Von Neumann group algebras and type analysis

CH Pretorius

orcid.org 0000-0003-0388-9776

Dissertation accepted in partial fulfilment of the requirements for the degree *Master of Science in Mathematics* at the North-West University

Supervisor: Prof LE Labuschagne

Graduation: July 2024

Acknowledgements

I offer my sincere gratitude to:

- My supervisor and mentor, Prof. Louis E Labuschagne, who without fail, was always able to find the time to patiently guide me. Our conversations in the past couple of years have been some of the most stimulating and enjoyable I have had to date. The knowledge and wisdom he has imparted to me will surely guide me for many years to come. To have been his student has been both an honour and a privilege.
- My parents Ian and Kobie, as well as my brother, Joseph, for all the love and support they have given me. Without them, none of this would be possible. I will never be able to thank them enough.
- My friends, Jaimé Goedhals, Thian Herbert, Herman Neveling, and Jonathan Troskie. They provided good company and helped me navigate the final stages of this study.
- The DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) for the generous financial support they provided through me 2022 and 2023.

Abstract

Any locally compact group G admits a left (resp. right) translation invariant measure, called a left (resp. right) Haar measure. These measures give access to a sensible theory of integration, as well as the Hilbert space $L^2(G)$. The unitary operators defined by $(\lambda(g)\xi)(k) = \xi(g^{-1}k)$ for all $g \in G$ and $\xi \in L^2(G)$, generate a von Neumann algebra, called a group von Neumann algebra, denoted $\mathrm{VN}_l(G)$. This study investigates group von Neumann algebras, with special attention paid to type analysis.

We start by investigating locally compact groups, their semidirect products and modular functions; and the Haar measure. Special attention is paid to the existence and uniqueness of the Haar measure.

Secondly, we briefly introduce von Neumann algebra theory and the associated direct integral theory. We also investigate the crossed products of von Neumann algebras, especially an interesting connection (unique to a group von Neumann algebra context) they have with semidirect products. We also develop what is known as the Plancherel weight and the canonical modular automorphism group associated to a group von Neumann algebra. To this end, we investigate some Tomita–Takesaki theory.

Next, we assume G is separable. In this context, we investigate structure theorems derived by Colin E. Sutherland ([**Sut78**]), that provides both necessary and sufficient conditions for a group von Neumann algebra to have a central summand of pure type III_λ ($\lambda \in [0, 1]$).

Lastly, we construct group von Neumann algebra examples of type III factors.

Keywords: locally compact group, group von Neumann algebra, crossed product, Plancherel weight, type III factors

Opsomming

Elke lokaalkompakte groep het ’n links (onderskeidelik regs) translasie-invariante maat, wat ons die links (onderskeidelik regs) Haar maat noem. Hierdie mate bied toegang tot ’n verstandige teorie van integrasie, asook die Hilbert ruimte $L^2(G)$. Die von Neumann algebra, voortgebring deur die unitêre operatore gedefinieer deur $(\lambda(g)\xi)(k) = \xi(g^{-1}k)$ vir alle $g \in G$ en $\xi \in L^2(G)$, word ’n groep von Neumann algebra genoem en word aangedui met $\mathrm{VN}_l(G)$. Hierdie studie, ondersoek groep von Neumann algebras en gee spesiale aandag aan die analise van hulle tipe.

Ons begin deur die lokaalkompakte groepe, hulle semidirekte produkte en modulêrefunksies, asook die Haar maat te ondersoek. Spesiale aandag word gegee aan die bestaan en die uniekheid van Haar mate.

Tweedens stel ons kortliks die teorie van von Neumann algebras, asook die geassosieerde teorie van direkte integrale voor. Ons ondersoek ook die kruisproduk van von Neumann algebras, veral ’n interresante konneksie (uniek tot die groep von Neumann algebra konteks) wat dit met semidirekte produkte het. Ons ontwikkel ook wat bekend staan as die Plancherel gewig en die kanoniese modulêre outomorfisme groep geassosieer met group von Neumann algebras. Daarom ondersoek ons dele van Tomita-Takesaki teorie.

Volgende veronderstel ons dat G separabel is. In hierdie konteks ondersoek ons struktuurstellings bewys deur Colin E. Sutherland ([**Sut78**]), wat nodig en voldoende kondisies gee, vir ’n groep von Neumann algebra om ’n sentraal sommand van tipe III_λ ($\lambda \in [0, 1]$) te hê.

Laastens konstrueer ons groep von Neumann algebra voorbeelde van tipe III faktore.

Sleutelwoorde: lokaalkompakte groep, groep von Neumann algebra, kruisproduk van von Neumann algebras, Plancherel gewig, tipe III faktore

Contents

Introduction

In 1933, Alfred Haar discovered the existence of a left translation (resp. right) invariant measure on any locally compact group G, [**Haa33**]. In his honour, such measures are called left (resp. right) Haar measures. The existence of a general Haar measure was a great achievement of 20th century mathematics, as experts at the time did not think it probable. “John von Neumann used afterwards to tell with a wry smile how he had tried to talk Haar out of Haar measure” [**Ped00**, p. 1].

The Haar measure is the cornerstone that connects the rewarding theory of locally compact groups with: Harmonic analysis, Ergodic theory, entropy, dynamical systems, and von Neumann algebras; to name but a few. This study is interested in the theory that arises from the intersection of the theory of locally compact groups and the theory of von Neumann algebras; aptly called the theory of group von Neumann algebras.

The theory of group von Neumann algebras becomes particularly interesting under the assumption that groups are nonabelian. Unfortunately, mathematical literature is not as densely populated with entries studying the nonabelian case as it is with entries of the abelian case. This review is interested in the structure theorems found in [**Sut78**]. We refer to [**Sut78**, Theorem 3.4 and Theorem 4.4] as Sutherland’s structure theorems. These theorems show that, subject to additional assumptions, we have access to a group von Neumann algebra case-specific description of the projections engendering the type decomposition of von Neumann algebras. Furthermore, the theorems provide both necessary and sufficient conditions for a group von Neumann algebra to have a central summand of pure type III_λ with $\lambda \in [0, 1]$.

In order to prove Sutherland’s structure theorems, we will need to introduce the relevant definitions and theorems from various other fields of mathematics. Sections 1.1, 2.3, and 3.1, while not a treatise, enjoy a greater depth of exposition, as the subject matter is more closely related to the theory of locally compact groups and group von Neumann algebras.

In the later chapters, we rely on Tomita-Takesaki theory to provide what is known as the Plancherel weight and modular automorphism group that is associated with a locally compact group, which we develop in Chapter 3. Tomita-Takesaki theory finds its origins in an unpublished work by Minoru Tomita, which was later revised by Masamichi Takesaki. See [**M70**]. These findings later influenced Alain Connes in developing his classification of type III factors [**CLvS10**, pp. 252-253], which we apply in the final chapter of this study. This study also makes use of an interesting intersection of modular theory and direct integral theory; direct intergal theory was introduced by von Neuman in [**vN49**].

In the last chapter, we conclude the study by constructing Sutherland's examples ([**Sut78**, Chapter 5]) of group von Neumann algebras which are factors of type III_λ, $\lambda \in [0,1]$. As it might be of independent interest, we construct both examples which are hyperfinite (approximately finite dimensional von Neumann algebras) and examples which are not. Hyperfinite algebras are especially interesting since they are found in certain areas of mathematical physics, see [**Pow67**].

The author presumes that the reader has a basic understanding of groups, topology, measure theory, von Neumann algebras, and tensor products of both Hilbert spaces and von Neumann algebras. In this study, a neighbourhood of a point will be defined as a set whose interior is an open set that contains the point.

CHAPTER 1

Locally compact groups

In this chapter we probe the theories of locally compact groups and semidirect products. This is done to establish elementary results that will be needed in the chapters hereafter. In particular, we establish the existence of and examine the Haar measure, which in turn is needed to establish the theory of group von Neumann algebras. This was done using [**Fol16**] and sometimes making some novel contributions thereon, including but not limited to a more thorough exposition of the existence of the Haar measure. The semidirect product is explored because it has an important relationship with the crossed product, which we will exploit in later chapters.

1.1. Locally compact groups and the Haar measure

Definition 1.1 (Topological group). A **topological group** is a group G endowed with a topological structure such that both inversion and the group operation are continuous. That is, $G \times G \to G : (g,k) \mapsto gk$ and $G \to G : g \mapsto g^{-1}$ are continuous.

Remark 1.2. For an abstract topological group, we shall adopt multiplication notation and use e_G for the identity - or simply e if it is clear from the context. Let $g \in G$ and A, B be subsets of G. We then also use the following notation:

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\}$$
$$A^{-1} = \{a^{-1} : a \in A\}, \quad AB = \{ab : a \in A, b \in B\}.$$

We shall forgo exponential notation; that is, writing A^2 for AA, to avoid confusion with the set $\{a^2 : a \in A\}$.

We recall the following facts regarding topological groups without proof:

Proposition 1.3 ([**Fol16**, 2.1 Proposition]). *Let G be a topological group*

(i) The topology of G is invariant under translations and inversion; that is, if U is open, then so are gU, Ug, and U^{-1} for any $g \in G$. Moreover, if U is open, then so are UA and AU for any $A \subset G$.

(ii) For every neighbourhood U of e there is a symmetric neighborhood V of e such that $VV \subset U$.

(iii) If H is a subgroup of G, so is $\overline{H}$.

(iv) Every open subgroup of G is closed.

(v) If K and W are compact sets in G, so is KW.

Definition 1.4 (Locally compact group). A **locally compact group** is a topological group G, such that the topology considered is locally compact, by which we mean that each point has a compact neighbourhood.

Example 1.5. Immediately we see that $\mathbb{R}$ is a locally compact group. Another easy example is $\mathbb{T}$, the circle group.

Proposition 1.6 ([**Fol16**, 2.2 Proposition]). *Let H be a subgroup of a topological group G. Endow G/H with its quotient topology.*

(i) If H is closed, G/H is Hausdorff.
(ii) If G is locally compact, so is G/H.
(iii) If H is normal, G/H is a topological group.

Corollary 1.7. *Let G be a topological group. If G satisfies the T_1 separation axiom, then G is Hausdorff. If not, then $G/\overline{\{e\}}$ is a Hausdorff topological group.*

Proof. If G is a T_1 topological space, then singletons are closed. So $\{e\}$ is a closed subgroup. We consider $G/\{e\}$, which is a group isomorphic copy of G. By Proposition 1.6 (i), G is Hausdorff. In the remaining case, we note that $\overline{\{e\}}$ is the smallest closed subgroup of G, since $\{e\}$ is contained in each subgroup. But $g\overline{\{e\}}g^{-1}$ is also a subgroup, which implies that $g\overline{\{e\}}g^{-1} \cap \overline{\{e\}}$ is a closed subgroup contained in $\overline{\{e\}}$. So we must have that $\overline{\{e\}}$ is normal. By Proposition 1.6 (i)-(iii) $G/\overline{\{e\}}$ is a Hausdorff topological group. □

Remark 1.8. In light of Corollary 1.7, assuming that locally compact groups are Hausdorff is virtually no restriction. Therefore, all locally compact groups are assumed to be Hausdorff spaces.

We begin to construct the Haar measure, which is very much at the core of group von Neumann algebra theory. In the approach we are taking, the Haar measure is known as a regular Borel measure:

Definition 1.9 (Regular Borel measure). Let X be a locally compact space with topology τ. Let $\mathscr{A}$ be a σ-algebra containing $\mathfrak{B}(X)$, the Borel σ-algebra of X. A positive measure μ on X is said to be **regular** if

(i) μ is finite when evaluated at any compact set in $\mathscr{A}$,
(ii) $\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ and } U \in \tau\}$ for all $A \in \mathscr{A}$, (Outer regularity)
(iii) $\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ compact}\}$ for all $U \in \tau$, (Inner regularity)

and **Borel** if the domain of μ is $\mathfrak{B}(X)$.

Definition 1.10 (Haar measure). Let a locally compact group G with Borel σ-algebra $\mathfrak{B}(G)$ be given. A regular Borel measure μ_G defined on $\mathfrak{B}(G)$ is called a **left Haar measure** if for any $g \in G$ and $A \in \mathfrak{B}(G)$ it is **left invariant**, i.e. if it satisfies

$$\mu_G(gA) = \mu_G(A).$$

Analogous to the definition above, one can also define a right Haar measure $\tilde{\mu_G}$ to be a regular Borel measures that is right invariant, i.e. satisfying $\tilde{\mu_G}(Kg) = \tilde{\mu_G}(K)$. It is not to difficult to show that each left Haar measure induces a right Haar measure on $\mathfrak{B}(G)$, but more on this in Definition 1.21.

We often refer to left Haar measures simply as Haar measures (since there is essentially no difference between the theories of left and right Haar measures). In Theorem 1.17 we show that a Haar measure of a group is unique up to multiplication by some positive number; hence, we are allowed to refer to a Haar measure of a group, as the Haar measure of that group, if the context allows it. To construct the Haar measure, we will make use of a particular type of function from G into the reals. Some definitions and notational agreements are in order.

Definition 1.11 (Function of compact support). Let ξ be a function defined on a topological space X, with target vector space Y. The **closed support** of ξ, denoted $\operatorname{supp}(\xi)$, is defined as the closure of the set that ξ maps to nonzero vectors. That is

$$\operatorname{supp}(\xi) = \overline{\xi^{-1}(\{y \in Y : y \neq 0\})}.$$

If $\operatorname{supp}(\xi)$ is compact, then we say ξ is of **compact support**.

Remark 1.12. We make the following notational arrangement: Let X be a locally compact Hausdorff space and Y a topological space. We will write $C(X, Y)$ for the spaces of continuous, Y-valued functions on X. If Y is also a vector space, we write $C_c(X, Y)$ for the space of continuous, Y-valued functions of compact support on X.

If the target space is $\mathbb{C}$, we instead write $C(X), C_c(X)$, identified with the notation above in the obvious manner. More specifically, we set

$$C_c^+(G) = \{\xi \in C_c(G) : \xi \geq 0 \text{ and } \xi \not\equiv 0\}.$$

Furthermore, suppose X is a locally compact Hausdorff space and U an open set of X. Given a function $\xi \in C_c(X)$ such that $\operatorname{supp}(\xi) \subseteq U$ and $0 \leq \xi(x) \leq \chi_U$ for all $x \in X$, we will write $\xi \prec U$.

Theorem 1.13. *Any locally compact group G, has a nonzero left Haar measure.*

Proof. We expand upon the proof of Folland [**Fol16**, 2.10 Theorem]. The idea of the proof is to construct a nonzero, positive linear functional on $C_c(G, \mathbb{R})$ and invoke the Riesz–Markov–Kakutani representation theorem [**Coh13**, Theorem 7.2.8], which gifts us an appropriate measure.

For any $g \in G$, and function ξ with domain G, we define the corresponding function $\lambda(g)\xi$ on G by the prescription

$$(\lambda(g)\xi)(s) = \xi(g^{-1}s).$$

This function is called the left translate of ξ through g.

Let $\xi, \eta \in C_c^+(G)$. We define

$$(\xi : \eta) = \inf \left\{ \sum_{j=1}^{n} c_j : \xi \leq \sum_{j=1}^{n} c_j \lambda(g_j)\eta,\ c_j > 0,\ g_j \in G \right\}.$$

We verify that this map is well defined. We know that both ξ and η have compact support; therefore, $\|\xi\|_{\text{sup}} < \infty$ and $\|\eta\|_{\text{sup}} < \infty$. Consider the open interval $A = \left(\frac{1}{2}\|\eta\|_{\text{sup}}, \infty\right)$. Since η is continuous, $\eta^{-1}(A)$ is open. Thus, there exists $g_1, g_2, \ldots, g_n$ in G such that $\text{supp}(\xi) \subseteq \cup_{j=1}^{n} g_j \eta^{-1}(A)$. Fix $s' \in \text{supp}(\xi)$. There exist g_m and $q \in \eta^{-1}(A)$ such that $s' = g_m q$. Consider the inequality

$$\xi(s') = \xi(s')\frac{\eta(q)}{\eta(q)} < \frac{2\|\xi\|_{\text{sup}}}{\|\eta\|_{\text{sup}}}\eta(q) = \frac{2\|\xi\|_{\text{sup}}}{\|\eta\|_{\text{sup}}}(\lambda(g_m)\eta)(s') \leq 2\sum_{i=1}^{n} \frac{\|\xi\|_{\text{sup}}}{\|\eta\|_{\text{sup}}}(\lambda(g_i)\eta)(s'),$$

which is the result of our choice of A. But clearly this inequality holds for any $s \in \text{supp}(\xi)$, for as we iterate though $1 \leq m \leq n$, simply choose $q \in \eta^{-1}(A)$ appropriately. Thus, we must have that $(\xi : \eta) \leq 2\sum_{i=1}^{n} \|\xi\|_{\text{sup}}/\|\eta\|_{\text{sup}}$. As the argument works for any $\xi, \eta \in C_c^+(G)$, the quantity $(\xi : \eta)$ is well defined for all $\xi, \eta \in C_c^+(G)$.

From its definition, the quantity $(\xi : \eta)$ enjoys the properties:

(i) $(\xi : \eta) = (\lambda(g)\xi : \eta)$ for any $g \in G$ (Left invariance)
(ii) $(\xi_1 + \xi_2 : \eta) \leq (\xi_1 : \eta) + (\xi_2 : \eta)$ for any $\xi_1, \xi_2 \in C_c^+(G)$ (Subadditivity)
(iii) $(c\xi : \eta) = c(\xi : \eta)$ for any $a > 0$ (Positive homogoneity)
(iv) $(\xi_1 : \eta) \leq (\xi_2 : \eta)$ given $\xi_1 \leq \xi_2$ (Monotonicity)
(v) $(\xi : \eta) \geq \|\xi\|_{\text{sup}}/\|\eta\|_{\text{sup}}$
(vi) $(\xi : \eta) \leq (\xi : \zeta)(\zeta : \eta)$ for any $\zeta \in C_c^+(G)$.

One can almost immediately deduce these properties, with the exception of (v) and (vi). Since $\text{supp}(\xi)$ is compact in a Hausdorff topology, it is also a closed set. Thus, there exists some $s \in G$ such that $\|\xi\|_{\text{sup}} = \xi(s)$. But by definition, $\xi(s) \leq (\xi : \eta)\|\eta\|_{\text{sup}}$ and we deduce (v).

Suppose $\xi \leq \sum_i a_i \lambda(g_i)\zeta$ and $\zeta \leq \sum_i b_j \lambda(g_j)\eta$. Then we must have that

$$\xi(s) \leq \sum_i a_i(\lambda(g_i)\zeta)(s) \leq \sum_i a_i \sum_j b_j(\lambda(g_j)\eta)(g_i^{-1}s) = \sum_{i,j} a_i b_j(\lambda(g_i g_j)\eta)(s).$$

This shows that $(\xi : \eta) \leq (\sum_i a_i)(\sum_j b_j)$ and we deduce (vi).

We now normalize the quantities $(\xi : \eta)$. To do this, fix $\xi_0 \in C_c^+(G)$ and define

$$I_\eta(\xi) = \frac{(\xi : \eta)}{(\xi_0 : \eta)} \quad \text{for } \xi, \eta \in C_c^+(G).$$

Of course for each η, we then have a functional I_η, which inherits properties (i)-(iv), i.e., I_η is left-invariant, subadditive, homogeneous and monotone.

The functionals of the form I_η are not linear, but do in fact allow us to construct a linear functional on $C_c(G, \mathbb{R})$ as follows:

First, we establish an inequality that will be relevant later in the proof. Using (vi) we obtain the inequalities

$$1 = \frac{(\xi_0 : \eta)}{(\xi_0 : \eta)} \leq (\xi_0 : \xi) I_\eta(\xi) \text{ and } I_\eta(\xi) \leq \frac{(\xi : \xi_0)(\xi_0 : \eta)}{(\xi_0 : \eta)} = (\xi : \xi_0).$$

That is to say

$$(\xi_0 : \xi)^{-1} \leq I_\eta(\xi) \leq (\xi : \xi_0). \tag{1.1}$$

We pause to prove the following technical lemma:

Lemma 1.14. *Given* $\xi_1, \xi_2 \in C_c^+(G)$ *and* $\epsilon > 0$*, there is a neighbourhood* V *of* e *such that* $I_\eta(\xi_1) + I_\eta(\xi_2) \leq I_\eta(\xi_1 + \xi_2) + \epsilon$*, if* $\operatorname{supp}(\eta) \subset V$*.*

Subproof. Fix $\epsilon > 0$. By Urysohn's lemma for locally compact Hausdorff spaces [**Rud87**, 2.12 Urysohn's Lemma], fix $\xi' \in C_c^+(G)$ such that $\xi' = 1$ on $\operatorname{supp}(\xi_1 + \xi_2)$. Let $\delta > 0$ and set $\zeta = \xi_1 + \xi_2 + \delta\xi'$. For $i = 1, 2$ we define a function $\zeta_i = \xi_i/\zeta$ on $\operatorname{supp}(\xi_i)$, and 0 elsewhere. Thus, $\zeta_i \in C_c^+(G)$ for $i = 1, 2$. But then there exists a neighbourhood $\mathfrak{U}_i$ of e in G, such that

$$|\zeta_i(h) - \zeta_i(g)| < \delta \quad \text{if } g^{-1}h \in \mathfrak{U}_i, \tag{1.2}$$

by [**Fol16**, 2.6 Proposition]. Set $\mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2$, so that inequality (1.2) holds for $i = 1, 2$. We proceed to prove that $\mathfrak{U}$ is a neighbourhood of e satisfying the hypothesis.

Suppose that $\eta \in C_c^+(G)$ and $\operatorname{supp}(\eta) \subset \mathfrak{U}$. If $\zeta \leq \sum_j c_j \lambda(g_j)\eta$, then

$$\xi_i(s) = \zeta(x)\zeta_i(s) \leq \sum_j c_j \eta(g_j^{-1}s)\zeta_i(s) \leq \sum_j c_j \eta(g_j^{-1}x)(\zeta_i(g_j) + \delta). \tag{1.3}$$

The last inequality follows from Equation (1.2) and by our assumption that $\operatorname{supp}(\eta) \subset \mathfrak{U}$.

Since $\zeta_1 + \zeta_2 \leq 1$, Equation (1.3) shows that

$$(\xi_1 : \eta) + (\xi_2 : \eta) \leq \sum_j c_j(\zeta_1(g_j) + \delta) + \sum_j c_j(\zeta_2(g_j) + \delta) \leq (1 + 2\delta)\sum_j c_j.$$

We take the infimum of all sums $\sum_j c_j$ such that $\zeta \leq \sum_j c_j \lambda(g_j)\eta$ to find that

$$(\xi_1 : \eta) + (\xi_2 : \eta) \leq (1 + 2\delta)(\zeta : \eta). \tag{1.4}$$

By the definition of ζ and properties (ii) and (iii) we also have

$$(\zeta : \eta) = (\xi_1 + \xi_2 + \delta\xi' : \eta) \leq (\xi_1 + \xi_2 : \eta) + \delta(\xi' : \eta). \tag{1.5}$$

Combining the results of inequalities 1.4 and 1.5 we obtain

$$I_\eta(\xi_1) + I_\eta(\xi_2) \leq (1 + 2\delta)I_\eta(\zeta) \leq (1 + 2\delta)[I_\eta(\xi_1 + \xi_2) + \delta I_\eta(\xi')] \tag{1.6}$$

via the definition of I_η. Under the guidance of Equation (1.1), choose δ small enough so that the inequality

$$2\delta(\xi_1 + \xi_2 : \xi_0) + \delta(1 + 2\delta)(\xi' : \xi_0) < \epsilon,$$

and observe that $\mathfrak{U}$ satisfies the hypothesis of Lemma (1.14).

■

We proceed with the proof of Theorem 1.13.

For each $\xi \in C_c^+(G)$, let X_ξ denote the closed interval $[(\xi_0 : \xi)^{-1}, (\xi : \xi_0)]$. We let X denote the Cartesian product of all such intervals, which is compact by Tychonoff's theorem. By definition, X is the space of all functions $f : C_c^+(G) \to \cup_{\xi \in C_c^+(G)} X_\xi$ such that $f(\xi) \in X_\xi$. By Equation (1.1), we have $I_\eta \in X$ for all $\eta \in C_c^+(G)$.

In general topology, it is known that a topological space Y is compact if and only if for each family of closed sets $\mathscr{C}$ from Y that has the finite intersection property, the intersection $\cap\mathscr{C}$ is nonempty.

For each neighbourhood V of e, let $\Omega(V)$ denote the closure of $\{I_\eta : \operatorname{supp}(\eta) \subset V\}$ in X. These sets enjoy the finite intersection property because $\cap_{j=1}^n \Omega(V_j) \supseteq \Omega(\cap_{j=1}^n V_j)$ and $\cap_{j=1}^n V_j$ is of course a nonempty neighbourhood of e. So there exists an element $I \in X$ than is in every $\Omega(V)$.

Let $B_{\epsilon,\xi}(I)$ denote the open ball of radius ϵ around the ξth coordinate of I and fix $\epsilon > 0$. For any $n \in \mathbb{N}$ choose any $\xi_1, \xi_2, \ldots, \xi_n \in C_c^+(G)$ and let V be a neighbourhood of e. We construct the neighbourhood $U = \prod U_\xi$ of I, where $U_\xi = B_{\epsilon,\xi}(I)$ if $\xi = \xi_j$ for some $1 \leq j \leq n$ and $U_\xi = X_\xi$ otherwise. By the argument above, every neighbourhood of I intersects $\Omega(V)$. So there exists an η with $\operatorname{supp}(\eta) \subset V$ such that $|I(\xi_j) - I_\eta(\xi_j)| \leq \epsilon$ for all $1 \leq j \leq n$. By (i) and (iii), I commutes with left translations and multiplication by positive scalars. If we consider neighbourhoods of the form $V \cap \mathfrak{U}$, then by 1.14 and property (ii), I is also additive.

Any $\xi \in C_c(G, \mathbb{R})$ can be expressed as $\xi = \xi^+ - \xi^-$ with $\xi^+, \xi^- \in C_c^+(G)$. We extend I from $C_c^+(G)$ to $C_c(G, \mathbb{R})$ by defining $I(\xi) := I(\xi^+) + I(\xi^-)$ if $0 \not\equiv \xi \in C_c(G, \mathbb{R})$ and $I(0) := 0$. This extension is positive, linear, and left-invariant. See [**Sal16**, Lemma 8.15] for a proof.

We obtain a regular Borel measure by invoking the Riesz–Markov–Kakutani representation theorem [**Coh13**, Theorem 7.2.8]. We will not discuss the details of this representation theorem, but merely verify that the measure it produces (defined below) is left invariant.

Define a function μ^* on the open subsets of G by

$$\mu^*(U) = \sup\{I(\xi) : \xi \in C_c(G) \text{ and } \xi \prec U\},$$

and then extend it to all $A \in \mathscr{P}(X)$ by the prescription

$$\mu^*(A) = \inf\{\mu^*(U) : U \text{ open and } A \subseteq U\}.$$

Our Haar measure μ_G is the restriction of μ^* to $\mathfrak{B}(G)$. We need only verify that $\mu^*|_{\mathfrak{B}(G)}$ is left-invariant. Let $V \in \mathfrak{B}(G)$ and $g \in G$. Then we have that

$$\begin{aligned}\mu^*(gV) &= \inf\{\mu^*(U) : U \text{ open and } gV \subset U\}\\ &= \inf\{\mu^*(gU) : gU \text{ open and } gV \subset gU\}.\\ &= \inf\{\mu^*(gU) : U \text{ open and } V \subset U\}.\end{aligned}$$

The previous equation encourages us to investigate $\mu^*(gU)$ for open sets $U \in \mathfrak{B}(G)$. We see that

$$\begin{aligned}\mu^*(gU) &= \sup\{I(\xi) : \xi \in C_c(G) \text{ and } \xi \prec gU\}\\ &= \sup\{I(\lambda(g)\xi) : \lambda(g)\xi \in C_c(G) \text{ and } \xi \prec U\}\\ &= \sup\{I(\xi) : \xi \in C_c(G) \text{ and } \xi \prec U\},\end{aligned}$$

since I is left-invariant. Thus we must have that $\mu^*(gV) = \inf\{\mu^*(U) : U \text{ open and } V \subset U\} = \mu^*(V)$, and $\mu^*|_{\mathfrak{B}(G)}$ is left-invariant. □

PROPOSITION 1.15. *Let μ be a Haar measure. Let $V \in \mathfrak{B}(G)$. Then we have $\int_G \lambda(g)\chi_V(s)d\mu(s) = \mu(gV) = \mu(V) = \int_G \chi_V(s)d\mu(s)$, where χ_V is the characteristic function of V. That is*

$$\int_G \lambda(g)\xi(s)\, d\mu(s) = \int_G \xi(s)\, d\mu(s),$$

if ξ is a characteristic function. Consequently, the same result holds for all simple functions by the linearity of the integral. By the monotone convergence theorem, we obtain the result for all Borel functions that are μ-integrable or nonnegative.

PROPOSITION 1.16 ([**Fol16**, 2.19 Proposition]). *Let μ be a Haar measure on a locally compact group G. Then $\mu(U) > 0$ for each nonempty open set U. Furthermore, $\int_G \xi\, d\mu > 0$ for each $\xi \in C_c^+(G)$.*

THEOREM 1.17. *Left Haar measures are unique up to positive scalar multiples.*

PROOF. We expand upon the proof of Folland [**Fol16**, 2.20 Theorem]. Let μ and ν be Haar measures on G. By Proposition 1.16, $\int_G \xi\, d\mu / \int_G \xi\, d\nu > 0$ is well defined for all $\xi \in C_c^+(G)$. Suppose $\int_G \xi\, d\mu / \int_G \xi\, d\nu = c$ for all $\xi \in C_c^+(G)$ and some $c > 0$. That is, $\int_G \xi\, d\mu = c \int_G \xi\, d\nu$ for all $\xi \in C_c^+(G)$. Reasoning similarly as in Proposition 1.15, this holds for all μ-integrable functions, implying that $\mu = c\nu$. The converse is immediate. Therefore, we prove that the ratio $\int_G \xi\, d\mu / \int_G \xi\, d\nu$ is fixed for all $\xi \in C_c^+(G)$.

