Solutions and conservation laws of MEW-Burgers and KdV-BBM equations

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Solutions and conservation laws of MEW-Burgers and KdV-BBM equations

by

INNOCENT SIMBANEFAYI (24536849)

Dissertation submitted for the degree of Master of Science in Applied Mathematics in the Department of Mathematical Sciences in the Faculty of Agriculture, Science and Technology at North-West University, Mafikeng Campus

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Declaration

I INNOCENT SIMBANEFAYI, student number 24536849, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Mr. Innocent Simbanefayi

Date: ......30 April 2018......

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: 30 April 2018..

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

I. Simbanefayi, C.M. Khalique, Travelling wave solutions and conservation laws for the Korteweg-de Vries- Bejamin-Bona-Mahony equation, accepted and to appear in Results in Physics

Chapter 3

I. Simbanefayi, C.M. Khalique, Exact solutions and conservation laws of a modified equal width Burgers equation, submitted for publication to Open Physics

Dedication

To my family.

Acknowledgements

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Abstract

In this work we study two nonlinear partial differential equations of fluid mechanics. The Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation, which describes the two-way propagation of waves, is studied first. We compute the optimal system of one-dimensional subalgebras and use it to perform symmetry reductions and find group-invariant solutions. Also, we derive conservation laws of the KdV-BBM equation using the multiplier approach. Secondly we study the modified equal width Burgers equation, which describes long wave propagation in nonlinear media with dispersion and dissipation. We construct exact solutions using its Lie point symmetries and invoking the $(G'/G)-$expansion method. Moreover we determine the conservation laws by employing the new conservation theorem due to Ibragimov.

Introduction

Over the years nonlinear partial differential equations (NLPDEs) have proven indispensable in the modelling of diverse nonlinear multidimensional systems which are manifest in countless and varied natural phenomena. Studying NLPDEs is fundamental to understanding the complex behaviours of these systems and continues to be explored by many researchers.

Many methods have been developed for obtaining exact solutions of NLPDEs. These include the homogeneous balance method [1], the ansatz method [2], the inverse scattering transform method [3], the Bäcklund transformation [4], the Darboux transformation [5], the Hirota bilinear method [6], the simplest equation method [7], the $(G'/G)-$expansion method [8], the Jacobi elliptic function expansion method [9], the Kudryashov method [10] and the Lie symmetry method [13–18], just to mention a few.

In the late 19th century, Sophus Lie, a brilliant mathematician from Norway, developed a revolutionary symmetry-based method for solving differential equations, known today as Lie group analysis. This has made it possible to obtain exact solutions of differential equations in a systematic way. A robust amount of research based on Lie's work has been published by various researchers [13–21].

Conservation laws are immutable laws of nature which have been encountered by researchers in diverse scientific fields. Some of the most common conservation laws are the conservation of mechanical energy in the absence of dissipative forces, conservation of linear momentum in an isolated system, conservation of energy, conservation of electric charge and many more. In applied mathematics, conservation laws are essential to determining the extent of integrability of differential equations, development of numerical schemes, reduction and solutions of partial differential equations amongst others. See, for example [22–29] and references therein.

The outline of this dissertation is as follows:

In Chapter one, we present important preliminaries regarding Lie point symmetries, the Kudryashov method, $(G'/G)-$expansion method, multiplier approach and conservation theorem due to Ibragimov.

In Chapter two, we apply the extended Jacobi elliptic function expansion method and the Kudryashov method to find the travelling wave solutions of the third-order KdV-BBM equation. Penultimately, we calculate an optimal system of one-dimensional subalgebras and use it to determine an optimal system of group-invariant solutions. Finally, conservation laws are derived using the multiplier approach.

In Chapter three, conservation laws of the MEW-Burgers equation are derived using Ibragimov's new conservation theorem. The $(G'/G)-$expansion method is used to obtain exact solutions of the equation.

In Chapter four, a summary of results is provided and future work proposed.

Bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter we give a synopsis of pertinent concepts that are essential to this dissertation. These include the algorithm to determine the Lie point symmetries of PDEs, new conservation theorem due to Ibragimov, the variational derivative approach and some methods for obtaining exact solutions of PDEs.

1.1 Introduction

In the late 19th century, Marius Sophus Lie, an eminent mathematician from Norway, developed a revolutionary symmetry-based method for solving differential equations. This method today is known as Lie group analysis and gives a systematic way to obtain exact solutions of differential equations. Recently several books on Lie group analysis have been published [13–17]. The definitions and results presented in this chapter are taken from the books mentioned above.

1.2 Continuous one-parameter groups

Suppose $x = (x^1, \cdots, x^n)$ is the independent variable with coordinates x^i and $u = (u^1, \cdots, u^m)$ is the dependent variable with coordinates u^α (n and m finite). We consider the following change of the variables x and u:

$$T_a : \bar{x}^i = f^i(x, u, a), \ \bar{u}^\alpha = \phi^\alpha(x, u, a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A continuous one-parameter (local) Lie group of transformations in the space of variables x and u is a set G of transformations (1.1) which satisfies the following properties:

(i) If $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) There exists $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that from (i) the associativity property is satisfied. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a. b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if the group composition law is additive, i.e. $\phi(a, b) = a + b$.

Theorem 1.1 For any composition law $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)},$$

where

$$w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where the operator of total differentiation is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Therefore the u_s^α are called differential variables.

Considering a pth-order PDE, namely

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

1.3.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u}^\alpha = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, finding the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations :

$$\bar{x}^i \approx x^i + a\,\xi^i(x,u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x,u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Consequently, we have

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

We now introduce the symbol of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

is known as the infinitesimal operator or generator of the group G. We say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5), if the group G is admitted by (1.5).

We now show how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots .$$

Let us now apply (1.10) and (1.6)

$$D_i(\phi^\alpha) \;=\; D_i(f^j)\bar{D}_j(\bar{u}^\alpha)$$

$$= \quad D_i(f^j)\bar{u}_j^\alpha. \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small a, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \qquad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $(x, u, u_{(1)})$ space given by (1.6) and (1.13) form a one-parameter group called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14). Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

1.3.2 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we obtain

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j). \quad \text{(sum on } j\text{)}.
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Similarly, one can obtain the second prolongation

$$\zeta_{ij}^\alpha \quad = \quad D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k\text{)}. \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} = D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i} \quad \text{(sum on } i, \alpha), \\ &\ \ \vdots \\ X^{[p]} &= X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \end{aligned} \tag{1.18}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}. \tag{1.19}$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{1.20}$$

is a Lie point symmetry of the pth-order PDE (1.5), if

$$X^{[p]} E\big|_{E=0} = 0, \tag{1.21}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 An equation (1.21) that determines all the infinitesimal symmetries of (1.5) is called the determining equation.

Definition 1.4 A one-parameter group G of continuous transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{u}, \bar{u}_{(1)}, \cdots, \bar{u}_{(p)}) = 0, \tag{1.22}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \tag{1.23}$$

identically in x, u and a.

Theorem 1.2 A necessary and sufficient condition for a function $F(x, q)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.24}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants. One can take, as basic invariants the left-hand side $n-1$ first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x,) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.25}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The commutator of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X,Z] + b[Y,Z], \ \ [X, aY + bZ] = a[X,Y] + b[X,Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X,Y] = -[Y,X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

1.7 Solution methods of differential equations

In this section we present some methods for finding exact solutions of differential equations.

1.7.1 The $(G'/G)-$expansion method

We provide a synopsis of the salient features of the $(G'/G)-$expansion method. This method was proposed by Wang et al. [8] in 2008. The function G in this method satisfies a second-order linear ordinary differential equation with constant coefficients. This method provides travelling wave solutions that are expressed as hyperbolic, trigonometric and rational functions. The $(G'/G)-$expansion method is simple to use and can be applied on nonlinear differential equations. Consider a NLPDE of the form

$$E(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0. \tag{1.26}$$

Firstly we transform the NLPDE (1.26) to a nonlinear ordinary differential equation (ODE) by making use of the substitution

$$u(t, x) = U(z), \quad z = x - ct. \tag{1.27}$$

Using the above substitutions, (1.26) is transformed into the nonlinear ODE

$$E(U, -cU', U', c^2U'', -cU'', U'', \cdots) = 0. \tag{1.28}$$

Secondly we assume that

$$U(z) = \sum_{i}^{m} A_i \left(\frac{G'(z)}{G(z)}\right)^i \tag{1.29}$$

where $A_0, A_1, \cdots A_m$ are to be determined. The balancing procedure [1] is used to find the value of m. a positive integer. Here $G(z)$ satisfies the second-order linear homogeneous ODE

$$G''(z) + \lambda G'(z) + \mu G(z) = 0 \tag{1.30}$$

with λ and μ arbitrary constants. In (1.29) (G'/G) is given by

Case 1. When $M = \lambda^2 - 4\mu > 0$

$$\frac{G'(z)}{G(z)} = \Delta_1 \frac{A\cosh(\Delta_1 z) + B\sinh(\Delta_1 z)}{A\sinh(\Delta_1 z) + B\cosh(\Delta_1 z)} - \frac{\lambda}{2}, \tag{1.31}$$

where $\Delta_1 = \sqrt{M}/2$.

Case 2. When $M = \lambda^2 - 4\mu < 0$

$$\frac{G'(z)}{G(z)} = \Delta_2 \frac{-A\sin(\Delta_2 z) + B\cos(\Delta_2 z)}{A\cos(\Delta_2 z) + B\sin(\Delta_2 z)} - \frac{\lambda}{2}, \tag{1.32}$$

where $\Delta_2 = \sqrt{-M}/2$

Case 3. When $M = \lambda^2 - 4\mu = 0$

$$\frac{G'(z)}{G(z)} = \frac{B}{Bz + A} - \frac{\lambda}{2} \tag{1.33}$$

In all three cases A, B and C are arbitrary constants. The substitution of (1.29) and (1.30) into (1.28) yields a polynomial in $(G'/G)^i$. Finally, equating the coefficients of $(G'/G)^i$ to zero and solving the resultant system of algebraic equations yields the required values of A_i.

1.7.2 The extended Jacobi elliptic function expansion method

We briefly outline the Jacobi elliptic function expansion method, another algorithm for determining the exact solutions of differential equations. There is extensive literature, some of which dates back several decades, covering different aspects of Jacobi elliptic functions [30–32] such as their derivation, interrelationships and applications. Several researchers [9, 33–35] have recently employed the properties of some of these elliptic functions to determine exact solutions of differential equations.

The procedure for implementing the extended Jacobi elliptic function expansion method is as follows: Firstly we transform the NLPDE (1.26) into the nonlinear ODE (1.28).

Secondly, we assume that our solutions can be expressed in the form

$$U(z) = \sum_{i=-M}^{M} A_i H(z)^i, \tag{1.34}$$

where M is a positive integer obtained by the balancing procedure and

$$H(z) = cn(z|\omega), \tag{1.35}$$

the cosine-amplitude function, is a solution to the first-order ODE [31, 33, 35]

$$H'(z) = -\sqrt{(1 - H^2(z))(1 - \omega + \omega H^2(z))}, \tag{1.36}$$

and the sine amplitude function

$$H(z) = sn(z|w) \tag{1.37}$$

is a solution to the first-order ODE

$$H'(z) = \sqrt{(1 - H^2(z))(1 - \omega H^2(z))}. \tag{1.38}$$

The third step of our procedure entails substituting (1.34) subject to (1.36) or (1.38) into (1.28) to obtain a polynomial in powers of $H(z)$. Separating coefficients with respect to like powers of $H(z)$ yields an algebraic system of equations. These can be solved to obtain the values of A_i, $i = 0, \pm 1. \pm 2, \cdots \pm M$.

1.7.3 The Kudryashov method

This method is used for finding exact solutions of NLPDEs and is described in [10]. See also, for example, the papers [36, 37]).

Consider the NLPDE

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{1.39}$$

The algorithm of Kudryashov method is recalled below:

Step 1. The substitution $u(x, t) = U(z)$. $z = kx + \omega t$, where k and ω are constants. reduces equation (1.39) to the ordinary differential equation

$$E_2(U, \omega U', kU', \omega^2 U'', k^2 U'', \cdots) = 0. \tag{1.40}$$

Step 2. Suppose that the exact solution of equation (1.40) is expressed as

$$U(z) = \sum_{n=0}^{N} a_n Q^n(z), \tag{1.41}$$

where the coefficients a_n $(n = 0, 1, 2, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(z)$ is the solution of the first-order nonlinear ODE

$$Q_z(z) = Q^2(z) - Q(z). \tag{1.42}$$

Equation (1.42) has the solution

$$Q(z) = \frac{1}{1+e^z}. \tag{1.43}$$

Step 3. We substitute the value for $U(z)$ into equation (1.40) and use equation (1.42) to obtain an equation involving powers of Q.

