Quasi-Newtonian Cosmological Models in Scalar-Tensor Theories of Gravity

Heba S.H. Abdulrahman

orcid.org/0000-0003-2917-5181

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Supervisor: Dr. Amare Abebe

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Student number: 29082722

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Abstract

Theoretical physics in general and cosmology in particular have faced some challenges due to the recent observations in cosmology and astronomy, such as the discovery of the accelerated cosmic expansion and the existence of dark matter and dark energy. Einstein's theory of General Relativity (GR) together with the standard model of cosmology fall short of giving any explanations for these problems. Higher-order gravitational theories such as $f(R)$ and scalar-tensor theories of gravity have been suggested to provide an answer to these shortcomings. In this dissertation, we investigate the cosmological implications of such theories such as the background expansion history as well as the evolution of cosmological density perturbations. We present the equivalence between $f(R)$ and scalar-tensor theories of gravity and we exploit this equivalence to discuss some simple procedures to reconstruct $f(R)$ gravity models from the Chaplygin scalar field in flat de Sitter spacetimes. We derive the Lagrangian densities for the $f(R)$ action and we get the exact $f(R)$ models and the corresponding scalar field potentials for asymptotically de Sitter spacetimes in early and late epochs. We also reconstruct $f(R)$ from the exact cosmological solutions of the Einstein field equations with a non-interacting classical scalar field-and-radiation with predetermined inflationary expansion scenarios of the Friedmann-Lemaître-Robertson-Walker (FLRW) background spacetime, and we get the corresponding potentials for the scalar field. We calculate the slow-roll parameters (the spectral index n_s and the tensor-to-scalar ratio r) to show how these parameters can be constrained by observational data. Although GR is a generalization of Newtonian gravity in the presence of strong gravitational fields, there is no properly defined Newtonian limit of GR on cosmological scales. Recently, general relativistic *quasi-Newtonian* cosmologies have been studied in the context of large-scale structure formation and nonlinear gravitational collapse in the late-time universe. This despite the general covariant inconsistency of these cosmological models except in some special cases such as the spatially homogeneous and isotropic, spherically symmetric, expanding FLRW spacetimes. $f(R)$ models have been shown to exhibit more shared properties with Newtonian gravitation than does GR. We investigate the existence and integrability conditions of quasi-Newtonian cosmological spacetimes of classes of shear-free cosmological dust models with irrotational fluid flows in the context of scalar-tensor theories of gravity. We also derive the covariant density and velocity propagation equations of such models and analyse the corresponding solutions to these perturbation equations.

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List of Abbreviations

BAO	Baryon Acoustic Oscillations
BD	Brans-Dicke theory
CG	Chaplygin Gas
CMB	Cosmic Microwave Background
EEP	Einstein Equivalence Principle
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General Relativity
GUT	Grand Unified Theory
LSS	Large-Scale Structure
ΛCDM	Lambda Cold Dark Matter
SEP	Strong Equivalence Principle
SN Ia	Type Ia Supernovae
WEP	Weak Equivalence Principle

Chapter 1

Introduction

Gravity is one of the fundamental interactions and one of the most ubiquitous phenomena related to our everyday experience. The explanation of gravity has gone through many permutations throughout history. Scientists have made a great attempt to get a scientific explanation of this phenomenon. However, it still remains mysterious. Galileo was the first to demonstrate the idea of gravity, by showing that falling objects accelerate at the same rate regardless of their weights.

The next step of our understanding of gravity came from Newton. He introduced the inverse-square gravitational force law which extended gravity beyond earth by combining celestial and terrestrial gravity (objects fall on earth for the same reason planets move in ellipses). Newton believed that gravitational force is responsible for each kind of motion. Newton's law of universal gravitation and his laws of motion were working perfectly in explaining the motion of the planets and the precession of the orbits and were able to give a good explanation of both terrestrial and celestial phenomena until something strange had been noticed about Mercury's orbit. The experimental results of observing and measuring the precession of Mercury's orbit was found to be 43 arc-seconds which was inconsistent with Newton's theory. The orbit of Mercury is an ellipse and according to Newton's theory of gravity, it should just keep following this ellipse and never change [1].

On the other hand, in 1893, Mach stated what was later called by Einstein "Mach's principle". This was the first attack to Newton's idea of absolute space: Inertia originates in a kind of interaction between bodies. This is obviously in contradiction with Newton's ideas, according to which inertia was always relative to the absolute frame of space. However, there was another clearer interpretation that the gravitational constant should be a function of the mass distribution in the universe. This is different from Newton's idea of the gravitational constant as being universal and unchanging [2–4]. Newton's theory of gravity needed to be modified since it was only limited to non-relativistic phenomena (low speeds and weak gravitational fields).

In 1905, Einstein came up with his theory of special relativity. He extended the range over which Newtonian theory of gravity was applied to include high speeds and non- inertial frames. According to Newton's gravitational theory, the notion of absolute time and absolute space were thought to be independent realities. Einstein's theory of special relativity discarded this notion of absolute time and space and replaced it by the idea of time being dependent on reference frames and spatial position. The theory of special relativity applies only when the curvature of spacetime due to gravity is neglected and it is only restricted to flat spacetime known as Einstein-Minkowski spacetime. Einstein completed his theory of GR, which is a generalization of special relativity, by extending the range over which special relativity is applied to include high gravitational fields and accelerated frames. He asserted the equivalence between inertial and gravitational forces of acceleration. He realized that a uniform gravitational field cannot be distinguishable from a uniform acceleration, according to Einstein these effects are the same. In practice this means that locally, a person cannot feel the difference between moving away in a rocket and standing on the surface of a gravitating object with the same acceleration [3].

There are different versions of the equivalence principle in Einstein's general relativity [1].

- The Weak Equivalence Principle (WEP): All freely falling objects in a given gravitational field, will follow the same trajectories regardless of their compositions.
- The Einstein Equivalence Principle (EEP): The WEP holds and in all freely falling frames, the results of any local non-gravitational experiment are independent of velocity or position.
- The Strong Equivalence Principle (SEP): The WEP holds for test particles and massive gravitating objects, and in all freely falling frames, the results of any local non-gravitational experiment are independent of velocity or positioı

As a consequence of the principle of equivalence, Einstein noted that the light coming from strong gravitational fields should have its frequency reduced or shifted to small values. He noted that the path of light is curved by a gravitational field and this is known as gravitational redshift. The observations of this redshift showed that spacetime must be curved. Hence, Einstein's theory of GR treats spacetime as four-dimensional manifold, and according to the principles of GR, locally the spacetime is flat and universally it is curved by massive objects.

GR together with quantum field theory are considered to be the backbones of modern physics. The theory is given in the language of differential geometry and was the first such mathematical physics theory, leading the way for other mathematical theories in physics such as gauge theories and string theories. One of the most astonishing facts about GR is that after almost an entire century it has not changed at all. How spacetime behaves on macroscopic scales is best described by Einstein's field equations [1]. According to Einstein, the curvature of spacetime is determined by the sources of the gravitational field, *i.e.* (the metric of spacetime is determined by the matter). This idea is expressed as 10 field equations that describe the gravitational effect produced by a given mass [1],

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor which is a four dimensional generalization of the stress tensor and its components describe the flux of the energy-momentum, $\kappa \equiv \dfrac{8\pi G}{c^4}$ is a constant known as the Einstein constant, G is Newton's gravitational constant, c is the speed of light in vacuum and $G_{\mu\nu}$ is the Einstein tensor which is a four-dimensional tensor that describes the geometry of a region of spacetime, we have used the convention $\kappa = 8\pi G = c = 1$. One remark about this tensor is that it is at most a second order partial derivative of the metric tensor, and it is a combination of the following geometrical quantities:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor and R is the scalar curvature (Ricci scalar) and it is defined as

$$R = R_{\mu\nu} g^{\mu\nu} \ , \tag{1.3}$$

and the Ricci tensor is defined as

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \ . \tag{1.4}$$

where $R^{\alpha}_{\beta\mu\nu}$ is the Riemann tensor and it is given in terms of the Christoffel symbols as

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} \ . \tag{1.5}$$

where $\Gamma^{\alpha}_{\beta\mu}$ is the Christoffel symbol of the second kind and it is defined as

$$\Gamma^{\alpha}_{\beta\mu} = \frac{1}{2} g^{\alpha\lambda} \Big(g_{\lambda\beta,\mu} + g_{\lambda\mu,\beta} + g_{\beta\mu,\lambda} \Big) . \tag{1.6}$$

where the commas here represent covariant differentiation.

Einstein's field equations have completely changed our way of understanding the Universe and how it evolves. These equations helped to predict the existence of gravitational waves and explained the behaviour of black holes and how they warp the space around them. GR has great achievements in our understanding of the Universe we live in and it is considered to be the basis of standard cosmology. Through GR and Einstein's field equations, it has been shown that our universe expands and this fact has shown the way to the discovery of the Big Bang. However, there have always been challenges to the unique status of GR as the ultimate theory of gravitation [5]. For example the recent observation of the accelerated expansion of the universe and the existence of dark matter and dark energy are not well explained by GR nor standard model of cosmology. In light of this, several alternatives to GR have been constructed to address these theoretical challenges.

Recently, higher-order theories of gravity have attracted great interest [5]. Some important classes of such modified gravity are the $f(R)$ and scalar-tensor theories of gravity [6–8]. The study of large-scale structure and cosmological perturbations in these alternative theories of gravity has become a point of interest in modern cosmology. This dissertation aims to study the possible implications to cosmology of these modified theories, and has been organised as follows:

- In Chapter 2, we give a general review of modern cosmology and the shortcomings of the concordance model of cosmology. We also give a review of the $1 + 3$ covariant approach and provide the field equations which are required to study the cosmological perturbations and will be applied to the scalar-tensor theory of gravity.
- In Chapter 3, we give an overview of $f(R)$ and scalar-tensor theories of gravity as a modification of GR and introduce the corresponding field equations. We also get the equivalence between $f(R)$ and scalar-tensor theories of gravity.
- In Chapter 4, we present some applications on the equivalence between $f(R)$ and scalar-tensor models, and show how can we reconstruct $f(R)$ models from different cosmological evolution scenarios.
- In Chapter 5, we study the quasi-Newtonian models in scalar-tensor theories of gravitation by using the correspondence between $f(R)$ theories and scalar-tensor theories. We derive the integrability conditions that describe a consistent evolution of the linearised field equations of the quasi-Newtonian universes. We also present the perturbation equations that arise as a result of the imposed integrability conditions. We introduce the exact and approximated solutions to these perturbation equations.

Chapter 2

The concordance model of cosmology

2.1 Introduction

The scientific theory of cosmology is concerned with the study of the large-scale structure of the observable region of the Universe, and its relation to local physics on the one hand and to the rest of the Universe on the other [9]. By studying the properties of our universe such as its origin, evolution (how did it form) and the ultimate fate of the Universe as a whole. Cosmology involves many hypotheses about the Universe just like any field of physics together with the experimental observations. Modern cosmology started when Einstein developed his theory of general relativity and tried to apply this new gravitational dynamics to the Universe as a whole. He discovered that his field equations cannot work for a static universe (the Universe is neither expanding or contracting). In order to fix this problem and to achieve a dynamically stable universe, Einstein proposed a modification term to his field equations called the cosmological constant, Λ, to avoid these dynamical effects of gravity and to construct static models of the Universe [10–12],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \ . \tag{2.1}$$

This cosmological constant represents the energy of empty space. A cosmological solution to Einstein's field equations is the FLRW metric [13] was obtained and this required an assumption of how the matter distributes in the Universe, by assuming that the matter distribution in the Universe is homogeneous and isotropic on a very large scales. The Universe is homogeneous means there is no preferred observing position in the Universe, and isotropic means there is no difference in the structure of the Universe as we look in different directions. Friedmann equations were the first to predict that our universe is expanding. In 1929, Hubble confirmed the expansion of our universe when he studied the nearby galaxies. He observed that they were moving away from us and the velocities of receding galaxies from us are proportional to their distance from us. He announced that our universe is in fact expanding and was also consistent with a cosmological solution to Einstein's field equations that was obtained by Friedmann. Therefore the need for a cosmological constant disappeared and this caused Einstein to abandon his cosmological constant. Nowadays we have a rather different view of the cosmological constant [12], currently is associated with a vacuum energy or dark energy in empty space according to the standard model of cosmology known as Lambda Cold Dark Matter (ΛCDM) model.
As we mentioned a cosmological solution to Einstein's field equations is the FLRW metric, which take the form:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \ , \tag{2.2}$$

where $a(t)$ is the cosmological scale factor which tells us about the relative expansion of the Universe and $d\sigma^2$ is the normalized metric that characterizes a 3-space of normalized constant curvature k whose spatial coordinates (r, θ, φ). It is given as [5]

$$d\sigma^2 = dr^2 + f^2(r)\Big(d\theta^2 + \sin^2\theta d\varphi^2\Big) . \tag{2.3}$$

where

$$f(r) = \begin{cases} \sin(r) & \text{for} \quad k = +1 , \\ r & \text{for} \quad k = 0 , \\ \sinh(r) & \text{for} \quad k = -1 , \end{cases}$$

where k is the curvature and takes the values 1, 0 or +1 which represent open, flat or closed universe, respectively. The homogeneity and isotropy of the FLRW metric yields the Friedmann and Raychaudhuri equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\mu}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} , \tag{2.4}$$

$$3\left(\frac{\ddot{a}}{a}\right) = -\frac{1}{2}\Big(\mu + 3p\Big) + \Lambda , \tag{2.5}$$

where μ and p represent the energy density and isotropic pressure, respectively.
Since the Universe is expanding, then it must have started from a very dense and hot point and this was a key evidence of the Big Bang theory. During the earliest phases of the Bing Bang, the Universe was expanding very rabidly and there was a special time after the Big Bang, as the Universe cooled to allow the protons and electrons to combine to form neutral hydrogen atoms through which the photons could freely travel through the space and this is known as photon decoupling which lead to cosmic microwave background radiation (CMB). The measurements of the CMB showed that there are some density fluctuations that occurred because of the exponential growth of the Universe and these fluctuations can be considered to be the early seeds that form the large-scale structure.

The Freidmann equations are also considered to be the basis of the standard model of cosmology. The observed universe from the CMB looks homogeneous and isotropic on large scales, which is not the case on small scales, where the Universe looks inhomogeneous and anisotropic, therefore, the cosmologists are trying to apply cosmological perturbations to the FLRW spacetime as a background and analyse the behaviour of these perturbations.

There is currently a consensus that in the FLRW universe, roughly 95% of the overall energy density must be dark in order to be compatible with observations. The current best fit model claims that about 25% of this dark material is in the form of a non-relativistic, non-interacting form of matter called dark matter, and that the remaining 70% is in the form of a non-clustering form of energy density with a negative equation of state known as dark energy [1, 14]. In 1998 the Supernova Cosmology Project and the High-Z Supernova Search Team observations of Type Ia supernova [15] showed that the present universe is in a state of accelerated expansion, considered to have started since the Universe entered its dark-energy-dominated era.

2.2 Cosmological perturbations

The Universe is not perfectly smooth: primordial fluctuations seeded the formation of structures on large-scales and hence the real universe is slightly lumpy. Theories of cosmological perturbations tell us how these small fluctuations grow and form the large-scale structures (galaxies, clusters and superclusters) in the real universe [5]. There are two approaches to these theories of cosmological perturbations namely, the metric based approach and the covariant approach. The metric approach was first developed by Lifshitz (1946) [16] and later building on Lifshitz's work, Bardeen (1980) [17] and Kodama and Sasaki (1984) [18] made other improvements. The metric approach is based on foliating a spacetime with hypersurfaces. It is completely dependent on the gauge choice. When we write down the perturbed metric, we implicitly choose a specific time slicing of the spacetime and define specific spatial coordinates on these time slices. Making a different choice of coordinates can change the values of the perturbation variables. Therefore, It is a non-local, linear and coordinate dependent approach and since it is a linear theory, the nonlinear effects are not accounted for [5, 17, 19–24].

The covariant formalism describes spacetime via covariantly defined variables with respect to a partial frame formalism such as $1+1+2$ or $1+3$. It is a way of describing the physics and geometry by tensor quantities. It is local and valid in all coordinate systems. The main interest of this approach is that, unphysical gauge modes do not appear here [5].

2.2.1 The $1+3$ covariant formalism

The 1+3 covariant approach was first developed by Ehlers (1961) [25] followed by Hawking (1966) [26] and Ellis and Bruni (1989) [27]. This approach provides a very convenient framework within which to study the exact intrinsic dynamics of cosmological models. It is formulated at the level of variables that are covariantly defined in the real universe, and are (exactly) gauge- invariant by construction [28].
In the $1+3$ covariant approach, space-time is split into space and time; it is divided into perpendicular 4-velocity field vector u^a and foliated hyper-surfaces. The vector field u^a is given as

$$u^a = \frac{dx^a}{d\tau}, \qquad u^a u_a = -1 \, ,$$

where τ is proper time measured along the worldlines. In this approach the metric g_{ab} is decomposed into the projected tensor h_{ab} as follows:

$$g_{ab} = h_{ab} - u_a u_b \, , \tag{2.6}$$

with

$$h^a_c h^c_b = h^a_b, \; h^a_a = 3, \; h_{ab}u^b = 0 \, .$$

In this approach, the kinematic quantities which describe all the kinematic features of the fluid flow can be obtained from irreducible parts of the decomposed $\nabla_a u_b$ as [5, 29]

$$\nabla_a u_b = \tilde{\nabla}_a u_b - u_a \dot{u}_b = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b \, , \tag{2.7}$$

where the volume rate of expansion of the fluid is given as

$$\Theta = 3H(t) = \tilde{\nabla}_a u^a \, , \tag{2.8}$$

where $H(t) = \frac{\dot{a}}{a}$ is the Hubble which represents the rate of expansion at any time. The rate of distortion of the matter flow is given as

$$\sigma_{ab} = \tilde{\nabla}_{\langle a} u_{b\rangle} \, . \tag{2.9}$$

and it is the trace-free symmetric rate of the shear tensor ($\sigma_{ab} = \sigma_{(ab)}, \sigma_{ab}u^b = 0, \sigma^a_a = 0$).
The relativistic acceleration vector is given as

$$\dot{u}^a = u^b \nabla_b u^a \, . \tag{2.10}$$

and the skew-symmetric vorticity tensor

$$\omega_{ab} = \tilde{\nabla}_{[a} u_{b]} \tag{2.11}$$

describes the rotation of the fluid relative to a non-rotating frame.

The energy-momentum tensor T_{ab} is decomposed using the $1+3$ covariant approach and it is given as [5]

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab} \, , \tag{2.12}$$

where q_a and π_{ab} are the heat flux and anisotropic pressure, respectively.

2.2.2 Covariant equations

Given a choice of 4-velocity field u^a, the Ehlers-Ellis approach [27,30] employs only fully covariant quantities and equations with transparent physical and geometric meaning [31]. In such a treatment for any scalar quantity X in the background we have

$$\tilde{\nabla}_a X = 0 \, ,$$

thus, by virtue of the Stewart-Walker Lemma [32], any quantity is considered to be gauge-invariant if it vanishes in the background. Therefore, the FLRW background is characterised by the following dynamics, kinematics and gravito-electromanetics [33]:

$$\tilde{\nabla}_a \mu = 0 = \tilde{\nabla}_a p, \qquad q_a = 0, \qquad \pi_{ab} = 0 \, , \tag{2.13}$$

$$\tilde{\nabla}_a \Theta = 0, \qquad A_a = 0 = \omega_a, \qquad \sigma_{ab} = 0 \, , \tag{2.14}$$

$$E_{ab} = 0 = H_{ab} \, , \tag{2.15}$$

where E_{ab} and H_{ab} are the gravito-electromagnetic fields responsible for tidal forces and gravitational waves. They are the "gravito-electric" and "gravito-magnetic" components of the Weyl tensor C_{abcd} defined from the Riemann tensor R^a_{bcd} as

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3} g^{[a}{}_{[c}g^{b]}{}_{d]} \, , \tag{2.16}$$

$$E_{ab} \equiv C_{agbh} u^g u^h, \qquad H_{ab} \equiv \frac{1}{2}\eta_{ae}{}^{gh} C_{ghbd} u^e u^d \, . \tag{2.17}$$

The covariant linearised evolution equations in the general case are given by [31,33,34]

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a \, , \tag{2.18}$$

$$\dot{\mu} = -\Big(\mu + p\Big)\Theta - \tilde{\nabla}^a q_a \, , \tag{2.19}$$

$$\dot{q}_a = -\frac{4}{3}\Theta q_a - \mu A_a \, , \tag{2.20}$$

$$\dot{E}^{\langle ab\rangle} = \eta^{cd\langle a}\tilde{\nabla}_c H^{\rangle b}_d - \Theta E^{ab} - \frac{1}{2}\dot{\pi}^{ab} - \frac{1}{2}\tilde{\nabla}^{\langle a} q^{b\rangle} - \frac{1}{6}\Theta\pi^{ab} \, , \tag{2.21}$$

$$\dot{H}^{\langle ab\rangle} = -\Theta H^{ab} - \eta^{cd\langle a}\tilde{\nabla}_c E^{\rangle b}_d + \frac{1}{2}\eta^{cd\langle a}\tilde{\nabla}_c \pi^{\rangle b}_d \, , \tag{2.22}$$

$$\dot{\omega}^{\langle a\rangle} = -\frac{2}{3}\Theta\omega^a - \frac{1}{2}\eta^{abc}\tilde{\nabla}_b A_c \, , \tag{2.23}$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta\sigma_{ab} - E_{ab} + \frac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a} A_{b\rangle} \, . \tag{2.24}$$

These evolution equations propagate consistent initial data on some initial ($t = t_0$) hypersurface S_0 uniquely along the reference time-like consistency [35]. They are constrained by the following linearised equations [31,33,34]:

$$C_0^{ab} \equiv E^{ab} - \tilde{\nabla}^{\langle a} A^{b\rangle} - \frac{1}{2}\pi^{ab} = 0 \, , \tag{2.25}$$

$$C_1^a \equiv \tilde{\nabla}_b \sigma^{ab} - \eta^{abc} \tilde{\nabla}_b \omega_c - \frac{2}{3} \tilde{\nabla}^a \Theta + q^a = 0 , \tag{2.26}$$

$$C_2 \equiv \tilde{\nabla}^a \omega_a = 0 . \tag{2.27}$$

$$C_3^{ab} \equiv \eta_{cd(} \tilde{\nabla}^c \sigma^d_{b)} + \tilde{\nabla}^{(a} \omega^{b)} - H^{ab} = 0 . \tag{2.28}$$

$$C_5^a \equiv \tilde{\nabla}_b E^{ab} + \frac{1}{2} \tilde{\nabla}_b \pi^{ab} - \frac{1}{3} \tilde{\nabla}^a \eta + \frac{1}{3} \Theta q^a = 0 , \tag{2.29}$$

$$C_b^a \equiv \tilde{\nabla}_b H^{ab} + (\mu + p) \omega^a + \frac{1}{2} \eta^{abc} \tilde{\nabla}_b q_a = 0 . \tag{2.30}$$

These constraints restrict the initial data to be specified and they must remain satisfied on any hypersurface S_t for all comoving time t.