Let $\xi_1, \xi_2 \in C_c^+(G)$ and by Proposition 1.3 choose a compact and symmetric neighbourhood V_0 of e. Define $V(\xi) := \operatorname{supp}(\xi)V_0 \cup V_0 \operatorname{supp}(\xi)$ for all $\xi \in C_c^+(G)$. Clearly, $V(\xi)$ is compact. Let $g \in V_0$, then by definition, for any ξ, we have that $\xi(sg) - \xi(gs)$ is supported on $V(\xi)$ as a function of s.

Fix $\epsilon > 0$. By [**Fol16**, 2.6 Proposition], there exists a symmetric neighbourhood $W \subset V_0$ of e, such that $|\xi_i(sg) - \xi_i(gs)| < \epsilon$ for all $s \in G$ if $g \in W$, for $i = 1, 2$. Let $\eta \in C_c^+(G)$ such that $\eta(s) = \eta(s^{-1})$ and $\operatorname{supp}(\eta) \subset W$. The existence of such functions is due to Proposition 1.3 and Urysohn's lemma [**Rud87**, 2.12 Urysohn's Lemma]. Choose a compact symmetric neighbourhood of e, say K such that $KK \subset W$. Choose $\zeta \in C_c^+(G)$ such that $\zeta \prec K$. Then set $\eta(s) = \zeta(s)\zeta(s^{-1})$.

Since ν is a Haar measure, we have the following.

$$\begin{aligned}\int_G \eta \, d\mu \int_G \xi_i \, d\nu &= \int_G \int_G \eta(g)\xi_i(s) \, d\nu(s) \, d\mu(g) \\ &= \int_G \int_G \eta(g)\xi_i(gs) \, d\nu(s) \, d\mu(g).\end{aligned}$$

By choice η is an even function and since μ is a Haar measure, similar reasoning then yields

$$\begin{aligned}\int_G \eta \, d\nu \int_G \xi_i \, d\mu &= \int_G \int_G \eta(g^{-1}s)\xi_i(g) \, d\nu(s) \, d\mu(g) \\ &= \int_G \int_G \eta(s^{-1}g)\xi_i(g) \, d\mu(g) \, d\nu(s) \qquad \text{(A)} \\ &= \int_G \int_G \eta(g)\xi_i(sg) \, d\nu(s) \, d\mu(g). \qquad \text{(B)}\end{aligned}$$

In (A) and (B) we have made use of a version of Fubini's theorem applicable to functions belonging to $C_c(G \times G, \mathbb{R})$, see [**Coh13**, Proposition 7.6.4]. Since $\operatorname{supp}(\eta) \subset W$, we have

$$\begin{aligned}\left| \int_G \eta \, d\nu \int_G \xi_i \, d\mu - \int_G \eta \, d\mu \int_G \xi_i \, d\nu \right| &= \left| \int_G \int_G \eta(g) \left[\xi_i(sg) - \xi_i(gs) \right] \, d\nu(s) \, d\mu(g) \right| \\ &\leq \epsilon \, \nu(V(\xi_i)) \int_G \eta \, d\mu,\end{aligned}$$

for $i = 1, 2$. By dividing this inequality with $\int_G \eta \, d\mu \int_G \xi_i \, d\mu$ and invoking the triangle inequality, we obtain

$$\begin{aligned}\left| \frac{\int_G \xi_1 \, d\nu}{\int_G \xi_1 \, d\mu} - \frac{\int_G \xi_2 \, d\nu}{\int_G \xi_2 \, d\mu} \right| &\leq \sum_{i=1}^{2} \left| \frac{\int_G \eta \, d\nu}{\int_G \eta \, d\mu} - \frac{\int_G \xi_i \, d\nu}{\int_G \xi_i \, d\mu} \right| \\ &\leq \epsilon \left(\frac{\nu(V(\xi_1))}{\int_G \xi_1 \, d\mu} + \frac{\nu(V(\xi_2))}{\int_G \xi_2 \, d\mu} \right).\end{aligned}$$

But ϵ does not depend on ξ_1 or ξ_2. That is, the ratio $\int_G \xi \, d\nu / \int_G \xi \, d\mu$ is constant for all $\xi \in C_c^+(G)$, which is what we set out to prove. □

THEOREM 1.18. *For every $1 \leq p < \infty, C_c(G)$ is dense in $L^p(G, \mu)$, the space of p-integrable functions.*

PROOF. Fix $\epsilon > 0$ and let $1 \leq p < \infty$. Let $V \in \mathscr{B}(G)$ such that $\mu(V) < \infty$. By the outer regularity of μ there exists an open set $U \supset V$ such that $\mu(V) - \mu(U) < \frac{1}{2}\epsilon$ and by Lusin's theorem [**Coh13**, Theorem 7.4.4] there exists a compact set $K \subset V$, such that $\mu(K) - \mu(V) < \frac{1}{2}\epsilon$. Of course, we have $\mu(K) - \mu(U) < \epsilon$.

Then $\mu(U \cup K) = \mu(K) + \mu(U \setminus K)$ so that $\mu(U \setminus K) = \mu(K) - \mu(U)$. We invoke Urhysohn's lemma for locally compact Hausdorff spaces and choose $\xi \in C_c^+(G)$ such that $0 \leq \xi \leq 1$ such that $\xi|_K = 1$ and $\operatorname{supp}(\xi) \subset U$. Then

$$\|\chi_V - \xi\|_p^p = \int_{U \setminus K} |\chi_V - \xi|^p \, d\mu < \epsilon.$$

But the integral is a linear functional, so we can approximate simple functions with functions from $C_c(G,\mathbb{R})$. Thus $C_c(G)$ is dense in $L^p(G,\mu)$. $\square$

Remark 1.19. Later we will have use of a similar result for the square Bochner integrable functions (see [**Coh13**, Appendix E]). More specifically, that $C_c(G,\mathfrak{H})$ is dense in $L^2(G,\mathfrak{H})$, where $\mathfrak{H}$ is a Hilbert space. The result can be deduced (similarly as above) from the general form of Lusin's theorem [**LT04**, Theorem 1 (Lusin)] and Bochner integration theory. Alternatively [**VD78**, Appendix A] provides a proof using different means.

We are now in a position to construct, what is to us, an important example of a locally compact group and its corresponding Haar measure.

Example 1.20. We look at some particular results found in Folland [**Fol16**, p. 46]. Let $\{G_\alpha\}_{\alpha\in A}$ be a family of normalized compact groups (i.e., $\mu_\alpha(G_\alpha)=1$, where μ_α denotes the Haar measure of G_α). Let $G=\prod_{\alpha\in A}G_\alpha$. Let $C_F(G)$ be the space of continuous functions on G that depend only on finitely many coordinates. Given $\xi\in C_F(G)$, there are coordinates $\alpha_1,\alpha_2,\dots,\alpha_n\in A$ such that ξ depends only on them. We define

$$I(\xi)=\int_{G_{\alpha_1}}\cdots\int_{G_{\alpha_n}}\xi(x_{\alpha_1},\dots,x_{\alpha_n})d\mu_{\alpha_1}(x_{\alpha_1})\dots d\mu_{\alpha_n}(x_{\alpha_n}).$$

Since I does not depend on the order of the coordinates (we have access to Fubini's theorem) nor the inclusion of additional coordinates on which ξ does not depend, we conclude that I is well defined. By the boundedness of continuous functions on compact sets, Theorem 1.13, Proposition 1.16 and the linearity of integrals, I is a left-invariant, positive linear functional on $C_F(G)$ such that $|I(\xi)|\leq\|\xi\|_{\sup}$. From the Stone-Weierstrass theorem, we determine that $C_F(G)$ is dense in $C(G,\mathbb{R})$. Therefore, I extends uniquely (by continuity) to a left-invariant positive functional on $C(G,\mathbb{R})$. We obtain a Haar measure via the Riesz-Markov-Kakutani representation theorem, as we did in Theorem 1.13.

We now consider the group $\{0,1\}^{\mathbb{Z}}$, where $\{0,1\}$ is to be understood as the integers mod 2. We make use of [**Fol16**, pp. 46-47]. Let the Haar measure ν on $\{0,1\}$ be the fair coin probability measure, i.e. $\nu(\{0\})=\nu(\{1\})=\frac{1}{2}$. Let $\mu_{\mathbb{Z}}$ be the Haar measure and $\mathfrak{B}(\mathbb{Z})$ the Borel σ-algebra associated with $\{0,1\}^{\mathbb{Z}}$. Similarly $\mu_{\mathbb{N}}$ is the Haar measure on $\{0,1\}^{\mathbb{N}}$ and $\mathfrak{B}(\mathbb{N}\times\mathbb{N})$ the Borel σ-algebra of $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$.

The elements of $(\alpha_i)_{i\in\mathbb{Z}}$ of $\{0,1\}^{\mathbb{Z}}$ are sequences of 0's and 1's. The map $(\alpha_i)_{i\in\mathbb{Z}}\mapsto(\alpha_i)_{i<0}\times(\alpha_i)_{i\geq0}$ is a bijection from $\{0,1\}^{\mathbb{Z}}$ to $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$. The map and its inverse are measurable when both groups are considered with their respective Borel σ-algebras. By Theorem 1.17, we may up to some scalar consider $(\{0,1\}^{\mathbb{Z}},\mathfrak{B}(\mathbb{Z}),\mu_{\mathbb{Z}})$ to be measure isomorphic to the space $(\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}},\mathfrak{B}(\mathbb{N}\times\mathbb{N}),\mu_{\mathbb{N}}\times\mu_{\mathbb{N}})$.

For the elements of $\{0,1\}^{\mathbb{Z}}$ we define the map $\Phi:(\alpha_i)_{i\in\mathbb{N}}\mapsto\sum_{i=0}\alpha_i2^{-i-1}$. Then for Φ we have that $\Phi^{-1}(B)$ is Borel whenever B is Borel in $[0,1]$ and $\mu_{\mathbb{N}}(\Phi^{-1}(B))=m(B)$, where m is the Lebesgue measure.

With respect to Φ, there exists a bijection $\tilde{\Phi}$ that is equal to Φ except for countably many points and is an isomorphism between $(\{0,1\}^{\mathbb{N}}, \mathfrak{B}(\mathbb{N}), \mu_{\mathbb{N}})$ and $([0,1], \mathfrak{B}([0,1]), m)$. For a precise development of Φ and $\tilde{\Phi}$ see [**Fre01**, 254K].

We may identify $(\{0,1\}^{\mathbb{Z}}, \mathfrak{B}(\mathbb{Z}), \mu_{\mathbb{Z}})$ with $([0,1]^2, \mathfrak{B}([0,1]^2), m \times m)$.

As previously mentioned, each left Haar measure gives rise to a right Haar measure. It is therefore natural to enquire about how the left and right Haar measures of a given locally compact group relate to one another. The answer is found in what is known as the modular function.

Just as we define the left translates $\lambda(g)\xi$, $(g \in G)$ of functions ξ defined on G in the proof of Theorem 1.13, we define a so called "right translate" of ξ (with domain G) through g to be the function

$$(\rho(g)\xi)(s) = \xi(sg).$$

Definition 1.21 (Modular function). Let G be a locally compact group and μ_G the resulting Haar measure on G. Let $g \in G$ and $E \in \mathfrak{B}(G)$, then Eg is measurable. We define another measure by $\mu_G^g(E) = \mu_G(Eg)$ for all measurable sets E. Since group operations are associative, we find that μ_G^g is again a Haar measure. Since Theorem 1.17 established uniqueness, there exists some number $\Delta_G(g) \in (0, \infty)$ such that $\Delta_G(g)\mu_G^g(E) = \mu_G(E)$.

We define the **modular function** $\Delta_G : G \to \mathbb{R}_+$ by $g \mapsto \Delta_G(g)$. If $\Delta_G \equiv 1$, we say the group G is unimodular. Sometimes we will write Δ for Δ_G when the group in question is understood.

Theorem 1.22. *The function $\Delta : G \to \mathbb{R}_+$ is a continuous homomorphism and for ξ a μ-integrable function, we have*

$$\int_G \rho(g)\xi(s)\, d\mu(s) = \Delta(g^{-1}) \int_G \xi \, d\mu. \tag{1.7}$$

Proof. Let $E \in \mathfrak{B}(G)$. We have $\mu_G^{gk}(E) = \mu_G(Egk) = \Delta(k)\mu_G(Eg) = \Delta(k)\Delta(g)\mu_G(E)$. That is, $\Delta(gk) = \Delta(k)\Delta(g)$. Since $\mathbb{R}_+$ is abelian, we conclude that Δ is a homomorphism.

Note that $\chi_E(gx) = \chi_{Ex^{-1}}(g)$. We obtain

$$\int_G \chi_E(sg)\, d\mu(s) = \mu_G(Eg^{-1}) = \Delta(g^{-1})\mu_G(E).$$

So that Equation (1.7) is satisfied for all characteristic functions. The general case holds by approximation with simple functions.

Finally, we prove continuity. Let $\{g_\alpha\}_{\alpha \in I}$ be a net in G converging to g, which implies that $g_\alpha g^{-1}$ converges to e. Let $\xi \in C_c(G)$, then $\rho(g)\xi \in C_c(G)$. Then

$$\|\rho(g_\alpha)\xi - \rho(g)\xi\|_{\sup} = \|\rho(g_\alpha g^{-1})\rho(g)\xi - \rho(g)\xi\|_{\sup} \to 0$$

as $g_\alpha \to g$, since $\rho(g)\xi$ enjoys a property called right uniform continuity by [**Fol16**, 2.6 Proposition].

As G is locally compact, there exists some compact neighbourhood K of e. There exists some $\alpha_0 \in I$ such that $gg_\alpha^{-1} \in K$ if $\alpha \geq \alpha_0$. Therefore

$$\begin{aligned}\left|\int_G (\rho(g_\alpha)\xi)(s) - (\rho(g)\xi)(s)\, d\mu(s)\right| &\leq \int_G \left|(\rho(g_\alpha g^{-1})\rho(g)\xi)(s) - (\rho(g)\xi)(s)\right| \, d\mu(s) \\ &\leq \mu\left(\operatorname{supp}(\xi) g^{-1} g g_\alpha^{-1}) \cup \operatorname{supp}(\xi)\right) \|\rho(g_\alpha)\xi - \rho(g)\xi\|_{\sup} \\ &< \epsilon\end{aligned}$$

for sufficient choices of α. Hence $g \mapsto \int_G \rho(g)\xi \, d\mu$ is continuous for all $\xi \in C_c(G)$, and therefore we must have that Δ is continuous due to Equation (1.7), by reasoning similarly to that of Theorem 1.17. □

Definition 1.23 (Pontryagin dual of a locally compact group). Let G be a locally compact abelian group. We call a continuous homomorphism from G into the circle group $\mathbb{T}$ a **character**. We let $\widehat{G}$ denote the set of all characters of G. The set $\widehat{G}$ is called the **Pontryagin dual** of G. Sometimes we will use the bracket notation of characters, i.e. $\langle g, \xi\rangle := \xi(g)$, where $\xi \in \widehat{G}$.

Proposition 1.24 (Pontryagin Duality Theorem, [**Fol16**, 4.32 Theorem]). *Let $\widehat{G}$ be as above. Then $\widehat{G}$ is a locally compact abelian group if it is endowed with the topology of compact convergence on G and the group operation is given by pointwise multiplication.*

The map $\Phi : G \to \widehat{\widehat{G}}$ defined by $\langle \xi, \Phi(g)\rangle = \langle g, \xi\rangle$, is an isomorphism of topological groups.

Definition 1.25 (The Fourier transform, [**Fol16**, p. 101]). We define a map $\mathscr{F} : L^1(G) \to C(\widehat{G})$ by

$$\mathscr{F}\xi(f) = \hat{\xi}(f) = \int \overline{\langle s, f\rangle} \xi(s) \, d\mu_G(s)$$

This map is the Fourier transform on G. (We denote the Fourier transform as an operator by $\mathscr{F}$, but we usually denote the Fourier transform of $\xi \in L^1(G)$ by $\hat{\xi}$ rather than $\mathscr{F}\xi$.)

Proposition 1.26 (The Plancherel theorem [**GL20**, Definition 6.45]). *The Fourier transform $\mathscr{F}$ preserves the L^2-norm on $L^1(G) \cap L^2(G)$ and uniquely extends to a unitary from $L^2(G)$ to itself. For every $f \in L^1(\widehat{G}) \cap L^2(\widehat{G})$, the map $\mathscr{F}^* = \mathscr{F}^{-1}$ agrees with the inverse Fourier transform defined by $\mathscr{F} f(g) = \int_{\widehat{G}} \langle g, \gamma\rangle f(\gamma)\, d\gamma$, where $f \in L^1(\widehat{G})$.*

We will later have use of the following invaluable existence theorem (see Chapter 4).

Theorem 1.27 ([**Sut78**, Proposition 2.1]). *Suppose G is a separable, locally compact group such that $\Delta_G(G) = \mathbb{R}_+$. Then there is a continuous one parameter subgroup $L = \{g_t : t \in \mathbb{R}\}$ of G with $\Delta_G(g_t) = e^t$.*

Proof. We expand upon the proof given by Sutherland. Suppose first that G is connected. Then by a theorem of Iwasawa [**MZ55**, §4.13, p.188-189] there are continuous one parameter subgroups $V_1, \cdots, V_r$ of G, and a maximal compact subgroup K of G such that $G = KV_1V_2, \cdots, V_r$.

Clearly $\Delta_G(K) = \{1\}$. Thus, there is a one parameter subgroup, which we may take as V_1, such that $\Delta_G(V_1) \neq \{1\}$. Since V_1 is connected, we will have $\Delta_G(V_1) = \mathbb{R}_+$. Write $V_1 = \{h_t : t \in \mathbb{R}\}$; since the map $t \in \mathbb{R} \to \Delta_G(h_t) \in \mathbb{R}_+$ is continuous, onto and $\Delta_G(h_{t+p}) = \Delta_G(h_t)\Delta_G(h_p)$, we have the classical result: $\Delta_G(h_t) = e^{at}$ for some $a \in \mathbb{R}$. But then if $g_t = h_{a^{-1}1_t}, \Delta_G(g_t) = e^t$ and $L = \{g_r : r \in \mathbb{R}\}$ is the desired subgroup.

If G is not connected, let G_0 denote the connected component of the identity of G. Since G_0 is a connected component, it is closed. Clearly $G \to G : x \mapsto pxq^{-1}$ is continuous for any $p, q \in G$. Hence pG_0p^{-1} and qG_0^{-1} are both connected and intersect G_0 for all $p \in G$ and $q \in G_0$. Therefore $pG_0p^{-1} \subseteq G_0$ for all $p \in G$ and $qG_0p^{-1} \subseteq G_0$ for all $q \in G_0$, i.e., G_0 is a normal subgroup of G. We note that G/G_0 is a locally compact group [**Fol16**, Proposition 2.2] and totally disconnected.

Now either $\Delta_G(G_0) = \{1\}$, or $\Delta_G(G_0) = \mathbb{R}_+$. In the latter case the result follows by replacing "G" with "G_0" in the argument above and following it verbatim. If on the other hand $\Delta_G(G_0) = \{1\}$; Δ_G induces a continuous homomorphism of G/G_0 onto $\mathbb{R}_+$ with $\Delta_G(xG_0) := \Delta_G(x)$. As G/G_0 is totally disconnected, it has a basis consisting of compact open subgroups [**Rud90**, Lemma 2.4.4]. Thus the prescribed homomorphism is locally constant on G/G_0. That is to say Δ_G is locally constant on G as G/G_0 is endowed with its quotient topology. But then $\Delta_G^{-1}(\{r\})$ is open for all $r \in \mathbb{R}_+$. Let $\mathscr{G} = \{\Delta_G^{-1}(\{r\}) : r \in \mathbb{R}_+\}$. It is easily checked that $\mathbb{R}_+ \to \mathscr{G} : r \mapsto \Delta_G^{-1}(\{r\})$ is a bijection. Since G is seperable, we have some $\{e_n\}_{n\in\mathbb{N}}$ dense in G. For any $s \in \mathscr{G}$, there exists a sequence e_j contained in $\{e_n\}_{n\in\mathbb{N}}$ and $N \in \mathbb{N}$ such that $e_j \in s$ for all $j \geq N$. But then $e_j \notin G \setminus s$ for all $j \geq N$. By the axiom of choice, we find $|\mathbb{R}_+| = |\mathscr{G}| \leq |\mathbb{N}|$. This is a contradiction; thus, the sought-after subgroup can be found within G_0. ☐

1.2. Semidirect products

We develop the theory of semidirect products based on the work of E. Kaniuth and K.F. Taylor in [**KT13**]. Let G be a locally compact group. Let $\mathrm{Aut}(G)$ denote the group automorphisms of G; i.e., if $\pi : G \to G$ is a homeomorphism and a group isomorphism, then $\pi \in \mathrm{Aut}(G)$.

Definition 1.28 (Continuous group action). Let G and H be locally compact groups. Equip $G \times H$ with the product topology. A homomorphism $\alpha : h \mapsto \alpha_h$ from H to $\mathrm{Aut}(G)$ such that $(g, h) \mapsto \alpha_h(g)$ is continuous from $G \times H$ to G, is called a **continuous group action** from H on G.

Remark 1.29. The definition above is a case-specific formulation of the much more general definition of a (left) action $\alpha : G \times X \to X$, of G on X where X is a set.

Later, we will encounter another important case-specific definition.

Let α be a continuous action from H on G. We define the binary operation

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1\alpha_{h_1}(g_2), h_1h_2) \tag{A}$$

on $G \times H$, which we will call “multiplication”.

We investigate this operation: Firstly,

$$\begin{aligned}
[(g_1, h_1)(g_2, h_2)](g_3, h_3) &= (g_1 \alpha_{h_1}(g_2)\alpha_{h_1 h_2}(g_3), h_1 h_2 h_3) \\
&= (g_1 \alpha_{h_1}(g_2 \alpha_{h_2}(g_3)), h_1 h_2 h_3) \\
&= (g_1, h_1)(g_2 \alpha_{h_2}(g_3), h_2 h_3) \\
&= (g_1, h_1)[(g_2, h_2)(g_3, h_3)].
\end{aligned}$$

That is, multiplication satisfies associativity. Secondly, let e_G and e_H be the identities of G and H respectively, then

$$\begin{aligned}
(e_G, e_H)(g, h)(e_G, e_H) &= (e_G \alpha_{e_H}(g)\alpha_{e_H h}(e_G), e_H h e_H) \\
&= (\alpha_{e_H}(g)\alpha_h(e_G), h) \\
&= (g, h).
\end{aligned}$$

So, there exists an identity element in $G \times H$ with respect to multiplication. Lastly, let $(g, h) \in G \times H$, we find that

$$\begin{aligned}
(\alpha_{h^{-1}}(g^{-1}), h^{-1})(g, h) &= ((\alpha_{h^{-1}}(g^{-1})\alpha_{h^{-1}}(g), h^{-1}h) \\
&= (e_G, e_H) \\
&= (g\alpha_h(\alpha_{h^{-1}}(g^{-1})), hh^{-1}) \\
&= (g, h)(\alpha_{h^{-1}}(g^{-1}), h^{-1}),
\end{aligned}$$

so that we see every element in $G \times H$ has an inverse with respect to multiplication and $G \times H$ together with multiplication is a group.

Definition 1.30 (Semidirect product). Given a triple (G, H, α), where G and H are locally compact groups, and α continuous action such that H acts on G through α, we define the **semidirect product** of (G, H, α), denoted $G \ltimes_\alpha H$, as the group engendered by $G \times H$ and the operation defined in Equation (A).

If the action is clear from the context, we simply write $G \ltimes H$.

Remark 1.31. The definition above makes sense even when the group action is not continuous or the groups are not subject to any topological requirements. We are not interested in such semidirect products; all of our semidirect products are assumed to satisfy the topological requirements of Definition 1.30.

Theorem 1.32. *Semidirect products are locally compact groups when they are endowed with their product topology.*

Proof. Suppose that the nets $(g_\lambda, h_\lambda)_\lambda$ and $(x_\gamma, y_\gamma)_\gamma$ of $N \ltimes H$ converge to (g, h) and (x, y) in $G \ltimes H$ respectively - which is of course true iff we have joint convergence. By definition $(g_\lambda, h_\lambda)(x_\gamma, y_\gamma) = (g_\lambda \alpha_{h_\lambda}(x_\gamma), h_\lambda y_\gamma)$. But then $(g_\lambda \alpha_{h_\lambda}(x_\gamma), h_\lambda y_\gamma)_\lambda \to (g\alpha_h(x), hy)$ since $(g, h) \mapsto \alpha_h(n)$ is continuous from $G \times H$ to G and using Proposition 1.3. Similarly $(\alpha_{h_\lambda^{-1}}(g_\lambda^{-1}), h_\lambda^{-1}) \to$

$(\alpha_{h^{-1}}(g^{-1}), h^{-1})$. We conclude that $G \ltimes H$ is a topological group, as both multiplication and inversion are continuous.

That the product topology of $G \ltimes H$ is locally compact (Hausdorff) follows from the definition of the product topology and the fact that G and H have locally compact topologies. $\square$

EXAMPLE 1.33. Let G be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Then $G \ltimes_s \mathbb{Z}$ is well defined if we let $s_n = \alpha^n$ for $n \in \mathbb{Z}$.

THEOREM 1.34. *Fix $h \in H$. Given a Haar measure μ_G of G we consider the map*

$$\mu_G^h : \mathfrak{B}(G) \to [0, \infty] := E \mapsto \mu_G(\alpha_h(E)).$$

Then μ_G^h is a left Haar measure on G.

PROOF. Fix $h \in H$. Set $\mathfrak{B}_h = \{\alpha_h(A) : a \in \mathfrak{B}(G)\}$. It is an exercise to see that $\mathfrak{B}_h$ is a σ-algebra. If U is open in G, then $\alpha_h^{-1}(U)$ is open. But then $U = \alpha_h\left(\alpha_h^{-1}(U)\right) \in \mathfrak{B}_h$. That is to say $\mathfrak{B}(G) \subseteq \mathfrak{B}_h$. Vice versa $\mathfrak{B}_h \subseteq \mathfrak{B}(G)$.

We have validated that μ_G^h does indeed have $\mathfrak{B}(G)$ as a domain. Next we verify that μ_G^h is a measure: Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of disjoint sets in $\mathfrak{B}(G)$, then

$$\begin{aligned}
\mu_N^h\Big(\bigsqcup_i^{\infty} A_i\Big) &= \mu_N\Big(\alpha_h\Big(\bigsqcup_i^{\infty} A_i\Big)\Big) \\
&= \sum_i^{\infty} \mu_N(\alpha_h(A_i)), \qquad \text{since } \alpha_h \text{ is bijective and } \mu_G \text{ is a measure,} \\
&= \sum_i^{\infty} \mu_N^h(\alpha_h(A_i)).
\end{aligned}$$

So μ_G^h is countably additive. Furthermore, $\mu_G^h(\emptyset) = \mu_G(\alpha_h(\emptyset)) = \mu_G(\emptyset) = 0$. Since μ_G is a regular Borel measure, we deduce that μ_G^h is a regular Borel measure.

Let $g \in G$ and $A \in \mathfrak{B}(G)$. Then $\mu_G(\alpha_h(gA)) = \mu_G(\alpha_h(g)\alpha_h(A)) = \mu_G(\alpha_h(A))$. So, μ_G^h defines a Haar measure on G. $\square$

DEFINITION 1.35 (Modulus). Let $h \in H$. As μ_G^h is a Haar measure, by the uniqueness of Haar measures, there exists a $\delta_\alpha(h) > 0$ such that $\mu_N = \delta_\alpha(h)\mu_N^h$. We define $\delta_\alpha : H \to \mathbb{R}^+$ by $h \mapsto \delta_\alpha(h)$. We call δ_α the **modulus** of the action α.

When the action is clear, we write δ for δ_α.

THEOREM 1.36. *$\delta : H \to \mathbb{R}^+$ is a continuous homomorphism and*

$$\int_G \xi \, d\mu = \delta(h) \int_G \xi\left(\alpha_h(s)\right) \, d\mu(s) \tag{1.8}$$

for any μ-integrable function ξ on G.

PROOF. Follows mutatis mutandis from the proof of Theorem 1.22. Note that [**Fol16**, 2.6 Proposition] (a key result used to prove Theorem 1.22) also requires modification. $\square$

THEOREM 1.37. *The left Haar integral of any non-negative measurable function* ξ *on* $G \ltimes H$ *is given by*

$$\int_{G\ltimes H} \xi(g,h)\, d(g,h) = \int_H \int_G \xi(g,h)\delta(h)^{-1}\, dg\, dh. \tag{1.9}$$

The modular function of $G \ltimes H$ *is given by*

$$\Delta_{G\ltimes H}(s,t) = \Delta_G(s)\Delta_H(t)\delta(t)^{-1}. \tag{1.10}$$

PROOF. Careful perusal of [**Coh13**, §7.6] reveals what we have access to Fubini's theorem [**Coh13**, Proposition 7.6.4]. In accordance with [**Fol16**, 2.9 Proposition], we verify the left invariance of Equation (1.9). Let $(s,t) \in G \ltimes H$, then

$$\begin{aligned}
\int_{G\ltimes H} \xi((s,t)(g,h))\, d(g,h) &= \int_H \int_G \xi(s\alpha_t(g), th)\delta(h)^{-1}\, dg\, dh \\
&= \int_H \int_G \xi(\alpha_{th}(g), th)\, dg\, dh \\
&= \int_G \int_H \xi(\alpha_h(g), h)\, dh\, dg \\
&= \int_H \int_G \xi(g,h)\delta(h)^{-1}\, dg\, dh.
\end{aligned}$$

This shows that Equation (1.9) defines a Haar measure of $G \ltimes H$. Lastly, we determine the modular function of $G \ltimes H$ via direct computation:

$$\begin{aligned}
\int_{G\ltimes H} \xi((g,h)(s,t)) d(g,h) &= \int_H \int_G \xi(g\alpha_h(s), ht)\delta(h)^{-1}\, dg\, dh \\
&= \int_H \int_G \xi(\alpha_h(\alpha_{h^{-1}}(g)s), ht)\delta(h)^{-1}\, dg\, dh \\
&= \int_H \int_G \xi(\alpha_h(gs), ht)\, dg\, dh \\
&= \int_H \int_G \xi(\alpha_{ht^{-1}}(g), h)\Delta_H(t^{-1})\Delta_G(s^{-1})\, dg\, dh \\
&= \delta(t)\, (\Delta_G(s)\Delta_H(t))^{-1} \int_{G\ltimes H} \xi(g,h)\, d(g,h),
\end{aligned}$$

which verifies Equation (1.10). □

CHAPTER 2

Von Neumann algebras

In this chapter we establish some introductory theory of von Neumann algebras. This is done to define and describe the structure of which a group von Neumann algebra is but a specific example. Direct integrals and central decompositions of von Neumann algebras are also briefly introduced, so that they do not appear alien when encountered in Chapters 4 and 5. The greater part of this chapter is concerned with crossed products and aims to establish Corollary 2.49, which highlights the relationship between semidirect products and crossed products, and is key to constructing examples in Chapter 6. The author has taken special care to integrate the facts that lead to Corollary 2.49 with the rest of the chapter, as they appear to be somewhat scattered in the literature. This was possible due to [**Rou81**].