Step 4. Equating different powers of Q to zero, we obtain the system of algebraic equations

$$P_n(a_N, a_{N-1}, \cdots, a_0, k, \omega, \cdots) = 0, \quad (n = 0, \cdots, N). \tag{1.44}$$

Step 5. The solution of the system of algebraic equations gives the values of coefficients $a_0, a_1, \cdots, a_{N-1}, a_N$ and relations for parameters of equation (1.40). As a result, we obtain exact solutions of equation (1.40) in the form (1.41).

1.8 Conservation laws

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \cdots, x^n)$ and m dependent variables $u = (u^1, u^2, \cdots, u^m)$, which are defined as

$$E_\alpha(x, u, u_{(1)}, u_{(2)} \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m, \tag{1.45}$$

where $u_{(i)}$ denotes the collection of all i-th-order partial derivatives of u. The n-tuple vector which is given by $T = (T^1, T^2, \ldots, T^n)$, $T^j \epsilon \mathcal{A}$, $j = 1, \ldots, n$, ($\mathcal{A}$ is the space of differential functions) is a conserved vector of (1.45) if T^i satisfies

$$D_i T^i|_{(1.45)} = 0. \tag{1.46}$$

1.8.1 The multiplier approach

This approach has been applied by many researchers. See for example [18,35,38–40]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers with each differential equation in the system, where the multipliers Λ_α are functions of the dependent and independent variables as well as of a finite number of derivatives with respect to the dependent variables of the system of differential equations. A multiplier $\Lambda_\alpha(x, u, u_1, \ldots)$ has the property that [15, 40]

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.47}$$

holds identically. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.48}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u^\alpha_{ij_1 \cdots j_s}}, \alpha = 1, \ldots, m. \tag{1.49}$$

Once the multipliers have been obtained from (1.48), we can determine our conserved vectors by invoking equation (1.47) as illustrated in [40].

1.8.2 The new conservation theorem due to Ibragimov

This relatively new method for deriving the conservation laws [41] has been applied by many researchers in recent years, for instance [38, 42–44]. The essence of the new conservation theorem due to Ibragimov is that we can obtain a conservation law from every Lie point, Lie-Bäcklund and non-local symmetry of a system of differential equations.

Consider the system of NLPDEs

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)}) = 0 \tag{1.50}$$

and its adjoint equations given by

$$E^*_\alpha(x, u, v, \ldots, v_{(s)}, u_{(s)}) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta E_\beta) = 0, \tag{1.51}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator (1.49) and m new dependent variables $v = (v^1, \ldots, v^m)$.

Theorem 1.4 Consider a system of m equations (1.50). The adjoint system given by (1.51), inherits the symmetries of the system (1.50). Namely, if the system (1.50) admits a point transformation group with a generator (1.20), then the adjoint system (1.51) admits the operator (1.20) extended to the variables v^α by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha} \tag{1.52}$$

with appropriately chosen $\eta_*^\alpha = \eta_*^\alpha(x, u, v)$.

In [41], the coefficients η_*^α in (1.52) are given by

$$\eta_*^\alpha = -\left[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\right], \tag{1.53}$$

where λ_β^α can be computed by utilising the equation

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \tag{1.54}$$

We can obtain a conserved vector, for instance, for a third-order Lagrangian by applying the formula

$$\begin{aligned} C^i = {} & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] \\ & + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \ldots\right] + D_j D_k(W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots, \end{aligned} \tag{1.55}$$

where $\mathcal{L}$ is the Lagrangian of the system E and E^* that is defined as

$$\mathcal{L} = v^\alpha E_\alpha \tag{1.56}$$

and W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \ \alpha = 1, \ldots, m. \tag{1.57}$$

1.9 Conclusion

In this chapter, a brief introduction to the Lie group analysis methods for finding the exact solutions of PDEs has been presented. A synopsis of two methods for deriving the conservation laws were discussed. The techniques deliberated upon in this section will be utilised throughout this dissertation.

Chapter 2

Travelling wave solutions and conservation laws for the KdV-BBM equation

In this chapter we study the third-order Korteweg de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation

$$u_t + u_x + \frac{3}{2}uu_x + \nu u_{xxx} - \left(\frac{1}{6} - \nu\right) u_{txx} = 0, \quad \nu \neq \frac{1}{6}, \tag{2.1}$$

This equation is a generalization of the celebrated KdV and BBM equations. When $\nu = 1/6$, equation (2.1) reduces to the KdV equation and when $\nu = 0$, equation (2.1) becomes the BBM equation. Equation (2.1) was investigated in [45] where the researchers presented conditions for the existence of hyperbolic, unbounded periodic and soliton solutions in terms of Weierstrass functions. Furthermore, an analysis for the initial value problem was developed together with a local and global well-posedness theory for (2.1).

In our study we obtain exact travelling wave solutions of (2.1) here by making use of different approaches. Moreover, we derive the conservation laws of the KdV-BBM equation by employing the variational derivative approach.

The contents of this chapter have been accepted for publication. See [46].

2.1 Exact solutions of (2.1)

In this section we present exact solutions of (2.1) by applying different methods.

2.1.1 Exact solutions of (2.1) by direct integration

Since the equation (2.1) has two independent variables t and x and one dependent variable $u(t,x)$, the vector field (1.20) can be written as

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{2.2}$$

where $\xi^i, i = 1,2$ and η depend on t, x and u. We recall that (2.2) is a Lie point symmetry of (2.1) if

$$\mathrm{pr}^{(3)}X\Delta|_{\Delta=0} = 0, \tag{2.3}$$

where

$$\Delta \equiv u_t + u_x + \frac{3}{2}uu_x + \nu u_{xxx} - \left(\frac{1}{6} - \nu\right) u_{txx},$$

and $\mathrm{pr}^{(3)}$ is the third prolongation [15] of (2.2) defined by

$$\mathrm{pr}^{(3)}X = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}}. \tag{2.4}$$

Here $\zeta_t, \zeta_x, \zeta_{xx}, \zeta_{txx}$ and ζ_{xxx} are determined as follows:

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta_{txx} &= D_x(\zeta_{tx}) - u_{txx} D_x(\xi^1) - u_{xxx} D_x(\xi^2),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2),
\end{aligned} \tag{2.5}$$

where the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots . \tag{2.6}$$

Expanding (2.3) we obtain

$$\begin{aligned}
&\frac{1}{6}\xi^2_{tuu}u_x^3 - \nu\xi^2_{uuu}u_x^4 - \nu u_t\xi^1_{uuu}u_x^3 - \nu\xi^2_{tuu}u_x^3 - \nu u_t\xi^2_{uuu}u_x^3 + \frac{1}{6}u_t\xi^2_{uuu}u_x^3 - 3\nu\xi^2_{xuu}u_x^3 \\
&+ \nu\eta_{uuu}u_x^3 - \frac{3}{2}u\xi^2_u u_x^2 - \xi^2_u u_x^2 - \nu u_{tt}\xi^1_{uu}u_x^2 + \frac{1}{6}u_{tt}\xi^1_{uu}u_x^2 - 3\nu u_{tx}\xi^1_{uu}u_x^2 - \nu u_{tt}\xi^1_{xx} \\
&- 3\nu u_{tx}\xi^2_{uu}u_x^2 + \frac{1}{2}u_{tx}\xi^2_{uu}u_x^2 - 6\nu u_{xx}\xi^2_{uu}u_x^2 - \nu u_t\xi^1_{tuu}u_x^2 + \frac{1}{6}u_t\xi^1_{tuu}u_x^2 - \nu u_t^2\xi^1_{uuu}u_x^2 \\
&+ \frac{1}{6}u_t^2\xi^1_{uuu}u_x^2 - 3\nu u_t\xi^1_{xuu}u_x^2 - 2\nu\xi^2_{txu}u_x^2 + \frac{1}{3}\xi^2_{txu}u_x^2 - 2\nu u_t\xi^2_{xuu}u_x^2 + \frac{1}{3}u_t\xi^2_{xuu}u_x^2 \\
&- 3\nu\xi^2_{xxu}u_x^2 + \nu\eta^1_{tuu}u_x^2 - \frac{1}{6}\eta_{tuu}u_x^2 + \nu u_t\eta_{uuu}u_x^2 - \frac{1}{6}u_t\eta_{uuu}u_x^2 + 3\nu\eta_{xuu}u_x^2 + \frac{3\eta u_x}{2} \\
&- \frac{3}{2}uu_t\xi^1_u u_x - u_t\xi^1_u u_x - \xi^2_t u_x - u_t\xi^2_u u_x - \frac{3}{2}u\xi^2_x u_x + \frac{3}{2}u\eta_u u_x - 2\nu u_t u_{xx}\xi^2_{xu} \\
&- 2\nu u_{tx}\xi^1_{tu}u_x + \frac{1}{3}u_{tx}\xi^1_{tu}u_x - 4\nu u_t u_{tx}\xi^1_{uu}u_x + \frac{2}{3}u_t u_{tx}\xi^1_{uu}u_x - 3\nu u_t u_{xx}\xi^1_{uu}u_x \\
&+ \frac{1}{6}u_{tt}\xi^1_{xx} - 2\nu u_{tt}\xi^1_{xu}u_x + \frac{1}{3}u_{tt}\xi^1_{xu}u_x - 6\nu u_{tx}\xi^1_{xu}u_x - 3\nu u_{xx}\xi^2_{tu}u_x + \frac{1}{2}u_{xx}\xi^2_{tu}u_x \\
&- 3\nu u_t u_{xx}\xi^2_{uu}u_x + \frac{1}{2}u_t u_{xx}\xi^2_{uu}u_x - 4\nu u_{tx}\xi^2_{xu}u_x + \frac{2}{3}u_{tx}\xi^2_{xu}u_x - 9\nu u_{xx}\xi^2_{xu}u_x - \xi^2_x u_x \\
&+ 2\nu u_{tx}\eta_{uu}u_x - \frac{1}{3}u_{tx}\eta_{uu}u_x + 3\nu u_{xx}\eta_{uu}u_x - 2\nu\xi^1_u u_{ttx}u_x + \frac{1}{3}\xi^1_u u_{ttx}u_x - 3\nu\xi^1_u u_{txx}u_x \\
&- 3\nu\xi^2_u u_{txx}u_x + \frac{1}{2}\xi^2_u u_{txx}u_x - 4\nu\xi^2_u u_{xxx}u_x - 2\nu u_t\xi^1_{txu}u_x + \frac{1}{3}u_t\xi^1_{txu}u_x - 2\nu u_t^2\xi^1_{xuu}u_x \\
&+ \frac{1}{3}u_t^2\xi^1_{xuu}u_x - 3\nu u_t\xi^1_{xxu}u_x + \frac{1}{6}\xi^2_{txx}u_x - \nu u_t\xi^2_{xxu}u_x - \nu\xi^2_{txx}u_x + \eta_t + \frac{1}{6}u_t\xi^2_{xxu}u_x \\
&+ 2\nu\eta_{txu}u_x - \frac{1}{3}\eta_{txu}u_x + 2\nu u_t\eta_{xuu}u_x - \frac{1}{3}u_t\eta_{xuu}u_x + 3\nu\eta_{xxu}u_x - 2\nu\xi^1_u u_{tx}^2 + \frac{1}{3}\xi^1_u u_{tx}^2 \\
&- 3\nu\xi^2_u u_{xx}^2 - u_t^2\xi^1_u - \frac{3}{2}uu_t\xi^1_x - u_t\xi^1_x + u_t\eta_u + \frac{3u\eta_x}{2} + \eta_x - \nu\xi^1_u u_{tt}u_{xx} - \nu\xi^2_{xxx}u_x \\
&+ \frac{1}{6}\xi^1_u u_{tt}u_{xx} - 3\nu\xi^1_u u_{tx}u_{xx} - 3\nu\xi^2_u u_{tx}u_{xx} + \frac{1}{2}\xi^2_u u_{tx}u_{xx} - \nu u_t u_{xx}\xi^1_{tu} + \frac{1}{3}u_{tx}\xi^1_{tx} \\
&+ \frac{1}{6}u_t u_{xx}\xi^1_{tu} - 2\nu u_{tx}\xi^1_{tx} - \nu u_t^2 u_{xx}\xi^1_{uu} + \frac{1}{6}u_t^2 u_{xx}\xi^1_{uu} - 4\nu u_t u_{tx}\xi^1_{xu} + \frac{2}{3}u_t u_{tx}\xi^1_{xu} \\
&- 3\nu u_t u_{xx}\xi^1_{xu} - 3\nu u_{tx}\xi^1_{xx} - 2\nu u_{xx}\xi^2_{tx} + \frac{1}{3}u_{xx}\xi^2_{tx} + \frac{1}{3}u_t u_{xx}\xi^2_{xu} - \nu u_{tx}\xi^2_{xx} + \frac{1}{6}u_{tx}\xi^2_{xx} \\
&+ \eta_u u_x + \nu u_{xx}\eta^1_{tu} - \frac{1}{6}u_{xx}\eta_{tu} + \nu u_t u_{xx}\eta_{uu} - \frac{1}{6}u_t u_{xx}\eta_{uu} + 2\nu u_{tx}\eta_{xu} - \frac{1}{3}u_{tx}\eta_{xu} \\
&+ 3\nu u_{xx}\eta_{xu} - 2\nu\xi^1_x u_{ttx} + \frac{1}{3}\xi^1_x u_{ttx} - \nu\xi^1_t u_{txx} + \frac{1}{6}\xi^1_t u_{txx} - 2\nu u_t\xi^1_u u_{txx} + \frac{1}{3}u_t\xi^1_u u_{txx} \\
&- 2\nu\xi^2_x u_{txx} + \frac{1}{3}\xi^2_x u_{txx} + \nu\eta_u u_{txx} - \frac{1}{6}\eta_u u_{txx} - \nu u_t\xi^1_u u_{xxx} - \nu\xi^2_t u_{xxx} + \frac{1}{6}\xi^2_t u_{xxx} \\
&- \nu u_t\xi^2_u u_{xxx} + \frac{1}{6}u_t\xi^2_u u_{xxx} - 3\nu\xi^2_x u_{xxx} + \nu\eta_u u_{xxx} - \nu u_t\xi^1_{txx} + \frac{1}{6}u_t\xi^1_{txx} - \nu u_t^2\xi^1_{xxu}
\end{aligned}$$