2.3 Shortcomings of the standard model

The standard model of cosmology with ΛCDM fits a number of observational data very well [36]. The Hot Big Bang model is by far the most successful cosmological paradigm. It explains key cosmic phenomena such as the expansion of the Universe, the origin of the CMB and the formation of galaxies and large-scale structure. But it also has its shortcomings, it leaves many unanswered and unexplained phenomena. In this section, we will introduce some of these shortcomings [37, 38].

2.3.1 The anti-matter problem

According to the standard Big Bang cosmology, the Universe should be made up equally of matter and anti-matter. The present universe is dominated by matter in such a way that there is asymmetry that caused most of anti-matter to annihilate with matter and leaving our present universe with very little matter around. In the Standard Model, the source of this asymmetry is still unknown, and is referred to as the anti-matter problem or the baryon asymmetry problem [39].

2.3.2 The magnetic monopole problem

During the hot early universe, the Universe was described by particle theories. One of these theories is the Grand Unified Theory (GUT), that predicts the existence of the magnetic monopole. The GUT predicted the formation of topological defects in space. These defects appeared due to the phase transition in particle models. It refers to these defects as magnetic monopoles. These monopoles became a problem in a hot Big Bang model since a large number of magnetic monopoles should be produced and detected but the fact is that no monopole has ever been observed to this day.

2.3.3 The dark energy and dark matter problems

Recent advances in precision cosmology have enabled us to determine the energy and matter content of the Universe, using techniques such as measurements of the temperature fluctuations in the CMB, distance-luminosity relations analyses in supernovae, and analyses of large-scale structure [5, 40, 41]. The dark matter and dark energy are about 95% of the matter-energy content of the Universe and both of them are not well understood.
Dark matter and ordinary matter are not behaving in the same way: the gravity of the ordinary matter is not strong enough to form galaxies and other complex structures. Therefore, there is another kind of matter that is able to interact with gravity even if it is not yet observed but its existence can be seen through its effect on the formation of galaxies. This kind of matter is called dark matter since it is invisible and does not emit or reflect light and it was discovered when people were analysing the rotational curve of stars inside galaxies. They expected that the rotational velocities of the stars will decrease with the distance from the center of galaxies but this was not observed. Therefore, it was concluded that there is a dark matter which is responsible for this phenomena.

Dark energy is probably the most extensively speculated candidate in recent years, but it is merely a phenomenological addendum whose existence has not been predicted from the Big Bang/inflationary cosmology. Its dynamical nature is hardly understood, and none of the variant models put forward have been convincingly viable so far [5].

2.3.4 The horizon problem

Also known as the *homogeneity problem*, this problem arises because of the difficulty in explaining different casually disconnected regions in the Universe sharing the same temperature [5].
According to the CMB, our observed universe is isotropic and homogeneous and the CMB displays a uniformity of temperature, the temperature is exactly the same in every region in the Universe which implies that all regions must have been connected with one another in the early radiation-dominated epoch. Contrary to the Big Bang model in the early universe, the temperature of the CMB should vary at different regions in the Universe. The Big Bang theory assumed that there has not been sufficient time for light to travel between distant regions of the Universe, so how can these different causally disconnected regions have the same temperature [42, 43]?

2.3.5 The flatness problem

This is a problem related to the fine-tuning of the initial conditions which would otherwise have greatly affected the geometry and expansion history of the Universe [5]. The geometry of our current universe appears to be flat. If there is enough matter in the Universe, the Universe is closed and it will collapse back on itself because the effect of gravity will overcome the expansion of the Universe. If there is not enough matter in the Universe, the Universe is open and it will expand forever. The flatness problem arises from the observational constraint on Ω which describes the ratio of the actual matter density of the Universe to the critical matter density. A universe with a critical density $\Omega = 1$ is globally flat [44]. If the Universe is in perfect balance and the matter density is exactly the critical density, the Universe will be flat and it will continue expanding at a slow rate. The problem arises because the matter density of the Universe measured so far appears to be exactly the critical density.

2.4 Some suggested solutions

The rapid development of observational cosmology has recently shown that the Universe has undergone two phases of cosmic accelerated expansion. The first one is the so-called inflation [45–47] which is understood to have occurred prior to the radiation-dominated era [48]. This phase is required not only to solve the flatness and horizon problems plagued in standard Big Bang cosmology, but also to explain a nearly flat spectrum of temperature anisotropies observed in the CMB [49]. The second accelerating expansion phase, since it has started after the matter-domination era, therefore, it implies that the pressure p and the energy density μ of the Universe should have violated the strong physical energy condition $\mu + 3p > 0$ [50]. Therefore, this late cosmic acceleration cannot be explained by the presence of standard matter whose equation of state $w = p/\mu$ satisfies the condition $w \geq 0$. In fact, we need a component of negative pressure with at least $w < -1/3$ to realise the acceleration of the Universe. This unknown smooth component responsible for this acceleration in the expansion rate is referred to as "dark energy" [51]. The need for the existence of dark energy has been confirmed by a number of observations such as type Ia Supernovae (SN Ia) [50, 52], the Large-Scale Structure (LSS) [53, 54], the Baryon Acoustic Oscillations (BAO) [55], and the CMB [56]. However, there are numerous recent attempts and alternatives to the standard Big Bang model proposed as solutions to those cosmological problems mentioned above to explain away dark energy. One of the most common such attempts comes in the form of the introduction of new matter or scalar field contributions to the action of GR, in this case, one suggestion is that the change in the behaviour of the missing energy density might be regulated by the change in the equation of state of the background fluid instead of the form of the potential [57, 58]. The second attempt comes in the form of modifications to the theory of gravity, as we will see in the following chapter. We will be focusing on $f(R)$ and scalar-tensor theories of gravity as modified theories of gravity.

Chapter 3

$f(R)$ and scalar-tensor models of gravitation

3.1 Introduction

GR is widely accepted as a fundamental theory to describe the geometric properties of spacetime. In a homogeneous and isotropic spacetime the Einstein's field equations give rise to the FLRW model that describes the observed homogeneous and isotropic universe on large scales. In fact, the standard Big Bang cosmology based on radiation and matter dominated epochs can be well described within the framework of GR [59].
However, the recent developments in observational cosmology and astronomy such as the apparent discovery of the accelerated expansion of the Universe and the existence of dark matter have put theoretical physics in general and cosmology in particular into crisis. Despite the progress made in our understanding of the Universe through GR and the standard model of cosmology, they fail to explain those highlighted problems and other problems arising from current observational cosmology. A study of higher-order theories of gravity have already been proposed to address these theoretical challenge. There are different alternatives to GR, namely modified theories of gravity such as Dvali-Gabadadze-Porrati gravity [60], Braneworld gravity, the Einstein-ether theory, $f(R)$ theories of gravity and scalar-tensor theory which are the best-known alternative to general relativity. Among these gravitational theories, we focus on scalar-tensor theories and $f(R)$ theories.

3.2 $f(R)$ theories of gravity

These are fourth-order theories of gravity considered to be one of the most studied alternative theories. They are based on the modification of the gravitational action. They are generally obtained by including higher-order curvature invariants in the Einstein-Hilbert action, or by making the action non-linear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R [5]. $f(R)$ theories were first proposed by Buchdal [61] in 1970 and can be divided into three formalisms based on which variational principle is used, the metric, Palatini and metric-affine $f(R)$ formalisms.

The Lagrangian in the Einstein-Hilbert action is given as [4, 62]

$$S_{EH} = \frac{1}{2}\int \sqrt{-g}\Big(R + 2\mathcal{L}_m(g_{\mu\nu}, \psi)\Big)d^4x \, . \tag{3.1}$$

where $\mathcal{L}_m$ is the matter Lagrangian, g is the determinant of the metric tensor $g_{\mu\nu}$ and ψ is the matter field. By replacing the Ricci scalar R with a function $f(R)$ in the Einstein-Hilbert action in

Eq. 3.1, we will get the action of $f(R)$ gravity as [5,63]

$$S_{f(R)} = \frac{1}{2\kappa}\int \sqrt{-g}\Big(f(R) + 2\mathcal{L}_m(g_{\mu\nu}, \psi)\Big)d^4x \,. \tag{3.2}$$

The idea being that, because of the arbitrary function introduced in the Lagrangian, there is more freedom to explain the observed cosmic acceleration and large-scale structure formation without the inclusion of exotic matter and energy. However, not all functional forms of these models can be viable cosmological models: a wide range of them can be ruled out based on observations, cosmological and astrophysical, while others can be rejected because of theoretical pathologies [5].

One can apply different variational principles to the Einstein-Hilbert action Eq. 3.1 in order to derive Einstein's field equations. In the coming subsections, we will go further into the discussion of such formalisms.

3.2.1 The metric $f(R)$ formalism

In the metric (or second-order) formalism, the action is varied with respect to the metric g_{ab} in order to obtain the field equations [1–3, 59, 64, 65]

$$T^R_{\mu\nu} = f^{'}(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla_\mu\nabla^\mu\Big)f^{'}(R)\,, \tag{3.3}$$

where $f = f(R)$ and $f^{'} = \dfrac{df}{dR}$.
The energy-momentum tensor of matter is given as

$$T^m_{\mu\nu} = -\frac{2\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\sqrt{-g}\delta g^{\mu\nu}}\,.$$

From the last two terms of Eqs. 3.3, we notice that the field equations obtained in $f(R)$ are of fourth-order partial differential equations in the metric. However, the fourth-order terms vanish when $f^{'}$ is a constant, i.e. for an action which is linear in R. Thus, it is straightforward for these equations to reduce to the Einstein equations once $f(R) = R$ [3].

3.2.2 The Palatini $f(R)$ formalism

In the Palatini (or first-order) formalism, the metric and the connection are assumed to be independent variables and the action varies with respect to both of them.
The action is formally the same as the action in Eq. 3.2 but now the Riemann tensor and the Ricci tensor are constructed with the independent connection $\Gamma^\gamma_{\mu\nu}$. We denote the Ricci tensor constructed with this independent connection as $\mathcal{R}_{\mu\nu}$ and the corresponding Ricci scalar is $\mathcal{R} = g_{\mu\nu}\mathcal{R}_{\mu\nu}$. The action now takes the form [1, 2, 4, 59, 62, 64, 65]:

$$S_{Pal} = \frac{1}{2}\int \sqrt{-g}\Big[f(\mathcal{R}) + 2\mathcal{L}_m(g_{\mu\nu}, \psi)\Big)\Big]d^4x \,. \tag{3.4}$$

The field equations are given as [3,5,62]

$$G_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{f'}\right) + \frac{1}{f'}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f^{'} - \frac{3}{2}\Big(\nabla_\mu f^{'}\nabla_\nu f^{'} - \frac{1}{2}g_{\mu\nu}(\nabla f^{'})^2\Big)\,. \tag{3.5}$$

We note that there are no second-order covariant derivatives of f, and that is why the Palatini formalism is known as a first-order approach [5].

3.2.3 The metric-affine $f(R)$ formalism

In the metric-affine formalism, the metric and the connection are independent, as in the case of the Palatini formalism. However, in this metric-affine theories of gravity there is a direct coupling between matter and connection [66, 67]. The action varies with respect to the metric and the affine connection neglecting the assumption that the matter action is independent of the connection.
Hence the matter action would be a function of the independent metric connection and the matter fields $\mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \Gamma^{\gamma}_{\mu\nu}, \psi)$. The action is therefore

$$S_{ma} = \frac{1}{2\kappa}\int \sqrt{-g}\Big(f(\mathcal{R}) + 2\mathcal{L}_m(g_{\mu\nu}, \Gamma^{\gamma}_{\mu\nu}, \psi)\Big)d^4x \,. \tag{3.6}$$

By varying the action independently with respect to the metric and the connection, we get the field equations as [62]

$$f^{'}(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = T_{\mu\nu} \,, \tag{3.7}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\left[-\tilde{\nabla}_{\lambda}\left(\sqrt{-g}f^{'}(\mathcal{R}g^{\mu\nu})\right) + \tilde{\nabla}_{\sigma}\left(\sqrt{-g}f^{'}(\mathcal{R})g^{\mu\sigma}\right)\right] \\ &+ 2f^{'}(\mathcal{R})g^{\mu\nu}S^{\sigma}_{\lambda\sigma} = \left[\Delta^{\mu\nu}_{\lambda} - \frac{2}{3}\Delta^{\sigma[\nu}_{\sigma}\delta^{\mu]}_{\lambda}\right] \,, \end{aligned} \tag{3.8}$$

$$S^{\sigma}_{\mu\sigma} = 0 \,, \tag{3.9}$$

where $\tilde{\nabla}_{\lambda}$ denotes the covariant derivative defined with the independent connection $\Gamma^{\mu}_{\lambda\nu}$ and $\Delta^{\mu\nu}_{\lambda}$ is called the hyper-momentum tensor and mimics the role of the energy momentum tensor. $S^{\sigma}_{\mu\sigma}$ is the Cartan-Torsion tensor which is connected to the antisymmetric part of the connection

$$S^{\sigma}_{\mu\sigma} = \Gamma^{\sigma}_{[\mu\sigma]} \,,$$

where $[\mu\sigma]$ denotes the anti-symmetrisation over the indices μ and σ respectively [68].

An important fact about $f(R)$ theories of gravity is that, not all of $f(R)$ models are viable or accepted. The accepted ones have to meet some conditions. They have to reduce to GR in special cases and exhibit correct behaviour of cosmological perturbations.
For any class of $f(R)$ gravity, the following are binding conditions on the Lagrangian for consistent cosmological viability on all scales [5, 69–74]:

- $f^{''} > 0$ for high curvatures [75]. This is the requirement for a classically stable high curvature regime and for the existence of a matter-dominated phase in the cosmological evolution [69].
- $f^{'} > 0$ for all Ricci scalar curvature values, so that the effective Newtonian gravitational constant to be positive at all times.
- $|f^{'} - 1| \ll 1$ at recent epochs and is not a necessary condition for the ongoing cosmic acceleration.

3.3 Scalar-tensor theories of gravity

The scalartensor theory of gravity is one of the most accepted alternatives to Einstein's theory of gravity [76]. It was invented first by Jordan [77] in 1955 and then taken over by Brans and Dicke [78] to try to explain the gravitational interaction through both a scalar field and a tensor field. It is a higher-order theory, where degrees of freedom to explain different scenarios are present. Some of those degrees of freedom are scalar field, the coupling constant and cosmological constant as well [79, 80]. The general form of the action of the scalar-tensor theory is given as follows:

$$S_{ST} = \frac{1}{2}\int d^4x\sqrt{-g}\Big[f(\phi)R - g(\phi)\nabla_{\mu}\phi\nabla^{\mu}\phi - 2\Lambda(\phi) + \mathcal{L}_m(g_{\mu\nu}\psi)\Big] \,. \tag{3.10}$$

where f, g and Λ are some arbitrary functions of the scalar field ϕ [1, 80–83]. By a redefinition of the scalar field ϕ we can now set $f(\phi) \to \phi$, without loss of generality. The Lagrangian density in Eq. 3.10 can then be written as [1, 80]

$$S_{ST} = \frac{1}{2}\int d^4x\sqrt{-g}\Big[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi) + \mathcal{L}_m(g_{\mu\nu}\psi)\Big] , \tag{3.11}$$

where $\omega(\phi)$ is an arbitrary function, often referred to as the coupling parameter.

The variation of the action in Eq. 3.11 with respect to the metric $g_{\mu\nu}$ gives the field equations as

$$\phi G_{\mu\nu} + \Big[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2 + \Lambda\Big]g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = T_{\mu\nu} . \tag{3.12}$$

Now, as well as the metric tensor $g_{\mu\nu}$, these theories also contain the dynamical scalar field ϕ, and so we must vary the action derived from Eq. 3.11 with respect to this additional degree of freedom. After eliminating R with the trace of Eq. 3.12, this yields [1]

$$(2\omega + 3)\Box\phi + \omega^{'}(\nabla\phi)^2 + 4\Lambda - 2\phi\Lambda^{'} = T , \tag{3.13}$$

where primes here denote differentiation with respect to ϕ. These are the field equations of the scalar-tensor theories of gravity.

3.3.1 The Brans-Dicke theory

The Brans-Dicke theory (BD) [78] is one of the special classes of the scalar-tensor theory, where the coupling parameter $\omega(\phi)$ is considered to be constant.
By setting $\omega(\phi)$ to be a constant and dropping the cosmological constant in the action given in Eq. 3.11, we write the BD action as

$$S_{BD} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + 2\mathcal{L}_m\right] . \tag{3.14}$$

The BD theory is based on certain assumptions. The matter Lagrangian $\mathcal{L}_m$ is assumed not to contain the scalar field ϕ and the scalar field is also assumed to be cosmic time dependent only [76]. When a homogeneous and isotropic universe is assumed, the Einstein-Hilbert fundamental constant of gravitational G is related to the gravitation constant G_1 in the BD theory by [84]

$$G_1 = \frac{4+2\omega}{3+2\omega}G . \tag{3.15}$$

When the coupling constant ω tends to infinity in Eq. 3.15, the BD theory is reduced to Einstein gravity [84].

3.4 $f(R)$ theory as scalar-tensor theory of gravitation

$f(R)$ theories have been shown to be a sub-class of the scalar-tensor theory of gravity [85]. A clear illustration of how $f(R)$ theories are classified as a subclass of a scalar-tensor theory is in the Brans-Dicke theory for the case of the coupling constant $\omega = 0$ in Eq. 3.14. $f(R)$ theory is related to scalar-tensor theory by defining the scalar field ϕ to be a function of Ricci scalar R, by which each $f(R)$ has a potential $U(\phi)$.
This work makes use of the equivalence between $f(R)$ theory in its metric formalism and the scalar-tensor theories of gravity.
In this equivalence the scalar-tensor action takes the form [5, 86, 87]

$$S_{f(\phi)} = \frac{1}{2\kappa}\int\sqrt{-g}\Big(f(\phi(R)) + 2\mathcal{L}_m(g_{\mu\nu}, \psi)\Big)d^4x , \tag{3.16}$$

where $f(\phi(R))$ is now a function of $\phi(R)$ and we define the scalar field ϕ to be

$$\phi = f' - 1 . \tag{3.17}$$

Here the prime indicates differentiation with respect to R and the scalar field should be invertible [1,62,80,85]. This definition allows ϕ to be an extra degree of freedom that vanishes in GR. Therefore, by comparing the scalar-tensor action in Eq. 3.16 to that action of the BD in Equation 3.14 for the case of vanishing coupling constant $\omega = 0$, we see that the action of the BD results in generalized field equations that describe the same cosmological dynamics as the action od scalar-tensor in Eq. 3.16. These two actions are dynamically equivalent, thus $f(R)$ theory is a special case of the scalar-tensor theory.
The field equations of the action in Eq. 3.17 can be now written as [5,80]

$$G_{ab} = \frac{T^m_{ab}}{(\phi+1)} + \frac{1}{(\phi+1)}\left[\frac{1}{2}g_{ab}\left(f-(\phi+1)R\right) + \nabla_a\nabla_b\phi - g_{ab}\nabla_c\nabla^c\phi\right] . \tag{3.18}$$

By decomposing the right hand side of Eq. 3.18 into two cosmological fluids (the standard matter and scalar field), Eq. 3.18 can be rewritten as

$$G_{ab} = \tilde{T}^m_{ab} + T^\phi_{ab} , \tag{3.19}$$

where $\tilde{T}^m_{ab} = \dfrac{T^m_{ab}}{(\phi+1)}$ is the energy-momentum tensor for the effective matter fluid and

$$T^\phi_{ab} = \frac{1}{(\phi+1)}\left[\frac{1}{2}g_{ab}\left(f-(\phi+1)R\right) + \nabla_a\nabla_b\phi - g_{ab}\nabla_c\nabla^c\phi\right] . \tag{3.20}$$

is the energy momentum tensor for the scalar field.