2.1. General von Neumann algebra theory

We give a brief overview of the theory and definitions needed from von Neumann algebra theory.

Definition 2.1 (Banach and C^*-algebras). A **Banach algebra** is a vector space A together with an associative binary operation $(a,b) \mapsto ab$ called multiplication, and a complete norm $\|\cdot\|$ that is submultiplicative, that is, $\|ab\| \leq \|a\|\|b\|$. If A is also equipped with an involution $a \mapsto a^*$, i.e.

(i) $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$, (Conjugate-linear)
(ii) $a^{**} = a$ (Period two)
(iii) $(ab)^* = b^* a^*$ (Anti-isomorphism)

then A is called an **involutive Banach algebra**. If A additionally satisfies

$$\|a^* a\| = \|a\|^2 \qquad (C^* \text{ property})$$

then we call A a **C^*-algebra**. Lastly, we call a C^*-algebra a **unital C^*-algebra** if it has a multiplicative unit $\mathbb{1}$ such that $\|\mathbb{1}\| = 1$.

Example 2.2. Let $\mathfrak{H}$ be a Hilbert (over $\mathbb{C}$) space and $\mathcal{B}(\mathfrak{H})$ be the set of bounded linear operators on $\mathfrak{H}$. Then $\mathcal{B}(\mathfrak{H})$ is a C^*-algebra.

Let $x, y \in \mathcal{B}(\mathfrak{H})$. It is well known that the operations of pointwise addition and scalar multiplication turn $\mathcal{B}(\mathfrak{H})$ into a Banach space under the operator norm. We note that the map defined by composition, i.e. $(x,y) \mapsto xy := x \circ y$, is an associative bilinear map that preserves continuity (hence boundedness of operators). Clearly $\|xy\| \leq \|x\|\|y\|$. As such, composition will serve as vector multiplication.

We verify that the map sending an operator to its Hilbert adjoint can serve as involution. From standard Hilbert space theory we know that $\|x^*\| = \|x\|$. It is therefore immediate that $\|x^*x\| \leq \|x\|^2$. We check the reverse inequality.

For any bounded linear operator x, it's norm is equal to $\sup\{|\langle x\xi, \eta\rangle| : \xi, \eta \in \mathfrak{H}, \|\xi\|, \|\eta\| \leq 1\}$. Consequently, we find that:

$$\begin{aligned} \|x\|^2 &= \sup_{\xi \in \mathfrak{H}, \|\xi\| \leq 1} \langle x\xi, x\xi\rangle \\ &\leq \sup_{\xi, \eta \in \mathfrak{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, x\eta\rangle| \\ &= \sup_{\xi, \eta \in \mathfrak{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x^*x\xi, \eta\rangle| = \|x^*x\| . \end{aligned}$$

We will not investigate von Neumann algebras in the more abstract sense as structures called W^*-algebras, but as operator algebras. To do so, we investigate the relevant operator topologies.

DEFINITION 2.3 (Seminorms). Let X be a vector space over a field $\mathbb{K}$ (the complex plane or the real line). A **seminorm** is a function from $\varrho : X \to \mathbb{R}$ that satisfies

(i) $\varrho(v + x) \leq \varrho(v) + \varrho(x)$ (Subadditivity)
(ii) $\varrho(sx) = |s|\varrho(x)$ for all $s \in \mathbb{K}$ (Absolute homogeneity)

for all $x, v \in X$.

DEFINITION 2.4 (Seminorm topologies). Let X be a topological vector space over a field $\mathbb{K}$ and $\mathcal{P}$ a family of seminorms on X. We define a subbase of X by letting it consist of subsets $V \subset X$, such that

$$V = \left\{ v \in X : \sup_{1 \leq i \leq n} \varrho_i(v - x) \leq \epsilon \right\}$$

with $x \in X$, $\{\varrho\}_{i=1}^n$ a n-tuple from $\mathcal{P}$ and $\epsilon > 0$. The topology on X, generated by the aforementioned subbase of X, is called the seminorm topology of X corresponding to $\mathcal{P}$.

It is known that every topology arising from Definition 2.4 is also a locally convex topology and that every locally convex topology is generated by a corresponding family of seminorms.

We discuss such topologies with respect to $\mathcal{B}(\mathfrak{H})$, the bounded operators on a Hilbert space $\mathfrak{H}$.

DEFINITION 2.5 (Strong operator topology). The locally convex topology corresponding to the family of seminorms on $\mathcal{B}(\mathfrak{H})$ given by

$$\{x \mapsto \|x\xi\| : \xi \in \mathfrak{H}\}$$

is called the **Strong Operator Topology** often abbreviated as **SOT**.

DEFINITION 2.6 (Weak operator topology). The topology corresponding to the family of seminorms on $\mathcal{B}(\mathfrak{H})$ given by

$$\{x \mapsto |\langle x\xi, \eta\rangle| : \xi, \eta \in \mathfrak{H}\}$$

is called the **weak operator topology** often abbreviated as **WOT**.

PROPOSITION 2.7. *From coarsest to finest we have, WOT $\prec$ SOT $\prec$ uniform.*

We are ready to define von Neumann algebras and summarise the relevant theory.

DEFINITION 2.8 (von Neumann algebras). A **von Neumann algebra** (over a Hilbert space $\mathfrak{H}$) is a *-subalgebra of $\mathcal{B}(\mathfrak{H})$ that contains $\mathbb{1}$ and is closed under the weak operator topology.

It is well known that the strong and weak operator closures of convex sets agree and that von Neumann algebras are convex sets. Hence we may also consider von Neumann algebras as strong operator closures of *-subalgebras.

EXAMPLE 2.9. Let G be a locally compact group. Then $L^\infty(G)$ acting as the algebra of left multiplication is a von Neumann algebra over $L^2(G)$. That is, for each $\eta \in L^\infty$ define m_η by $[m_\eta(\xi)](s) = \eta(s)\xi(s)$ for all $\xi \in L^2(G)$. These multiplication operators then constitute a von Neumann algebra. We will write η for m_η.

The following is is our most important example of a von Neumann algebra:

EXAMPLE 2.10 (**Group von Neumann algebras**). Let G be a locally compact group. We will show in Theorem 2.34 that the left translates of functions $\xi \in L^2(G)$ (as defined in Theorem 1.13) through elements $g \in G$, defines a unitary representation of G. That is, we may talk of the unitary operators $\lambda(g)$ $(g \in G)$ defined by

$$\lambda(g)\xi(s) = \xi(g^{-1}s) \text{ for all } \xi \in L^2(G).$$

The von Neumann algebra generated by $\{\lambda(g) : g \in G\}$ is called the **(left) group von Neumann algebra** of G and is denoted by $\mathrm{VN}_l(G)$, i.e. $\mathrm{VN}_l(G) := \lambda(G)''$.

Clearly we can also define a right group von Neumann algebra by substituting the left translates for right translates in the argument above. More on this following Theorem 2.34.

DEFINITION 2.11 (Factors and the commutant of a family of operators). Let A be a set of bounded operators on a Hilbert space $\mathfrak{H}$. Then A' denotes the set of all bounded operators on $\mathfrak{H}$ that commutes with every operator in A. We call A' the **commutant** of A.

Let $\mathcal{M}$ be a von Neumann algebra. We denote by $\mathcal{Z}(\mathcal{M})$ the subspace of $\mathcal{M}$ equal to $\mathcal{M} \cap \mathcal{M}'$. We call $\mathcal{Z}(\mathcal{M})$ the **centre** of $\mathcal{M}$. If a von Neumann algebra has a centre consisting only of scalar multiples of $\mathbb{1}$, we call $\mathcal{M}$ a **factor**.

Proposition 2.12 (Bicommutant theorem, [**KR83**, 5.3.1. Theorem]). *Originally due to John von Neumann, this theorem, also called the double commutant theorem, asserts that: A self-adjoint algebra* $\mathcal{M} \subseteq \mathcal{B}(\mathfrak{H})$ *that contains the identity operator is a von Neumann algebra iff* $\mathcal{M} = \mathcal{M}'' := (\mathcal{M}')'$.

Definition 2.13 (Positive cone of a von Neumann algebra). Let $\mathcal{M}$ be a von Neumann algebra and $x \in \mathcal{M}$ be self-adjoint, that is, $x = x^*$. If there exists some $y \in \mathcal{M}$ such that $x = y^*y$, then we call x positive. The set $\mathcal{M}_+$ of all positive elements of $\mathcal{M}$ is called the **positive cone** of $\mathcal{M}$, as it satisfies (the properties of a cone [**KR83**, p. 212]):

(i) If both x and $-x$ belong to $\mathcal{M}_+$ then $x = 0$.
(ii) If $c \in \mathbb{R}_+$ and $x \in \mathcal{M}_+$ then $cx \in \mathcal{M}_+$.
(iii) If $x, y \in \mathcal{M}_+$ then $x + y \in \mathcal{M}_+$.

Definition 2.14 (Weights and traces). A **weight** on a von Neumann algebra $\mathcal{M}$ is a functional $\phi : \mathcal{M}_+ \to [0, \infty]$ satisfying the following properties:

(i) $\phi(x + y) = \phi(x) + \phi(y)$. (Additivity)
(ii) $\phi(cx) = c\phi(x)$, $c \geq 0$. (Positive homogeneity)

A weight may also be:

(a) Semifinite: if $\operatorname{span}\{a \in \mathcal{M}_+ : \phi(a) < \infty\}$ is weak* dense in $\mathcal{M}$.
(b) Faithful: if $\phi(x) = 0$ iff $x = 0$.
(c) Normal: if $\phi(\sup x_\gamma) = \sup \phi(x_\gamma)$ for every bounded increasing net $\{x_\gamma\} \subseteq \mathcal{M}_+$.

If additionally we have that $\phi(x^*x) = \phi(xx^*)$ for all $x \in \mathcal{M}$, we call ϕ a **trace**.

It is well known that every von Neumann algebra admits a faithful, normal, semifinite weight; see [**Tak03a**, VII, Theorem 2.7].

Definition 2.15 (Murray–von Neumann equivalence). Let $\mathcal{P}(\mathcal{M})$ denote the set of projections of the von Neumann algebra $\mathcal{M}$, i.e. $p \in \mathcal{P}(\mathcal{M})$ if and only if $p = p^2 = p^*$. Given $p, q \in \mathcal{M}$, if there exists a partial isometry v such that $v^*v = p$ and $vv^* = q$ we say p and q are **(Murray–von Neumann) equivalent** and we write $p \sim q$.

Proposition 2.16 (Compression of von Neumann algebras, [**KR83**, 5.5.7. Corollary]). *Let* $\mathcal{M}$ *be a von Neumann algebra over* $\mathfrak{H}$. *Let* $p \in \mathcal{M}$ *be a projection, then* $p\mathcal{M}p$ *is a von Neumann algebra over* $p(\mathfrak{H}) := \{p\xi : \xi \in \mathfrak{H}\}$. *Moreover, the commutant of* $p\mathcal{M}p$ *in* $\mathcal{B}(p(\mathfrak{H}))$ *is equal to* $\mathcal{M}'p$.

Definition 2.17 ([**Pet13**, pp. 44-45]). Given a von Neumann algebra $\mathcal{M}$, we call a projection p:

(i) **minimal** if $p \neq 0$, and the only subprojections are 0 and p.

(ii) **abelian** if $p\mathcal{M}p$ is abelian.
(iii) **finite** if $q \leq p$ and $q \sim p$, imply that $q = p$.
(iv) **semifinite** if there are pairwise orthogonal finite projections $p_\alpha \in \mathcal{P}(\mathcal{M})$ such that $p = \sum_\alpha p_\alpha$.
(v) **purely infinite** if $p \neq 0$ and there is no nonzero finite projection $q \leq p$.
(vi) **properly infinite** if $p \neq 0$ and zp is not-finite for any nonzero central projection $z \in \mathcal{P}(\mathcal{M})$.

Remark 2.18. We call a $\mathcal{M}$ finite, semifinite, purely infinite, or properly infinite if $\mathbb{1}$ has the corresponding property as a projection of $\mathcal{M}$.

We consider the different types of von Neumann algebras.

Definition 2.19 (Types of von Neumann algebras, [**Pet13**, p. 91]). A von Neumann algebra $\mathcal{M}$ is said to be of type

(i) I if every nonzero projection has a nonzero abelian subprojection.
(ii) II if it is semifinite and has no nonzero abelian projections.
    (a) if $\mathcal{M}$ is also finite then $\mathcal{M}$ is of type II_1.
    (b) if $\mathcal{M}$ is properly infinite then $\mathcal{M}$ is of type II_∞.
(iii) III if it has no nonzero finite projections.

Proposition 2.20 (Type decomposition of von Neumann algebras, [**Pet13**, Theorem 5.3.1]). *Let $\mathcal{M}$ be a von Neumann algebra. Then there exist unique central projections $p_I, p_{II_1}, p_{II_\infty}$, and p_{III} of $\mathcal{M}$ such that $\mathcal{M}p_I, \mathcal{M}p_{II_1}$, $\mathcal{M}p_{II_\infty}$, and $\mathcal{M}p_{III}$ are of type I, II_1, II_∞, and III respectively, and such that $p_I + p_{II_1} + p_{II_\infty} + p_{III} = \mathbb{1}$.*

2.2. Direct integral theory

We briefly introduce the definitions and theorems of direct integral theory, as done by Blackadar [**Bla06**, III.1.6] and Folland [**Fol16**, pp. 238-240], which will be required to develop Sutherland's structure theorems. For a more in depth probe of the applicable theory, the reader may consult Takesaki [**Tak03a**, VI, §3].

Definition 2.21 (Direct integral of Hilbert spaces). Let (X, μ) be a standard measure space. For almost all $x \in X$, let $\mathfrak{H}_x$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_x$. A vector subspace $\Gamma \subseteq \prod_x \mathfrak{H}_x$ such that:

(i) Γ closed under multiplication by $L^\infty(X, \mu)$,
(ii) $x \mapsto \langle \xi(x), \eta(x) \rangle_x$ is measurable for all $\xi, \eta \in \Gamma$,
(iii) for any $\xi \in \Gamma$ we have that

$$\int_X \langle \xi(x), \xi(x) \rangle_x \, d\mu(x) < \infty;$$

which is generated as an $L^\infty(X,\mu)$-module by some $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ such that $\overline{\operatorname{span}}\{\xi_n(x)\} = \mathfrak{H}_x$ for almost all x, together with $\prod_x \mathfrak{H}_x$ is called a **measurable field of Hilbert spaces**. We call a map η on X such that $\eta(x) \in \mathfrak{H}_x$ for all $x \in X$ a **vector field**.

Note that (iii) implies that

$$\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle_x \, d\mu(x)$$

is a pre-inner product on Γ, and an inner product on the space of vector fields from Γ agreeing a.e. Given a measurable field of Hilbert spaces $(\{\mathfrak{H}_x\}, \{\xi_n\} \subseteq \Gamma)$ on X, a vector field η on X will be called a **measurable vector field** if $\langle \eta(x), \xi_n(x) \rangle_x$ is a measurable function on X for each x.

The completion of Γ (with vector fields agreeing a.e. identified) is a separable Hilbert space $\mathfrak{H}$ which can be identified with a space of equivalence classes of measurable vector fields of the field $(\mathfrak{H}_x)$; we write

$$\mathfrak{H} = \int_X^\oplus \mathfrak{H}_x \, d\mu(x)$$

and call $\mathfrak{H}$ the **direct integral of the Hilbert spaces** $\mathfrak{H}_x$.

Definition 2.22 (Operators of direct integral Hilbert spaces, [**Bla06**, III.1.6.2]). If $T_x \in \mathcal{B}(\mathfrak{H}_x)$, then (T_x) is a **measurable field of bounded operators** if $(T_x\xi(x))$ is a measurable vector field for each measurable vector field ξ.

Suppose $\|T_x\|$ is uniformly bounded by $C > 0$, then $\|T_x\xi(x)\|_x \leq C\|\xi(x)\|_x$. Thus (T_x) defines an operator $T \in \mathcal{B}(\mathfrak{H})$ and $\|T\|$ is the essential supremum of $\|T_x\|$. We call such a T **decomposable** and we write

$$T = \int_X^\oplus T_x \, d\mu(x).$$

Furthermore, $L^\infty(X,\mu)$ acts through decomposable operators by the prescription:

$$f \mapsto \int_X^\oplus f(x) I_x \, d\mu(x). \tag{2.1}$$

The image of $L^\infty(X,\mu)$ under (2.1) is called the **algebra of diagonalizable operators** of the field.

Proposition 2.23 (Central decomposition of a von Neumann algebra [**Bla06**, III.1.6.3-III.1.6.4]). *Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $\mathfrak{H}$. Then there exists a standard measure space (X,μ), a measurable field of Hilbert spaces $(\mathfrak{H}_x)$ over (X,μ), and a unitary*

$$U : \mathfrak{H} \to \int_X^\oplus \mathfrak{H}_x \, d\mu(x)$$

carrying $\mathcal{Z}(\mathcal{M})$ onto the set of diagonalizable operators. Furthermore,

$$UTU^* = \int_X^\oplus T_x \, d\mu(x)$$

is measurable for each $T \in \mathcal{Z}(\mathcal{M})' \supseteq \mathcal{M}$. For almost every $x \in X$, let $\mathcal{M}_x$ be the von Neumann subalgebra of $\mathcal{B}(\mathfrak{H}_x)$ generated by $\{T_x : T \in \mathcal{M}\}$, then $\mathcal{M}_x$ is a factor and a measurable operator

$$S = \int_X^\oplus S_x \, d\mu(x)$$

satisfies $U^*SU \in \mathcal{M}$ *if and only if* $S_x \in \mathcal{M}_x$ *for almost all* x. *We write*

$$\mathcal{M} = \int_X^{\oplus} \mathcal{M}_x \, d\mu(x).$$

2.3. Crossed products

Understanding crossed products and how they relate to semidirect products will be important in proving Sutherland's structure theorems and in constructing examples of factors of type III, prompting further investigation. This section is primarily a summary of A. van Daele [**VD78**], S. Goldstein and L.E. Labuschagne [**GL20**].

We once again require a case-specific definition of a group action, as in Definition 1.28.

Definition 2.24 (Group acting on a von Neumann algebra). Let G be a locally compact group and $\mathcal{M}$ a von Neumann algebra over a Hilbert space $\mathfrak{H}$.

A homomorphism α from G into the group of *-automorphisms of $\mathcal{M}$ is called a **continuous action** of G on $\mathcal{M}$ if for each $x \in \mathcal{M}$, $s \mapsto \alpha_s(x)$ from G into $\mathcal{M}$ is continuous when $\mathcal{M}$ is considered with its strong operator topology.

We say α is **unitarily implemented** if there exists a group of unitary operators $\{\nu_s \in \mathcal{B}(\mathfrak{H})\}_{s\in G}$ such that

$$\alpha_s(x) = \nu_s^* x \nu_s, \text{ for all } x \in \mathcal{M}.$$

Any such tuple $(\mathcal{M}, G, \alpha)$ will be referred to as a **covariant system**.

Henceforth, G will be a locally compact group and $\mathcal{M}$ a von Neumann algebra over the Hilbert space $\mathfrak{H}$. Consider the tensor product $\mathfrak{H} \otimes L^2(G)$ of Hilbert spaces. This will be the Hilbert space on which the crossed product will act. Kadison and Ringrose provides easy access to the theory of tensor products of Hilbert spaces, see [**KR83**, §2.6]. As such, we investigate $\mathfrak{H} \otimes L^2(G)$.

Lemma 2.25. *Let* $\xi, \eta \in C_c(G, \mathfrak{H})$, *then* $s \mapsto \langle \xi(s), \eta(s)\rangle$ *is of compact support and continuous. Furthermore,*

$$\langle \xi, \eta\rangle := \int_G \langle \xi(s), \eta(s)\rangle \, d\mu(s) \tag{2.2}$$

defines an inner product on $C_c(G, \mathfrak{H})$. *We denote the completion of* $C_c(G, \mathfrak{H})$ *by* $L^2(G, \mathfrak{H})$.

Proof. Clearly $\operatorname{supp}(s \mapsto \langle \xi(s), \eta(s)\rangle) = \operatorname{supp}(\xi) \bigcap \operatorname{supp}(\eta)$ and is compact because both $\operatorname{supp}(\xi)$ and $\operatorname{supp}(\eta)$ are compact.

Continuity follows from the fact that the functions ξ, η and the inner product are continuous. We verify that the map $\langle \cdot, \cdot \rangle$ defined by (2.2) is an inner product on $C_c(G, \mathfrak{H})$. Since $s \mapsto \langle \xi(s), \eta(s)\rangle$ is of compact support and continuous, we deduce that $\langle \cdot, \cdot \rangle$ is well defined.

Clearly $\langle \cdot, \cdot \rangle$ is linear in the first argument. We have $\langle \xi(s), \xi(s)\rangle > 0$ if $\xi \neq 0$. So $\langle \xi, \xi\rangle > 0$ by Proposition 1.16 since the function $s \mapsto \langle \xi(s), \xi(s)\rangle$ belongs to $C_c^+(G, \mathfrak{H})$. In the case where

$\langle \xi, \xi \rangle = 0$, we have $\int_G |\xi(s)|^2 \, d\mu(s) = 0$. Since the function $s \mapsto |\xi(s)|^2$ belongs to $C_c^+(G, \mathfrak{H})$, we must have $\xi \equiv 0$.

Lastly, we have

$$\begin{aligned}\overline{\int_G \langle \eta(s), \xi(s) \rangle ds} &= \int_G \overline{\langle \eta(s), \xi(s) \rangle} \, d\mu(s) \\ &= \langle \xi, \eta \rangle\end{aligned}$$

so that $\langle \cdot, \cdot \rangle$ is conjugate symmetric. ☐

The completion of $C_c(G, \mathfrak{H})$ is a copy of the space of square Bochner-integrable functions from G to $\mathfrak{H}$. This fact justifies our choice to denote the completion of $C_c(G, \mathfrak{H})$ by $L^2(G, \mathfrak{H})$. For a proof, see [**VD78**, Appendix A]. In Theorem 2.26 we consider a more useful isomorphic copy of the space.

THEOREM 2.26. *There is an isomorphism U of $\mathfrak{H} \otimes L^2(G)$ onto $L^2(G, \mathfrak{H})$ such that*

$$\left(U\left(\xi_0 \otimes f\right)\right)(s) = f(s)\xi_0$$

for all $\xi_0 \in \mathfrak{H}$ and $f \in C_c(G)$.

PROOF. Let $\xi_1, \xi_2, \ldots, \xi_n \in \mathfrak{H}$ and $f_1, f_2, \ldots, f_n \in C_c(G)$. Define $\xi : G \to \mathfrak{H}$ by

$$\xi(s) = \sum_{i=1}^{n} f_i(s)\xi_i.$$

Clearly $\xi \in C_c(G, \mathfrak{H})$. As such, we can determine the norm of ξ.

$$\begin{aligned}\int_G \langle \xi(s), \xi(s) \rangle \, d\mu(s) &= \sum_{i,j=1}^{n} \int_G f_i(s)\overline{f_j(s)} \langle \xi_i, \xi_j \rangle \, d\mu(s) \\ &= \sum_{i,j=1}^{n} \langle \xi_i, \xi_j \rangle \langle f_i, f_j \rangle_{L^2} \\ &= \left\langle \sum_{i=1}^{n} \xi_i \otimes f_i, \sum_{j=1}^{n} \xi_j \otimes f_j \right\rangle \\ &= \left\| \sum_{i=1}^{n} \xi_i \otimes f_i \right\|^2\end{aligned}$$

We define the linear operator U from the algebraic tensor product of $\mathfrak{H}$ with $C_c(G)$ into $C_c(G, \mathfrak{H})$ by the prescription

$$U\left(\sum_{i=1}^{n} \xi_i \otimes f_i\right) = \xi.$$

We note that $C_c(G)$ is dense in $L^2(G)$, therefore $\mathfrak{H} \otimes C_c(\mathfrak{H})$ is dense in $\mathfrak{H} \otimes L^2(\mathfrak{H})$. As U is isometric it extends uniquely to a linear operator (also denoted U) from the Hilbert space tensor product $\mathfrak{H} \otimes L^2(G)$ into $L^2(G, \mathfrak{H})$.

It remains to show the surjectivity of U. Appropriately, we show that the functions defined as above are dense in $C_c(G, \mathfrak{H})$.

Let $\xi_0 \in C_c(G,\mathfrak{H})$, $K = \operatorname{supp}(\xi_0)$ and V be an open set of finite Haar measure, such that $K \subseteq V$. Take $\epsilon > 0$. For each $s \in K$, choose a neighbourhood V_s of s such that $V_s \subseteq V$ and $\|\xi_0(t) - \xi_0(s)\| < \epsilon$ for all $t \in V_s$.

By the compactness of K, choose $s_1, s_2, \dots, s_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{s_i}$. Reason similarly as in Lemma 1.14 and let $h_1, h_2, \dots, h_n \in C_c^+(G)$ such that $0 \le \sum_{j=1}^n h_j(s) \le 1$ for all $s \in G$ and $\sum_{j=1}^n h_j(s) = 1$ for all $s \in K$. Define

$$\xi(\cdot) = \sum_{j=1}^n h_j(\cdot)\xi_0(s_j).$$

Then

$$\begin{aligned}\|\xi(s) - \xi_0(s)\| &= \left\| \sum_{j=1}^n h_j(s)\xi_j(s_j) - \sum_{j=1}^n h_j(s)\xi_0(s) \right\| \\ &\le \sum_{j=1}^n h_j(s)\|\xi_0(s_j) - \xi_0(s)\| \\ &< \epsilon\end{aligned}$$

for every $s \in G$. Since $\operatorname{supp}(\xi)$ and $\operatorname{supp}(\xi_0)$ are contained in V we deduce that $\|\xi - \xi_0\| \le \epsilon\mu_G(V)^{\frac{1}{2}}$. That is to say, U is surjective. ☐

Remark 2.27. In the sequel, by abuse of notation we sometimes write $(\xi_0 \otimes f)(s)$ for $\xi(s) := f(s)\xi_0$ with $\xi_0 \in \mathfrak{H}$ and $f \in L^2(G)$. This is a useful convention, as it simplifies and highlights the interplay between operators defined on $\mathfrak{H} \otimes L^2(G)$ and $L^2(G,\mathfrak{H})$.

We turn our attention to defining the operators on $\mathfrak{H} \otimes L^2(G)$ that will generate the crossed product. Henceforth, we shall suppose that we are given a covariant system $(\mathcal{M}, G, \alpha)$.

Theorem 2.28. *For all $x \in \mathcal{M}$ and $\xi \in C_c(G,\mathfrak{H})$, the function ξ_1 defined by $\xi_1(s) = \alpha_{s^{-1}}(x)\xi(s)$ belongs to $C_c(G,\mathfrak{H})$. Moreover $\|\xi_1\| \le \|\xi\|\|x\|$.*

Proof. Note that ξ_1 indeed maps G to $\mathfrak{H}$. We verify continuity:

$$\begin{aligned}\|\xi_1(s) - \xi_1(s_0)\| &= \|\alpha_{s^{-1}}(x)\xi(s) - \alpha_{s_0^{-1}}(x)\xi(s_0)\| \\ &\le \|\alpha_{s^{-1}}(x)(\xi(s) - \xi(s_0))\| + \|[\alpha_{s^{-1}}(x) - \alpha_{s_0^{-1}}(x)]\xi(s_0)\| \\ &\le \|x\|\|\xi(s) - \xi(s_0)\| + \|(\alpha_{s^{-1}}(x) - \alpha_{s_0^{-1}}(x))\xi(s_0)\| \\ &\to 0 \text{ as } s \to s_0.\end{aligned}$$

by the continuity of $s \mapsto \xi(s)$, $s \mapsto \alpha_s(x)\xi(s_0)$ and $s \mapsto s^{-1}$.

Suppose s is in the closed support of ξ_1. Then there exists a net (s_λ) such that $s_\lambda \to s$ and $\xi_1(s_\lambda) = \alpha_{s_\lambda^{-1}}(x)\xi(s_\lambda) \ne 0$. Therefore $\xi(s_\lambda) \ne 0$, and $s \in \operatorname{supp}(\xi)$. That is, $\operatorname{supp}(\xi_1) \subseteq \operatorname{supp}(\xi)$. Therefore, $\operatorname{supp}(\xi_1)$ is compact.

Finally, we have

$$\begin{aligned}\|\xi_1\|^2 = \int_G \|\xi_1(s)\|^2 ds &= \int_G \|\alpha_{s^{-1}}(x)\xi(s)\|^2 ds \\ &\leq \int_G \|\alpha_{s^{-1}}(x)\|^2 \|\xi(s)\|^2 ds \\ &= \|x\|^2 \int_G \|\xi(s)\|^2 ds \\ &= \|x\|^2 \|\xi\|^2.\end{aligned}$$

□

Definition 2.29. For every $x \in \mathcal{M}$, we define a bounded operator on $L^2(G, \mathfrak{H})$ by the prescription

$$(\pi_\alpha(x)\xi)(s) := \alpha_{s^{-1}}(x)\xi(s), \ \xi \in C_c(G, \mathfrak{H}).$$

We adopt the following convention: If the group action α is clearly understood, we write π for π_α.

