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Cyclic pairs and common best proximity points in uniformly convex Banach spaces

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Abstract: In this article, we survey the existence, uniqueness and convergence of a common best proximity point for a cyclic pair of mappings, which is equivalent to study of a solution for a nonlinear programming problem in the setting of uniformly convex Banach spaces. Finally, we provide an extension of Edelstein's fixed point theorem in strictly convex Banach spaces. Examples are given to illustrate our main conclusions.

Keywords: Common best proximity point, Best proximity pair, Cyclic contraction, Uniformly convex Banach space

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In 2003, an interesting extension of *Banach contraction principle* was given as below.

Theorem 0.1 ([1]). *Let A and B be nonempty and closed subsets of a complete metric space (X, d) . Suppose that $T : A \cup B \rightarrow A \cup B$ is a cyclic mapping, that is, $T(A) \subseteq B$ and $T(B) \subseteq A$, such that $d(Tx, Ty) \leq \alpha d(x, y)$ for some $\alpha \in [0, 1]$ and for all $x \in A, y \in B$. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

If in Theorem 0.1 $A \cap B = \emptyset$, then the fixed point equation $Tx = x$ does not have any solution. In this situation we have the following notion. We should mention that the existence of best approximant for non-self mappings was first studied by Ky Fan as below.

Theorem 0.2 ([2]). *Let A be a nonempty, compact and convex subset of a normed linear space X and $T : A \rightarrow X$ be a continuous mapping. Then there exists a point $x^* \in A$ such that*

$$\|x^* - Tx^*\| = \text{dist}(\{Tx^*\}, A).$$

We now state the notion of *best proximity points* for cyclic mappings.

Definition 0.3. *Let A, B be nonempty subsets of a metric space (X, d) and $T : A \cup B \rightarrow A \cup B$ be a cyclic mapping. A point $p \in A \cup B$ is called a best proximity point of T if*

$$d(p, Tp) = \text{dist}(A, B),$$

where $\text{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

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Notice that best proximity point results have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A \cup B} d(x, Tx), \quad (1)$$

has at least one solution, where T is a cyclic mapping defined on $A \cup B$.

In 2006, a class of cyclic mappings was introduced in [3] as follows.

Definition 0.4 ([3]). *Let A and B be nonempty subsets of a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a cyclic contraction provided that T is cyclic on $A \cup B$ and*

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \text{dist}(A, B)$$

for some $\alpha \in [0, 1[$ and for all $(x, y) \in A \times B$.

After that in 2009, a generalized class of cyclic contractions was introduced as below.

Definition 0.5 ([4]). *Let A and B be nonempty subsets of a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a cyclic φ -contraction if T is cyclic on $A \cup B$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing function and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(\text{dist}(A, B)),$$

for all $(x, y) \in A \times B$.

It is remarkable to note that the class of cyclic φ -contraction mappings contains the class of cyclic contractions as a subclass by considering $\varphi(t) = (1 - \alpha)t$ for $t \geq 0$ and for some $\alpha \in [0, 1[$.

Next theorem guarantees the existence, uniqueness and convergence of a best proximity point for cyclic φ -contractions in uniformly convex Banach spaces.

Theorem 0.6 (Theorem 8 of [4]). *Let A and B be nonempty subsets of a uniformly convex Banach space X such that A is closed and convex, and let $T : A \cup B \rightarrow A \cup B$ be a cyclic φ -contraction mapping. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $p \in A$ such that $x_{2n} \rightarrow p$ and $\|p - Tp\| = \text{dist}(A, B)$.*

Recently, many authors have studied the existence of best proximity points for various classes of cyclic mappings which one can refer to [5-18] for more information.

In the current paper, we discuss sufficient conditions which ensure the existence and uniqueness of a solution for a nonlinear programming problem. Then we obtain a similar result of Theorem 0.6 for another class of cyclic mappings in uniformly convex Banach spaces. We also study the existence of best proximity pairs for noncyclic contractive mappings in strictly convex Banach spaces and so we present a generalization of Edelstein's fixed point theorem.

1 Preliminaries

In this section, we recall some notions which will be used in our main discussions.

Definition 1.1. *A Banach space X is said to be*

- (i) *uniformly convex if there exists a strictly increasing function $\delta : [0, 2] \rightarrow [0, 1]$ such that for every $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$, the following implication holds:*

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geq r \end{cases} \Rightarrow \left\| \frac{x + y}{2} - p \right\| \leq (1 - \delta(\frac{r}{R}))R;$$

(ii) *strictly convex if for every $x, y, p \in X$ and $R > 0$, the following implication holds:*

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \left\| \frac{x + y}{2} - p \right\| < R.$$

It is well known that Hilbert spaces and l^p spaces ($1 < p < \infty$) are uniformly convex Banach spaces. Also, the Banach space l^1 with the norm

$$\|x\| = \sqrt{\|x\|_1 + \|x\|_2}, \quad \forall x \in l^1,$$

where, $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms on l^1 and l^2 , respectively, is strictly convex which is not uniformly convex (see [19] for more details).

Let A and B be nonempty subsets of a normed linear space X . We shall say that a pair (A, B) satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex. We define

$$\Psi := \{\psi : [0, \infty[\rightarrow [0, \infty[; \psi \text{ is upper semi-continuous from the right and } 0 \leq \psi(t) < t, \forall t > 0\},$$

$$\|x - y\|^* := \|x - y\| - \text{dist}(A, B), \quad \forall (x, y) \in A \times B,$$

$$A_0 := \{x \in A : \|x - y\| = \text{dist}(A, B), \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : \|x - y\| = \text{dist}(A, B), \text{ for some } x \in A\}.$$

Notice that if (A, B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X , then (A_0, B_0) is also nonempty, closed and convex pair in X . We say that the pair (A, B) is *proximal* if $A = A_0$ and $B = B_0$. Also, the *metric projection operator* $\mathcal{P}_A : X \rightarrow 2^A$ is defined as $\mathcal{P}_A(x) := \{y \in A : \|x - y\| = \text{dist}(\{x\}, A)\}$, where 2^A denotes the set of all subsets of A . It is well known that if A is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X , then the metric projection $\mathcal{P}_A$ is single valued from X to A .

