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RESEARCH PAPER

On the Conservation Laws and Traveling Wave Solutions of a Nonlinear Evolution Equation that Accounts for Shear Strain Waves in the Growth Plate of a Long Bone

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Abstract
A nonlinear partial differential equation that accounts for shear strain waves in a long bone's development plate is investigated. Lie group classification is performed on the underlying equation and it is found that the principal Lie algebra consists of a single time translation symmetry. The underlying analysis prompts that the principal algebra is extended by a single space translation and consequentially it is found the Lie algebra is two dimensional. A linear combination of the translation symmetries results in highly fourth-order nonlinear ordinary differential. Courtesy of an ansatz method this highly fourth-order nonlinear ordinary differential results in obtaining series of hyperbolic and trigonometric traveling wave solutions. Finally, we compute conservation laws courtesy of the invariance and multiplier technique. The findings can well mimic complex waves and their dealing dynamics in a long bone's development.

Keywords Nonlinear partial differential equation · Conservation laws · Lie point symmetries · Extended tanh method · Extended tan method

1 Introduction

Nonlinear partial differential equations (Gao 2015; Jiang et al. 2010; Gao 2015a, b; Zhen et al. 2015; Sun et al. 2015; Xie et al. 2015; Ablowitz and Clarkson 1991; Wazwaz 2005, 2010a, b, 2012, 2013, 2014; Duan et al. 2013; Sebogodi et al. 2023; Podile et al. 2022; Boussinesq 1877; Clarkson and Kruskal 1989; Wazwaz 2022; Adem 2016; Ma et al. 2010; Ma 2021, 2022, 2023; Ye et al. 2021; Muatjetjeja 2017a, b; Muatjetjeja and Adem 2017; Moretlo et al. 2021a, b; Adem and Lü 2016; Adem 2017; Osman et al. 2020; Adem 2016) play a crucial role in various fields of engineering, physical sciences, and even everyday applications. They provide a powerful framework for modeling and simulating complex phenomena that cannot be adequately described by linear equations. One major reason for the importance of nonlinear PDEs is their ability to capture the behavior of real-world systems that exhibit nonlinearity. Linear equations, while useful for many simple systems, fail to accurately represent the dynamics of many complex phenomena. Nonlinear PDEs, on the other hand, allow for more accurate and realistic modeling of these systems by accounting for nonlinear relationships and interactions between variables. For instance, in biomedical engineering, nonlinear PDEs are essential for modeling biological processes that involve intricate interactions and feedback loops. Another critical implication of nonlinear PDEs lies in their ability to simulate and predict the behavior of complex physical systems.

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These equations provide a mathematical framework for understanding the dynamics of fluid flow, heat transfer, wave propagation, and other physical phenomena. By solving nonlinear PDEs, researchers and engineers can gain insights into the behavior of these systems, make predictions, and design more efficient and effective solutions

Motivated by the importance of nonlinear partial differential equations, we investigate a nonlinear partial differential equation

$$\rho u_{tt} - (\mu(x)u)_{xx} - \lambda(u^3)_{xx} - \eta u_{txx} - Qu_{xxxx} = 0, \tag{1}$$

that accounts for shear strain waves $u(x, t)$ in a long bone's development plate (Porubov 2016; Porubov and Maugin 2011).

Ossification, which changes the value of a cartilage's μ_c to a bone's μ_b, is what causes the variances in μ

$$\mu = \kappa(x)\mu_c + (1 - \kappa(x))\mu_b. \tag{2}$$

It should be noted that the parameter κ accounts for biological processes (Fasano et al. 2010; Herrero and Lopez 2005).

$$\kappa = \frac{1}{2}(1 + \tanh(mx)). \tag{3}$$

In a wide range of scientific disciplines, particularly in biology, solid state physics, plasma physics, plasma waves, and fluid mechanics, nonlinear evolution equations (Abdelsalam and Zobaer 2018; Mogorosi and Muatjetjeja 2019; Ma et al. 2021; Chen et al. 2021; LüX et al. 2021; He et al. 2021; Zhou et al. 2021; Lü and Chen 2021) such as (1) are frequently employed as models to reflect physical events. Therefore, it is crucial to find exact solutions to these nonlinear evolution equations. It is laborious to calculate the precise solutions of nonlinear evolution equations, and only in a small number of exceptional circumstances can one plainly write down their solutions. However, over the past few decades, significant progress has been made, and numerous strong and efficient approaches for getting accurate solutions to nonlinear evolution equations have been proposed in the literature (Wazwaz 2020; Hirota 1971; Wazwaz 2005, 2018, 2012).

