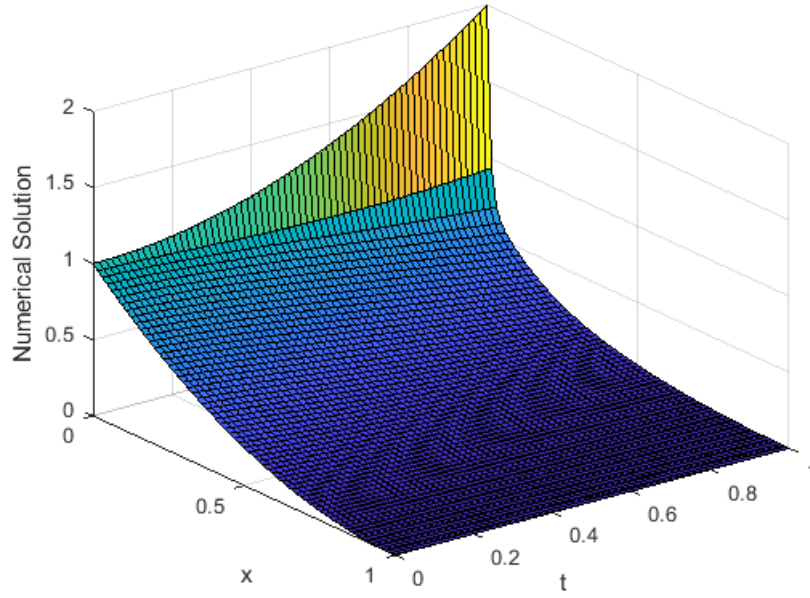


Figure 5.1: Surface plot of the numerical solution of Example 5.7.1 for $p = 1$, $n = 64$ and $\varepsilon = 10^{-24}$.

5.8 Conclusion

A second order numerical method was proposed to solve a parabolic convection diffusion problem with degenerate coefficient. The scheme employed the Crank Nicolson finite difference scheme and a fitted operator finite difference scheme to solve the aforementioned problem via the Rothe's method. It is important to note that the fitted operator finite difference scheme which was designed was based on the midpoint rule. Some properties of the discrete problem which ensured the stability of the scheme were presented and used to analyse the scheme for convergence. This analysis resulted in a full second order in both the space and the time variables. To test the efficiency of the scheme, numerical experiments were conducted and the results were displayed in Table 5.1. The uniform convergence rate can be observed from the table. Again surface plots of the numerical solution can be observed in Figure 5.1. As shown from the Figure 5.1, the left boundary layer in the solution can be observed.

In the next chapter, the fitted operator finite difference scheme which was considered in this chapter is employed along with the implicit Euler finite difference scheme to solve a modified Burgers equation.

Chapter 6

A Second Order Fitted Operator Finite Difference Scheme for a Modified Burgers Equation

6.1 Introduction

In the previous chapters we designed a fitted operator finite difference scheme to solve different forms of linear singularly perturbed problems. In this chapter, we employ this fitted operator finite difference scheme along with the implicit Euler finite difference scheme to solve a non-linear modified Burgers equation via the Rothe's method and analyse for convergence.

A one-dimensional non-linear modified Burger's equation

$$\mathcal{L}_\varepsilon u(x, t) \equiv u_t(x, t) - \varepsilon u_{xx}(x, t) + u^2 u_x(x, t) = 0, \quad (x, t) \in Q \quad (6.1)$$

subject to the initial and the boundary conditions

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T] \quad (6.2)$$

is considered in this study. Equation (6.1) comprises of an unsteady term u_t , a non-linear convection term $u^2 u_x(x, t)$, a viscous dissipation $u_{xx}(x, t)$ and a perturbation parameter $\varepsilon \in (0, 1]$ which is the dissipation coefficient and the inverse of an effective Reynolds number. Thus for large Reynolds number problem (6.1)–(6.2) can be classified as a one-dimensional time-dependent singularly perturbed non-linear convection diffusion problem. In this instance, its solution possesses a multiscale character in some parts of the domain. Therefore, classical numerical method can not serve as a good approximation technique to the exact solution especially in the parts of the domain where the rapid oscillations occur, thereby producing unsatisfactory results.

In this study, we seek to address the issue of unsatisfactory results of these classical numerical methods by modifying the classical finite difference method. We replace the denomi-

nator function in the classical finite difference method with a new positive function which reflects the analytical properties of the problem under study to design a fitted operator finite difference scheme. Notice that the fitted operator finite difference scheme is based on the midpoint upwind scheme. Using the Rothe's method or the transversal method of lines procedure, the backward Euler finite difference scheme is employed along with this fitted operator finite difference scheme to obtain the numerical solution.

The rest of the chapter is organised as follows: In Section 6.2, we integrate the non-linear problem in time and then analyse it for convergence. The resulting systems of semi-discrete non-linear boundary value problems are linearised and analysed for convergence in Section 6.3. In Section 6.4 we present a priori estimate of the solution of the semi-discrete boundary value problems and its derivatives. The fitted operator finite difference scheme is designed in Section 6.5. In Section 6.6 we study the stability of the scheme and analyse it for convergence in Section 6.7. Numerical experiments are conducted to in Section 6.8 whilst a summary of the main result in this Chapter is presented in the last section which is the conclusion part.

6.2 Time Discretisation

Below we transform the problem (6.1)–(6.2) into semi-discrete boundary value problems via the discretisation of the time variable. At this stage the spatial domain is held continuous. Using the backward Euler finite difference scheme on a uniform mesh, we integrate problem (6.1)–(6.2) in time to obtain the semi-discrete problem

$$\begin{aligned}\mathcal{L}_\varepsilon^{**}u^{k+1}(x) &\equiv \frac{u^{k+1} - u^k}{\Delta t} - \varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2(x)u_x^{k+1}(x) + b^{k+1}(x)u^{k+1}(x) \\ &= 0, \quad k = 0, 1, 2, \dots, m,\end{aligned}\tag{6.3}$$

along with the initial and boundary conditions,

$$u^0 = u(x, 0) = \varphi(x), \quad u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0.\tag{6.4}$$

The scheme (6.3)–(6.4) is rewritten as

$$u(x, 0) = \varphi(x),\tag{6.5}$$

$$\mathcal{L}_\varepsilon^m u^{k+1}(x) \equiv -\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x)u^{k+1}(x) = \frac{u^k(x)}{\Delta t},\tag{6.6}$$

$$u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0,\tag{6.7}$$

where

$$d^{k+1}(x) = 1/\Delta t + b^{k+1}(x), \quad d^{k+1}(x) \geq \gamma,$$

or

$$u(x, 0) = \varphi(x),\tag{6.8}$$

$$\begin{aligned}\mathcal{L}_\varepsilon^* u^{k+1}(x) &\equiv \Delta t(-\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x)u^{k+1}(x)) \\ &= u^k(x),\end{aligned}\tag{6.9}$$

$$u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0,\tag{6.10}$$

where

$$d^{k+1}(x) = 1/\Delta t + b^{k+1}(x), \quad d^{k+1} \geq \gamma.$$

Clearly the operator $\mathcal{L}_\varepsilon^*$ defined by the scheme (6.3)–(6.4) satisfies a discrete maximum principle which ensures the stability of the temporal semi-discretisation process.

Lemma 6.2.1. *Maximum principle*

Let $\Psi^{k+1}(x) \in \mathcal{C}^4(\bar{\Omega})$. If $\Psi^{k+1}(x) \geq 0$, $x \in \partial\Omega$ and $\mathcal{L}_\varepsilon^* \Psi^{k+1}(x) \geq 0$, $x \in \Omega$ then, $\Psi^{k+1}(x) \geq 0$, $x \in \bar{\Omega}$.

Proof. Suppose there exist $x^* \in \bar{\Omega}$ such that

$$\Psi^{k+1}(x^*) = \min_{x \in \bar{\Omega}} \Psi^{k+1}(x) < 0.$$

It is evident that $x^* \neq 0$, nor 1, but $x^* \in \Omega$. Now applying the differential operator $\mathcal{L}_\varepsilon^*$ on Ψ^{k+1} at the point x^* gives

$$\mathcal{L}_\varepsilon^* \Psi^{k+1}(x^*) = \Delta t(-\varepsilon \Psi_{xx}^{k+1}(x^*) + (\Psi^{k+1}(x^*))^2 \Psi_x^{k+1}(x^*) + d(x^*) \Psi^{k+1}(x^*)).$$

Since $\Psi_{xx}^{k+1}(x^*) \geq 0$, and $\Psi_x^{k+1}(x^*) = 0$, $\mathcal{L}_\varepsilon^* \Psi^{k+1}(x^*) \leq 0$, which is a contradiction. Thus $\Psi^{k+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$, as required. $\square$

6.2.1 Error Estimate

In this section, we analyse the time discretisation process for convergence.

Lemma 6.2.2. *The local truncation error of the temporal semi-discretisation process satisfies*

$$\|e^{k+1}\| \leq C^*(\Delta t)^2. \quad (6.11)$$

Proof. The local truncation error is defined as

$$e^{k+1} = u^{k+1}(x) - \bar{u}^{k+1}(x), \quad (6.12)$$

where $u^{k+1}(x)$, is the exact solution of (6.8)–(6.10) and $\bar{u}^{k+1}(x)$, is the computed solution of

$$\bar{u}(x, 0) = \varphi(x), \quad (6.13)$$

$$\begin{aligned} \mathcal{L}_\varepsilon^* \bar{u}^{k+1}(x) &\equiv \Delta t(-\varepsilon \bar{u}_{xx}^{k+1}(x) + (\bar{u}^{k+1})^2 \bar{u}_x^{k+1}(x) + d^{k+1}(x) \bar{u}^{k+1}(x)) \\ &= \bar{u}^k(x), \end{aligned} \quad (6.14)$$

$$\bar{u}^{k+1}(0) = 0, \quad \bar{u}^{k+1}(1) = 0. \quad (6.15)$$

A truncated Taylor series expansion of $u(x, t_k)$ takes the form

$$u^k(x) = u^{k+1}(x) - \Delta t u_t^{k+1}(x) + \frac{\Delta t^2}{2} u_{tt}^{k+1}(x) + \mathcal{O}(\Delta t)^3. \quad (6.16)$$

From Equation (6.16), we have

$$\frac{u^{k+1}(x) - u^k(x)}{\Delta t} = u_t^{k+1}(x) + \mathcal{O}(\Delta t),$$

which yields

$$\mathcal{L}_\varepsilon^* u^{k+1}(x) = \Delta t \left(-\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x) u^{k+1}(x) \right) + \mathcal{O}(\Delta t)^2. \quad (6.17)$$

Using Equations (6.13)–(6.15) and (6.17), the local truncation error satisfies

$$\mathcal{L}_\varepsilon^* e^{k+1} = -\varepsilon e_{xx}^{k+1}(x) + (e^{k+1})^2 e_x^{k+1}(x) + d^{k+1}(x) e^{k+1} = \mathcal{O}((\Delta t)^2), \quad (6.18)$$

$$e^{k+1}(0) = e^{k+1}(1) = 0. \quad (6.19)$$

The same argument which was used to bound the error in the proof of Lemma 5.3.2 in Chapter 5 can be employed here to obtain the result

$$|e^{k+1}(x)| \leq C(\Delta t)^2.$$

□

Lemma 6.2.3. *The global error $\mathcal{E}^m$ satisfies*

$$\|\mathcal{E}^m\| \leq C^*(\Delta t). \quad (6.20)$$

Proof. The global error of the time-discretisation satisfies the result below

$$\|\mathcal{E}^m\| \leq \left| \sum_{k=1}^m e^{k+1}(x) \right| \leq Cm(\Delta t)^2 = C\Delta t. \quad (6.21)$$

□

6.3 Quasilinearization

In this section, the semi-discrete non-linear modified Burgers equation (6.3) is transformed into a sequence of linear convection diffusion problems by the quasilinearization technique in Bellman and Kalaba (1965). The linearization of the non-linear term $(u^{k+1}(x))^2$ in Equation (6.3) is done by choosing $u_0^{k+1}(x)$ to be the initial approximation of the function $u^{k+1}(x)$ in the $(u^{k+1}(x))^2$. Now we expand $(u^{k+1}(x))^2$ around $u_0^{k+1}(x)$ in Taylor series to obtain

$$[u_1^{k+1}(x)]^2 = [u_0^{k+1}(x)]^2 + 2[u_0^{k+1}(x)][u_1^{k+1}(x) - u_0^{k+1}(x)] + \dots \quad (6.22)$$

Using $j = 0, 1, 2, \dots$, as the iteration index, equation (6.22) can be written as

$$[u_{j+1}^{k+1}(x)]^2 = [u_j^{k+1}(x)]^2 + 2[u_j^{k+1}(x)][u_{j+1}^{k+1}(x) - u_j^{k+1}(x)] + \dots \quad (6.23)$$

Truncating equation (6.23) and utilizing it in (6.6) yields

$$\begin{aligned} -\varepsilon \frac{\partial^2}{\partial x^2} u_{j+1}^{k+1}(x) &+ \left(u_j^{k+1}(x) \right)^2 + 2[u_j^{k+1}(x)][u_{j+1}^{k+1}(x) - u_j^{k+1}(x)] \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) \\ &+ d^{k+1}(x) u_{j+1}^{k+1}(x) = \frac{u_{j+1}^k}{\Delta t}. \end{aligned} \quad (6.24)$$

Further simplification results in

$$\begin{aligned} -\varepsilon \frac{\partial^2}{\partial x^2} u_{j+1}^{k+1}(x) &+ \left(u_j^{k+1}(x) \right)^2 \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) + \left(2u_j^{k+1}(x) \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) + d^{k+1}(x) \right) u_{j+1}^{k+1} \\ &= \frac{u_{j+1}^k}{\Delta t} + 2 \left(u_j^{k+1} \right)^2 \frac{\partial}{\partial x} u_{j+1}^{k+1}(x), \quad x \in \Omega, \quad k \geq 0, \quad j = 0, 1, \dots, \end{aligned} \quad (6.25)$$

along with the initial and boundary condition

$$u_{j+1}^0(x) = \varphi(x), \quad u_{j+1}^{k+1}(0) = 0, \quad u_{j+1}^{k+1}(1) = 0. \quad (6.26)$$

For notational simplicity, we let $u_{j+1}(x) = \tilde{u}(x)$ so that the sequence of semi-discrete linear problems (6.25)–(6.26) becomes

$$\mathcal{L}_\varepsilon^{m,l} \tilde{u}^{k+1} \equiv -\varepsilon \frac{\partial^2}{\partial x^2} \tilde{u}^{k+1}(x) + \tilde{a}(x) \frac{\partial}{\partial x} \tilde{u}^{k+1}(x) + \tilde{c}(x) \tilde{u}^{k+1}(x) = \tilde{f}(x), \quad (6.27)$$

where the coefficient functions $\tilde{a}(x)$, $\tilde{c}(x)$ and the source term $\tilde{f}(x)$ are given by

$$\tilde{a}(x) = \left(\tilde{u}^{k+1}(x) \right)^2, \quad \tilde{c}(x) = \left(2\tilde{u}^{k+1}(x) \frac{\partial}{\partial x} \tilde{u}^{k+1}(x) + d^{k+1}(x) \right)$$

and

$$\tilde{f}(x) = \frac{\tilde{u}^k(x)}{\Delta t} + 2 \left(\tilde{u}^{k+1}(x) \right)^2 \frac{\partial \tilde{u}^{k+1}(x)}{\partial x}$$

respectively. Now we assume the functions $\tilde{a}(x)$, $\tilde{c}(x)$ and $\tilde{f}(x)$ are smooth, with $\tilde{a}(x)$ and $\tilde{c}(x)$ satisfying

$$\tilde{a}(x) \geq \alpha > 0, \quad x \in \bar{\Omega}$$

and

$$\tilde{c}(x) \geq c_0 > 0, \quad x \in \bar{\Omega}$$

respectively.

Notice that the equation (6.25)–(6.26) is a sequence of linear second order differential equation. Thus instead of solving the non-linear semi-discrete problem (6.5)–(6.7), we solve the sequence of linear problem (6.25)–(6.26). Theoretically, the solution $u^{k+1}(x)$ of the semi-discrete non-linear problem (6.5)–(6.7) must satisfy

$$\lim_{k \rightarrow \infty} u_j^{k+1}(x) = u^{k+1}(x), \quad x \in \bar{\Omega}, \quad (6.28)$$

and numerically

$$\left| \tilde{u}^{k+1}(x) - u_j^{k+1}(x) \right| < \epsilon, \quad x \in \bar{\Omega}, \quad (6.29)$$

is required. The parameter ϵ is a small tolerance imposed to terminate the iteration.