Lemma 2.30. *For every $x \in \mathcal{M}$, the operator $\pi(x)$ is well-defined.*

Proof. The linearity $\pi(x)$ stems from the fact that each $x \in \mathcal{M}$ is linear. By Theorem 2.28, we have that $\pi(x)$ is bounded. Therefore $\pi(x)$ extends uniquely to an operator on $L^2(G, \mathfrak{H})$ by continuity. □

Remark 2.31. In the above theorem, $\pi(x)$ was defined on the dense subspace $C_c(G, \mathfrak{H})$ and extended by continuity to $L^2(G, \mathfrak{H})$. So we obtain the algebraic expression $[\pi(x)(\xi_0 \otimes f)](s) = f(s)\alpha_{s^{-1}}(x)\xi_0$ for all simple tensors in $\mathfrak{H} \otimes L^2(G)$ (not just those from $\mathfrak{H} \otimes C_c(G)$).

Theorem 2.32. *The map*

$$\pi_\alpha : \mathcal{M} \to \mathcal{B}(L^2(G, \mathfrak{H})) := x \mapsto \pi_\alpha(x), \quad x \in \mathcal{M}$$

*is a faithful normal *-representation of $\mathcal{M}$ in $L^2(G, \mathfrak{H})$.*

Proof. To show that π is a *-representation of $\mathcal{M}$ in $L^2(G, \mathfrak{H})$, we must show it is a *-homomorphism from $\mathcal{M}$ into $\mathcal{B}(L^2(G, \mathfrak{H}))$. This follows immediately from the fact that $\alpha_{s^{-1}}$ is *-automorphism of $\mathcal{M}$ for each $s \in G$. We show π is faithful, that is $\ker(\pi) = \{0\}$.

Suppose $0 \neq x \in \mathcal{M}$. Let $\xi \in C_c(G, \mathfrak{H})$ such that $\xi(e_G) \neq 0 \in \mathfrak{H}$. In this case, the quantity $\langle x\xi(e_G), x\xi(e_G)\rangle \neq 0$. The preimage of $\mathbb{C} \setminus \{0\}$ under the continuous function $s \in G \mapsto \langle \alpha_{s^{-1}}(x)\xi(s), x\xi(e_G)\rangle$, is an open set containing e_G. As such, there exists a compact neighbourhood K of e_G, such that $\langle \alpha_{s^{-1}}(x)\xi(s), x\xi(e_G)\rangle \neq 0$ for all $s \in K$. This implies $\alpha_{s^{-1}}(x)\xi(s) \neq 0$ for all $s \in K$.

We deduce that for some $\epsilon > 0$ we have that $\|\alpha_{s^{-1}}(x)\xi(s)\| \geq \epsilon$ on K. For suppose contrariwise that for each $\epsilon > 0$ there exists an $s_\epsilon \in K$ such that $\|\alpha_{s_\epsilon^{-1}}(x)\xi(s_\epsilon)\| \leq \epsilon$. Then (s_ϵ) defines a net. Since K is compact, (s_ϵ) has a convergent subnet. Without loss of generality, assume

$s_\epsilon \to s' \in K$ as $\epsilon \to 0$. But then $\lim_{\epsilon\to 0} \|\alpha_{s_\epsilon^{-1}}(x)\xi(s_\epsilon)\| \leq \lim_{\epsilon\to 0}\epsilon$. By the continuity of $s \mapsto \alpha_s(x)\xi(s)$ we have that $\|\alpha_{(s')^{-1}}(x)\xi(s')\| = 0$. A contradiction.

By the arguments hitherto made, we find that

$$\begin{aligned}\|\pi(x)\xi\|^2 &= \int_G \|\alpha_{s^{-1}}(x)\xi(s)\|^2 \, d\mu(s)\\ &\geq \int_K \|\alpha_{s^{-1}}(x)\xi(s)\|^2 \, d\mu(s)\\ &\geq \epsilon\mu(K).\end{aligned}$$

That is $\|\pi(x)\xi\| > 0$ and therefore $\pi(x) \neq 0$. So $\ker(\pi) = \{0\}$.

Lastly, we prove that π is normal. Let $\{x_i\}_{i\in I}$ be a bounded, increasing net in $\mathcal{M}_+$ with x as supremum. Since π is a *-representation, $\pi(x_i)$ is a bounded and increasing net. As such it increases to some operator on $L^2(G,\mathfrak{H})$, say $\tilde{y}$. As $x_i \leq x$ for all i we also have $\pi(x_i) \leq \pi(x)$ so that $\tilde{y} \leq \pi(x)$. We establish that π is normal by showing that $\tilde{y} = \pi(x)$.

Let $f \in C_c(G)$, $\xi_0 \in \mathfrak{H}$, and $\xi = \xi_0 \otimes f$. By definition we have

$$\langle \pi(x_i)\xi, \xi\rangle = \int_G \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0\rangle \, |f(s)|^2 \, d\mu(s).$$

Now $s \mapsto \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0\rangle$ is a net of continuous positive functions (as x_i is positive and π is a *-representation), increasing to the function $s \mapsto \langle \alpha_{s^{-1}}(x)\xi_0, \xi_0\rangle$ which is also continuous. By Dini's theorem [**Kel17**, Chapter 7, Problem E], we have uniform convergence on compact sets. Particularly, we have uniform convergence on $\operatorname{supp}(f)$. Hence the integrals will converge. That is to say $\langle \pi(x_i)\xi, \xi\rangle \to \langle \pi(x)\xi, \xi\rangle$.

By a monotone convergence type theorem, we find

$$\langle \pi(x)\xi, \xi\rangle = \sup_{i\in I} \int_G \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0\rangle |f(s)|^2 \, d\mu(s) = \int_G \sup_{i\in I} \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0\rangle |f(s)|^2 \, d\mu(s).$$

In other words $\langle \pi(x)\xi, \xi\rangle = \langle \tilde{y}\xi, \xi\rangle$. Now as $\tilde{y} \leq \pi(x)$ it follows from $\langle (\pi(x)-\tilde{y})\xi, \xi\rangle = 0$ that also $(\pi(x)-\tilde{y})\xi = 0$ or $\pi(x)\xi = \tilde{y}\xi$. This implies $\pi(x) = \tilde{y}$ because functions of the form $\xi = \xi_0 \otimes f$ with $\xi_0 \in \mathfrak{H}$ and $f \in C_c(G)$ span a dense subspace. ☐

DEFINITION 2.33 (Shift operators). Let G be a locally compact group and $\mathfrak{H}$ a Hilbert space. For every $g \in G$, we define the **left shift operator** $\lambda_{\mathfrak{H}}^G(g) : L^2(G,\mathfrak{H}) \to L^2(G,\mathfrak{H})$ by the prescription

$$[\lambda_{\mathfrak{H}}^G(g)\vartheta](s) = \vartheta(g^{-1}s), \quad \text{for all } \vartheta \in L^2(G,\mathfrak{H}). \tag{2.3}$$

If the group and/or Hilbert space is clearly understood, we reserve the right to write $\lambda(g)$, $\lambda^G(g)$ or $\lambda_{\mathfrak{H}}(G)$ for $\lambda_{\mathfrak{H}}^G(g)$. Similarly, for all $g \in G$ we define **the right shift operators** $\rho_{\mathfrak{H}}^G(g)$ from $L^2(G,\mathfrak{H})$ onto itself by the prescription

$$(\rho^G(g)\vartheta)(s) = \Delta_G^{\frac{1}{2}}(g)\vartheta(sg), \quad \text{for all } \vartheta \in L^2(G,\mathfrak{H}). \tag{2.4}$$

THEOREM 2.34. *The map $g \mapsto \lambda_{\mathfrak{H}}^{G}(g)$ is a unitary representation of G on $L^2(G, \mathfrak{H})$.*

PROOF. It is an easy exercise to determine that $g \mapsto \lambda(g)$ is well-defined and that $\lambda(g)$ is a linear operator for each $g \in G$. Let $\vartheta \in L^2(G, \mathfrak{H})$, then we find that $\|\lambda(g)\vartheta\|^2 = \int_G \|\vartheta(g^{-1}s)\|^2 \, d\mu(s)$ $= \|\vartheta\|^2$. So for each $g \in G$ we have that $\lambda(g)$ is both isometric and $\lambda(g) \in \mathcal{B}(L^2(G, \mathfrak{H}))$.

We verify that $g \mapsto \lambda(g)$ is indeed a unitary representation. The map $g \mapsto \lambda(g)$ is a group homomorphism, for $(\lambda(gh)\vartheta)(s) = \vartheta(h^{-1}g^{-1}s) = (\lambda(g)\lambda(h)\vartheta)(s)$ for all $\vartheta \in L^2(G, \mathfrak{H})$.

Furthermore, let $\vartheta, \varphi \in L^2(G, \mathfrak{H})$, then:

$$\begin{aligned} \langle \lambda(g)\vartheta, \varphi \rangle &= \int_G \langle \vartheta(g^{-1}s), \varphi(s) \rangle \, d\mu(s) \\ &= \int_G \langle \vartheta(s), \varphi(gs) \rangle \, d\mu(s) \\ &= \int_G \langle \vartheta(s), (\lambda(g^{-1})\varphi)(s) \rangle \, d\mu(s) \\ &= \langle \vartheta, \lambda(g^{-1})\varphi \rangle. \end{aligned}$$

by the left-invariance of Haar measures. Since this holds for all $\vartheta, \varphi \in C_c(G, \mathfrak{H})$, and $C_c(G, \mathfrak{H})$ is dense in $L^2(G, \mathfrak{H})$, we conclude that $\lambda(g)^* = \lambda(g^{-1}) = \lambda(g)^{-1}$.

We argue that $g \mapsto \lambda_{\mathfrak{H}}^{G}(g)$ is strongly continuous: Let $\vartheta = \xi_0 \otimes f \in C_c(G, \mathfrak{H})$, then $\lambda_{\mathfrak{H}}^{G}(g)\vartheta(s) = f(g^{-1}s)\xi = [\xi \otimes \lambda_{\mathbb{C}}^{G}(g)f](s)$. The preceding fact shows that $\|\lambda_{\mathfrak{H}}^{G}(g)\vartheta - \vartheta\|_2 = \|\xi_0\| \|\lambda_{\mathbb{C}}^{G}(g)f - f\|_2$. From Haar measure theory, we know that $\|\lambda_{\mathbb{C}}^{G}(g)f - f\|_2 \to 0$ as g approaches the group unit e_G (see the proof of Theorem 1.22). Lastly, we appropriately approximate elements of $L^2(G, \mathfrak{H})$ with linear combinations of simple tensors. As such, we have proved strong continuity and duly showed that $g \mapsto \lambda_{\mathfrak{H}}^{G}(g)$ is a unitary representation. □

REMARK 2.35 (Right group von Neumann algebra). Let G be a group and $\mathfrak{H}$ a Hilbert space. That $g \in G \mapsto \rho_{\mathfrak{H}}^{G}(g)$ also defines a unitary representation of G follows mutatis mutandis from the proof of Theorem 2.32. It is this representation that we use to define the **right group von Neumann algebra** of G, i.e.,$\mathrm{VN}_r(G) := \rho_{\mathbb{C}}^{G}(G)''$.

THEOREM 2.36. *The set of linear combinations of operators of the form $\pi(x)\lambda(g)$ with $x \in \mathcal{M}$ and $g \in G$ is a *-algebra.*

PROOF. Let $g \in G$, $x \in \mathcal{M}$ and $\xi \in L^2(G, \mathfrak{H})$. We find that

$$\begin{aligned} (\lambda(g)\pi(x)\lambda(g)^*\xi)(s) &= (\pi(x)\lambda(g)^*\xi)(g^{-1}s) \\ &= [\alpha_{s^{-1}g}(x)\lambda(g)^*\xi](g^{-1}s) \\ &= [\alpha_{s^{-1}}(\alpha_g(x))\xi](s) \\ &= [\pi(\alpha_g(x))\xi](s). \end{aligned}$$

So we have shown that $\lambda(g)\pi(x)\lambda(g)^* = \pi(\alpha_g(x))$. Also let $y \in \mathcal{M}$ and $s \in G$, then by our preceding calculation

$$\begin{aligned}\pi(x)\lambda(g)\pi(y)\lambda(s) &= \pi(x)\lambda(g)\pi(y)[\lambda(g)^*\lambda(g)]\lambda(s)\\ &= \pi(x)\pi\left(\alpha_g(y)\right)\lambda(gs)\\ &= \pi(x\alpha_g(y))\lambda(gs)\end{aligned}$$

and

$$\begin{aligned}(\pi(x)\lambda(g))^* &= \lambda(g)^*\pi(x)^*\\ &= \lambda(g^{-1})\pi(x^*)[\lambda(g^{-1})^*\lambda(g^{-1})]\\ &= \pi(\alpha_{g^{-1}}(x^*))\lambda(g^{-1}).\end{aligned}$$

□

Definition 2.37. The **crossed product** of $\mathcal{M}$ by the action α of G is the von Neumann algebra generated by the operators $\{\pi(x), \lambda(g) : x \in \mathcal{M}, g \in G\}$ and is denoted by $\mathcal{M} \rtimes_\alpha G$. By the preceding theorem it is the WOT closure of the *-algebra of linear combinations of products $\pi(x)\lambda(g)$ with $x \in \mathcal{M}$ and $g \in G$.

2.3.1. Uniqueness of the crossed product.

Theorem 2.38. *Given a covariant system* $(\mathcal{M}, G, \alpha)$*, we have that* $\mathcal{M} \rtimes_\alpha G \subseteq \mathcal{M}\bar{\otimes}\mathcal{B}\left(L^2(G)\right)$.

Proof. Under the identification of $\mathfrak{H} \otimes L^2(G)$ with $L^2(G, \mathfrak{H})$, we identify each $\lambda(g) \in \mathcal{B}(L^2(G, \mathfrak{H}))$ with a corresponding operator in $\mathcal{M}\bar{\otimes}\mathcal{B}(L^2(G))$. Let $(\xi_0 \otimes f) \in C_c(G, \mathfrak{H})$, then $[\lambda_{\mathfrak{H}}(g)(\xi_0 \otimes f)](s) = [\xi_0 \otimes \lambda_{\mathbb{C}}(g)f](s)$. So we identify $\lambda_{\mathfrak{H}}(g)$ with $(\mathbb{1} \otimes \lambda_{\mathbb{C}}(g))$. We have already made use of this fact in Theorem 2.34.

Let $x' \in \mathcal{M}'$. Then the simple tensor $(x'\xi_0 \otimes f) = (x' \otimes \mathbb{1})(\xi_0 \otimes f)$ is identified with $\eta(\cdot) := f(\cdot)x(\xi_0) \in C_c(G, \mathfrak{H})$. We note that x' and $\alpha_s^{-1}(x)$ commute, since $x' \in \mathcal{M}'$ and $\alpha_{s^{-1}}$ is a *-isomorphism of $\mathcal{M}$. This shows that $\alpha_s^{-1}(x)\eta(s) = f(s)[\alpha_s^{-1}(x)x'](\xi_0) = f(s)[x'\alpha_s^{-1}(x)](\xi_0)$.

Then $[\pi(x)(x' \otimes \mathbb{1})(\xi_0 \otimes f)](s) = f(s)[x'\alpha_{s^{-1}}(x)](\xi_0) = [(x' \otimes \mathbb{1})\pi(x)(\xi_0 \otimes f)](s)$. Since our arguments hold for all $C_c(G, \mathfrak{H})$, they indicate that $\pi(x)$ commutes with $\mathcal{M}'\bar{\otimes}\,\mathbb{C}$. So the operator in $\mathcal{B}(\mathfrak{H})\bar{\otimes}\mathcal{B}(L^2(G))$ corresponding to $\pi(x)$ belongs to $\mathcal{M}\bar{\otimes}\mathcal{B}(L^2(G))$ by the Bicommutant theorem and [**Tak01**, VI, Theorem 5.9]. □

Theorem 2.39 (Theorem 6.44. [**GL20**]). *Let* $\mathcal{M} \subseteq \mathcal{B}(\mathfrak{H})$ *and* $\mathcal{N} \subseteq \mathcal{B}(\mathfrak{A})$ *be two von Neumann algebras and* $\mathcal{J}$ *a* **-isomorphism from* $\mathcal{M}$ *onto* $\mathcal{N}$*. Let* G *be a locally compact group admitting actions* α *and* β *on* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively. If for all* $g \in G$ *and* $x \in \mathcal{M}$ *we have that* $\mathcal{J}\left(\alpha_g(x)\right) = \beta_g(\mathcal{J}(x))$*, then* $\widetilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}$ *is a* **-isomorphism* $\widetilde{\mathcal{J}}$ *from* $\mathcal{M} \rtimes_\alpha G$ *onto* $\mathcal{N} \rtimes_\beta G$*, for which we have that* $\widetilde{\mathcal{J}}\left(\pi_\alpha(x)\right) = \pi_\beta(\mathcal{J}(x))$ *for all* $x \in \mathcal{M}$*. Furthermore,* $\widetilde{\mathcal{J}}\left(\lambda(g)\right) = \tilde{\lambda}(g)$ *for all* $g \in G$ *where* $\lambda(g)$ *and* $\tilde{\lambda}(g)$ *respectively denote the left-shift operators corresponding to* $\mathcal{M} \rtimes_\alpha G$ *and* $\mathcal{N} \rtimes_\beta G$*.*

Proof. The standard theory of von Neumann algebra tensor products states that $\widetilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}$ is a *-isomorphism from $\mathcal{M} \bar{\otimes} \mathcal{B}(L^2(G))$ to $\mathcal{N} \bar{\otimes} \mathcal{B}(L^2(G))$. We need to show that $\widetilde{\mathcal{J}}$ maps the subspace $\mathcal{M} \rtimes_\alpha G$ onto the subspace $\mathcal{N} \rtimes_\beta G$, in accordance with the hypothesis.

For each $g \in G$, we know from Theorem 2.38 that $\lambda^G_{\mathfrak{H}}(g) \in \mathcal{M} \rtimes_\alpha G$ corresponds to $\mathbb{1}_{\mathcal{M}} \otimes \lambda^G_{\mathbb{C}}(g) \in \mathcal{M} \bar{\otimes} \mathcal{B}(L^2(G))$. Similarly $\lambda^G_{\mathfrak{A}}(g) \in \mathcal{N} \rtimes_\alpha G$ corresponds to $\mathbb{1}_{\mathcal{N}} \otimes \lambda^G_{\mathbb{C}}(g) \in \mathcal{N} \bar{\otimes} \mathcal{B}(L^2(G))$. We clearly have that $\widetilde{\mathcal{J}}(\lambda^G_{\mathfrak{H}}(g)) = \lambda^G_{\mathfrak{A}}(g)$ for all $g \in G$.

To complete the proof, we need to show that $\widetilde{\mathcal{J}}\,(\pi_\alpha(x)) = \pi_\beta(\mathcal{J}(x))$ for all $x \in \mathcal{M}$, since then $\widetilde{\mathcal{J}}$ will clearly map the von Neumann algebra generated by $\{\pi_\alpha(x), \lambda(t) : x \in \mathcal{M}, t \in G\}$ (i.e. $\mathcal{M} \rtimes_\alpha G$) onto the von Neumann algebra generated by $\{\pi_\beta(y), \tilde{\lambda}(t) : y \in \mathcal{N}, t \in G\}$ (that is $\mathcal{N} \rtimes_\beta G$).

Here, we need the fact that any element of $\mathcal{M} \bar{\otimes} \mathcal{B}\left(L^2(G)\right)$ may be represented as some sort of (infinite) matrix. See [**GL20**, Definition 1.87], [**GL20**, Proposition 1.88] and [**GL20**, Proposition 1.90]. Select an orthonormal basis $\{f_i\}$ of $L^2(G)$. Any element $\widetilde{x} \in \mathcal{M} \bar{\otimes} \mathcal{B}\left(L^2(G)\right)$ may be written as the sum $\sum_{i,j \in I} \left(\mathbb{1} \otimes e_i\right) \widetilde{x} \left(\mathbb{1} \otimes e_j\right)$ where the e_i's are the projections $\langle \cdot, f_i \rangle_2 f_i$, and convergence is in the σ-strong topology. We investigate the specific case where $\tilde{x} = \pi_\alpha(x)$ for some $x \in \mathcal{M}$.

First we show that $(\mathbb{1} \otimes e_i)\pi_\alpha(x)\left(\mathbb{1} \otimes e_j\right) = \left(\pi_\alpha(x)_{ij} \otimes u_{i,j}\right)$, where $u_{i,j} = \langle \cdot, f_j \rangle_2 f_i$ and

$$\pi_\alpha(x)_{ij}(\cdot) := \int_G \overline{f_i(s)} f_j(s) \alpha_{s^{-1}}(x)(\cdot)\, d\mu(s),$$

by which we mean $\pi_\alpha(x)_{ij}(\cdot)$ is the pointwise defined operator which satisfies the relation

$$\langle \pi_\alpha(x)_{ij}(\xi_0), \eta_0 \rangle = \int_G \overline{f_i(s)} f_j(s) \langle \alpha_s^{-1} \xi_0, \eta_0 \rangle\, ds.$$

The $u_{i,j}$'s determine a set of matrix units, that is $u^*_{i,j} = u_{j,i}$ and $u_{i,j}u_{j,k} = u_{j,k}$ with $\sum_{i \in I} u_{i,i} = \sum_{i \in I} e_i = \mathbb{1}_{\mathcal{B}(L^2(G))}$ (where the sum converges in the σ-strong* topology).

Let $\xi, \eta \in C_c(G, \mathfrak{H})$ such that $\xi = \xi_0 \otimes f$ and $\eta = \eta_0 \otimes z$. Then

$$\begin{aligned}
\langle (\mathbb{1} \otimes e_i)\pi_\alpha(x)(1 \otimes e_j)\xi, \eta \rangle &= \langle \pi_\alpha(x)(1 \otimes e_j)\xi, (\mathbb{1} \otimes e_i)\eta \rangle \\
&= \int_G \langle \langle f, f_j \rangle_2 f_j(s) \alpha_s^{-1} \xi_0, \langle z, f_i \rangle_2 f_i(s) \eta_0 \rangle\, ds \\
&= \langle f, f_j \rangle_2 \langle f_i, z \rangle_2 \int_G \langle f_j(s) \overline{f_i(s)} \alpha_s^{-1} \xi_0, \eta_0 \rangle\, ds \\
&= \langle \langle f, f_j \rangle_2 f_i, z \rangle_2 \langle \pi_\alpha(x)_{ij}(\xi_0), \eta_0 \rangle \\
&= \langle \pi_\alpha(x)_{ij}(\xi_0), \eta_0 \rangle \langle u_{i,j} f, z \rangle_2 \\
&= \langle (\pi_\alpha(x)_{ij} \otimes u_{i,j})(\xi_0 \otimes f), \eta_0 \otimes z \rangle.
\end{aligned}$$

The final equality follows from the definition of inner product defined for tensor products of Hilbert spaces. So we have shown that $(\mathbb{1} \otimes e_i)\pi_\alpha(x)\left(\mathbb{1} \otimes e_j\right) = \left(\pi_\alpha(x)_{ij} \otimes u_{i,j}\right)$.

By the discussion preceding the above calculations, we must have $\pi_\alpha(x) = \sum_{i,j} \pi_\alpha(x)_{ij} \otimes u_{i,j}$ for all $x \in \mathcal{M}$.

For $x \in \mathcal{M}$, the normality of $\widetilde{\mathcal{J}}$ (*-isomorphisms between von Neumann algebras are normal) then ensures that $\widetilde{\mathcal{J}}\left(\pi_\alpha(x)\right) = \sum_{i,j} \widetilde{\mathcal{J}}\left(\pi_\alpha(x)_{ij} \otimes u_{i,j}\right) = \sum_{i,j}\left(\mathcal{J}\left(\pi_\alpha(x)_{ij}\right) \otimes u_{i,j}\right)$. Observe that for all $i, j \in I$ we have that

$$\begin{aligned}\mathcal{J}\left((\pi_\alpha(x))_{ij}\right) &= \int_G \mathcal{J}\left(\alpha_{s^{-1}}(x)\right)\overline{f_i}(s) f_j(s)\, ds \\ &= \int_G \beta_{s^{-1}}(\mathcal{J}(x))\overline{f_i}(s) f_j(s)\, ds \\ &= \left(\pi_\beta(\mathcal{J}(x))\right)_{ij}.\end{aligned}$$

We therefore have that $\widetilde{\mathcal{J}}\left(\pi_\alpha(x)\right) = \pi_\beta(\mathcal{J}(x))$. □

Suppose we are given the covariant system $(\mathcal{M}, G, \alpha)$. We have encountered the dual group $\widehat{G}$ of G. We establish the very important fact that there is a canonical action of $\widehat{G}$ acting on $\mathcal{M} \rtimes_\alpha G$. We follow the construction as developed in [**GL20**, p. 205].

Let $v_\gamma : L^2(G) \to L^2(G)$ for $\gamma \in \widehat{G}$ be the operator defined by $v_\gamma(f)(s) = \overline{\gamma(s)} f(s)$ for each $f \in L^2(G)$ and each $s \in G$. Note that the unimodularity of the numbers $\overline{\gamma(s)}$, ensures that each v_γ is a unitary belonging to $\mathcal{B}(L^2(G))$.

Then the maps $w_\gamma := \mathbb{1} \otimes v_\gamma$ on $\mathfrak{H} \otimes L^2(G) \cong L^2(G, \mathfrak{H})$ are also unitaries. We use these maps to define an action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$.

Let $\xi \in L^2(G, H)$ be the limit of the net $(\xi^{(i)} \otimes f^{(i)})$. Then $\lim_i(\mathbb{1} \otimes v_\gamma)(\xi^{(i)} \otimes f^{(i)}) = \overline{\gamma(s)}\xi(s)$. That is, we have $[w_\gamma(\xi)](s) = \overline{\gamma(s)}\xi(s)$.

For each simple tensor $\xi \otimes f$ ($\xi \in \mathfrak{H}, f \in L^2(G)$), we have that

$$\int_G \left\|\left(\mathbb{1} - w_\gamma\right)(\xi \otimes f)(s)\right\|^2 ds = \|\xi\|^2 \int_G |1 - \overline{\gamma(s)}|^2 |f(s)|^2 ds.$$

We recall that $\widehat{G}$ has a topology of compact convergence, where $A := \{\Gamma \in \widehat{G} : \sup_{s \in \mathrm{supp}(f)} |1 - \Gamma(s)| < \epsilon\}$ is a neighbourhood of the identity of $\widehat{G}$. If (γ_i) is a net converging to the identity, then $\gamma_i \in A$ for all i great enough. Therefore

$$\int_G |1 - \gamma_i(s)|^2 |f(s)|\, d\mu(s) \le \int_{\mathrm{supp}(f)} |f(s)|^2\, d\mu(s)\, \epsilon^2,$$

for i great enough. So $\int_G |1 - \overline{\gamma_i(s)}|^2 |f(s)|^2\, d\mu(s) \to 0$ as γ_i tends to the group unit of $\widehat{G}$.

Hence for each simple tensor $\gamma \mapsto w_\gamma(\xi \otimes f)$ is continuous in the $L^2(G, \mathfrak{H})$-norm. By appropriately approximating elements of $L^2(G, \mathfrak{H})$ with the span of simple tensors, we find that $\gamma \mapsto w_\gamma$ is strongly continuous on all of $L^2(G, \mathfrak{H})$.

Computing $w_{\gamma_1} w_{\gamma_2}(\xi \otimes f)$ shows that $w_{\gamma_1} w_{\gamma_2} = w_{\gamma_1 \gamma_2}$ for each $\gamma_1, \gamma_2 \in \widehat{G}$. If for each $x \in \mathcal{B}\left(L^2(G, \mathfrak{H})\right)$ we define $\widehat{\alpha}_\gamma(x) := w_\gamma x w_\gamma^*$, then the maps $\widehat{\alpha}_\gamma$ define a group action of $\widehat{G}$ on $\mathcal{B}\left(L^2(G, \mathfrak{H})\right)$.

Furthermore, for $x \in \mathcal{M}$ we have

$$\begin{aligned} [w_\gamma \pi(x) w_\gamma^*(\xi_0 \otimes f)](s) &= \overline{\gamma(s)}[\pi(x)(\xi_0 \otimes v_\gamma^* f)](s) \\ &= [\pi(x)(\xi_0 \otimes f)](s). \end{aligned} \tag{P1}$$

Similarly

$$\widehat{\alpha}(\lambda(g)) = \overline{\gamma(g)}\lambda(g) \text{ for each } g \in G. \tag{P2}$$

As such, the maps $\widehat{\alpha}_\gamma$ map the generators of $\mathcal{M} \rtimes_\alpha G$ back into $\mathcal{M} \rtimes_\alpha G$. Thus, we must have that action $\widehat{\alpha}$ restricts to an action on $\mathcal{M} \rtimes_\alpha G$.

Definition 2.40 (Dual action). The restriction of $\widehat{\alpha}$ to $\mathcal{M} \rtimes_\alpha G$ (also denoted $\widehat{\alpha}$) is defined to be the **dual action** of $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$. We say the action $\widehat{\alpha}$ of $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$ is dual to the action α of G on $\mathcal{M}$.

Theorem 2.41. *Properties (P1) and (P2) uniquely determine the action dual to the action α of G on $\mathcal{M}$.*

Proof. Obvious. ☐

2.3.2. Relating the semidirect product to the crossed product. In this section we investigate a very useful result. We show that for any (continuous) group action α of G on A, where G and A are locally compact groups, there exists a group action β of G on $\mathrm{VN}_l(A)$ such that $\mathrm{VN}_l(A) \rtimes_\beta G \cong \mathrm{VN}_l(A \ltimes_\alpha G)$. We do so by proving a more general result due to R. Rousseau [**Rou81**].

Suppose you are given the covariant system $(\mathcal{M}, \beta, A \ltimes_\alpha G)$ and suppose that β is unitarily implemented (see Definition 2.24). That is,

$$\beta_{(a,s)}(x) = \nu_{(a,s)} x \nu_{(a,s)}^*, \quad x \in \mathcal{M}. \tag{2.5}$$

Insisting that β be unitarily implemented is not really a restriction, as one can always find a faithful normal representation of $\mathcal{M}$ in which the action is unitarily implemented. See [**VD78**, p. 8] and [**GL20**, Remark 6.42].