3.5 Background thermodynamics of scalar-tensor gravity

The linearised thermodynamic quantities for the scalar field are given by

$$\mu_\phi = \frac{1}{(\phi+1)}\left[\frac{1}{2}\Big(R(\phi+1) - f\Big) - \Theta\dot{\phi} + \tilde{\nabla}^2\phi\right] . \tag{3.21}$$

$$p_\phi = \frac{1}{(\phi+1)}\left[\frac{1}{2}\Big(f - R(\phi+1)\Big) + \ddot{\phi} - \frac{\dot{\phi}\dot{\phi}'}{\phi'} + \frac{\phi''\dot{\phi}^2}{\phi'^2} + \frac{2}{3}(\Theta\dot{\phi} - \tilde{\nabla}^2\phi)\right] , \tag{3.22}$$

$$q^\phi_a = -\frac{1}{(\phi+1)}\left[\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right]\tilde{\nabla}_a\phi , \tag{3.23}$$

$$\pi^\phi_{ab} = \frac{\phi'}{(\phi+1)}\left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\Big(\frac{\dot{\phi}}{\phi'}\Big)\right] . \tag{3.24}$$

where μ_m and p_m are the energy density and isotropic pressure of standard matter, respectively. The Ricci scalar is given by

$$R = 2\dot{\Theta} + \frac{4}{3}\Theta^2 . \tag{3.25}$$

The total (*effective*) energy density, isotropic pressure, anisotropic pressure and heat flux of standard matter and scalar field combination are defined as

$$\mu \equiv \frac{\mu_m}{(\phi+1)} + \mu_\phi, \quad p \equiv \frac{p_m}{(\phi+1)} + p_\phi, \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{(\phi+1)} + \pi^\phi_{ab}, \quad q_a \equiv \frac{q^m_a}{(\phi+1)} + q^\phi_a . \tag{3.26}$$

In the FLRW spacetime universe, the Friedmann and Raychaudhuri Eqs. 2.4 and 2.5 that govern the expansion history of the Universe can be wrirtten as follows due to the resulting non-trivial field equations [88]:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2(\phi+1)}\left[\mu_m + 3p_m + f - R(\phi+1) + \Theta\dot{\phi} + 3\phi''\left(\frac{\dot{\phi}^2}{\phi'^2}\right) + 3\ddot{\phi} - 3\frac{\dot{\phi}\dot{\phi}'}{\phi'}\right] . \quad (3.27)$$

$$\Theta^2 = \frac{3}{(\phi+1)}\left[\mu_m + \frac{R(\phi+1) - f}{2} + \Theta\dot{\phi}\right] . \quad (3.28)$$

Chapter 4

Some applications of the $f(R)$-scalar tensor equivalence

In this chapter we present some applications to the reconstruction techniques for models of $f(R)$ gravity through the scalar-tensor theory of gravity, using the equivalence between $f(R)$ theory in its metric formalism and the scalar-tensor theories of gravity in the Brans-Dicke theory.

4.1 Reconstructing $f(R)$ gravity from a Chaplygin scalar field in de Sitter spacetimes

The Chaplygin Gas (CG) model in cosmology has been considered as another alternative to the cosmological FLRW universe models with a perfect fluid equation of state and negative pressure [89, 90]. The model provides interesting features of the cosmic expansion history consistent with a smooth transition between an inflationary phase and a matter-dominated decelerating era. In addition, the late-time accelerated de Sitter phase of cosmic expansion can be achieved [91–94]. In the following subsections, we consider the treatment of the Chaplygin gas as a scalar fluid. Using the equivalence between scalar-tensor theory (BD theory) and $f(R)$ theory of gravity, we obtain the energy density of the Chaplygin gas in terms of the scalar field, and from the Chaplygin gas property, we obtain the Chaplygin gas pressure. The reason behind this treatment is that a lot of work has been done that suggests that the Chaplygin gas can be treated as a unified model for dark matter and dark energy [89–94]. The model has, however, some subtle and unsolved problems such as the negative sound speed of the fluid, which in turn implies the instability of density perturbations, or the fact that the sound speed behaves differently depending on the epoch. In [95, 96], it has been shown that the Chaplygin gas model for early times has a negligible sound speed and superluminal speeds in the late universe. Because the model shows such issues if we treat it as a form of matter, we reconstruct a gravity model that explains the same cosmic expansion history, but is a manifestation of geometry instead.

4.1.1 The original Chaplygin gas model

In the original treatment, the negative pressure associated with the Chaplygin gas model is related to the (positive) energy density through the EoS [89, 93, 97]

$$p = -\frac{A}{\mu}\,. \tag{4.1}$$

where A is a positive constant. Using this equation of state in the conservation equation of μ given by

$$\dot{\mu} + 3\frac{\dot{a}}{a}(\mu + p) = 0 , \tag{4.2}$$

we find that the energy density of the Chaplygin gas evolves with respect to the scale factor $a(t)$ as

$$\mu(a) = \sqrt{A + \frac{B}{a^6}} , \tag{4.3}$$

where B is a constant of integration. One can see that for large $a(t)$, the energy density becomes independent of the scale factor for positive B. This can be thought of as an empty universe with a cosmological constant (see more detailed analysis in Ref. [93]). In Ref. [93], it has been pointed out that for the early universe, the approximated expression of energy density can represent a universe that contains pressureless dust matter. From the Friedmann equation, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\mu}{3} - \frac{k}{a^2} . \tag{4.4}$$

where k stands for curvature and can be either $-1, 0$ or 1, depending on the spacetime geometry. If we restrict ourselves to the case where the Universe is flat, i.e., $k = 0$, then

$$\dot{a} = a\sqrt{\frac{\mu}{3}} . \tag{4.5}$$

The energy density of the scalar field and pressure are given as [89, 98]

$$\mu_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) , \tag{4.6}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) . \tag{4.7}$$

Adding Eqs. 4.6 and 4.7 and using Eq. 4.3, we get

$$\dot{\phi}^2 = \frac{B}{a^3\sqrt{Aa^6 + B}} . \tag{4.8}$$

Using the following simple trick, where in this case, a prime denotes partial differentiation w.r.t the scale factor a,

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\partial\phi}{\partial a}\frac{\partial a}{\partial t} = \phi'\dot{a} . \tag{4.9}$$

and substituting the expression of Eq. 4.5, we obtain

$$\dot{\phi} = \phi' a\sqrt{\frac{\mu}{3}} . \tag{4.10}$$

Eq. 4.8 can therefore be rewritten as

$$\phi'^2 = \frac{3B}{a^2\left(Aa^6 + B\right)} \implies \phi' = \pm\frac{\sqrt{3B}}{a\sqrt{Aa^6 + B}} . \tag{4.11}$$

This equation can be integrated with respect to a to yield

$$\phi(a) = \pm\frac{1}{2\sqrt{3}}\ln\left(\frac{Aa^6}{Aa^6 + 2B + 2\sqrt{ABa^6 + B^2}}\right) + C_1 , \tag{4.12}$$

where C_1 is a constant of integration. Our next step will be to reconstruct the $f(R)$ functional which produces ϕ per Eq. 3.17. To do so, we first relate the Ricci scalar R and the scale factor a using the trace equation

$$R = \mu - 3p = \frac{4A + \frac{B}{a^6}}{\sqrt{A + \frac{B}{a^6}}} , \tag{4.13}$$

which can also be rearranged as

$$Ra^6\sqrt{A\left(1 + \frac{B}{Aa^6}\right)} - 4Aa^6 - B = 0 . \tag{4.14}$$

From the equation describing energy density μ_ϕ and pressure p_ϕ, we solve for the potential as

$$V(\phi) = \frac{\mu_{\dot\phi} - p_\phi}{2} = \frac{2Aa^6 + B}{2a^3\sqrt{Aa^6 + B}} . \tag{4.15}$$

The de Sitter universe is a solution of the Einstein field equations with no standard matter sources but a positive cosmological constant or a scalar field. The Friedmann equation for such a universe reduces to

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\Lambda}{3} , \tag{4.16}$$

with the scale factor exponentially evolving as [98, 99]

$$a(t) = De^{\sqrt{\frac{\Lambda}{3}}t} = De^{mt} . \tag{4.17}$$

Using this solution for the scale factor at the early stages of the Universe's evolution, one can obtain the characteristic potential that is solely dependent on the Universe's cosmic time.

4.1.2 Reconstruction of $f(R)$ gravity from the original Chaplygin gas model

4.1.2.1 Case 1: early universe

For the early universe, we assume that the scale factor a is small enough for the approximation $Aa^6/B << 1$ to hold, that is, we can make the following treatment

$$\sqrt{B^2\left(1 + \frac{Aa^6}{B}\right)} \approx B . \tag{4.18}$$

Replacing this in Eq. 4.12, we have

$$\phi(a) = \mp\frac{1}{2\sqrt{3}}\ln\left(1 + \frac{4B}{Aa^6}\right) + C_1 . \tag{4.19}$$

From Eq. 4.14, we have

$$Ra^3\sqrt{B} - B = 0 . \tag{4.20}$$

This means that we can write

$$a^3 = \frac{\sqrt{B}}{R} , \tag{4.21}$$

and hence replacing Eq. 4.21 in Eq. 4.19, we have

$$\phi(R) = \pm\frac{1}{2\sqrt{3}}\ln\left(1 + \frac{4R^2}{A}\right) + C_1 . \tag{4.22}$$

By substituting Eq. 4.22 into Eq. 3.17 and integrating, we obtain

$$f(R) = \pm\left[\frac{\sqrt{3}R}{6}\ln\left(\frac{4R^2+A}{A}\right) + \frac{\sqrt{3A}}{6}\arctan\left(\frac{2R}{4A}\right) - \frac{\sqrt{3}R}{3} + C_1 R\right] + R + C_2\,, \qquad (4.23)$$

where C_2 is a constant of integration, see Fig. 4.1.

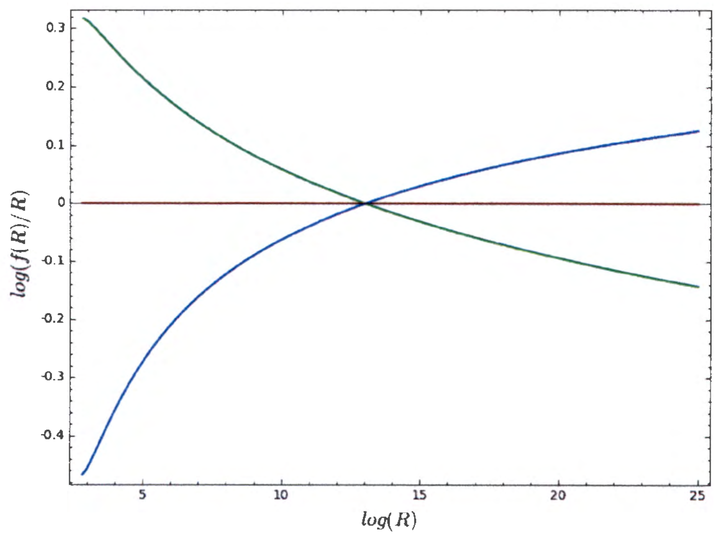


Figure 4.1: Plot of $f(R)/R$ against Ricci scalar R for the early universe, setting $A = 0.5$, $C_2 = 0$ and $C_1 = -0.4$. The red line represents the GR case, while the green and blue curves depict negative and positive $f(R)$ solutions respectively.

Combining Eqs. 4.6 and 4.7 and use the fact that

$$\sqrt{A + \frac{B}{a^6}} = \frac{1}{a^3}\sqrt{B}\sqrt{1 + \frac{Aa^6}{B}} \approx \frac{\sqrt{B}}{a^3}\,, \qquad (4.24)$$

we obtain the potential $V(a)$ as

$$V(a) = \frac{\sqrt{B}}{a^3}\,. \qquad (4.25)$$

We get $a(\phi)$ from Eq. 4.19 as

$$a^3 = \frac{2\sqrt{B}}{\sqrt{A}(e^{\pm 2\sqrt{3}(\phi - C_1)} - 1)^{1/2}}\,. \qquad (4.26)$$

Therefore, the potential is given as

$$V(\phi) = \frac{\sqrt{A}}{2}\left(e^{\pm 2\sqrt{3}(\phi - C_1)} - 1\right)^{\frac{1}{2}}\,. \qquad (4.27)$$

We set the constants such that $V(\phi)$ is given as

$$V(\phi) = \frac{1}{2}\left(e^{\pm 2\sqrt{3}\phi} - 1\right)^{\frac{1}{2}}\,. \qquad (4.28)$$

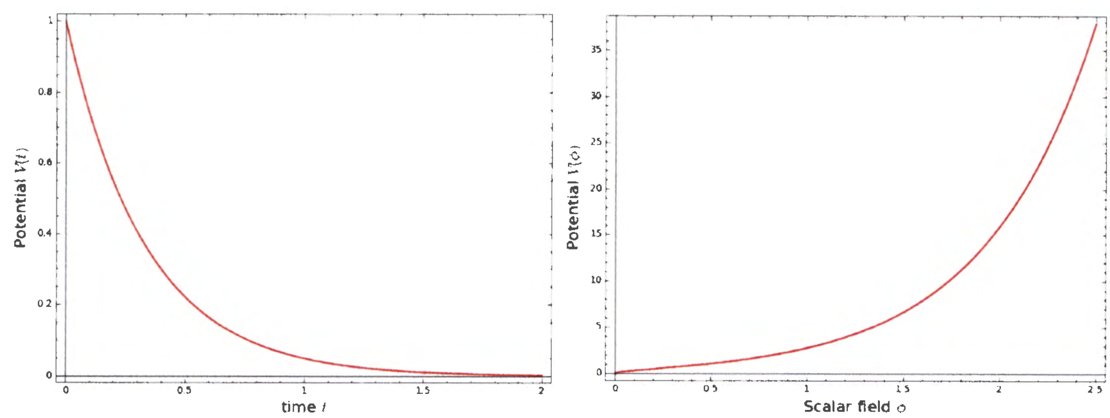
Then the time-dependent potential can be obtained as

$$V(t) = \frac{\sqrt{B}}{D^3 e^{3mt}} \,. \tag{4.29}$$

By considering positive potential from Eq. 4.28, one can write

$$V(\phi) = \frac{\sqrt{A}}{2}\left(e^{2\sqrt{3}\phi} - 1\right)^{\frac{1}{2}} \,. \tag{4.30}$$

Fig. 4.2 shows how the potential changes with scalar field and time.



(a) (b)

Figure 4.2: Plots of the potential V as a function of ϕ and t from Eqs. 4.30 and 4.29 respectively for values $A = B = m = D = 1$ and $C = 0$.

4.1.2.2 Case 2: late universe

The term on the denominator 4.12 can be approximated with the assumption that for the late universe, the scale factor a is large enough such that $B/Aa^6 << 1$ holds. Therefore, we have the term as

$$2\sqrt{ABa^6 - B^2} = 2\sqrt{ABa^6\Big(1 - \frac{B}{Aa^6}\Big)} \approx 2\sqrt{AB}a^3 \,. \tag{4.31}$$

With this approximation, Eq. 4.12 becomes

$$\phi(a) = \pm\frac{1}{2\sqrt{3}}\ln\Big(1 + \frac{2B}{Aa^6} + \frac{2\sqrt{B}}{\sqrt{A}a^3}\Big) + C_1 \,. \tag{4.32}$$

Therefore Eq. 4.32 takes the form

$$\phi(a) = \pm\frac{1}{2\sqrt{3}}\ln\Big(\frac{2\sqrt{B}}{\sqrt{A}a^3}\Big) + C_1 \,. \tag{4.33}$$

Eq. 4.14 results in

$$a^6 = \frac{\sqrt{B}}{\sqrt{R\sqrt{A} - 4A}} \,, \quad R \neq 4\sqrt{A}. \tag{4.34}$$

Therefore, we update Eq. 4.32 as

$$\phi(R) = \pm\frac{1}{2\sqrt{3}}\ln\Big(1 + \frac{2}{\sqrt{A}}\sqrt{R\sqrt{A} - 4A}\Big) + C_1 \,. \tag{4.35}$$

Replacing Eq. 4.35 in Eq. 3.17, we have $f(R)$ as

$$\begin{aligned} f(R) = \pm\Big[&\frac{\sqrt{3}}{6}\Big(R - \frac{17\sqrt{A}}{4}\Big)\ln\Big(1 + \frac{2\sqrt{R\sqrt{A} - 4A}}{\sqrt{A}}\Big) + \frac{\sqrt{3}}{12}\sqrt{R\sqrt{A} - 4A} + \Big(-\frac{\sqrt{3}}{12} + C_1\Big)R \\ &+ \frac{19\sqrt{3A}}{48}\Big] + R + C_3 \,. \end{aligned} \tag{4.36}$$

where C_3 is a constant of integration, see Fig. 4.3.

By combining Eqs. 4.6 and 4.7 and using the large-scale-factor approximation, we have the potential as

$$V(\phi) = \sqrt{A} \,. \tag{4.37}$$

As a combination of Eqs. 4.6 and 4.7 results in

$$V(a) = \frac{2A + \frac{B}{a^6}}{2\sqrt{A + \frac{B}{a^6}}} \,, \tag{4.38}$$

one can start from dependence of scale factor a on scalar field ϕ to have the potential $V(\phi)$ as

$$V(\phi) = \frac{2A + \frac{A}{4}\Big(e^{\mp 2\sqrt{3}\phi - C_1}\Big)^2}{2\sqrt{A + \frac{A}{4}\Big(e^{\mp 2\sqrt{3}\phi - C_1}\Big)^2}} \,. \tag{4.39}$$

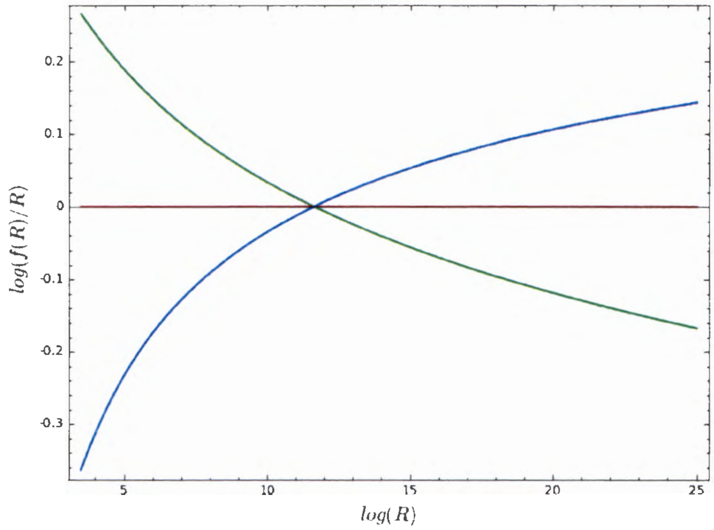


Figure 4.3: Plot of $f(R)/R$ against Ricci scalar R for the late universe, setting $A = 0.4$, $C_3 = 0$ and $C_1 = -0.4$. The red line represents the GR case, while the green and blue curves depict negative and positive $f(R)$ solutions respectively.

In principle, we can constrain the scalar field ϕ by equating Eqs. 4.37 and 4.39. Once this is done, one has a constant scalar field given by

$$\phi = \pm\frac{C_1}{2\sqrt{3}}\,. \tag{4.40}$$

So far, we have considered the original model of the Chaplygin gas and have obtained expressions for $f(R)$ that correspond to the early and late universe along with their corresponding potentials. In the following, we present a similar analysis for the generalized Chaplygin model.

4.1.3 Reconstruction of $f(R)$ gravity from the generalized Chaplygin gas model

For the generalized Chaplygin gas, we have the pressure as [89, 93, 97]

$$p = -\frac{A}{\mu^{\alpha}}\,. \tag{4.41}$$

where $0 \leq \alpha \leq 1$ and A is a constant.

$$\mu(a) = \left(A + Ba^{-3(\alpha+1)}\right)^{\frac{1}{1+\alpha}}\,. \tag{4.42}$$

For a flat universe, $k = 0$, we can write the Friedmann equation as

$$\dot{a} = a\sqrt{\frac{\mu}{3}}\,. \tag{4.43}$$

Combining Eqs. 4.6 and 4.7 results in

$$\dot{\phi}^2 = \frac{Ba^{-3(\alpha+1)}}{\left(A + Ba^{-3(\alpha+1)}\right)^{\frac{\alpha}{1+\alpha}}}\,. \tag{4.44}$$

Therefore, we can write

$$\phi' = \pm\sqrt{\frac{3B}{Aa^{3\alpha+5} + Ba^2}} \ . \tag{4.45}$$

4.1.3.1 Case 1: early universe

For the early universe, the scale factor is assumed to be small enough, therefore, we can write Eq. 4.45 as

$$d\phi = \pm\frac{\sqrt{3B}}{\sqrt{Ba^2(1+\frac{Aa^{3(\alpha+1)}}{B})}}da \ . \tag{4.46}$$

then using the fact that $(1+x)^n \approx 1 + nx + O(x^2)$, for $x << 1$, we have

$$d\phi = \pm\frac{\sqrt{3}}{a(1+\frac{Aa^{3(\alpha+1)}}{2B})}da \ . \tag{4.47}$$

Integrating this equation, we have

$$\phi(a) = \pm\frac{\sqrt{3}}{3(\alpha+1)}\ln\left(\frac{2B}{a^{3(\alpha+1)}} + A\right) + C_1 \ . \tag{4.48}$$

we have $a(R)$ as

$$a(R) = \frac{B^{\frac{1}{3(\alpha+1)}}}{R^{1/3}} \ . \tag{4.49}$$

Therefore, we have $\phi(R)$ as

$$\phi(R) = \pm\frac{\sqrt{3}}{3(\alpha+1)}\ln(2R^{\alpha+1} + A) + C_1 \ . \tag{4.50}$$

Substituting Eq .4.50 into Eq. 3.17 and integrating, we obtain

$$f(R) = \pm\left[\frac{\sqrt{3}\Big(R(\alpha+1) - R\ln\left(2R^{\alpha+1}+A\right)\Big)}{3(\alpha+1)} - \frac{\sqrt{3}A}{3}\int\frac{dR}{2R^{\alpha+1}+A} + C_1 R\right] + R + C_4 \ , \tag{4.51}$$

where C_4 is a constant the integration, see Fig. 4.4.

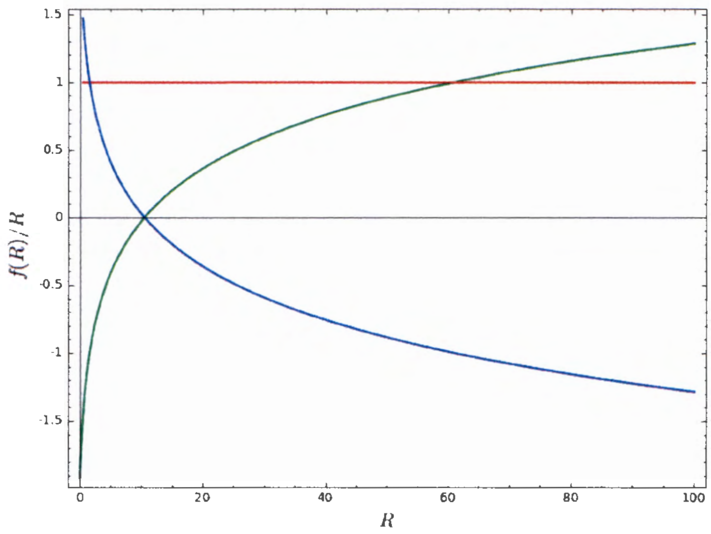


Figure 4.4: Plot of $f(R)/R$ against Ricci scalar R for the early universe, setting $A = 0.7$, $C_4 = 0$, $C_1 = 0.4$ and $\alpha = 0.2$. The red line represents the GR case, while the green and blue curves depict negative and positive $f(R)$ solutions respectively.