Definition 1.2 ([20]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if and only if*

$$\begin{cases} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It was announced in [21] that every nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the P-property.

Next two lemmas will be used in the sequel.

Lemma 1.3 ([3]). *Let A be a nonempty, closed and convex subset and B be a nonempty and closed subset of a uniformly convex Banach space X . Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and let $\{y_n\}$ be a sequence in B such that*

- (i) $\|z_n - y_n\| \rightarrow \text{dist}(A, B)$,
 - (ii) *for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ so that for all $m > n > N_0$, $\|x_m - y_n\| \leq \text{dist}(A, B) + \varepsilon$.*
- Then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\|x_m - z_n\| \leq \varepsilon$ for any $m > n > N_1$.*

Lemma 1.4 ([3]). *Let A be a nonempty, closed and convex subset and B be a nonempty and closed subset of a uniformly convex Banach space X . Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and let $\{y_n\}$ be a sequence in B satisfying*

- (i) $\|x_n - y_n\| \rightarrow \text{dist}(A, B)$,
- (ii) $\|z_n - y_n\| \rightarrow \text{dist}(A, B)$.

Then $\|x_n - z_n\| \rightarrow 0$.

2 A nonlinear programming problem: common best proximity point

Let (A, B) be a nonempty pair in a normed linear space X and $T, S : A \cup B \rightarrow A \cup B$ be two cyclic mappings. A point $p \in A \cup B$ is called a *common best proximity point* for the cyclic pair $(T; S)$ provided that

$$\|p - Tp\| = \text{dist}(A, B) = \|p - Sp\|.$$

In view of the fact that

$$\min\{\|x - Tx\|, \|x - Sx\|\} \geq \text{dist}(A, B), \quad \forall x \in A \cup B,$$

the optimal solution to the problem of

$$\min_{x \in A \cup B} \{\|x - Tx\|, \|x - Sx\|\} \quad (2)$$

will be the one for which the value $\text{dist}(A, B)$ is attained. Thereby, a point $p \in A \cup B$ is a common best proximity point for the cyclic pair $(T; S)$ if and only if that is a solution of the minimization problem (2).

In this section, we provide some sufficient conditions in order to study the existence of a solution for (2). We begin with the following result.

Theorem 2.1. *Let (A, B) be a nonempty, closed, and convex pair in a uniformly convex Banach space X and $(T; S)$ be a cyclic pair defined on $A \cup B$ such that*

- (i) $S(A) \subseteq T(A) \subseteq B$ and $S(B) \subseteq T(B) \subseteq A$,
- (ii) $\|Sx - Sy\|^* \leq \psi(\|Tx - Ty\|^*)$, for all $(x, y) \in A \times B$ where $\psi \in \Psi$,
- (iii) S and T commute,
- (iv) $T|_A$ is continuous.

Then $(T; S)$ has a unique common best proximity point in B .

Proof. Choose $x_0 \in A$. Since $S(A) \subseteq T(A)$, there exists $x_1 \in A$ such that $Sx_0 = Tx_1$. Again, by the fact that $S(A) \subseteq T(A)$, there exists $x_2 \in A$ such that $Sx_1 = Tx_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A such that $Sx_n = Tx_{n+1}$. It follows from the conditions (ii) and (iii) that

$$\begin{aligned} \|Sx_n - SSx_n\|^* &\leq \psi(\|Tx_n - TSx_n\|^*) = \psi(\|Tx_n - STx_n\|^*) \\ &= \psi(\|Sx_{n-1} - SSx_{n-1}\|^*) \leq \|Sx_{n-1} - SSx_{n-1}\|^*, \end{aligned}$$

that is, $\{\|Sx_n - SSx_n\|^*\}$ is a decreasing sequence of nonnegative real numbers and hence it converges. Let r be the limit of $\|Sx_n - SSx_n\|^*$. We claim that $r = 0$. Suppose that $r > 0$. Then

$$\limsup_{n \rightarrow \infty} \|Sx_n - SSx_n\|^* \leq \limsup_{n \rightarrow \infty} \psi(\|Sx_{n-1} - SSx_{n-1}\|^*),$$

which implies that $r \leq \psi(r)$ which is a contradiction. So, $\|Sx_n - SSx_n\| \rightarrow \text{dist}(A, B)$. Moreover,

$$\|Sx_{n+1} - SSx_n\|^* \leq \psi(\|Tx_{n+1} - TSx_n\|^*) = \psi(\|Sx_n - SSx_{n-1}\|^*),$$

$$\|Sx_n - SSx_{n+1}\|^* \leq \psi(\|Tx_n - TSx_{n+1}\|^*) = \psi(\|Sx_{n-1} - SSx_n\|^*).$$

By a similar argument we conclude that

$$\|Sx_{n+1} - SSx_n\| \rightarrow \text{dist}(A, B), \quad \|Sx_n - SSx_{n+1}\| \rightarrow \text{dist}(A, B).$$

Let us prove that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $m > n > N_0$,

$$\|Sx_m - SSx_n\|^* < \varepsilon.$$

Suppose the contrary. Then there exists $\xi > 0$ such that for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ for which

$$\|Sx_{m_k} - SSx_{n_k}\|^* \geq \varepsilon, \quad \|Sx_{m_k-1} - SSx_{n_k}\|^* < \varepsilon.$$

We now have

$$\varepsilon \leq \|Sx_{m_k} - SSx_{n_k}\|^* \leq \|Sx_{m_k} - SSx_{m_k-1}\| + \|Sx_{m_k-1} - SSx_{n_k}\|^*.$$