$$+\frac{1}{6}u_t^2\xi^1_{xxu}-\nu u_t\xi^1_{xxx}+\nu\eta^1_{txx}-\frac{1}{6}\eta_{txx}+\nu u_t\eta_{xxu}-\frac{1}{6}u_t\eta^1_{xxu}-3\nu u_{xx}\xi^2_{xx}-3\nu\xi^1_x u_{txx}$$
$$+\nu\eta_{xxx}-u_t\xi^1_t=0$$

and separating with respect to the derivatives of u yields the following system of linear partial differential equations:

$$\begin{aligned}
&\xi^1_u=0,\quad \xi^2_u=0,\quad \eta_{uu}=0,\quad \xi^1_x=0,\\
&2\eta_{xu}-\xi^2_{xx}=0,\\
&\eta_{xxu}+2\xi^2_x=0,\\
&3\nu\eta_{xu}-3\nu\xi^2_{xx}+\left(\nu-\frac{1}{6}\right)\eta_{tu}-2\left(\nu-\frac{1}{6}\right)\xi^2_{tx}=0,\\
&\nu\eta_{xxx}+\eta_t+\left(\nu-\frac{1}{6}\right)\eta_{txx}+\left(\frac{3u}{2}+1\right)\eta_x=0,\\
&\nu\xi^1_t-3\nu\xi^2_x-\left(\nu-\frac{1}{6}\right)\xi^2_t-\left(\nu-\frac{1}{6}\right)\nu\eta_{xxu}=0,\\
&\frac{3}{2}\eta-\nu\xi^2_{xxx}-\xi^2_t+2\left(\nu-\frac{1}{6}\right)\eta_{txu}+\left(\frac{3u}{2}+1\right)\xi^1_t\\
&-\left(\nu-\frac{1}{6}\right)\xi^2_{txx}+\left(2\nu+\frac{1}{6}-\frac{3}{2}u\left(\nu-\frac{1}{6}\right)\right)\eta_{xxu}-\left(\frac{3u}{2}+1\right)\xi^2_x=0.
\end{aligned}$$

Solving the above system we obtain

$$\xi^1(t,x,u)=C_1t+C_2, \tag{2.7}$$
$$\xi^2(t,x,u)=\frac{6C_1\nu t}{6\nu-1}+C_3, \tag{2.8}$$
$$\eta\,(t,x,u)=\frac{C_1\,(2-3(6\nu-1)u)}{18\nu-3}, \tag{2.9}$$

where C_1, C_2 and C_3 are arbitrary constants. Therefore the infinitesimal generators corresponding to (2.7–2.9) are

$$X_1=\frac{\partial}{\partial t}, \tag{2.10}$$
$$X_2=\frac{\partial}{\partial x}, \tag{2.11}$$
$$X_3=3t(6\nu-1)\frac{\partial}{\partial t}+18\nu t\frac{\partial}{\partial x}+(2-3(6\nu-1)u)\,\frac{\partial}{\partial u}. \tag{2.12}$$

A linear combination [47] of the translation symmetries (2.10) and (2.11)

$$X=X_1+cX_2, \tag{2.13}$$

yields the two invariants

$$z = x - ct \quad \text{and} \quad U = u, \tag{2.14}$$

where c is a constant, which give the group invariant solution $U = U(z)$. Using this result, and with z as our new independent variable, (2.1) is transformed into the nonlinear ordinary differential equation (ODE)

$$\left(\frac{c}{6} - c\nu + \nu\right) U'''(z) + (1-c)U'(z) + \frac{3}{2}U(z)U'(z) = 0. \tag{2.15}$$

Integrating (2.15) once with respect to z, we obtain

$$\left(\frac{c}{6} - c\nu + \nu\right) U''(z) + (1-c)U(z) + \frac{3}{4}U^2(z) + C_0 = 0, \tag{2.16}$$

where C_0 is an arbitrary constant of integration.The nonlinear ODE (2.16) can be rewritten as

$$U''(z) + \frac{9}{2\,(c - 6c\nu + 6\nu)}U^2(z) + \frac{6(1-c)}{c - 6c\nu + 6\nu}U(z) + \frac{6C_0}{c - 6c\nu + 6\nu} = 0. \tag{2.17}$$

By introducing a new variable $y = y(z)$ in (2.17) defined by

$$U(z) = \frac{2}{3}\,(c - 6c\nu + 6\nu)\, y(z), \quad c \neq \frac{6\nu}{6\nu - 1}, \tag{2.18}$$

equation (2.17) can be re-written in terms of $y(z)$ as

$$y'' + 3y^2 - \omega y + C_1 = 0, \tag{2.19}$$

where C_1 and ω are given by

$$C_1 = \frac{6C_0}{c - 6c\nu + 6\nu}, \quad \omega = -\frac{6(1-c)}{c - 6c\nu + 6\nu}. \tag{2.20}$$

The solutions of (2.17) can be expressed via the Jacobi elliptic function [30, 48]. We now turn our attention to (2.19). Multiplying (2.19) with y' and integrating once with repsect to z yields the first-order linear ODE

$$y'^2 + 2y^3 - \omega y^2 + 2C_1 y + 2C_2 = 0, \tag{2.21}$$

where C_2 is an arbitrary constant of integration. Since the expression

$$2y^3 - \omega y^2 + 2C_1 y + 2C_2 \tag{2.22}$$

in (2.21) is a cubic function in $y(z)$, it is reasonable to assume that (2.21) can be written as

$$y'^2 = -2(y-\lambda_1)(y-\lambda_2)(y-\lambda_3). \tag{2.23}$$

where λ_1, λ_2 and λ_3 are factors of (2.22) ordered so that $\lambda_3 \geq \lambda_2 \geq \lambda_1$. The general solution of (2.19) can thus be expressed in terms of the Jacobi elliptic cosine amplitude function as [30, 48]

$$y(z) = \lambda_2 + (\lambda_3 - \lambda_2)cn^2\left(\sqrt{\frac{\lambda_3-\lambda_1}{2}}z, k^2\right), \quad k^2 = \frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}. \tag{2.24}$$

where $0 \leq k^2 \leq 1$ and $(\lambda_3 - \lambda_1)/2 \geq 0$.

Reverting to the original variables, the solution of (2.1) is

$$u(t,x) = \frac{2}{3}(c - 6c\nu + 6\nu)\left\{\lambda_2 + (\lambda_3-\lambda_2)cn^2\left(\sqrt{\frac{\lambda_3-\lambda_1}{2}}(x-ct), k^2\right)\right\}. \tag{2.25}$$

By letting $c = 1, \nu = 0.083, \lambda_1 = 0.1, \lambda_2 = 0.2$ and $\lambda_3 = 0.4$, we render the graphical representation of (2.25) in Figure 2.1.

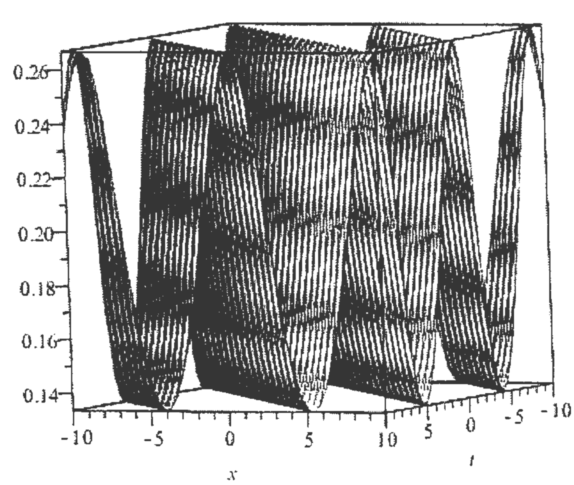

Figure 2.1: Graphical representation of solution (2.25)

2.1.2 Exact solutions of (2.1) using the Kudryashov method

In this section we invoke the Kudryashov method [10] to determine exact solutions of (2.1).

We begin by assuming that the solutions to the third-order ODE (2.15) can be written in the form

$$U(z) = \sum_{i=0}^{M} A_i Q^i(z), \tag{2.26}$$

where $Q(z)$ satisfies the Riccati equation

$$Q'(z) = Q^2(z) - Q(z), \tag{2.27}$$

whose solution is

$$Q(z) = \frac{1}{1+e^z}. \tag{2.28}$$

The value of M in (2.26) can be determined by using the balancing procedure [10] and A_i $i = 0, 1, \cdots, M$ are constants which we will determine. We balance the highest order derivative with the nonlinear term, that is, $U'''(z)$ and $U(z)U'(z)$ respectively. This gives $M = 2$. Thus the solution (2.26) can be written as

$$U(z) = A_0 + A_1 Q(z) + A_2 Q^2(z). \tag{2.29}$$

Substituting (2.29) into (2.15) and invoking (2.27) we obtain the following equation in $Q(z)$:

$$\begin{aligned}
&12c\nu A_1 Q^3(z) - 6c\nu A_1 Q^4(z) - 7c\nu A_1 Q^2(z) + c\nu A_1 Q(z) - 24c\nu A_2 Q^5(z) \\
&+ 54c\nu A_2 Q^4(z) - 38c\nu A_2 Q^3(z) + 8c\nu A_2 Q^2(z) - 2A_2 Q^2(z) - 3{A_2}^2 Q^4(z) \\
&+ 3{A_2}^2 Q^5(z) - A_1 Q(z) + \frac{3}{2}{A_1}^2 Q^3(z) - \frac{3}{2}{A_1}^2 Q^2(z) + A_1 Q^2(z) \\
&+ 2A_2 Q^3(z) - 9cA_2 Q^4(z) + 4cA_2 Q^5(z) - 8\nu A_2 Q^2(z) + cA_1 Q^4(z) \\
&+ \frac{5}{6}cA_1 Q(z) + \frac{13}{3}cA_2 Q^3(z) - 2cA_1 Q^3(z) + \frac{2}{3}cA_2 Q^2(z) + 7\nu A_1 Q^2(z) \\
&- \nu A_1 Q(z) + 24\nu A_2 Q^5(z) + \frac{1}{6}cA_1 Q^2(z) - 54\nu A_2 Q^4(z) + 38\nu A_2 Q^3(z) \\
&- 3A_0 A_2 Q^2(z) + \frac{9}{2}A_1 Q^4(z) A_2 + 3A_0 A_2 Q^3(z) - 12\nu A_1 Q^3(z) - \frac{3}{2}A_0 A_1 Q(z) \\
&- \frac{9}{2}A_1 Q^3(z) A_2 + \frac{3}{2}A_0 A_1 Q^2(z) + 6\nu A_1 Q^4(z) = 0.
\end{aligned} \tag{2.30}$$