The potential is given by

$$V(\phi) = \frac{\mu_\phi - p_\phi}{2} \ . \tag{4.52}$$

Considering energy density as

$$\mu = \left(A + Ba^{-3(\alpha+1)}\right)^{\frac{1}{\alpha+1}} \tag{4.53}$$

and pressure as

$$p = -\frac{A}{\mu^\alpha} \ . \tag{4.54}$$

we have

$$V(a) = \frac{2A + \frac{B}{a^{3(\alpha+1)}}}{2(A + Ba^{-3(\alpha+1)})^{\alpha/(\alpha+1)}} \ . \tag{4.55}$$

By treating the denominator, we have

$$(A + Ba^{-3(\alpha+1)})^{\frac{\alpha}{(\alpha+1)}} = \left(\frac{B}{a^{3(\alpha+1)}}\left(\frac{a^{3(\alpha+1)}}{B} + 1\right)\right)^{\frac{\alpha}{\alpha+1}} \approx \left(\frac{B}{a^{3(\alpha+1)}}\right)^{\frac{\alpha}{\alpha+1}} \ . \tag{4.56}$$

where we have considered that $a^{3(\alpha-1)}$ is small enough in the early universe. So the potential becomes

$$V(a) = \frac{2A + \frac{B}{a^{3(\alpha+1)}}}{2\left(\frac{B}{a^{3(\alpha+1)}}\right)^{\frac{\alpha}{\alpha+1}}} \ . \tag{4.57}$$

From Eq. 4.48, we have

$$a^{3(\alpha+1)} = \frac{2B}{e^{\mp\frac{3\phi(\alpha+1)}{\sqrt{3}}} - A} \ . \tag{4.58}$$

If we substitute Eq. 4.58 into Eq. 4.57, we get the potential V as a function of ϕ as

$$V(\phi) = \frac{1}{2^{\frac{\alpha+2}{\alpha+1}}B}\left(2A + \frac{1}{2}(e^{\pm\frac{3\phi(\alpha+1)}{\sqrt{3}}} - A)\right)\left(e^{\pm\frac{3\phi(\alpha+1)}{\sqrt{3}}} - A\right)^{\frac{1}{\alpha+1}} \ . \tag{4.59}$$

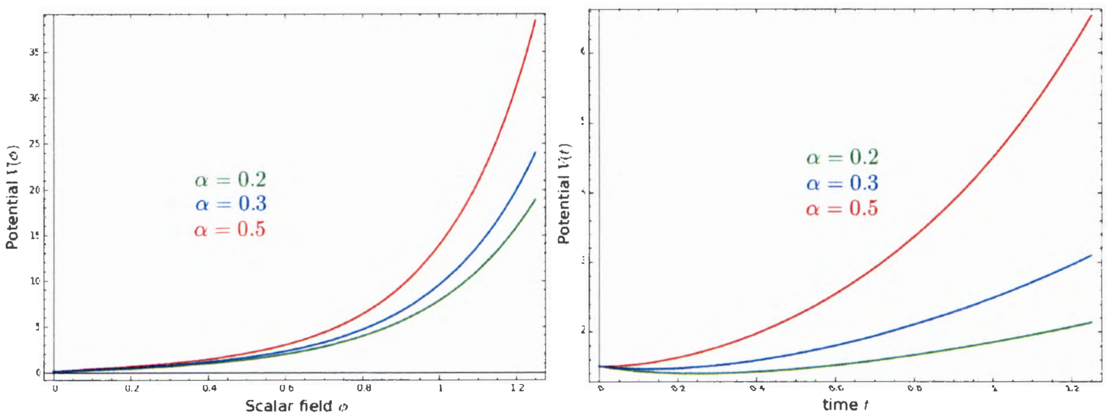
We only consider positive potential as

$$V(\phi) = \frac{1}{2^{\frac{\alpha-2}{\alpha-1}}B}\Big(2A + \frac{1}{2}(e^{\frac{3\phi(\alpha+1)}{\sqrt{3}}} - A)\Big)\Big(e^{\frac{3\phi(\alpha+1)}{\sqrt{3}}} - A\Big)^{\frac{1}{\alpha+1}} . \tag{4.60}$$

Subsequently, Eqs. 4.57 and 4.17 give rise to

$$V(t) = \frac{1}{2B^{\frac{\alpha}{\alpha+1}}}\Big(2A + B(De^{mt})^{\ 3(\alpha+1)}\Big)(De^{mt})^{3\alpha} . \tag{4.61}$$

Fig. 4.5, shows how the potential changes with scalar field and time.



(a) (b)

Figure 4.5: Plots of the potential V as a function of ϕ and t from Eqs. 4.60 and 4.61 respectively, setting $A = B = m = D = 1$ and $\alpha = 0.5$ (red), $\alpha = 0.3$ (blue) and $\alpha = 0.2$ (green).

4.1.3.2 Case 2: late universe

For the late universe, we have the following assumption

$$Ba^2 << Aa^{3\alpha-5} . \tag{4.62}$$

Thus we have

$$d\phi \approx \sqrt{\frac{3B}{Aa^{3\alpha+5}}}da . \tag{4.63}$$

We perform integration to get

$$\phi(a) = \pm\frac{2\sqrt{B}}{\sqrt{3}\sqrt{A}(\alpha+1)a^{3(\alpha+1)/2}} + C_1 . \tag{4.64}$$

The next step is to get $\phi(R)$. Using the trace equation once again yields

$$R = \frac{Ba^{-3(\alpha+1)} + 4A}{\Big(A + Ba^{-3(\alpha+1)}\Big)^{\frac{\alpha}{1+\alpha}}} . \tag{4.65}$$

By manipulating Eq 4.65, we obtain

$$a(R) = \Big(\frac{RA^{\frac{\alpha}{1+\alpha}} - 4A}{B(1 - \frac{\alpha R}{1+\alpha}A^{\frac{-1}{\alpha-1}})}\Big)^{-\frac{1}{3(\alpha+1)}} . \tag{4.66}$$

Thus from Eq. 4.64, we have

$$\phi(R) = \pm\frac{2\sqrt{B}}{\sqrt{3A}(\alpha+1)}\left(\frac{RA^{\frac{\alpha}{1+\alpha}} - 4A}{B\left(1 - \frac{\alpha R}{1+\alpha}A^{\frac{1}{\alpha+1}}\right)}\right)^{\frac{1}{2}} . \tag{4.67}$$

Substituting this expression of $\phi(R)$ in Eq. 3.17, we get

$$\begin{aligned} f(R) = \pm &\left[\frac{A^2(3\alpha - 1)^2 m_1(A,\alpha,R)\arcsin\left(\frac{2\alpha A^{\frac{\alpha}{\alpha+1}}R - 5A\alpha - A}{A(3\alpha-1)}\right) + m_2(A,\alpha,R)}{4\sqrt{3}\alpha^{3/2}(\alpha+1)^{3/2}m_1(A,\alpha,R)} - C_1 R\right] \\ &+ R - C_5 . \end{aligned} \tag{4.68}$$

where C_5 is a constant of integration. Fig 4.6 shows how the Lagrangian presented in 4.68 changes with Ricci scalar R.

$$m_1(A,\alpha,R) = \sqrt{-\alpha A^{\frac{\alpha}{\alpha+1}}\left(\alpha R - (\alpha+1)A^{\frac{1}{\alpha+1}}\right)\left(RA^{\frac{\alpha}{\alpha+1}} - 4A\right)} , \tag{4.69}$$

and

$$\begin{aligned} m_2(A,\alpha,R) = &- 4\alpha^3 A^{\frac{3\alpha}{\alpha+1}}R^3 + 6(5\alpha+1)\alpha^2 A^{\frac{3\alpha+1}{\alpha+1}}R^2 - 2\alpha(33\alpha^2 - 18\alpha + 1)A^{\frac{3\alpha+2}{\alpha+1}}R \\ &- (40\alpha^3 - 48\alpha^2 - 18\alpha)A^3 . \end{aligned} \tag{4.70}$$

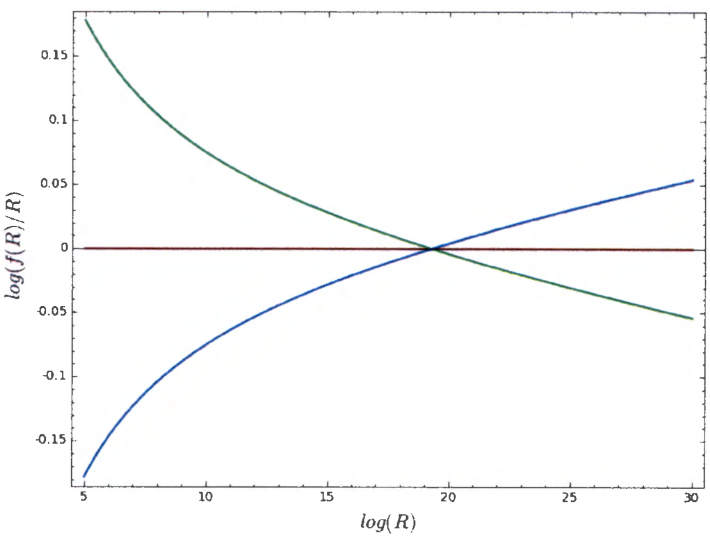


Figure 4.6: Plot of $f(R)/R$ against Ricci scalar R for the late universe, setting $A = 10^{-4}$, $C_1 = 2\times10^{-3}$, $C_5 = 1$ and $\alpha = 3\times10^{-5}$. The red line represents the GR case, while the green and blue curves depict negative and positive $f(R)$ solutions respectively.

From Eq. 4.42 in the late universe, that is together with its corresponding pressure, we have potential $V(a)$ from Eq. 4.15 as

$$V(a) = \frac{Ba^{-3(\alpha+1)} - A}{2\left(A + Ba^{-3(\alpha+1)}\right)^{\frac{\alpha}{\alpha+1}}} . \tag{4.71}$$

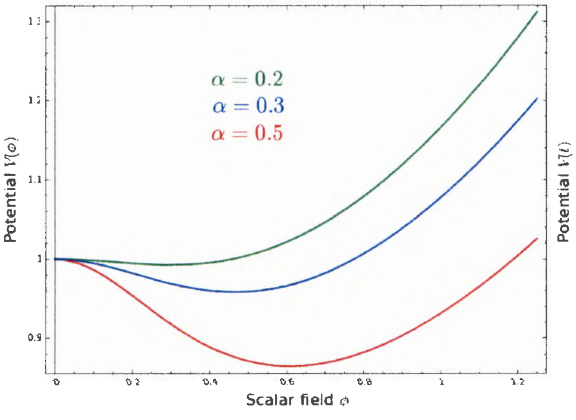
Replacing Eq. 4.64 in Eq. 4.71 we get

$$V(\phi) = \frac{3A(\alpha+1)^2\phi^2 + 8AB}{8B(A + 3A(\alpha+1)^2\phi^2)^{\frac{\alpha}{\alpha+1}}} . \tag{4.72}$$

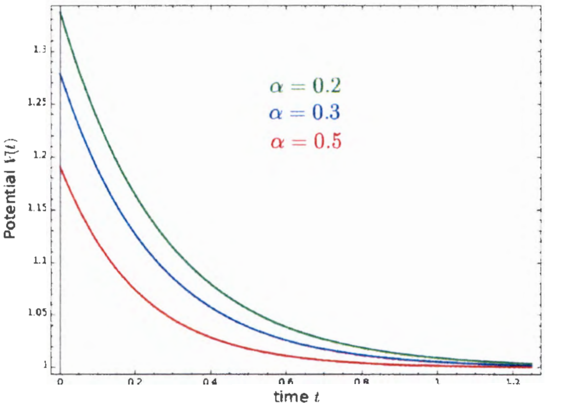
We replace the scale factor Eq. 4.17 in Eq. 4.71 to get the potential as a function of time as

$$V(t) = \frac{B(De^{mt})^{-3(\alpha+1)} + 2A}{2\left(A + B(De^{mt})^{-3(\alpha+1)}\right)^{\frac{\alpha}{\alpha+1}}} . \tag{4.73}$$

Fig. 4.7 shows how the potential changes with scalar field and time.



(a)



(b)

Figure 4.7: Plots of the potential V as a function of ϕ and t from Eqs. 4.72 and 4.73, respectively, setting $A = B = m = D = 1$, and $\alpha = 0.5$ (red), $\alpha = 0.3$ (blue) and $\alpha = 0.2$ (green).

For the early universe, we plotted the potential dependence on the scalar field and one can easily observe that the potential increases with an increase in ϕ. However, for the late universe, this dependence is realized after a decrease to a certain minimum. It can be observed that as α increases the position of the minimum point gets lowered. For the time-dependent potential in the early universe, an increase in time results in an increase in the potential. Conversely, the potential decreases as time increases for the late universe. This trend has been obtained through the use of approximated solutions. The actual behaviour of the potential under consideration may be obtained through acquiring the exact solutions of the scalar field.

4.2 Inflationary $f(R)$ cosmologies

As we mentioned the cosmological inflation was first introduced to solve the horizon and flatness problems [38]. In inflation theory there are several types of potentials that have different behaviors [100]. The nature of the potential dependence on the scalar field shows how slow-roll situation affects the scalar field [101]. In principle for a given potential, one can obtain the expressions for the tensor-to-scalar ratio r and spectral index n_s. These parameters can be determined and compared to the available observations [101–103]. In this section we point out that for a given Lagrangian, one could actually have the potential with values of parameters that can be constrained by r and n_s values from observational data. It is widely understood today that a scalar field, which dominated the earliest moments after the Big Bang propelled the Universe into an exponential expansion, and eventually decayed into radiation. In [98], exact potentials for different expansion models were obtained where the radiation contribution is neglected for de Sitter spacetimes, but it was indicated that one can, in principle, generalize the study to include radiation. In this work, we do include the radiation contribution and we focus our attention to two scale factor expansion models namely, exponential and linear. The potentials that correspond to the expansion models under consideration are used to obtain the parameters like r and n_s after the calculations of $f(R)$ Lagrangians. One could see how the inclusion of radiation contribution makes the calculations complicated. The comparison with the Planck Survey results is made where the ranges of the parameters are taken into account.

4.2.1 Matter description

We consider the FLRW background filled with a non-interacting combination of a classical scalar field and radiation such that the energy-momentum tensor is given in terms of the total energy density μ and isotropic pressure p as [98]

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \,. \tag{4.74}$$

From Eqs. 4.6 and 4.7, for the total fluid of the cosmic medium, one has

$$\mu = \mu_r + \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \tag{4.75}$$

$$p = \frac{\mu_r}{3} + \frac{1}{2}\dot{\phi}^2 - V(\phi) \,. \tag{4.76}$$

whereas for radiation, $\mu_r = \frac{M}{a^4}$, and $p_r = 1/3\mu_r$, $a = a(t)$ being the cosmological scale factor and M, a constant of time.

The scalar field obeys the Klein-Gordon equation [98] given as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \,. \tag{4.77}$$

The field equations and conservation equations of the background spacetime are given as

$$3H^2 + 3K = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \mu_r \,, \tag{4.78}$$

$$3\dot{H} + 3H^2 = V(\phi) - \dot{\phi}^2 - \mu_r \,, \tag{4.79}$$

$$\dot{\mu} + 3H(\mu + p) = 0 \,. \tag{4.80}$$

Since we are assuming that the background cosmic medium is a non-interacting mixture of the scalar field and radiation, the energy densities of the scalar field and radiation evolve independently as

$$\dot{\mu}_r + 4H\mu_r = 0 \,, \tag{4.81}$$

$$\dot{\mu}_\phi + 3H\dot{\phi}^2 = 0 \,, \tag{4.82}$$

whereas Eq. 4.80 describes the total energy conservation equation for the total fluid mixture.

We combine Eqs. 4.78 and 4.79 to solve for the potential and the scalar field as

$$V(\phi) = 3H^2 + 2K + \dot{H} - \frac{\mu_r}{3} \, . \tag{4.83}$$

and

$$\dot{\phi}^2 = 2K - 2\dot{H} - \frac{4}{3}\mu_r \, . \tag{4.84}$$

The above two equations 4.83 and 4.84 can be solved once one has the expressions for H and $\dot{H}$ with the specification of the geometry of the spacetime, k. In the following two Sub-sections, we consider two different expansion laws and obtain the solutions of the scalar field ϕ and the corresponding potential. The reconstruction of $f(R)$ Lagrangian densities will need the definition of the Ricci scalar given as

$$R = 6(\dot{H} + 2H^2 + K) \, . \tag{4.85}$$

For a specified scale factor $a(t)$, one would solve Eq. 4.84 to get the momentum of the scalar field $\dot{\phi}$. Its integration with respect to time gives the expression of the scalar field. To connect the scalar field ϕ with the Ricci scalar R, we have to get $t(R)$ from Eq. 4.85. Then after establishing $\phi(R)$, we can use the definition defined in Eq. 3.17 to obtain the $f(R)$ Lagrangian. One can get the potential $V(t)$ from Eq. 4.83 with the specification of the scale factor and we first get $t(\phi)$, then we get $V(\phi)$. In the following, we consider two expansion models namely, exponential and linear models.

4.2.2 Exponential expansion

For the exponential expansion, one has the scale factor given as [98]

$$a(t) = Ae^{wt}, \text{ where } A, w > 0. \tag{4.86}$$

So we write Eq. 4.84 as

$$\dot{\phi}^2 = 2K - \frac{4M}{3a^4} \, . \tag{4.87}$$

For $k = 0$ and $k = -1$ we have a complex scalar field since both a and M are positive. For $k = 1$ we can write Eq. 4.87 as

$$\dot{\phi} = \pm \left(\frac{2}{a^2}\right)^{1/2} \left(1 - \frac{2M}{3a^2}\right)^{1/2} \, . \tag{4.88}$$

If $\frac{2M}{3a^2} > 1$, we would have a complex term. We only consider the case where $0 < \frac{2M}{3a^2} << 1$, this is because during inflation epoch radiation has little contribution. We therefore have

$$\dot{\phi} = \pm \left(\frac{2}{a^2}\right)^{1/2} \left(1 - \frac{M}{3a^2}\right) \, . \tag{4.89}$$

By using Eq. 4.86 in Eq. 4.89 and performing integration, we get

$$\phi(t) = \pm \left(-\frac{\sqrt{2}}{Aw} e^{-wt} + \frac{\sqrt{2}M}{9A^3 w} e^{-3wt} - \phi_0 \right) \, . \tag{4.90}$$

where ϕ_0 is the constant of integration. One can recover the scalar field solution obtained in [98] by simply setting $M = 0$ which means a radiation-less universe consideration (de Sitter universe).

4.2.2.1 Reconstruction of $f(R)$ for exponential expansion models

Combining Eqs. 4.85 and 4.86, we write the Ricci scalar as

$$R = 12w^2 + \frac{6}{A^2 e^{2wt}} \, . \tag{4.91}$$

Thus we have time as function of R given by

$$t(R) = \frac{1}{2w}\ln\left(\frac{6}{A^2(R-12w^2)}\right) . \tag{4.92}$$

In Eq.4.92, we should have $R > 12w^2$ to avoid negative expression inside the braket. Replacing Eq. 4.92 in Eq. 4.90 we have

$$\phi(R) = \pm\Big[-\frac{\sqrt{2}}{Aw}\left(\frac{6}{A^2(R-12w^2)}\right)^{-1/2} + \frac{\sqrt{2}M}{9A^3w}\left(\frac{6}{A^2(R-12w^2)}\right)^{-3/2} - \phi_0\Big] . \tag{4.93}$$

Therefore, $f(R)$ can be written as

$$f(R) = \pm\Big[\frac{2}{3\sqrt{3}w}(R-12w^2)^{3/2} + \frac{2\sqrt{2}M}{45\times 6^{3/2}w}\left(R-12w^2\right)^{5/2} - \phi_0 R\Big] + R + C_1 , \tag{4.94}$$

where C_1 is the constant of integration. For some limiting cases, this equation reduces to $f(R) = R$, which is GR. see Fig. 4.8

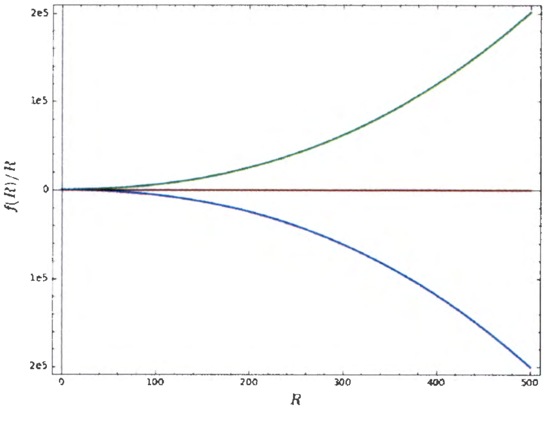


Figure 4.8: Plot of $\frac{f(R)}{R}$ versus R from Eq. 4.94 for $M = 1$, $w = 0.11$, $\phi_0 = 1$ and $C_1 = 0$ for the exponential expansion. where the green and blue lines refer to the positive and negative Lagrangians for $f(R)$ while the red line refers to the Lagrangian for GR. One can see that the deviations from GR occur at higher curvature regimes.