6.3.1 Convergence Analysis of the Quasilinearization Process

To analyse the quasilinearization process for convergence, first we consider the problem

$$\begin{aligned} \varepsilon \frac{\partial u^{k+1}}{\partial x^2}(x) &= G(u^{k+1}), \quad x \in \Omega, \quad k \geq 0, \\ u^{k+1}(0) &= 0 \quad u^{k+1}(1) = 0, \quad k \geq 0. \end{aligned} \quad (6.30)$$

To quasilinearize $G(u)$ we take the first order Taylor series expansion to obtain

$$\varepsilon \frac{\partial u_{j+1}^{k+1}}{\partial x^2}(x) \approx G(u_j^{k+1}) + (u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1}, \quad x \in \Omega \quad k \geq 0, \quad (6.31)$$

$$u_{j+1}^{k+1}(0) = 0, \quad u_{j+1}^{k+1}(1) = 0, \quad k \geq 0. \quad (6.32)$$

Using u_0^{k+1} as an initial approximation in equation (6.30) yields

$$\varepsilon \frac{\partial u_j^{k+1}}{\partial x^2}(x) \approx G(u_{j-1}^{k+1}) + (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u_{j-1}^{k+1}} u_{j-1}^{k+1}, \quad x \in \Omega \quad k \geq 0, \quad (6.33)$$

$$u_{j+1}^{k+1}(0) = 0, \quad u_{j+1}^{k+1}(1) = 0, \quad k \geq 0. \quad (6.34)$$

Subtracting equation (6.33) from (6.31), we obtain

$$\begin{aligned} \varepsilon \left(\frac{\partial u_{j+1}^{k+1}}{\partial x^2} - \frac{\partial u_j^{k+1}}{\partial x^2} \right)(x) &= G(u_j^{k+1}) - G(u_{j-1}^{k+1}) - (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u_{j-1}^{k+1}} u_{j-1}^{k+1} \\ &\quad + (u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1}. \end{aligned} \quad (6.35)$$

The equation (6.35) is a second order differential equation for $u_{j+1}^{k+1} - u_j^{k+1}$. Using Green's function, equation (6.35) is transformed into the integral function

$$\begin{aligned} \varepsilon (u_{j+1}^{k+1} - u_j^{k+1})(x) &= \int_0^1 \mathcal{G}(x, s) \left[G(u_j^{k+1}) - G(u_{j-1}^{k+1}) - (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u_{j-1}^{k+1}} u_{j-1}^{k+1} \right. \\ &\quad \left. + (u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1} \right] ds. \end{aligned} \quad (6.36)$$

Here $\mathcal{G}(x, s)$ is the Greens function given by

$$\mathcal{G}(x, s) = \begin{cases} (x-1)s, & 0 \leq s \leq x \leq 1, \\ x(s-1), & 0 \leq x \leq s \leq 1. \end{cases}$$

For $s \leq x$, we have

$$\begin{aligned} |\mathcal{G}(x, s)| &= |x-1||s| \leq (1-x)x \\ &= -(x^2 - x) = - \left[\left(x - \frac{1}{2} \right)^2 - \frac{1}{4} \right] = \frac{1}{4} - \left(x - \frac{1}{2} \right)^2 \leq \frac{1}{4}. \end{aligned}$$

Similarly, $x \leq s$, also yields

$$\begin{aligned} |\mathcal{G}(x, s)| &= |x||s-1| \leq (s-1)s \\ &= -(s^2 - s) = - \left[\left(s - \frac{1}{2} \right)^2 - \frac{1}{4} \right] = \frac{1}{4} - \left(s - \frac{1}{2} \right)^2 \leq \frac{1}{4}. \end{aligned}$$

Thus

$$|\mathcal{G}(x, s)| \leq \frac{1}{4}, \quad \forall x, s \in [0, 1]. \quad (6.37)$$

From the mean value theorem, we have

$$G(u_j^{k+1}) - G(u_{j-1}^{k+1}) = (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u^{k+1}} u_j^{k+1} + \frac{(u_j^{k+1} - u_{j-1}^{k+1})^2}{2} \frac{\partial^2 G(\theta)}{\partial (u^{k+1})^2}, \quad (6.38)$$

where $u_{j-1}^{k+1} \leq \theta \leq u_j^{k+1}$. Now substituting (6.38) into (6.36) yields

$$\varepsilon (u_{j+1}^{k+1} - u_j^{k+1})(x) = \int_0^1 \mathcal{G}(x, s) \left[(u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1} + \frac{(u_j^{k+1} - u_{j-1}^{k+1})^2}{2} \frac{\partial^2 G(\theta)}{\partial (u^{k+1})^2} \right] ds. \quad (6.39)$$

Let

$$\max_{|u| \leq 1} \left| \frac{\partial G}{\partial u^{k+1}} u^{k+1} \right| = s_1, \quad \max_{|u| \leq 1} \left| \frac{\partial^2 G}{\partial (u^{k+1})^2} u^{k+1} \right| = s_2. \quad (6.40)$$

From (6.37), (6.40), equation (6.39) can be written as

$$\left| (u_{j+1}^{k+1} - u_j^{k+1})(x) \right| = \left| \frac{1}{4\varepsilon} \int_0^1 \left[(u_{j+1}^{k+1} - u_j^{k+1}) s_1 + \frac{(u_j^{k+1} - u_{j-1}^{k+1})^2}{2} s_2 \right] ds \right|. \quad (6.41)$$

Further simplification of equation (6.41) yields

$$(8 - 2s_1) \left| (u_{j+1}^{k+1} - u_j^{k+1})(x) \right| \leq s_2 (u_j^{k+1} - u_{j-1}^{k+1})^2$$

which leads to

$$\left| (u_{j+1}^{k+1} - u_j^{k+1}) \right| \leq \frac{s_2}{(8\varepsilon - 2s_1)} \left| u_j^{k+1} - u_{j-1}^{k+1} \right|^2.$$

Therefore the sequence of (u_j^{k+1}) converges quadratically.

In the next section, we estimate the solution of the semi-discrete problem and its derivatives.

6.4 A Priori Estimate of the Linear Semi-discrete Problem

In this section bounds of the solution of the semi-discrete problem and its derivatives are presented.

Lemma 6.4.1. *Maximum principle*

Suppose $\tilde{\Psi}^{k+1}(x)$ is a smooth function satisfying $\tilde{\Psi}^{(k+1)}(x) \geq 0, \quad \forall x \in \partial\Omega$. Then $\mathcal{L}_\varepsilon^{m,l} \tilde{\Psi}^{(k+1)}(x) \geq 0, \quad \forall x \in \Omega$, implies that $\tilde{\Psi}^{(k+1)}(x) \geq 0, \quad \forall x \in \bar{\Omega}$.

Proof. Let $\tilde{\Psi}^{k+1}(x^*) < 0$, $x^* \in \bar{\Omega}$, and

$$\tilde{\Psi}^{k+1}(x^*) = \min_{x \in \bar{\Omega}} \tilde{\Psi}^{k+1}(x).$$

It is clear that $x^* \notin \partial\Omega$, and $\tilde{\Psi}_x^{k+1}(x^*) = 0$ and $\tilde{\Psi}_{xx}^{k+1}(x^*) \geq 0$. Thus, $\mathcal{L}_\varepsilon^{m,l} \tilde{\Psi}^{k+1}(x^*) \leq 0$, which is a contradiction. Thus $\tilde{\Psi}^{k+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$. $\square$

Lemma 6.4.2. *Stability estimate*

Let $\tilde{u}^{k+1}(x)$ be the solution of the linear problem (6.5)–(6.7). Then we have

$$|\tilde{u}^{k+1}(x)| \leq c_0^{-1} \|f\|, \quad k = 0, 1, 2, \dots \quad (6.42)$$

Proof. Let $(\tilde{\xi}^{k+1}(x))^\pm$ be smooth comparison functions

$$(\tilde{\xi}^{k+1}(x))^\pm = p \pm \tilde{u}^{k+1}(x), \quad p = c_0^{-1} \|f\|.$$

At the boundaries $x = 0, 1$

$$\begin{aligned} (\xi^{k+1}(x))^\pm &= p \pm \tilde{u}^{k+1}(0) \geq 0, \\ (\xi^{k+1}(x))^\pm &= p \pm \tilde{u}^{k+1}(1) \geq 0. \end{aligned}$$

Further on the domain Ω ,

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l}(\tilde{\xi}^{k+1})^\pm(x) &= -\varepsilon(\tilde{\xi}^{k+1}(x))_{xx}^\pm + \tilde{a}(x)(\tilde{\xi}^{k+1}(x))_x^\pm + \tilde{c}(x)(\tilde{\xi}^{k+1}(x))^\pm \\ &= \tilde{c}(x)p \pm \mathcal{L}_\varepsilon^{m,l}\tilde{u}^{k+1}(x) \\ &= \tilde{c}(x)(c_0^{-1}\|f\|) \pm \tilde{f}(x) > 0. \end{aligned}$$

From Lemma 6.4.1, $(\tilde{\xi}^{k+1}(x))^\pm \geq 0, \forall x \in \bar{\Omega}$ as required. $\square$

Lemma 6.4.3. Let $\tilde{u}_x^{k+1}(x)$ be the solution of the linear semi-discrete problem (6.5)–(6.7), then it satisfies $|\tilde{u}_x^{k+1}(x)| \leq C(1 + \varepsilon^{-1} \exp(-\alpha(1-x)/\varepsilon))$, $x \in \bar{\Omega}$.

Proof. Equation (6.27) is written as

$$-\varepsilon \tilde{u}_{xx}^{k+1} + \tilde{a}(x) \tilde{u}_x^{k+1} = h_1(x), \quad (6.43)$$

where $h_1(x) = \tilde{f}(x) - \tilde{c}(x) \tilde{u}^{k+1}(x)$. We multiply through by the integration factor $A(x) = \int_0^x a(\gamma) d\gamma$, to obtain

$$\exp(-\varepsilon^{-1} A(x)) \tilde{u}_{xx}^{k+1} - \varepsilon^{-1} \tilde{a}(x) \exp(-\varepsilon A(x)) \tilde{u}_x^{k+1} = -\varepsilon^{-1} \exp(-\varepsilon^{-1} A(x)) h_1(x),$$

that is,

$$\frac{d}{dx} \left(\exp(-\varepsilon^{-1} A(x)) \frac{d\tilde{u}^{k+1}}{dx} \right) = -\varepsilon^{-1} \exp(-\varepsilon^{-1} A(x)) h_1(x).$$

Integrating this equation from x to 1 yields

$$\begin{aligned} \exp(-\varepsilon^{-1} A(1)) \tilde{u}_x^{k+1}(1) - \exp(-\varepsilon^{-1} A(x)) \tilde{u}_x^{k+1}(x) &= -\varepsilon^{-1} \int_x^1 h_1(\gamma) \exp(-\varepsilon^{-1} A(\gamma)) d\gamma, \\ \tilde{u}_x^{k+1}(x) &= \tilde{u}_x^{k+1}(1) \exp(-\varepsilon^{-1} (A(1) - A(x))) + z_1(x), \end{aligned} \quad (6.44)$$

where $z_1(x)$ is given by

$$z_1(x) = \varepsilon^{-1} \int_x^1 h_1(\gamma) \exp(-\varepsilon^{-1}A(\gamma) - A(x)) d\gamma.$$

To derive the bound of $\tilde{u}_x(1)$ in (6.44), we integrate (6.44) from x to 1, to obtain

$$\tilde{u}^{k+1}(1) - \tilde{u}^{k+1}(x) = \tilde{u}_x^{k+1}(1) \int_x^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds + \int_x^1 z_1(s) ds.$$

Evaluating the above equation at $x = 0$ leads to

$$\tilde{u}_x^{k+1}(1) = - \frac{\int_0^1 z_1(s) ds}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds}, \quad (6.45)$$

since $\tilde{u}^{k+1}(1) = \tilde{u}^{k+1}(0) = 0$. Substituting (6.45) into equation (6.44) gives

$$\tilde{u}_x^{k+1}(x) = - \frac{\exp(-\varepsilon^{-1}(A(1) - A(x)))}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds} \int_0^1 z_1(s) ds + z_1(x). \quad (6.46)$$

Now we take the norm on both sides of (6.46) to obtain

$$\begin{aligned} |\tilde{u}_x^{k+1}(x)| &\leq \frac{|\exp(-\varepsilon^{-1}(A(1) - A(x)))|}{|\int_0^1 \exp(\varepsilon^{-1}(A(1) - A(s))) ds|} \left| \int_0^1 z_1(s) ds \right| + |z_1(x)| \\ &\leq \frac{|\exp(-\varepsilon^{-1}A(1))| |\exp(\varepsilon^{-1}A(x))|}{|\exp(-\varepsilon^{-1}A(1))| |\int_0^1 \exp(\varepsilon^{-1}A(s)) ds|} \int_0^1 |z_1(s)| ds + |z_1(x)| \\ &= \frac{|\exp(\varepsilon^{-1}A(x))|}{|\int_0^1 \exp(\varepsilon^{-1}A(s)) ds|} \int_0^1 |z_1(s)| ds + |z_1(x)| \\ &\leq C \left(1 + \frac{|\exp(\varepsilon^{-1}\bar{\alpha}x)|}{|\int_0^1 \exp(\varepsilon^{-1}\alpha s) ds|} \right), \end{aligned} \quad (6.47)$$

where $\bar{\alpha}$ is an upper bound of $a(x)$ over $[0, 1]$ and C is also an upper bound of $|z_1(x)|$ over $\bar{\Omega}$. Further simplification of (6.47) results in

$$\begin{aligned} |\tilde{u}_x^{k+1}(x)| &\leq C \left(1 + \frac{\exp(\bar{\alpha}x\varepsilon^{-1})}{\varepsilon\alpha^{-1}(\exp(\alpha\varepsilon^{-1}) - 1)} \right) \\ &= C \left(1 + \frac{\alpha\varepsilon^{-1} \exp(\bar{\alpha}\varepsilon^{-1}x) \exp(-\bar{\alpha}\varepsilon^{-1})}{\exp(\alpha\varepsilon^{-1}) - 1} \right). \end{aligned} \quad (6.48)$$

Since $0 < \varepsilon < 1$, we can find ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and hence $\exp(\bar{\alpha}\varepsilon^{-1}) < \exp(\bar{\alpha}\varepsilon_0^{-1})$. Thus the estimate (6.48) can be written as

$$|\tilde{u}_x^{k+1}(x)| \leq C \exp(\bar{\alpha}\varepsilon_0^{-1}) \left(1 + \frac{\alpha\varepsilon^{-1} \exp(-\bar{\alpha}(1-x)\varepsilon^{-1})}{\exp(\alpha\varepsilon^{-1}) - 1} \right). \quad (6.49)$$

Let f be a real valued function defined by $f(\alpha) = \exp(\alpha\varepsilon^{-1})$ over the interval $[0, 1]$. Now expanding $f(\alpha)$ in Maclaurin series yields $\alpha \leq (\exp(\alpha\varepsilon^{-1}) - 1)$ and then the estimate (6.49) reduces to

$$|\tilde{u}_x^{k+1}(x)| \leq C_5(1 + \varepsilon^{-1} \exp(-\bar{\alpha}(1-x)/\varepsilon)), \quad (6.50)$$

where $C_5 = C \exp(\bar{\alpha}\varepsilon_0^{-1})$. □

The proof of the higher order derivatives can be derived in a similar manner.

Theorem 6.4.1. *The solution of the sequence of linear problems (6.25)–(6.26) and its derivatives satisfy*

$$\left| \frac{\partial^i}{\partial x^i} \tilde{u}^{k+1}(x) \right| \leq C(1 + \varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon)), \quad x \in \bar{\Omega},$$

where i is a non-negative integer which satisfies $0 \leq i \leq 6$.