This paper examines (1), where $u = u(x, t)$ is a real function of x and t and $\mu(x)$ is an arbitrary function of x. A multiplier analysis procedure leads to the conservation laws of (1) and the Lie symmetry analysis of (1) will performed, and the arbitrary function appearing in underlying equation is systematically determined. This then naturally leads to a symmetry reduction that consists of a highly nonlinear ordinary differential equation. This nonlinear ordinary differential equation leads a series of traveling wave solutions with the aid of various ansatz methods. Although various attempts have been made to find numerous methods to solve (integrable or non-integrable) nonlinear evolution equations, there is no unified method. To the best of our knowledge, this is for the first time that the methods in this paper are being shown for this Eq. (1).

2 Conservation Laws of the Nonlinear Evolution Eq. (1)

Conservation laws (Sebogodi et al. 2023; Podile et al. 2022) are an important concept in the study of partial differential equations (PDEs). They describe the mathematical relationships between the properties of physical systems and how they change over time. The basic idea behind conservation laws is that the total amount of a particular property, such as mass, energy, or momentum, remains constant within a closed system. In PDEs, conservation laws are represented as equations that describe how certain physical quantities change in response to changes in other quantities. For example, the conservation of mass equation for a fluid is described by the continuity equation, which states that the rate of change of the fluid's density is proportional to the rate of change of its volume. Similarly, the conservation law of energy equation for a thermodynamic system is described by the first law of thermodynamics, which states that the change in the internal energy of a system is equal to the heat added to the system minus the work done on the system. Another important concept in conservation laws is the concept of the flux of a physical property. The flux is a measure of the flow of a property from one region of a system to another. For example, in fluid dynamics, the flux of mass is represented by the velocity of the fluid; while, the flux of energy is represented by the heat transfer rate. There are several methods used to derive conservation laws from PDEs. One common method is the use of symmetry considerations, where the conservation law is derived from the symmetries of the physical system. Another method is the use of Noether's theorem, which states that the conservation laws can be derived from the invariances of the system under certain transformations (Muatjetjeja 2017a, b).

In summary, conservation laws play a crucial role in the study of partial differential equations. They provide mathematical descriptions of the relationships between physical quantities and how they change over time, and they are derived from the symmetries and invariances of physical systems. Understanding conservation laws is essential for a wide range of scientific and engineering applications, including fluid dynamics, thermodynamics, and solid mechanics, among others.

This section aims to investigate conservation laws of the nonlinear evolution Eq. (1). Consider a differential equation $E = 0$, and Λ being the characteristic function. ΛE is

divergent if and only if $E_u(QE) = 0$, where E_u is the Euler–Lagrange operator. Without loss of generality, we can now state the following theorems.

Theorem *The nonlinear evolution Eq. (1) formally admits a unique characteristic function, namely*

$$\Lambda(t,x,u) = (k_1x + k_2)t + k_3x + k_4, \tag{4}$$

where $k_i,\ \ i = 1\cdots 4$ are arbitrary constants of integration.

Proof A straightforward but lengthy computation can be carried out from $\varepsilon_u(\Lambda E) = 0$. The expansion of this equation leads to an overdetermined system of linear differential equations in the unknown characteristic function Λ. Solving these equations, one obtains the unique characteristic function (4).