4.2.2.2 Potential $V(\phi)$ for exponential expansion models

From Eq. 4.90, if we set

$$x = e^{-wt} . \tag{4.95}$$

then we write Eq. 4.90 considering only the positive root with the understanding that similar analysis can be done with the negative root as well:

$$\phi + \phi_0 = -\frac{\sqrt{2k}}{Aw}x + \frac{\sqrt{2}M}{9\sqrt{k}A^3w}x^3 . \tag{4.96}$$

The above equation has two complex solutions and one real solution given as

$$
\begin{aligned}
x =\; & 3^{\frac{2}{3}} A w^{\frac{2}{3}} M^{\frac{1}{3}} \left[3\sqrt{2} A w (\varphi + \varphi_0) + \sqrt{6}\big(3w^2(\varphi^2 + \varphi_0^2) + 6w^2 \varphi \phi_0 + \frac{8A^2}{M}\big)^{1/2} \right]^{1/3} \\
& - 6^{\frac{2}{3}} A^{\frac{1}{3}} M^{\frac{2}{3}} \left[3\sqrt{2} A w (\phi + \varphi_0) + \sqrt{6}\big(3w^2(\varphi^2 + \phi_0^2) + 6w^2 \phi\varphi_0 + \frac{8A^2}{M}\big)^{1/2} \right]^{1/3} .
\end{aligned}
\tag{4.97}
$$

From Eq. 4.95, we can write

$$ t = -\frac{\ln x}{w} . \tag{4.98} $$

Here $\ln x$ is constrained to be negative such that we do not experience negative time parameter. This is because the parameter w is a positive number by construction. The potential defined in Eq. 4.83 reads

$$ V(t) = 3w^2 - \frac{2}{A^2} e^{-wt} - \frac{M}{3A^4} e^{-4wt} . \tag{4.99} $$

so that we have $V(\varphi)$ as

$$ V(\varphi) = 3w^2 + \frac{2}{A^2} x - \frac{M}{3A^4} x^4 . \tag{4.100} $$

This potential and its first and second derivatives with respect to the scalar field φ will help in constructing the slow-roll parameters in Sub-section 4.2.4. Fig. 4.9 below shows how the potential depends on the scalar field φ.

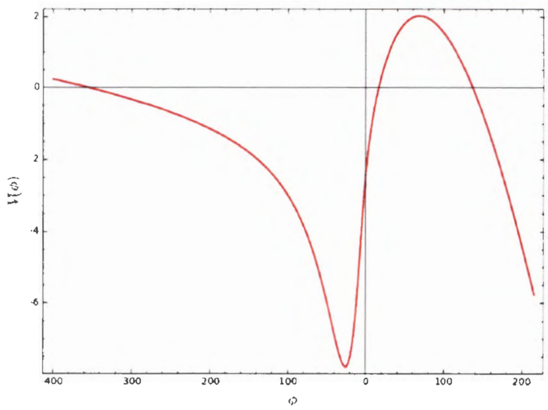


Figure 4.9: Plot of $V(\varphi)$ versus φ from Eq. 4.100 for $M = 1$, $A = 0.9$, $w = 0.12$ and $\varphi_0 = 0.5$.

4.2.3 Linear expansion

For linear expansion, one has a scale factor given as [98]

$$ a = At. \tag{4.101} $$

where $A > 0$. By replacing Eq. 4.101 in Eq. 4.84 we get

$$ \dot{\varphi} = \pm \left(\frac{2(k + A^2)}{a^2} \right)^{1/2} \left(1 - \frac{2M}{a^2(k + A^2)} \right)^{1/2} . \tag{4.102} $$

If $\frac{2M}{a^2(k+A^2)} > 1$, we would have a complex scalar field. We therefore consider the case where $\frac{2M}{a^2(k+A^2)} << 1$. This is assumed with the assumption that a is small and the constant A is big. This leads to the approximation of the above equation as

$$\dot{\phi} = \pm\left[\left(\frac{2(k+A^2)}{a^2}\right)^{1/2} - \frac{\sqrt{2}M}{a^2\sqrt{(k+A^2)}}\right] . \tag{4.103}$$

Replacing the expression for scale factor and performing integration, we have the scalar field given as

$$\phi(t) = \pm\left[\frac{\sqrt{2(k+A^2)}}{A}\ln(t) + \frac{\sqrt{2}M}{A^2\sqrt{k+A^2}t} - \phi_0\right] . \tag{4.104}$$

4.2.3.1 Reconstruction of $f(R)$ for linear inflation models

In order to construct $f(R)$ models from scalar field formulations obtained in Eq. 4.104, we need to have $\phi(R)$. To achieve this we start from the definition of the Ricci scalar given in Eq.4.85, so that we can write

$$R = \frac{6(A^2-k)}{A^2t^2} . \tag{4.105}$$

From this equation, one has $t(R)$ given as

$$t(R) = \left(\frac{6(A^2+k)}{A^2R}\right)^{1/2} . \tag{4.106}$$

Therefore, the scalar field in Eq. 4.104 has the form

$$\phi(R) = \pm\left[\frac{\sqrt{2(A^2+k)}}{A}\ln\left(\frac{6(A^2+k)}{A^2R}\right)^{1/2} + \frac{\sqrt{2}M}{A^2\sqrt{A^2+k}}\left(\frac{6(A^2+k)}{A^2R}\right)^{-1/2} - \phi_0\right] . \tag{4.107}$$

Integrating this equation using the definition of Eq. 3.17 leads to

$$\begin{aligned} f(R) = \pm\Big\{&\frac{\sqrt{2(A^2+k)}}{2A(A^2+k)}\left[A^2R\ln\left(\frac{6(A^2+k)}{A^2R}\right) + kR\ln\left(\frac{6(A^2+k)}{A^2R}\right) + A^2R + kR\right] \\ &+ \frac{\sqrt{12}MR^{3/2}}{9A(A^2-k)} - \phi_0R\Big\} + R + C_2 , \end{aligned} \tag{4.108}$$

where C_2 is a constant of integration. This Lagrangian takes different form depending on the geometry of the spacetime, as Fig. 4.10 depicts.

4.2.3.2 Potential $V(\phi)$ for linear inflation models

Using Eq. 4.101 together with the definition in Eq. 4.83, the potential in terms of cosmic time t is given as

$$V(t) = \frac{2}{t^2} + \frac{2k}{A^2t^2} - \frac{M}{3A^4t^4} . \tag{4.109}$$

We need to get $t(\phi)$ in order to get $V(\phi)$. From Eq. 4.104 with the consideration of its positive root, when one considers flat universe ($k=0$), we have

$$t = -\frac{M}{A^3\mathbf{W}(x)} , \tag{4.110}$$

where $x \equiv -\frac{Me^{-\frac{\sqrt{2}(\phi+\phi_0)}{2}}}{A^3}$ and $\mathbf{W}(x)$ is Lambert's function. The Taylor expansion for this function is given as

$$\mathbf{W}(x) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n \approx x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + O(x^5) . \tag{4.111}$$

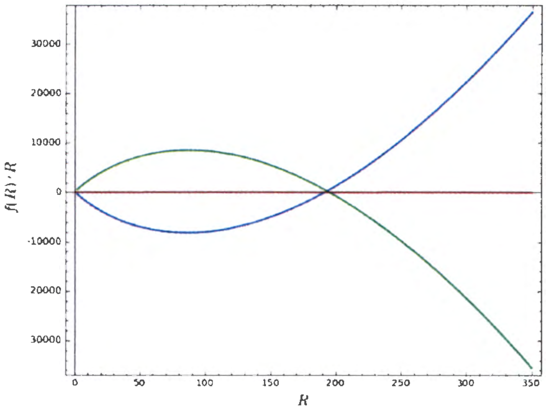


Figure 4.10: Plot of $\frac{f(R)}{R}$ vs R from Eq. 4.108 for $M = 1$, $C_1 = -100$, $\phi_0 = 50$, $k = 1$ and $A = 0.11$ for linear expansion, where the blue and green lines refer to the positive and negative Lagrangians for $f(R)$ while the red line refers to the Lagrangian for GR.

Thus up to quadratic order, one has

$$\mathbf{W}(x) = -\frac{Me^{-\frac{\sqrt{2}(\phi+\phi_0)}{2}}}{A^3} - \frac{M^2 e^{-\sqrt{2}(\phi+\phi_0)}}{A^6} + \dots \,. \tag{4.112}$$

Replacing Eq. 4.112 in Eq. 4.110, we have $t(\phi)$ as

$$t(\phi) = \frac{M}{A^3}\Big[\frac{Me^{-\frac{\sqrt{2}(\phi+\phi_0)}{2}}}{A^3} + \frac{M^2 e^{-\sqrt{2}(\phi+\phi_0)}}{A^6}\Big]^{-1} \,. \tag{4.113}$$

Replacing $t(\phi)$ in Eq. 4.109, we have

$$V(\phi) = 2\Big[e^{-\frac{\sqrt{2}(\phi+\phi_0)}{2}} + \frac{Me^{-\sqrt{2}(\phi+\phi_0)}}{A^3}\Big]^2 - \frac{M}{3}\Big[\frac{e^{-\frac{\sqrt{2}(\phi-\phi_0)}{2}}}{A^2} + \frac{Me^{-\sqrt{2}(\phi+\phi_0)}}{A^5}\Big]^4 \,. \tag{4.114}$$

For $k = \pm 1$, we have

$$V(t) = \frac{2(A^2+1)}{A^2 t^2} - \frac{M}{3A^4 t^4} \,. \tag{4.115}$$

From Eq. 4.104, we have

$$t(\phi) = \exp\Big[\frac{A(\phi_0+\phi)}{\sqrt{2(A^2\pm 1)}} + \mathbf{W}(x)\Big]. \tag{4.116}$$

where $x = \frac{Me^{-\frac{A(\phi+\phi_0)}{\sqrt{2(A^2\pm1)}}}}{A(A^2\pm1)}$ and the new $\mathbf{W}(x)$ is expanded as

$$\mathbf{W}(x) = -\frac{M}{A(A^2\pm1)}e^{-\frac{A(\phi+\phi_0)}{\sqrt{2(A^2\pm1)}}} - \frac{M^2}{A^2(A^2\pm1)^2}e^{-2\frac{A(\phi+\phi_0)}{\sqrt{2(A^2\pm1)}}} + \dots \,. \tag{4.117}$$

Thus, one has $t(\phi)$ given as

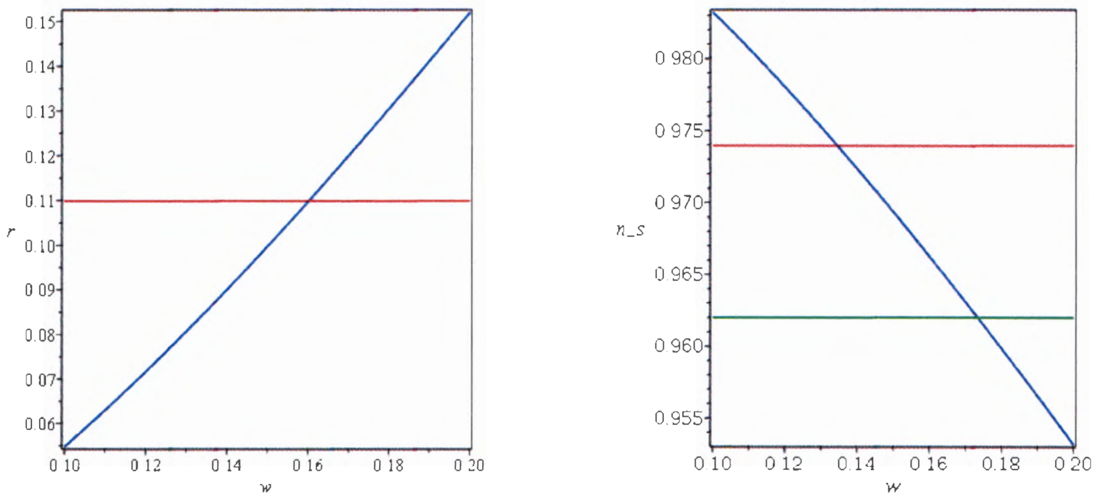
$$t(\phi) = \exp\Big[\frac{A(\phi_0+\phi)}{\sqrt{2(A^2\pm1)}} - \frac{M}{A(A^2\pm1)}e^{-\frac{A(\phi+\phi_0)}{\sqrt{2(A^2\pm1)}}} - \frac{M^2}{A^2(A^2\pm1)^2}e^{-2\frac{A(\phi+\phi_0)}{\sqrt{2(A^2\pm1)}}}\Big] \,. \tag{4.118}$$

Replacing this expression for $t(\phi)$ in Eq. 4.115, we have

$$
\begin{aligned}
V(\phi) = \frac{2(A^2 \pm 1)}{A^2} \exp\Big\{ -2\Big[\frac{A(\phi_0 + \phi)}{\sqrt{2(A^2 \pm 1)}} - \frac{M}{A(A^2 \pm 1)} e^{-\frac{A(\phi+\phi_0)}{\sqrt{2(A^2 \mp 1)}}} - \frac{M^2}{A^2(A^2 \pm 1)^2} e^{-2\frac{A(\phi+\phi_0)}{\sqrt{2(A^2 \mp 1)}}} \Big]\Big\} \\
- \frac{M}{3A^4} \exp\Big\{ -4\Big[\frac{A(\phi_0 + \phi)}{\sqrt{2(A^2 \pm 1)}} - \frac{M}{A(A^2 \pm 1)} e^{-\frac{A(\phi+\phi_0)}{\sqrt{2(A^2 \pm 1)}}} - \frac{M^2}{A^2(A^2 \pm 1)^2} e^{-2\frac{A(\phi+\phi_0)}{\sqrt{2(A^2 \pm 1)}}} \Big]\Big\} .
\end{aligned}
\tag{4.119}
$$

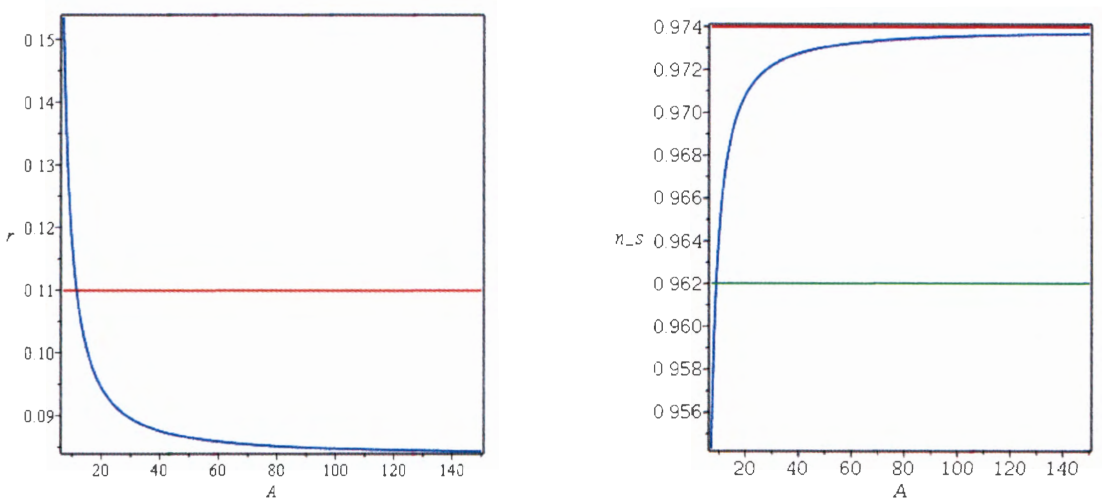
In the following section, we apply the results obtained so far to the inflation epoch especially on slow-roll approximation to see how the models respond to inflation parameters n_s and r.

4.2.4 Application to inflation epoch



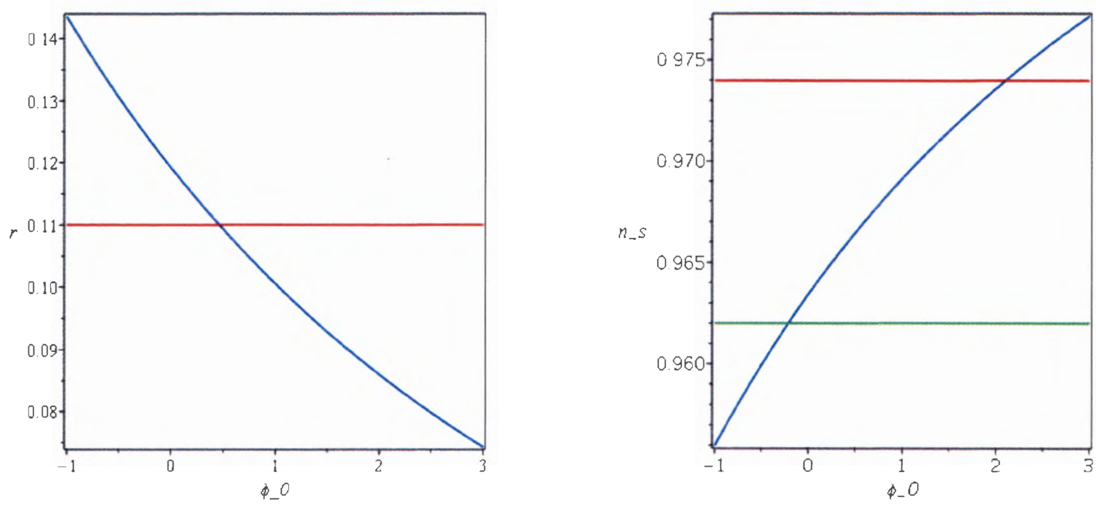
(a) (b)

Figure 4.11: Plots of $r(w)$ and $n_s(w)$ respectively for $M = 1$, $A = 15$, $\phi_0 = 1$ and $\phi = 1.2$, for exponential expansion. The red and green lines refer to the upper and lower bounds of Planck data respectively.



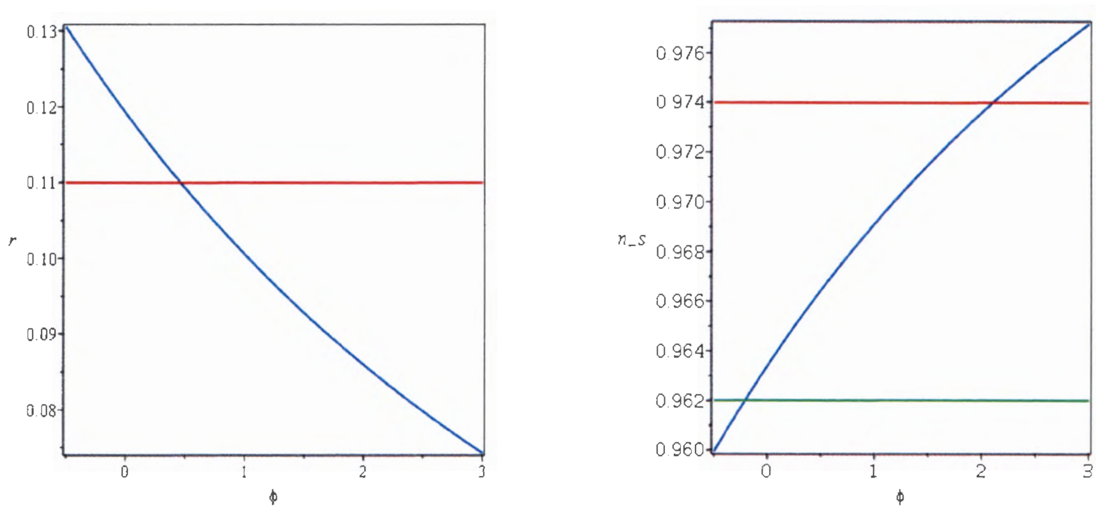
(a) (b)

Figure 4.12: Plots of $r(A)$ and $n_s(A)$ respectively for $M = 1$, $w = 0.15$, $\phi_0 = 1$ and $\phi = 1.5$, for exponential expansion. The red and green lines refer to the upper and lower bounds of Planck data respectively.



(a) (b)

Figure 4.13: Plots of $r(\phi_0)$ and $n_s(\phi_0)$ respectively for $M = 1$, $w = 0.15$, $A = 15$ and $\phi = 1.15$, for exponential expansion. The red and green lines refer to the upper and lower bounds of Planck data respectively.



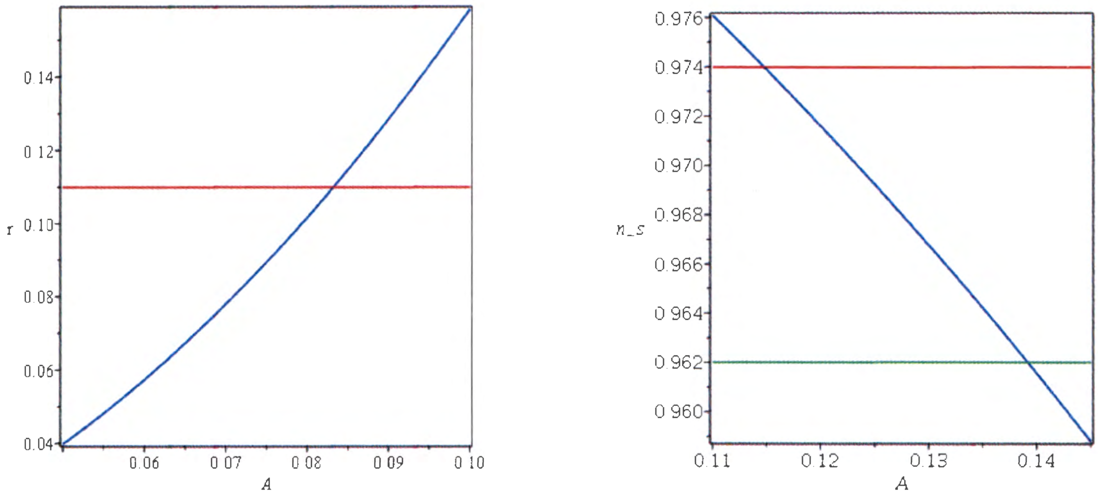
(a) (b)

Figure 4.14: Plots of $r(\phi)$ and $n_s(\phi)$ respectively for $M = 1$, $w = 0.15$, $A = 15$ and $\phi_0 = 1.15$, for exponential expansion. The red and green lines refer to the upper and lower bounds of Planck data respectively.