Letting $k \rightarrow \infty$ we obtain $\|Sx_{m_k} - SSx_{n_k}\|^* \rightarrow \varepsilon$. Besides,

$$\begin{aligned} \|Sx_{m_k} - SSx_{n_k}\|^* &\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \|Sx_{m_k+1} - SSx_{n_k+1}\|^* + \|SSx_{n_k+1} - SSx_{n_k}\| \\ &\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \psi(\|Tx_{m_k+1} - TSx_{n_k+1}\|^*) + \|SSx_{n_k+1} - SSx_{n_k}\| \\ &= \|Sx_{m_k} - Sx_{m_k+1}\| + \psi(\|Sx_{m_k} - SSx_{n_k}\|^*) + \|SSx_{n_k+1} - SSx_{n_k}\|. \end{aligned}$$

Therefore,

$$\limsup_{k \rightarrow \infty} \|Sx_{m_k} - SSx_{n_k}\|^* \leq \limsup_{k \rightarrow \infty} \psi(\|Sx_{m_k} - SSx_{n_k}\|^*),$$

and from the upper semi-continuity of ψ we have $\varepsilon \leq \psi(\varepsilon)$ which is a contradiction. Using Lemma 1.3, $\{Sx_n\}$ is a Cauchy sequence and converges to $q \in B$. So $Tx_n \rightarrow q$. By this reality that $T|_A$ is continuous, $STx_n = TSx_n \rightarrow Tq$ and so $TTx_n \rightarrow Tq$. We have

$$\|STx_n - Sx_n\|^* \leq \psi(\|TTx_n - Tx_n\|^*).$$

Letting $\limsup$ in above relation when $n \rightarrow \infty$, then by the fact that ψ is upper semi-continuous from the right, we obtain $\|Tq - q\|^* \leq \psi(\|Tq - q\|^*)$, which implies that $\|q - Tq\|^* = 0$. Also,

$$\|Sx_n - Sq\|^* \leq \psi(\|Tx_n - Tq\|^*),$$

which concludes that $\|q - Sq\|^* \leq \psi(\|q - Tq\|^*) = \psi(0) = 0$. Thus

$$\|q - Tq\| = \text{dist}(A, B) = \|q - Sq\|,$$

and so $q \in B$ is a common best proximity point for the cyclic pair $(T; S)$. Now assume that $q' \in B$ is another common best proximity point for the cyclic pair $(T; S)$. Then $\|q' - Tq'\| = \text{dist}(A, B) = \|q' - Sq'\|$. By the fact that (A, B) has the P-property, $Tq = Sq$ and $Tq' = Sq'$. We have

$$\|STq - Sq'\|^* \leq \psi(\|TTq - Tq'\|^*) = \psi(\|TSq - Tq'\|^*) = \psi(\|STq - Sq'\|^*),$$

which implies that $\|STq - Sq'\|^* = 0$. Equivalently, $\|STq - Sq\|^* = 0$. Therefore,

$$\|q - Sq\| = \|STq - Sq\| = \text{dist}(A, B) = \|q' - Sq'\| = \|STq - Sq'\|.$$

Again since (A, B) has the P-property, $q = STq = q'$ and the proof is complete. $\square$

The following corollary is the main result of [22].

Corollary 2.2. *Let (A, B) be a nonempty, closed, and convex pair in a uniformly convex Banach space X and $(T; S)$ be a cyclic pair defined on $A \cup B$ such that*

- (i) $S(A) \subseteq T(A) \subseteq B$ and $S(B) \subseteq T(B) \subseteq A$,
- (ii) $\|Sx - Sy\|^* \leq k\|Tx - Ty\|^*$, for some $k \in [0, 1)$ and for all $(x, y) \in A \times B$,
- (iii) S and T commute,
- (iv) $T|_A$ is continuous.

Then $(T; S)$ has a unique common best proximity point in B .

Proof. It is sufficient to consider $\psi(t) = kt$ in Theorem 2.1. $\square$

Remark 2.3. We mention that Theorem 2.1 can be proved in complete metric spaces by using a geometric notion of property UC on closed pairs, which is a property for closed and convex pairs in uniformly convex Banach spaces (see Theorem 3.9 of [22]). Since we will use the other geometric notions of uniformly convex Banach spaces, we prefer to prove Theorem 2.1 in uniformly convex Banach spaces.

The following best proximity point theorem is a different version of Theorem 0.6.

Theorem 2.4. Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X and $S : A \cup B \rightarrow A \cup B$ be a cyclic mapping such that

$$\|Sx - Sy\|^* \leq \psi(\|x - y\|^*),$$

for all $(x, y) \in A \times B$ where $\psi \in \Psi$. Then S has a unique best proximity point in B .

Proof. As we mentioned, (A_0, B_0) is nonempty, closed and convex. Note that the mapping S is cyclic on $A_0 \cup B_0$. Indeed, if $x \in A_0$ then there exists a unique $y \in B_0$ such that $\|x - y\| = \text{dist}(A, B)$ or $\|x - y\|^* = 0$. Thus $\|Sx - Sy\|^* \leq \psi(\|x - y\|^*) = \psi(0) = 0$ and so, $\|Sx - Sy\| = \text{dist}(A, B)$ which implies that $Sx \in B_0$, that is, $S(A_0) \subseteq B_0$. Similarly, $S(B_0) \subseteq A_0$. Now consider the mapping $\mathcal{P} : A_0 \cup B_0 \rightarrow A_0 \cup B_0$ defined with

$$\mathcal{P}x = \begin{cases} \mathcal{P}_{B_0}(x) & \text{if } x \in A_0, \\ \mathcal{P}_{A_0}(x) & \text{if } x \in B_0. \end{cases}$$

It is clear that $\mathcal{P}$ is cyclic on $A_0 \cup B_0$. We have two following observations.

- $\mathcal{P}$ is surjective:

Let $y \in B_0$. Then there exists a unique element $x \in A_0$ such that $\|x - y\| = \text{dist}(A, B)$. Therefore,

$$\|x - y\| = \text{dist}(A, B) \leq \|x - \mathcal{P}x\| = \|x - \mathcal{P}_{B_0}x\|,$$

which implies that $y = \mathcal{P}x$ by the uniform convexity of X . Thus $\mathcal{P}(A_0) = B_0$. Similarly, we can see that $\mathcal{P}(B_0) = A_0$.