Proof. The proof of this Theorem for $0 < i \leq 1$, follows from the Lemma 6.4.3. The proof of the higher order derivatives, that is $1 < i \leq 6$, can be obtained analogously. $\square$

6.5 Spatial Discretisation

In this section we construct a fitted operator finite difference scheme to solve the sequence of linear problems. We adapt the notation $U_i \equiv U(x_i)$ as the numerical solution of $\tilde{u}^{k+1}(x_i)$. Notice that we have dropped the superscript index for notational simplicity. Now we perform the discretisation as follows:

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l,n} U_i &\equiv -\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi_i^2} + \hat{a}_{i-1/2} \frac{U_i - U_{i-1}}{h} + \hat{c}_{i-1/2} U_i = \hat{f}_{i-1/2}, \\ i &= 1, 2, \dots, n-1, \end{aligned} \tag{6.51}$$

$$U(0) = 0, \quad U(1) = 0, \tag{6.52}$$

where the denominator function, the coefficient functions and the source term are given by

$$\phi_i^2(h, \varepsilon) = \frac{\varepsilon h}{\hat{a}_i} \left(\exp \left(\frac{\hat{a}_{i-1/2} h}{\varepsilon} \right) - 1 \right),$$

$$\hat{a}_{i-1/2} = \frac{a_i + a_{i-1}}{2},$$

$$\hat{c}_{i-1/2} = \frac{c_i + c_{i-1}}{2},$$

and

$$\hat{f}_{i-1/2} = \frac{f_i + f_{i-1}}{2},$$

respectively. The denominator function satisfies

$$\phi_i^2(h, \varepsilon) = h^2 + \mathcal{O}(h^3) \tag{6.53}$$

In matrix notation, the scheme (6.51)–(6.52) is a triadiagonal systems of linear equation

$$AU = F,$$

where A is a $((n-1) \times (n-1))$ square matrix whilst U and F are vectors of size $(n-1)$. Their entries are given by

$$A_{ij} = r_i^-, \quad i = 2, 3, \dots, n-1, \quad j = i-1,$$

$$A_{ij} = r_i^c, \quad i = 1, 2, \dots, n-1, \quad j = i,$$

$$A_{ij} = r_i^+, \quad i = 1, 2, \dots, n-2, \quad j = i+1,$$

$$F_i = \hat{f}_{i-1/2}, \quad i = 1, 2, \dots, n-1,$$

with

$$r_i^- = \frac{-\varepsilon}{\phi_i^2} - \frac{\hat{a}_{i-1/2}}{h}, \quad r_i^c = \frac{2\varepsilon}{\phi_i^2} + \frac{\hat{a}_{i-1/2}}{h} + \hat{c}_{i-1/2}, \quad r_i^+ = \frac{-\varepsilon}{\phi_i^2}.$$

6.6 Stability of the Scheme

Here we establish the stability of the scheme (6.51)–(6.52).

Lemma 6.6.1. *Discrete maximum principle*

The operator $\mathcal{L}_\varepsilon^{m,l,n}$ defined by the scheme (6.51)–(6.52) satisfies a discrete maximum principle. That is, if $\xi(x_i)$ is a mesh function which satisfies $\xi(x_i) \geq 0$, $\forall x_i \in \partial\Omega^n$ and $\mathcal{L}_\varepsilon^{m,l,n}\xi(x_i) > 0$, $\forall x_i \in \Omega^n$, then $\xi(x_i) \geq 0$, $\forall x_i \in \bar{\Omega}^n$.

Proof. Let s be an index such that

$$\xi_s = \min_{x_i \in \bar{\Omega}^n} \xi_i,$$

holds and assume $\xi_s < 0$. Clearly $s \neq 0$, $s \neq n$. We have $\xi_{s+1} - \xi_s \geq 0$, and $\xi_s - \xi_{s-1} \geq 0$. It follows that

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l,n}\xi_s &\equiv -\frac{\varepsilon}{\phi_i^2}(\xi_{s+1} - 2\xi_s + \xi_{s-1}) + \frac{\hat{a}_{s-1/2}}{h}(\xi_s - \xi_{s-1}) + \hat{c}_{s-1/2}\xi_s, \\ &= -\frac{\varepsilon}{\phi_i^2}[(\xi_{s+1} - \xi_s) - (\xi_s - \xi_{s-1})] + \frac{\hat{a}_{s-1/2}}{h}(\xi_s - \xi_{s-1}) + \hat{c}_{s-1/2}\xi_s \leq 0, \end{aligned}$$

which is a contradiction. Therefore, $\xi_i \geq 0$, $\forall x_i \in \bar{\Omega}^n$. $\square$

Next we show that the scheme (6.51)–(6.52) satisfies a uniform stability estimate below.

Lemma 6.6.2. *Let u_i be the solution of the discrete problem (6.51)–(6.52). Then it satisfies*

$$|u_i| \leq c_0^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n}u_i|.$$

Proof. Let

$$z = c_0^{-1} \max_{(x_i) \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n}u_i|,$$

and define the mesh function $\Psi_i^\pm$ by

$$\Psi_i^\pm = z \pm u_i.$$

At $i = 0$ and $i = n$, we have

$$\Psi_0^\pm = z \pm u_0 = z \pm 0 \geq 0$$

and

$$\Psi_n^\pm = z \pm u_n = z \pm 0 \geq 0,$$

respectively. Further on the domain $x_i \in \Omega^n$, we obtain

$$\begin{aligned}\mathcal{L}_\varepsilon^{m,l,n}\Psi_i^\pm &= \varepsilon \left(\frac{z+u_{i+1}-2(z+u_i)+z \pm u_{i-1}}{\phi_i^2(h,\varepsilon)} \right) + \hat{a}_{i-1/2} \left(\frac{z \pm u_i - (z \pm u_{i-1})}{h} \right) + \hat{c}_{i-1/2}(z \pm u_i), \\ &= \hat{c}_{i-1/2}z \pm \mathcal{L}_\varepsilon^{m,l,n}u_i, \\ &= \hat{c}_{i-1/2} \left(c_0^{-1} \max_{(x_i) \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n}u_i| \right) \pm \hat{f}_{i-1/2} \geq 0.\end{aligned}$$

From Lemma 6.6.1, $\Psi_i^\pm \geq 0$, $\forall x_i \in \bar{\Omega}^n$, as required. $\square$

6.7 Error Estimate

In this section the spatial discretisation is analysed for convergence.

Lemma 6.7.1. *The error associated with the fitted operator finite difference scheme based on midpoint satisfies*

$$|u_i - U_i| \leq Ch^2.$$

Proof.

$$\begin{aligned}\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) &= \mathcal{L}_\varepsilon^{m,l,n}u(x_i) - \mathcal{L}_\varepsilon^{m,l,n}U(x_i) = \mathcal{L}_\varepsilon^{m,n}u(x_i) - \hat{f}_{i-1/2}, \\ &= \mathcal{L}_\varepsilon^{m,l,n}u(x_i) - \mathcal{L}_\varepsilon^{m,l}u(x_{i-1/2}), \\ &= -\varepsilon (\delta^2 u_i - u''(x_{i-1/2})) + \hat{a}_i (D^- u_i - u'(x_{i-1/2})).\end{aligned}$$

Using a truncated Taylor series expansion of the terms u_{i-1} , u_i and ϕ_i^2 and simplifying further gives

$$\begin{aligned}\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) &= -\frac{\varepsilon}{h^2} \left(h^2 u'' + \frac{h^4}{8.3!} u^{iv} + \frac{h^6}{32.5!} u^{vi} + \dots \right) + \varepsilon u'' \\ &\quad + \frac{\hat{a}_i h^2}{8.3!} u''' + \frac{\hat{a}_i h^4}{32.5!} u^v + \dots\end{aligned}\tag{6.54}$$

Further simplification results in

$$\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) = \left(\frac{\varepsilon}{8.3!} u^{iv} + \frac{\hat{a}_{i-1/2}}{8.3!} u^{iii} \right) h^2 + \left(\frac{\varepsilon}{32.5!} u^{vi} + \frac{\hat{a}_i}{32.5!} u^v \right) h^4 + \dots$$

From Theorem 6.4.1 and noticing that the exponential terms vanish as $\varepsilon \rightarrow 0$ see Lemma 2.4.3 for proof, yields the estimate

$$\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) = \left(\frac{\varepsilon}{8.3!} + \frac{\hat{a}_{i-1/2}}{8.3!} \right) h^2 + \left(\frac{\varepsilon}{32.5!} + \frac{\hat{a}_i}{32.5!} \right) h^4 + \dots\tag{6.55}$$

Using the relation $h^2 > h^4 > \dots$ and taking the norm on both sides of Equation (6.55) yields

$$|\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i))| \leq Ch^2.\tag{6.56}$$

Now let $\mathcal{R}_i$ be a mesh function defined by

$$\mathcal{R}_i = Ch^2, \quad 0 \leq i \leq n.$$

Notice that $\mathcal{R}_0 = \mathcal{R}_n = Ch^2$ and

$$\mathcal{L}_\varepsilon^{m,l,n}\mathcal{R}_i \equiv -\frac{\varepsilon}{\phi_i^2}(\mathcal{R}_{i+1} - 2\mathcal{R}_i + \mathcal{R}_{i-1}) + \frac{\hat{a}_{i-1/2}}{h}(\mathcal{R}_i - \mathcal{R}_{i-1}) + \hat{c}_{i-1/2}\mathcal{R}_i = \hat{c}_{i-1/2}Ch^2.$$

Suppose $\mathcal{H}_i$ is also another function which given by

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i|.$$

At $i = 0$ and $i = n$, we have

$$\mathcal{H}_0 = \mathcal{R}_0 - |u_0 - U_0| = Ch^2 > 0,$$

and

$$\mathcal{H}_n = \mathcal{R}_n - |u_n - U_n| = Ch^2 > 0,$$

respectively. Further on $1 \leq i \leq n-1$, we have

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l,n}\mathcal{H}_i &= \mathcal{L}_\varepsilon^{m,l,n}\mathcal{R}_i - \mathcal{L}_\varepsilon^{m,l,n}|u_i - U_i| \\ &= \hat{c}_{i-1/2}Ch^2 - \mathcal{L}_\varepsilon^{m,l,n}|u_i - U_i| > 0. \end{aligned}$$

From Lemma 6.6.1, we obtain

$$\mathcal{H}_i = \mathcal{R}_i - |u_i - U_i| \geq 0,$$

and hence

$$|u_i - U_i| \leq \mathcal{R}_i,$$

which leads to

$$|u_i - U_i| \leq Ch^2.$$

□

Theorem 6.7.1. (*Uniform Convergence*)

Let u be the exact solution of the continuous problem (6.1)-(6.2) and U be the approximate solution. The error associated with the proposed numerical scheme satisfies

$$\max_{0 \leq i \leq n; 0 \leq k \leq K} |u(x_i, t_k) - U(x_i, t_k)| \leq C(h^2 + \Delta t).$$

Proof. The proof of this theorem follows from Lemma 6.2.3 and 6.7.1. □

Next we apply the method on test problem to validate the theoretical estimate.

6.8 Numerical Results

In this section, a test problem is simulated numerically using the proposed scheme to demonstrate the method in practise. Similar to the example considered in Chapter 5, the exact solution of the test problem is not known, thus to calculate the maximum pointwise errors we use the formula

$$E_{n,\Delta t}^1 = \max_{0 \leq i \leq n; 0 \leq k \leq m} |U_{i:n}^{k:K} - U_{i:2n}^{k:m}|. \quad (6.57)$$

The $2n$ and m have been used because the proposed scheme is second order accurate in space and first order in time. Also, the numerical rates of convergence are calculated using the formula

$$r_1 = \log_2 \left(\frac{E_{n,\Delta t}^1}{E_{2n,\Delta t}^1} \right). \quad (6.58)$$

To evaluate the ε uniform maximum errors we use 5.64 with the corresponding ε -uniform rates of convergence obtained using 5.65.

Example 6.8.1. *Gupta and Kadalbajoo (2010)*

Consider the problem

$$u_t(x, t) - \varepsilon u_{xx}(x, t) + u^2 u_x(x, t) = 0, \quad (x, t) \in Q,$$

$$u(x, 0) = \sin(\pi x), \quad u(0, t) = u(1, t) = 0, \quad t \in [0, T].$$

Table 6.1: Maximum pointwise error and rate of convergence for Example 6.8.1.

ε	n=32	64	128	256	512	1024
10^{-4}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9924	1.9962	
10^{-6}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
10^{-8}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
10^{10}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
E^1	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
R_1	1.9262	1.9667	1.9842	1.9923	1.9962	

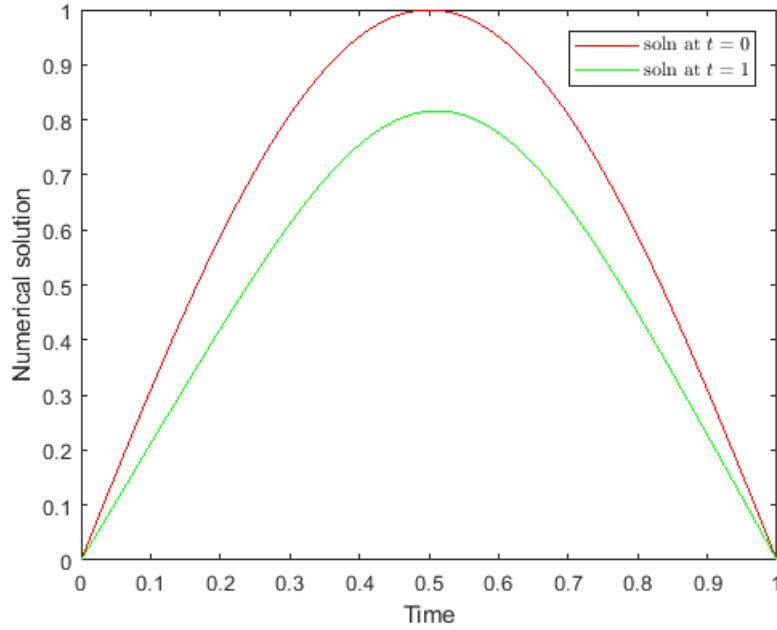


Figure 6.1: Solution profile of Example 6.8.1 for $n = 64$ and $\varepsilon = 10^{-24}$.

6.9 Conclusion

A numerical scheme was proposed to solve a modified Burgers' equation in this paper. The scheme was a combination of the implicit Euler finite difference scheme and a fitted operator finite difference scheme. More specifically, the implicit Euler finite difference scheme was first used to discretise the time variable. This resulted in a sequence of non-linear boundary value problems which was then quasilinearized to sequence of linear problems. A fitted operator finite difference scheme based on the midpoint upwind scheme was then designed to solve the boundary value problems. Convergence analysis led to a first order accuracy in time and second order in space. To confirm the theoretical findings, numerical simulations were conducted and the results were in conformity with Theorem 6.7.1. Also surface plot of the numerical solution can be observed in Figure 6.1 and the right boundary layer can be observed.

Chapter 7

Global Dynamics for a Model of Amyloid Fibril Formation in Pancreatic Islet Beta Cells Subjected to a Therapy

7.1 Introduction

The previous chapters saw to the numerical solutions of various forms of singularly perturbed problems, in this chapter we study the dynamics of a model of Amyloid fibril formation in pancreatic islet beta cells and also conduct numerical simulations to support the theory.

Below, an existing model describing the nucleated polymerisation's dynamics of the human amyloid polypeptide within the pancreatic islet β -cells under the impacts of therapeutic interventions as formulated in Murali and Raman (2013) which takes the form

$$\begin{aligned}\frac{d}{dt}M(t) &= A - gM(t) - rM(t) \int_{x_0}^{\infty} p(x, t)dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)p(y, t)dydx, \\ \frac{\partial}{\partial t}p(x, t) &= -rM(t) \frac{\partial}{\partial x}p(x, t) - (m(x) + b(x) + k_2D(t))p(x, t) \\ &\quad + 2 \int_x^{\infty} b(y)f(x, y)p(y, t)dy, \\ \frac{d}{dt}D(t) &= -k_0D(t) + k_1D(t) \int_{x_0}^{\infty} p(x, t)dx,\end{aligned}\tag{7.1}$$

along with the initial conditions

$$M(0) = M_0, \quad D(0) = M_0, \quad p(x, 0) = p_0(x), \quad (x_0 < x < \infty)\tag{7.2}$$

and the boundary conditions

$$p(x_0, t) = 0, \quad t \geq 0, \quad (7.3)$$

is investigated. This model is made up of the rates of changes of the islet amyloid polypeptide (IAPP) monomers population size, IAPP polymers of length x population size and the amount of drug in the system. This is a hybrid system of non-linear coupled ordinary and partial differential equations, where $M(t)$, $p(x, t)$ and $D(t)$ denote the IAPP monomers population size, the IAPP polymers of length x population density and the amount of drug in the system, respectively, at time t . The variables M_0 and $p_0(x)$ are IAPP monomers population and IAPP polymers of length x initial sizes, D_0 is an initial amount of drug in the system and x_0 is the critical length of the IAPP polymer below which the IAPP polymer breaks down into normal IAPP monomers. All parameters in (7.1) are all positive (constant or function) which account for the following:

Table 7.1: The definitions of the parameters in (7.1).