The existence of this characteristic function prompts the following theorem. □

Theorem *The nonlinear evolution Eq. (1) admits conservation laws corresponding to the unique characteristics* $\Lambda(t,x,u) = (k_1x + k_2)t + k_3x + k_4$, *namely*

$$\begin{aligned}
T_1^t &= -\rho xu + \frac{1}{3}\eta tu_x - \frac{1}{3}\eta txu_{xx} + \rho txu_t,\\
T_1^x &= -\frac{2}{3}\eta u + \lambda tu^3 + \mu tu\\
&\quad + \frac{1}{3}\eta xu_x + Qtu_{xx} + \frac{1}{3}\eta tu_t - \mu txu_x - Qtxu_{xxx}\\
&\quad - \frac{2}{3}\eta txu_{tx} - 3\lambda txu^2u_x - tx\mu_x u;\\
T_2^t &= -\rho u - \frac{1}{3}\eta tu_{xx} + \rho tu_t,\\
T_2^x &= -3\lambda tu^2u_x - \mu tu_x - Qtu_{xxx} - \frac{2}{3}\eta tu_{tx}\\
&\quad + \frac{1}{3}\eta u_x - t\mu_x u;\\
T_3^t &= \frac{1}{3}\eta u_x - \frac{1}{3}\eta xu_{xx} + \rho xu_t,\\
T_3^x &= -Qxu_{xxx} - \frac{2}{3}\eta xu_{tx} - 3\lambda xu^2u_x + \mu u\\
&\quad + Qu_{xx} + \frac{1}{3}\eta u_t + \lambda u^3 - \mu_{xx}u - \mu_x u_x;\\
T_4^t &= -\frac{1}{3}\eta u_{xx} + \rho u_t,\\
T_4^x &= -3\lambda u^2u_x - \mu uu_x - Qu_{xxx} - \frac{2}{3}\eta u_{tx} - \mu_x u,
\end{aligned}$$

respectively.

Proof The proof of Theorem 2 is straightforward but long. It consists of applying the divergence equation $\partial_t T^t + \partial_x T^x = 0$, which vanishes for all solutions of (1). □

3 Point Symmetries of Eq. (1)

Lie symmetry analysis (Bluman and Kumei 1989; Olver 1993; Ibragimov 1994) is a powerful mathematical technique used to study and analyze differential equations. By utilizing Lie symmetry analysis, researchers can uncover hidden symmetries and transformations that simplify the equations and enable deeper insights into their behavior. Lie symmetry analysis, developed by Sophus Lie in the 19th century, offers a systematic approach to study the symmetries of differential equations. These symmetries provide valuable information about the structure and properties of the solutions. They can help to identify and derive invariant solutions, which are solutions that remain unchanged under certain transformations. For instance, Lie symmetry analysis allows for the derivation of exact or invariant solutions of nonlinear differential equations. Lie symmetry analysis has been widely applied in various scientific fields, including mathematics and physics, over the past four decades. It has proven to be a valuable tool for solving nonlinear problems characterized by differential equations in areas such as fluid dynamics, quantum mechanics, and general relativity. The theory and application of Lie symmetry methods have revolutionized the study of nonlinear problems characterized by differential equations in various scientific fields, including mathematics, physics, and engineering. In the field of mathematics, Lie symmetry analysis has been used to study a wide range of mathematical equations, including ordinary differential equations and partial differential equations. In the field of physics, Lie symmetry analysis has been employed to analyze and solve equations that describe the behavior of physical systems. For example, in the study of fluid dynamics, Lie symmetry analysis has been used to derive exact solutions for the Navier–Stokes equations, which describe the motion of viscous fluids. Lie symmetry analysis is a systematic approach that allows for the identification and derivation of invariant solutions of differential equations, which remain unchanged under certain transformations. Therefore, it would be worthwhile to apply Lie symmetry analysis to investigate the solutions of Einstein field equations. Lie symmetry analysis is a powerful tool for investigating and solving nonlinear problems characterized by differential equations in various scientific disciplines. The Lie symmetry analysis has proven to be a powerful and systematic approach for studying the symmetries of differential equations. It allows for the identification and derivation of invariant solutions, which are solutions that remain unchanged under certain transformations. These transformations can include changes in variables, coordinate systems, or parameter values. By applying Lie symmetry analysis to the Einstein field equations, researchers can

potentially uncover invariant solutions that remain unchanged under certain transformations.

This section aims to compute the Lie point symmetries of a generalized nonintegrable nonlinear Eq. (1) and later use them to transform a nonintegrable nonlinear Eq. (1) into ordinary differential equations (ODEs). Thereafter we construct exact solutions of Eq. (1).