- $\mathcal{P}$ is an isometry:

Assume that $(x, y) \in A_0 \times B_0$. Then we have $\|x - \mathcal{P}x\| = \text{dist}(A, B) = \|y - \mathcal{P}y\|$. In view of the fact that (A, B) has the P-property, $\|x - y\| = \|\mathcal{P}x - \mathcal{P}y\|$ and the result follows.

- S and $\mathcal{P}$ commute on $A_0 \cup B_0$:

Suppose $x \in A_0$. Then there exists a unique $y \in B_0$ such that $\|x - y\| = \text{dist}(A, B)$. Thus $x = \mathcal{P}y$ and $y = \mathcal{P}x$. Hence, $\|Sx - Sy\| = \text{dist}(A, B)$ which implies that $Sy = \mathcal{P}Sx$ and so, $S\mathcal{P}x = \mathcal{P}Sx$. Similar argument holds when $x \in B_0$, that is, S and $\mathcal{P}$ are commuting.

- $\mathcal{P}|_{A_0}$ is continuous:

Let $\{x_n\}$ be a sequence in A_0 such that $x_n \rightarrow x \in A_0$. We have

$$\|x_n - \mathcal{P}x\| \leq \|x_n - x\| + \|x - \mathcal{P}x\| \rightarrow \text{dist}(A, B),$$

$$\|x_n - \mathcal{P}x_n\| = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

Now using Lemma 1.4 we conclude that $\|\mathcal{P}x_n - \mathcal{P}x\| \rightarrow 0$, or $\mathcal{P}x_n \rightarrow \mathcal{P}x$.

Finally, we note that

$$\|Sx - Sy\|^* \leq \psi(\|x - y\|^*) = \psi(\|\mathcal{P}x - \mathcal{P}y\|^*),$$

for any $(x, y) \in A_0 \times B_0$. Thereby, all of the assumptions of Theorem 2.1 are satisfied and then the cyclic pair $(S; \mathcal{P})$ has a unique common best proximity point such as $q \in B_0$ and this completes the proof. $\square$

Let us illustrate Theorem 2.1 with the following example.

Example 2.5. Suppose $X = l^2$ and let $A = \{te_1 + e_2 : 0 \leq t \leq \frac{1}{4}\}$ and $B = \{e_2 + se_3 : 0 \leq s \leq \frac{1}{4}\}$. Define the cyclic pair $(T; S)$ as below

$$S(te_1 + e_2) = e_2 + t^2e_3 \quad \text{and} \quad S(e_2 + se_3) = s^2e_1 + e_2, \quad t, s \in [0, \frac{1}{4}],$$

$$T(te_1 + e_2) = e_2 + te_3 \quad \text{and} \quad T(e_2 + se_3) = se_1 + e_2, \quad t, s \in [0, \frac{1}{4}].$$

It is clear that $S(A) \subseteq T(A) = B$ and $S(B) \subseteq T(B) = A$. Also,

$$TS(te_1 + e_2) = T(e_2 + t^2e_3) = t^2e_1 + e_2 = S(e_2 + te_3) = ST(te_1 + e_2),$$

$$TS(e_2 + se_3) = T(s^2e_1 + e_2) = e_2 + s^2e_3 = T(s^2e_1 + e_2) = TS(e_2 + se_3),$$

that is, T and S are commuting. Now define the function $\psi \in \Psi$ with

$$\psi(r) = \begin{cases} r^2 & 0 \leq r < 1, \\ \frac{r}{r+1} & 1 \leq r. \end{cases}$$

For $\mathbf{x} := te_1 + e_2 \in A$ and $\mathbf{y} := e_2 + se_3 \in B$ we have

$$\|S\mathbf{x} - S\mathbf{y}\|^* = \sqrt{s^4 + t^4} \leq s^2 + t^2 = \psi(\sqrt{s^2 + t^2}) = \psi(\|T\mathbf{x} - T\mathbf{y}\|^*).$$

Therefore, all of the assumptions of Theorem 2.1 hold and so, the cyclic pair $(T; S)$ has a unique common best proximity point in B and this point is $p = e_2$ which is a common fixed point of the mappings T and S in this case.

The following example shows that the uniformly convexity condition of the Banach space X in Theorem 2.4 is sufficient but not necessary.

Example 2.6. Let X be the real Banach space l_2 renormed according to

$$\|x\| = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\},$$

where, $\|x\|_\infty$ denotes the l_∞ -norm and $\|x\|_2$ the l_2 norm. Assume $\{e_n\}$ is a canonical basis of l_2 . Note that for any $x \in X$ we have $\|x\|_2 \leq \|x\| \leq \sqrt{2}\|x\|_2$ which implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_2$ and so, $(X, \|\cdot\|)$ is a reflexive Banach space. Moreover, in view of the fact that l_∞ is not strictly convex, X is not uniformly convex. Put

$$A = \{x = (x_n); x_2 = 1, \|x\| \leq \sqrt{2}\} \quad \text{and} \quad B = \{y := 2e_2\}.$$

Then (A, B) is a bounded, closed and convex pair in X and $\text{dist}(A, B) = \sqrt{2}$. Moreover, $A_0 = A$ and $B_0 = B$. Define the cyclic mapping $S : A \cup B \rightarrow A \cup B$ with

$$Sx = y \quad (\forall x \in A) \quad \text{and} \quad Sy = \frac{1}{2}e_1 + e_2.$$

For all $x \in A$ and $r \in (0, 1)$ we have

$$\|Sx - Sy\| = \|e_2 - \frac{1}{2}e_1\| = \max\{\sqrt{\frac{1}{4} + 1}, \sqrt{2}\} = \sqrt{2} \leq r\|x - y\| + (1 - r)\text{dist}(A, B),$$

that is, S is cyclic contraction. We note that y is a unique best proximity point of S in B .