We focus our interests on slow-roll approximations. The slow-roll approximation has the condition that the kinetic energy of the scalar field is much less than its potential [104]. This leads to two conditions as

$$\dot{\phi}^2 < V(\phi) \,. \tag{4.120}$$

$$2|\ddot{\phi}| < |V'(\phi)| \,. \tag{4.121}$$



(a) (b)

Figure 4.15: Plots of $r(A)$ and $n_s(A)$ respectively or $M = 1$, $\phi_0 = 750$ and $\phi = 1.05$, for linear expansion. The red and green lines refer to the upper and lower bounds of Planck data respectively.

where here the apostrophe ($'$) indicates differentiation with respect to the scalar field ϕ. Then we can define two slow-roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ [104, 105]

$$\epsilon(\phi) = \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 , \tag{4.122}$$

and

$$\eta(\phi) = \frac{1}{\kappa^2}\left(\frac{V''(\phi)}{V(\phi)}\right) . \tag{4.123}$$

The spectral index n_s and the tensor-to-scalar ratio r are defined, respectively, as [104, 105]

$$n_s = 1 - 6\epsilon + 2\eta , \tag{4.124}$$

$$r = 16\epsilon . \tag{4.125}$$

The numerical computation of these two parameters is done for both exponential and linear expansion models. The observational values of n_s and r are taken from the Planck data [106].

The above figures summarize the change of the parameters w, A, ϕ_0 and ϕ to get both spectral index n_s and tensor-to-scalar ratio r close to the observational values for the linear law of expansion for the case when $k = 1$. Where $k = 0$ and $k = -1$, one has negative and complex values for the spectral index n_s and the tensor-to-scalar ratio r, this evidently makes the corresponding $f(R)$ Lagrangian being ruled out from the viable ones.

It is clear from the figures presented in this work that the observational results of n_s and r constrain the ranges of the parameters under variation. In Fig. 4.11a one can notice that r increases monotonically as function of w until it surpasses the upper limit of Planck data which constrain that $r < 0.11$. On the other side in Fig. 4.11b the spectral index decreases as w changes and the only acceptable values can be obtained between $w = 0.14$ and $w = 0.177$. In Fig. 4.12a, after n_s crosses the observational value, it asymptotically varies with increasing A. On the other side, in Figure 4.12b, n_s increases and saturates to a finite amplitude after it crosses the lower bound from the Planck data. In Fig. 4.13a, the tensor-to-scalar ratio r is decreasing with increasing ϕ_0 and the values that are compatible with observation are achieved after passing $\phi_0 = 0.42$. In Fig. 4.13b, the spectral index n_s increases with increasing ϕ_0 until it crosses the lower and the upper bounds of the observational data. The same behavior is manifested in Fig. 4.14a and Fig. 4.14b for r and n_s for changing Φ.

For linear expansion model, Figs. 4.15a and 4.15b show how r and n_s change as function of A. The tensor-to-scalar ratio increases as function of A and n_s decreases as function of A. This makes n_s cross the upper bound first and later it crosses the lower bound.

Chapter 5

Quasi-Newtonian universes

5.1 Background description

The space-time which obeys Einstein's field equations has been constrained to be irrotational and shear-free. Space-time according to Newton does not rotate and does not experience shear. Actually the physics used in astrophysical observations (like the motion of the stars, inside galaxies, motion of galaxies in the universe, discovery of dark matter and the behaviour of rotational curves), have been made by considering Newtonian physics.
Despite there being no proper Newtonian limit for GR on cosmological scales, recent works on so-called quasi-Newtonian cosmologies [31, 107, 108] have shown that gravitational physics can be studied to a good approximation [5]. The importance of investigating the Newtonian limit for general relativity on cosmological contexts is that, there is a viewpoint that cosmology is essentially a Newtonian affair, with the relativistic theory only needed for examination of some observational relations. Most of the astrophysical calculations on the formation of large-scale structure in the universe rely on such a limit [108]. In [108], a covariant approach to cold matter universes in quasi-Newton has been developed and it has been applied and extended in [107] in order to derive and solve the equations governing density and velocity perturbations. This approach revealed the existence of integrability conditions in GR. In this chapter, we derive the evolution of the velocity and density perturbations in the comoving (Lagrangian) and quasi-Newtonian frames. We investigate the existence of integrability conditions of a class of irrotational and shear-free perfect fluid cosmological models in the context of scalar-tensor gravity. Such work has been done in the context of $f(R)$ gravity [5], where some models of $f(R)$ gravity have been shown to exhibit Newtonian behaviour in the shear-free regime.

5.2 Quasi-Newtonian spacetimes

If a comoving 4-velocity $\tilde{u}^a$ is chosen such that, in the linearised form

$$\tilde{u}^a = u^a + v^a, \quad v_a u^a = 0, \quad v_a v^a << 1 . \tag{5.1}$$

the dynamical, kinematic and gravito-electromagnetic quantities Eqs. 2.13, 2.14 and 2.15 undergo transformation.
Here v^a is the relative velocity of the comoving frame with respect to the observer in the quasi-Newtonian frame, defined such that it vanishes in the background. In other words, it is a non-relativistic peculiar velocity. Quasi-Newtonian cosmological models are irrotational, shear-free dust

spacetimes characterised by [31, 33]:

$$p_m = 0\,,\quad q_a^m = \mu_m v_a\,,\quad \pi_{ab}^m = 0\,. \tag{5.2}$$

$$\omega_a = 0\,,\quad \sigma_{ab} = 0\,. \tag{5.3}$$

Therefore, the evolution equations 2.18-2.22 for this class of spacetimes can be written as

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a\,, \tag{5.4}$$

$$\dot{\mu}_m = -\mu_m\Theta - \tilde{\nabla}^a q_a^m\,, \tag{5.5}$$

$$\dot{q}_a^m = -\frac{4}{3}\Theta q_a^m - \mu_m A_a\,, \tag{5.6}$$

$$\dot{E}^{\langle ab\rangle} = \eta^{cd\langle a}\tilde{\nabla}_c H_d^{\rangle b} - \Theta E^{ab} - \frac{1}{2}\dot{\pi}^{ab} - \frac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} - \frac{1}{6}\Theta\pi^{ab}\,, \tag{5.7}$$

$$\dot{H}^{\langle ab\rangle} = -\Theta H^{ab} - \eta^{cd\langle a}\tilde{\nabla}_c E_d^{\rangle b} + \frac{1}{2}\eta^{cd\langle a}\tilde{\nabla}_c\pi_d^{\rangle b}\,. \tag{5.8}$$

Due to the vanishing of the shear in the quasi-Newtonian frame, Eq. 2.24 is turned into a new constraint

$$E_{ab} - \frac{1}{2}\pi_{ab}^{\phi} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0\,, \tag{5.9}$$

and using the identity in Eq. A.1 for any scalar φ, Eq. 2.23 can be simplified as

$$\eta^{abc}\tilde{\nabla}_a A_c = 0 \Rightarrow A_a = \tilde{\nabla}_a\varphi\,. \tag{5.10}$$

In this case φ is the covariant relativistic generalisation of the Newtonian potential. The gravito-magnetic constraint Eq. 2.28 and the shear-free and irrotational condition Eq. 5.3 show that the gravito-magnetic component of the Weyl tensor automatically vanishes:

$$H^{ab} = 0\,. \tag{5.11}$$

The vanishing of this quantity implies no gravitational radiation in quasi-Newtonian cosmologies, and Eq. 2.30 together with Eq. 5.3 show that q_a^m is irrotational:

$$\eta^{abc}\tilde{\nabla}_b q_a = 0 = \eta^{abc}\tilde{\nabla}_b v_a\,. \tag{5.12}$$

Since the vorticity vanishes, there exists a velocity potential such that

$$v_a = \tilde{\nabla}_a\Phi\,. \tag{5.13}$$

5.3 Integrability conditions

A constraint equation $C_A = 0$ evolves consistently with the evolution equations in the sense that [31, 33, 107, 109]

$$\dot{C}_A = F_B^A C^B + G_{Ba}^A D^a C^B\,, \tag{5.14}$$

where F and G depend on the kinematic, dynamical and gravito-electromagnetic quantities but not their derivatives [31, 33]. The non-linear models have been shown to be generally inconsistent [31, 33, 110] if one imposes the constraint Eq. 5.11, but the linearised models are consistent. Thus, a simple approach to the integrability conditions for quasi-Newtonian cosmologies follows from showing that these models are in fact a subclass of the linearised silent models. This can happen by using the transformation between the quasi-Newtonian and comoving frames.

The transformed linearised kinematics, dynamics and gravito-electromagnetic quantities from the quasi-Newtonian frame to the comoving frame are given as follows [31,33,110]:

$$\tilde{\Theta} = \Theta + \tilde{\nabla}^a v_a \ . \tag{5.15}$$

$$\tilde{A}_a = A_a + \dot{v}_a + \frac{1}{3}\Theta v_a \ . \tag{5.16}$$

$$\tilde{\omega}_a = \omega_a - \frac{1}{2}\eta_{abc}\tilde{\nabla}^b v^c \ . \tag{5.17}$$

$$\tilde{\sigma}_{ab} = \sigma_{ab} + \tilde{\nabla}_{\langle a}v_{b\rangle} \ . \tag{5.18}$$

$$\tilde{\mu} = \mu, \quad \tilde{p} = p, \quad \tilde{\pi}_{ab} = \pi_{ab}, \quad \tilde{q}^{\phi}_a = q^{\phi}_a \ . \tag{5.19}$$

$$\tilde{q}^m_a = q^m_a - (\mu_m + p_m)v_a \ . \tag{5.20}$$

$$\tilde{E}_{ab} = E_{ab}, \quad \tilde{H}_{ab} = H_{ab} \ . \tag{5.21}$$

It follows from the Eqs. 5.2 – 5.21 that

$$\tilde{p} = 0, \quad \tilde{q}_a = 0, \quad \tilde{\pi}_{ab} = 0 \ , \tag{5.22}$$

$$\tilde{A}_a = 0, \quad \tilde{\omega}_a = 0, \quad \tilde{\sigma}_{ab} = \tilde{\nabla}_{\langle a}v_{b\rangle} \ . \tag{5.23}$$

$$\tilde{E}_{ab} = E_{ab}, \quad \tilde{H}_{ab} = 0 \ . \tag{5.24}$$

These equations describe the linearised silent universe except that the restriction on the shear in Eq. 5.23 results in the integrability conditions for the quasi-Newtonian models.

5.3.0.1 First integrability condition

Since Eq. 5.9 is a new constraint, we need to ensure its consistent propagation at all epochs and in all spatial hypersurfaces. Differentiating it with respect to cosmic time t and using Eqs. 3.24, 5.7 and 2.26 together with the identity A.2, one obtains

$$\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\Big[\dot{\varphi} + \frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big] + \Big[\dot{\varphi} + \frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big]\tilde{\nabla}_a\tilde{\nabla}_b\varphi = 0 \ . \tag{5.25}$$

which is the first integrability condition for quasi-Newtonian cosmologies in scalar-tensor theory of gravitation and it is a generalisation of the one obtained in [31], *i.e.*, Eq. 5.25 reduces to an identity for the generalized van Elst-Ellis condition [31,33,108]

$$\dot{\varphi} + \frac{1}{3}\Theta = -\frac{\dot{\phi}}{(\phi+1)} \ . \tag{5.26}$$

Using Eq. 5.5 with the time evolution of the modified van Elst-Ellis condition, we obtain the covariant modified Poisson equation in scalar-tensor gravity as follows:

$$\tilde{\nabla}^2\varphi = \frac{\mu_m}{2(\phi+1)} - (3\ddot{\varphi}+\Theta\dot{\varphi}) + \frac{1}{2(\phi+1)}\Big[f - R(\phi+1) - \Big(\frac{3\dot{\phi}'}{\phi'} - \Theta\Big)\dot{\phi} - 3\ddot{\phi} + 3\Big(\frac{2}{(\phi+1)} - \frac{\phi''}{\phi'^2}\Big)\dot{\phi}^2 - \tilde{\nabla}^2\phi\Big] \ . \tag{5.27}$$

By taking the gradient of Eq. 5.26, one gets

$$\frac{1}{3}\tilde{\nabla}_a\Theta + \tilde{\nabla}_a\dot{\varphi} = \Big[-\frac{\dot{\phi}'}{(\phi+1)} + \frac{\dot{\phi}}{(\phi+1)^2}\Big] \ . \tag{5.28}$$

using the identity A.3, Eq. 5.28 can be written as

$$(\tilde{\nabla}_a\varphi)^{\cdot} = -\frac{1}{3}\tilde{\nabla}_a\Theta - \frac{\dot{\phi}'}{\phi'(\phi+1)}\tilde{\nabla}_a\phi + \frac{\dot{\phi}}{(\phi+1)^2}\tilde{\nabla}_a\phi - \frac{1}{3}\Theta\tilde{\nabla}_a\varphi + \dot{\varphi}A_a \ , \tag{5.29}$$

using Eq. 5.10, one can write Eq. 5.29 as

$$\dot{A}_a = -\frac{1}{3}\tilde{\nabla}_a\Theta - \frac{\dot{\phi}'}{\phi'(\phi+1)}\tilde{\nabla}_a\phi + \frac{\dot{\phi}}{(\phi+1)^2}\tilde{\nabla}_a\phi - \frac{1}{3}\Theta A_a + \dot{\varphi}A_a \ , \tag{5.30}$$

and from the shear-free constraint Eq. 2.26, one has

$$q_a = \frac{q^m_a}{(\phi+1)} + q^\phi_a = \frac{\mu_m v_a}{(\phi+1)} - \frac{1}{(\phi+1)}\Big(\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\Big)\tilde{\nabla}_a\phi \ . \tag{5.31}$$

By substituting Eq. 5.31 into Eq. 5.30, one can obtain the evolution equation of the 4-acceleration A_a as

$$\dot{A}_a + \Big[\frac{2}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big]A_a = -\frac{1}{2(\phi+1)}\Big[\mu_m v_a + \Big(\frac{1}{3}\Theta + \frac{\dot{\phi}'}{\phi'} - \frac{2\dot{\phi}}{(\phi+1)}\Big)\tilde{\nabla}_a\phi\Big] \ . \tag{5.32}$$

This evolution equation arises as a consequence of the modified van Elst-Ellis condition Eq. 5.26 and the shear-free condition.

5.3.0.2 Second integrability condition

There is a second integrability condition arising by checking for the consistency of the constraint Eq. 5.9 on any spatial hyper-surface of constant time t. By taking the divergence of Eq. 5.9 and by using the following identity:

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}A_{b\rangle} = \frac{1}{2}\tilde{\nabla}^2 A_a + \frac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^c A_c) + \frac{1}{3}(\mu - \frac{1}{3}\Theta^2)A_a \ , \tag{5.33}$$

which holds for any projected vector A_a, and by using Eq. 5.10 with the identity A.10 it follows that:

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\varphi = \frac{2}{3}\tilde{\nabla}_a(\tilde{\nabla}^2\varphi) + \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\tilde{\nabla}_a\varphi \ . \tag{5.34}$$

By using Eqs. 5.34, 2.26 and 2.29, one obtains:

$$\begin{aligned}&\tilde{\nabla}_a\mu_m - \Big[\dot{\phi} + \frac{2}{3}(\phi+1)\Theta\Big]\tilde{\nabla}_a\Theta + \frac{1}{(\phi+1)}\Big[\frac{f}{2} - \mu_m + \Theta\dot{\phi} - \frac{\Theta\dot{\phi}'(\phi+1)}{\phi'}\Big]\tilde{\nabla}_a\phi \\ &- 2(\phi+1)\tilde{\nabla}^2(\tilde{\nabla}_a\varphi) - 2\Big[\mu_m + \frac{R(\phi+1)}{2} - \frac{f}{2} - \Theta\dot{\phi} - \frac{\Theta^2(\phi+1)}{3}\Big]\tilde{\nabla}_a\varphi - \tilde{\nabla}^2(\tilde{\nabla}_a\phi) = 0 \ ,\end{aligned} \tag{5.35}$$

which is the second integrability condition and in general it appears to be independent of the first integrability condition Eq. 5.25.
We can rewrite Eq. 5.35 as

$$\begin{aligned}&\tilde{\nabla}_a\mu_m - \frac{2}{3}\Theta(\phi+1)\tilde{\nabla}_a\Theta - 2(\phi+1)\tilde{\nabla}^2(\tilde{\nabla}_a\varphi) - 2\Big[\mu_m - \frac{\Theta^2(\phi+1)}{3}\Big]\tilde{\nabla}_a\varphi = \dot{\phi}\tilde{\nabla}_a\Theta \\ &- \frac{1}{(\phi+1)}\Big[\frac{f}{2} - \mu_m + \Theta\dot{\phi} - \frac{\Theta\dot{\phi}'(\phi+1)}{\phi'}\Big]\tilde{\nabla}_a\phi + \tilde{\nabla}^2(\tilde{\nabla}_a\phi) + 2\Big[\frac{R(\phi+1)}{2} - \frac{f}{2} - \Theta\dot{\phi}\Big]\tilde{\nabla}_a\varphi \ .\end{aligned} \tag{5.36}$$

The left and right hand sides of Eq. 5.36 refer to GR and non-GR contributions. For GR, the right hand side vanishes and this matches Maartens'result [31] for the second integrability condition.

By taking the gradient of Eq. 5.26 and using Eq. 2.26, one can obtain the peculiar velocity:

$$v_a = -\frac{1}{\mu_m}\Big[2(\phi+1)\tilde{\nabla}_a\dot{\varphi} + \Big(\frac{\dot{\phi}'}{\phi'} - \dot{\varphi} - \frac{3\dot{\phi}}{(\phi+1)}\Big)\tilde{\nabla}_a\phi\Big] \ . \tag{5.37}$$

By virtue of Eqs. 5.5 and 5.6, v_a evolves according to

$$\dot{v}_a + \frac{1}{3}\Theta v_a = -A_a \,. \tag{5.38}$$

The coupled evolution Eqs. 5.32 and 5.38 decouple to produce the second-order propagation equation of the peculiar velocity v_a. By using Eqs. 3.27 and 3.28 in Eq. 5.38 one obtains:

$$\begin{aligned}
&\ddot{v}_a + \Big[\Theta + \frac{\dot\phi}{(\phi+1)}\Big]\dot{v}_a + \Big[\frac{1}{9}\Theta^2 - \frac{1}{6(\phi+1)}(5\mu_m - f - 4\Theta\dot\phi)\Big]v_a \\
&+ \frac{1}{(\phi+1)}\Big[\frac{\dot\phi}{(\phi+1)} - \frac{\dot\phi''}{2\phi'} - \frac{\Theta}{6} - \frac{\ddot\phi'}{2\phi'} + \frac{\phi''\dot\phi}{2\phi'^2}\Big]\tilde\nabla_a\phi = 0 \,.
\end{aligned} \tag{5.39}$$

By substituting Eq. 5.37 into Eq. 5.38 one obtains

$$\begin{aligned}
&2(\phi+1)\tilde\nabla_a\ddot\varphi + 2\Big[\dot\phi + \Theta(\phi+1)\Big]\tilde\nabla_a\dot\varphi - \Big[\mu_m - 2(\phi+1)\ddot\varphi + \ddot\phi - \frac{\dot\phi\dot\phi'}{\phi'} - \frac{\dot\phi^2\phi''}{\phi'^2} + \dot\varphi\phi' \\
&+ \frac{3\dot\phi^2}{(\phi+1)}\Big]\tilde\nabla_a\varphi + \Big[\frac{3\phi''\dot\phi\dot\phi'}{\phi'^3} + \frac{\ddot\phi'}{\phi'} - \frac{\dot\phi'^2}{\phi'^2} - \frac{\dot\phi\dot\phi''}{\phi'^2} + \frac{\Theta\dot\phi'}{\phi'} - \frac{\dot\varphi\dot\phi'}{\phi'} - \frac{2\phi''\dot\phi^2}{\phi'^4} - \frac{3\dot\phi^2\phi''}{\phi'^2(\phi+1)} \\
&- \Theta\dot\varphi - \ddot\varphi + \frac{3\dot\phi}{(\phi+1)^2} + \frac{\phi'''\dot\phi^2}{\phi'^3} - \frac{3\ddot\phi}{(\phi+1)} - \frac{3\Theta\dot\phi}{(\phi+1)}\Big]\tilde\nabla_a\phi = 0 \,.
\end{aligned} \tag{5.40}$$

By using Eq. 5.5 together with Eq. 5.25, one can show that the acceleration potential φ satisfies

$$\ddot\varphi = \frac{1}{9}\Theta^2 + \frac{1}{6(\phi+1)}\Big[\mu_m + f - R(\phi+1) - 3\ddot\phi + \Theta\dot\phi + \frac{3\dot\phi\dot\phi'}{\phi'} - \frac{3\dot\phi^2\phi''}{\phi'^2} + \frac{6\dot\phi^2}{(\phi+1)} - \tilde\nabla^2\phi\Big] - \frac{1}{3}\tilde\nabla^2\varphi \,. \tag{5.41}$$

Now we can substitute Eq. 5.41 into Eq. 5.40 to arrive at the linearised equation

$$\begin{aligned}
&2(\phi+1)\tilde\nabla_a\ddot\varphi + \Big[2\Theta(\phi+1) + 2\dot\phi\Big]\tilde\nabla_a\dot\varphi + \frac{1}{3}\Big[f - R(\phi+1) + 2\Theta\dot\phi - 2\mu_m + \frac{2}{3}\Theta^2(\phi+1)\Big]\tilde\nabla_a\varphi \\
&+ \Big[\frac{\ddot\phi'}{\phi'} - \frac{\dot\phi\dot\phi''}{\phi'^2} + \frac{7f}{6} - \frac{\mu_m}{3(\phi+1)} - \frac{R}{3} + \frac{3\dot\phi\dot\phi'\phi''}{\phi'^3} - \frac{\dot\phi'^2}{\phi'^2} + \frac{2\dot\phi^2}{(\phi+1)^2} + \frac{\dot\phi^2\phi'''}{\phi'^3} - \frac{\dot\phi\Theta}{3(\phi+1)} \\
&+ \frac{4\Theta\dot\phi'}{3\phi'} - \frac{2\dot\phi\dot\phi'}{\phi'(\phi+1)} - \frac{2\dot\phi^2\phi''^2}{\phi'^4}\Big]\tilde\nabla_a\phi = 0 \,.
\end{aligned} \tag{5.42}$$

We can get the second dynamical equation for the acceleration potential φ, by taking the gradient of the modified Poisson Eq. 5.27 and using Eq. 5.26:

$$\begin{aligned}
&2(\phi+1)\tilde\nabla_a(\tilde\nabla^2\varphi) = \tilde\nabla_a\mu_m - 6(\phi+1)\tilde\nabla_a\ddot\varphi - 2\Theta(\phi+1)\tilde\nabla_a\dot\varphi - \Big[2\dot\varphi(\phi+1) + \dot\phi\Big]\tilde\nabla_a\Theta \\
&+ \Big[\frac{3\dot\phi'^2}{\phi'^2} - \frac{3\ddot\phi'}{\phi'} + \frac{3\dot\phi\dot\phi''}{\phi'^2} - \frac{5\dot\phi\dot\phi'\phi''}{\phi'^3} + \frac{12\dot\phi\dot\phi'}{\phi'(\phi+1)} - \frac{\Theta\dot\phi'}{\phi'} + \frac{6\dot\phi^2\phi''^2}{\phi'^4} - \frac{6\dot\phi^2}{(\phi+1)^2} \\
&- \frac{3\dot\phi^2\phi'''}{\phi'^3} - 2\Theta\dot\varphi - 6\ddot\varphi - R\Big]\tilde\nabla_a\phi - \tilde\nabla^2(\tilde\nabla_a\phi) \,.
\end{aligned} \tag{5.43}$$

If one uses the preceding Eq. 5.35, the resulting equation becomes

$$\begin{aligned}
&6(\phi+1)\tilde\nabla_a\ddot\varphi + 6\Big[\dot\phi + (\phi+1)\Theta\Big]\tilde\nabla_a\dot\varphi + \Big[2\mu_m - \frac{2}{3}\Theta^2(\phi+1) + R(\phi+1) - f - 2\Theta\dot\phi\Big]\tilde\nabla_a\varphi \\
&+ \Big[\frac{3\ddot\phi'}{\phi'} - \frac{3\dot\phi^2}{\phi'^2} - \frac{3\dot\phi\dot\phi''}{\phi'^2} + \frac{12\phi''\dot\phi\dot\phi'}{\phi'^3} - \frac{6\dot\phi\dot\phi'}{\phi'(\phi+1)} + \frac{4\Theta\dot\phi'}{\phi'} - \frac{6\phi''^2\dot\phi^2}{\phi'^4} + \frac{7f}{6(\phi+1)} \\
&- \frac{\mu_m}{(\phi+1)} - R - \frac{\Theta\dot\phi}{(\phi+1)} + \frac{3\phi'''\dot\phi^2}{\phi'^3} + \frac{6\dot\phi^2}{(\phi+1)^2} - \frac{3\dot\phi\dot\phi'}{\phi'^3}\Big]\tilde\nabla\phi = 0 \,,
\end{aligned} \tag{5.44}$$

which is identically satisfied by virtue of Eq. 5.42.