The equation under study arises to account for shear strain waves $u(x, t)$ in a growth plate of a long bone (1) is invariant under the group with the generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{5}$$

if and only if

$$X^{[4]}\Big(\rho u_{tt} - (\mu(x)u)_{xx} - \lambda(u^3)_{xx} - \eta u_{txx} - Qu_{xxxx} = 0\Big)\Big|_{(1)} = 0 \tag{6}$$

with $X^{[4]}$ be the fourth extension of the generator (5). Applying $X^{[4]}$ onto Eq. (1) and expand, thereafter separate the monomials, we get an overdetermined system partial differential equations:

$$\tau_x = 0, \tag{7}$$

$$\xi_u = 0, \tag{8}$$

$$\tau_u = 0, \tag{9}$$

$$\eta_{uu} = 0, \tag{10}$$

$$2\xi_x - \tau_t = 0, \tag{11}$$

$$-\eta\xi_{xx} + 2\eta\eta_{xu} + 2\rho\xi_t = 0, \tag{12}$$

$$\rho\tau_{tt} - 2\rho\eta_{tu} + \eta\eta_{xxu} = 0, \tag{13}$$

$$-6\xi_{xx}Q + 4\eta_{xu}Q - \eta\xi_t\eta_u u + \eta + 2\xi_x u = 0. \tag{14}$$

$$\xi\mu_x - 2\eta\xi_{tx} + 2\xi_x\mu - 4\xi_{xxx}Q + 6\eta_{xxu}Q + 6\eta\lambda u + 6\xi_x\lambda u^2 + \eta\eta_{tu} = 0 \tag{15}$$

$$\begin{aligned}&\eta\mu_{xx} + \eta_{xx}\mu + \eta_{xxxx}Q + \xi\mu_{xxx}u - \rho\eta_{tt}\\ &+ 2\eta_x\mu_x + 4\xi_x\mu_{xx}u - \eta_u\mu_{xx}u\\ &+ 3\eta_{xx}\lambda u^2 + \eta\eta_{txx}2\mu\eta_{xu} - \mu\xi_{xx} - \eta\xi_{txx}\\ &+ 2\eta\eta_{txu} + 2\xi\mu_{xx} + 6\xi_x\mu_x + 12\eta_x\lambda u\\ &- \xi_{xxxx}Q + 4\eta_{xxxu}Q + \rho\xi_{tt} - 3\xi_{xx}\lambda u^2 + 6\eta_{xu}\lambda u^2 = 0\end{aligned} \tag{16}$$

It is feasible to observe that the following two scenarios arise when the above overdetermined system of equations is solved.

Case 1. $\mu(x)$ is an arbitrary function of x. In this cases, the equation understudy admits only a translation symmetry given by

$$\Gamma_1 = \frac{\partial}{\partial t} \tag{17}$$

and this point symmetry is referred to as the principal Lie algebra.

Case 2. $\mu(x) =$ constant. In this cases, the principal algebra extends by one. Hence the Lie algebra is two dimensional. It is spanned by two translation symmetries, viz.,

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial t}, && \text{time translation}\\ \Gamma_2 &= \frac{\partial}{\partial x} && \text{space translation.}\end{aligned} \tag{18}$$

We now invoke the symmetry combination of $X = \delta X_1 + \delta\omega X_2$, to get the invariant:

$$u = F(z) \tag{19}$$

where $z = \delta x - \delta\omega t$.

Substituting Eq. (19) into Eq. (1) and simplifying yields

$$\begin{aligned}&-Q\delta^4 F''''(z) + \delta^3\eta\omega F'''(z) + \delta^2\rho\omega^2 F''(z) - \delta^2\sigma F''(z)\\ &- \lambda\Big(3\delta^2 F(z)^2 F''(z) + 6\delta^2 F(z)F'(z)^2\Big) = 0\end{aligned} \tag{20}$$

3.1 Exact Solutions Using the Extended tan Method

We use the extended tan function method which was presented by Wazwaz (2006), Adem (2017). The elementary idea in this technique is to assume that the solution of (20) can be written in the form

$$F(z) = \sum_{i=-M}^{M} A_i H(z)^i, \tag{21}$$

where $H(z)$ satisfies the Riccati equation

$$H'(z) = 1 + H^2(z), \tag{22}$$

whose solution is

$$H(z) = \tan(z).$$

The positive integer M will be computed by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (20). A_i is a parameter to be computed.