3 A generalization of Edelstein fixed point theorem

We begin the main results of this section by stating the well known Edelstein's fixed point theorem.

Theorem 3.1 ([23]). Let (X, d) be a compact metric space and T be a mapping on X such that

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in X \quad \text{with } x \neq y.$$

Then T has a unique fixed point and for any $x_0 \in X$ the iterate sequence $\{T^n x_0\}$ converges to the fixed point of T .

Let (A, B) be a nonempty pair in a normed linear space X . A mapping $T: A \cup B \rightarrow A \cup B$ is said to be noncyclic provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* for the noncyclic mapping T if

$$p = Tp, \quad q = Tq \quad \text{and} \quad \|p - q\| = \text{dist}(A, B).$$

It is interesting to note that the existence of best proximity pairs for noncyclic mappings is equivalent to the existence of a solution of the following minimization nonlinear problem:

$$\min_{x \in A} \|x - Tx\|, \quad \min_{y \in B} \|y - Ty\|, \quad \text{and} \quad \min_{(x, y) \in A \times B} \|x - y\|. \quad (3)$$

The existence of best proximity pairs was first studied by Eldred et al. in [24] using a geometric notion of *proximal normal structure* on nonempty, weakly compact and convex pairs in strictly convex Banach spaces for noncyclic relatively nonexpansive mappings.

Definition 3.2 ([24]). A convex pair (A, B) in a Banach space X is said to have proximal normal structure if for any bounded, closed, convex and proximal pair $(K_1, K_2) \subseteq (A, B)$ for which $\delta(K_1, K_2) > \text{dist}(K_1, K_2)$ and $\text{dist}(K_1, K_2) = \text{dist}(A, B)$, there exists $(x_1, x_2) \in K_1 \times K_2$ such that

$$\max\{\delta_{x_1}(K_2), \delta_{x_2}(K_1)\} < \delta(K_1, K_2).$$

Since every nonempty, compact and convex pair in a Banach space X has proximal normal structure (Proposition 2.2 of [24]), the following result concludes.

Theorem 3.3 (Theorem 2.2 of [24]). Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X and T be a noncyclic relatively nonexpansive mapping, that is, T is noncyclic and $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$. Then T has a best proximity pair.

Motivated by Theorem 3.3, we study the convergence results of best proximity pairs for *noncyclic contractive mappings* in strictly convex Banach spaces.

Definition 3.4. Let (A, B) be a nonempty pair in a normed linear space X . A mapping $T: A \cup B \rightarrow A \cup B$ is said to be a *noncyclic contractive mapping* if T is noncyclic on $A \cup B$ and

$$\|Tx - Ty\| < \|x - y\|, \quad \text{for all } (x, y) \in A \times B, \quad \text{with } \|x - y\| > \text{dist}(A, B).$$

Next lemma describes the relation between noncyclic relatively nonexpansive mappings and noncyclic contractive mappings in uniformly convex Banach spaces.

Lemma 3.5. Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X and $T: A \cup B \rightarrow A \cup B$ be a noncyclic contractive mapping. Then T is noncyclic relatively nonexpansive.

Proof. We only have to prove that $\|Tx - Ty\| = \text{dist}(A, B)$ whenever $\|x - y\| = \text{dist}(A, B)$. So let $\|x - y\| = \text{dist}(A, B)$. Choose a sequence $(\{x_n\}, \{y_n\})$ in $A \times B$ such that $\|x_n - y_n\| > \text{dist}(A, B)$ and $x_n \neq x, y_n \neq y$ for any $n \in \mathbb{N}$. By the compactness condition of the pair (A, B) , we may assume that $\lim_{n \rightarrow \infty} x_n = x \in A$ and $\lim_{n \rightarrow \infty} y_n = y \in B$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = \text{dist}(A, B)$. Notice that if $\|x_{n_0} - y\| = \text{dist}(A, B)$ for some $n_0 \in \mathbb{N}$, then by the strictly convexity of X we must have $x_{n_0} = x$ which is a contradiction. Thus

$$\text{dist}(A, B) \leq \|P_A(Ty) - Ty\| \leq \|Tx_n - Ty\| < \|x_n - y\|.$$

Therefore, $\|Tx_n - Ty\| \rightarrow \text{dist}(A, B)$. Since $\|P_A(Ty) - Ty\| \leq \|Tx_n - Ty\|$ and $Ty \in B_0$,

$$Tx_n \rightarrow P_A(Ty).$$

Similarly we can see that $Ty_n \rightarrow P_B(Tx)$. In view of the fact that $\|Tx_n - Ty_n\| \rightarrow \text{dist}(A, B)$, we obtain $\|P_A(Ty) - P_B(Tx)\| = \text{dist}(A, B)$. Again, using the strict convexity of X ,

$$Tx = P_A(Ty), \quad \text{and} \quad Ty = P_B(Tx).$$

Thereby, $\|Tx - Ty\| = \text{dist}(A, B)$ and the result follows. $\square$

Next example shows that the strictly convexity of the Banach space X in Lemma 3.5 is a necessary condition.

Example 3.6. Let $X = \{\mathbb{R}^2, \|\cdot\|_\infty\}$ and let $A = \{(0, s) : 0 \leq s \leq 2\}$ and $B = \{(1, t) : 0 \leq t \leq 2\}$. It is clear that $\text{dist}(A, B) = 1$. Define the noncyclic mapping $T : A \cup B \rightarrow A \cup B$ by

$$T(0, s) = (0, \frac{s}{1+s}), \quad T(1, t) = (1, \frac{t}{1+t}).$$

For $(x, y) \in A \times B$ if $\|x - y\|_\infty > \text{dist}(A, B)$, then

$$\begin{aligned} \|Tx - Ty\|_\infty &= \|(0, \frac{s}{1+s}) - (1, \frac{t}{1+t})\|_\infty \\ &= \max\{1, \frac{|s-t|}{(1+s)(1+t)}\} \\ &< |s-t| \\ &= \|x - y\|_\infty, \end{aligned}$$

which implies that T is noncyclic contractive. Besides, $\|Tx - Ty\|_\infty = \text{dist}(A, B)$, when $\|x - y\|_\infty = \text{dist}(A, B)$. Hence, T is noncyclic relatively nonexpansive.