Definition 2.42. Given a covariant system $(\mathcal{M}, \beta, A \ltimes_\alpha G)$, we define a map κ from A to the *-automorphisms of $\mathcal{M}$ by

$$\kappa_a = \beta_{(a,e)} \text{ for all } a \in A. \tag{2.6}$$

Theorem 2.43. *The map $\kappa : a \mapsto \beta_{a,e}$ is a group action of A on $\mathcal{M}$.*

Proof. It is clear that $\{(a, e) \mid a \in A\}$ is a subgroup of $A \ltimes_\alpha G$ isomorphic to A. Since $\kappa_a = \beta_{(a,e)}$ for all $a \in A$ and β is a group action of $A \ltimes_\alpha G$ on $\mathcal{M}$, we deduce that κ is a group action of A on $\mathcal{M}$. ☐

Therefore $(\mathcal{M}, \kappa, A)$ becomes a covariant system and we construct the crossed product $\mathcal{M} \rtimes_\kappa A$ acting on $L^2(A, \mathfrak{H})$.

Theorem 2.44. *The prescription*

$$(u_s \xi)(a) = \delta_\alpha(s)^{-\frac{1}{2}} \xi(\alpha_s^{-1}(a)), \tag{2.7}$$

with $\xi \in L^2(A), s \in G, a \in A$ and $\delta_\alpha(s)$ as in Definition 1.35 defines a unitary operator in $\mathcal{B}(L^2(A))$.

Proof. That u_s is linear is clear. We obtain

$$\|u_s\xi\|^2 = \int_A |\xi(\alpha_s^{-1}(a))|^2 \delta_\alpha^{-1}(s)\, d\mu(a) = \int_A |\xi(a)|^2\, d\mu(a) = \|\xi\|_2^2$$

via Theorem 1.36. Let $\eta \in L^2(A)$. Similarly

$$\begin{aligned} \langle u_s\xi, \eta\rangle &= \int_A \delta_\alpha(s)^{-\frac{1}{2}} \xi(\alpha_{s^{-1}}(a)) \overline{\eta(a)}\, d\mu(a) \\ &= \int_A \delta_\alpha(s)^{\frac{1}{2}} \xi(a) \overline{\eta(\alpha_s(a))}\, d\mu(a) \\ &= \langle \xi, \delta_\alpha^{\frac{1}{2}}(s) \eta \circ \alpha_s\rangle. \end{aligned}$$

Since this holds for all ξ, η, we find that $u_s^*\xi = \delta_\alpha^{\frac{1}{2}}(s)\xi \circ \alpha_s$. Clearly u_s defines a unitary operator in $\mathcal{B}(L^2(A))$. ☐

Theorem 2.45. *Let u_s be as in Theorem 2.44. Then*

$$\bar{\alpha}_s : \mathcal{B}(L^2(A)) \to \mathcal{B}(L^2(A)) : y \to u_s y u_s^*$$

defines a strongly continuous action of G on $\mathcal{B}(L^2(A))$. In particular we have: $\bar{\alpha}_s(\lambda(a)) = \lambda(\alpha_s(a))$, where λ is the left regular representation of A. So $\bar{\alpha}$ is an action of G on $\mathrm{VN}_l(A)$, the group von Neumann algebra of A.

Proof. Take $\xi \in L^2(A), b \in A$; then

$$\begin{aligned} (u_s\lambda(b)u_s^* f)(a) &= \delta_\alpha(s)^{-\frac{1}{2}} (\lambda(b)u_s^*\xi)(\alpha_{s^{-1}}(a)) \\ &= \delta_\alpha(s)^{-\frac{1}{2}} (u_s^*\xi)\left(b^{-1}\alpha_{s^{-1}}(a)\right) \\ &= \xi\left(\alpha_s(b^{-1})a\right) \\ &= \lambda(\alpha_s(a))\xi(a). \end{aligned}$$

The claim regarding the continuity of $\bar{\alpha}$ follows from the fact that $s \in G \mapsto u_s \in \mathcal{B}(L^2(A))$ is a unitary representation of G (see [**Rou77**, Lemma 4.2, p. 414]) and that $\bar{\alpha}$ is then by definition a continuous group action of G on $\mathcal{B}(L^2(A))$, [**VD78**, p. 3]. ☐

Definition 2.46. We define a strongly continuous action γ of G on $\mathcal{M}\bar{\otimes}\mathcal{B}(L^2(A))$ by

$$\gamma_s = \beta_{(e,s)} \otimes \bar{\alpha}_s; \quad s \in G.$$

See [**VD78**, I3.3, p. 14] for a proof that γ is indeed a strongly continuous action. We recall Equation (2.5) and see that γ_s is unitarily implemented by $\nu_{(e,s)} \otimes u_s$.

LEMMA 2.47. *For $x \in \mathcal{M}, s \in G, a \in A$ we have:*

$$\gamma_s\left(\pi_\kappa(x)\right) = \pi_\kappa(\beta_{(e,s)}(x))$$
$$\gamma_s(\tilde{\lambda}(a)) = \tilde{\lambda}\left(\alpha_s(a)\right),$$

where $\tilde{\lambda}(a), \pi_\kappa(x)$ are the operators generating $\mathcal{M} \rtimes_\kappa A$. So γ is an action of G on $\mathcal{M} \rtimes_\kappa A$.

PROOF. Let $\xi \in C_c(A, \mathfrak{H}), b \in A$. Recall that γ_s is unitarily implemented. We shall freely make use of the isomorphism between $L^2(A, \mathfrak{H})$ and $\mathfrak{H} \otimes L^2(A)$. We obtain:

$$\begin{aligned}(\gamma_s(\pi_\kappa(x))\xi)(b) &= ((\nu_{(e,s)} \otimes u_s)\pi_\kappa(x)(\nu^*_{(e,s)} \otimes u^*_s)\xi)(b)\\ &= \nu_{(e,s)}((\pi_\kappa(x)(\nu^*_{(e,s)} \otimes u^*_s)\xi)(\alpha_s^{-1}(b)))\delta_\alpha(s)^{-\frac{1}{2}}\\ &= \nu_{(e,s)}\beta_{(\alpha_s^{-1}(b^{-1}),e)}(x)(((\nu^*_{(e,s)} \otimes u^*_s)\xi)(\alpha_s^{-1}(b)))\delta_\alpha(s)^{-\frac{1}{2}}\\ &= \nu_{(e,s)}\beta_{(\alpha_s^{-1}(b^{-1}),e)}(x)\nu^*_{(e,s)}(\xi(b))\\ &= \beta_{(\alpha_s^{-1}(b^{-1}),s)}(x)(\xi(b))\\ &= \beta_{(b^{-1},e)}(\beta_{(e,s)}(x))(\xi(b))\\ &= (\pi_\kappa(\beta_{(e,s)}(x))\xi)(b)\end{aligned}$$

where we have made use of $\beta_{(e,s)}(x) = \nu_{(e,s)}x\nu^*_{(e,s)}$. Also

$$\begin{aligned}\gamma_s(\tilde{\lambda}(a)) &= (\nu_{(e,s)} \otimes u_s)(\mathbb{1} \otimes \lambda(a))(\nu^*_{(e,s)} \otimes u^*_s)\\ &= \mathbb{1} \otimes \bar{\alpha}_s(\lambda(a))\\ &= \mathbb{1} \otimes \lambda(\alpha_s(a))\\ &= \tilde{\lambda}(\alpha_s(a)).\end{aligned}$$

□

THEOREM 2.48. *The crossed products $\mathcal{M} \rtimes_\beta (A \ltimes_\alpha G)$ and $(\mathcal{M} \rtimes_\kappa A) \rtimes_\gamma G$ are spatially isomorphic.*

PROOF. The Hilbert spaces $L^2\left(A \ltimes_\alpha G, \mathfrak{H}\right)$ and $L^2\left(G, L^2(A, \mathfrak{H})\right)$ are isometrically isomorphic by

$$((Vf)(s))(a) = f(a,s)\delta_\alpha(s)^{-\frac{1}{2}}; \quad f \in C_c\left(A \ltimes_\alpha G, \mathfrak{H}\right).$$

We verify the claim. Consider $s \mapsto f(\cdot, s)\delta_\alpha(s)^{-\frac{1}{2}}$. We check that for any $s \in G$, this map is an element in $C_c\left(G, L^2(A, \mathfrak{H})\right)$ This map is continuous and has compact support since $f \in C_c\left(A \ltimes_\alpha G, \mathfrak{H}\right)$ and $\delta : H \to \mathbb{R}^+$ is continuous. So we must show that for all $s \in G$, the map $a \mapsto f(a,s)\delta_\alpha(s)^{-\frac{1}{2}}$ is in $L^2(A, \mathfrak{H})$.

Since $f \in C_c\left(A \ltimes_\alpha G, \mathfrak{H}\right)$ and $\delta : H \to \mathbb{R}^+$ is continuous, we have that $a \mapsto f(a,s)\delta_\alpha(s)^{-\frac{1}{2}}$ is in $C_c\left(A, \mathfrak{H}\right)$ which is dense in $L^2(A, \mathfrak{H})$.

So the operator V is well-defined. We show that V is an isometry,

$$
\begin{aligned}
\|Vf\|^2 &= \int_G \langle Vf(s), Vf(s)\rangle ds \\
&= \int_G \int_A \langle (Vf(s))(a), (Vf(s))(a)\rangle dads \\
&= \int_G \int_A \langle f(a,s), f(a,s)\rangle \delta_\alpha(s)^{-1} dads \\
&= \int_{A \ltimes_\alpha G} \langle f(a,s), f(a,s)\rangle d(a,s) \\
&= \|f\|^2.
\end{aligned}
$$

Furthermore $C_c(A\ltimes_\alpha G, \mathfrak{H})$ is dense in $L^2(A\ltimes_\alpha G, \mathfrak{H})$ and $C_c\left(G, L^2(A,\mathfrak{H})\right)$ is dense in $L^2\left(G, L^2(A,\mathfrak{H})\right)$. So V is an isometry of $L^2(A \ltimes_\alpha G, \mathfrak{H})$ into $L^2\left(G, L^2(A,\mathfrak{H})\right)$.

It remains to show surjectivity.

Let $g = \psi(\cdot)\otimes\gamma$, with $\psi \in L^2(A,\mathfrak{H})$ and $\gamma \in C_c(G)$. Clearly $g \in C_c(G, L^2(A,\mathfrak{H})) \subseteq L^2(G, L^2(A,\mathfrak{H})) \cong L^2(A,\mathfrak{H}) \otimes L^2(G)$.

Define $f'(a,s) = \delta_\alpha(s)^{\frac{1}{2}}\gamma(s)\psi(a)$. Then since $\psi \in L^2(A,\mathfrak{H})$, there exists a net (ψ_j) contained in $C_c(A,\mathfrak{H})$ such that $(\psi_j) \to \psi$.

Define $f'_j(a,s) = \delta_\alpha(s)^{\frac{1}{2}}\gamma(s)\psi_j(a)$. Note that since $\psi_j \in C_c(A,\mathfrak{H})$ for all j, we must have $f'_j \in C_c(A \ltimes_\alpha G, \mathfrak{H})$.

We show that $f'_j \to f'$. By the preceding facts we find that

$$
\begin{aligned}
\|f' - f'_j\| &= \int_{A\ltimes_\alpha G} \langle f'(a,s) - f'_j(a,s), f'(a,s) - f'_j(a,s)\rangle d(a,s) \\
&= \int_G \int_A |\gamma(s)|^2 \langle \psi(a) - \psi_j(a), \psi(a) - \psi_j(a)\rangle dads \\
&= \int_G |\gamma(s)|^2 ds \int_A \langle \psi(a) - \psi_j(a), \psi(a) - \psi_j(a)\rangle da \\
&\to 0 \text{ as } j \text{ increases.}
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
(Vf'(s)(a)) &= f'(a,s)\delta_\alpha(s)^0 \\
&= \gamma(s)\psi(a) \\
&= (g(s))(a).
\end{aligned}
$$

So for all simple tensors $g \in L^2(A,\mathfrak{H}) \otimes C_c(G)$, there exists a function $f \in L^2(A \ltimes G, \mathfrak{H})$ such that $(Vf(s))(a) = (g(s))(a)$.

We show that $L^2(A,\mathfrak{H})\otimes C_c(G)$ is dense in $L^2(A,\mathfrak{H})\otimes L^2(G)$. Let $\psi\otimes\gamma \in L^2(A,\mathfrak{H})\otimes L^2(G)$. Then there is a net γ_j contained in $C_c(G)$ such that $\gamma_j \to \gamma$. Then $\|\psi\otimes\gamma - \psi\otimes\gamma_j\| = \|\psi\otimes(\gamma-\gamma_j)\| = \|\psi\|\|\gamma-\gamma_j\| \to 0$ as j increases. The span of simple tensors $\psi\otimes\gamma \in L^2(A,\mathfrak{H})\otimes L^2(G)$ is dense in $L^2(A,\mathfrak{H})\otimes L^2(G)$ and as V has closed range, we have proved surjectivity.

Now, $\mathcal{M} \rtimes_\beta (A \ltimes_\alpha G)$ is generated by operators of the form $\pi_\beta(x), \tilde{\lambda}(a,e)$ and $\tilde{\lambda}(e,s); x \in \mathcal{M}, a \in A, s \in G$; while $(\mathcal{M} \rtimes_\kappa A) \rtimes_\gamma G$ is generated by operators of the form $\pi_\gamma\left(\pi_\kappa(x)\right), \pi_\gamma(\tilde{\lambda}(a))$ and $\tilde{\lambda}(s); x \in \mathcal{M}, a \in A, s \in G$. For $f \in C_c\left(A \ltimes_\alpha G, \mathfrak{H}\right), t \in G, b \in A$ we have:

$$\begin{aligned}
\left(\left(\pi_\gamma\left(\pi_\kappa(x)\right) V f\right)(t)\right)(b) &= \left(\gamma_t^{-1}\left(\pi_\kappa(x)\right)(Vf)(t)\right)(b) \\
&= \left(\pi_\kappa\left(\beta_{(e,t)}^{-1}(x)\right)(Vf)(t)\right)(b) \\
&= \beta_{(b^{-1},e)(e,t^{-1})}(x)((Vf)(t))(b) \\
&= \beta_{(\alpha_t(b^{-1}),t^{-1})}(x)(f(b,t)\delta(t)^{-\frac{1}{2}}) \\
&= \beta_{(b,t)}^{-1}(x)(f(b,t))\delta(t)^{-\frac{1}{2}} \\
&= \left(\pi_\beta(x)f\right)(b,t)\delta(t)^{-\frac{1}{2}} \\
&= \left(\left(V\left(\pi_\beta(x)f\right)\right)(t)\right)(b),
\end{aligned}$$

where we made use of Lemma 2.47. Therefore

$$V^*\pi_\gamma\left(\pi_\kappa(x)\right)V = \pi_\beta(x), \text{ for every } x \in \mathcal{M}.$$

Furthermore

$$\begin{aligned}
\left(\left(\pi_\gamma(\tilde{\lambda}(a))Vf\right)(t)\right)(b) &= \left(\gamma_t^{-1}(\tilde{\lambda}(a))(Vf)(t)\right)(b) \\
&= \left(\tilde{\lambda}\left(\alpha_t^{-1}(a)\right)(Vf)(t)\right)(b) \\
&= (Vf)(t)\left(\alpha_t^{-1}\left(a^{-1}\right)b\right) \\
&= f\left(\alpha_t^{-1}\left(a^{-1}\right)b, t\right)\delta(t)^{-\frac{1}{2}} \\
&= f\left(\left(a^{-1}, e\right)(b,t)\right)\delta(t)^{-\frac{1}{2}} \\
&= (\tilde{\lambda}(a,e)f)(b,t)\delta(t)^{-\frac{1}{2}} \\
&= ((V\tilde{\lambda}(a,e)f)(t))(b),
\end{aligned}$$

where we once again made use of Lemma 2.47. We see that

$$V^*\pi_\gamma(\tilde{\lambda}(a))V = \tilde{\lambda}(a,e), \text{ for every } a \in A.$$

Lastly

$$\begin{aligned}
((\tilde{\lambda}(s)Vf)(t))(b) &= \left((Vf)\left(s^{-1}t\right)\right)(b) \\
&= f\left(b, s^{-1}t\right)\delta(s^{-1}t)^{-\frac{1}{2}} \\
&= f\left(\left(e, s^{-1}\right)(b,t)\right)\delta(s^{-1}t)^{-\frac{1}{2}} \\
&= (\tilde{\lambda}(e,s)f)(b,t)\delta(s^{-1})^{-\frac{1}{2}}\delta(t)^{-\frac{1}{2}} \\
&= ((V\tilde{\lambda}(e,s)f)(t))(b)\delta(s^{-1})^{-\frac{1}{2}}
\end{aligned}$$

so that

$$V^*\tilde{\lambda}(s)V = \tilde{\lambda}(e,s)\delta(s^{-1})^{-\frac{1}{2}} \text{ for every } s \in G.$$

This proves the theorem. □

Corollary 2.49. *$\mathrm{VN}_l(A) \rtimes_{\bar{\alpha}} G$ is spatially isomorphic with $\mathrm{VN}_l\left(A \ltimes_\alpha G\right)$.*

PROOF. This holds as $\mathbb{C} \rtimes_\beta G$ is spatially isomorphic with $\mathrm{VN}_l(G)$ for any locally compact group G, where β is necessarily trivial. We verify this claim.

$\mathbb{C} \rtimes_\beta G$ acts in $L^2(G, \mathbb{C}) \cong \mathbb{C} \otimes L^2(G)$, the space of square Bochner-integrable functions from G to $\mathbb{C}$. But this is exactly the space $L^2(G)$ in which $\mathrm{VN}_l(G)$ acts.

Now consider $\pi_\beta(x) \in \mathbb{C} \rtimes_\beta G$ and $\xi \in L^2(G, \mathbb{C})$. If β is trivial then direct computations show

$$(\pi_\beta(x)\xi)(s) = \pi_\beta(x)\xi(s) = \xi(s) = (\lambda(e_G)\xi)(s)$$

which is just an entry of the von Neumann algebra generated by the left regular representation of G on $L^2(G)$. Therefore the claim holds.

Therefore, if we have a semidirect product $G \ltimes_\alpha H$, then $\mathrm{VN}_l(G \ltimes_\alpha H) \cong \mathbb{C} \rtimes_\beta (G \ltimes_\alpha H)$ with β necessarily trivial. Invoking Theorem 2.48 we find that $\mathbb{C} \rtimes_\beta (G \ltimes_\alpha H) \cong (\mathbb{C} \rtimes_\kappa G) \rtimes_\gamma H$. But since β is trivial, we have that κ is trivial by definition. Therefore

$$\mathrm{VN}_l(G \ltimes_\alpha H) \cong (\mathbb{C} \rtimes_\kappa G) \rtimes_\gamma H \cong \mathrm{VN}_l(G) \rtimes_\gamma H.$$

And once again since β is trivial, $\mathrm{VN}_l(G) \rtimes_\gamma H \cong \mathrm{VN}_l(G) \rtimes_{\bar{\alpha}} H$ where $\bar{\alpha}$ is as defined in Theorem 2.45. Similarly, with β necessarily trivial and given $\mathrm{VN}_l(G) \rtimes_{\bar{\alpha}} H$ we deduce that $\mathrm{VN}_l(G) \rtimes_{\bar{\alpha}} H \cong \mathrm{VN}_l(G \ltimes_\alpha H)$. ☐

CHAPTER 3

Modular theory of group von Neumann algebras

In the chapter we explore modular theory with respect to the left Hilbert algebra $C_c(G)$, where G is a locally compact group. Highlights are the existence of the canonical modular automorphism group and Plancherel weight of a locally compact group, and the existence of the Modular spectrum. The contributions of the author boil down mainly to a case-specific exposition of the relevant modular theory from [**Tak03a**]. Without the constructs of this chapter, establishing Sutherland's structure theorems would be virtually impossible. For example, the hypotheses of Theorems 4.6 and 4.13 point to the importance of this chapter.

3.1. The Plancherel weight and modular operator

We will need some basic results from Modular theory for Sutherland's structure theorems. The necessary theory and definitions are briefly summarised from [**Tak03a**] and [**GL20**], which may be consulted for more details. This section aims to establish the existence of the Plancherel weight, as well as to define the modular automorphism groups and the modular spectrum for factors of type III.

Definition 3.1 (Hilbert algebra, [**Tak03a**, VI, Definition 1.1]). An involutive algebra $\mathfrak{A}$ over $\mathbb{C}$ with involution $\xi \in \mathfrak{A} \mapsto \xi^\sharp \in \mathfrak{A}$ (resp. $\xi \mapsto \xi^b$) is called a left (resp. right) Hilbert algebra if $\mathfrak{A}$ admits an inner product satisfying the following postulates:

(i) Each fixed $\xi \in \mathfrak{A}$ gives rise to a bounded operator $\pi_\ell(\xi) : \eta \in \mathfrak{A} \mapsto \xi \cdot \eta \in \mathfrak{A}$ (resp. $\pi_r(\xi) : \eta \in \mathfrak{A} \mapsto \eta \cdot \xi \in \mathfrak{A}$) by multiplying from the left (resp. right);
(ii) $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\sharp \zeta \rangle$ (resp. $\langle \xi\eta, \zeta \rangle = \langle \xi, \zeta\eta^\flat \rangle$);
(iii) The involution: $\xi \in \mathfrak{A} \mapsto \xi^\sharp \in \mathfrak{A}$ (resp. $\xi \in \mathfrak{A} \mapsto \xi^b \in \mathfrak{A}$) is closable;
(iv) The subalgebra, denoted $\mathfrak{A}^2$, spanned linearly by all possible products $\xi \cdot \eta$ $(\xi, \eta \in \mathfrak{A})$, is dense in $\mathfrak{A}$ with respect to the inner product.

For any left Hilbert algebra $\mathfrak{A}$, we call the von Neumann algebra $\mathscr{R}_\ell(\mathfrak{A}) := \pi_\ell(\mathfrak{A})''$ the **left von Neumann algebra** of $\mathfrak{A}$.

If S is the closure of $\xi \mapsto \xi^\sharp$, the operator given by $\Delta = |S|^2$ is called the **modular operator** of $\mathfrak{A}$. The anti-linear isometry J arising from the polar decomposition $S = J\Delta^{1/2}$ is called **modular conjugation** of $\mathfrak{A}$.

Example 3.2. Let G be a locally compact group with left Haar measure μ. The dense space $C_c(G) \subset L^2(G)$ of all continuous functions on G with compact support is a left Hilbert algebra with respect to the following product, involution and inner product:

$$\left.\begin{aligned} \xi * \eta(s) &= \textstyle\int_G \xi(t)\eta\left(t^{-1}s\right)\, dt; \\ \xi^\sharp(s) &= \Delta_G\left(s^{-1}\right)\overline{\xi\left(s^{-1}\right)}, \quad s \in G; \\ \langle \xi, \eta\rangle &= \textstyle\int_G \xi(s)\overline{\eta(s)}\, ds. \end{aligned}\right\}$$

If we define the involution by

$$\xi^\flat(s) = \overline{\xi\left(s^{-1}\right)}$$

then $C_c(G)$ is a right Hilbert algebra.

Proof. Firstly,

$$\begin{aligned} \|\pi_\ell(\xi)\eta\| &= \left(\int_G \left|\int_G \xi(t)\eta(t^{-1}s)dt\right|^2 ds\right)^{\frac{1}{2}} \\ &\leq \int_G \left(\int_G |\xi(t)||(\lambda(t)\eta)(s)|^2 ds\right)^{\frac{1}{2}} dt \\ &= \|\xi\|_1\|\eta\|. \end{aligned}$$

Here the first inequality is what is known as Minkowski's inequality for integrals, see [**Fol16**, 2.40 Proposition]. So $\pi_\ell(\xi)$ does indeed define a bounded operator for each $\xi \in C_c(G)$. Secondly,

$$\begin{aligned} \langle \xi * \eta, \zeta\rangle &= \int_G \xi * \eta(s)\overline{\zeta(s)}ds \\ &= \int_G \left(\int_G \xi(t^{-1})\eta(ts)\Delta_G(t^{-1})dt\right)\overline{\zeta(s)}ds \\ &= \int_G \xi(t^{-1})\Delta_G(t^{-1})\left(\int_G \eta(s)\overline{\zeta(t^{-1}s)}ds\right)dt \\ &= \int_G \eta(s)\left(\overline{\int_G \Delta_G(t^{-1})\overline{\xi(t^{-1})}\zeta(t^{-1}s)dt}\right)ds \\ &= \langle \eta, \xi^\sharp\zeta\rangle. \end{aligned}$$

Here the second equality follows from an identical form of the product $\xi * \eta$ ($\xi, \eta \in C_c(G)$), see [**Fol16**, p. 56]. Thirdly, let $(\xi_n)_{n\in\mathbb{N}} \subset C_c(G)$ such that $\xi_n \to 0$, then

$$\begin{aligned} \langle \xi_n^\sharp, \eta\rangle &= \int_G \Delta_G(s^{-1})\overline{\xi(s^{-1})\eta(s)}\, d\mu(s) \\ &= \int_G \overline{\xi(s^{-1})\eta(s)}\, d\mu(s^{-1}) \\ &= \langle \eta^\flat, \xi_n\rangle. \end{aligned}$$

Since $\langle \eta^\flat, \xi_n\rangle \to 0$ as $n \to \infty$ for all $\eta \in C_c(G)$, we have that $\xi_n^\sharp \to 0$. This shows $\xi \mapsto \xi^\sharp$ is in fact closable.

Finally, let $\xi \in C_c(G)$ and fix $\epsilon > 0$. Let $\{\psi_U\} \subseteq C_c(G)$ be an approximate identity (see [**Fol16**, 2.4.4 Proposition]) by invoking Urysohn's lemma. Then there exists some $\psi_{U'} \in \{\psi_U\}$ such that $\|\psi_{U'} * \xi - \xi\| \leq \epsilon$. That is, any $\xi \in C_c(G)$ is in the closure of $\{\xi * \eta : \xi, \eta \in C_c(G)\}$.

The proof of the right Hilbert algebra case where $\xi \mapsto \xi^\sharp$ is replaced with $\xi \mapsto \xi^\flat$ follows mutatis mutandis. ☐

Remark 3.3. The product defined in Example 3.2 is known as "convolution" and has been well studied in the literature. For an introduction, see [**Fol16**, pp. 55-60].

The von Neumann algebra $\mathscr{R}_\ell(C_c(G))$ will be important in obtaining a prescription of the Plancherel weight. We proceed to investigate the notion of full left Hilbert algebras.

Suppose we are given a left Hilbert algebra $\mathfrak{A}$ with completion $\mathfrak{H}$. Let S with domain $\mathfrak{D}(S)$ be the closure of $\xi \in \mathfrak{A} \mapsto \xi^\sharp \in \mathfrak{A}$ (it is a closed anti-linear operator). We call a vector $\eta \in \mathfrak{H}$ right bounded if

$$\sup\{\|\pi_\ell(\xi)\eta\| : \xi \in \mathfrak{A}, \|\xi\| \leq 1\} = c < +\infty.$$

We denote the set of all right bounded vectors by $\mathfrak{B}'$. Note that $\eta \in \mathfrak{B}'$ if and only if there exists an operator $\Lambda_r(\eta) \in \mathcal{B}(\mathfrak{H})$ uniquely determined by η, such that $\Lambda_r(\eta)\xi = \pi_\ell(\xi)\eta$, if $\xi \in \mathfrak{A}$. We let

$$\mathfrak{D}^\flat := \{\eta \in \mathfrak{H} : \xi \in \mathfrak{D}(S) \mapsto \langle \eta, S\xi \rangle \text{ is bounded }\};$$
$$\mathfrak{A}' := \mathfrak{B}' \cap \mathfrak{D}^\flat.$$

Similarly, we call a vector $\xi \in \mathfrak{H}$ called left bounded if

$$\sup\{\|\Lambda_r(\eta)\xi\| : \eta \in \mathfrak{A}', \|\eta\| \leq 1\} = c < +\infty.$$

We denote by $\mathfrak{B}$ the set of all left bounded vectors. Note that $\xi \in \mathfrak{B}$ if and only if there exists an operator $\Lambda_\ell(\xi) \in \mathcal{B}(\mathfrak{H})$ uniquely determined by ξ, such that $\Lambda_\ell(\xi)\eta = \Lambda_r(\eta)\xi$, for $\eta \in \mathfrak{A}'$.

The preceding facts show that we can extend the product of $\mathfrak{A}$ to a wider class of functions of $\mathfrak{H}$ by the prescription $\xi \cdot \eta := \Lambda_\ell(\xi)\eta$ with $\eta \in \mathfrak{B}$ and $\eta \in \mathfrak{H}$. We let

$$\mathfrak{A}'' := \mathfrak{B} \cap \mathfrak{D}(S).$$

Proposition 3.4 ([**VD74**, Theorem 5.2]). *$\mathfrak{A}''$ is a left Hilbert algebra with the product defined by $\xi \cdot \eta := \Lambda_\ell(\xi)\eta$ with $\eta \in \mathfrak{B}$ and $\eta \in \mathfrak{H}$ and involution S. Furthermore, $\mathscr{R}_\ell(\mathfrak{A}) = \mathscr{R}_\ell(\mathfrak{A}'')$ and $\mathfrak{A} \subseteq \mathfrak{A}'' = (\mathfrak{A}'')''$.*

Definition 3.5 (Full left Hilbert algebra). We say a left Hilbert algebra $\mathfrak{A}$ is **full**, if $\mathfrak{A} = \mathfrak{A}''$. Some authors also call them **achieved** ([**VD74**, Definition 5.2]).

The left von Neumann algebra of such a left Hilbert algebra admits a normal, faithful, semifinite weight via:

Proposition 3.6 ([**Tak03a**, Theorem 2.5]). *Let $\mathfrak{A}$ be a full left Hilbert algebra with left von Neumann algebra $\mathscr{R}_\ell(\mathfrak{A})$, then*

$$\varphi_\ell(x) = \begin{cases} \|\xi\|^2 & \text{if } x^{1/2} = \pi_\ell(\xi),\ \xi \in \mathfrak{A}; \\ +\infty & \text{otherwise.} \end{cases}$$

for all $x \in \mathscr{R}_\ell(\mathfrak{A})_+$ defines a normal, faithful, semifinite weight.