Parameter	Description
A	constant rate of production of IAPP monomers in the pancreatic beta cells
g	constant rate of degradation of IAPP monomers due to metabolism processes
r	constant polymerization rate or the conversion rate of IAPP monomers to IAPP polymers
k_0	rate at which the drug is degraded from the system due to metabolic processes
k_1	rate at which the drug increases in the system
k_2	rate at which the polymer population reduces due to the presence of the drug
$b(x)$	binary splitting rate of the IAPP polymers of length x
$m(x)$	degradation rate of the IAPP polymers due to metabolism
$f(x, y)$	probability density function that a polymer of length y splits into two polymers of length x and $y - x$

On the right-hand side of the first equation in (7.1), the last term accounts for the number of IAPP monomers resulting from the splitting of an IAPP polymer which also produces at least one IAPP polymer of length lesser than x_0 . It is assumed that such an IAPP polymer dissociates from IAPP monomers. The factor 2 in this term explains the binary splitting of an IAPP polymer of length x into two polymers whenever $x > x_0$. It is also assumed that the length of IAPP polymers picks up continuous values for mathematical analysis purposes. Moreover, on the right-hand side of the second equation in (7.1), the first term (the transport term) and the last term represent, respectively, the loss in the amount of IAPP polymers of length x as a result of lengthening and the gain of IAPP polymers of length x due to splitting of lengthy polymers.

The functions

$$P(t) = \int_{x_0}^{\infty} p(x, t) dx \quad \text{and} \quad Q(t) = \int_{x_0}^{\infty} xp(x, t) dx$$

will be used to denote the total amount of IAPP monomers and IAPP polymers of length x , respectively. Clearly, for a $t \geq 0$,

$$P(t) < Q(t). \quad (7.4)$$

Under the following assumptions

$$m(x) = m, \quad b(x) = bx \quad \text{and} \quad f(x, y) = \begin{cases} y^{-1}, & \text{if } y > x_0, \\ 0, & \text{if } y \leq x_0 \text{ or } y \leq x. \end{cases}$$

Murali and Raman (2013) transformed the model (7.1)–(7.3) into a system of ordinary differential equations given by

$$\begin{aligned} \frac{d}{dt}M(t) &= A - gM(t) + bx_0^2P(t) - rM(t)P(t), \\ \frac{d}{dt}P(t) &= -mP(t) + bQ(t) - 2bx_0P(t) - k_2D(t)P(t), \\ \frac{d}{dt}Q(t) &= -mQ(t) - bx_0^2P(t) - k_2D(t)Q(t) + rM(t)P(t), \\ \frac{d}{dt}D(t) &= -k_0D(t) + k_1D(t)P(t), \end{aligned} \quad (7.5)$$

subject to the initial conditions

$$M(0) = M_0, \quad P(0) = P_0, \quad Q(0) = Q_0, \quad D(0) = D_0, \quad (7.6)$$

with $M_0, P_0, Q_0, D_0 \geq 0$, for simplicity. These authors carried out the asymptotic behaviour of the system of equations (7.5)–(7.6) under the aforementioned assumptions and obtained the following results. A disease-free state equilibrium, $E_1 = (\tilde{M}, \tilde{P}, \tilde{Q}, \tilde{D}) = (A/g, 0, 0, 0)$ and a disease state equilibrium $E_2 = (M^0, P^0, Q^0, D^0)$, where

$$M^0 = \frac{(bx_0 + m)^2}{br}, \quad P^0 = \frac{brA - g(bx_0 + m)^2}{mr(2bx_0 + m)}, \quad Q^0 = \frac{brA - g(bx_0 + m)^2}{mbr}, \quad D^0 = 0,$$

with $brA > g(bx_0 + m)^2$, of the system of equations (7.5)–(7.6) were found after nullifying therapeutic interventions effects; while a coexistence state equilibrium, $E_3 = (M^*, P^*, Q^*, D^*)$, where

$$M^* = \frac{A + bx_0^2P^*}{g + rP^*}, \quad P^* = \frac{k_0}{k_1}, \quad Q^* = \frac{(rM^* - bx_0^2)P^*}{m + k_2D^*}, \quad D^* = \frac{\sqrt{brM^*} - (mbx_0)}{k_2},$$

with $rAM^* > \max\{bx_0^2, b^{-1}(bx_0 + m)^2\}$ of the system of equations (7.5)–(7.6) is found after taking into account the therapeutic interventions. The local stability of equilibria is proven by the mean of the Routh-Hurwitz criterion and revealed that E_1 (respectively E_2) is locally asymptotically stable if and only if $brA < g(m + bx_0)^2$ (respectively $brA > g(m + bx_0)^2$ and $P^0 < k_0/k_1$) and E_3 is locally asymptotically stable if some conditions (as given in [Murali and Raman (2013), p. 6]) are satisfied. Next, Murali and Raman used

some numerical simulations based on the system of equations (7.5)–(7.6) to confirm the results from the theoretical analysis. In contrast to the results from Murali and Raman (2013) and from other authors in references therein, here we consider arbitrary (bounded or unbounded) functions $b(\cdot)$ and $m(\cdot)$. Furthermore, for the sake of handling the hybrid system of equations (7.5)–(7.6), we make the following assumptions:

A1: $k_1 = k_2$,

A2: functions $b(\cdot)$ and $m(\cdot)$ satisfy $\max\{(x+4)b(x), rA/g\} < m(x)$, where $1 < m(x) + b(x)$, for any $x \geq x_0$,

A3: $f(x, y)$ satisfies the natural constraints, see Simonett and Walker (2006); Walker (2007),

$$f(x, y) = f(y - x, y), \quad y > x_0, \quad 0 < x < y, \quad (7.7)$$

which accounts for the binary splitting of the IAPP polymers, and

$$\int_0^y f(x, y) dx = 1, \quad y > x_0. \quad (7.8)$$

These two constraints, namely (7.7)–(7.8) yields

$$2 \int_0^y x f(x, y) dx = y, \quad y > x_0, \quad (7.9)$$

i.e. there is the conservation of number of monomers during the splitting process. The problem consisting of equations (7.1)–(7.3) subjected to assumptions **A1**–**A3** is regarded as novel and have not yet been investigated to our best knowledge. Such a problem is mathematically challenging but allows to provide some relevant features of the model. We make use of the mathematical tool developed in M’pika Massoukou et al. (2018) to carry out the steady states and stability analysis of equations (7.1)–(7.3).

The rest of the chapter is structured as follows. Section 7.2 deals with some preliminary results and more importantly the existence result of global attractors for the continuous semi-flow defined by the system of equations (7.1)–(7.3). The steady states of the system of equations (7.1)–(7.3) are computed in Section 7.3. Three equilibria are obtained: a disease-free state equilibrium, a disease state equilibrium and a coexistence state equilibrium. The next section, namely Section 7.4, is devoted to the uniform persistence of the coexistence state equilibrium. Sections 7.5 and 7.6 deal, respectively, with local and global stability analysis of the disease-free state equilibrium and disease state equilibrium. Numerical simulations are carried out in Section 7.7 whilst the concluding remarks are presented in Section 7.8.

7.2 Preliminaries

Let X_1 be defined by $X_1 = L^1([x_0, \infty), x dx) = \{\phi : \|\phi\|_1 = \int_{x_0}^{\infty} x |\phi(x)| dx\}$. We consider the Banach space $\mathbb{X} = \mathbb{R} \times X_1 \times \mathbb{R}$ endowed with the norm $\|(u, \phi, v)\|_{\mathbb{X}} = |u| + \|\phi\|_1 + |v|$,

for any $(u, \phi, v) \in \mathbb{X}$. Let us denote by $\mathbb{X}_+$ the positive cone of the Banach space $\mathbb{X}$ such that $\mathbb{X}_+ = \mathbb{R}_+ \times X_{1,+} \times \mathbb{R}_+$, where $X_{1,+} = L^1_+[x_0, \infty), xdx)$.

For any initial value $\mathbf{X}_0 = (M_0, p_0(\cdot), D_0) \in \mathbb{X}_+$ satisfying the following condition $p_0(0) = 0$, the system of equations (7.1)–(7.3) is well-posed. Indeed, the well-posedness of the system of equations (7.1)–(7.3), without therapeutic interventions (that is, $D(\cdot) \equiv 0$), is established in Simonett and Walker (2006), where the existence and uniqueness of global classical solutions for bounded functions $b(\cdot)$ and $m(\cdot)$ and the existence and uniqueness of global weak solutions for unbounded functions $b(\cdot)$ and $m(\cdot)$ are shown. So, this result can easily be extended to the solvability of the system of equations (7.1)–(7.3) with therapeutic interventions. So, we obtain a continuous semi-flow $\Phi : \mathbb{R}_+ \times \mathbb{X}_+ \rightarrow \mathbb{X}_+$ defined by the system of equations (7.1)–(7.3) so that

$$\Phi(t, \mathbf{X}_0) = \Phi_t(\mathbf{X}_0) = (M(t), p(\cdot, t), D(t)), \quad t \in \mathbb{R}_+, \quad \mathbf{X}_0 \in \mathbb{X}_+. \quad (7.10)$$

Since the components of $\Phi_t(\mathbf{X}_0)$ are positive, using the above notation, we obtain

$$\|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} = M(t) + \|p(\cdot, t)\|_1 + D(t). \quad (7.11)$$

Differentiating (7.11) with respect to t yields

$$\frac{d}{dt} \|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} = \frac{d}{dt} M(t) + \frac{d}{dt} \|p(\cdot, t)\|_1 + \frac{d}{dt} D(t). \quad (7.12)$$

Next, we show that

$$\frac{d}{dt} \|p(\cdot, t)\|_1 = \frac{d}{dt} \int_{x_0}^{\infty} xp(x, t)dx = \int_{x_0}^{\infty} x \frac{\partial}{\partial t} p(x, t)dx. \quad (7.13)$$

Substituting the second equation of (7.1) into (7.13) leads to

$$\begin{aligned} \frac{d}{dt} \|p(\cdot, t)\|_1 &= \int_{x_0}^{\infty} x \left(-rM(t) \frac{\partial}{\partial t} p(x, t) - (m(x) + b(x) + k_2 D(t)) p(x, t) \right) dx \\ &\quad + 2 \int_{x_0}^{\infty} x \int_x^{\infty} b(y) f(x, y) p(y, t) dy dx. \end{aligned} \quad (7.14)$$

Expanding the integrand of the first integral, changing the order of integration of the last integral in (7.14) and using assumption **A1**, we get

$$\begin{aligned} \frac{d}{dt} \|p(\cdot, t)\|_1 &= -rM(t) \int_{x_0}^{\infty} x \frac{\partial}{\partial t} p(x, t) dx - \int_{x_0}^{\infty} x(m(x) + b(x)) p(x, t) dx \\ &\quad - k_1 D(t) \int_{x_0}^{\infty} xp(x, t) dx + \int_{x_0}^{\infty} \left(2 \int_{x_0}^y xf(x, y) dx \right) b(y) p(y, t) dy. \end{aligned}$$

Combining this equation, the first and third equations of (7.1), yields

$$\begin{aligned} \frac{d}{dt} M(t) + \|p(\cdot, t)\|_1 + D(t) &= A - \left(gM(t) + \frac{r}{g} A \|p(\cdot, t)\|_1 + k_0 D(t) \right) \\ &\quad - k_1 Q((t) - P(t)) D(t). \end{aligned} \quad (7.15)$$

Let η be such that

$$\sqrt{rA} < \eta < \inf \left\{ g, k_0 \frac{r}{g} A \right\}. \quad (7.16)$$

Using (7.4), we have

$$\frac{d}{dt}(M(t) + \|p(\cdot, t)\|_1 + D(t)) \leq A - \eta(M(t) + \|p(\cdot, t)\|_1 + D(t)), \quad (7.17)$$

that is $(d/dt)\|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq A - \eta\|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}}$. Thus, we obtain

$$\|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq \frac{A}{\eta} - e^{-\eta t} \left(\frac{A}{\eta} - \|\mathbf{X}_0\|_{\mathbb{X}} \right), \text{ where } \|\Phi_0(\mathbf{X}_0)\|_{\mathbb{X}} = \|\mathbf{X}_0\|_{\mathbb{X}}. \quad (7.18)$$

Let Γ be the state of space (7.1) defined by

$$\Gamma = \left\{ (u, \phi, v) \in \mathbb{X}_+ : \|(u, \phi, v)\|_{\mathbb{X}} < \frac{A}{\eta} \right\}.$$

From (7.18), we get $\|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq A/\eta$ for any $t \geq 0$, whenever $\mathbf{X}_0 \in \Gamma$. Moreover, $\limsup_{t \rightarrow \infty} \|\Phi_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq A/\eta$ for any $\mathbf{X}_0 \in \mathbb{X}_+$. Then, we obtain the following result.

Lemma 7.2.1. *The set Γ is positively invariant for Φ ; that is $\Phi_t(\mathbf{X}_0) \in \Gamma, \forall t \geq 0, \mathbf{X}_0 \in \Gamma$. Moreover, Φ is point dissipative.*

As we aim to make use of the Lasalle's Invariance Principle, we are required to establish the relative compactness of the orbit $\{\Phi_t(\mathbf{X}_0) : t \geq 0\}$ in $\mathbb{X}_+$ due to the infinite-dimensional Banach space $\mathbb{X}$. This is done as in M'pika Massoukou et al. (2018) by introducing two operators Θ and Ψ , $(\Theta, \Psi : \mathbb{R}_+ \times \mathbb{X}_+ \rightarrow \mathbb{X}_+)$, defined by

$$\Theta_t(\mathbf{X}_0) = (0, p(x, t), 0) \text{ and } \Psi_t(\mathbf{X}_0) = (M(t), 0, D(t)), \quad (7.19)$$

where $\Theta(t, \mathbf{X}_0) = \Theta_t(\mathbf{X}_0)$ and $\Psi(t, \mathbf{X}_0) = \Psi_t(\mathbf{X}_0)$, so that

$$\Phi_t(\mathbf{X}_0) = \Theta_t(\mathbf{X}_0) + \Psi_t(\mathbf{X}_0), \text{ for any } t \geq 0 \quad (7.20)$$

and by obtaining from the proof of [Webb et al. (1985), Proposition 3.13] and Lemma 7.19, the result below.

Theorem 7.2.1. *For $\mathbf{X}_0 \in \Gamma$, the orbit $\{\Phi_t(\mathbf{X}_0) : t \geq 0\}$ has a compact closure in $\mathbb{X}_+$ if the following conditions are satisfied*

- (i) There exists a function $\Delta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $r > 0$, $\lim_{t \rightarrow \infty} \Delta(t, r) = 0$ and if $\mathbf{X}_0 \in \Gamma$ with $\|\mathbf{X}_0\|_{\mathbb{X}} \leq r$, then $\|\Theta_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq \Delta(t, r)$ for any $t \geq 0$.
- (ii) For any $t \geq 0$, $\Psi_t(\cdot)$ maps any bounded sets of Γ into set with compact closure in $\mathbb{X}_+$.

Conditions (i) and (ii) of Theorem 7.2.1 are verified through the following lemmas.