But if we modify $T : A \rightarrow A$ as $T(0, s) = (0, \frac{s}{1+s})$ for $0 < s \leq 2$ and $T(0, 0) = (0, 1.1)$, then T is still noncyclic contractive. To see T this, let $\mathbf{x} := (0, 0)$. Since

$$\{\mathbf{y} \in B : \|\mathbf{x} - \mathbf{y}\|_\infty > 1\} = \{(1, t) : 1 < t \leq 2\},$$

we have

$$\|T\mathbf{x} - T\mathbf{y}\| = \max\{1, 1.1 - \frac{t}{1+t}\} = 1 < \|\mathbf{x} - \mathbf{y}\|_\infty,$$

whenever $1 < t \leq 2$. On the other hand, if $\mathbf{v} := (1, 0)$, then

$$\|T\mathbf{x} - T\mathbf{v}\| = \max\{1.1, 1\} > 1 = \|\mathbf{x} - \mathbf{v}\|_\infty,$$

that is, T is not a noncyclic relatively nonexpansive mapping.

The following theorem is an extension of Edelstein's fixed point theorem in strictly convex Banach spaces.

Theorem 3.7. Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X and $T : A \cup B \rightarrow A \cup B$ be a noncyclic contractive mapping. Then T has a unique best proximity pair. Moreover, for any $(x_0, y_0) \in A_0 \times B_0$ if we define $x_{n+1} := Tx_n$ and $y_{n+1} := Ty_n$ then the sequence $\{(x_n, y_n)\}$ converges to the best proximity pair of T .

Proof. It follows from Lemma 3.5 that T is a noncyclic relatively nonexpansive mapping. Since the pair (A, B) is compact and convex, the existence of a best proximity pair for the mapping T is concluded from Theorem 3.3. Suppose $(p, q) \in A \times B$ is a best proximity pair of the mapping T . Then $p = Tp, q = Tq$ and $\|p - q\| = \text{dist}(A, B)$. It is worth noticing that the fixed point sets of T in A_0 and B_0 are singleton. Indeed, if $p' \in A_0$ such that $p' = Tp'$ and $p \neq p'$ then from the strictly convexity of X we have $\|p' - q\| > \text{dist}(A, B)$. Therefore,

$$\|p' - q\| = \|Tp' - Tq\| < \|p' - q\|,$$

which is impossible. Equivalently, we can see that the fixed point set in B_0 is singleton. This implies that T has a unique best proximity pair in $A \times B$. Let $x_0 \in A_0$ and $x_{n+1} = Tx_n$. Assume that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightarrow z \in A_0$. Thus

$$\begin{aligned} d(x_n, \mathcal{P}_B p) &= d(Tx_{n-1}, \mathcal{P}_B(Tp)) \\ &= d(Tx_{n-1}, T(\mathcal{P}_B p)) < d(x_{n-1}, \mathcal{P}_B p) \end{aligned}$$

Hence, $d(z, \mathcal{P}_B p) = \lim_{k \rightarrow \infty} d(x_{n_k}, \mathcal{P}_B p)$. From Proposition 3.4 of [25] T is continuous on $A_0 \cup B_0$. Suppose $d(z, \mathcal{P}_B p) > \text{dist}(A, B)$. We now have

$$\begin{aligned} d(z, \mathcal{P}_B p) &= \lim_{k \rightarrow \infty} d(x_{n_{k+1}}, \mathcal{P}_B p) \\ &= \lim_{k \rightarrow \infty} d(Tx_{n_k}, \mathcal{P}_B p) \\ &= d(Tz, \mathcal{P}_B p) \\ &= d(Tz, \mathcal{P}_B(Tp)) \\ &= d(Tz, T(\mathcal{P}_B p)) \\ &< d(z, \mathcal{P}_B p), \end{aligned}$$

which is a contradiction and so we must have $d(z, \mathcal{P}_B p) = \text{dist}(A, B)$. Then $z = p$. Since any convergent subsequence of $\{x_{n_k}\}$ converges to p , the sequence itself converges to p . Similarly we can prove the convergence of $\{y_n\}$ to the point q and this completes the proof. $\square$

Remark 3.8. The notion of noncyclic contractive mappings was introduced in [25] as below (see Definition 3.2 and Theorem 4.6 of [25]): Let (A, B) be a nonempty pair in a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is called noncyclic contractive if T is noncyclic on $A \cup B$ and

- (i) $d(Tx, Ty) < d(x, y)$ whenever $d(x, y) > \text{dist}(A, B)$ for $x \in A$ and $y \in B$,
- (ii) $d(Tx, Ty) = d(x, y)$ whenever $d(x, y) = \text{dist}(A, B)$ for $x \in A$ and $y \in B$.

Then the existence result of a unique best proximity pair for such mappings was established using a notion of projectional property (Theorem 4.6 of [25]). It is remarkable to note that under the assumptions of Theorem 3.7 the condition (ii) on the noncyclic mapping T holds naturally.

At the end of this section, we study the existence of a unique common best proximity point for a cyclic pair of commuting mappings under a contractive condition.

We begin with the following lemma.