Here, the balancing procedure gives $M = 1$ and so the solutions of (20) are of the form

$$F(z) = A_{-1}H^{-1} + A_0 + A_1 H, \tag{23}$$

With the aid of Mathematica, we obtain the following three cases (Figs. 1, 2).

Case 1

$$Q = -\frac{\lambda A_1^2}{2\delta^2},$$
$$\eta = \frac{3\lambda A_0 A_1}{\delta\omega},$$
$$\sigma = -3\lambda A_0^2 + \lambda A_1^2 + \rho\omega^2,$$
$$A_{-1} = 0.$$

Case 2

$$Q = -\frac{\lambda A_{-1}^2}{2\delta^2},$$
$$\eta = -\frac{3\lambda A_{-1} A_0}{\delta\omega},$$
$$\sigma = \lambda A_{-1}^2 - 3\lambda A_0^2 + \rho\omega^2,$$
$$A_1 = 0$$

Case 3

$$Q = -\frac{\lambda A_{-1}^2}{2\delta^2},$$
$$\eta = -\frac{3\lambda A_{-1} A_0}{\delta\omega},$$
$$\sigma = 4\lambda A_{-1}^2 - 3\lambda A_0^2 + \rho\omega^2,$$
$$A_1 = -A_{-1}$$

Thus, a solution of (1) is

$$u(x,t) = A_{-1}\cot(z) + A_0 + A_1\tan(z), \tag{24}$$

where $z = \delta(x - \omega t)$.

The graphical simulations of solutions (24) and (28) are given in Figs. 1 and 2 respectively.