5.4 Perturbations

In the previous section, we showed how imposing special restrictions to the linearized perturbations of FLRW universes in the quasi-Newtonian setting result in the integrability conditions. These integrability conditions imply velocity and acceleration propagation equations resulting from the generalised van Elst-Ellis condition for the acceleration potential in scalar-tensor theories. In this section, we show how one can obtain the velocity and density perturbations via these propagation equations, thus generalizing GR results obtained in [31].

5.4.1 Definition of vector gradient variables

The covariant vector gradient variable D^m_a for the total matter fluid can be defined as

$$D^m_a = \frac{a\tilde{\nabla}_a\mu_m}{\mu_m} \, . \tag{5.45}$$

The comoving acceleration is defined as

$$\mathcal{A}_a = aA_a \, . \tag{5.46}$$

The covariant vector gradient variable V^m_a for the velocity inhomogeneity of the matter is defined as

$$V^m_a = av_a \, . \tag{5.47}$$

The comoving rate of the volume expansion gradient can be given as

$$Z_a = a\tilde{\nabla}_a\Theta \, , \tag{5.48}$$

and by using the constraint Eq. 2.26 together with Eq. 5.31, one can rewrite Eq. 5.48 as

$$Z_a = \frac{3}{2(\phi+1)}\mu_m V^m_a - \frac{3}{2(\phi-1)}\left[\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right]\Phi_a \, . \tag{5.49}$$

We also define another gradient variable Φ_a for the scalar fluid as

$$\Phi_a = a\tilde{\nabla}_a\phi \, . \tag{5.50}$$

The gradient variable Ψ_a for the inhomogeneity of scalar field momentum is given as

$$\Psi_a = a\tilde{\nabla}_a\dot{\phi} \, , \tag{5.51}$$

and the gradient variable C_a

$$C_a = a^3\tilde{\nabla}_a\tilde{R} \, , \tag{5.52}$$

where $\tilde{R}$ is the three dimension Ricci scalar, is defined as

$$\tilde{R} = 2\mu - \frac{2}{3}\Theta^2 \, .$$

5.4.1.1 Linear evolution equations

The system of equations governing the evolution of the variables defined in the previous subsection are given as follows using the linear covariant identity A.3 together with Eq. 5.49, one can write the evolution of the total matter fluid Eq. 5.45 as

$$\dot{D}^m_a + \Theta\mathcal{A}_a - \frac{3}{2(\phi+1)}\Big[\Big(\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\Big)\Phi_a - \mu_m V^m_a\Big] - \tilde{\nabla}^2 V^m_a = 0\ . \tag{5.53}$$

Using Eq. 5.32, the evolution of Eq. 5.46 is given as

$$\dot{\mathcal{A}}_a + \Big[\frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big]\mathcal{A}_a + \frac{1}{2(\phi+1)}\mu_m V^m_a + \frac{1}{2(\phi+1)}\Big[\frac{1}{3}\Theta + \frac{\dot{\phi}'}{\phi'} - \frac{2\dot{\phi}}{(\phi+1)}\Big]\Phi_a = 0\ . \tag{5.54}$$

The evolution of Eqs. 5.47, 5.50 and 5.51 are given as

$$\dot{V}^m_a + \mathcal{A}_a = 0\ , \tag{5.55}$$

$$\dot{\Phi}_a - \frac{\ddot{\phi}'}{\dot{\phi}'}\Phi_a - \dot{\phi}\mathcal{A}_a = 0\ , \tag{5.56}$$

$$\dot{\Psi}_a - \left(\dot{\phi}' - \frac{2\dot{\phi}'\phi''}{\phi'} + \frac{\dot{\phi}'}{\phi'}\right)\Psi_a - \left(\frac{\phi''\dot{\phi}^2}{\phi'^3} - \frac{\phi'''\dot{\phi}^2}{\phi'^2} - \frac{\dot{\phi}\dot{\phi}'\phi''}{\phi'^3} + \frac{\dot{\phi}\dot{\phi}''}{\phi'} + \frac{\dot{\phi}''\dot{\phi}}{\phi'^2}\right)\Phi_a - \ddot{\phi}\mathcal{A}_a = 0\ . \tag{5.57}$$

The evolution of Eq. 5.49 is given as

$$\begin{aligned}\dot{Z}_a &+ \left[\frac{\dot{\phi}}{(\phi+1)} + \Theta\right]Z_a - \frac{3\mu_m}{2(\phi+1)}\dot{V}^m_a + \frac{3}{2(\phi+1)}\left(\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right)\dot{\Phi}_a - \Bigg[-\frac{3\ddot{\phi}'}{2\phi'(\phi+1)} \\ &+ \frac{3\dot{\phi}'^2}{4\phi'^2(\phi+1)} + \frac{\Theta^2}{3(\phi+1)} - \frac{\mu_m}{2(\phi+1)} + \frac{f}{4(\phi+1)^2} + \frac{\Theta\dot{\phi}}{2(\phi+1)^2} - \frac{\tilde{\nabla}^2\phi}{2(\phi+1)} - \frac{3\dot{\phi}'\Theta}{2\phi'(\phi+1)}\Bigg]\Phi_a = 0\ .\end{aligned} \tag{5.58}$$

These linear evolution equations are constraint by

$$\begin{aligned}\frac{C_a}{a^2} &= \left(-\frac{2\mu_m}{(\phi+1)^2} + \frac{f}{(\phi+1)^2} - \frac{5\Theta\dot{\phi}}{3(\phi+1)^2} - \frac{2\tilde{\nabla}^2\phi}{3(\phi+1)^2}\right)\Phi_a \\ &+ \frac{2D^m_a}{(\phi+1)} - \frac{2\Theta}{(\phi+1)}\Psi_a - \Big(\frac{2\dot{\phi}}{(\phi+1)} + \frac{4}{3}\Theta\Big)Z_a - \frac{2}{(\phi+1)}\tilde{\nabla}^2\Phi_a\ .\end{aligned} \tag{5.59}$$

5.4.1.2 Second-order equations

If one time propagates Eq. 5.55, the resulting second-order evolution equation for the velocity inhomogeneity of the matter will be given as

$$\ddot{V}^m_a + \left[\frac{\Theta}{3} + \frac{\dot{\phi}}{(\phi+1)}\right]\dot{V}^m_a - \frac{1}{2(\phi+1)}\left[\mu_m V^m_a + \left(\frac{1}{3}\Theta + \frac{\dot{\phi}'}{\phi'} - \frac{2\dot{\phi}}{(\phi+1)}\right)\Phi_a\right] = 0\ . \tag{5.60}$$

Using Eq. 5.60 together with Eqs. 5.56 and 5.50 in the time propagated equation of Eq. 5.53, one obtains

$$\begin{aligned}\ddot{D}^m_a &+ \frac{\dot{\phi}}{(\phi+1)}\dot{D}^m_a - \frac{3\mu_m}{2(\phi+1)}D^m_a - \Theta\ddot{V}^m_a + \Big[\frac{\Theta^2}{3} + \frac{5\mu_m}{2(\phi+1)} - \frac{f}{2(\phi+1)} + \frac{\tilde{\nabla}^2\phi}{(\phi+1)} \\ &+ \frac{3\dot{\phi}\dot{\phi}'}{2\phi'(\phi+1)} - \frac{3\Theta\dot{\phi}}{2(\phi+1)}\Big]\dot{V}^m_a - \tilde{\nabla}^2\dot{V}^m_a + \left[\frac{2}{3}\Theta - \frac{\dot{\phi}}{(\phi+1)}\right]\tilde{\nabla}^2V^m_a + \frac{\Theta}{(\phi+1)}\dot{\Phi}_a \\ &+ \frac{1}{2(\phi+1)}\left[\frac{2f}{(\phi+1)} - \frac{3\ddot{\phi}'}{\phi'} - \frac{4\mu_m}{(\phi+1)} + \frac{2}{3}\Theta^2 + \frac{4\Theta\dot{\phi}}{(\phi+1)} - \frac{\tilde{\nabla}^2\phi}{(\phi+1)} - \frac{7\Theta\dot{\phi}'}{\phi'}\right]\Phi_a = 0\,.\end{aligned} \tag{5.61}$$

The second evolution equation for the scalar fluid Eq. 5.50 can be obtained by using Eq. 5.54 as

$$\begin{aligned}\ddot{\Phi}_a &- \left(\frac{\dot{\phi}'}{\phi'}\right)\dot{\Phi}_a - \left[\frac{\ddot{\phi}'}{\phi'} - \frac{\dot{\phi}'^2}{\phi'^2} - \frac{\Theta\dot{\phi}}{6(\phi+1)} - \frac{\dot{\phi}\dot{\phi}'}{2\phi'(\phi+1)} + \frac{\dot{\phi}^2}{(\phi+1)^2}\right]\Phi_a - \\ &\left[\frac{\Theta\dot{\phi}}{3} + \frac{\dot{\phi}^2}{(\phi+1)} - \ddot{\phi}\right]\dot{V}^m_a + \frac{\dot{\phi}\mu_m}{2(\phi+1)}V^m_a = 0\,.\end{aligned} \tag{5.62}$$

5.4.2 Definition of scalar gradient variables

We study formation of structures on large scales through the density fluctuation which is a scalar quantity. The scalar perturbations are the ones responsible for these fluctuations. The quantities we have defined so far contain both a scalar and a vector part. therefore, we extract the scalar parts of the perturbation vectorial gradients of these quantities and we do this by applying a local decomposition [111]

$$a\tilde{\nabla}aX = X_{ab} = \frac{1}{3}h_{ab}X + \Sigma^X_{ab} + X_{[ab]}\,, \tag{5.63}$$

where $\Sigma^X_{ab} = X_{(ab)} - 3h_{ab}X$ describes shear whereas $X_{[ab]}$ describes the vorticity. When extracting the scalar contribution the vorticity term vanishes.
Based on the above decomposition, by applying the comoving differential operator $a\tilde{\nabla}_a$ to those quantities we have defined in the previous subsection, our scalar variables can be given to linear order as

$$\Delta_m = a\tilde{\nabla}^a D^m_a = \frac{a^2\tilde{\nabla}^2\mu_m}{\mu_m}\,. \tag{5.64}$$

$$Z = a\tilde{\nabla}^a Z_a = a^2\tilde{\nabla}^2\Theta\,, \tag{5.65}$$

$$V^m = a\tilde{\nabla}V^m_a = a^2\tilde{\nabla}^a v_a\,, \tag{5.66}$$

$$\Psi = a\tilde{\nabla}^a\Psi_a = a^2\tilde{\nabla}^2\dot{\phi}\,, \tag{5.67}$$

$$\mathcal{A} = a\tilde{\nabla}^a\mathcal{A}_a = a^2\tilde{\nabla}^a A_a\,, \tag{5.68}$$

$$\Phi = a\tilde{\nabla}^a\Phi_a = a^2\tilde{\nabla}^2\phi\,. \tag{5.69}$$

5.4.2.1 Scalar equations

The scalar variables evolve according to

$$\dot{\Delta}^m + \Theta\mathcal{A} - \frac{3}{2(\phi+1)}\left[\left(\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right)\Phi - \mu_m V^m\right] - \tilde{\nabla}^2V^m = 0\,. \tag{5.70}$$

$$\dot{V}^m + \mathcal{A}_a = 0\,. \tag{5.71}$$

$$\dot{\Phi} - \frac{\dot{\phi}'}{\phi'}\Phi - \dot{\phi}\mathcal{A} = 0\,, \tag{5.72}$$

$$\dot{\mathcal{A}}+\Big(\frac{1}{3}\Theta+\frac{\dot{\phi}}{(\phi+1)}\Big)\mathcal{A}+\frac{1}{2(\phi+1)}\mu_m V^m+\frac{1}{2(\phi+1)}\left(\frac{1}{3}\Theta+\frac{\dot{\phi}'}{\phi'}-\frac{2\dot{\phi}}{(\phi+1)}\right)\Phi=0\,, \tag{5.73}$$

$$\dot{\Psi}-\left(\dot{\phi}'-\frac{2\dot{\phi}'\phi''}{\phi'}+\frac{\dot{\phi}'}{\phi'}\right)\Psi-\left(\frac{\phi''\dot{\phi}^2}{\phi'^3}-\frac{\phi'''\dot{\phi}^2}{\phi'^2}-\frac{\dot{\phi}\dot{\phi}'\phi''}{\phi'^3}+\frac{\dot{\phi}\dot{\phi}''}{\phi'}+\frac{\dot{\phi}''\dot{\phi}}{\phi'^2}\right)\Phi-\ddot{\phi}\mathcal{A}=0\,, \tag{5.74}$$

$$\dot{Z}-\left[\frac{\dot{\phi}}{(\phi+1)}+\Theta\right]Z-\frac{3\mu_m}{2(\phi+1)}V^m+\frac{3}{2(\phi+1)}\Big(\frac{\dot{\phi}'}{\phi'}-\frac{1}{3}\Theta\Big)\dot{\Phi}-\Big[-\frac{3\ddot{\phi}'}{2\phi'(\phi+1)} \tag{5.75}$$

$$+\frac{3\dot{\phi}'^2}{4\phi'^2(\phi+1)}+\frac{\Theta}{3(\phi+1)}-\frac{\mu_m}{2(\phi+1)}+\frac{f}{4(\phi+1)^2}+\frac{\Theta\dot{\phi}}{2(\phi+1)^2}-\frac{\tilde{\nabla}^2\phi}{2(\phi+1)}-\frac{3\dot{\phi}'\Theta}{2\phi'(\phi+1)}\Big]\Phi=0\,.$$

5.4.2.2 Second-order equations

The second-order equations are given as

$$\ddot{V}^m+\left(\frac{\Theta}{3}+\frac{\dot{\phi}}{(1+\phi)}\right)\dot{V}^m-\frac{1}{2(\phi+1)}\left[\mu_m V^m+\left(\frac{1}{3}\Theta+\frac{\dot{\phi}'}{\phi'}-\frac{2\dot{\phi}}{(\phi+1)}\right)\Phi\right]=0\,, \tag{5.76}$$

$$\ddot{D}^m+\left[\frac{\dot{\phi}}{(\phi+1)}\right]\dot{D}^m-\Big[\frac{3\mu_m}{2(\phi+1)}\Big]D^m-\Theta\dot{V}^m+\Big[\frac{\Theta^2}{3}+\frac{5\mu_m}{2(\phi-1)}-\frac{f}{2(\phi+1)} \tag{5.77}$$

$$+\frac{\tilde{\nabla}^2\phi}{(\phi+1)}+\frac{3\dot{\phi}\dot{\phi}'}{2\phi'(\phi+1)}-\frac{3\Theta\dot{\phi}}{2(\phi+1)}\Big]\dot{V}^m-\tilde{\nabla}^2\dot{V}^m+\left(\frac{2}{3}\Theta-\frac{\dot{\phi}}{(\phi+1)}\right)\tilde{\nabla}^2V^m$$

$$+\frac{\Theta}{(\phi+1)}\dot{\Phi}+\frac{1}{2(\phi+1)}\Big[\frac{2f}{(\phi+1)}-\frac{3\ddot{\phi}'}{\phi'}-\frac{4\mu_m}{(\phi+1)}+\frac{2}{3}\Theta^2+\frac{4\Theta\dot{\phi}}{(\phi+1)}-\frac{\tilde{\nabla}^2\phi}{(\phi+1)}-\frac{7\Theta\dot{\phi}'}{\phi'}\Big]\Phi=0\,.$$

$$\ddot{\Phi}-\frac{\dot{\phi}'}{\phi'}\dot{\Phi}-\Big[\frac{\ddot{\phi}'}{\phi'}\frac{\dot{\phi}'^2}{\phi'^2}-\frac{\Theta\dot{\phi}}{6(\phi+1)}-\frac{\dot{\phi}\dot{\phi}'}{2\phi'(\phi+1)}+\frac{\dot{\phi}^2}{(\phi+1)^2}\Big]\Phi-\Big[\frac{\Theta\dot{\phi}}{3}+\frac{\dot{\phi}^2}{(\phi+1)}-\ddot{\phi}\Big]\dot{V}^m \tag{5.78}$$

$$+\frac{\dot{\phi}\mu_m}{2(\phi+1)}V^m=0\,.$$

5.4.3 Harmonic decomposition

The above evolution equations can be thought of as a coupled system of harmonic oscillator differential equations of the form [5]

$$\ddot{X}+A\dot{x}+BX=C(Y,\dot{Y})\,, \tag{5.79}$$

where A, B and C are independent of X and they represent friction (damping), restoring and source forcing terms respectively.