Lemma 3.9. Let (A, B) be a nonempty, closed, and convex pair in a normed linear space X and $(T; S)$ be a cyclic pair defined on $A \cup B$ such that

- (i) $S(A) \subseteq T(A) \subseteq B$ and $S(B) \subseteq T(B) \subseteq A$,
- (ii) $T(A)$ and $T(B)$ are compact subsets of B and A respectively,
- (iii) $\|Sx - Sy\| < \|Tx - Ty\|$, for all $(x, y) \in A \times B$ such that $\|Sx - Sy\| > \text{dist}(A, B)$,

Then

$$\text{dist}(S(A), S(B)) = \text{dist}(T(A), T(B)) = \text{dist}(A, B).$$

Proof. Clearly

$$\text{dist}(S(A), S(B)) \geq \text{dist}(T(A), T(B)) \geq \text{dist}(A, B). \quad (4)$$

If $\text{dist}(S(A), S(B)) = \text{dist}(A, B)$, then there is nothing to prove. Suppose $\text{dist}(S(A), S(B)) > \text{dist}(A, B)$. By the assumption (iii),

$$\text{dist}(S(A), S(B)) \leq \text{dist}(T(A), T(B)).$$

Therefore, $\text{dist}(S(A), S(B)) = \text{dist}(T(A), T(B))$ and so $\text{dist}(T(A), T(B)) > \text{dist}(A, B)$. Let $a' \in T(B)$, $b' \in T(A)$ be such that $\text{dist}(T(A), T(B)) = \|a' - b'\| > \text{dist}(A, B)$. Assume $a' = T(b)$ and $b' = T(a)$ for some $(a, b) \in A \times B$. Since

$$\|S(a) - S(b)\| \geq \text{dist}(S(A), S(B)) > \text{dist}(A, B),$$

we have

$$\|S(a) - S(b)\| < \|T(a) - T(b)\| = \text{dist}(T(A), T(B)),$$

and this is a contradiction with (4) and the result follows. $\square$

Theorem 3.10. Let (A, B) be a nonempty, closed and convex pair in a strictly convex Banach space X . Let $(T; S)$ be a cyclic pair defined on $A \cup B$ such that

- (i) $S(A) \subseteq T(A) \subseteq B$ and $S(B) \subseteq T(B) \subseteq A$
- (ii) $T(A)$ and $T(B)$ are compact and convex subsets of B and A respectively.
- (iii) $\|Sx - Sy\| < \|Tx - Ty\|$ for all $(x, y) \in A \times B$ such that $\|Sx - Sy\| > \text{dist}(A, B)$
- (iv) S and T commute.

Then $(T; S)$ has a unique common best proximity point in B .

Proof. Let

$$[T(A)]_0 = \{y \in T(A) : d(x, y) = \text{dist}(T(A), T(B)), \text{ for some } x \in T(B)\},$$

and

$$[T(B)]_0 = \{x \in T(B) : d(x, y) = \text{dist}(T(A), T(B)), \text{ for some } y \in T(A)\}.$$

Notice that from Lemma 3.9, $\text{dist}(T(A), T(B)) = \text{dist}(A, B)$. To show ST^{-1} is singleton, let $x \in [T(B)]_0$. Then there exists $y \in [T(A)]_0$ such that $\|x - y\| = \text{dist}(A, B)$. For any $z \in ST^{-1}x \subseteq T(B)$ and $w \in ST^{-1}y \subseteq T(A)$,

$$\|z - w\| = \|Sx' - Sy'\|$$

where $z = Sx'$ and $w = Sy'$ for some $x' \in T^{-1}x$ and $y' \in T^{-1}y$. If $\|Sx' - Sy'\| > \text{dist}(A, B)$, then

$$\begin{aligned} \|Sx' - Sy'\| &< \|Tx' - Ty'\| \\ &= \|TT^{-1}x - TT^{-1}y\| \\ &= \|x - y\| \\ &= \text{dist}(A, B), \end{aligned}$$

which is impossible. So

$$\|z - w\| = \|Sx' - Sy'\| = \text{dist}(A, B),$$

for any $z \in ST^{-1}x$ and $w \in ST^{-1}y$. It now follows from the strict convexity of X that $ST^{-1}x$ and $ST^{-1}y$ are singleton. Also, it is clear that $ST^{-1}([T(A)]_0) \subseteq [T(A)]_0$ and $ST^{-1}([T(B)]_0) \subseteq [T(B)]_0$, that is, $ST^{-1} : [T(A)]_0 \cup [T(B)]_0 \rightarrow [T(A)]_0 \cup [T(B)]_0$ is a noncyclic mapping. Let $(x, y) \in [T(A)]_0 \times [T(B)]_0$ be such that $\|x - y\| > \text{dist}(T(A), T(B)) (= \text{dist}(A, B))$. If $\|ST^{-1}x - ST^{-1}y\| > \text{dist}(A, B)$, then

$$\|ST^{-1}x - ST^{-1}y\| < \|TT^{-1}x - TT^{-1}y\| = \|x - y\|,$$

which implies that ST^{-1} is a noncyclic contractive mapping on a compact and convex pair $([T(A)]_0, [T(B)]_0)$. Now using Theorem 3.7 for any $x \in [T(A)]_0 \cup [T(B)]_0$ the sequence $\{(ST^{-1})^n x\}_{n \geq 1}$ converges to the unique fixed point z of ST^{-1} . Since S and T commutes,

$$Tz = T(ST^{-1}z) = Sz = S(ST^{-1}z)$$

Suppose $\|Sz - T(Sz)\| > \text{dist}(A, B)$. Thus

$$\begin{aligned} \|Sz - T(Sz)\| &= \|Sz - S(Tz)\| \\ &< \|Tz - T(Tz)\| \\ &= \|Sz - T(Sz)\|, \end{aligned}$$

which is a contradiction. Hence $\|Sz - T(Sz)\| = \text{dist}(A, B)$. On the other hand, if $\|Sz - S(Sz)\| > \text{dist}(A, B)$, then

$$\begin{aligned} \|Sz - S(Sz)\| &< \|Tz - T(Sz)\| \\ &= \|Sz - T(Sz)\| \\ &= \|Sz - S(Tz)\| \end{aligned}$$

$$= \|Sz - S(Sz)\|,$$

which is impossible. Thereby $\|Sz - S(Sz)\| = \text{dist}(A, B)$ and so the point Sz is a common best proximity point for the cyclic pair $(T; S)$. Uniqueness of the common best proximity point follows as in the proof of Theorem 2.1. $\square$