Equating the coefficients of like powers of $Q(z)$ in equation (2.30) we obtain the following five algebraic equations in terms of A_0, A_1 and A_2:

$$
\begin{aligned}
&\frac{9}{2}A_1A_2 + 6\nu A_1 - 54\nu A_2 + cA_1 - 9cA_2 - 6c\nu A_1 + 54c\nu A_2 - 3{A_2}^2 = 0,\\
&3A_0A_2 - \frac{9}{2}A_1A_2 - 12\nu A_1 + 38\nu A_2 + \frac{13}{3}cA_2 - 2cA_1 + 12c\nu A_1 - 38c\nu A_2\\
&+ 2A_2 + \frac{3}{2}{A_1}^2 = 0,\\
&4cA_2 + 24\nu A_2 + 3{A_2}^2 - 24c\nu A_2 = 0,\\
&\frac{3}{2}A_0A_1 - 3A_0A_2 + 7\nu A_1 - 8\nu A_2 + \frac{1}{6}cA_1 + \frac{2}{3}cA_2 - 7c\nu A_1 + 8c\nu A_2\\
&+ A_1 - \frac{3}{2}{A_1}^2 - 2A_2 = 0,\\
&\frac{3}{2}A_0A_1 + \nu A_1 + A_1 - \frac{5}{6}cA_1 - c\nu A_1 = 0.
\end{aligned}
\tag{2.31}
$$

Below we give one solution of interest obtained from solving (2.31).

$$
\begin{aligned}
A_0 &= \frac{1}{9}(6c\nu + 5c - 6\nu - 6),\\
A_1 &= \frac{1}{3}(24\nu - 24c\nu + 4c),\\
A_2 &= -\frac{1}{3}(24\nu - 24c\nu + 4c).
\end{aligned}
\tag{2.32}
$$

Thus, from (2.14), (2.29) and (2.32) we can write the solution of (2.1) as

$$
u(t,x) = \frac{1}{9}(6c\nu + 5c - 6\nu - 6) + \frac{1}{3}(24\nu - 24c\nu + 4c)\frac{e^{x-ct}}{(1+e^{x-ct})^2}. \tag{2.33}
$$

One possible graphical representation of (2.33) is shown in Figure 2.2.

As expected [30], the solution (2.33) is contained in the general solution (2.25). This can be seen if we let $\lambda_1 = \lambda_2$ in (2.25), which yields a soliton whose outline is akin to Figure 2.2.

2.1.3 Cnoidal wave solutions of (2.1) using the extended Jacobi elliptic function expansion method

In this subsection we employ the extended Jacobi elliptic cosine amplitude function expansion method to obtain cnoidal wave solutions of (2.1). As before, the balancing

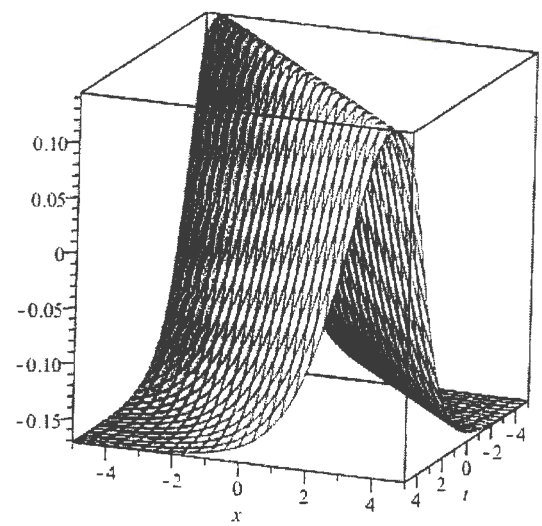

Figure 2.2: Graphical representation of solution (2.33) for $c = 0.9$ and $\nu = 0.083$.

procedure gives $M = 2$ and hence the solution is of the form

$$U(z) = A_{-2}H^{-2}(z) + A_{-1}H^{-1}(z) + A_0 + A_1H(z) + A_2H^2(z), \tag{2.34}$$

where $A_i, i = \{-2, -1, \cdots, 2\}$ are constants to be determined and $H(z)$ satisfies the first-order ODE (1.36). We now proceed to substitute (2.34) into (2.15) and use (1.36) which yields the following equation in $H(z)$:

$$\begin{aligned}
&96A_2c\nu\omega H(z)^6 - 144A_2c\nu\omega H(z)^8 - 36A_1c\nu\omega H(z)^7 + 12A_1c\nu\omega H(z)^5 \\
&+ 6A_{-1}c\nu H(z)^3 - 12A_{-1}c\nu\omega H(z)^3 - 96A_{-2}c\nu\omega H(z)^2 + 36A_{-1}c\nu\omega H(z) \\
&- 48A_2c\nu H(z)^6 - 6A_1c\nu H(z)^5 + 48A_{-2}c\nu H(z)^2 - 36A_{-1}c\nu H(z) \\
&+ 24A_2c\omega H(z)^8 + 6A_1c\omega H(z)^7 - 16A_2c\omega H(z)^6 - 2A_1c\omega H(z)^5 \\
&+ 2A_{-1}c\omega H(z)^3 + 16A_{-2}c\omega H(z)^2 - 6A_{-1}c\omega H(z) + 20A_2cH(z)^6 \\
&+ 7A_1cH(z)^5 - 7A_{-1}cH(z)^3 - 20A_{-2}cH(z)^2 + 6A_{-1}cH(z) + 144A_{-2}c\nu\omega \\
&- 144A_{-2}c\nu - 24A_{-2}c\omega + 24A_{-2}c + 144A_2\nu\omega H(z)^8 + 36A_1\nu\omega H(z)^7 \\
&- 96A_2\nu\omega H(z)^6 - 12A_1\nu\omega H(z)^5 + 12A_{-1}\nu\omega H(z)^3 + 96A_{-2}\nu\omega H(z)^2 \\
&- 36A_{-1}\nu\omega H(z) + 48A_2\nu H(z)^6 + 6A_1\nu H(z)^5 - 6A_{-1}\nu H(z)^3 - 48A_{-2}\nu H(z)^2
\end{aligned}$$

$$+36A_{-1}\nu H(z) - 18A_2^2H(z)^8 - 27A_1A_2H(z)^7 - 9A_1^2H(z)^6 - 18A_0A_2H(z)^6$$
$$-12A_2H(z)^6 - 9A_0A_1H(z)^5 - 6A_1H(z)^5 - 9A_{-1}A_2H(z)^5 + 6A_{-1}H(z)^3$$
$$+9A_{-1}A_0H(z)^3 + 9A_{-2}A_1H(z)^3 + 9A_{-1}^2H(z)^2 + 12A_{-2}H(z)^2 + 18A_{-2}A_0H(z)^2$$
$$+27A_{-2}A_{-1}H(z) - 144A_{-2}\nu\omega + 144A_{-2}\nu + 18A_{-2}^2 = 0. \tag{2.35}$$

Equating the coefficients of like-powers of $H(z)$ in (2.35) to zero, we obtain the following overdetermined system of eight algebraic equations in A_{-2}, A_{-1}, A_0, A_1 and A_2:

$$12A_1c\nu\omega - 2A_1c\omega - 12A_1\nu\omega + 9A_2A_1 = 0,$$
$$24A_2c\nu\omega - 4A_2c\omega - 24A_2\nu\omega + 3A_2^2 = 0.$$
$$24A_{-2}c\nu\omega - 24A_{-2}c\nu - 4A_{-2}c\omega + 4A_{-2}c - 24A_{-2}\nu\omega + 24A_{-2}\nu$$
$$+3A_{-2}^2 = 0,$$
$$12A_{-1}c\nu\omega - 12A_{-1}c\nu - 2A_{-1}c\omega + 2A_{-1}c - 12A_{-1}\nu\omega + 12A_{-1}\nu$$
$$+9A_{-2}A_{-1} = 0,$$
$$48A_{-2}c\nu - 96A_{-2}c\nu\omega + 16A_{-2}c\omega - 20A_{-2}c + 96A_{-2}\nu\omega - 48A_{-2}\nu + 9A_{-1}^2$$
$$+12A_{-2} + 18A_{-2}A_0 = 0,$$
$$12A_{-1}c\nu\omega - 9A_{-2}A_1 - 6A_{-1}c\nu - 2A_{-1}c\omega - 12A_{-1}\nu\omega + 6A_{-1}\nu - 9A_0A_{-1}$$
$$+7A_{-1}c - 6A_{-1} = 0,$$
$$12A_1c\nu\omega - 6A_1c\nu - 2A_1c\omega - 12A_1\nu\omega + 6A_1\nu - 9A_0A_1 - 6A_1 - 9A_{-1}A_2$$
$$+7A_1c = 0,$$
$$+48A_2c\nu + 16A_2c\omega - 20A_2c - 96A_2c\nu\omega - 48A_2\nu + 9A_1^2 + 18A_0A_2 + 12A_2$$
$$+96A_2\nu\omega = 0.$$

One plausible solution of the above system is

$$A_{-2} = \frac{1}{3}(24\nu\omega + 24c\nu + 4c\omega - 4c - 24c\nu\omega - 24\nu),$$
$$A_{-1} = A_1 = 0,$$
$$A_0 = \frac{1}{9}(48c\nu\omega - 24c\nu - 8c\omega + 10c - 48\nu\omega + 24\nu - 6),$$
$$A_2 = \frac{1}{3}(24\nu\omega - 24c\nu\omega + 4c\omega). \tag{2.36}$$

Hence the solution for equation (2.1) that corresponds to (2.36) is

$$u(t,x) = A_{-2}cn^{-2}(z,\omega) + A_0 + A_2 cn^2(z,\omega), \tag{2.37}$$

where $z = x - ct$ and $0 \leq \omega \leq 1$. We now give a solution profile of (2.37) for $c = 0.5$, $\nu = 0.1, \omega = 0.01$ in Figure 2.3.