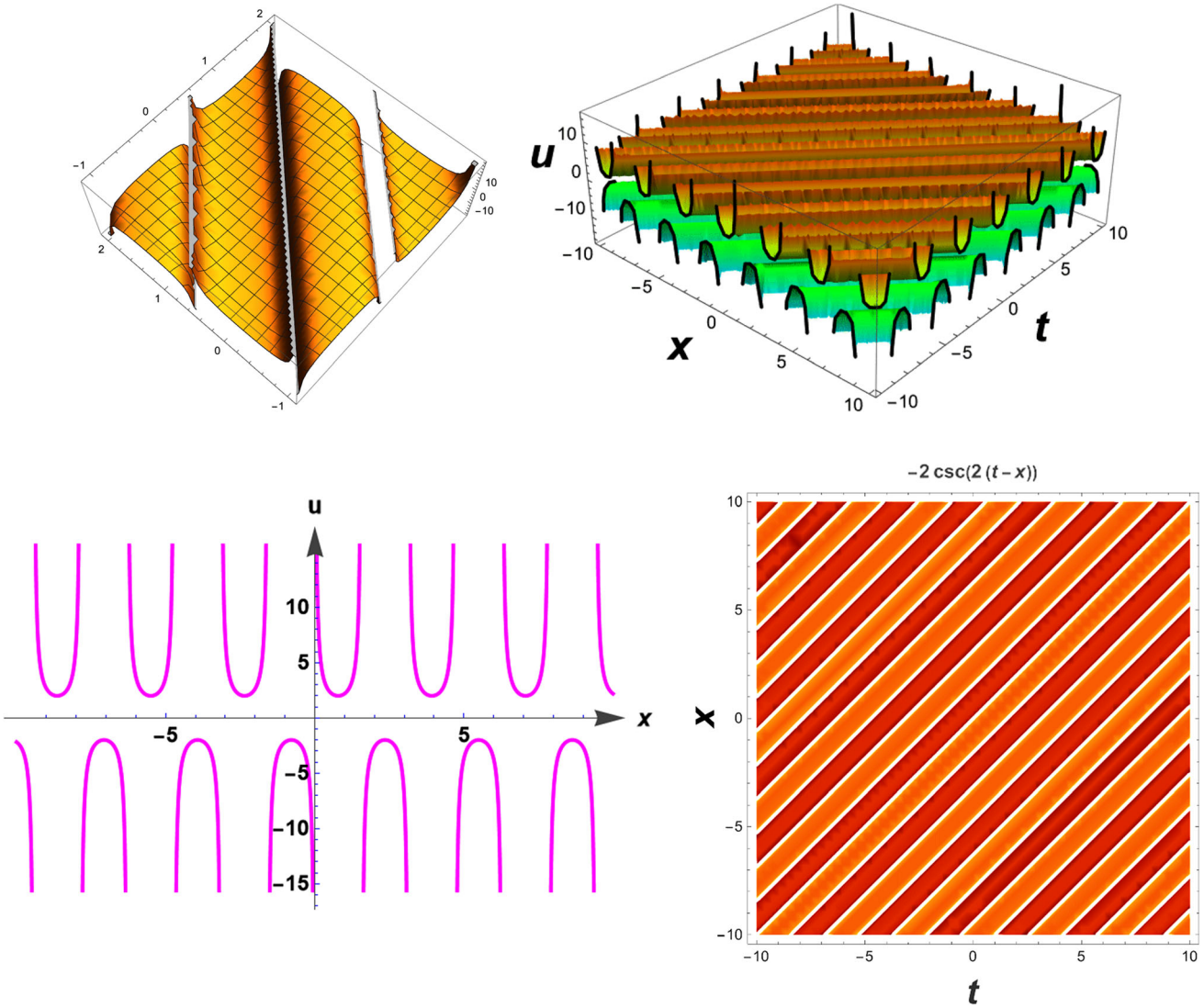


Fig. 1 Profiles of solution (24) with $\delta = 1$, $\omega = 1$, $A_{-1} = 1$, $A_0 = 0$, $A_1 = 1$