By Proposition 3.4, we obtain the full left Hilbert algebra $C_c(G)''$. Let $\xi \in C_c(G)''$, that is to say ξ is left bounded and $\xi \in \mathfrak{D}(S)$. By definition, $\xi \in L^2(G)$ is left bounded if for all $\eta \in C_c(G)'$, that is all η such that

$$\zeta \mapsto \zeta * \eta \text{ is bounded for all } \zeta \in C_c(G) \tag{3.1}$$

$$\zeta \mapsto \langle \eta, S\zeta \rangle \text{ is bounded for all } \zeta \in \mathfrak{D}(S) \tag{3.2}$$

we have that $\eta \mapsto \Lambda_r(\eta)\xi$ is bounded. If we choose $\xi \in \mathfrak{D}(S)$ then $\Lambda_r(\eta)\xi = \Lambda_\ell(\xi)\eta$. Since $C_c(G) \subseteq C_c(G)''$, we can by the preceding fact, obtain each operator $\Lambda_r(\eta)$ $(\eta \in C_c(G)')$ as the norm limit of operators $\Lambda_r(\eta_n)$ with $(\eta_n) \subseteq C_c(G)''$. So we can give a more intuitive description of $C_c(G)''$; namely, $\xi \in C_c(G)''$ if and only if $\eta \mapsto \Lambda_r(\eta)\xi$ is bounded for all $\eta \in C_c(G)$.

Any unitary representation ϖ of a locally compact group G, with representation space $\mathfrak{H}_\varpi$, also allows us to define a representation of $L^1(G)$ on $\mathfrak{H}_\varpi$. For every $\xi \in L^1(G)$, we define a bounded operator $\varpi(\xi)$ on $\mathfrak{H}_\varpi$ pointwise as follows: For any vector $u \in \mathfrak{H}_\varpi$ we define $\varpi(\xi)u$ by describing its inner product with respect to any vector $v \in \mathfrak{H}_\varpi$; specifically, we define $\varpi(\xi)$ by

$$\langle \varpi(\xi)u, v \rangle = \int_G \xi(s) \langle \varpi(s)u, v \rangle ds. \tag{3.3}$$

Indeed, Equation (3.3) ensures that the map $u \mapsto \varpi(\xi)u$ is linear and that $\|\varpi(\xi)\| \leq \|\xi\|_1$. The sceptical reader may consult [**Fol16**, Appendix 4, Unitary representations]. We write

$$\varpi(\xi) = \int_G \xi(s)\varpi(s)\, ds$$

for such operators. It is known that operators of the form $\varpi(\xi)$ $\xi \in C_c(G)$ are strong limits of sums $\sum_i \xi(s)\varpi(s_i)\mu_G(E_i)$ where $\{E_i\}$ partitions $\operatorname{supp}(\xi)$ and $s_i \in E_i$, see [**Fol16**, 3.12 Theorem]. These sums are contained in the algebra generated by $\varpi(G)$.

We consider an example that is relevant to our current setting.

Example 3.7. Let G be a locally compact group with left regular representation $g \mapsto \lambda^G(g)$. Then

$$\lambda^G(\xi)\eta = \int_G \xi(t)\lambda^G(t)\eta(\cdot)\, dt = \xi * \eta(\cdot), \tag{3.4}$$

with convolution predefined in Example 3.2. Note that (3.4) may be interpreted pointwise, or as a L^2-valued integral [**Fol16**, Appendix 4, Unitary representations].

Example 3.7 along with the facts preceding it shows that

$$\pi_\ell(\xi) = \int_G \xi(s)\lambda(s)\, ds,$$

and that $\mathscr{R}_\ell(C_c(G)) \subseteq \mathrm{VN}_l(G)$.

Let $g, s \in G$, $\xi \in C_c(G)$ and $\zeta \in L^2(G)$. Then for $\xi * \zeta \in L^2(G)$

$$\xi * \zeta(g^{-1}s) = \int_G \xi(g^{-1}st^{-1})\zeta(t)\Delta_G(t^{-1})\, dt = \lambda^G(g)\xi * \zeta(s),$$

where the first equality is due to an equivalent form of the convolution of functions ([**Fol16**, p. 56]). Fix $\zeta \in C_c(G)'$. Recall that $\Lambda_r(\zeta)\xi = \xi * \zeta$ for all $\xi \in C_c(G)$. Let $g \in G$, $\xi, \eta \in L^2(G)$ and $(\xi_n) \subset C_c(G)$ such that $\xi_n \to \xi$. Since $\Lambda(\zeta) \in \mathcal{B}(L^2(G))$, we find that

$$\begin{aligned}\lim_n \langle \lambda^G(g)\Lambda_r(\zeta)\xi_n, \eta\rangle &= \lim_n \int_G \xi_n * \zeta(g^{-1}s)\overline{\eta(s)}\, ds\\ &= \lim_n \int_G \lambda^G(g)\xi_n * \zeta(s)\overline{\eta(s)}\, ds\\ &= \lim_n \langle \Lambda_r(\zeta)\lambda^G(g)\xi_n, \eta\rangle\\ &= \langle \Lambda_r(\zeta)\lambda^G(g)\xi, \eta\rangle.\end{aligned}$$

That is to say $\Lambda_r(\zeta)$ and $\lambda^G(g)$ commute. Thus, $\mathrm{VN}_l(G)' \supseteq \Lambda_r(C_c(G)')''$. But by [**Tak03a**, VI, Lemma 1.14], we have $\mathscr{R}_\ell(C_c(G))' = \Lambda_r(C_c(G)')''$. So we have shown that $\mathrm{VN}_l(G) \subseteq \mathscr{R}_\ell(C_c(G))$.

We summarise the facts hitherto obtained in the following theorem:

Theorem 3.8. *Let G be a locally compact group. Then $\mathrm{VN}_l(G) = \mathscr{R}_\ell(C_c(G)'')$, where the left Hilbert algebra $C_c(G)''$ is full. Hence $\mathrm{VN}_l(G)$ admits a normal, faithful, semifinite weight called the Plancherel weight via the prescription*

$$\phi^G(x^*x) = \begin{cases} \|\xi\|_2^2 & \textit{if } x = \Lambda_\ell(\xi),\ \xi \in C_c(G)'';\\ +\infty & \textit{otherwise.}\end{cases}$$

Theorem 3.9. *Given locally compact groups G and N, let α be a continuous group action of N on G and $\bar\alpha$ be the corresponding action of N on $\mathrm{VN}_l(G)$. Then $\phi^G(\bar\alpha_n(x)) = \delta_\alpha(n)^{-1}\phi^G(x)$; $x \in \mathrm{VN}_l(G)_+$.*

Proof. Consider the left Hilbert algebra $C_c(G)$. It is an exercise to show that $\pi_\ell(\xi)^*\pi_\ell(\xi) = \pi_\ell(\xi^\sharp * \xi)$, and even more specifically that $\pi_\ell(\xi)^* = \pi_\ell(\xi^\sharp)$. We note that in order to prove the hypothesis, we need only prove $\phi^G(\bar\alpha(x)) = \delta_\alpha(1)^{-1}\phi^G(\bar\alpha(x))$ for $x \in \mathrm{VN}_l(G)_+$ of the form $x = \pi_\ell(\xi^\sharp * \xi)$, $(\xi \in C_c(G))$. This follows from the definition of the Plancherel weight and the fact that the product defined on $C_c(G)''$ is an extension of convolution (on $C_c(G)$), so that arguing for the hypothesis in greater generality would in essence be identical to the proof below.

Let $\xi \in C_c(G)$, then

$$\begin{aligned}[\bar\alpha_n(\pi_\ell(\xi))\eta](s) &= [u_\alpha(n)\pi_\ell(\xi)u_\alpha(n)^*\eta](s)\\ &= \delta_\alpha(n)^{-1/2}[\xi * u_\alpha(n)^*\eta](\alpha_n^{-1}(s))\\ &= \delta_\alpha(n)^{-1/2}\int_G \xi(t)u_\alpha(n)^*\eta(t^{-1}\alpha_n^{-1}(s))\, dt\\ &= \delta_\alpha(n)^{-1}\int_G \delta_\alpha(n)\xi(t)\eta(\alpha_n(t^{-1})s)\, dt\\ &= \delta_\alpha(n)^{-1/2}\int_G \delta_\alpha(n)^{-1/2}\xi(\alpha_n^{-1}(t))\eta(t^{-1}s)\, dt\\ &= \delta_\alpha(n)^{-1/2}[u_\alpha(n)\xi * \eta](s)\\ &= \delta_\alpha(n)^{-1/2}[\pi_\ell(u_\alpha(n)\xi)\eta](s).\end{aligned}$$

That is to say, $\bar{\alpha}_n(\pi_\ell(\xi^\sharp * \xi)) = \delta_\alpha(n)^{-1}\pi_\ell((u_\alpha(n)\xi)^\sharp * u_\alpha(n)\xi)$. So by the definition of the Plancherel weight:

$$\begin{aligned}\phi^G(\bar{\alpha}_n(\pi_\ell(\xi^\sharp * \xi))) &= \delta_\alpha(n)^{-1}\phi^G(\pi_\ell((u_\alpha(n)\xi)^\sharp * u_\alpha(n)\xi)) \\ &= \delta_\alpha(n)^{-1}\|u_\alpha(n)\xi\|_2^2 \\ &= \delta_\alpha(n)^{-1}\|\xi\|_2^2 \\ &= \delta_\alpha(n)^{-1}\phi^G(\pi_\ell(\xi^\sharp * \xi)).\end{aligned}$$

□

A left Hilbert algebra $\mathfrak{A}$, when equipped with a complex one parameter group $\{U(c) : c \in \mathbb{C}\}$, can potentially be considered a Tomita algebra (as defined in [**Tak03a**, VI, Definition 2.1]). Tomita algebras enjoy the property that their modular operator Δ is given by the closure of $U(-i)$, see [**Tak03a**, Theorem 2.2 (ii)].

Definition 3.10 (Canonical modular operator of G). Let G be a locally compact group. We define an operator on $L^2(G)$ by the prescription

$$(\Delta_G\, \xi)(g) = \delta_G(g)\xi(g),\ \xi \in \mathfrak{D}(\Delta_G),$$

where we set

$$\mathfrak{D}(\Delta_G) = \left\{\xi \in L^2(G) : \int_G \delta_G(g)^2|\xi(g)|^2\, \mathrm{d}\, g < \infty\right\}.$$

We call this operator the **(canonical) modular operator** of G.

The definition above defines a non-singular, self-adjoint operator Δ_G on $L^2(G)$. It is an exercise to show that the left Hilbert algebra $C_c(G)$ equipped with complex one parameter group $\{\Delta_G^{ic}$: $c \in \mathbb{C}\}$, is a Tomita algebra. We do not prove this, but accept the statement by the authority of Takesaki, [**Tak03a**, VII, Proposition 3.1]. (Note, there is a typo in [**Tak03a**, VII, Proposition 3.1], "Δ_G^α" should read "$\Delta_G^{i\alpha}$".) The immediately preceding argument shows that Δ_G is in fact the modular operator of $C_c(G)$. The uniqueness of the polar decomposition ensures that the modular conjugation of $C_c(G)$ is given by

$$j = \Delta_G(s^{-1})^{1/2}\overline{\xi(s^{-1})}.$$

From [**Tak03a**, VI, Theorem 1.19] we know that $\Delta_G^{it}\ \mathrm{VN}_l(G)\ \Delta_G^{-it} = \mathrm{VN}_l(G)$ for all $t \in \mathbb{R}$. So the prescription $x \in \mathrm{VN}_l(G) \mapsto \Delta_G^{it}\ x\ \Delta_G^{-it}$ defines a (strong) continuous automorphism of $\mathrm{VN}_l(G)$ for each $t \in \mathbb{R}$.

Definition 3.11 (Canonical modular automorphism group). We let $\sigma_t(x) := \Delta_G^{it}\ x\ \Delta_G^{-it}$ where $x \in \mathrm{VN}_l(G)$ and $t \in \mathbb{R}$. Then $\sigma : t \mapsto \sigma_t(\cdot)$ defines a group action of $\mathbb{R}$ on $\mathrm{VN}_l(G)$ in the sense of Definition 2.24. We call $\{\sigma_t\}_{t\in\mathbb{R}}$ the **(canonical) modular automorphism group** of G.

We reserve Δ_G and σ_t for the purpose of denoting the modular operator and modular automorphism group respectively.

3.2. Modular automorphism groups

Let $\mathcal{M} \subset \mathcal{B}(\mathfrak{H})$ be a von Neumann algebra. Suppose that $\mathcal{M}$ is equipped with a weight φ. We let

$$\begin{aligned} \mathfrak{p}_\varphi &:= \{x \in \mathcal{M} : \varphi(x) < \infty\}, \\ \mathfrak{n}_\varphi &:= \{x \in \mathcal{M} : x^*x \in \mathfrak{p}_\varphi\}, \\ N_\varphi &:= \{x \in \mathcal{M} : \varphi(x^*x) = 0\}. \end{aligned}$$

Then N_φ is a left ideal of $\mathfrak{n}_\varphi$. Via the canonical quotient map $\eta_\varphi : x \mapsto \eta_\varphi(x) = x + N_\varphi \in \mathfrak{n}_\varphi/N_\varphi$, we define the sesquilinear functional

$$\langle \eta_\varphi(x), \eta_\varphi(y)\rangle = \varphi(y^*x), \text{ for all } x, y \in \mathfrak{n}_\varphi, \tag{A}$$

which makes $\mathfrak{n}_\varphi/N_\varphi$ a pre-Hilbert space. Let $\mathfrak{H}_\varphi$ denote the completion thereof.

We now make the additional assumptions that φ is also normal and semifinite. Then there exists a normal, semifinite *-representation π_φ of $\mathcal{M}$ realising it as a subalgebra of $\mathcal{B}(\mathfrak{H}_\varphi)$, such that $\pi_\varphi(a)\eta_\varphi(x) = \eta_\varphi(ax)$. For a proof see [**Tak03a**, VII, Proposition 1.4].

If we insist that φ be faithful, then $N_\varphi = \{0\}$, and π_φ becomes a *-isomorphism. Now, it turns out that if we equip the subspace $\eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ with the operations:

$$(\eta_\varphi(x), \eta_\varphi(y)) \mapsto \eta_\varphi(xy), \tag{B}$$

$$\eta_\varphi(x) \mapsto \eta_\varphi(x^*), \tag{C}$$

then prescriptions (B) and (C) together with (A) turn $\eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ into a full left Hilbert algebra, see [**Tak03a**, VII, Theorem 2.6]. We denote by S the closed extension of $\eta_\varphi(x) \mapsto \eta_\varphi(x^*)$.

Definition 3.12. Given a von Neumann algebra $\mathcal{M}$ with a normal, semifinite, faithful weight φ. Let S be as above, then the **modular operator** Δ_φ associated with $(\mathcal{M}, \varphi)$ is given by $\Delta_\varphi = |S|^2$. The anti-linear isometry J_φ arising from the polar decomposition $S = J_\varphi\ \Delta_\varphi^{1/2}$, is called the **modular conjugation** associated with $(\mathcal{M}, \varphi)$.

Two important results from Takesaki, [**Tak03a**, VI, Theorem 1.19] and [**Tak03a**, VII, Theorem 2.6] imply that $\Delta_\varphi^{it}\ \pi_\varphi(\mathcal{M})\ \Delta_\varphi^{-it} = \pi_\varphi(\mathcal{M})$ for all $t \in \mathbb{R}$. Therefore the prescription

$$\sigma_t^\varphi(x) = \pi_\varphi^{-1}(\Delta_\varphi^{it}\ \pi_\varphi(x)\ \Delta_\varphi^{-it}), \text{ for } x \in \mathcal{M},\ t \in \mathbb{R},$$

makes sense.

Definition 3.13 (Modular automorphism group). The map $t \mapsto \sigma_t^\varphi$ defines a continuous action of $\mathbb{R}$ on $\mathcal{M}$ in the sense of Definition 2.24. We say that $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ is the **modular automorphism group** associated with the pair $(\mathcal{M}, \varphi)$.

Definition 3.14 (Modular spectrum [**Tak03a**, XII, Definition 1.5]). Let $\mathcal{M}$ be a factor of type III. Let $\mathcal{W}$ be the set of faithful, normal, semifinite weights on $\mathcal{M}$. We define

$$S(\mathcal{M}) = \bigcap_{\varphi\in\mathcal{W}} \mathrm{Sp}(\Delta_\varphi),$$

where $\mathrm{Sp}(\Delta_\varphi)$ is the spectrum of Δ_φ . We call $S(\mathcal{M})$ the **modular spectrum** of $\mathcal{M}$.

The modular spectrum is important because it grants us access to Connes' classification of type III factors.

Proposition 3.15 (Connes classification of type III factors, [**Con75b**]). *Any factor $\mathcal{M}$ belongs to one of the following three classes:*

(i) III_λ, $\lambda \in (0,1)$ *i.e.* $S(\mathcal{M}) = \overline{\{\lambda^n : n \in \mathbb{Z}\}}$;
(ii) III_0, *i.e.* $S(\mathcal{M}) = \{0,1\}$;
(iii) III_1 *i.e.* $S(\mathcal{M}) = [0,+\infty)$.

Definition 3.16 (Strictly ergodic, [**erg**]). Let X be a topological space, G a topological group, and equip $G \times X$ with the product topology. Let α be a continuous left action from G on X (i.e. $\alpha : G \times X \to X$ is continuous). We call the triple (X, G, α) **strictly ergodic** if:

(i) it has a unique, invariant, normalized, regular, Borel measure μ;
(ii) $\mu(U) > 0$ for any non-empty open set U;
(iii) for any bounded continuous function f its time averages along any trajectory tend to $\int f d\mu$.

Building on the above, work of Connes and Takesaki ([**Con75b**, p. 89]) yielded the following structural classification.

Proposition 3.17 (Structural classification). *Any factor $\mathcal{M}$ belongs to one of the following three classes:*

(i) The crossed product $N(\theta)$ of a factor N of type II_∞ by an automorphism θ scaling traces by λ is a factor of type III_λ, and any factor of type III_λ is obtained this way.
(ii) Let N be a von Neumann algebra of type II_∞, and $\theta \in \mathrm{Aut}(N)$ be a strict contraction with respect to some trace and strictly ergodic on the centre $\mathcal{Z}(N)$ of N. Then the crossed product $N(\theta)$ is a factor of type III_0. Any factor of type III_0 arises this way
(iii) Let N be a factor of type II_∞, $(\theta_t)_{t\in\mathbb{R}}$ a one-parameter group of automorphisms of N, with $\tau \circ \theta_t = e^{-t}\tau$ for any normal trace τ; then the continuous crossed product of N by $(\theta_t)_{t\in\mathbb{R}}$ is a factor of type III_1. Any factor of type III_1 arises this way and the decomposition is unique (as for factors of type III_λ).

3.3. Comparing modular operators

After the preceding developments in this chapter, we now ask, given a locally compact group G, how Δ_G compares to Δ_{ϕ_G} . Fortunately, this is not difficult to answer via the well-developed theory of the standard form of von Neumann algebras.

Definition 3.18 (Standard form [**Haa75**, Theorem 1.6]). Consider the quadruple $(\mathcal{M}, \mathfrak{H}, J, \mathscr{P})$, where $\mathcal{M}$ is a von Neumann algebra on a Hilbert space $\mathfrak{H}$, with antilinear isometric involution $J : \mathfrak{H} \to \mathfrak{H}$ and a selfdual cone $\mathscr{P}$ in $\mathfrak{H}$. We call $(\mathcal{M}, \mathfrak{H}, J, \mathscr{P})$ a **standard form** of $\mathcal{M}$ if it satisfies the following properties:

(i) $JzJ = z^*$ for all z in the centre of $\mathcal{M}$,
(ii) $J\mathcal{M}J = \mathcal{M}'$ (the commutant of $\mathcal{M}$),
(iii) $J\xi = \xi$ for all $\xi \in \mathscr{P}$,
(iv) $x(JxJ)\mathscr{P} \subseteq \mathscr{P}$ for all $x \in \mathcal{M}$.

(Note that when we say the cone $\mathscr{P}$ (Definition 2.13) is selfdual, we mean that $\xi \in \mathscr{P}$ if and only if $\langle \xi, \zeta \rangle \geq 0$ for all $\zeta \in \mathscr{P}$.)

Proposition 3.19 ([**Haa75**, Theorem 2.3]). *The standard form of a von Neumann algebra $\mathcal{M}$ is unique in the following sense: If*

$$(\mathcal{M}, \mathfrak{H}, J, \mathscr{P}) \text{ and } \left(\widetilde{\mathcal{M}}, \tilde{\mathfrak{H}}, \tilde{J}, \widetilde{\mathscr{P}}\right)$$

*are both standard forms, and there exists some *-isomorphism $\alpha : \mathcal{M} \to \widetilde{\mathcal{M}}$, then there exists a unique unitary operator $u : H_0 \to \widetilde{H}_0$ such that*

(i) $\alpha(x) = uxu^*$ *for* $x \in \pi_0(\mathcal{M})$;
(ii) $\tilde{J} = uJu^*$;
(iii) $\widetilde{\mathscr{P}} = u\mathscr{P}$.

Let $\mathfrak{A}$ be a left von Neumann algebra with modular conjugation. Haagerup showed that ([**Haa75**, Theorem 1.1]) the set defined by

$$\mathscr{P}_J = \overline{\{\xi \cdot J\xi : \xi \in \mathfrak{A}\}},$$

where the closure is taken in the completion of $\mathfrak{A}$, is in fact a selfdual cone.

So Proposition 3.19 is applicable to the standard forms

$$\left(\mathrm{VN}_l(G), L^2(G), j, \mathscr{P}_j\right) \text{ and } \left(\pi_{\phi^G}(\mathrm{VN}_l(G)), \mathfrak{H}_{\phi^G}, J_{\phi^G}, \mathscr{P}_{J_{\phi^G}}\right),$$

with the *-isomorphism of course given by π_{ϕ^G}. Then

$$uJ_{\phi^G}u^*u\,\Delta_{\phi^G}^{1/2}\,u^* = ju\,\Delta_{\phi^G}^{1/2}\,u^*.$$

The uniqueness of the polar decomposition then ensures that $\Delta_G^{1/2} = u\,\Delta_{\phi^G}^{1/2}\,u^*$. Equivalently $\Delta_G = u\,\Delta_{\phi^G}\,u^*$.

CHAPTER 4

Sutherland's first structure theorem

In this chapter, we will be looking at the first of Sutherland's two type structure theorems of group von Neumann algebras [**Sut78**, Theorem 3.4]. As such, the work presented here is a summary of the work done in [**Sut78**, pp. 229-239] using more contemporary notation (especially notation relating to crossed products) and attempts to simplify and expand upon Sutherland's exposition. For example, Theorem 4.20 was not proved in [**Sut78**]. The author has taken care to make sure that most of the necessary knowledge, to establish the first type structure theorem, is available; if not, the relevant references (of which [**Sut78**] is void) have been made.

Let G be a separable locally compact group. Throughout this chapter, we insist that $\Delta_G(G) = \mathbb{R}_+$. Note that Theorem 1.27 is applicable in this context; that is, there exists a subgroup $L = \{g_t : t \in \mathbb{R}\}$ such that $\Delta_G(g_t) = e^t$.

We reserve H to denote $\ker \Delta_G$. We define the action α of G on H by the prescription

$$\alpha_g(h) = ghg^{-1}. \tag{4.1}$$

We will denote left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ by

$$t \mapsto v(t). \tag{4.2}$$

We also define a bounded operator $u(t)$ ($t \in \mathbb{R}$) on $L^2(\mathbb{R})$ via the prescription

$$(u(t)\xi)(s) = e^{-ist}\xi(s) \text{ for all } \xi \in L^2(\mathbb{R}). \tag{4.3}$$

Some notational conventions: Let G be a locally compact group. For an element x of a von Neumann algebra $\mathcal{M}$, we define an automorphism by the prescription

$$[\mathrm{Ad}(x)](y) = xyx^* \text{ for all } y \in \mathcal{M}.$$

If β is a group action of G on a group A. Then $\bar{\beta}$ denotes the corresponding group action of G on $\mathrm{VN}_l(A)$ defined in Theorem 2.45. Lastly $\tilde{\beta}$ denotes the restriction of $\bar{\beta}$ to the centre of $\mathrm{VN}_l(A)$ denoted $\mathcal{Z}(A)$.

If we consider the action α of G on H (Equation (4.1)), we see that $\tilde{\alpha}_g$ is equal to the identity automorphism, if $g \in H$. Therefore, we think of $\tilde{\alpha}$ as an action of G/H on H. We also write $\bar{\alpha}_t$ for $\bar{\alpha}_{g_t}$ on $\mathrm{VN}_l(H)$, and $\tilde{\alpha}_t$ for $\tilde{\alpha}_{g_t}$ on $\mathcal{Z}(H)$.

Definition 4.1 (Fixed point subalgebra). Given a continuous group action β of G on $\mathcal{M}$, we define

$$\mathcal{M}^\beta := \{x \in \mathcal{M} : \beta_g(x) = x \text{ for all } g \in G\}.$$

The subspace $\mathcal{M}^{\beta}$ is a von Neumann algebra in its own right and is called the **fixed point subalgebra** of $\mathcal{M}$ with respect to β.

We summarise some facts from [**Tat72**]. Let G be a separable locally compact group, then it is true that $\mathcal{Z}(G)$ is equal to the centre of $\mathrm{VN}_r(G)$. This has the particular consequence that there exists a standard measure space (Γ_G, μ_G) such that we have access to the central decomposition

$$\{L^2(G), \lambda^G, \rho^G, J_G\} \cong \int_{\Gamma_G}^{\oplus} \{\mathfrak{H}(\omega), \lambda_{\omega}^G, \rho_{\omega}^G, J_G(\omega)\}\, d\mu_G(\omega),$$

where J_G is an involution on $L^2(G)$ defined by $(J_G f)(h) = \overline{f(h^{-1})}$.

Definition 4.2 (Reduced quasi-dual). The measure space (Γ_G, μ_G) is called the **reduced quasi-dual of** G and may be considered as a set of factor representations $\omega \equiv \{\mathcal{H}(\omega), \lambda_{\omega}^G\}$ of G.

Let $\lambda^G = \int_{\Gamma_G}^{\oplus} \lambda_{\omega}^G\, d\mu_G(\omega)$ and $\lambda^H = \int_{\Gamma_H}^{\oplus} \lambda_{\gamma}^G\, d\mu_H(\gamma)$ denote the central decompositions of λ^G and λ^H respectively. Given any representation π of H, we define $\check{\alpha}_g \pi(h) = \pi(g^{-1}hg)$ for $h \in H$ and $g \in G$. In the current context, we define

$$\check{\alpha}_g \lambda^H(h) = \lambda^H(ghg^{-1}) \text{ for } g \in G,\ h \in H. \tag{4.4}$$

Since $\check{\alpha}_g \lambda^H$ is unitarily equivalent with λ^H, we may consider $\check{\alpha}_g$ as a group action on (Γ_H, μ_H). We write $\check{\alpha}_t$ for $\check{\alpha}_{g_t}$.

We will need a result known as the Ergodic decomposition of a measure. Heuristically, it asserts that every invariant measure on a dynamical system is a mixture of ergodic measures. We make this more precise.

Definition 4.3 (Disintegration and measurable families of probability measures, [**Hoc13**, pp. 24-26]). Given a measurable space $(X, \mathfrak{B})$, a family $\{\nu_x\}_{x \in X}$ of probability measures on $(Y, \mathcal{C})$ is measurable if for every $E \in \mathcal{C}$ the map $x \mapsto \nu_x(E)$ is **measurable** (with respect to $\mathfrak{B}$). Equivalently, for every bounded measurable function $f : Y \to \mathbb{R}$, the map $x \mapsto \int f(y) d\nu_x(y)$ is measurable.

We define an equivalence relation on X with respect to the σ-algebra $\mathfrak{B}$ by letting $x \sim_{\mathfrak{B}} y$ iff $\chi_E(x) = \chi_E(y)$ for all $E \in \mathfrak{B}$. We denote by $\mathfrak{B}(x)$ the equivalence class of $x \in X$. Let $\mathcal{E} \subseteq \mathfrak{B}$ be a sub-σ-algebra. If there exists a $\mathcal{E}$-measurable family of probability measures such that μ_y is supported on $\mathcal{E}(y)$ and

$$\mu = \int \mu_y\, d\mu(y),$$

then the representation $\mu = \int \mu_y d\mu(y)$ is called a **disintegration** of μ over $\mathcal{E}$.

Proposition 4.4 (Ergodic decomposition, [**Hoc13**, Theorem 3.8.1]). *Let $(X, \mathfrak{B}, \mu, T)$ be a measure preserving system (see* [**Hoc13**, Definition 2.1.1]*) on a Borel space. Let $\mathcal{I} \subseteq \mathfrak{B}$ denote the σ-algebra of T-invariant measurable sets. If $L^1(X, \mathcal{I}, \mu)$ is separable, there exists a countably generated μ-dense sub-σ-algebra $\mathcal{I}_0$ of $\mathcal{I}$, such that $L^1(X, \mathcal{I}_0, \mu) = L^1(X, \mathcal{I}, \mu)$. Furthermore, there exists an $\mathcal{I}_0$-measurable (and in particular $\mathcal{I}$-measurable) disintegration $\mu = \int \mu_x d\mu(x)$ of*

μ such that a.e. μ_y is T-invariant, ergodic, and supported on $\mathcal{I}_0(y)$. Furthermore, the representation is unique in the sense that if $\{\mu'_y\}$ is any other family with the same properties then $\mu_y = \mu'_y$ for μ-a.e. y.

The following theorem will be crucial to later developments.