Lemma 7.2.2. *For any $\delta > 0$, let $\Delta(t, \delta) = e^{-\eta t} \delta$. Then $\lim_{t \rightarrow \infty} \Delta(t, \delta) = 0$. Then (i) of Theorem 7.2.1 holds.*

Proof. We have a $\lim_{t \rightarrow \infty} \Delta(t, \delta) = 0$. We choose the initial condition $\mathbf{X}_0 \in \Gamma$ such that $\|\mathbf{X}_0\|_{\mathbb{X}} \leq r$. For $t \geq 0$, we have $\|\Theta_t(\mathbf{X}_0)\|_{\mathbb{X}} = |0| + \|p(\cdot, t)\|_1 + |0| = \|p(\cdot, t)\|_1$. Using (7.4)–(7.9)–(7.15)–(7.16), assumption **A1** and **A2**, we have

$$\begin{aligned} \frac{d}{dt} \|p(\cdot, t)\|_1 &\leq \frac{r}{\eta} A \|p(\cdot, t)\|_1 - \eta \int_{x_0}^{\infty} x m(x) p(x, t) dx \\ &= \left(\frac{r}{\eta} A - \eta \right) \|p(\cdot, t)\|_1 \end{aligned} \quad (7.21)$$

and hence $\|p(\cdot, t)\|_1 \leq e^{((r/\eta)A - \eta)t} \|p_0(\cdot)\|_1 \leq e^{(r/\eta)A - \eta)t} (|M_0| + \|p_0(\cdot)\|_1 + |d_0|)$; that is $\|p(\cdot, t)\|_1 \leq e^{((r/\eta)A - \eta)t} \|\mathbf{X}_0\|_{\mathbb{X}}$. Therefore, we get $\|\Theta_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq e^{-\eta t} = \Delta(t, \delta)$, where $\delta = e^{(r/\eta)At} \|\mathbf{X}_0\|_{\mathbb{X}} \geq \|\mathbf{X}_0\|_{\mathbb{X}}$. Moreover, since $(r/\eta)A - \eta = (\sqrt{rA} + \eta)(\sqrt{rA} - \eta)/\eta$, using (7.16), it is easy to see that $\Delta(t, \delta) \rightarrow 0$ as $t \rightarrow \infty$. $\square$

Lemma 7.2.3. *For $t \geq 0$, $\Psi_t(\cdot)$ maps any bounded set Γ into a set with a compact closure in $\mathbb{X}_+$.*

Proof. Since $M(t)$ and $D(t)$ remain in the compact set $[0, A/\eta]$ by Lemma 7.2.1, $\Psi_t(\cdot) : \Gamma \rightarrow \Psi_t(\Gamma)$, where

$$\Psi_t(\Gamma) = \left\{ \Psi_t(\mathbf{X}_0) \in \mathbb{X}_+ : \|\Psi_t(\mathbf{X}_0)\|_{\mathbb{X}} \leq \frac{A}{\eta} \right\}. \quad (7.22)$$

We clearly see, from (7.22), that the set $\Psi_t(\Gamma)$ is bounded. To establish the closedness of the set $\Psi_t(\Gamma)$, we consider α and σ such that $\alpha = \lim_{t \rightarrow \infty} M(t)$ and $\sigma = \lim_{t \rightarrow \infty} D(t)$. By the continuity of the norm $\|\cdot\|_{\mathbb{X}}$, we have $\|(\alpha, 0, \sigma)\|_{\mathbb{X}} \leq A/\eta$, that is $\lim_{t \rightarrow \infty} \Psi_t(\mathbf{X}_0) \in \Psi_t(\Gamma)$ for all $\mathbf{X}_0 \in \Gamma$. Hence, the set $\Psi_t(\Gamma)$ is closed. By definition, $\overline{\Psi_t(\Gamma)}$ is the smallest closed set that contains $\Psi_t(\Gamma)$, that is $\Psi_t(\Gamma) \subset \overline{\Psi_t(\Gamma)}$. Moreover, $\overline{\Psi_t(\Gamma)}$ is also a closed set that contains $\Psi_t(\Gamma)$. So, $\overline{\Psi_t(\Gamma)} \subset \Psi_t(\Gamma)$. Therefore, $\overline{\Psi_t(\Gamma)} = \Psi_t(\Gamma)$.

By way of contradiction, suppose that $\overline{\Psi_t(\Gamma)}$ is not compact in $\mathbb{X}_+$, that is $\overline{\Psi_t(\Gamma)}$ is not totally bounded, since $\overline{\Psi_t(\Gamma)}$ is complete as a closed set of a Banach space $\mathbb{X}$, which is a complete space. This means that for some positive ϵ and for any finite $\mathcal{K} \subset \overline{\Psi_t(\Gamma)}$ we can find $\mathbf{U} \in \overline{\Psi_t(\Gamma)}$ such that $\mathbf{V} - \mathbf{U} \in \mathbb{X}_+$ and $\|\mathbf{V} - \mathbf{U}\|_{\mathbb{X}} \geq \epsilon$ for every $\mathbf{V} \in \mathcal{K}$. Let $(\mathbf{V}_n)$ be a sequence of points in $\mathcal{K}$, that is $\|\mathbf{V}_n\|_{\mathbb{X}} \leq A/\eta$ for any $n \in \mathbb{N}$. Thus $\mathbf{V}_n - \mathbf{U} \in \mathbb{X}_+$ and $\|\mathbf{V}_n - \mathbf{U}\|_{\mathbb{X}} > \epsilon$ for any $n \in \mathbb{N}$. It follows that $\|\mathbf{V}_n\|_{\mathbb{X}} \geq \|\mathbf{V}_n - \mathbf{U}\|_{\mathbb{X}}$, and hence $\|\mathbf{V}_n\|_{\mathbb{X}} > \epsilon$ for any $n \in \mathbb{N}$. Choosing $\epsilon = A/\eta$ yields a contradiction and therefore $\overline{\Psi_t(\Gamma)}$ is a totally bounded and complete set. Hence, $\overline{\Psi_t(\Gamma)}$ is a compact set in $\mathbb{X}_+$. $\square$

Therefore, from Lemma 7.2.1 and Theorem 7.2.1, the existence result of global attractors (see Hale (2010)) follows.

Theorem 7.2.2. *The semi-flow $\{\Phi_t(\mathbf{X}_0) : t \geq 0\}$ has a global attractor in $\mathbb{X}_+$, which attracts any bounded subset of $\mathbb{X}_+$.*

7.3 Equilibria

The equilibria of the system (7.1) are given by the following equation

$$\begin{aligned} 0 &= A - (g + rP^*)M + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)p(y)dydx, \\ 0 &= -rM \frac{dp}{dx}(x) - (m(x) + b(x) + k_2D)p(x) + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)p(y)dydx, \\ 0 &= (-k_0 + k_1P^*)D, \end{aligned} \quad (7.23)$$

where $P^* = \int_{x_0}^{\infty} p(x)dx$. From the last equation in (7.23), it follows that $D = 0$ or $P^* = k_0/k_1$. If $D = 0$, the system (7.23) reduces to

$$\begin{aligned} 0 &= A - (g + rP^*)M + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)p(y)dydx, \\ 0 &= -rM \frac{dp}{dx}(x) - (m(x) + b(x))p(x) \\ &\quad + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)p(y)dydx. \end{aligned} \quad (7.24)$$

Now assuming $p(x) = 0$ in (7.24), we obtain the disease-free steady state E^0 of the system (7.1) given by $E^0 = (M^0, p^0(x), D^0)$, where $M^0 = A/g$, $p^0(x) = 0$, $D^0 = 0$. However, assuming $p(x) \neq 0$ in (7.24), from the last equation in (7.24) we have

$$2 \int_x^{\infty} b(y)f(x, y)p(y)dy = rM \frac{dp}{dx}(x) + (m(x) + b(x))p(x) \quad (7.25)$$

and substituting (7.25) into first equation of (7.24) yields

$$A - (g + rP^*)M + 2 \int_0^{x_0} x \int_{x_0}^x b(y)f(x, y)p(y)dydx = 0,$$

that is,

$$A - (g + rP^*)M - 2 \int_0^{x_0} x \int_x^{x_0} b(y)f(x, y)p(y)dydx = 0.$$

Now changing the order of integration in the last equation leads to

$$A - (g + rP^*)M - \int_0^{x_0} \left(2 \int_0^y xf(x, y)dx \right) b(y)p(y)dy = 0. \quad (7.26)$$

Using (7.9), equation (7.26) gives

$$A - (g + rP^*)M - \int_0^{x_0} yb(y)p(y)dy = 0, \quad (7.27)$$

which implies that $A - (g + rP^*)M = 0$. Thus, a disease steady state E^* of the system (7.1) is given by $E^* = (M^*, p^*(x), D^*)$, where $M^* = A/(g + rP^*)$, $D^* = 0$ and $p^*(x)$ is the solution of the non-linear ordinary differential equation

$$\frac{dp}{dx}(x) + \frac{m(x) - b(x)}{rM^0}p(x) = \frac{p(x)}{A} \int_{x_0}^{\infty} p(x)dx. \quad (7.28)$$

If $P^* = k_0/k_1$, the system (7.23) reduces to

$$\begin{aligned} 0 &= A - (g + r \frac{k_0}{k_1})M + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x,y)p(y)dydx, \\ 0 &= -rM \frac{dp}{dx}(x) - (m(x) + b(x) + k_2D)p(x) + 2 \int_{x_0}^{\infty} b(y)f(x,y)p(y)dy. \end{aligned}$$

From the latter equation, we have

$$2 \int_x^{\infty} b(y)f(x,y)p(y)dy = rM \frac{dp}{dx}(x) + (m(x) + b(x) + k_2D)p(x) \quad (7.29)$$

and the first equation of (7.29) is equivalent to

$$\begin{aligned} 0 = A - & \left(g + r \frac{k_0}{k_1} \right) M - 2 \int_0^{x_0} x \int_x^{x_0} b(y)f(x,y)p(y)dydx \\ & + 2 \int_0^{x_0} x \int_x^{\infty} b(y)f(x,y)p(y)dydx. \end{aligned} \quad (7.30)$$

Substituting (7.29) into (7.30) leads to

$$0 = A - \left(g + r \frac{k_0}{k_1} \right) M - 2 \int_0^{x_0} x \int_x^{x_0} b(y)f(x,y)p(y)dydx \quad (7.31)$$

and after changing the order of integration and using (7.9) in (7.31), we obtain,

$$0 = A - \left(g + r \frac{k_0}{k_1} \right) M - \int_x^{x_0} yb(y)p(y)dydx, \quad (7.32)$$

which reduces to

$$0 = A - \left(g + r \frac{k_0}{k_1} \right) M. \quad (7.33)$$

Thus, a coexistence steady state $\check{E}$ of the system (7.1) is given by $\check{E} = (\check{M}, \check{p}(x), \check{D})$, where $\check{M} = k_1A(k_0r + k_1g)$, $\check{D} \neq 0$ and $\check{p}(x)$ is the solution of the linear ordinary differential equation

$$\frac{dp}{dx}(x) = \frac{k_0r + k_1g}{k_1Ar} (m(x) - b(x) + k_2D)p(x), \quad (7.34)$$

where $D = \check{D}$ and $k_2 = k_1$, (see **A1**). Defining $\bar{x}_0 = \inf\{x|p(x) \neq 0, \text{ for } x > x_0\}$ and integrating (7.34) over $[\bar{x}_0, x]$ leads to $\check{p}(x) = p(\bar{x}_0)\Omega(x)$, where

$$\Omega(x) = \exp \left\{ -\frac{k_0r + k_1g}{k_1rA} \left(k_1\check{D}(x - \bar{x}_0) + \int_{\bar{x}_0}^x (m(s) - b(s))ds \right) \right\}.$$

Thus, the equation

$$\int_{x_0}^{\infty} p(x)dx = \frac{k_0}{k_1}$$

yields

$$p(\bar{x}_0) = k_0/k_1.1 / \int_{\bar{x}_0}^{\infty} \Omega(x)dx$$

and hence

$$p(x) = k_0/k_1.\Omega(x) / \int_{\bar{x}_0}^{\infty} \Omega(x)dx,$$

that is,

$$\check{p}(x) = k_0/k_1.\Omega(x) / \int_{\bar{x}_0}^{\infty} \Omega(x)dx.$$

7.4 Uniform Persistence

In this section, we investigate the uniform persistence of the system (7.1) at the disease state equilibrium E^* , that is under the condition

$$\frac{k_1}{k_0} \int_{x_0}^{\infty} p^*(x) dx < 1. \quad (7.35)$$

To do this, we consider a function $\rho : X \rightarrow \mathbb{R}_+$ defined by

$$\rho(u, \zeta, \nu) = \frac{k_1}{k_0} \int_{x_0}^{\infty} \zeta(x) dx, \quad \text{where } (u, \zeta, \nu) \in \mathbb{X}_+. \quad (7.36)$$

Let $\mathbb{X}_0$ be a set defined by $\mathbb{X}_0 = \{\mathbf{X}_0 \in \mathbb{X}_+ : \rho(\Phi_t(\mathbf{X}_0)) > 0 \text{ for some } t_0 \geq 0\}$ so that $\Phi_t(\mathbf{X}_0) \rightarrow E^0$ as $t \rightarrow \infty$ whenever $\mathbf{X}_0 \in \mathbb{X}_+ \setminus \mathbb{X}_0$.

Definition 7.4.1 (Smith and Thieme (2011), p.61). *Φ is uniformly weakly ρ -persistent (respectively, uniformly (strongly) ρ -persistent) if there exists a positive ϵ , not depending on the initial conditions, so that $\lim_{t \rightarrow \infty} \sup \rho(\Phi_t(\mathbf{X}_0)) > \epsilon$ (respectively $\lim_{t \rightarrow \infty} \inf \rho(\Phi_t(\mathbf{X}_0)) > \epsilon$) for any $\mathbf{X}_0 \in \mathbb{X}_+$.*

Theorem 7.4.1. *Let (7.35) be satisfied. Then Φ is uniformly (strongly) ρ -persistent.*

Proof. By way of contradiction, for all $\epsilon > 0$ there exist $\mathbf{X}^\epsilon \in \mathbb{X}_0$ such that $\lim_{t \rightarrow \infty} \inf \rho(\Phi_t(\mathbf{X}^\epsilon)) \leq \epsilon$. In particular, there exists $\mathbf{X}^{\epsilon_0} \in \mathbb{X}_0$ such that

$$\liminf_{t \rightarrow \infty} \rho(\Phi_t(\mathbf{X}^{\epsilon_0})) \leq \epsilon, \quad (7.37)$$

where $0 < \epsilon_0 < 1$. Choosing $\epsilon_0 = k_1 A / k_0 \eta$ and using (7.4) yields

$$\frac{k_1}{k_0} \int_{x_0}^{\infty} p^{\epsilon_0}(x) dx < \frac{k_1}{k_0} \int_{x_0}^{\infty} x p^{\epsilon_0}(x) dx. \quad (7.38)$$

It follows that

$$\frac{k_1}{k_0} \int_{x_0}^{\infty} x p^{\epsilon_0}(x) dx < \frac{k_1}{k_0} \|p^{\epsilon_0}(\cdot, t) - p^*(\cdot)\|_1 + \frac{k_1}{k_0} \int_{x_0}^{\infty} p^*(x) dx \quad (7.39)$$

and hence

$$\rho(\Phi_t(\mathbf{X}^{\epsilon_0})) < \frac{k_1}{k_0} \|\Phi_t(\mathbf{X}^{\epsilon_0}) - E^*\|_x + \frac{k_1}{k_0} \int_{x_0}^{\infty} p^*(x) dx. \quad (7.40)$$

Since $\Phi(\mathbf{X}_0) - \check{E} \in \Gamma$, we can make use of (7.35) so that

$$\rho(\Phi_t(\mathbf{X}^{\epsilon_0})) < \frac{k_1 A}{k_0 \eta} + 1, \quad (7.41)$$

that is,

$$\rho(\Phi_t(\mathbf{X}^{\epsilon_0})) < \epsilon_0 + 1, \quad \text{for all } t \geq 0. \quad (7.42)$$

Therefore

$$\liminf_{t \rightarrow \infty} \rho(\Phi_t(\mathbf{X}^{\epsilon_0})) \leq \limsup_{t \rightarrow \infty} \rho(\Phi_t(\mathbf{X}^{\epsilon_0})) < \epsilon_0 + 1. \quad (7.43)$$

This contradicts (7.37). $\square$

7.5 Local Stability of Equilibria

The local stability for the equilibria of the system (7.1) will be investigated through its linearization around each equilibrium, see M'pika Massoukou et al. (2018). For the disease-free steady state E^0 , the following theorem holds.