Example 3.11. Let $X = \{\mathbb{R}^2, \|\cdot\|_2\}$ and let $A = \{(0, s) : 0 \leq s \leq \frac{1}{2}\}$ and $B = \{(1, t) : 0 \leq t \leq \frac{1}{2}\}$. Thus $\text{dist}(A, B) = 1$. Define the cyclic pair $(T; S)$ on $A \cup B$ as follows:

$$S(0, s) = (1, s^3), \quad S(1, t) = (0, t^3) \quad \text{and} \quad T(0, s) = (1, s^2), \quad T(1, t) = (0, t^2).$$

Then

$$S(A) = \{(1, s) : 0 \leq s \leq \frac{1}{8}\}, \quad S(B) = \{(0, t) : 0 \leq t \leq \frac{1}{8}\},$$

$$T(A) = \{(1, s) : 0 \leq s \leq \frac{1}{4}\}, \quad T(B) = \{(0, t) : 0 \leq t \leq \frac{1}{4}\}.$$

Therefore, $S(A) \subseteq T(A) \subseteq B$ and $S(B) \subseteq T(B) \subseteq A$. Also, $T(A)$ and $T(B)$ are compact and convex subsets of B and A , respectively. Moreover,

$$ST(0, s) = S(1, s^2) = (0, s^6) = T(1, s^3) = TS(0, s),$$

$$ST(1, t) = S(0, t^2) = (1, t^6) = T(0, t^3) = TS(1, t),$$

and so, S and T are commuting. Finally, if $\|S(0, s) - S(1, t)\| > \text{dist}(A, B)$, we conclude that $s \neq t$. Hence,

$$\|S(0, s) - S(1, t)\| = \sqrt{1 + (|s^3 - t^3|)^2} < \sqrt{1 + (|s^2 - t^2|)^2} = \|T(0, s) - T(1, t)\|,$$

whenever $s, t \geq 0$ and $s + t \leq 1$. Thereby all of the assumptions of Theorem 3.10 hold and the cyclic pair $(T; S)$ has a unique common best proximity point in B and this point is $p = (0, 0)$.

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References

- [1] Kirk W.A., Srinivasan P.S., Veeramani P., Fixed points for mappings satisfying cyclic contractive conditions, *Fixed Point Theory*, 2003, 4, 79-86
- [2] Fan K., Extensions of two fixed point theorems of F.E. Browder, *Math. Z.*, 1969, 112, 234-240
- [3] Eldred A.A., Veeramani P., Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, 2006, 323, 1001-1006
- [4] Al-Thagafi M.A., Shahzad N., Convergence and existence results for best proximity points, *Nonlinear Anal.*, 2009, 70, 3665-3671
- [5] Abkar A., Gabeleh M., Best proximity points for asymptotic cyclic contraction mappings, *Nonlinear Anal.*, 2011, 74, 7261-7268
- [6] De la Sen M., Agarwal R.P., Some fixed point-type results for a class of extended cyclic self-mappings with a more general contractive condition, *Fixed Point Theory Appl.*, 2011, 2011:59
- [7] De la Sen M., Agarwal R.P., Common fixed points and best proximity points of two cyclic self-mappings, *Fixed Point Theory and Appl.*, 2012, 2012:136
- [8] Derafshpour M., Rezapour Sh., Shahzad N., Best Proximity Points of cyclic ϕ -contractions in ordered metric spaces, *Topological Methods in Nonlinear Anal.*, 2011, 37, 193-202
- [9] Di Bari C., Suzuki T., Vetro C., Best proximity points for cyclic Meir-Keeler contractions, *Nonlinear Anal.*, 2008, 69, 3790-3794
- [10] Gabeleh M., Best proximity points and fixed point results for certain maps in Banach spaces, *Numer. Funct. Anal. Optim.*, 2015, 36, 1013-1028
- [11] Gabeleh M., Semi-normal structure and best proximity pair results in convex metric spaces, *Banach J. Math. Anal.*, 2014, 8, 214-228
- [12] Maria Felicit J., Eldred A.A., Best proximity points for cyclical contractive mappings, *Appl., General Topol.*, 2015, 16, 119-126.
- [13] Rus I.A., Cyclic representations and fixed points, *Ann. T. Popoviciu, Seminar Funct. Eq. Approx. Convexity*, 2005, 3, 171-178
- [14] Sadiq Basha S., Best proximity points : global optimal approximate solutions, *J. Glob. Optim.*, 2011, 49, 15-21
- [15] Suzuki T., Kikkawa M., Vetro C., The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal.*, 2009, 71, 2918-2926

- [16] Włodarczyk K., Plebaniak R., Banach A., Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces, *Nonlinear Anal.*, 2009, 70, 3332-3342
- [17] Włodarczyk K., Plebaniak R., Banach A., Erratum to: Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces, *Nonlinear Anal.*, 2009, 71, 3583-3586
- [18] Włodarczyk K., Plebaniak R., Obczyński C., Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces, *Nonlinear Anal.*, 2010, 72, 794-805
- [19] Zizler V., On some rotundity and smoothness properties of Banach spaces, *Dissertationes Mat.*, 1971, 87, 1-33
- [20] Sankar Raj V., A best proximity point theorem for weakly contractive non-self-mappings, *Nonlin. Anal.*, 2011, 74, 4804-4808
- [21] Abkar A., Gabeleh M., Global optimal solutions of noncyclic mappings in metric spaces, *J. Optim. Theory Appl.*, 2012, 153, 298-305
- [22] Sadiq Basha S., Shahzad N., Common best proximity point theorems: Global minimization of some real-valued multi-objective functions, *J. Fixed Point Theory Appl.*, 2016, 18, 587-600
- [23] Edelstein M., On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, 1962, 37, 74-79
- [24] Eldred A.A., Kirk W.A., Veeramani P., Proximal normal structure and relatively nonexpansive mappings, *Studia Math.*, 2005, 171, 283-293
- [25] Sankara Raju Kosuru G., Extensions of Edelstein's theorem on contractive mappings, *Numer. Funct. Anal. Optim.*, 2015, 36, 887-900