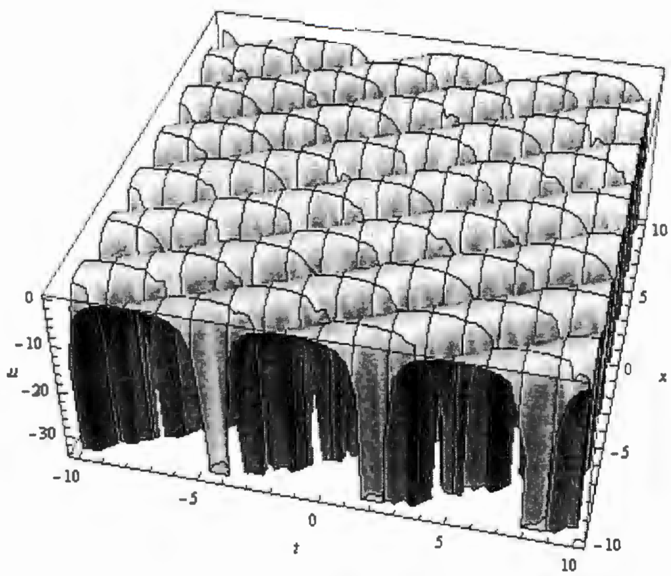

Figure 2.3: Graphical representation of solution (2.37)

2.1.4 Snoidal wave solutions of (2.1) using the extended Jacobi elliptic function expansion method

To obtain the snoidal wave solutions of (2.1) using the extended Jacobi elliptic function expansion method we assume the solution of the third-order ODE (2.15) of the form

$$U(z) = \sum_{i=-M}^{M} A_i H(z)^i,$$

where $H(z)$ satisfies the first order ODE

$$H'(z) = \sqrt{(1 - H^2(z)(1 - \omega H^2(z))}.$$

The solution of the above equation is the sine amplitude function

$$H(z) = sn(z|w).$$

The balancing procedure yields $M = 2$ and so following the same procedure as in the previous subsection we obtain the following equation in $H(z)$:

$$\begin{aligned}
&144A_2c\nu\omega H(z)^8 + 36A_1c\nu\omega H(z)^7 - 48A_2c\nu\omega H(z)^6 - 6A_1c\nu\omega H(z)^5 \\
&+ 6A_{-1}c\nu\omega H(z)^3 + 48A_{-2}c\nu\omega H(z)^2 - 48A_2c\nu H(z)^6 - 6A_1c\nu H(z)^5 \\
&+ 6A_{-1}c\nu H(z)^3 + 48A_{-2}c\nu H(z)^2 - 36A_{-1}c\nu H(z) - 24A_2c\omega H(z)^8 \\
&- 6A_1c\omega H(z)^7 + 8A_2c\omega H(z)^6 + A_1c\omega H(z)^5 - A_{-1}c\omega H(z)^3 \\
&- 8A_{-2}c\omega H(z)^2 + 20A_2cH(z)^6 + 7A_1cH(z)^5 - 7A_{-1}cH(z)^3 \\
&- 20A_{-2}cH(z)^2 + 6A_{-1}cH(z) - 144A_{-2}c\nu + 4A_{-2}c - 144A_2\nu\omega H(z)^8 \\
&- 36A_1\nu\omega H(z)^7 + 48A_2\nu\omega H(z)^6 + 6A_1\nu\omega H(z)^5 - 6A_{-1}\nu\omega H(z)^3 \\
&- 48A_{-2}\nu\omega H(z)^2 + 48A_2\nu H(z)^6 + 6A_1\nu H(z)^5 - 6A_{-1}\nu H(z)^3 \\
&- 48A_{-2}\nu H(z)^2 + 36A_{-1}\nu H(z) - 18A_2^2H(z)^8 - 27A_1A_2H(z)^7 - 9A_1^2H(z)^6 \\
&- 18A_0A_2H(z)^6 - 12A_2H(z)^6 - 9A_0A_1H(z)^5 - 6A_1H(z)^5 - 9A_{-1}A_2H(z)^5 \\
&+ 6A_{-1}H(z)^3 + 9A_{-1}A_0H(z)^3 + 9A_{-2}A_1H(z)^3 + 9A_{-1}^2H(z)^2 + 12A_{-2}H(z)^2 \\
&+ 18A_{-2}A_0H(z)^2 + 27A_{-2}A_{-1}H(z) + 144A_{-2}\nu + 18A_{-2}^2 = 0. \qquad (2.38)
\end{aligned}$$

The next step entails collecting and equating the coefficients of $H^i(z)$ to zero yielding eight algebraic equations

$$\begin{aligned}
&24A_{-2}c - 144A_{-2}c\nu + 144A_{-2}\nu + 18A_{-2}^2 = 0, \\
&6A_{-1}c - 36A_{-1}c\nu + 36A_{-1}\nu + 27A_{-2}A_{-1} = 0, \\
&48A_{-2}c\nu\omega + 48A_{-2}c\nu - 8A_{-2}c\omega - 20A_{-2}c - 48A_{-2}\nu\omega - 48A_{-2}\nu \\
&+ 9A_{-1}^2 + 12A_{-2} + 18A_{-2}A_0 = 0, \\
&6A_{-1}c\nu\omega + 6A_{-1}c\nu - A_{-1}c\omega - 7A_{-1}c - 6A_{-1}\nu\omega - 6A_{-1}\nu + 9A_0A_{-1} \\
&+ 6A_{-1} + 9A_{-2}A_1 = 0, \\
&A_1c\omega - 6A_1c\nu\omega - 6A_1c\nu + 7A_1c + 6A_1\nu\omega + 6A_1\nu - 9A_0A_1 - 6A_1 \\
&- 9A_{-1}A_2 = 0,
\end{aligned}$$

$$8A_2c\omega - 48A_2c\nu\omega - 48A_2c\nu + 20A_2c + 48A_2\nu\omega + 48A_2\nu - 9A_1^2 - 18A_0A_2$$
$$- 12A_2 = 0,$$
$$36A_1c\nu\omega - 6A_1c\omega - 36A_1\nu\omega - 27A_2A_1 = 0,$$
$$144A_2c\nu\omega - 24A_2c\omega - 144A_2\nu\omega - 18A_2^2 = 0.$$

One set of solutions for the above system of algebraic equations is

$$\begin{aligned} A_{-2} &= \frac{1}{3}\left(24c\nu - 4c - 24\nu\right), \\ A_{-1} &= A_1 = 0, \\ A_0 &= \frac{1}{9}\left(24\nu\omega + 10c + 24\nu - 24c\nu\omega - 24c\nu + 4c\omega - 6\right), \\ A_2 &= \frac{1}{3}\omega\left(24c\nu - 4c - 24\nu\right). \end{aligned} \tag{2.39}$$

Thus, in light of (2.39) the solution of (2.1) is

$$u(t,x) = A_0 + \frac{1}{\omega}A_2\{sn^{-2}(z|\omega) + \omega sn^2(z|\omega)\}, \tag{2.40}$$

where A_0 and A_2 are given in (2.39). The corresponding graphical representation is given for $c = 0.5$, $\nu = 0.1$, and $\omega = 0.01$ in Figure 2.4.

2.1.5 Symmetry reductions using optimal system of one-dimensional subalgebras

We now calculate the optimal system of one-dimensional subalgebras for equation (2.1) This will then be used to find the optimal system of group-invariant solutions for equation (2.1). To obtain the optimal system of one-dimensional subalgebras we use the method presented in [15]. Recall that the adjoint transformations are given by

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2}\epsilon^2[X_i, [X_i, X_j]] - \cdots, \tag{2.41}$$

where $[X_i, X_j]$ is the commutator defined by

$$[X_i, X_j] = X_iX_j - X_jX_i. \tag{2.42}$$

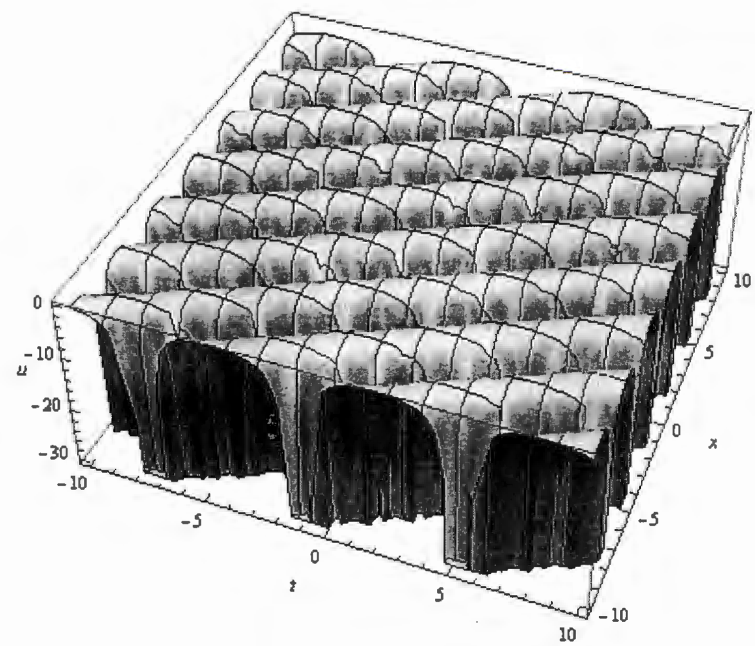

Figure 2.4: Graphical representation of solution (2.40)

We first compute all the commutators of the Lie symmetries and present in Table 1. Then using the commutator table and (2.41) we construct the adjoint representations of the symmetry group of (2.1) and present it in Table 2. Following the procedure given in [15], it turns out that an optimal system of one-dimensional subalgebras is given by $\{X_1, aX_1 + X_2, bX_1 + X_3\}$, where $a, b \in \mathbb{R}$.

Table 1. Commutator table of the Lie algebra of equation (2.1)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$18\nu X_1 + 3(6\nu - 1)X_2$
X_3	0	$-18\nu X_1 - 3(6\nu - 1)X_2$	0

Table 2. Adjoint table of the Lie algebra of equation (2.1)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$-18\varepsilon\nu X_1 - 3\varepsilon(6\nu - 1)X_2 + X_3$
X_3	X_1	$\frac{6\nu}{6\nu-1}(e^{3\varepsilon(6\nu-1)} - 1)X_1 + e^{3\varepsilon(6\nu-1)}X_2$	X_3

2.1.6 Symmetry reductions and exact solutions of (2.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated in the previous subsection to obtain symmetry reductions and exact solutions of the KdV-BBM equation.

Case 1. X_1

The symmetry X_1 gives rise to the two invariants

$$z = t, \quad f = u \tag{2.43}$$

and using these the KdV-BBM equation (2.1) transforms to

$$f'(z) = 0, \tag{2.44}$$

which integrates to $f(z) = K$, where K is a constant of integration. Hence the group-invariant solution of (2.1) under X_1 is given by $u(t, x) = K$.

Case 2. $aX_1 + X_2$

The symmetry $aX_1 + X_2$ gives rise to the following two invariants:

$$z = x - at, \quad f = u. \tag{2.45}$$

We note that these two invariants are identical to the invariants given in equation (2.14) with $a = c$ and hence the group-invariant solution, in this case, is given by (2.25), which is the most general solution we can obtain for equation (2.1).

Case 3. $bX_1 + X_3$

The symmetry $bX_1 + X_3$ yields the two invariants

$$z = \frac{b\ln(3t) + 3(6\nu t - 6\nu x + x)}{3 - 18\nu}, \quad f = t\left(\frac{2}{3 - 18\nu} + u\right). \tag{2.46}$$

By treating f as the new dependent variable and z as a new independent variable, the KdV-BBM equation (2.1) reduces to third-order ODE

$$(6\nu - 1)\{bf''' + 3(6\nu - 1)f''\} - 9(6\nu - 1)(3f' - 2)f(z) + 6bf' = 0. \tag{2.47}$$

2.2 Conservation laws of (2.1)

In this section we construct conservation laws for the KdV-BBM equation (2.1) using the multiplier approach whose algorithm has already been outlined in the previous chapter. For our equation (2.1), the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_x^3\frac{\partial}{\partial u_{xxx}}-D_tD_x^2\frac{\partial}{\partial u_{txx}}. \tag{2.48}$$

where D_t and D_x are the total derivatives as in (2.6). In this work, we consider second-order multipliers, i.e., Λ depends on t, x, u_x and u_{xx}. We now proceed to derive the conservation laws of (2.1) using the multiplier approach.