Theorem 7.5.1. *The disease-free steady state E^0 is stable.*

Proof. To prove this we will perturb the equilibria as follows. We introduce new functions $u, v(x, \cdot)$ and w such that

$$u(t) = M(t) - M^0, \quad v(x, t) = p(x, t), \quad w(t) = D(t).$$

Differentiating functions $u, v(x, \cdot)$ and w with respect to time t , making use of equations in (7.1) and omitting terms of order higher than or equal to two, leads to the following system

$$\begin{aligned} \frac{du}{dt}(t) &= -gu(t) - rM^0 \int_{x_0}^{\infty} v(x, t) dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y) f(x, y) v(x, t) dy dx, \\ \frac{\partial v}{\partial t}(x, t) &= -rM^0 \frac{\partial v}{\partial x}(x, t) - (m(x) + b(x))v(x, t) + 2 \int_x^{\infty} b(y) f(x, y) v(y, t) dy, \\ \frac{dw}{dt}(t) &= -k_0 w(t). \end{aligned} \quad (7.44)$$

In order to derive the spectrum of the linear operator associated to the linearised system (7.44), we consider exponential solution of the form

$$u(t) = \bar{u}e^{\lambda t}, \quad v(x, t) = \bar{v}(x)e^{\lambda t}, \quad w(t) = \bar{w}e^{\lambda t}, \quad (7.45)$$

where $\bar{u}$ and $\bar{w}$ are arbitrary constants, $\bar{v}(x)$ is an x -depending function and λ is a complex number. Substituting (7.45) into (7.44) yields

$$\begin{aligned} \lambda \bar{u} &= -g\bar{u} - rM^0 \int_{x_0}^{\infty} \bar{v}(x) dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y) f(x, y) \bar{v}(y) dy dx, \\ \lambda \bar{v}(x) &= -rM^0 \frac{d\bar{v}}{dx}(x) - (m(x) + b(x))\bar{v}(x) + 2 \int_x^{\infty} b(y) f(x, y) \bar{v}(y) dy, \\ \lambda &= -k_0. \end{aligned} \quad (7.46)$$

One eigenvalue of (7.46) is $\lambda = -k_0 < 0$. The remaining eigenvalues will be derived later. Using the second equation in (7.46), we have

$$2 \int_x^{\infty} b(y) f(x, y) \bar{v}(y) dy = rM^0 \frac{d\bar{v}}{dx}(x) + (\lambda + m(x) + b(x))\bar{v}(x) \quad (7.47)$$

and the first equation in (7.46) is equivalent to

$$\begin{aligned} \lambda \bar{u} &= -g\bar{u} - rM^0 \int_{x_0}^{\infty} \bar{v}(x) dx + 2 \int_0^{x_0} x \int_{x_0}^x b(y) f(x, y) \bar{v}(y) dy \\ &\quad + 2 \int_0^{x_0} x \int_x^{\infty} b(y) f(x, y) \bar{v}(y) dy. \end{aligned} \quad (7.48)$$

Substituting (7.47) into (7.48) reduces the latter equation to $\lambda = -(g + rM^0/\bar{u}) \int_{x_0}^{\infty} \bar{v}(x) dx < 0$. It follows that the spectral bound of (7.46) is negative and thus stable. $\square$

Likewise, by considering the disease steady state E^* , we have the following theorem.

Theorem 7.5.2. *The disease steady state E^* is locally asymptotically stable if $P^* < k_0/k_1$ and unstable if $P^* > k_0/k_1$.*

Proof. We define the functions u , $v(x, \cdot)$ and w by

$$u(t) = M(t) - M^*, \quad v(x, t) = p(x, t) - p^*(x), \quad w(t) = D(t)$$

and we obtain the following system

$$\begin{aligned} \frac{du}{dt}(t) &= -(g + rP^*)u(t) - rM^* \int_{x_0}^{\infty} v(x, t)dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)v(x, t)dydx, \\ \frac{\partial v}{\partial t}(x, t) &= -rM^* \frac{\partial v}{\partial x}(x, t) - (m(x) + b(x))v(x, t) + 2 \int_x^{\infty} b(y)f(x, y)v(y, t)dy \\ &\quad - ru(t) \frac{dp^*}{dx}(x) - k_2w(t)p^*(x), \\ \frac{dw}{dt}(t) &= -k_0w(t) + k_2w(t)P^*. \end{aligned} \quad (7.49)$$

To derive the characteristic equation of the system (7.49), the function u , $v(x, \cdot)$ and w defined in (7.45) are used into (7.49) so that

$$\begin{aligned} \lambda \bar{u} &= -(g + rP^*)\bar{u} - rM^* \int_{x_0}^{\infty} \bar{v}(x)dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x, y)\bar{v}(y)dydx, \\ \lambda \bar{v}(x) &= -rM^* \frac{d\bar{v}}{dx}(x) - (m(x) + b(x))\bar{v}(x) + 2 \int_x^{\infty} b(y)f(x, y)\bar{v}(y)dy \\ &\quad - r\bar{u} \frac{dP^*}{dx}(x) - k_2\bar{w}p^*(x), \\ \lambda &= -k_0 + kP^*. \end{aligned} \quad (7.50)$$

One eigenvalue of (7.50) is $\lambda = -k_0 + k_1P^*$, which is negative if $P^* < k_0/k_1$ and positive if $P^* > k_0/k_1$. The remaining eigenvalues will be derived later. Using the second equation in (7.50), we have

$$2 \int_x^{\infty} b(y)f(x, y)\bar{v}(y)dy = rM^* \frac{d\bar{v}}{dx}(x) + (\lambda + m(x) + b(x))\bar{v}(x) + r\bar{u} \frac{dP^*}{dx}(x) + k_2\bar{w}p^*(x) \quad (7.51)$$

and the first equation in (7.50) is equivalent to

$$\begin{aligned} \lambda \bar{u} &= -(g + rP^*)\bar{u} - rM^* \int_{x_0}^{\infty} \bar{v}(x)dx + 2 \int_0^{x_0} x \int_{x_0}^x b(y)f(x, y)\bar{v}(y)dydx \\ &\quad + 2 \int_0^{x_0} x \int_x^{\infty} b(y)f(x, y)\bar{v}(y)dy. \end{aligned} \quad (7.52)$$

Substituting (7.51) into (7.52) reduces the latter to $\lambda = -(g + rP^* + (rM^*/\bar{u}) \int_{x_0}^{\infty} \bar{v}(x)dx) < 0$. It follows that the spectral bounds of (7.50) is negative if $P^* < k_0/k_1$. $\square$

In a similar way, by considering the coexistence steady state $\check{E}$, the following theorem holds.

Theorem 7.5.3. *The coexistence steady state $\check{E}$ is unstable if it exists.*

Proof. We define the functions u , $v(x, \text{cot})$ and w by

$$u(t) = M(t) - \check{M}, \quad v(x, t) = p(x, t) - \check{p}(x), \quad w(t) = D(t) - \check{D}$$

and we obtain the following system

$$\begin{aligned} \frac{du}{dt}(t) &= -\left(g + r \frac{k_0}{k_1}\right) u(t) - r\check{M} \int_{x_0}^{\infty} v(x, t) dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y) f(x, y) v(x, t) dy dx, \\ \frac{\partial v}{\partial t}(x, t) &= -r\check{M} \frac{\partial v}{\partial x}(x, t) - (m(x) + b(x) + k_2 \check{D}) v(x, t) + 2 \int_x^{\infty} b(y) f(x, y) v(y, t) dy \\ &\quad - ru(t) \frac{d\check{p}}{dx}(x) - k_2 w(t) \check{p}(x), \\ \frac{dw}{dt}(t) &= -k_1 \check{D} \int_{x_0}^{\infty} v(x, t) dx. \end{aligned} \quad (7.53)$$

To derive the characteristic equation of the system (7.53), the function u , $v(x, t)$ and w defined in (7.45) are used in (7.53) so that

$$\begin{aligned} \lambda \bar{u} &= -\left(g + r \frac{k_0}{k_1}\right) \bar{u} - r\check{M} \int_{x_0}^{\infty} \bar{v}(x) dx + 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y) f(x, y) \bar{v}(y) dy dx, \\ \lambda \bar{v}(x) &= -r\check{M} \frac{d\bar{v}}{dx}(x) - (m(x) + b(x) + k_2 \check{D}) \bar{v}(x) + 2 \int_x^{\infty} b(y) f(x, y) \bar{v}(y) dy \\ &\quad - r\bar{u} \frac{d\check{p}}{dx}(x) - k_2 \bar{w} \check{p}(x), \\ \lambda &= \frac{k_1}{\bar{w}} \int_{x_0}^{\infty} \bar{v}(x) dx. \end{aligned} \quad (7.54)$$

From the last equation in (7.54), it follows that the spectral bound of (7.54) is positive. $\square$

7.6 Global Stability of Equilibria

This section focuses on the global asymptotic stability analysis of the system (7.1) at the equilibria E^0 and E^* . This is achieved by means of appropriate Volterra-type functions of the form $G(X) = X - 1 - \ln X$ and the following result is obtained.

Theorem 7.6.1. *The disease-free steady state E^0 is globally asymptotically stable.*

Proof. We consider the Lyapunov function $L^0(t) = L_1^0(t) + L_2^0(t)$, where $L_1^0(t) = M^0 G(M(t)/M^0)$, $L_2^0(t) = D(t) + Q(t)$. Differentiating $L_1^0(t)$, making use of the first equation in (7.1) and $A = gM^0$ and changing the order of integration, we have

$$\begin{aligned} \frac{dL_1^0}{dt}(t) &= gM^0 \left(2 - \frac{M^0}{M(t)} - \frac{M(t)}{M^0}\right) + rM^0 \left(1 - \frac{M(t)}{M^0}\right) P(t) \\ &\quad + \left(1 - \frac{M^0}{M(t)}\right) \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x f(x, y) dx\right) b(y) p(y, t) dy. \end{aligned} \quad (7.55)$$

We now differentiate $L_2^0(t)$, make use of the second and the last equation in (7.1) and change the order of integration so that

$$\begin{aligned} \frac{dL_2^0}{dt}(t) &= -k_0 D + (k_1 D + rM) P - \int_{x_0}^{\infty} x(m(x) + b(x) + k_2 D) p(x, t) dx \\ &\quad + \int_{x_0}^{\infty} \left(2 \int_{x_0}^y x f(x, y) dx\right) b(y) p(y, t) dy. \end{aligned} \quad (7.56)$$

Considering (7.55) and (7.56) and using (7.9) of **A3**, we get

$$\begin{aligned} \frac{dL^0}{dt}(t) = & -k_0D + gM^0 \left(2 - \frac{M^0}{M} - \frac{M}{M^0} \right) + \left(rM^0P - \int_{x_0}^{\infty} x(m(x)p(x,t)dx \right) \\ & + (k_1P - k_2Q)D - \frac{M^0}{M} \int_{x_0}^{\infty} \left(2 \int_0^{x_0} xf(x,y)dx \right) b(y)p(y,t)dy. \end{aligned} \quad (7.57)$$

Clearly, the first, second and last terms in (7.57) are negative. Using assumption **A1** and condition (7.4), we get

$$(k_1P(t) - k_2Q(t))D(t) = k_1D(t)Q(t) \left(\frac{P(t)}{Q(t)} - 1 \right) < 0 \quad (7.58)$$

and by assumption **A2** and condition (7.4), we obtain

$$rM^0P(t) - \int_{x_0}^{\infty} xm(x)p(x,t)dx \leq rM^0Q(t) \left(\frac{P(t)}{Q(t)} - 1 \right) < 0. \quad (7.59)$$

Hence, $dL^0/dt < 0$ is satisfied for any $M(t), p(x,t), D(t) \in \mathbb{R}_+$ and $dL^0/dt = 0$ for $(M(t), p(x,t), D(t)) = (M^0, p^0(x), D^0)$. Furthermore, it can be verified that $\{(M(t), p(x,t), D(t)) : L^0(t) = 0\} = \{E^0\}$. Therefore, it follows from Lasalle's Invariance Theorem [Auger et al. (2008), p. 200] that E^0 is globally asymptotically stable. $\square$

Theorem 7.6.2. *The disease steady state E^* is globally asymptotically stable if $P^* < k_0/k_1$.*

Proof. We consider the Lyapunov function $L^*(t) = L_1^*(t) + L_2^*(t)$, where

$$L_1^*(t) = M^*G(M(t)/M^*), \quad L_2^*(t) = D(t) + \int_{x_0}^{\infty} p^*(x)G(p(x,t)/p^*(x))dx.$$

Differentiating $L_1^*(t)$ and then making use of the first equation in (7.1), and then substituting A by the following expression

$$A = (g + rP^*)M^* - 2 \int_0^{x_0} x \int_{x_0}^{\infty} b(y)f(x,y)p^*(y)dydx$$

and changing the order of integration, we obtain

$$\begin{aligned} \frac{dL_1^*}{dt}(t) = & -\frac{(M - M^*)^2}{2}(g + rP^*) - rM \int_{x_0}^{\infty} p^*(x) \left(1 - \frac{M^*}{M} \right) \left(1 - \frac{p^*(x)}{p(x, \cdot)} \right) dx \\ & + \int_{x_0}^{\infty} \left(2 \int_0^{x_0} xf(x,y)dx \right) \left(1 - \frac{M^*}{M} \right) \left(1 - \frac{p^*(y)}{p(y, \cdot)} \right) b(y)p(y,t)dy. \end{aligned} \quad (7.60)$$

We now differentiate $L_2^*(t)$, make use of the second and the last equation in (7.1) and

$$-rM^* \frac{dp^*}{dx}(x) - (m(x) + b(x))p^*(x) + 2 \int_x^{\infty} b(y)f(x,y)p^*(y)dy = 0$$

and change the order of integration so that

$$\begin{aligned}
\frac{dL_2^*}{dt}(t) = & -(k_0 - k_1 P^*)D(t) - (k_2 - k_1)D(t) \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)}\right) p(x, t) dx \\
& + rM^* \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)}\right) \frac{dp^*}{dx}(x) dx \\
& - rM(t) \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)}\right) \frac{\partial p^*}{\partial x}(x, t) dx \\
& - \int_{x_0}^{\infty} (m(x) + b(x))p(x, t) \left(1 - \frac{p^*(x)}{p(x, t)}\right)^2 dx \\
& + 2 \int_{x_0}^{\infty} \left(\int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx \right) \left(1 - \frac{p^*(y)}{p(y, t)}\right) b(y)p(y, t) dy.
\end{aligned}$$

Expanding terms in the latter equation and making use of conditions $p(x_0, t) = p^*(x_0) = 0$ and $\lim_{x \rightarrow \infty} p(x, t) = \lim_{x \rightarrow \infty} p^*(x) = 0$, we get

$$\begin{aligned}
\frac{dL_2^*}{dt}(t) = & -k_1 \left(\frac{k_0}{k_1} - p^* \right) D - k_2 \left(1 - \frac{k_1}{k_2} \right) D \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)} \right) p(x, t) dx \\
& + \int_{x_0}^{\infty} \left(\frac{M}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) \left(-rM^* \frac{dp^*}{dx}(x) \right) dx \\
& - \int_{x_0}^{\infty} (m(x) + b(x))p(x, t) \left(1 - \frac{p^*(x)}{p(x, t)} \right)^2 dx \\
& + \int_{x_0}^{\infty} \left(\int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)} \right) f(x, y) dx \right) \left(1 - \frac{p^*(y)}{p(y, t)} \right) b(y)p(y, t) dy. \quad (7.61)
\end{aligned}$$