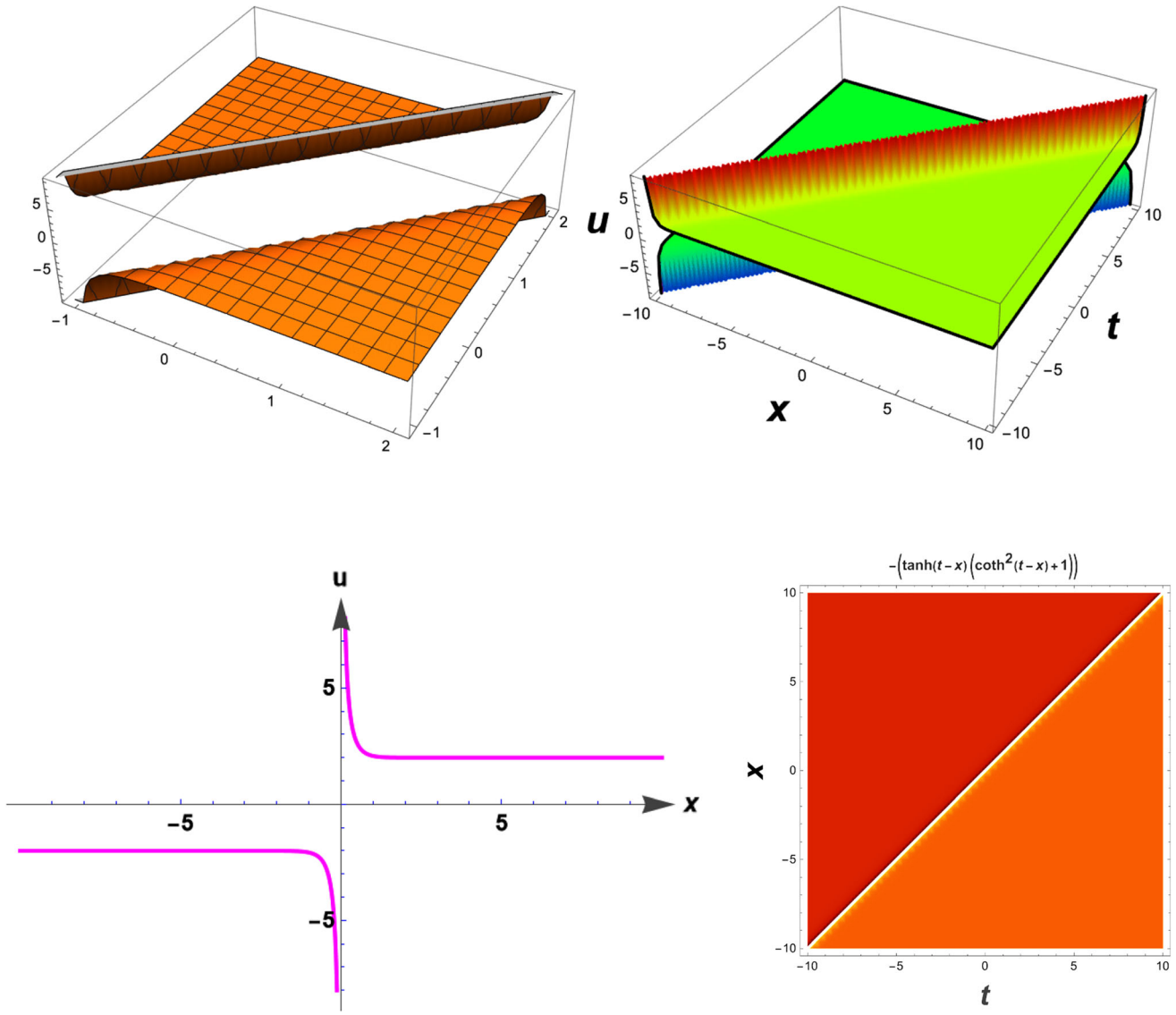


Fig. 2 Profiles of solutions (28) with $\delta = 1$, $\omega = 1$, $A_{-1} = 1$, $A_0 = 0$, $A_1 = 1$

3.2 Exact Solutions Using the Extended tanh Method

We use the extended tanh function method which was presented by Wazwaz (2006), Adem (2017). The elementary idea in this technique is to assume that the solution of (20) can be written in the form

$$F(z) = \sum_{i=-M}^{M} A_i H(z)^i, \tag{25}$$

where $H(z)$ satisfies the Riccati equation

$$H'(z) = 1 - H^2(z), \tag{26}$$

whose solution is

$H(z) = \tanh(z)$.

Here, the balancing procedure also gives $M = 1$ and so the solutions of (20) are of the form

$$F(z) = A_{-1}H^{-1} + A_0 + A_1 H, \tag{27}$$

Again, we obtain the following three cases with the aid of Mathematica. **Case 1**

$$Q = -\frac{\lambda A_1^2}{2\delta^2},$$
$$\eta = -\frac{3\lambda A_0 A_1}{\delta\omega},$$
$$\sigma = -3\lambda A_0^2 - \lambda A_1^2 + \rho\omega^2,$$
$$A_{-1} = 0.$$

Case 2

$$Q = -\frac{\lambda A_{-1}^2}{2\delta^2},$$
$$\eta = -\frac{3\lambda A_{-1} A_0}{\delta\omega},$$
$$\sigma = -\lambda A_{-1}^2 - 3\lambda A_0^2 + \rho\omega^2,$$
$$A_1 = 0.$$

Case 3

$$Q = -\frac{\lambda A_{-1}^2}{2\delta^2},$$
$$\eta = -\frac{3\lambda A_{-1} A_0}{\delta\omega},$$
$$\sigma = -4\lambda A_{-1}^2 - 3\lambda A_0^2 + \rho\omega^2,$$
$$A_1 = A_{-1}$$

Accordingly, a solution of (1) is

$$u(x,t) = A_{-1}\coth(z) + A_0 + A_1\tanh(z), \tag{28}$$

where $z = \delta(x - \omega t)$.

4 Concluding Remarks

A nonlinear partial differential equation that accounts for shear strain waves in a long bone's development plate was investigated. Lie group classification was performed on the underlying equation and it was found that the principal Lie algebra consisted of a single time translation symmetry. The underlying analysis prompted that the principal algebra was extended by a single space translation and consequentially it was found the Lie algebra was two dimensional. A linear combination of the translation symmetries resulted in highly fourth-order nonlinear ordinary differential. Courtesy of ansatz methods this highly fourth-order nonlinear ordinary differential resulted in obtaining series of hyperbolic and trigonometric traveling wave solutions. Finally, we computed conservation laws courtesy of the invariance and multiplier technique. These findings can well mimic complex waves and their dealing dynamics in a long bone's development. It is projected that the conservation laws and exact solutions calculated in this paper can be used as a point of reference for numerical simulations of the underlying equation. In future we intend to extract further solutions of the underlying equation by utilizing the conservation laws obtained in this paper and these results will be reported elsewhere.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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