2.2.1 Conservation laws of (2.1) using the multiplier approach

We consider the second-order multiplier $\Lambda=\Lambda(t,x,u,u_x,u_{xx})$. The determining equation for the multiplier is given by

$$\frac{\delta}{\delta u}\left[\Lambda(t,x,u,u_x,u_{xx})\left\{u_t+u_x+\frac{3}{2}uu_x+\nu u_{xxx}-\left(\frac{1}{6}-\nu\right)u_{txx}\right\}\right]=0. \tag{2.49}$$

Expanding (2.49) and splitting the resultant equation on derivatives of u yields the fifteen determining equations

$$3\nu\Lambda_{u_xu_{xx}}-\frac{1}{2}\Lambda_{u_xu_{xx}}=0,$$
$$\nu\Lambda_{u_xu_{xx}u_{xx}}-\frac{1}{6}\Lambda_{u_xu_{xx}u_{xx}}=0,$$
$$\frac{1}{3}\Lambda_{u_x}-2\nu\Lambda_{u_x}=0.$$
$$\frac{1}{6}\Lambda_{u_xu_{xx}}-\nu\Lambda_{u_xu_{xx}}=0,$$
$$\Lambda_{u_{xx}u_{xx}}+\frac{1}{6}\Lambda_{uu_{xx}}-\nu\Lambda_{uu_{xx}}=0.$$
$$\Lambda_{u_{xx}u_{xx}u_{xx}}+\frac{1}{6}\Lambda_{uu_{xx}u_{xx}}-\nu\Lambda_{uu_{xx}u_{xx}}=0,$$
$$\frac{1}{2}\Lambda_{u_xu_x}+\frac{1}{2}u_x\Lambda_{uu_x}+\frac{1}{2}\Lambda_{xu_x}-3\nu\Lambda_{xu_x}-3\nu\Lambda_{uu_x}u_x-3\nu u_{xx}\Lambda_{u_xu_x}=0,$$
$$\frac{3}{2}uu_x\Lambda_{u_{xx}u_{xx}}-\nu\Lambda_{xu_{xx}}+u_x\Lambda_{u_{xx}u_{xx}}-\nu\Lambda_{tu_{xx}}-2\nu\Lambda_{u_xu_{xx}}-\nu u_x\Lambda_{uu_{xx}}$$

$$+\frac{1}{6}\Lambda_{tu_{xx}}=0,$$

$$\frac{3}{2}u_x\Lambda_{u_{xx}u_{xx}u_{xx}}+\frac{1}{6}\Lambda_{tu_{xx}u_{xx}}-\nu u_{xx}\Lambda_{u_xu_{xx}u_{xx}}-\nu\Lambda_{uu_{xx}u_{xx}}u_x+u_x\Lambda_{u_{xx}u_{xx}u_{xx}}$$

$$-\nu\Lambda_{tu_{xx}u_{xx}}-3\nu\Lambda_{u_xu_{xx}}-\nu\Lambda_{xu_{xx}u_{xx}}=0,$$

$$\frac{1}{3}u_x\Lambda_{uuu_{xx}}+\frac{1}{3}\Lambda_{uu_xu_{xx}}-2\nu\Lambda_{xuu_{xx}}-\nu\Lambda_{uu_x}+2u_x\Lambda_{uu_{xx}u_{xx}}+2u_{xx}\Lambda u_xu_{xx}u_{xx}$$

$$+2\Lambda_{xu_{xx}u_{xx}}+\frac{1}{3}\Lambda_{xuu_{xx}}+\frac{1}{6}\Lambda_{uu_x}-2u_x\nu\Lambda_{uuu_{xx}}-2u_{xx}\nu\Lambda_{uu_xu_{xx}}=0,$$

$$\frac{1}{3}u_x\Lambda_{u_{xx}u_xu}-2\nu\Lambda_{uu_{xx}}+\frac{1}{3}u_{xx}\Lambda_{u_xu_xu_{xx}}-2\nu\Lambda_{xu_{xx}u_x}-\nu\Lambda_{u_xu_x}+2\Lambda_{u_{xx}u_{xx}}$$

$$+\frac{1}{3}\Lambda_{xu_{xx}u_x}+\frac{1}{6}\Lambda_{u_xu_x}+\frac{1}{3}\Lambda_{uu_{xx}}-2\nu u_x\Lambda_{uu_xu_{xx}}-2\nu u_{xx}\Lambda_{u_xu_xu_{xx}}=0,$$

$$\frac{1}{3}u_xu_{xx}\Lambda_{uu_xu_x}-\nu {u_x}^2\Lambda_{uu_xu_x}-2\nu u_x\Lambda_{xuu_x}-2\nu\Lambda_{uu}u_x-3\nu u_{xx}\Lambda_{uu_x}$$

$$+\frac{1}{6}\Lambda_{xxu_x}-\nu {u_{xx}}^2\Lambda_{u_xu_xu_x}-2\nu u_{xx}\Lambda_{xu_xu_x}+\frac{1}{3}\Lambda_{xu}+2u_x\Lambda_{uu_{xx}}+2\Lambda_{u_xu_{xx}}u_{xx}$$

$$+\frac{1}{3}u_{xx}\Lambda_{xu_xu_x}-\nu\Lambda_{xxu_x}+\frac{1}{6}{u_{xx}}^2\Lambda_{u_xu_xu_x}+\frac{1}{3}u_x\Lambda_{xuu_x}+\frac{1}{3}\Lambda_{uu}$$

$$+\frac{1}{6}{u_x}^2\Lambda_{uuu_x}-2\nu\Lambda_{xu}+\frac{1}{2}\Lambda_{uu_x}+2\Lambda_{xu_{xx}}-2\nu u_xu_{xx}\Lambda_{uu_xu_x}-2\Lambda_{u_x}=0,$$

$$\frac{1}{3}\Lambda_{uuu_x}-\nu {u_x}^2\Lambda_{uuu}-2\nu\Lambda_{xuu}u_x-\nu {u_{xx}}^2\Lambda_{uu_xu_x}-\nu u_{xx}\Lambda_{uu}-2\nu u_{xx}\Lambda_{xuu_x}$$

$$+2\Lambda_{uu_xu_{xx}}u_x+\frac{1}{6}\Lambda_{xxu}-\Lambda_{uu_x}u_x-u_{xx}\Lambda_{u_xu_x}+{u_x}^2\Lambda_{uuu_{xx}}+2u_x\Lambda_{xuu_{xx}}$$

$$+{u_{xx}}^2\Lambda_{u_xu_xu_{xx}}+u_{xx}\Lambda_{uu_{xx}}+2u_{xx}\Lambda_{xu_xu_{xx}}-\nu\Lambda_{xxu}+\frac{1}{3}u_{xx}\Lambda_{xuu_x}+\frac{1}{6}u_{xx}\Lambda_{uu}$$

$$+\frac{1}{6}{u_{xx}}^2\Lambda_{uu_xu_x}+\frac{1}{3}u_x\Lambda_{xuu}+\frac{1}{6}{u_x}^2\Lambda_{uuu}-2\nu u_xu_{xx}\Lambda_{uuu_x}-\Lambda_{xu_x}$$

$$+\Lambda_{xxu_{xx}}=0,$$

$$3uu_xu_{xx}\Lambda_{u_xu_{xx}u_{xx}}-4\nu u_xu_{xx}\Lambda_{uu_xu_{xx}}-2\nu u_x\Lambda_{tuu_{xx}}-2\nu u_{xx}\Lambda_{tu_xu_{xx}}-4\nu u_{xx}\Lambda_{xu_xu_{xx}}$$

$$+3u{u_x}^2\Lambda_{uu_{xx}u_{xx}}-2{u_{xx}}^2\nu\Lambda_{u_xu_xu_{xx}}-2\nu {u_x}^2\Lambda_{uuu_{xx}}+3uu_x\Lambda_{xu_{xx}u_{xx}}-4\nu u_x\Lambda_{xuu_{xx}}$$

$$+2u_xu_{xx}\Lambda_{u_xu_{xx}u_{xx}}-4\nu u_x\Lambda_{uu_x}-4\nu u_{xx}\Lambda_{u_xu_x}-2\nu\Lambda_{uu_{xx}}+3uu_{xx}\Lambda_{u_{xx}u_{xx}}$$

$$+\frac{1}{3}\Lambda_{xtu_{xx}}+\frac{1}{6}\Lambda_{tu_x}+2u_{xx}\Lambda_{u_{xx}u_{xx}}+\frac{1}{3}\Lambda_{tu_xu_{xx}}+\frac{1}{3}u_x\Lambda_{tuu_{xx}}-\nu\Lambda_{tu_x}$$

$$+3{u_x}^2\Lambda_{u_{xx}u_{xx}}-2\nu\Lambda_{xxu_{xx}}-2\nu\Lambda_{txu_{xx}}+2\Lambda_{xu_{xx}u_{xx}}u_x-4\nu\Lambda_{xu_x}+2{u_x}^2\Lambda_{uu_{xx}u_{xx}}=0.$$

$$\frac{9}{2}u_xu_{xx}\Lambda_{u_{xx}}-3uu_{xx}\Lambda_{u_x}+\frac{1}{6}{u_x}^2\Lambda_{tuu}-\nu\Lambda_{xxx}+\frac{1}{3}u_x\Lambda_{txu}+2{u_{xx}}^2\Lambda_{u_xu_{xx}}$$

$$+\Lambda_{xxu_{xx}}u_x-\Lambda_{xu_x}u_x-\nu\Lambda_{xxt}+2u_{xx}\Lambda_{xu_{xx}}-{u_x}^2\Lambda_{uu_x}+2{u_x}^2\Lambda_{xuu_{xx}}$$

$$+3{u_x}^2\Lambda_{xu_{xx}}+\frac{1}{6}\Lambda_{tu_xu_x}+{u_x}^3\Lambda_{uuu_x}+\frac{1}{6}\Lambda_{tu}+3{u_x}^3\Lambda_{uu_{xx}}+\frac{1}{3}\Lambda_{txu_x}+\frac{1}{6}\Lambda_{txx}$$

$$
\begin{aligned}
&+\frac{1}{3}\Lambda_{tuu_x}-\nu{u_x}^3\Lambda_{uuu}-\nu u_{xx}\Lambda_{tu}-2\nu u_{xx}\Lambda_{txu_x}-\nu{u_{xx}}^3\Lambda_{u_xu_xu_x}-3\nu u_{xx}\Lambda_{xxu_x}\\
&+\frac{3}{2}uu_x\Lambda_{xxu_{xx}}-\frac{3}{2}uu_x\Lambda_{xu_x}-3\nu u_x\Lambda_{xxu}-3\nu{u_{xx}}^2\Lambda_{xu_xu_x}-3\nu{u_x}^2\Lambda_{xuu}\\
&+3u{u_x}^2\Lambda_{xuu_{xx}}-3\nu{u_{xx}}^2\Lambda_{uu_x}+\frac{3}{2}u{u_x}^3\Lambda_{uuu_{xx}}-\frac{3}{2}u{u_x}^2\Lambda_{uu_x}-u_xu_{xx}\Lambda_{u_xu_x}\\
&+3u_xu_{xx}\Lambda_{uu_{xx}}+2{u_x}^2u_{xx}\Lambda_{uu_xu_{xx}}+3{u_x}^2u_{xx}\Lambda_{u_xu_{xx}}+3u{u_{xx}}^2\Lambda_{u_xu_{xx}}+u_x{u_{xx}}^2\Lambda_{u_xu_xu_{xx}}\\
&+3uu_{xx}\Lambda_{xu_{xx}}+2u_xu_{xx}\Lambda_{xu_xu_{xx}}-\nu{u_{xx}}^2\Lambda_{tu_xu_x}-3\nu u_{xx}\Lambda_{xu}-\nu{u_x}^2\Lambda_{tuu}-2\nu\Lambda_{txu}\\
&+3u{u_x}^2u_{xx}\Lambda_{uu_xu_{xx}}-\frac{3}{2}u\Lambda_x-2u_{xx}\Lambda_{u_x}-\Lambda_t-\Lambda_x-\frac{3}{2}{u_x}^2\Lambda_{u_x}-6\nu u_x\Lambda_{xuu_x}u_{xx}\\
&+\frac{9}{2}u_xu_{xx}\Lambda_{uu_{xx}}-3\nu u_x{u_{xx}}^2\Lambda_{uu_xu_x}-3\nu{u_x}^2u_{xx}\Lambda_{uuu_x}-3\nu u_xu_{xx}\Lambda_{uu}-2\nu uu_xu_{xx}\Lambda_{tuu_x}\\
&+\frac{3}{2}{u_{xx}}^2uu_x\Lambda_{u_xu_xu_{xx}}+3uu_xu_{xx}\Lambda_{xu_xu_{xx}}-\frac{3}{2}u_{xx}uu_x\Lambda_{u_xu_x}=0.
\end{aligned}
$$

Solving the above system we obtain the multiplier

$$
\Lambda=c_1u-c_2 \tag{2.50}
$$

with c_1 and c_2 arbitrary constants. We now recall and apply the property of a multiplier given in (1.47), that is,

$$
\begin{aligned}
&(c_1u-c_2)\left\{u_t+u_x+\frac{3}{2}uu_x+\nu u_{xxx}-\left(\frac{1}{6}-\nu\right)u_{txx}\right\}=T^t_t+u_tT^t_u+u_{tx}T^t_{u_x}\\
&+u_{txx}T^t_{u_{xx}}+T^x_x+u_xT^x_u+u_{xx}T^x_{u_x}+u_{xxx}T^x_{u_{xx}}.
\end{aligned} \tag{2.51}
$$

The solution to (2.51) yields the following two conservation laws:

$$
\begin{aligned}
T^t_1&=\frac{1}{2}u^2-\frac{1}{18}uu_{xx}+\frac{1}{3}\nu uu_{xx}-\frac{1}{6}\nu u_x^2+\frac{1}{36}u_x^2,\\
T^x_1&=\frac{1}{2}u^2+\frac{1}{2}u^3+\nu uu_{xx}-\frac{1}{9}uu_{tx}+\frac{2}{3}\nu uu_{tx}-\frac{1}{2}\nu u_x^2+\frac{1}{18}u_xu_t-\frac{1}{3}\nu u_xu_t;
\end{aligned}
$$

$$
\begin{aligned}
T^t_2&=u-\frac{1}{18}u_{xx}+\frac{1}{3}\nu u_{xx},\\
T^x_2&=u+\frac{3}{4}u^2+\nu u_{xx}+\frac{2}{3}\nu u_{tx}-\frac{1}{9}u_{tx}.
\end{aligned}
$$

2.3 Concluding remarks

In this chapter we presented the exact solutions of the KdV-BBM equation (2.1). Of note is that from direct integration we obtained Jacobi elliptic type solutions

(2.25) which were cnoidal in nature. The Kudryashov and extended Jacobi elliptic function expansion methods were also used to obtain exact solutions. Furthermore, we computed the optimal system of one-dimensional subalgebras and then used it to perform symmetry reductions and find group-invariant solutions for equation (2.1). We also noted that the solution (2.25) was the most general solution for (2.1) and the other solutions were contained in (2.25). Finally, we derived the conservation laws of the KdV-BBM equation by using the multiplier approach, obtaining two multipliers and consequently, two conserved vectors.