Using

$$-rM^*(dp^*/dx)(x) = (m(x) + b(x))p^*(x) - 2 \int_x^{\infty} b(y)f(x, y)p^*(y)dy$$

in equation (7.61) yields

$$\begin{aligned}
\frac{dL_2^*}{dt}(t) = & -k_1 \left(\frac{k_0}{k_1} - p^* \right) D(t) - k_2 \left(1 - \frac{k_1}{k_2} \right) D(t) \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
& + \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
& - 2 \int_{x_0}^{\infty} \left(\frac{M}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) \int_x^{\infty} f(x, y)b(y)p^*(y)dy dx \\
& - \int_{x_0}^{\infty} (m(x) + b(x))p(x, t) \left(1 - \frac{p^*(x)}{p(x, t)} \right)^2 dx \\
& + 2 \int_{x_0}^{\infty} \left(\int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)} \right) f(x, y) dx \right) \left(1 - \frac{p^*(y)}{p(y, t)} \right) b(y)p(y, t) dy
\end{aligned}$$

and therefore, we obtain

$$\begin{aligned}
\frac{dL^*}{dt}(t) = & -\frac{(M(t) - M^*)^2}{M(t)}(g + rP^*) - k_1 \left(\frac{k_0}{k_1} - P^* \right) D(t) \\
& - k_2 \left(1 - \frac{k_1}{k_2} \right) D(t) \int_{x_0}^{\infty} \left(1 - \frac{p^*(x)}{p(x, t)} \right) p(x, t) dx \\
& - rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)} \right) \left(1 - \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
& + \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M(t)}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
& - 2 \int_{x_0}^{\infty} \left(\frac{M(t)}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) \int_x^{\infty} f(x, y) b(y) p^*(y) dy dx \\
& - \int_{x_0}^{\infty} (m(x) + b(x)) \left(1 - \frac{p^*(x)}{p(x, t)} \right)^2 p(x, t) dx \\
& + \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x f(x, y) dx \right) \left(1 - \frac{M^*}{M(t)} \right) \left(1 - \frac{p^*(y)}{p(y, t)} \right) b(y) p(y, t) dy \\
& + \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)} \right) f(x, y) dx \right) \left(1 - \frac{p^*(y)}{p(x, y)} \right) b(y) p(y, t) dy.
\end{aligned} \tag{7.62}$$

From the right-hand side in equation (7.62), it is clear that the first term is negative, the second term is also negative since $P^* < k_0/k_1$ and the third term vanishes by assumption **A1**. Most of the remaining terms on the right-hand side in equation (7.62) will be expanded in the following manner. The fourth term in equation (7.62) becomes

$$\begin{aligned}
-rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)} \right) \left(1 - \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
= -rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)} \right) p^*(x) dx \\
+ rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)} \right) \left(\frac{p^*(x)}{p(x, t)} \right)^2 p(x, t) dx.
\end{aligned} \tag{7.63}$$

The fifth term in (7.62) is expanded as

$$\begin{aligned}
& \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M(t)}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx \\
= & \frac{M(t)}{M^*} \int_{x_0}^{\infty} (m(x) + b(x)) (\ln p(x, t) - p(x, t)) p^*(x) dx \\
& + \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M(t)}{M^*} p(x, t) + \frac{p^*(x)}{p(x, t)} \right) p^*(x) dx,
\end{aligned} \tag{7.64}$$

where the first term on the right-hand side in equation (7.64) is negative. Next, we transform the sixth term in equation (7.62) into

$$\begin{aligned}
& -2 \int_{x_0}^{\infty} \left(\frac{M(t)}{M^*} \ln p(x, t) + \frac{p^*(x)}{p(x, t)} \right) \int_x^{\infty} f(x, y) b(y) p^*(y) dy dx \\
= & -\frac{M(t)}{M^*} \int_{x_0}^{\infty} (\ln p(x, t) + p(x, t)) \int_x^{\infty} f(x, y) b(y) p^*(y) dy dx \\
& + \int_{x_0}^{\infty} 2 \left(\int_{x_0}^y \left(\frac{M(t)}{M^*} p(x, t) - \frac{p^*(x)}{p(x, t)} \right) f(x, y) dx \right) b(y) p^*(y) dy,
\end{aligned}$$

where the first term on the right-hand side of this equation is negative. We also expand the seventh term in (7.62) as

$$\begin{aligned} & - \int_{x_0}^{\infty} (m(x) + b(x)) \left(1 - \frac{p^*(x)}{p(x, t)}\right)^2 p(x, t) dx \\ &= \int_{x_0}^{\infty} (m(x) + b(x)) \left[-1 + 2 \frac{p^*(x)}{p(x, t)} - \left(\frac{p^*(x)}{p(x, t)}\right)^2\right] p(x, t) dx. \end{aligned} \quad (7.65)$$

The eight term in (7.62) yields

$$\begin{aligned} & \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x f(x, y) dx\right) \left(1 - \frac{M^*}{M(t)}\right) \left(1 - \frac{p^*(y)}{p(y, t)}\right) b(y) p(y, t) dy \\ &= \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x f(x, y) dx\right) \left(1 - \frac{M^*}{M(t)}\right) b(y) p(y, t) dy \\ &\quad - \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x x f(x, y) dx\right) \left(1 - \frac{M^*}{M(t)}\right) b(y) p^*(y) dy. \end{aligned} \quad (7.66)$$

Finally, we transform the last term in (7.62) into

$$\begin{aligned} & \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx\right) \left(1 - \frac{p^*(y)}{p(y, t)}\right) b(y) p(y, t) dy \\ &= \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx\right) b(y) p(y, t) dy \\ &\quad - \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx\right) b(y) p^*(y) dy. \end{aligned} \quad (7.67)$$

By adding up the last terms in equation (7.65) and (7.67), it follows that

$$\begin{aligned} & \int_{x_0}^{\infty} 2 \left(\int_{x_0}^y \left(\frac{M(t)}{M^*} p(x, t) - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx\right) b(y) p^*(y) dy \\ &\quad - \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x, t)}\right) f(x, y) dx\right) b(y) p^*(y) dy \\ &= -2 \int_{x_0}^{\infty} \left(\int_{x_0}^y \left(1 - \frac{M(t)}{M^*} p(x, t)\right) f(x, y) dx\right) b(y) p^*(y) dy, \end{aligned} \quad (7.68)$$

which is negative provided that $p(x, t) < M^*/M(t)$ for any $t \geq 0$ and $x > x_0$. In the sequel, we shall investigate the sign of the remaining terms under the conditions $p(x, t) < M^*/M(t) < 1$ and $1 < p(x, t) < M^*/M(t)$. First, we assume that $p(x, t) < M^*/M(t) < 1$. Clearly, we see in equation (7.63) that the first term on right-hand side is negative, while the second term can be expanded as

$$\begin{aligned} & rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)}\right) \left(\frac{p^*(x)}{p(x, t)}\right)^* 2p(x, t) dx \\ &= rM(t) \int_{x_0}^{\infty} \left(\frac{p^*(x)}{p(x, t)}\right)^2 p(x, t) dx \\ &\quad - rM^* \int_{x_0}^{\infty} \left(\frac{p^*(x)}{p(x, t)}\right)^2 p(x, t) dx, \end{aligned} \quad (7.69)$$

$$(7.70)$$

where the second term on the right-hand side is negative, and the first term on the right-hand side in equation (7.67) can be expanded as

$$\begin{aligned} & \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x,t)} \right) f(x,y) dx \right) b(y) p(y,t) dy \\ &= \int_{x_0}^{\infty} \left(2 \int_{x_0}^y f(x,y) dy \right) b(y) p(y,t) dy \\ & - \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \frac{p^*(x)}{p(x,t)} f(x,y) dx \right) b(y) p(y,t) dy, \end{aligned} \quad (7.71)$$

where the second term on the right-hand side is negative. Adding up the second term on the right hand side in (7.69) and the first term on the right-hand side in (7.71), we have

$$\begin{aligned} & -rM^* \int_{x_0}^{\infty} \left(\frac{p^*(x)}{p(x,t)} \right)^2 p(x,t) dx + \int_{x_0}^{\infty} \left(2 \int_{x_0}^y f(x,y) dy \right) b(y) p(y,t) dy \\ & \leq \int_{x_0}^{\infty} m(x) \left(1 - \frac{\sqrt{rM^*}}{\sqrt{m(x)}} \frac{p^*(x)}{p(x,t)} \right) \left(1 + \frac{\sqrt{rM^*}}{\sqrt{m(x)}} \frac{p^*(x)}{p(x,t)} \right) p(x,t) dx < 0, \end{aligned} \quad (7.72)$$

provided that $1 < \sqrt{m(x)}/\sqrt{rM^*} < p^*(x)/p(x,t)$ for any $t \geq 0$ and $x > x_0$. Moreover, the second term on the right-hand side in (7.64), equation (7.65), the first term on the right-hand side in (7.66) and the first term on the right-hand side in (7.69) yields

$$\begin{aligned} & \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M(t)}{M^*} p(x,t) + \frac{p^*(x)}{p(x,t)} \right) p^*(x) dx \\ & + \int_{x_0}^{\infty} (m(x) + b(x)) \left(-1 + 2 \frac{p^*(x)}{p(x,t)} - \left(\frac{p^*(x)}{p(x,t)} \right)^2 \right) p(x,t) dx \\ & + \int_{x_0}^{\infty} \left(2 \int_{x_0}^y x f(x,y) dx \right) \left(1 - \frac{M^*}{M(t)} \right) b(y) p(y,t) dy \\ & + rM(t) \int_{x_0}^{\infty} \left(\frac{p^*(x)}{p(x,t)} \right)^2 p(x,t) dx, \\ & \leq \int_{x_0}^{\infty} \left[(m(x) + b(x)) \left(3 \frac{p^*(x)}{p(x,t)} - \frac{M^*}{M(t)} \right) + rM(t) \left(\frac{p^*(x)}{p(x,t)} \right)^2 \right] p(x,t) dx, \end{aligned} \quad (7.73)$$

since $p(x,t) < M^*/M(t)$. It is not difficult to show that

$$rM(t) \left(\frac{p^*(x)}{p(x,t)} \right)^2 + 3(m(x) + b(x)) \frac{p^*(x)}{p(x,t)} - (m(x) + b(x)) \frac{M^*}{M(t)} < 0$$

if $p^*(x)/p(x,t) < \sqrt{m(x) + b(x)}\sqrt{rM}/rM$, that is $p^*(x)/p(x,t) < \sqrt{m(x) + b(x)}$ for any $t \geq 0$ and $x > x_0$. Therefore $\sqrt{m(x)}/\sqrt{rM^*} < p^*(x)/p(x,t) < \sqrt{m(x) + b(x)}$ and hence $dL^*/dt < 0$ is satisfied for $(M(t), p(x,t), D(t))$ such that

$$p(x,t) < \frac{M^*}{M(t)} < 1, \quad 1 < \frac{\sqrt{m(x)}}{\sqrt{rM^*}} < \frac{p^*(x)}{p(x,t)} < \sqrt{m(x) + b(x)}, \quad D(t) \in \mathbb{R}_+$$

and $dL^*/dt = 0$ for $(M(t), p(x,t), D(t)) = (M^*, p^*(x), D^*)$. Finally, we assume that $1 < p(x,t) < M^*/M(t)$. It follows that the second term on the right-hand side in (7.63) and

the first term on the right-hand side in (7.66) are both negative, while the remaining terms on the right-hand side in (7.63) and (7.66) can be transformed, respectively, as follows

$$-rM(t) \int_{x_0}^{\infty} \left(1 - \frac{M^*}{M(t)}\right) p^*(x) dx = -rP^*M(t) + \int_{x_0}^{\infty} rM^* \frac{p^*(x)}{p(x,t)} p(x,t) dx \quad (7.74)$$

and

$$\begin{aligned} & - \int_{x_0}^{\infty} \left(2 \int_0^{x_0} x f(x,y) dx\right) \left(1 - \frac{M^*}{M(t)}\right) b(y) p^*(y) dy \\ & \leq - \int_{x_0}^{\infty} y b(y) p^*(y) dy + \int_{x_0}^{\infty} \frac{M^*}{M(t)} y b(y) p^*(y) dy, \end{aligned} \quad (7.75)$$

where the first terms on the right-hand side in (7.74) and (7.75) are negative. After using **A2**, the second terms in (7.74) and (7.75) yield, respectively,

$$\int_{x_0}^{\infty} rM^* \frac{p^*(x)}{p(x,t)} p(x,t) dx < \int_{x_0}^{\infty} m(x) \frac{p^*(x)}{p(x,t)} p(x,t) dx \quad (7.76)$$

and

$$\int_{x_0}^{\infty} \frac{M^*}{M(t)} y b(y) p^*(y) dy < \int_{x_0}^{\infty} \frac{M^*}{M(t)} m(y) p^*(y) dy. \quad (7.77)$$

The first term on the right-hand side in (7.67) can also be transformed into

$$\begin{aligned} & \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(1 - \frac{p^*(x)}{p(x,t)}\right) f(x,y) dx\right) b(y) p(y,t) dy \\ & \leq \int_{x_0}^{\infty} 2b(y) p(y,t) dy - \int_{x_0}^{\infty} \left(2 \int_{x_0}^y \left(\frac{p^*(x)}{p(x,t)}\right) f(x,y) dx\right) b(y) p(y,t) dy, \end{aligned}$$

where the second term on the right-hand side is negative, while the first term on the right-hand side yields

$$\int_{x_0}^{\infty} 2b(y) p(y,t) dy < \int_{x_0}^{\infty} m(y) p(y,t) dy. \quad (7.78)$$

From (7.77)–(7.78), we get

$$\begin{aligned} & \int_{x_0}^{\infty} \frac{M^*}{M(t)} y b(y) p^*(y) dy + \int_{x_0}^{\infty} 2b(y) p(y,t) dy \\ & < \int_{x_0}^J \left[\frac{M^*}{M(t)} + \frac{1}{2} \left(\frac{M^*}{M(t)}\right)^2 + \frac{1}{2} \left(\frac{p^*}{p(x,y)}\right)^2 \right] m(y) p(y,t) dy. \end{aligned} \quad (7.79)$$

Combining the second term on the right-hand in (7.64)–(7.65), (7.76) and (7.79), we obtain

$$\begin{aligned}
& - \int_{x_0}^{\infty} (m(x) + b(x)) \left(1 - \frac{p^*(x)}{p(x, t)}\right)^2 p(x, t) dx \\
& + \int_{x_0}^{\infty} (m(x) + b(x)) \left(\frac{M(t)}{M^*} p(x, t) + \frac{p^*(x)}{p(x, t)} p^*(x) \right) dx \\
& + \int_{x_0}^{\infty} \frac{M^*}{M(t)} y b(y) p^*(y) dy + \int_{x_0}^{\infty} 2b(y) p(y, t) dy \\
& + \int_{x_0}^{\infty} r M^* \frac{p^*(x)}{p(x, t)} p(x, t) dx \\
& < \int_{x_0}^{\infty} (m(x) + b(x)) \left[-1 + 3 \frac{p^*(x)}{p(x, t)} + \frac{M(t)}{M^*} p^*(x) + \frac{M^*}{M(t)} \right. \\
& \quad \left. + \frac{1}{2} \left(\frac{M^*}{M(t)} \right)^2 + \frac{1}{2} \left(\frac{p^*(y)}{p(y, t)} \right)^2 \right] p(x, t) dx, \\
& < \int_{x_0}^{\infty} (m(x) + b(x)) \left[-1 + 4p^*(x) + \frac{M^*}{M(t)} + \frac{1}{2} \left(\frac{M^*}{M(t)} \right)^2 + \frac{1}{2} \left(\frac{p^*(y)}{p(y, t)} \right)^2 \right] \\
& \quad \times p(x, t) dx, \\
& < \int_{x_0}^{\infty} (m(x) + b(x)) \left[-1 + 4p^*(x) + \frac{p^*(x)}{2} + \frac{M^*}{M(t)} + \frac{1}{2} \left(\frac{M^*}{M(t)} \right)^2 \right] p(x, t) dx,
\end{aligned}$$

since $1 < p(x, t) < M^*/M(t)$. Writing the integrand in the last inequality as

$$\begin{aligned}
& -1 + 4p^*(x) + \frac{p^*(x)}{2} + \frac{M^*}{M(t)} + \frac{1}{2} \left(\frac{M^*}{M(t)} \right)^2 \\
& = \frac{1}{2} \left[(p^*(x) + 4)^2 + \left(\frac{M^*}{M(t)} + 1 \right)^2 - 19 \right],
\end{aligned} \tag{7.80}$$

it is not difficult to see that (7.80) is negative provided that

$$(p^*(x) + 4)^2 + \left(\frac{M^*}{M(t)} + 1 \right)^2 < (\sqrt{19})^2, \tag{7.81}$$

for any $t \geq 0$ and $x > x_0$, that is $p^*(x) + 4 < \sqrt{19}$ and $M^*/M(t) + 1 < \sqrt{19}$ and then

$$p^*(x) < \frac{9}{25} < 1, \text{ and } \frac{M^*}{M(t)} < \frac{17}{5} < 4. \tag{7.82}$$