Proposition 4.5. *Let σ denote the canonical modular automorphism of* $\mathrm{VN}_l(G)$, *then:*

- *(i) The fixed point subalgebra $\mathrm{VN}_l(G)^\sigma$ of $\mathrm{VN}_l(G)$ under $\{\sigma_t : t \in \mathbb{R}\}$ is $\left\{\lambda^G(h) : h \in H\right\}''$. Thus $\mathcal{Z}(G) \subset \left\{\lambda^G(h) : h \in H\right\}''$.*
- *(ii) The map $(g, \gamma) \in G \times \Gamma_H \mapsto \check{\alpha}_g(\gamma) \in \Gamma_H$ is Borel. Under the identification of $L^\infty(\Gamma_H, \mu_H)$ with $\mathcal{Z}(H)$, $\check{\alpha}_g$ is a point realization of $\tilde{\alpha}_g$. (That is to say $\check{\alpha}_g$ realises $\tilde{\alpha}_g$ modulo null sets, see* [**Mac62**, p. 327]*)*
- *(iii) There is an algebraic isomorphism κ carrying $\mathrm{VN}_l(G)^\sigma$ to $\mathrm{VN}_l(H)$ such that*
  - *(a) $\kappa(\lambda^G(h)) = \lambda^H(h)$,*
  - *(b) $\kappa(\mathcal{Z}(G)) \subset \mathcal{Z}(H)$, and $\kappa(\mathcal{Z}(G)) = \mathcal{Z}(H)^{\tilde{\alpha}}$.*
- *(iv) Let $\mu_H = \int_X^\oplus \mu_\zeta^H dm(\zeta)$ be the ergodic decomposition of μ_H (with respect to $\{\check{\alpha}_g : g \in G\}$) and $\lambda_\zeta^H = \int_{\Gamma_H}^\oplus \lambda_\gamma^H d\mu_\zeta^H(\gamma)$. Then $\lambda^G = \int_X^\oplus \mathrm{Ind}_H^G \lambda_\zeta^H \, dm(\zeta)$ is the central decomposition of λ^G, so that (X, m) is measure isomorphic with (Γ_G, μ_G).*

Proposition 4.5 is a summary of facts from [**Sut78**, p. 128] (originally proved by Takesaki [**M70**]). The construction $\mathrm{Ind}_H^G \lambda_\zeta^H$ is described in Theorem 4.13.

Theorem 4.6 ([**Sut78**, Theorem 3.1]). *Suppose $\Delta_G(G) = \mathbb{R}_+$. Then there is an algebraic isomorphism of the crossed product $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ with $\mathrm{VN}_l(H) \bar{\otimes} \mathcal{B}(L^2(\mathbb{R}))$, which carries the automorphism group $\{\theta_t : t \in \mathbb{R}\}$ dual to $\{\sigma_t : t \in \mathbb{R}\}$ to the automorphism group $\{\bar{\alpha}_t \otimes \mathrm{Ad}\, v(t) : t \in \mathbb{R}\}$. Thus the restriction $\left\{\tilde{\theta}_t : t \in \mathbb{R}\right\}$ of $\{\theta_t : t \in \mathbb{R}\}$ to the centre of $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ is equivalent to the action $\{\tilde{\alpha}_t : t \in \mathbb{R}\}$ of $\mathbb{R}$ on $\mathcal{Z}(H)$.*

Proof. Our strategy to find the isomorphism in question is as follows. Starting with $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$, we show that it is *-isomorphic to a fixed point subalgebra of some action β of $\mathbb{R}$ on $\mathrm{VN}_l(G) \bar{\otimes} \mathcal{B}(L^2(\mathbb{R}))$. We then search for a finite sequence of unitaries, say $x_1, x_2 \ldots, x_n$ such that they transform β into $\sigma \otimes \mathcal{I}$ (where $\mathcal{I}$ is the identity automorphism). More precisely, such that $\sigma \otimes \mathcal{I} = x_n \beta_n x_n^*$, where $\beta_n = x_{n-1} \beta_{n-1} x_{n-1}^*$, $\beta_0 = \beta$ and $x_0 = \mathbb{1}$. The action $\sigma \otimes \mathcal{I}$ is subject to Theorem 4.5 (iii), which produces the desired result. We proceed to make this rigorous.

By a result of Takesaki, [**Tak73**, Theorem 4.5] as well as Definition 2.40, it is a fact that $(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \widehat{\mathbb{R}}$ is *-isomorphic with $\mathrm{VN}_l(G) \bar{\otimes} \mathcal{B}(L^2(\mathbb{R}))$. From Theorem 2.32 we have that $\pi_\theta(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R})$ is a *-isomorphic copy of $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$.

It is a fact that $\pi_\theta(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R})$ is also *-isomorphic to $((\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \mathbb{R})^{\widehat{\theta}}$, see [**GL20**, Theorem 6.50]. We invoke another result of Takesaki, [**Tak73**, Theorem 4.6], which says that the action $\widehat{\theta}$ of $\mathbb{R}$ on $(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \widehat{\mathbb{R}}$ is transformed into the action $\{\sigma \otimes \mathrm{Ad}\, v(\cdot)^*\}$ of $\mathbb{R}$ on $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$, by the same map identifying $(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \widehat{\mathbb{R}}$ with $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

Therefore we have obtained that $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ is *-isomorphic to $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}\left(L^2(\mathbb{R})\right)^{\sigma\otimes\mathrm{Ad}\, v(\cdot)^*}$.

We will now show that $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ is isomorphic with $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}\left(L^2(\mathbb{R})\right)$. Consider the automorphism group $\{\sigma_t \otimes \mathrm{Ad}\, v(t)^*\}$ on $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

From the statement of Theorem 2.48, let $\mathfrak{H} = \mathbb{C}$, $A = \mathbb{R}$ and α be the identity group action of G on $\mathbb{R}$. The opening argument of the proof of Theorem 2.48 then shows that $L^2(\mathbb{R} \times G, \mathbb{C})$ is isometrically isomorphic to $L^2(G, L^2(\mathbb{R}, \mathbb{C}))$. From Theorem 2.26 we ascertain that $L^2(G, L^2(\mathbb{R}, \mathbb{C}))$ is isometrically isomorphic to $L^2(\mathbb{R})\otimes L^2(G)$. Similarly we also have that $L^2(G)\otimes L^2(\mathbb{R})$ is an isometrically isomorphic copy of $L^2(G \times \mathbb{R})$ via the now identified isomorphism Σ which enjoys the property:

$$\Sigma(\xi \otimes f)(s,r) = \xi(s)f(r), \text{ for all } \xi \in L^2(G), f \in L^2(\mathbb{R}).$$

We consider the operator $\mathbb{1} \otimes \mathscr{F}$ on $L^2(G) \otimes L^2(\mathbb{R})$, where $\mathscr{F}$ is the Fourier transform on $L^2(\mathbb{R})$. It is clear that $\mathbb{1} \otimes \mathscr{F}$ is a unitary. It is an easy exercise to determine that $[v(t)^*f](r) = f(r+t)$ and $[u(t)^*f](r) = e^{irt}f(r)$, hence

$$\begin{aligned}(\mathscr{F}v(t)^*f)(\gamma) &= \int_{\mathbb{R}} e^{-ir\gamma} f(r+t)\, d\mu_{\mathbb{R}}(r) \\ &= \int_{\mathbb{R}} e^{it\gamma}e^{-ir\gamma} f(r) d\mu_{\mathbb{R}}(r) \\ &= (u(t)^*\mathscr{F}\xi)(\gamma).\end{aligned}$$

That is $\mathscr{F}v(t)^*\mathscr{F}^* = u(t)^*$. Since $\mathbb{1} \otimes \mathscr{F}$ is unitary, we of course have that

$$(\mathbb{1} \otimes \mathscr{F}) \left[\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))\right] (\mathbb{1} \otimes \mathscr{F}^*) = \mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R})).$$

We are therefore urged to investigate the action $\{\sigma \otimes \mathrm{Ad}\, u(\cdot)^*\}$ of $\mathbb{R}$ on $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

We attempt to define a linear operator $W \in \mathcal{B}(L^2(G \times \mathbb{R}))$. For any simple tensor we define $(W(f \otimes g))(s,t) = \xi \otimes f(g_t^{-1}s, t) = \lambda(g_t)\xi \otimes f(s,t)$ (see Theorem 1.27 for g_t). We show that W is a unitary. Clearly it is linear. We verify that it is bounded. Let $\xi \otimes f \in L^2(G) \otimes L^2(\mathbb{R})$, then

$$\begin{aligned}\|W(\xi \otimes f)\|_2^2 &= \int_{G\times\mathbb{R}} |(\xi \otimes f)(g_t^{-1}s, t)|^2\, d(s,t) \\ &= \int_{\mathbb{R}} |f(t)|^2 \left(\int_G |\xi(g_t^{-1}s)|^2\, d\mu_G(s)\right) d\mu_{\mathbb{R}}(t) \\ &= \int_{\mathbb{R}} |f(t)|^2\, d\mu_{\mathbb{R}}(t) \int_G |\xi(s)|^2\, d\mu_G(s) \\ &= \|(\xi \otimes f)\|_2^2\end{aligned}$$

by the properties of the Haar measure of G. Using similar reasoning we also determine W^*:

$$\begin{aligned}\langle W(\xi \otimes f), \eta \otimes h\rangle &= \int_{G\times\mathbb{R}} \xi(g_t^{-1}s) f(t)\overline{\eta(s)h(t)} d(s,t) \\ &= \int_{G\times\mathbb{R}} \xi(s)f(t)\overline{\eta(g_t s)h(t)} d(s,t).\end{aligned}$$

So we must have $(W^*(\xi \otimes f))(s,t) = \xi \otimes f(g_t s, t)$. Indeed $W^*W = WW^* = \mathbb{1}$. Therefore our unitary W extends to an unitary on $L^2(G \times \mathbb{R})$ (also denoted W), by continuity. The unitary W can of course be defined by

$$W\xi(s,t) = \xi(g_t^{-1}s, t), \text{ for all } \xi \in L^2(G \times \mathbb{R}).$$

Let ρ^G denote the right regular representation as in Definition 2.33. Consider the following:

$$\begin{aligned}(W(\rho^G(g) \otimes \mathbb{1})\xi)(s,t) &= \xi(g_t^{-1}sg, t)\\ &= ((\rho^G(g) \otimes \mathbb{1})W\xi)(s,t).\end{aligned}$$

This implies that W belongs to $(\mathrm{VN}_r(G)\bar{\otimes}\,\mathbb{C})'$. But $\mathrm{VN}_r(G) = \mathrm{VN}_l(G)'$, [**Tak03a**, VII Proposition 3.1]. Thus, we must have that W belongs to $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}\left(L^2(G)\right)$ by both the bicommutant theorem and [**Tak01**, VI, Theorem 5.9] and that $W\,\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))W^* = \mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$. Let Δ_G be the canonical modular operator of $\mathrm{VN}_l(G)$, that we identified in Definition 3.10.

We then find that

$$\begin{aligned}\left(W\left(\Delta_G^{it} \otimes u(t)^*\right)W^*\xi\right)(g,p) &= \left(\left(\Delta_G^{it} \otimes u(t)^*\right)W^*\xi\right)(g_p^{-1}g, p)\\ &= \Delta_G(g_p^{-1})^{it}\Delta_G(g)^{it}e^{itp}\left(W^*\xi\right)(g_p^{-1}g,p)\\ &= e^{-itp}e^{itp}\Delta_G(g)^{it}\xi(g,p)\\ &= \left(\left(\Delta_G^{it} \otimes \mathbb{1}\right)\xi\right)(g,p).\end{aligned}$$

Thus $W\left(\Delta_G^{it} \otimes u(t)^*\right)W^* = \Delta_G^{it} \otimes \mathbb{1}$. Suppose $x \in \mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ is fixed under $\sigma_t \otimes \mathrm{Ad}\, u(t)^*$. This is true iff

$$W\left(\Delta_G^{it} \otimes u(t)^*\right)W^*WxW^*W\left(\Delta_G^{-it} \otimes u(t)\right)W^* = WxW^*.$$

That is to say WxW^* is fixed under $\sigma \otimes \mathcal{I}$. By Theorem 4.5 (i) and (iii) we have that the fixed point subalgebra of $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ with respect to $\sigma\otimes\mathcal{I}$ is isomorphic with $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

Let "$\cong$" indicate that spaces are *-isomorphic. Then we have obtained that

$$\begin{aligned}\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R} &\cong \mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))^{\sigma\otimes\mathrm{Ad}\, v(\cdot)^*}\\ &\cong (\mathbb{1} \otimes \mathscr{F})[\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))](\mathbb{1} \otimes \mathscr{F}^*)^{\sigma\otimes\mathrm{Ad}\, u(\cdot)^*}\\ &\cong (W[\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))]W^*)^{\sigma\otimes\mathcal{I}} \qquad (4.5)\\ &\cong \mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R})).\end{aligned}$$

It remains to identify the automorphism $\{\theta_t : t \in \mathbb{R}\}$ on $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

Let $\pi_\sigma(x) : L^2(\mathbb{R}, L^2(G)) \to L^2(\mathbb{R}, L^2(G))$ for $x \in \mathrm{VN}_l(G)$ be as in Definition 2.29 and $\xi \in C_c\left(\mathbb{R}, L^2(G)\right)$. For $t \in \mathbb{R}$ and $s, g \in G$, we see that

$$\begin{aligned}
[(\pi_\sigma(\lambda^G(g))\xi)(t)](s) &= [\sigma_{t^{-1}}(\lambda^G(g))\xi(t)](s) \\
&= [\Delta_G^{-it}\,\lambda^G(g)\,\Delta_G^{it}\,\xi(t)](s) \\
&= \Delta_G(s)^{-it}[\Delta_G^{it}\,\xi(t)](g^{-1}s) \\
&= [\Delta_G(g)^{-it}\lambda^G(g)\xi(t)](s).
\end{aligned} \tag{4.6}$$

If we let $r \in \mathbb{R}$ and $\lambda_1(t)$ for $t \in \mathbb{R}$ denote the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R}, L^2(G))$ as in Definition 2.33, then we see that the generators of $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ are the operators $\pi_1(g)$ and $\lambda_1(t)$ defined by

$$\begin{cases}
[(\pi_1(g)\xi)(r)](\cdot) := \Delta_G(g)^{-ir}\lambda^G(g)[\xi(r)](\cdot); & g \in G \\
[(\lambda_1(t)\xi)(r)](\cdot) := [\xi(r-t)](\cdot) & t \in \mathbb{R}.
\end{cases} \tag{4.7}$$

By construction, $(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \mathbb{R}$, acts on $L^2(\mathbb{R}, L^2(\mathbb{R}, G)) \cong L^2(\mathbb{R} \times \widehat{\mathbb{R}}, G)$. It is then an exercise to repeat the calculations deriving Equation (4.6), with π_θ substituted for π_σ and x chosen from the operators defined by Equations (4.7). Doing so shows that$(\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \widehat{\mathbb{R}}$ is generated by the following three types of operators:

$$\begin{cases}
[(\pi_2(g)\xi)(r,p)](\cdot) := \Delta_G(g)^{-ir}\lambda^G(g)[\xi(r,p)](\cdot); & g \in G \\
[(\lambda_2(t)\xi)(r,p)](\cdot) := e^{-it}[\xi(r-t,p)](\cdot); & t \in \mathbb{R} \\
[(u_2(q)\xi)(r,p)](\cdot) := [\xi(r,p-q)](\cdot); & q \in \mathbb{R}.
\end{cases} \tag{4.8}$$

We follow in the footsteps of Takesaki, [**Tak73**, p. 258], and consider, the von Neumann algebra $(\mathbb{1} \otimes \mathscr{F})\left((\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}) \rtimes_\theta \widehat{\mathbb{R}}\right)(\mathbb{1} \otimes \mathscr{F}^*)$. That is, we consider the Fourier transform in the second coordinate. We obtain the generators:

$$\begin{cases}
[(\pi_3(g)\xi)(r,p)](\cdot) := \Delta_G(g)^{-ir}\lambda^G(g)[\xi(r,p)](\cdot) \\
[(\lambda_3(t)\xi)(r,p)](\cdot) := [\xi(r-t,p-t)](\cdot) \\
[(u_3(q)\xi)(r,p)](\cdot) := e^{-iqp}[\xi(r,p)](\cdot)
\end{cases} \tag{4.9}$$

Let $\mathscr{N}$ be the algebra generated by the operators described in (4.9), and $\mathscr{P}$ the subalgebra generated by $\{\lambda_3(t), u_3(q) : t, q \in \mathbb{R}\}$. We define the map $\pi^\star : \mathrm{VN}_l(G) \to \mathcal{B}(L^2(\mathbb{R} \times \widehat{\mathbb{R}}, L^2(G)))$ by the prescription

$$(\pi^\star(x)\xi)(r,p) := \sigma_{r-p}^{-1}\xi(r,p). \tag{4.10}$$

We pause to consider some technical results we will need.

Proposition 4.7 ([**Tak73**] Lemma 4.3 & 4.4). *The map* $\pi^\star : x \in \mathrm{VN}_l(G) \mapsto \pi^\star(x) \in \mathcal{B}(L^2(\mathbb{R} \times \widehat{\mathbb{R}}, L^2(G)))$*, as brought about by Equation (4.10), is a normal* **-isomorphism of* $\mathrm{VN}_l(G)$ *into* $\mathscr{N} \cap \mathscr{P}'$*. Hence* $\pi^\star(\mathrm{VN}_l(G))$ *is a* **-isomorphic copy of* $\mathrm{VN}_l(G)$.

We also have that the von Neumann algebra $\mathscr{N}$ *is generated by* $\pi^\star(\mathrm{VN}_l(G))$ *and* $\mathscr{P}$*. Furthermore*

$$\mathscr{N} \cong \mathrm{VN}_l(G)\bar{\otimes}\mathscr{P}.$$

We proceed with the proof. Let $\chi(g) := \ln \Delta_G(g)$ and $s \in G$, then we have that

$$\begin{aligned}
\Big[(\pi^\star(\lambda^G(g))\xi)(r,p)\Big](s) &= [\sigma_{r-p}^{-1}(\lambda^G(g))\xi(r,p)](s) \\
&= [\boldsymbol{\Delta}_G^{-i(r-p)}\ \lambda^G(g)\ \boldsymbol{\Delta}_G^{i(r-p)}\ \xi(r,p)](s) \\
&= [\Delta_G(s)^{-i(r-p)}\Delta_G(g^{-1}s)^{i(r-p)}\xi(r,p)](g^{-1}s) \\
&= [(\pi_3(g)u_3(\chi(g))\xi)(r,p)](s).
\end{aligned}$$

Now, there exists a copy of $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ in $\mathscr{N}$. By Proposition 4.7 and the arguments hitherto made, we must have that this copy is generated by the operators

$$\begin{cases} \pi_3(g)u_3(\chi(g)) & g \in G \\ \lambda_3(t) & t \in \mathbb{R}. \end{cases} \tag{4.11}$$

As Takesaki states in [**Tak73**, p. 258], $\mathscr{P}$ is *-isomorphic to $\mathcal{B}\left(L^2(\mathbb{R})\right)$ such that $\lambda_3(t) \mapsto v(t)$ and $u_3(\chi(g)) \mapsto u(\chi(g))$. (See [**Mac49**],[**Loo52**] and [**NU61**].) So if we identify $\mathscr{N}$ with $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$, then by the immediately preceding argument applied to the operators from (4.11), the first crossed product is generated by the operators

$$\begin{cases} \lambda^G(g) \otimes u(\chi(g)), & g \in G \\ \mathbb{1} \otimes v(t), & t \in \mathbb{R}. \end{cases} \tag{4.12}$$

It's easy to show that $u_3(t)\lambda_3(r)u_3(t)^* = e^{-itr}\lambda_3(r)$ and that $u_3(t)u_3(\chi(g))u_3(t)^* = u_3(\chi(g))$. So by Theorem 2.41 we know that the dual automorphism of the first crossed product, understood as a subalgebra of $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$, is given by $\mathrm{Ad}(\mathbb{1} \otimes u(t))$. Having identified the dual action we must, as was done in the first part of the proof, subject it to the unitaries $(\mathbb{1} \otimes \mathscr{F})$ and W.

We apply the Fourier transform in the second coordinate as per the first part of the proof. The generators in (4.12) become $\lambda^G(g) \otimes v(\chi(g))\,(g \in G)$ and $\mathbb{1} \otimes u(t)\,(t \in \mathbb{R})$. Furthermore, the dual automorphism group is then given by $\{\mathrm{Ad}(\mathbb{1} \otimes v(t)) : t \in \mathbb{R}\}$.

Naturally, we must now subject $\mathrm{Ad}(\mathbb{1} \otimes v(t))$ to W similarly to what was just done for $(\mathbb{1} \otimes \mathscr{F})$. First we compute as follows:

$$\begin{aligned}
\left(W(\mathbb{1} \otimes v(t))W^*\xi\right)(s,r) &= \xi(g_{r-t}g_r^{-1}s, r-t) \\
&= \xi(g_t^{-1}s, r-t) \\
&= (\lambda^G(g_t) \otimes v(t)\xi)(s,r).
\end{aligned}$$

Let $w(t) := W(\mathbb{1} \otimes v(t))W^* = \lambda^G(g_t) \otimes v(t)$. Thus by Equation (4.5) the dual automorphism group on $\mathrm{VN}_l(G)^\sigma\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ is given by $\mathrm{Ad}\,w(t)$. We identify $\mathrm{VN}_l(G)^\sigma$ with $\mathrm{VN}_l(H)$ via $\kappa : \mathrm{VN}_l(G)^\sigma \to \mathrm{VN}_l(H)$ from Proposition 4.5 (iii) (a). Then the automorphism group brought about by $\mathrm{Ad}[\kappa(\lambda^G(g_t)) \otimes v(t)]$ is precisely $\{\bar{\alpha}_t \otimes \mathrm{Ad}\, v(t) : t \in \mathbb{R}\}$ as required. We note that $\mathcal{B}\left(L^2(\mathbb{R})\right)$ is a factor of type I, therefore the final conclusion follows. $\square$

Definition 4.8 (Ergodic actions[**Tak03a**], X, Definition 3.13). A covariant system $\{\mathcal{M}, G, \alpha\}$ is called **ergodic** if $\mathcal{M}^\alpha = \mathbb{C}$; **centrally ergodic** if α is ergodic on the centre $\mathcal{Z}$, i.e. $\mathcal{Z}^\alpha = \mathbb{C}$.

To proceed, we will need to enhance Theorem 4.6 with direct integral theory. The following are facts summarized by Sutherland in [**Sut78**] from his articles [**Sut74**] and [**Sut77**].

Proposition 4.9. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra with separable predual, and $\{\phi_t\}_{t\in\mathbb{R}}$ an arbitrary modular automorphism group. Suppose $\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}(\omega)\, d\mu(\omega)$ constitutes the central decomposition of $\mathcal{M}$, then:*

- *(i) the modular automorphism group $\{\phi_t\}_{t\in\mathbb{R}}$ has decomposition $\phi_t = \int_\Gamma^\oplus \phi_{t,\omega}\, d\mu(\omega)$ such that $\{\phi_t\}_{t\in\mathbb{R}}$ is a modular automorphism group on $\mathcal{M}(\omega)$.*
- *(ii) there is a canonical isomorphism of $\mathcal{M} \rtimes_{\phi_t} \mathbb{R}$ with $\int_\Gamma^\oplus \mathcal{M} \rtimes_{\phi_{t,\omega}} \mathbb{R}\, d\mu(\omega)$.*
- *(iii) With respect to to the decomposition in (ii), the dual automorphism group $\{\hat{\phi}_t\}_{t\in\mathbb{R}}$ has decomposition $\hat{\phi}_t = \int_\Gamma^\oplus \hat{\phi}_{t,\omega} d\mu(\omega)$, and $\{\hat{\phi}_{t,\omega}\}_{t\in\mathbb{R}}$ is dual to $\{\phi_{t,\omega}\}_{t\in\mathbb{R}}$ on $\mathcal{M} \rtimes_{\phi_{t,\omega}} \mathbb{R}$.*
- *(iv) the diagonal subalgebra of the decomposition in (ii) is the fixed point subalgebra of the centre of $\mathcal{M} \rtimes_{\phi_t} \mathbb{R}$ under $\{\hat{\phi}_{t,\omega}\}_{t\in\mathbb{R}}$.*

Remark 4.10. Claim (iv) is proved in neither [**Sut74**] nor [**Sut77**]. We observe that $\{\hat{\phi}_{t,\omega}\}_{t\in\mathbb{R}}$ is ergodic on the centre of $\mathcal{M} \rtimes_{\phi_{t,\omega}} \mathbb{R}$ so that (iii) describes the ergodic decomposition of the covariant system $(\mathcal{Z}(\mathcal{M} \rtimes_{\phi_t} \mathbb{R}), \mathbb{R}, \hat{\phi}|_{\mathcal{Z}})$. The diagonal subalgebra must then coincide with $\mathcal{Z}(\mathcal{M} \rtimes_{\phi_t} \mathbb{R})^{\hat{\phi}|_{\mathcal{Z}}}$. For more on this see [**Tak03a**, X, Chapter 3].

Recall the central decomposition preceding Definition 4.2,

$$\left\{\mathcal{M}(G), \lambda^G\right\} = \int_{\Gamma_G}^\oplus \left\{\mathcal{M}(G)(\omega), \lambda^G_\omega\right\} d\mu_G(\omega).$$

Thus Proposition 4.5 (iii), Proposition 4.9 (ii) and Theorem 4.6 arms us with a decomposition of $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ and hence of λ^H, with diagonal subalgebra $\kappa(\mathcal{Z}(G)) \subset \mathcal{Z}(H)$. By Theorem 4.6, the dual automorphism group on $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ is given by $\bar{\alpha}_t \otimes \mathrm{Ad}\, v(t)$. Let $\lambda^H = \int_{\Gamma_G}^\oplus \lambda^H_\omega d\mu_G(\omega)$ be the decomposition of λ^H over $\kappa(\mathcal{Z}(G))$ and let $\bar{\alpha}_t = \int_{\Gamma_G}^\oplus \bar{\alpha}_{t,\omega} d\mu_G(\omega)$ be the corresponding decomposition of $\{\bar{\alpha}_t\}_{t\in\mathbb{R}}$.

In what follows we investigate the description of the central decomposition of λ^G given in Proposition 4.5 (iv) in light of the preceding discussion.

Definition 4.11 ([**Tak73**, Proposition 3.5]). Let $(\mathcal{M}, \mathfrak{H})$ be a von Neumann algebra equipped with two continuous actions α and β of a locally compact group G. Suppose there exists a σ-strongly continuous function $u : g \in G \mapsto u_g \in \mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M})$ denotes the group of all unitaries in $\mathcal{M}$, such that

$$u_{gh} = u_g \alpha_g\left(u_h\right), \quad g, h \in G;$$

and β is unitarily implemented in the sense that

$$\beta_g(x) = u_g \alpha_g(x) u_g^*, \quad x \in m.$$

If the conditions above are satisfied we write $\alpha \sim \beta$, and say α is equivalent to β, as "$\sim$" defines a equivalence relation.

Definition 4.12 (Quasi-equivalent representations). Let ϖ_1 and ϖ_2 be two unitary representations of a locally compact group G, in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. We say ϖ_1 and ϖ_2 are quasi-equivalent if they satisfy one of the following:

(i) No nonzero subrepresentation ([**Fol16**, p. 76]) of ϖ_1 is disjoint from ϖ_2, and no nonzero subrepresentations of ϖ_2 is disjoint from ϖ_1;
(ii) There exists a *-isomorphism $\mathcal{J}$ of the von Neumann algebra $\varpi_1(G)''$ onto the von Neumann algebra $\varpi_2(G)''$ such that $\mathcal{J}(\varpi_1(g)) = \varpi_2(g)$ for all $g \in G$.
(iii) There exist unitarily equivalent representations ([**Fol16**, p. 75]) ϱ_1 and ϱ_2 such that ϱ_1 is a multiple of ϖ_1 and ϱ_2 is a multiple of ϖ_2;
(iv) ϖ_2 is unitarily equivalent to a subrepresentation of some multiple representation ϱ_1 of ϖ_1 that has central support $\mathbb{1}$;

For a proof that the conditions above are equivalent, see [**Dix77**, 13.1.4, p. 280 and 5.3.1. Proposition].

Theorem 4.13 ([**Sut78**, Proposition 3.2]). *Let* $\lambda^G := \int_{\Gamma_G}^{\oplus} \lambda^G_\omega \, d\mu_G(\omega)$ *be the central decomposition of* λ^G*, and* $\lambda^H = \int_{\Gamma_G}^{\oplus} \lambda^H_\omega \, d\mu_G(\omega)$ *be the decomposition of* λ^H *arising from the isomorphism of* $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ *with* $\mathrm{VN}_l(H) \bar{\otimes} \mathcal{B}(L^2(\mathbb{R}))$*. Then the representations* λ^G_ω *and* $\mathrm{Ind}_H^G \lambda^H_\omega$ *are quasi-equivalent.*

Proof. We expand upon Sutherland's proof. Let $\pi^G_\omega = \mathrm{Ind}_H^G \lambda^H_\omega$; we first write down generators for the von Neumann algebra $\{\pi^G_\omega(G)\}''$. Recall that π^G_ω acts on the Hilbert space $\mathfrak{H}(\omega)$ of Borel functions $\eta : G \to \mathscr{H}(\omega)$ (the space of the representation λ^H_ω) and which satisfy the properties:

(i) $\eta(gh) = \lambda^H_\omega(h)^{-1}\eta(g); h \in H$
(ii) $\int_{G/H} \|\eta(g)\|^2_{\mathscr{H}(\omega)} \, d\dot{g} < \infty$

where $d\dot{g}$ is an G-invariant measure on G/H (see [**Fol16**, 2.51 Theorem] and [**Fol16**, §6.1]). But by Theorem 1.27 we have identified G/H with L. Since L is a continuous one paramater subgroup of G, we may consider the measure on G/H to be, up to some scalar constant, the Lebegue measure on $\mathbb{R}$.