Chapter 3

Exact solutions and conservation laws of a modified equal width Burgers equation

In this chapter we study the modified equal width (MEW) Burgers equation [49]

$$u_t + \alpha u^2 u_x + \omega u_{xx} - \beta u_{txx} = 0, \tag{3.1}$$

which describes long wave propagation in nonlinear media with dispersion and dissipation. Here α, β are positive parameters and ω is a damping parameter.

The original Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{3.2}$$

a model for turbulent flow, first appeared in academic circles in [50, 51]. Over the years, various researchers have applied modifications to (3.2) and used many different methods to study the equation. For example, Arora et al. [52] performed the reduced differential transform method to find the numerical solution of the equal width wave equation and the exact analytical solution of the inviscid Burgers equation with initial conditions. The equal width equation

$$u_t - uu_x - u_{txx} = 0 \tag{3.3}$$

was first introduced in [53]. It describes amongst others, nonlinear dispersive waves in plasmas. Extensive work was done on equation (3.3) in constructing numerical solutions [54, 55]. In [56], quadratic B-splines were used on the generalised equal width equation

$$u_t + au^m u_x - bu_{txx} = 0 \tag{3.4}$$

to approximate the solution $u(x,t)$. Several researchers, for example [57, 58] have studied equation

$$u_t + au^2 u_x - bu_{txx} = 0, \tag{3.5}$$

which is a modified equal width equation, with slight variations to it.

In [49] the bifurcation behaviour and an external perturbation of the MEW-Burgers equation (3.1) was studied.

In this chapter we begin by determining the Lie point symmetries of MEW-Burgers equation (3.1) and then construct exact solutions by invoking the $(G'/G)-$expansion method. Furthermore, we derive the conservation laws by applying the new conservation theorem due to Ibragimov.

This work has been submitted for publication [59].

3.1 Lie point symmetries and similarity reduction of (3.1)

In this section we determine the Lie point symmetries and perform similarity reduction of (3.1). We begin the solution of this problem by determining its Lie point symmetries. Since equation (3.1) is a third-order NLPDE, we apply the third prolongation

$$X^{[3]}\{u_t + \alpha u^2 u_x + \omega u_{xx} - \beta u_{txx}\}|_{u_t+\alpha u^2 u_x + \omega u_{xx} - \beta u_{txx}=0} = 0, \tag{3.6}$$

where $X^{[3]}$ is the third prolongation given by (2.4). Expanding (3.6) yields the equation

$$\eta_t + 2u_x u_t \xi^2_u + 2\beta u_{ttx}\xi^1_t - 2\mu u_{t,x}\xi^1_x + 2\beta u^2_{tx}\xi^1_u + \beta u_{tx}\xi^2_{xx} - \mu u_x \xi^2_{xx} + \beta u_{tt}\xi^1_{xx}$$

$$
\begin{aligned}
&+ 2\mu u_x\eta_{xu} - \mu u_t\xi^1_{xx} - 2\beta u_{t,x}\eta_{xu} + \alpha u^2\eta_x + \beta\xi^1_{txx}u_t + \beta\xi^2_{txx}u_x + 2\beta\xi^1_{tx}u_{t,x} \\
&+ 2\beta u_{xx}\xi^2_{tx} - \beta u_t\eta_{xxu} - \beta\eta_{tuu}{u_x}^2 - \beta\eta_{tu}u_{xx} + \beta\xi^1_{xxu}{u_t}^2 + {u_x}^3\beta\xi^2_{tuu} \\
&+ 2\beta\xi^2_{txu}{u_x}^2 - 2\beta u_x\eta_{txu} - 2\mu{u_x}^2\xi^2_{xu} - \mu{u_x}^3\xi^2_{uu} + \mu{u_x}^2\eta_{uu} + \mu u_{xx}\xi^1_t + \beta u_{xxx}\xi^2_t \\
&+ 2\xi^2_x u_t - \xi^2_t u_x + \xi^1_u{u_t}^2 + \mu\eta_{xx} - \beta\eta_{txx} + \beta u_t u_x\xi^2_{uxx} + 3\beta u_x u_{xx}\xi^2_{tu} + \beta u_t{u_x}^2\xi^1_{tuu} \\
&+ 2\beta u_t u_x\xi^1_{txu} - \beta u_t{u_x}^2\eta_{uuu} + 2\alpha u u_x\eta + \beta{u_t}^2{u_x}^2\xi^1_{uuu} + \beta u_{xx}u_t\xi^1_{tu} + 2u_x u_{tx}\beta\xi^1_{tu} \\
&+ 4\beta u_t u_{tx}\xi^1_{xu} + 2\beta u_x u_{tt}\xi^1_{xu} - 2\mu u_t u_x\xi^1_{xu} + 2u_{xx}u_t\beta\xi^2_{xu} \\
&+ 4u_x\beta u_{tx}\xi^2_{xu} + 3\beta{u_x}^2u_{tx}\xi^2_{uu} - \beta u_{xx}u_t\eta_{uu} - 2\beta u_{tx}\eta_{uu} + \beta{u_t}^2u_{xx}\xi^1_{uu} + \beta{u_x}^2u_{tt}\xi^1_{uu} \\
&- \mu u_t{u_x}^2\xi^1_{uu} + \alpha u^2u_x\xi^2_x + \alpha u^2u_x\xi^1_t + 2\xi^2_u\alpha u^2{u_x}^2 + \xi^2_u\beta u_t u_{xxx} + 3\xi^2_u\beta u_{tx}u_{xx} \\
&- \alpha u^2u_t\xi^1_x + 2\beta u_{ttx}u_x\xi^1_u + \beta u_{xx}u_{tt}\xi^1_u + \mu u_{xx}u_t\xi^1_u - 2\mu u_x u_{tx}\xi^1_u + u_t{u_x}^3\beta\xi^2_{uuu} \\
&+ 2\beta{u_t}^2u_x\xi^1_{xuu} + 2\beta u_t{u_x}^2\xi^2_{xuu} - 2\beta u_t u_x\eta_{xuu} + 3\beta u_t u_x u_{xx}\xi^2_{uu} + 4\beta u_t u_x u_{tx}\xi^1_{uu} \\
&+ \alpha u^2u_t u_x\xi^1_u = 0.
\end{aligned} \tag{3.7}
$$

Separating (3.7) with respect to derivatives of u yields the system of ten linear partial differential equations

$$\xi^1_x = 0, \tag{3.8}$$

$$\xi^1_u = 0, \tag{3.9}$$

$$\xi^2_t = 0, \tag{3.10}$$

$$\xi^2_u = 0, \tag{3.11}$$

$$\eta_{uu} = 0, \tag{3.12}$$

$$\xi^2_{xx} - 2\eta_{xu} = 0 \tag{3.13}$$

$$\mu\xi^1_t - \beta\eta_{tu} = 0, \tag{3.14}$$

$$2\xi^2_x - \beta\eta_{xxu} = 0, \tag{3.15}$$

$$\mu\eta_{xx} - \beta\eta_{txx} + \eta_t + \alpha u^2\eta_x = 0. \tag{3.16}$$

$$\alpha u^2\xi^2_x + \alpha u^2\xi^1_t + 2\alpha u\eta + 2\mu\eta_{xu} - \mu\xi^2_{xx} - 2\beta\eta_{txu} = 0. \tag{3.17}$$

When we solve (3.8)–(3.17) for ξ^1, ξ^2 and η we obtain the values

$$\xi^1 = C_1, \tag{3.18}$$

$$\xi^2 = C_2, \tag{3.19}$$

$$\eta = 0. \tag{3.20}$$

We re-write (3.18)–(3.20) in vector form to obtain the infinitesimal generators

$$X_1 = \frac{\partial}{\partial t}, \tag{3.21}$$

$$X_2 = \frac{\partial}{\partial x}. \tag{3.22}$$

A linear combination of the Lie point symmetries (3.21) and (3.22) $X = X_1 + cX_2$ produces the associated Lagrange equations

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} \tag{3.23}$$

which upon solving yields the two invariants $z = x - ct$ and $U = u$, and thus the group-invariant solution $u = U(z)$. Taking U and z as the new dependent and independent variables respectively, we have

$$u_t = -cU'(z), \tag{3.24}$$

$$u_x = U'(z), \tag{3.25}$$

$$u_{xx} = U''(z), \tag{3.26}$$

$$u_{txx} = -cU'''(z). \tag{3.27}$$

The equations (3.24) – (3.27) transform (3.1) into the third-order nonlinear ODE

$$\beta cU'''(z) - cU'(z) + \omega U''(z) + \alpha U(z)^2 U'(z) = 0. \tag{3.28}$$

3.1.1 Application of the $(G'/G)-$expansion method on (3.28)

We now implement the $(G'/G)-$expansion method on (3.28).

The balancing procedure yields $M = 1$, thus (1.29) becomes

$$U(z) = A_0 + A_1 \left(\frac{G'(z)}{G(z)} \right). \tag{3.29}$$

Substituting (3.29) into (3.28) and using (1.30 results in the polynomial in (G'/G)

$$A_1\lambda\mu\omega + A_1 c\mu - 2A_1\beta c\mu^2 - \alpha A_0^2 A_1 \mu - A_1 \beta c \lambda^2 \mu$$

$$+\left(A_1c\lambda - 8A_1\beta c\lambda\mu - A_1\beta c\lambda^3 + 2A_1\mu\omega + A_1\lambda^2\omega - \alpha A_0^2A_1\lambda - 2\alpha A_0A_1^2\mu\right)\left(\frac{G'(z)}{G(z)}\right)$$
$$+\left(A_1c - 7A_1\beta c\lambda^2 - 8A_1\beta c\mu - \alpha A_0^2A_1 - 2\alpha A_0A_1^2\lambda - \alpha A_1^3\mu + 3A_1\lambda\omega\right)\left(\frac{G'(z)}{G(z)}\right)^2$$
$$+\left(2A_1\omega - 2\alpha A_0A_1^2 - \alpha A_1^3\lambda - 12A_1\beta c\lambda\right)\left(\frac{G'(z)}{G(z)}\right)^3$$
$$-\left(6A_1\beta c + \alpha A_1^3\right)\left(\frac{G'(z)}{G(z)}\right)^4 = 0. \tag{3.30}$$