Since $1 < p(x, t) < M^*/M(t)$, thus $0 < p^*(x)/p(x, t) < 1$ and $1 < M^*/M(t) < 4$. Hence, $dL^*/dt < 0$ is satisfied for $(M(t), p(x, t), D(t))$ such that

$$1 < p(x, t) < \frac{M^*}{M(t)} < \frac{17}{5} < 4, \quad 0 < \frac{p^*(x)}{p(x, t)} < \frac{9}{25} < 1, \quad D(t) \in \mathbf{R}_+ \tag{7.83}$$

and $dL^*/dt = 0$ for $(M(t), p(x, t), D(t)) = (M^*, p^*(x), D^*)$. In summary, $dL^*/dt < 0$ if $(M(t), p(x, t), D(t))$ satisfies either of the following cases:

- (1) $p(x, t) < M^*/M(t) < 1$, $1 < \sqrt{m(x)}/\sqrt{rM^*} < p^*(x)/p(x, t) < \sqrt{m(x) + b(x)}$,
 $D(t) \in \mathbf{R}_+$.
- (2) $1 < p(x, t) < M^*/M(t) < \frac{17}{5} < 4$, $0 < p^*(x)/p(x, t) < \frac{9}{25} < 1$, $D(t) \in \mathbf{R}_+$

and $dL^*/dt = 0$ for $(M(t), p(x, t), D(t)) = (M^*, p^*(x), D^*)$. Furthermore, it can be verified that $\{(M(t), p(x, t), D(t)) : L^*(t) = 0\} = \{E^*\}$. Therefore, it follows from Lasalle's Invariance Theorem [Auger et al. (2008), p. 200] that $\{E^*\}$ is globally asymptotically stable if $P^* < k_0/k_1$. $\square$

7.7 Numerical Simulation

Here numerical illustrations will be provided in order to explain the results stated in Theorem 7.6.1 and also investigate the impact of the drug therapy. Substantial similarities regarding the outcome of the steady states and the local stability analysis from the system of equations (7.1)–(7.3) and the system of equations (7.5)–(7.6) can be identified, namely the existence of the same number of steady states of the same type (disease-free steady state, disease steady state and coexistence steady state) with the same stability for the disease-free steady state and disease steady state; more importantly the stability condition, $P^* < k_0/k_1$, of the disease steady state is the same for these two systems. It can be observed that, the total number of IAPP polymers from the system of equations (7.1)–(7.3) converges to the total number of IAPP polymers from the system of equation (7.5)–(7.6) if we choose the parameters such that $-m(x) + m + b(x) + (1 - 2(x - x_0))b < \epsilon$ ($\epsilon > 0$). As a consequence the system of equations (7.5)–(7.6) can be used without any alteration, to perform the numerical simulations related to the total amount of IAPP polymers. To achieve this, we consider the following parameter values: $A = 4400$ per day, $r = 0.3$ fibrils per square unit per day, $b = 0.0001$ fibrils per square unit per day, as given in Murali and Raman (2013). In addition to these, we also make use of the following initial values: $M_0 = 1000$, $P_0 = 50$, $I_0 = 500$ and $D_0 \in \{0, 1000, 2000, 4000, 5000, 6000, 7000, 8000\}$ (units). Using the above parameter values and initial values, the system of equation (7.5)–(7.6) is simulated numerically to obtain the total population of IAPP polymers as shown in the Figures 7.1–7.8.

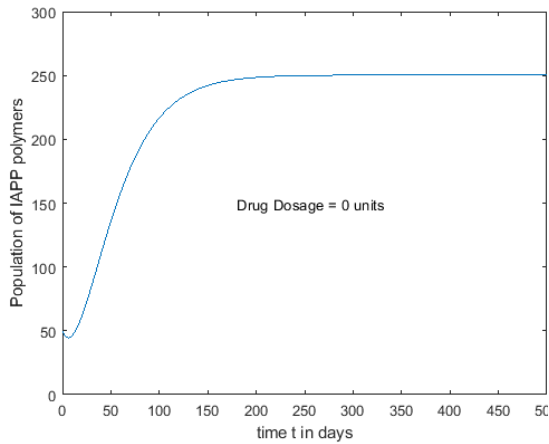


Figure 7.1: Drug dosage = 0 Unit

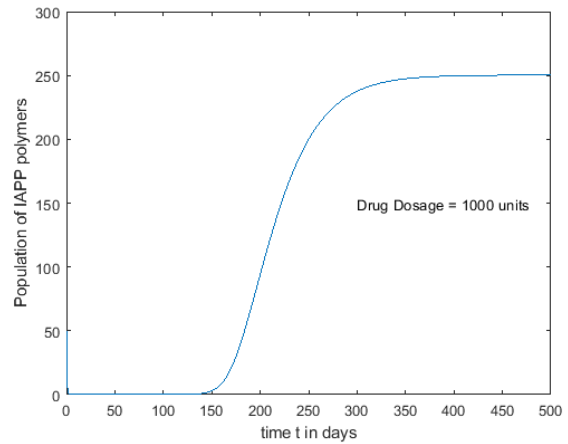


Figure 7.2: Drug dosage = 1000 Units

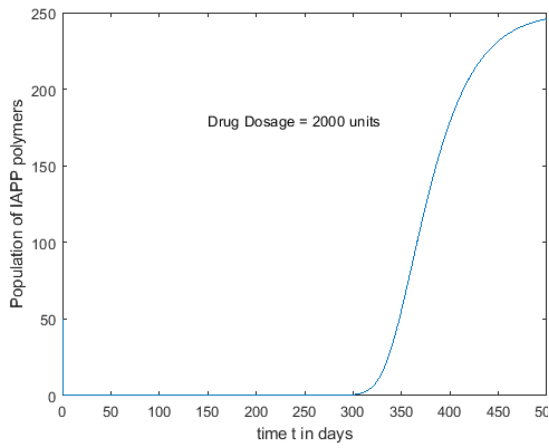


Figure 7.3: Drug dosage = 2000 Units

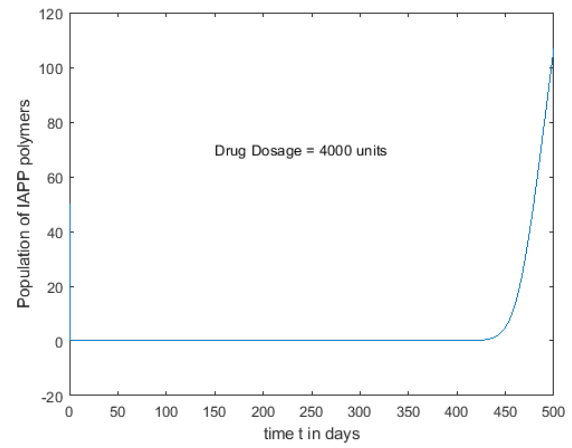


Figure 7.4: Drug dosage = 4000 Units

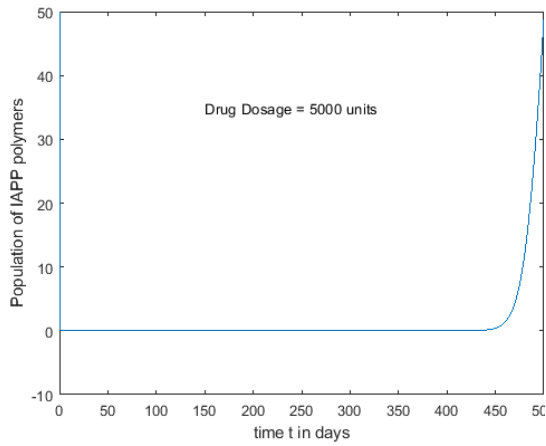


Figure 7.5: Drug dosage = 5000 Units

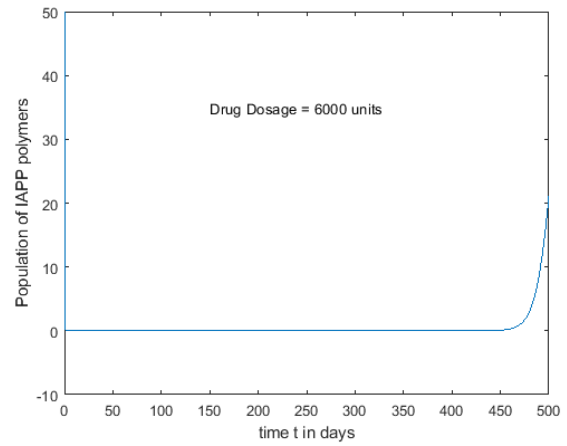


Figure 7.6: Drug dosage = 6000 Units

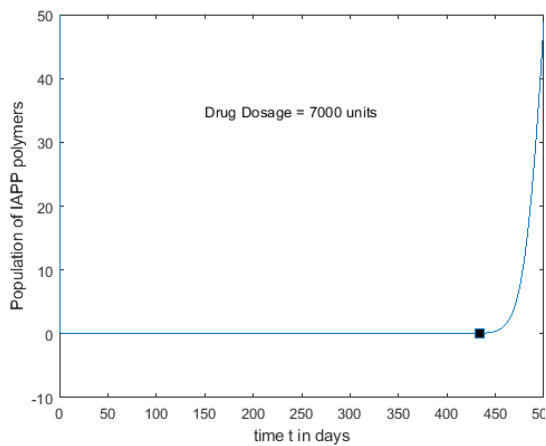


Figure 7.7: Drug dosage = 7000 Units

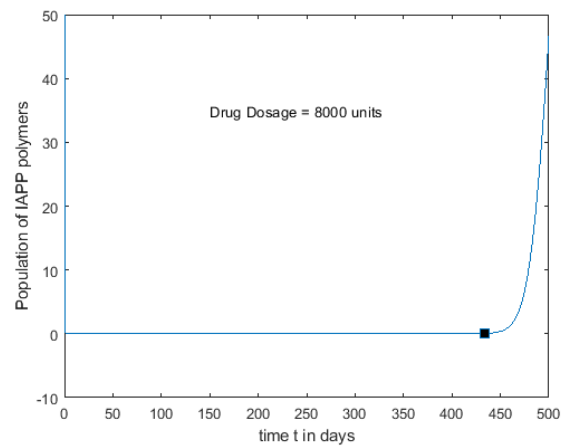


Figure 7.8: Drug dosage = 8000 Units

Figures 7.1–7.8 show that the total population of IAPP polymers will never exceed k_0/k_1 , that is 250. Moreover, the plateauing effect occurring in all these figures shows that

the disease steady state is stable as long as the total population of IAPP polymers does not exceed k_0/k_1 . This confirms the theoretical results from Theorem 7.5.2 and 7.6.1. From Figure 7.1, the total population IAPP decays slightly and briefly before rising and then stabilizes while in Figures 7.2–7.8 drug effect lowers severely the population of IAPP polymers to disappearance and then preserves the state free of IAPP polymers to a long period of time (in days) proportional to the drug dosage. Thus, the drug effect is two folds, namely instigating de-polymerisation of IAPP polymers and inhibiting polymerisation of IAPP monomers. The latter prevents the IAPP fibril formation, which accounts for the occurrence and development of T2DM disease.

7.8 Conclusion

Let $\mathcal{C}_0$ be a parameter defined by $\mathcal{C}_0 = (k_1/k_0)P^*$. Since, from the results of Theorems 7.5.2 and 7.6.2, the disease state equilibrium E^* is stable if $\mathcal{C}_0 < 1$ and unstable if $\mathcal{C}_0 > 1$ and the (unstable) coexistence state equilibrium $\check{E}$ exists if $\mathcal{C}_0 = 1$, thus $\mathcal{C}_0$ can be regarded as an essential threshold parameter in the steady state and stability analysis for the dynamics of amyloid fibril formation defined by the system of equations (7.1). Often, such a parameter can play a crucial role in implementing therapeutic interventions strategies. Clearly, $\mathcal{C}_0 \rightarrow 0$ as $P^* \rightarrow 0$, that is the development of T2DM disease is stable if the total population of polymers, P^* , is insignificant, and $\mathcal{C}_0 \rightarrow \infty$ as $P^* \rightarrow \infty$, that is the development of T2DM is unstable if the total population of polymers, P^* , is extremely high. Moreover, $\mathcal{C}_0 \rightarrow \infty$ as $k_0 \rightarrow 0$, that is the development of T2DM disease cannot stabilize if k_0 is negligible, and $\mathcal{C}_0 \rightarrow 0$ as $k_0 \rightarrow \infty$, that is development of T2DM disease stabilizes if k_0 is extremely large. However, the following

$$\frac{\partial \mathcal{C}_0}{\partial k_0} = -\frac{k_1}{k_0^2}P^* = -\frac{\mathcal{C}_0}{k_0} < 0$$

shows that increasing the rate at which the drug is degraded from the system due to metabolic processes, k_0 , results in a beneficial impact on the control of amyloid fibril formation, that is, rising of k_0 breaks the development of T2DM disease. Therefore, k_0 is a substantial parameter in elaborating an efficient therapeutic strategy aiming at controlling and stopping the development of T2DM disease.

Chapter 8

Conclusion

This thesis contributes to the literature on robust numerical techniques to solve various forms of singularly perturbed problems. Our objective was to construct parameter uniform numerical methods to solve such problems. Apart from the construction of the method, theories concerning their convergence and error estimate were also presented. To validate these theories, extensive numerical experiments were conducted. For us to be able to use less mesh points to achieve a better accuracy, Richardson extrapolation was applied on the methods with first order accuracies to enhance them to second order. The main results in this thesis are outlined below.

In Chapter 2, a uniformly convergent full discretisation (discretising all independent variable at once) scheme was proposed to solve a one-dimensional singularly perturbed Robin type convection-diffusion problem. The chapter began by first studying the properties of the exact solution and its derivatives, followed by the construction of the scheme. The scheme employed the classical finite difference scheme on appropriate positive denominator functions and the classical backward Euler to solve both the mixed boundary conditions and the one-dimensional convection diffusion problem. Some properties of the fully discrete scheme which imply the stability of the scheme were presented and used to analyse the scheme for convergence. A first order accuracy independent of the perturbation parameter in both the space and the time variables was established. Richardson extrapolation was then applied on the scheme to enhance the accuracy to second order accuracy.

Further the concept which was used for the spatial discretisation in Chapter 2 was combined with the Crank Nicolson finite difference scheme to solve a delayed parabolic problem via the framework of method of lines in Chapter 3.

To verify whether the scheme can be applicable to integro differential equations, we studied a singularly perturbed volterra integro differential equation in Chapter 4. Similar to the early chapters, an estimate of the solution of the continuous problem was presented, followed by the design of the fitted operator finite difference scheme along with the Simpson's rule to solve the differential and the integral parts respectively. Analysis of the scheme

resulted in a first order accuracy and this was improved to second order by Richardson extrapolation.

In Chapter 5, a fitted operator based on a midpoint method was constructed along with the Crank Nicolson finite difference scheme to solve a parabolic convection-diffusion with degenerate coefficient via the Rothe's method. The fitted operator scheme designed in Chapter 5 is employed with the backward Euler to solve a linearised Burgers equation in Chapter 6.

Finally, in Chapter 7 which deals with the contributed paper, a mathematical model was used to investigate the effects of drug therapy on the dynamics of amyloid fibril formation (or amyloid plaques). The model is a hybrid system of differential equations consisting of ODEs and a (hyperbolic) PDE. The (global) stability analysis of the model is established through the construction of suitable Lyapunov functionals and application of Lasalle's invariance principle. Moreover, stability of the disease steady state is fully determined by a threshold condition, which is based on the total population of IAPP polymers.

In this thesis, our objective was to construct, analyse and implement parameter uniform numerical methods for some singularly perturbed problems in the context of finite difference schemes. From literature, it is evident that fitted mesh finite difference scheme has been investigated tremendously unlike the fitted operator finite difference schemes. Thus this thesis attempted to develop fitted operator finite difference schemes in the context of Non-standard finite differences schemes to solve various forms of singularly perturbed problems.

One problem we encountered was the fact that a larger number of mesh point was needed before the scheme could resolve the layers for the degenerate parabolic problem we considered in Chapter 5, during the simulation. Thus, the scheme was computationally expensive for that problem.

Therefore, for future work, the proposed scheme in this thesis will be combined with the fitted mesh finite difference scheme (since it enjoys the layer resolving property for all types of SPPs) to generate a hybrid scheme to solve different kinds of singularly perturbed problems, specifically higher order problems and problems with non-smooth data. Also, a new direction of problems to consider is various forms (partial differential equations, ordinary differential equations, integro-differential equations) of higher order singularly perturbed three parameter problems.

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