To solve Eq. 5.79, a separation of variables is applied such that

$$X(x,t)=X(\vec{x})X(t),\qquad Y(x,t)=Y(\vec{x})Y(t)\,.$$

Since the evolution equations obtained so far are too complicated to be solved, the harmonic decomposition approach is applied to these equations using the eigenfunctions and the corresponding wave number for these equations, therefore we write

$$X=\sum_k X^k Q_k(\vec{x})\,,\qquad Y=\sum_k Y^k(t)Q_k(\vec{x})\,, \tag{5.80}$$

where $Q_k(x)$ are the eigenfunctions of the covariantly defined spatial Laplace-Beltrami operator [5, 112], such that

$$\tilde{\nabla}^2 Q=-\frac{k^2}{a^2}Q\,. \tag{5.81}$$

The order of the harmonic (wave number) is given by

$$k = \frac{2\pi a}{\lambda} , \tag{5.82}$$

where λ is the physical wavelength of the mode. The eigenfunctions Q are covariantly constant, *i.e.*

$$\dot{Q}_k(\vec{x}) = 0 . \tag{5.83}$$

5.4.4 Scalar perturbations

Applying the harmonic decomposition, the linear evolution equations 5.70-5.75 can be rewritten as

$$\dot{\Delta}^k_m + \Theta\mathcal{A}^k - \frac{3}{2(\phi+1)}\Big(\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\Big)\Phi^k + \Big[\frac{3\mu_m}{2(\phi+1)} + \frac{k^2}{a^2}\Big]\dot{V}^k_m = 0 , \tag{5.84}$$

$$\dot{V}^k_m + \mathcal{A}^k = 0 , \tag{5.85}$$

$$\dot{\Phi}^k - \frac{\ddot{\phi}'}{\phi'}\Phi^k - \dot{\phi}\mathcal{A}^k = 0 . \tag{5.86}$$

$$\dot{\mathcal{A}}^k + \Big[\frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big]\mathcal{A}^k + \frac{1}{2(\phi+1)}\mu_m \dot{V}^k_m + \frac{1}{2(\phi+1)}\Big[\frac{1}{3}\Theta + \frac{\dot{\phi}'}{\phi'} - \frac{2\dot{\phi}}{(\phi+1)}\Big]\Phi^k = 0 . \tag{5.87}$$

$$\dot{\Psi}^k - \Big(\ddot{\phi}' - \frac{2\ddot{\phi}'\phi''}{\phi'} + \frac{\ddot{\phi}'}{\phi'}\Big)\Psi^k - \Big(\frac{\phi''\dot{\phi}^2}{\phi'^3} - \frac{\phi'''\dot{\phi}^2}{\phi'^2} - \frac{\dot{\phi}\dot{\phi}'\phi''}{\phi'^3} + \frac{\dot{\phi}\dot{\phi}''}{\phi'} + \frac{\dot{\phi}''\dot{\phi}}{\phi'^2}\Big)\Phi^k \tag{5.88}$$
$$-\ddot{\phi}\mathcal{A}^k = 0 .$$

$$\dot{Z}^k + \Big[\frac{\dot{\phi}}{(\phi+1)} + \Theta\Big]Z^k - \frac{3\mu_m}{2(\phi+1)}\dot{V}^k_m + \frac{3}{2(\phi+1)}\left(\frac{\ddot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right)\dot{\Phi}^k \tag{5.89}$$
$$-\Big[-\frac{3\dddot{\phi}'}{2\phi'(\phi+1)} + \frac{3\ddot{\phi}'^2}{4\phi'^2(\phi+1)} + \frac{\Theta}{3(\phi+1)} - \frac{\mu_m}{2(\phi+1)} + \frac{f}{4(\phi+1)^2}$$
$$+\frac{\Theta\dot{\phi}}{2(\phi+1)^2} - \frac{\tilde{\nabla}^2\phi}{2(\phi+1)} - \frac{3\ddot{\phi}'\Theta}{2\phi'(\phi+1)}\Big]\Phi^k = 0 .$$

The second-order evolution equations 5.76-5.92 can be rewritten as

$$\ddot{V}^k_m + \left(\frac{\Theta}{3} + \frac{\dot{\phi}}{(1+\phi)}\right)\dot{V}^k_m - \frac{1}{2(\phi+1)}\left[\mu_m \dot{V}^k_m + \left(\frac{1}{3}\Theta + \frac{\ddot{\phi}'}{\phi'} - \frac{2\dot{\phi}}{(\phi+1)}\right)\Phi^k\right] = 0 . \tag{5.90}$$

$$\ddot{\Delta}^k_m + \frac{\dot{\phi}}{(\phi+1)}\dot{\Delta}^m - \frac{3\mu_m}{2(\phi+1)}\Delta^k_m - \Theta\ddot{V}^k_m + \Big[\frac{\Theta^2}{3} + \frac{5\mu_m}{2(\phi+1)} - \frac{f}{2(\phi+1)} - \frac{k^2\phi}{a^2(\phi+1)} \tag{5.91}$$
$$+\frac{3\dot{\phi}\ddot{\phi}'}{2\phi'(\phi+1)} - \frac{3\Theta\dot{\phi}}{2(\phi+1)} + \frac{k^2}{a^2}\Big]\dot{V}^k_m - \Big[\frac{2k^2}{3a^2}\Theta - \frac{\dot{\phi}k^2}{(\phi+1)a^2}\Big]\dot{V}^k_m + \frac{\Theta}{(\phi+1)}\dot{\Phi}^k$$
$$+\frac{1}{2(\phi+1)}\Big[\frac{2f}{(\phi+1)} - \frac{3\dddot{\phi}'}{\phi'} - \frac{4\mu_m}{(\phi+1)} + \frac{2}{3}\Theta^2 + \frac{4\Theta\dot{\phi}}{(\phi+1)} - \frac{\tilde{\nabla}^2\phi}{(\phi-1)} - \frac{7\Theta\ddot{\phi}'}{\phi'}\Big]\Phi^k = 0 .$$

$$\ddot{\Phi}^k - \frac{\ddot{\phi}'}{\phi'}\dot{\Phi}^k - \Big[\frac{\dddot{\phi}'}{\phi'} - \frac{\ddot{\phi}'^2}{\phi'^2} - \frac{\Theta\dot{\phi}}{6(\phi+1)} - \frac{\dot{\phi}\ddot{\phi}'}{2\phi'(\phi+1)} + \frac{\dot{\phi}^2}{(\phi+1)^2}\Big]\Phi^k \tag{5.92}$$
$$-\left(\frac{\Theta\dot{\phi}}{3} + \frac{\dot{\phi}^2}{(\phi+1)} - \ddot{\phi}\right)\dot{V}^k_m + \frac{\dot{\phi}\mu_m}{2(\phi+1)}\dot{V}^k_m = 0 .$$

5.5 Solutions

In this section, we consider R^n model, one of the $f(R)$ toy models that are considered to be the simplest and widely studied form of higher order $f(R)$ gravitational theories.
The Lagrangian density of such models is given as

$$f(R) = \beta R^n \,, \tag{5.94}$$

where β represents the coupling parameter and an arbitrary constant $n \neq 1$ is considered for exploring cosmological models. In [113], it has been shown, using the cosmological dynamical systems approach, that the scale factor $a(t)$ admits an exact solution of the form

$$a = a_0 t^{\frac{2n}{3(1+w)}} \,, \tag{5.95}$$

with $w = 0$ and normalized coefficients β and a_0. One can obtain the following expressions for the expansion, the Ricci scalar and the effective matter energy density respectively:

$$\Theta = \frac{2n}{t} \,. \tag{5.96}$$

$$R = \frac{4n(4n-3)}{3t^2} \,, \tag{5.97}$$

$$\mu_m = n\left(\frac{3}{4}\right)^{1-n}\left(\frac{4n^2-3n}{t^2}\right)^{n-1}\left(\frac{-16n^2+26n-6}{3t^2}\right) . \tag{5.98}$$

Therefore we have the perturbation equations 5.90, 5.91 and 5.88 as

$$\ddot{V}^k_m + \left(\frac{(6-4n)}{3t}\right)\dot{V}^k_m - \left(\frac{8n^2-13n+3}{3t^2}\right)V^k_m - \left[\frac{4t^{2n-3}}{3\left(\frac{4n(4n-3)}{3}\right)^{n-1}}\right]\Phi_k = 0 \,, \tag{5.99}$$

$$\begin{aligned}
&\ddot{\Delta}^k_m - \left[\frac{2(1-n)}{t}\right]\dot{\Delta}^k_m + \left(\frac{3+13n-8n^2}{t^2}\right)\Delta^k_m - \left(\frac{2n}{t}\right)\ddot{V}^k_m + \\
&\left[\frac{(62n^2-127n+27)}{3t^2} + \frac{k^2 t^{\frac{2(n-3)}{3}}}{n\left(\frac{4n(4n-3)}{3}\right)^{n-1}} + 6n^2 - 6n\right]\dot{V}^k_m - \left[\frac{2k^2(3-n)}{3t^{\frac{(4n-3)}{3}}}\right]V^k_m \\
&+ \left[\frac{2t^{(2n-3)}}{\left(\frac{4n(4n-3)}{3}\right)^{n-1}}\right]\dot{\Phi}_a + \frac{t^{2n}}{n\left(\frac{4n(4n-3)}{3}\right)^{n-1}}\left[\frac{(28n^2-8n)}{3t^4} - \frac{6(2n^2-7n+6)}{t^3}\right. \\
&\left. + \frac{k^2}{t^{\frac{(4n-6)}{3}}} - \frac{k^2}{nt^{\frac{(12-2n)}{3}} n\left(\frac{4n(4n-3)}{3}\right)^{n-1}}\right]\Phi_k = 0 \,.
\end{aligned} \tag{5.100}$$

$$\begin{aligned}
&\ddot{\Phi}_k - \left(\frac{(4-2n)}{t}\right)\dot{\Phi}_k - \left(\frac{(8n^2-8n+12)}{3t^2}\right)\Phi_k \\
&-(-2n+2)\left(\frac{4n(4n-3)}{3}\right)^{n-1}\left(\frac{2n^2}{3t^2} + 2n^2 - 3n + 2\right)t^{-2n}\dot{V}^k_m \\
&+n(1-n)\left(\frac{4}{3}\right)^{1-n}\left(\frac{4n^2-3n}{t^2}\right)^{n-1}\left(\frac{16n^2+26n-6}{3t^3}\right)\dot{V}^k_m = 0 \,.
\end{aligned} \tag{5.101}$$

5.5.1 Exact solutions

In this part of the chapter we will solve the perturbations equations we obtained so far. The exact solutions of the density and velocity perturbation equations are found in the comoving frame, using the $f(R)$ solutions in [88] and a simple workaround. A simple alternative is then to work in the comoving frame and apply the transformation from the comoving frame to the quasi-Newtonian frame using the following identity [31]

$$\tilde{D}_a f = \tilde{\nabla}_a f + \dot{f} v_a \,. \tag{5.102}$$

The comoving perturbation variables are defined as

$$\tilde{\Delta}^m_a = \frac{a\tilde{D}_a \mu_m}{\mu_m} \,, \tag{5.103}$$
$$\tilde{Z}_a = a\tilde{D}_a \Theta \,, \tag{5.104}$$
$$\tilde{\Phi}_a = a\tilde{D}_a \phi \,, \tag{5.105}$$
$$\tilde{\Psi}_a = a\tilde{D}_a \dot{\phi} \,. \tag{5.106}$$

By using the identity 5.102, the comoving perturbation variables can be rewritten as

$$\tilde{\Delta}^m_a = \Delta^m_a - \Theta V^m_a \,, \tag{5.107}$$
$$\tilde{Z}_a = Z_a - \left[\frac{1}{3}\Theta^2 + \frac{1}{2(\phi+1)}\left(2\mu_m - f - 2\Theta\dot{\phi} + 2\tilde{\nabla}^2\phi\right)V^m_a\right] \,. \tag{5.108}$$
$$\tilde{\Phi}_a = \Phi_a + \dot{\phi} V^m_a \,, \tag{5.109}$$
$$\tilde{\Psi}_a = \Psi_a + \ddot{\phi} V^m_a \,. \tag{5.110}$$

The second-order evolution equation of the density perturbation in the comoving frame admits a general solution of the form [88]

$$\tilde{\Delta}^k_m = C_1 t^{-1} + C_2 t^{\alpha_+} + C_3 t^{\alpha_-} - C_4 C_0 t^{2-\frac{4n}{3}} \,, \tag{5.111}$$

where C_1, C_2, C_3 and C_4 are constants, $\alpha_\pm$ is given as

$$\alpha_\pm = -\frac{1}{2} \pm \frac{\sqrt{(n-1)(256n^3 - 608n^2 + 417n - 81)}}{6(n-1)} \,, \tag{5.112}$$

and C_0 is the conserved value for the gradient variable C_a.
Therefore, by using Eqs. 5.107 and 5.111, the general solution of the density perturbation Eq. 5.91 in the quasi-Newtonian frame can be written as

$$\Delta^k_m = C_1 t^{-1} + C_2 t^{\alpha_-} + C_3 t^{\alpha_-} - C_4 C_0 t^{2-\frac{4n}{3}} + \frac{2n}{t} V^k_m \,. \tag{5.113}$$

The gradient variable Φ_a is equivalent to $\mathcal{R}_a$ defined in $f(R)$ theory [5], such that

$$\phi_a = \phi' \mathcal{R}_a \,.$$

The solution of $\mathcal{R}$ has been obtained in the comoving frame [88] and it has the form

$$\mathcal{R} = C_5 t^{-3} + C_6 t^{\beta_-} + C_7 t^{\beta_-} - C_8 C_0 t^{-\frac{4n}{3}} \,. \tag{5.114}$$

Therefore, the solution of the second-order perturbation Eq. 5.101 in the comoving frame can be written as

$$\tilde{\Phi}_k = n(n-1)\left(\frac{4n(4n-3)}{3t^2}\right)^{n-2}\left[C_5 t^{-3} + C_6 t^{\beta_+} + C_7 t^{\beta_-} - C_8 C_0 t^{-\frac{4n}{3}}\right] . \quad (5.115)$$

Therefore, by using Eq. 5.109, the general solution of the perturbation Eq. 5.101 in the quasi-Newtonian frame is given as

$$\Phi_k = n(n-1)\left(\frac{4n(4n-3)}{3t^2}\right)^{n-2}\left(C_5 t^{-3} + C_6 t^{\beta_+} + C_7 t^{\beta_-} - C_8 C_0 t^{-\frac{4n}{3}} + \left(\frac{8n(4n-3)}{3t^3}\right) V_m^k\right) . \quad (5.116)$$

Using Eq. 5.116, the general solution of the perturbation Eq. 5.90 in the quasi- Newtonian frame is given as

$$V_m^k = (K_2 + K_3(B_1 - B_2))\, t^{\frac{2n}{3}+\gamma_-} + (K_1 + K_3(B_3 + B_4))\, t^{\frac{2n}{3}+\gamma_+} , \quad (5.117)$$

where K_1, K_2 are two arbitrary constants and

$$K_3 = \frac{3(n-1)\left(\frac{16n^2}{3} - 4n\right)^n}{(4n-3)\sqrt{-8n^2 - 48n(2n+2) + 132n - 27}} , \quad (5.118)$$

$$\gamma_\pm = -\frac{1}{2} \pm \frac{\sqrt{-80n^2 - 48n(2n+2) + 132n - 27}}{6} , \quad (5.119)$$

$$B_1 = C_1 \int t^{\left(-\frac{2n}{3} - \frac{3}{2}\right)+\gamma_-} dt , \quad (5.120)$$

$$B_3 = C_1 \int t^{\left(-\frac{2n}{3} - \frac{3}{2}\right)+\gamma_-} dt , \quad (5.121)$$

where B_2 and B_4 are all functions of time and their expressions are rather complicated.
Thus, we get the full set of exact solutions for the density and velocity perturbation equations in the quasi-Newtonian frame.

5.5.2 Approximate solutions

In this subsection, we apply a quasi-static approximation to the evolution equations 5.91 and 5.90. In this approximation, terms involving time derivatives for gravitational potential are neglected and only those terms involving density perturbation are kept [5, 114, 115]. In [5], a quasi-static approximation for the matter perturbation has been introduced for both radiation and dust dominated epochs. A quasi-static approximation was taken such that the time evolutions of $\mathcal{R}_a$ are neglected, $\dot{\mathcal{R}}_a \simeq 0$ and $\ddot{\mathcal{R}}_a \simeq 0$.

According to Eq. 5.5.1, the time variations in Φ_a are neglected, *i.e.* $\dot{\Phi}_a \simeq 0$ and $\ddot{\Phi}_a \simeq 0$.
In this approximation, one can write the linear evolution Eq. 5.86 as

$$\Phi^k = \frac{\dot{\phi}'\phi}{\dot{\phi}'} \dot{V}_m^k = 0 , \quad (5.122)$$

which implies

$$\Phi^k = \left[\frac{n(-1-n)}{2-n}\left(\frac{4n(4n-3)}{3}\right)^{n-1} t^{(-2n+2)}\right] \dot{V}_m^k . \quad (5.123)$$

By substituting Eq. 5.123 into Eq. 5.99, we get

$$\ddot{V}^k_m + \left(\frac{48n^2 - 108n + 7n}{(36 - 18n)t}\right)\dot{V}^k_m + \left(\frac{(8n^2 - 13n + 3)}{3t^2}\right)V^k_m = 0\,. \tag{5.124}$$

This second-order equation admits a general solution of the form

$$V^k_m(t) = C_9 t^{D+E_+} + C_{10} t^{D+E_-}\,, \tag{5.125}$$

where

$$D = \frac{\sqrt{-32n^4 + 300n^3 - 723n^2 + 588n - 108}}{6(n-2)}\,,$$

and

$$E_\pm = \pm\frac{(8n^2 - 15n + 6)}{6(n-2)}\,.$$

Based on this solution in Eq. 5.125 and its first and second time derivative, the general solution of Eq. 5.100 is

$$\Delta(t) = C_{11} t^{-\frac{1}{2}+n+L_+} + C_{12} t^{-\frac{1}{2}+n+L_-}\,, \tag{5.126}$$

where C_{11} and C_{12} are arbitrary constants and

$$L_\pm = \pm\frac{\sqrt{36n^2 - 56n - 11}}{2}\,.$$

There are some other solutions to Eq. 5.100 which are rather too complicated to be presented.

Chapter 6

Conclusions and outlook

The recent observations in cosmology and astrophysics have shown that our universe is expanding at an accelerating rate. It has been shown that the standard model of cosmology and Einstein's general theory of relativity have failed to explain these phenomena and this has motivated cosmologists to deviate from these axioms of GR to focus their research on alternative theories of gravity. In this dissertation, our main goal has been the study of the cosmological perturbation in the context of one of these modified theories of gravity to draw some conclusions about the implications of gravity to the accelerated expansion of the universe.

In Chapter 1, we presented a detailed review of gravity and its historical evolution and we reviewed the main principles of GR, which is the most successful theory of gravity and how it is considered to be the main basis of modern cosmology. In Chapter 2, we reviewed the concordance model of cosmology and we discussed some of its shortcomings such as the accelerated cosmic expansion of the universe, the unknown forms and origins of dark energy and dark matter and discussed how cosmologists started to think of the alternative theories of gravity based on these shortcomings. We gave an overview of the theories of cosmological perturbations and discussed the two approaches of applying these perturbations in cosmology and we focused on the review of the $1+3$ covariant formalism. We presented the covariant linearised evolution and constraint equations. In Chapter 3, we reviewed two of these alternative theories of gravity, namely $f(R)$ and scalar-tensor theories. We presented different formulations of $f(R)$ such as metric formalism, Palatini formalism and the metric-affine formalism and their corresponding field equations. We showed that $f(R)$ theories of gravity are classified to be a sub-class of the scalar-tensor theories.

In Chapter 4, based on the equivalence between $f(R)$ and scalar-tensor theories of gravity, we presented some applications on reconstructing $f(R)$ models from different cosmological expansion scenarios such as the Chaplygin scalar field in flat de Sitter spacetimes. Based on this equivalence, the analysis made use of a combination of energy density, pressure, kinetic and potential terms of the scalar field to obtain relationships between the scalar field, the Ricci scalar and the scale factor. As it has been shown from the reconstructed functionals and their corresponding plots, we presented the $f(R)$ versus the GR Lagrangians density ($f(R) = R$). These expressions have GR as their respective limiting solutions. The results have been applied to a de Sitter universe and the time-dependent expressions for the potential have been obtained for both the original and generalized cases of the Chaplygin gas model. This was made possible by two asymptotic assumptions, early- and late-universe epochs of cosmic evolution. The behaviours of the potential for both the original and generalized cases show that the amplitude of the potential gets weaker with time in a de Sitter universe except for the early universe consideration in the generalized Chaplygin gas model where the potential grows with time. However, the dependence of the potential on the scalar field shows that for larger values of the scalar field, the potential is also large, but for the generalized case within a late universe, the potential is observed to have a valley before it follows the trend observed for other potentials. We also studied cosmological fluid systems composed of radiation and a scalar field and with predetermined inflationary expansion scenarios of the FLRW background spacetime. Using the scalar field as the extra degree of freedom that appears in generic $f(R)$ gravity models, we reconstructed $f(R)$ Lagrangians and the corresponding potentials for the scalar field. The reconstructed $f(R)$ actions have GR and ΛCDM actions as limiting cases. We applied the results to slow-roll approximations to show that actually, the parameters that are involved in the Lagrangian reconstruction can be constrained by observational data.

In Chapter 5, we investigated classes of shear-free cosmological dust models with irrotational fluid flows in the context of scalar-tensor theories. We presented the integrability conditions that describe a consistent evolution of the linearised field equations of quasi-Newtonian universes. We defined the gradient variables that characterize the cosmological perturbations and derived the first- and second-order evolution equations of these variables. The harmonic decomposition approach is applied to these equations in order to solve this complicated system of differential equations. After getting a complete set of the perturbation equations, we solved these equations by considering R^n models to get the exact solutions for the density and velocity perturbations. We introduced the so-called quasi-static approximation to admit the approximated solutions on small scales. Solving the whole system numerically has been left for future work since a more detailed analysis and certain assumptions of the initial conditions, which are currently not well understood, are needed.

Appendix A

Useful linearised differential identities

Some of the following linearised identities which hold for all scalars f, vectors V_a and tensors $S_{ab} = S_{\langle ab\rangle}$, have been used throughout this dissertation:

$$\eta^{abc}\tilde{\nabla}_b\tilde{\nabla}_c f = 0\,, \tag{A.1}$$

$$\left(\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}f\right)^{\cdot} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{f} - \frac{2}{3}\Theta\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}f + \dot{f}\tilde{\nabla}_{\langle a}A_{b\rangle}\,, \tag{A.2}$$

$$\left(\tilde{\nabla}_a f\right)^{\cdot} = \tilde{\nabla}_a\dot{f} - \frac{1}{3}\Theta\tilde{\nabla}_a f + \dot{f}A_a\,. \tag{A.3}$$

$$\left(\tilde{\nabla}_a S_{b\ldots}\right)^{\cdot} = \tilde{\nabla}_a\dot{S}_{b\ldots} - \frac{1}{3}\Theta\tilde{\nabla}_a S_{b\ldots}\,, \tag{A.4}$$

$$\left(\tilde{\nabla}^2 f\right)^{\cdot} = \tilde{\nabla}_a\dot{f} - \frac{2}{3}\Theta\tilde{\nabla}f + \dot{f}\tilde{\nabla}^a A_a\,, \tag{A.5}$$

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}V_c = \frac{1}{3}\left(\frac{1}{3}\Theta^2 - \mu\right)V_{[a}h_{b]c}\,. \tag{A.6}$$

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}S^{cd} = \frac{2}{3}\left(\frac{1}{3}\Theta^2 - \mu\right)S_{[a}^{(c}h_{b]}^{d)}\,. \tag{A.7}$$

$$\tilde{\nabla}^a\left(\eta_{abc}\tilde{\nabla}^b V^c\right) = 0\,, \tag{A.8}$$

$$\tilde{\nabla}_b\left(\eta^{cd\langle a}\tilde{\nabla}_c S_d^{b\rangle}\right) = \frac{1}{2}\eta^{abc}\tilde{\nabla}_b\left(\tilde{\nabla}_d S_c^d\right)\,. \tag{A.9}$$

$$\tilde{\nabla}^2(\tilde{\nabla}_a f) = \tilde{\nabla}_a(\tilde{\nabla}^2 f) + \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\tilde{\nabla}_a f + 2\dot{f}\eta_{abc}\tilde{\nabla}^b\omega^c\,. \tag{A.10}$$

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