Equating the coefficients of $(G'/G)^i$ yields the overdetermined system of five algebraic equations:

$$\left(\frac{G'(z)}{G(z)}\right)^4 : \alpha A_1^3 + 6A_1\beta c = 0,$$
$$\left(\frac{G'(z)}{G(z)}\right)^3 : \alpha A_1^3\lambda + 2\alpha A_0A_1^2 + 12A_1\beta c\lambda - 2A_1\omega = 0,$$
$$\left(\frac{G'(z)}{G(z)}\right)^2 : 2\alpha A_0A_1^2\lambda + \alpha A_1^3\mu + \alpha A_0^2A_1 + 7A_1\beta c\lambda^2 + 8A_1\beta c\mu - A_1c - 3A_1\lambda\omega = 0,$$
$$\left(\frac{G'(z)}{G(z)}\right)^1 : \alpha A_0^2A_1\lambda + 2\alpha A_0A_1^2\mu + A_1\beta c\lambda^3 + 8A_1\beta c\lambda\mu - A_1c\lambda - A_1\lambda^2\omega - 2A_1\mu\omega = 0,$$
$$\left(\frac{G'(z)}{G(z)}\right)^0 : A_1\lambda\mu\omega + A_1c\mu - \alpha A_0^2A_1\mu - A_1\beta c\lambda^2\mu - 2A_1\beta c\mu^2 = 0.$$

Using Mathematica two solutions of the above system of algebraic equations are

Solution set 1

$$\begin{aligned} A_0 &= \sqrt{\frac{-2\lambda\omega}{3\alpha}}, \\ A_1 &= \sqrt{\frac{-6\omega}{\lambda\alpha}}, \\ c &= \frac{2}{3\lambda}(3\mu - \lambda^2). \end{aligned} \tag{3.31}$$

Solution set 2

$$\begin{aligned} A_0 &= -\sqrt{\frac{2\omega\sqrt{\mu}}{\alpha}}, \\ A_1 &= 3\sqrt{\frac{2\omega}{\alpha\sqrt{\mu}}}, \\ \beta &= \frac{3}{4\mu}, \\ c &= \frac{-4\omega\sqrt{\mu}}{3}. \end{aligned} \tag{3.32}$$

Thus corresponding to the solution set 1 above, we have the following three types of solutions for our MEW-Burgers equation (3.1)

Case 1.1 $M = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u(t,x) = \left(1 + \frac{3}{\lambda}\right)\sqrt{\frac{-2\lambda\omega}{3\alpha}}\left(\Delta_1 \frac{A\cosh(\Delta_1 z) + B\sinh(\Delta_1 z)}{A\sinh(\Delta_1 z) + B\cosh(\Delta_1 z)} - \frac{\lambda}{2}\right), \tag{3.33}$$

where $\Delta_1 = \sqrt{M}/2$, $z = x - 2(3\mu - \lambda^2)t/(3\lambda)$ and A and B are constants.

Case 1.2 $M = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$u(t,x) = \left(1 + \frac{3}{\lambda}\right)\sqrt{\frac{-2\lambda\omega}{3\alpha}}\left(\Delta_2 \frac{A\sin(\Delta_2 z) + B\cos(\Delta_2 z)}{A\cos(\Delta_2 z) + B\sin(\Delta_2 z)} - \frac{\lambda}{2}\right), \tag{3.34}$$

where $\Delta_2 = \sqrt{-M}/2$, $z = x - 2(3\mu - \lambda^2)t/(3\lambda)$ and A and B are constants.

Case 1.3 $M = \lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(t,x) = \left(1 + \frac{3}{\lambda}\right)\sqrt{\frac{-2\lambda\omega}{3\alpha}}\left(\frac{B}{Bz + A} - \frac{\lambda}{2}\right). \tag{3.35}$$

where $z = x - 2(3\mu - \lambda^2)t/(3\lambda)$ and A and B are constants.

Similarly, considering the solution set 2 we obtain the following three types of solutions:

Case 2.1 $M = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u(t,x) = (1 + 4\beta\lambda)\sqrt{\frac{2\sqrt{\mu}\omega}{\alpha}}\left(\frac{\lambda}{2} - \Delta_1 \frac{A\cosh(\Delta_1 z) + B\sinh(\Delta_1 z)}{A\sinh(\Delta_1 z) + B\cosh(\Delta_1 z)}\right),$$

where $\Delta_1 = \sqrt{M}/2$, $z = x + 4\sqrt{\mu}\omega t/3$ and A and B are constants.

Case 2.2 $M = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$u(t,x) = (1 + 4\beta\lambda)\sqrt{\frac{2\sqrt{\mu}\omega}{\alpha}}\left(\frac{\lambda}{2} - \Delta_2 \frac{A\sin(\Delta_2 z) + B\cos(\Delta_2 z)}{A\cos(\Delta_2 z) + B\sin(\Delta_2 z)}\right),$$

where $\Delta_2 = \sqrt{-M}/2$, $z = x + 4\sqrt{\mu}\omega t/3$ and A and B are constants.

Case 2.3 $M = \lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(t,x) = (1 + 4\beta\lambda)\sqrt{\frac{2\sqrt{\mu}\omega}{\alpha}}\left(\frac{B}{Bz + A} - \frac{\lambda}{2}\right),$$

where $z = x + 4\sqrt{\mu}\omega t/3$ and A and B are constants.

The solution profile of (3.33) for $\lambda = -0.09, A = 0, B = 1, \mu = 0.01, \omega = 0.03, \alpha = 1$ is presented in Figure 3.1, whereas the solution profile of (3.35) for $\lambda = -0.8, A = 0.4, B = 3, \mu = 0.2, \omega = 0.01, \alpha = 0.1$ is given in Figure 3.2.

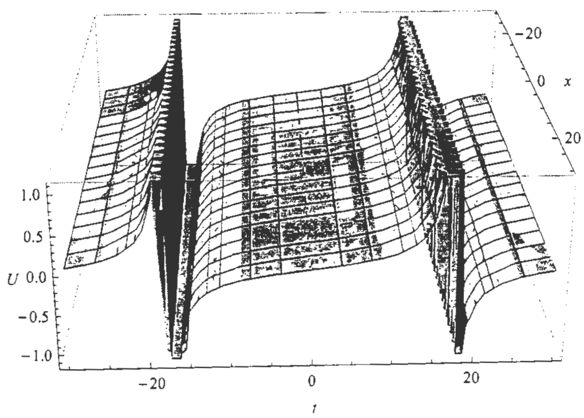

Figure 3.1: Profile of solution (3.33)

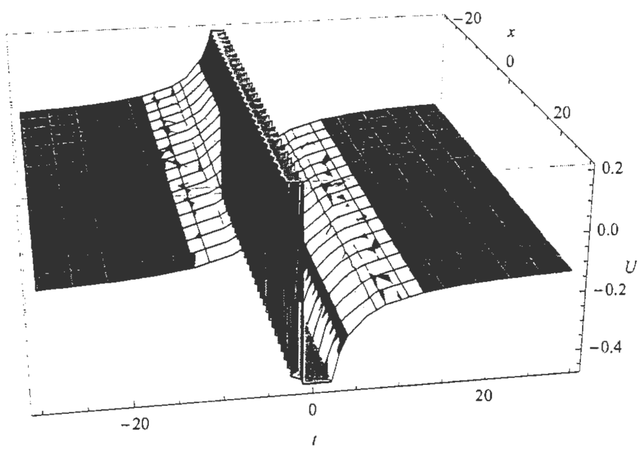

Figure 3.2: Profile of solution (3.35)

3.2 Conservation laws of (3.1)

In this section we derive conservation laws for MEW-Burgers equation (3.1) using the new conservation theorem due to Ibragimov [41].

We begin by determining the adjoint equation using the formula

$$F^* \equiv \frac{\delta}{\delta u}\{v(u_t + \alpha u^2 u_x + \omega u_{xx} - \beta u_{txx})\} = 0. \tag{3.36}$$

In this case the Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_t D_x^2 \frac{\partial}{\partial u_{txx}}, \tag{3.37}$$

where the total differential operators D_t and D_x are

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t}\cdots.$$

Equation (3.36) yields the adjoint equation

$$F^* \equiv v_t + \alpha u^2 v_x - \omega v_{xx} - \beta v_{txx} = 0. \tag{3.38}$$

We now consider the system (3.1) and its adjoint (3.38). The Lagrangian of this system is

$$\mathcal{L} = v(u_t + \alpha u^2 u_x + \omega u_{xx} - \beta u_{txx}) \tag{3.39}$$

which is equivalent to the second-order Lagrangian

$$\mathcal{L} = vu_t + \alpha v u^2 u_x + \omega v u_{xx} + \beta v_t u_{xx}. \tag{3.40}$$

According to the Lagrangian (3.40) and (1.55), the expanded formulae for obtaining the conserved vectors corresponding to an infinitesimal generator are

$$C^t = \xi^1\mathcal{L} + W^1\frac{\partial\mathcal{L}}{\partial u_t} + W^2\frac{\partial\mathcal{L}}{\partial v_t}, \tag{3.41}$$
$$C^x = \xi^2\mathcal{L} + W^1\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_x\frac{\partial\mathcal{L}}{\partial u_{xx}}\right] + D_x(W^1)\frac{\partial\mathcal{L}}{\partial u_{xx}}. \tag{3.42}$$

Recall that equation (3.1) has the two translation symmetries

$$X_1 = \frac{\partial}{\partial t}, \tag{3.43}$$
$$X_2 = \frac{\partial}{\partial x}. \tag{3.44}$$

Let us first consider the infinitesimal generator (3.43). It can be easily shown that extending the generator X_1 to the new variable v produces the same translation symmetry $X_1 = \partial/\partial t$. In order to determine the value of λ we apply the equation

$$X(F) = \lambda(F), \tag{3.45}$$

which yields $\lambda = 0$. Again,

$$D_t(\xi^1) = 0. \tag{3.46}$$

Consequently, $\eta^* = 0$ and the operator admitted by the adjoint equation (3.38) is

$$Y = \frac{\partial}{\partial t}. \tag{3.47}$$

We now use (3.47) to compute the Lie characteristic functions

$$W^1 = -u_t, \quad W^2 = -v_t.$$

The conserved vector for the system (3.1) and (3.38) corresponding to the infinitesimal generator (3.43) can thus be obtained by using (3.41) and (3.42). The following conserved vector is obtained

$$\begin{aligned} C_1^t &= \alpha v u^2 u_x + \omega v u_{xx}, \\ C_1^x &= \omega v_x u_t + \beta v_{tx} u_t - \omega v u_{xt} - \alpha v u^2 u_t - \beta u_{xt} v_t. \end{aligned}$$

Similarly, we compute the conserved vector corresponding to the infinitesimal generator (3.44). In this case $W^1 = -u_x$ and $W^2 = -v_x$ and the conserved vector thus rendered by applying (3.41) and (3.42) are

$$\begin{aligned} C_2^t &= -v u_x - \beta v_x u_{xx}, \\ C_2^x &= v u_t + \omega v_x u_x + \beta v_{tx} u_x. \end{aligned}$$

3.3 Conclusion

In this chapter we determined the Lie point symmetries of the MEW-Burgers equation (3.1). We then used the Lie point symmetries along with the $(G'/G)-$expansion method to obtain travelling wave solutions. The solutions obtained were hyperbolic, trigonometric and rational function solutions.

We proceeded further and derived two conservation laws of energy and momentum using the new conservation theorem due to Ibragimov.

Chapter 4

Concluding remarks

Many real-world physical systems, which include fluid mechanics, elasticity, thermodynamics, biology and gas dynamics are modelled by nonlinear partial differential equations. It is therefore important to study such systems from the point of view of finding exact solutions and conservation laws. In this dissertation we studied two nonlinear partial differential equations namely the Korteweg-de Vries-Bejamin-Bona-Mahony equation and the modified equal width Burgers equation.

In Chapter one we presented the relevant literature which was used in this dissertation. Various methods for finding the exact solutions of PDEs were described and two methods for deriving conservation laws were discussed.

Chapter two studied the Korteweg-de Vries-Bejamin-Bona-Mahony equation. Of note is that from direct integration we obtained Jacobi elliptic type solutions (2.25) which were cnoidal in nature. The Kudryashov and extended Jacobi elliptic function expansion methods were also used to obtain exact solutions. Furthermore, we computed the optimal system of one-dimensional subalgebras and then used it to obtain symmetry reductions and group-invariant solutions. Finally, we derived the conservation laws of the KdV-BBM equation by using the multiplier approach, which resulted in two multipliers and consequently, two conserved vectors.

In Chapter three. the modified equal width Burgers equation was investigated. Lie symmetry analysis along with the $(G'/G)-$expansion method were employed to find exact travelling wave solutions of the equation. In addition, we derived the conservation laws of energy and momentum of the underlying equation using the new conservation theorem due to Ibragimov.

In future work, we plan to study the modified equal width Burgers equation with the power law nonlinearity.

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