Fix $s \in G$. The action of π^G_ω on $\mathfrak{H}(\omega)$ is given by

$$(\pi^G_\omega(g)\eta)(s) = \eta(g^{-1}s); \quad \eta \in \mathfrak{H}(\omega), \quad g \in G.$$

We define $U : \mathfrak{H}(\omega) \to L^2(\mathbb{R}, \mathscr{H}(\omega))$ by $(U\eta)(t) = \eta(g_t)$ in an attempt to identify $\mathfrak{H}(\omega)$ with $L^2(\mathscr{H}(\omega))$. We draw the following conclusions:

$$\begin{aligned}\langle U\eta, U\zeta\rangle_{L^2(\mathbb{R},\mathscr{H}(\omega))} &= \int_{\mathbb{R}} \langle \eta(g_t), \zeta(g_t)\rangle_{\mathscr{H}(\omega)}\, d(t) \\ &= \int_{G/H} \langle \eta(g), \zeta(g)\rangle_{\mathscr{H}(\omega)}\, d\dot{g} \\ &= \langle \eta, \zeta\rangle_{\mathfrak{H}(\omega)}.\end{aligned} \tag{4.13}$$

Let $A : L^2(\mathbb{R}, \mathscr{H}(\omega)) \to \mathfrak{H}(\omega)$ be defined by $(Af)(g) = \lambda^H_\omega(g^{-1}g_{\chi(g)})[f(\chi(g))]$. Then for $f \in L^2(\mathbb{R}, \mathscr{H}(\omega))$ and $\eta \in \mathfrak{H}(\omega)$ we have

$$\begin{aligned}(UAf)(t) &= (Af)(g_t) \\ &= \lambda^H_\omega(g_t^{-1}g_{\chi(g_t)})[f(\chi(g_t))] \\ &= \lambda^H_\omega(e_G)[f(t)] \\ &= f(t)\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}(AU\eta)(g) &= \lambda^H_\omega(g^{-1}g_{\chi(g)})[(U\eta)(\chi(g))] \\ &= \lambda^H_\omega(g^{-1}g_{\chi(g)})[\eta(g_{\chi(g)})] \\ &= \eta(g),\end{aligned} \tag{4.15}$$

where we made use of property (i) to obtain Equation (4.15). By equations (4.13)-(4.14), we have that $\langle U\eta, f\rangle_{L^2(\mathbb{R},\mathscr{H}(\omega))} = \langle \eta, Af\rangle_{\mathfrak{H}(\omega)}$ for all $\eta \in \mathfrak{H}(\omega)$ and $f \in L^2(\mathbb{R}, \mathscr{H}(\omega))$. Equation (4.15) then allows us to conclude that U is an unitary identifying $\mathfrak{H}(\omega)$ with $L^2(\mathbb{R}, \mathscr{H}(\omega))$ such that $U^* = A$. Next we compute

$$\begin{aligned}(U\pi^G_\omega(g)U^*\xi)(t) &= (\pi^G_w(g)U^*f)(g_t) \\ &= (U^*f)(g^{-1}g_t) \\ &= \lambda^H_\omega(g_t^{-1}gg_{t-\chi(g)})f(t-\chi(g)).\end{aligned}$$

But $g = gg_{-\chi(g)}g_{\chi(g)}$ with $gg_{-\chi(g)} \in H$ and $g_{\chi(g)} \in L$. So we have that

$$(U\pi^G_\omega(gg_{-\chi(g)})\pi^G_\omega(g_{\chi(g)})U^*)f(t).$$

Therefore, if we let $\{\bar{\pi}^G_\omega(g) : g \in G\}$ be the set of operators on $L^2(\mathbb{R}, \mathscr{H}(\omega))$ engendered by

$$\bar{\pi}^G_\omega(h) := U\pi^G_\omega(h)U^*$$

and specialize to the case where $g = h \in H$ or $g = g_{\mathrm{s}} \in L$; we obtain generators for $\{\pi^G_\omega(G)\}''$ on $L^2(\mathbb{R}, \mathscr{H}(\omega))$ of the form

$$\begin{cases}(\bar{\pi}^G_\omega(h)\eta)(t) = \lambda^H_\omega(g_t^{-1}hg_t)\eta(t); & h \in H \\ (\bar{\pi}^G_\omega(g_s)\eta)(t) = \eta(t-s); & s \in \mathbb{R}.\end{cases} \tag{4.16}$$

Recall that

$$\int^{\oplus}_{\Gamma_g} \bar{\alpha}_{t,\omega}(\lambda^H_\omega(h))\xi(\omega)\, d\mu_G(\omega) = \bar{\alpha}_t(\lambda^H(h))\xi = \lambda^H(g_thg_t^{-1}).$$

The action of the automorphism group $\{\bar{\alpha}_{t,\omega}; t \in \mathbb{R}\}$ on $\mathcal{M}(H)(\omega) := \{\lambda^H_\omega(H)\}''$ is described by $\bar{\alpha}_{t,\omega}(\lambda^H_\omega(h)) = \lambda^H_\omega(g_t h g_t^{-1})$, so that the crossed product $\mathcal{M}(H)(\omega) \rtimes_{\bar{\alpha}_\omega} \mathbb{R}$ has generators on $L^2(\mathbb{R}, \mathscr{H}(\omega))$ given by

$$\begin{cases} (\bar{\lambda}(h)\xi)(t) = \lambda^H_\omega(g_t^{-1} h g_t)\xi(t); & h \in H \\ (\lambda(s)\xi)(t) = \xi(t-s); & s \in \mathbb{R}. \end{cases} \tag{4.17}$$

If we compare (4.16) and (4.17), it is immediately seen that $\{\bar{\pi}^G_\omega(G)\}''$ and $\mathcal{M}(H)(\omega) \rtimes_{\bar{\alpha}_\omega} \mathbb{R}$ have identical generators; therefore, $\{\pi^G_\omega(G)\}''$ and $\mathcal{M}(H)(\omega) \rtimes_{\bar{\alpha}_\omega} \mathbb{R}$ are *-isomorphic. We consider the dual automorphism $\{\theta_{t,\omega} : t \in \mathbb{R}\}$ of $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$. We know that $\theta_{t,\omega}(x) = (\mathbb{1} \otimes v(t))((\bar{\alpha}_{t,\omega} \otimes \mathcal{I})(x))(\mathbb{1} \otimes v(t)^*)$; furthermore, $(\mathbb{1} \otimes v(t))(\bar{\alpha}_{t,\omega} \otimes \mathcal{I})(\mathbb{1} \otimes v(r)) = \mathbb{1} \otimes v(t+r)$. So $\{\theta_{t,\omega}\}_{t\in\mathbb{R}} \sim \{\bar{\alpha}_{t,\omega} \otimes \mathcal{I}\}_{t\in\mathbb{R}}$ in the sense of Definition 4.11. Let π^θ_ω, $\pi^{\bar{\alpha}}_\omega$ denote the canonical inclusions of $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ in the respective crossed products, made precise by Theorem 2.29.

Now $x \mapsto x \otimes \mathbb{1}$ defines a *-isomorphism of $\mathcal{M}(H)(\omega)$ with $\mathcal{M}(H)(\omega)\bar{\otimes}\mathbb{C}$, such that $\bar{\alpha}_{t,\omega}(x) \mapsto (\bar{\alpha}_{t\omega} \otimes \mathcal{I})(x \otimes \mathbb{1})$. Consequently, by [**Tak73**, Proposition 3.4] and the identification of $\{\pi^G_\omega(G)\}''$ with $\mathcal{M}(H)(\omega)\rtimes_{\bar{\alpha}_\omega}\mathbb{R}$, we obtain a normal *-isomorphism ψ of $\{\pi^G_\omega(G)\}''$ into $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))\rtimes_{\bar{\alpha}_\omega\otimes\mathcal{I}} \mathbb{R}$ such that

$$\begin{cases} \psi(\pi^G_\omega(h)) = \pi^{\bar{\alpha}}_\omega(\lambda^H_\omega(h) \otimes \mathbb{1}) \\ \psi(\pi^G_\omega(g_s)) = \lambda(s). \end{cases}$$

But from the duality theorem, [**Tak73**, Theorem 4.5], $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R})) \rtimes_{\bar{\alpha}_\omega\otimes\mathcal{I}} \mathbb{R}$ is isomorphic with $\mathcal{M}(G)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

We compute the images of $\pi^{\bar{\alpha}}_\omega(\lambda^H_\omega(h) \otimes \mathbb{1})$ and $\lambda(s)$ in $\mathcal{M}(G)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

Since $\{\theta_{t,\omega}\}_{t\in\mathbb{R}} \sim \{\bar{\alpha}_{t,\omega} \otimes \mathcal{I}\}_{t\in\mathbb{R}}$, we are granted access to [**Tak73**, Proposition 3.5]; providing us with a normal isomorphism ψ^1, of $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))\rtimes_{\bar{\alpha}_\omega\otimes\iota}\mathbb{R}$ with $\mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))\rtimes_{\theta_\omega}\mathbb{R}$ carrying $\pi^\alpha_\omega(x)$ to $\pi^\theta_\omega(x)$ $(x \in \mathcal{M}(H)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R})))$ and $\lambda(s)$ to $\lambda(s)$.

We note that W (defined in the proof of Theorem 4.6) is decomposable over $\mathcal{Z}(G)\bar{\otimes}\,\mathbb{C}$. The decomposition of W together with the operators of (4.11), show that the images of $\pi^\theta_\omega(\lambda^H_\omega(h)\otimes\mathbb{1})$ and $\lambda(s)$ in $\mathcal{M}(G)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ can be identified with

$$\begin{cases} (\bar{\lambda}^H_\omega(h)\xi)(p) = \lambda^G_\omega(g_p^{-1} h g_p)\xi(p) \\ (\bar{\lambda}(s)\xi)(p) = \xi(p-s). \end{cases}$$

Lastly, we define an operator V_ω on $\mathscr{G}(\omega) \otimes L^2(\mathbb{R})$ by the prescription

$$(V_\omega\xi)(p) = \lambda^G_\omega(g_p)\xi(p),$$

where $\mathscr{G}(\omega)$ is the representation space of λ^G_ω. It is an exercise to show that V_ω is a unitary and that

$$\begin{cases} (V_\omega\bar{\lambda}^H_\omega(h)V^*_\omega\xi)(p) = \lambda^G_\omega(h)\xi(p) \\ (V_\omega\bar{\lambda}(s)V^*_\omega\xi)(p) = (\lambda^G_\omega(g_s)\xi(p-s)). \end{cases} \tag{4.18}$$

From the proof of Theorem 4.6, we see that (4.12) yields an isomorphism δ_ω, of $\mathcal{M}(G)(\omega)$ into $\mathcal{M}(G)(\omega)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ with $\delta_\omega(\lambda^G_\omega(g)) = \lambda^G_\omega(g) \otimes v(\chi(g))$. Note that we applied the Fourier transform in the second coordinate of the operators from (4.12).

Equivalently the representation of G determined by Equations (4.18) is quasi-equivalent with λ^G_ω. □

We will need to identify the "smooth" part of the quasi-dual (Γ_H, μ_H) to describe some of the projections in the structure theorem, Theorem 4.26. We investigate the necessary theory.

Definition 4.14. If S is a Borel space and $\mathfrak{F}$ is a family of Borel subsets of S, we shall say that $\mathfrak{F}$ separates S or is a separating family if for any two points $p, q \in S$ such that $p \neq q$ there exists $E \in \mathfrak{F}$ with $p \in E$ and $q \notin E$. If $\mathfrak{F}$ is countable, we say S is **countably separated**.

Definition 4.15 (Borel Function, [**Mac57**, p. 136]). Let S_1 and S_2 be Borel spaces and let f be a function defined on S_1 and having values in S_2. If $f^{-1}(E)$ is a Borel subset of S_1 whenever E is a Borel subset of S_2 we shall say that f is a **Borel function**. If f is one-to-one and onto, and if f and f^{-1} are both Borel functions we shall say that f is a Borel isomorphism of S_1 onto S_2, and that S_1 and S_2 are isomorphic as Borel spaces.

Definition 4.16 (Quotient Borel space, [**Mac57**, p. 136]). Let S be a Borel space and let r be an equivalence relation in S. Let $\tilde{S}$ denote the set of all equivalence classes, and let $r(x)$ for each $x \in S$ denote the equivalence class to which x belongs. (So $r : x \mapsto r(X)$ is the quotient map.) Let $\mathfrak{F}$ be the set of all $E \subseteq \tilde{S}$ such that $r^{-1}(E)$ is a Borel set. Obviously $\mathfrak{F}$ is a Borel structure for $\tilde{S}$ and is the largest such that $x \mapsto r(x)$ is a Borel function. We call $\tilde{S}$ equipped with the Borel structure $\mathfrak{F}$, the **quotient Borel space** of S by the equivalence relation r.

Definition 4.17 (Orbit equivalence). Let X be a set and α a group action of G on X. Let $x \in X$ and set

$$G(x) = \{\alpha_g(x) : g \in G\}.$$

We call this set the **orbit** of x. Let $y \in G(x)$, then $x = \alpha_g^{-1}(y)$ for some $g \in G$. We have shown that $x \in G(y)$. That is to say that $\{G(x) : x \in X\}$ partitions X.

Definition 4.18. We let X/G be the set of orbits of X, endowed with the quotient topology.

Definition 4.19 (Smooth sets, [**Sut78**, p. 235]). Let G be a locally compact separable group acting on a standard measure space (X, μ) so that the map $(g, x) \in G \times X \to \beta_g(x)$ is Borel. (Here $\beta : G \to \mathrm{Aut}(X)$ is a homorphism.) An invariant Borel set $B \subset X$ (invariant meaning $\beta_g(B) = B$ for all $g \in G$) will be said to be **smooth** for μ if there is an invariant μ null Borel set $B_1 \subset X$ such that $(B - B_1)/G$ is countably separated in the quotient Borel structure.

Theorem 4.20 ([**Sut78**, Lemma 3.3]). *Let G and (X, μ) be as above. Then there is an invariant Borel set $B \subset X$ which is smooth for μ, and such that if B_1 is any other invariant Borel set which is smooth for μ, then $\mu(B_1 - B) = 0$.*

Proof. Given standard measure space (X, μ) we show that any family of disjoint invariant Borel sets of positive measure in X is countable.

Since (X, μ) is σ-finite, so $X = \bigsqcup X_n$, $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Let $\mathfrak{U}$ be a family of disjoint invariant Borel sets of positive measure in X. That is, $\mathfrak{U} = \{U \subset X : U \in \mathfrak{U}\}$. Let $\mathfrak{U}_n = \{U \in \mathfrak{U} : \mu(U \cap X_n) > 0\}$. We have $U = \bigsqcup_n (U \cap X_n)$, therefore $\mu(U) = \sum_n \mu(U \cap X_n) > 0$. As such, each U at least belongs to some $\mathfrak{U}_n$.

We show that each $\mathfrak{U}_n$ is countable. If $\mathfrak{U}_n$ is uncountable, then $\exists \epsilon > 0$ such that $\{U : \mu(U \cap X_n) \geq \epsilon\}$ is of infinite cardinality.

Thus $\exists (U_m)_{m \in \mathbb{N}} \subset \mathfrak{U}$ such that $\mu(U_m \cap X_n) \geq \epsilon$. That is

$$\infty = \sum_{m=1}^{\infty} \mu(u_m \cap X_n) \leq \mu(X_n) < \infty.$$

A contradiction. Hence $\mathfrak{U}_n$ is countable for all n and thus so is $\mathfrak{U}$.

Clearly $\emptyset$ is vacuously a smooth set. Let $\mathcal{Q}(X)$ be the set all families of disjoint smooth sets in X. We let $\mathcal{Q}(X)$ be partially ordered by inclusion. Consider a chain c of $\mathcal{Q}(X)$. Then we consider $s = \bigcup_{x \in c} x$. The set s is again a family of disjoint smooth sets.

Thus, by Zorn's lemma, $\mathcal{Q}(X)$ has a maximal element. Let us denote this family (which is the maximal element) by $(s_n)_{n \in \mathbb{N}}$, as all such families are countable. We show that $s = \bigsqcup_{n \in \mathbb{N}} s_n$ is smooth and satisfies the hypotheses of the lemma.

Clearly, s is Borel. We have $s_n = \beta_g^{-1}(s_n)$, then $s = \bigsqcup_{n \in \mathbb{N}} s_n = \bigsqcup_{n \in \mathbb{N}} \beta_g^{-1}(s_n) = \beta_g^{-1}\left(\bigsqcup_{n \in \mathbb{N}} s_n\right)$, since s is the union of disjoint, invariant sets. So we have shown that s is invariant.

For each $s_m \in (s_n)_{n \in \mathbb{N}}$, there exists s'_m such that $(s_m - s'_m)/G$ is countably separated in the quotient Borel structure. Let $s' = \bigcup_{n \in \mathbb{N}} s'_n$, which is a Borel null set. Then $(s - s')/G$ is countably separated in the quotient Borel structure. For each $n \in \mathbb{N}$, there exists a countable family $\mathscr{F}_n$ of Borel subsets of $(s_n - s'_n)/G$ that separates $(s_n - s'_n)/G$. Let $\mathscr{F} = \bigcup_{n \in \mathbb{N}} \mathscr{F}_n^{\cap}$ where $\mathscr{F}_n^{\cap} := \{E \cap (s - s')/G : E \in \mathscr{F}_n\}$. We note that $\mathscr{F}$ is countable. Let $G(x), G(y) \in (s - s')/G$ such that $G(x) \neq G(y)$. Thus there exists an $n \in \mathbb{N}$ such that $x \in s_n - s$. Since $s_n - s \subseteq s_n - s'_n$, we have $G(x) \in (s_n - s'_n)/G \cap (s - s')/G$ which is countably separated by $\mathscr{F}_n^{\cap} \subset \mathscr{F}$. So we must only verify the case where $G(y) \notin (s_n - s'_n)/G \cap (s - s')/G$. In this case it is easily seen that any $E \in \mathscr{F}_n \subset \mathscr{F}$ such that $G(x) \in E$ separates $G(x)$ and $G(y)$ as $E \subset (s_n - s'_n)/G \cap (s - s')/G$. Consequently, we have shown that s is smooth for μ.

Suppose contrariwise that s does not satisfy the hypotheses. Then there exists an invariant Borel set t smooth for μ such that $\mu(t - s) > 0$. That is $t \nsubseteq s$. Since $t - s \subseteq t$, we have that $((t - s) - t')/G$ is countably separated using similar reasoning as above. It can be verified that $t - s$ is also invariant. Clearly $t - s$ is Borel. So $t - s$ is smooth for μ.

But then $(s_n)_{n \in \mathbb{N}} \subseteq (s_n)_{n \in \mathbb{N}} \cup \{t - s\}$. By maximality of $(s_n)_{n \in \mathbb{N}}$ we must have $(s_n)_{n \in \mathbb{N}} = (s_n)_{n \in \mathbb{N}} \cup \{t - s\}$. A contradiction. Hence s must satisfy the hypotheses of the lemma. □

Definition 4.21 (Borel cross-section). A **Borel cross-section** is a Borel subset that intersects every orbit at precisely one point.

We apply Theorem 4.20 to the standard measure space (Γ_H, μ_H) (see Definition 4.2) with action $\check{\alpha}$ (see Equation (4.4)) to obtain a smooth Borel set $B_H \subseteq \Gamma_H$. We call B_H the smooth part of $(\Gamma_H, \mu_H, \check{\alpha})$.

We discuss results we will need before proceeding to hypothesize and prove Theorem 4.26. The following is proved by Takesaki in [**Tak73**] and summarised by Sutherland [**Sut78**].

Proposition 4.22. *Let $\mathcal{N}$ be a property infinite semfinite von Neumann algebra, and $\{\phi_t\}_{t\in\mathbb{R}}$ a continuous one-parameter automorphism group on $\mathcal{N}$. Let $\mathcal{M} = \mathcal{N} \rtimes_{\phi} \mathbb{R}$. Additionally, we also suppose that $\mathcal{N}$ admits a faithful normal semifinite trace τ such that $\tau\circ \phi_t = e^{-t}\tau$. Then:*

- *(i) $\mathcal{M}$ is a factor if and only if $\{\phi_t\}$ is ergodic on the centre $\mathcal{Z}(\mathcal{N})$ of $\mathcal{N}$.*
- *(ii) if $\mathcal{M}$ is a factor and $t_0 \in R$, then $e^{t_0} \in S(\mathcal{M})$ if and only if ϕ_{t_0} is the identity on $\mathcal{Z}(\mathcal{N})$.*
- *(iii) if $\mathcal{M}$ is a factor, then it is semifinite if and only if the action of $\{\phi_t\}$ on $\mathcal{Z}(\mathcal{N})$ is equivalent to translation on $L^\infty(\mathbb{R})$.*

If $\mathcal{M}$ is a type III factor and $\{\phi_t\}_{t\in\mathbb{R}}$ is any modular automorphism group on $\mathcal{M}$, then $\mathcal{M} \rtimes_{\phi} \mathbb{R}$ is properly infinite, semifinite, and admits a faithful normal trace τ such that $\tau\circ\hat{\phi}_t = e^{-t}\tau$. Note that if $\mathcal{M}$ is not (necessarily) of type III, but is still a factor, then the restriction of $\{\hat{\phi}\}_{t\in\mathbb{R}}$ to $\mathcal{Z}(\mathcal{M} \rtimes_{\phi} \mathbb{R})$ is still ergodic.

Remark 4.23. As is the convention in [**Sut78**], we will refer to Proposition 4.22 as Takesaki's type criterion.

Proposition 4.24 ([**Sut78**], Proposition 3.5). *Let $(Y, \bar{\mu})$ be a standard measure space, and $\{\beta_t : t \in \mathbb{R}\}$ be an action of $\mathbb{R}$ on Y with $(y,t) \mapsto \check{\beta}_t(y)$ Borel, and $\beta_t\circ\bar{\mu} \sim \bar{\mu}$. Let $\bar{\mu} = \int_X^{\oplus} \mu_\omega d\mu(\omega)$ be the ergodic decomposition of $\bar{\mu}$ with respect to β. Suppose there is a Borel function $\lambda : X \mapsto (0,\infty)$ such that for almost all $\omega \in X$, the action $\{\beta_t : t \in \mathbb{R}\}$ on (Y, μ_ω) is equivalent to the canonical action of $\mathbb{R}$ on $\mathbb{R}/\lambda(\omega)\mathbb{Z}$. Then Y is smooth for $\bar{\mu}$ under $\widehat{\alpha}_t$.*

We will also need Table 1.

Definition 4.25 (Isotropy subgroup). Suppose a group G is acting on a set X. Fix $x \in X$. Then $G_x := \{g \in G : g\cdot x = x\}$ is called the **isotropy group** of x (with respect to G or a group action of G).

Finally we reach the crux of the matter.

Theorem 4.26 ([**Sut78**, Theorem 3.4]). *Let G be separable locally compact group with $\Delta_G(G) = \mathbb{R}_+$. We have:*

TABLE 1. Table showing the type of $\mathscr{R}\bar{\otimes}\mathscr{S}$, [**KR86**, p. 830]

	Type of $\mathscr{S}$				
Type of $\mathscr{R}$	I_n, n finite	I_n, n infinite	II_1	II_∞	III
I_m, m finite	I_{mn}	I_{mn}	II_1	II_∞	III
I_m, m infinite	I_{mn}	I_{mn}	II_∞	II_∞	III
II_1	II_1	II_∞	II_1	II_∞	III
II_∞	II_∞	II_∞	II_∞	II_∞	III
III	III	III	III	III	III

(i) The maximal central projection p_I of type I in $\mathcal{Z}(H)$, and the maximal central projection q_I of type I in $\mathcal{Z}(G)$ satisfy $\kappa(q_I) = p_I$. Thus $\mathrm{VN}_l(G)$ is of type I if and only if $\mathrm{VN}_l(H)$ is of type I.

(ii) The maximal central semifinite projection q_{II} in $\mathcal{Z}(G)$ corresponds to the set

$$P_{II} = \{\gamma \in B_H : \textit{the isotropy subgroup of } \gamma \textit{ under } \{\check{\alpha}_t : t \in \mathbb{R}\} \textit{ is trivial}\}.$$

(iii) For $\lambda \in (0,1]$ the maximal central projection q_λ in $\mathcal{Z}(G)$ of pure type III_λ corresponds to the set

$$P_\lambda = \{\gamma \in B_H : \gamma \textit{ has period precisely } \ln\lambda \textit{ under } \{\check{\alpha}_t : t \in \mathbb{R}\}\}.$$

(iv) The maximal central projection q_0 of pure type III_0 in corresponds to the set

$$P_0 = \Gamma_H - B_H.$$

REMARK 4.27 ([**Sut78**, p. 236]). We take note of the following:

(i) Considering the action $\{\check{\alpha}_g : g \in G\}$ of G on (Γ_H, μ_H) (see 4.4 and the discussion following it) then we have the following alternative definitions of P_{II} and P_λ :

$$P_{II} = \{\gamma \in B_H : G_\gamma = H\},$$

$$P_\lambda = \{\gamma \in B_H : \Delta_G(G_\gamma) = \{\lambda^n : n \in \mathbb{Z}\}\} \quad \lambda \in (0,1],$$

where G_γ is the isotropy subgroup of γ.

(ii) According to the theorem, the reduced quasi-dual of G, Γ_H (identified with $\mathcal{Z}(H)$, see Proposition 4.5) is split in four distinct parts. Firstly, the non-smooth part (III_0). Secondly, the fixed points under $\{\check{\alpha}_t : t \in \mathbb{R}\}$, ($\mathrm{III}_1$). Thirdly, the points of B_H whose orbits are "loops", (III_λ). Lastly, the points with orbits which are copies $\mathbb{R}$, that make up the semifinite part of $\mathrm{VN}_l(G)$.

PROOF. We give an overview of Sutherland's proof [**Sut78**, pp. 236-238]. We also freely make use of the theory hitherto obtained. Let E be a Borel cross-section for the action of $\{\check{\alpha}_t : t \in \mathbb{R}\}$ on B_H. We let $\lambda^G = \int_{\Gamma_G}^{\oplus} \lambda^G_\omega d\mu_G(\omega)$ and $\lambda^H = \int_{\Gamma_H}^{\oplus} \lambda^H_f d\mu_H(\gamma)$ be the central decompositions of λ^G and λ^H respectively.

(i) Let p_I, q_I be as described in the hypothesis of the Theorem 4.26, their existence ensured by the type decomposition of von Neumann algebras (Proposition 2.20). Let $\bar{\alpha}_t^I, \sigma_t^I$ be the restrictions of $\bar{\alpha}_t, \sigma_t$ to $\mathrm{VN}_l(H)_I := \mathrm{VN}_l(H)p_I$ and $\mathrm{VN}_l(G)_I := \mathrm{VN}_l(G)q_I$ respectively.

Since σ_t^I is inner (unitarily implemented), we know that $\mathrm{VN}_l(G)_I$ is semifinite; as $\mathrm{VN}_l(G)_I$ is semifinite, we have that $\mathrm{VN}_l(G)_I \rtimes_{\sigma_t^I} \mathbb{R}$ is *-isomorphic to $\mathrm{VN}_l(G)_I \bar{\otimes} L^\infty(\mathbb{R})$. See [**GL20**, Theorem 6.26 and Theorem 6.74]. Since both $\mathrm{VN}_l(G)_I$ and $L^\infty(\mathbb{R})$ are of type I, we know that $\mathrm{VN}_l(G)_I \rtimes_{\sigma_t^I} \mathbb{R}$ is of type I by Table 1.

We recall from Theorem 4.6 that $\mathrm{VN}_l(G) \rtimes_{\sigma_t} \mathbb{R}$ is *-isomorphic to $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$, which in turn is *-isomorphic to $\mathrm{VN}_l(G)_0\bar{\otimes}\mathscr{B}(L^2(\mathbb{R}))$. We note that $q_I \in \mathrm{VN}_l(G)_0$, thus q_I is identified with the projection $(q_I \otimes \mathbb{1})$ as an entry in $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$. So by the work of [**Wat88**, p. 508] and Proposition 4.5 we have $\mathrm{VN}_l(G)q_I \rtimes_{\sigma_t} \mathbb{R} \cong \mathrm{VN}_l(G)_0 q_I \bar{\otimes}\mathcal{B}(L^2(\mathbb{R})) \cong \mathrm{VN}_l(H)\kappa(q_I)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$.

Since both $\mathrm{VN}_l(H)\kappa(q_I)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ and $\mathscr{B}(L^2(\mathbb{R}))$ are of type I, we must have that $\mathrm{VN}_l(H)\kappa(q_I)$ is of type I by Table 1. So by the definition of p_I, we have $\kappa(q_I) \le p_I$.

The von Neumann algebra $\mathrm{VN}_l(H)_I$ is also semifinite, so by reasoning as above we see that $\mathrm{VN}_l(H)_I \rtimes_{\bar{\alpha}_t^I} \mathbb{R}$ is *-isomorphic with $\mathrm{VN}_l(H)_I \bar{\otimes} L^\infty(\mathbb{R})$.

In Theorem 4.13 we have shown that λ_ω^G and $\mathrm{ind}_H^G \lambda_\omega^H$ are quasi-invariant. That is, $\mathrm{VN}_l(G)(\omega)$ is *-isomorphic $\{\pi_\omega^G(G)\}''$ (see Definition 4.12), which in turn is unitarily equivalent to $\{\overline{\pi}_\omega^G(G)\}''$. See Theorem 4.13 for details. In Theorem 4.13 we also remarked that $\{\overline{\pi}_\omega^G(G)\}''$ and $\mathrm{VN}_l(H)(\omega) \rtimes_{\overline{\alpha}_{t,\omega}} \mathbb{R}$ have the same generators.

Consequently $\int_\Gamma^\otimes \mathrm{VN}_l(G)(\omega)\,\mathrm{d}\mu(\omega) \cong \int_\Gamma^\otimes \mathrm{VN}_l(H)(\omega) \rtimes_{\overline{\alpha}_{t,\omega}} \mathbb{R}\,\mathrm{d}\mu(\omega)$. Hence we have that $\mathrm{VN}_l(G) \cong \mathrm{VN}_l(H) \rtimes_{\bar{\alpha}_t} \mathbb{R}$.

Once again [**Wat88**, p. 508] and Proposition 4.5 shows that $\mathrm{VN}_l(G)\kappa^{-1}(p_I) \cong \mathrm{VN}_l(H)_I \rtimes_{\bar{\alpha}_t^I} \mathbb{R}$. As such, the same reasoning that showed $\kappa(q_I) \le p_I$ now shows $\kappa^{-1}(p_I) \le q_I$. We therefore have that $\kappa(q_I) = p_I$.

(ii) Let P_{II} be as in the hypothesis of the theorem. We note that P_{II} is Borel in the sense of [**AM66**], since the map $\gamma \mapsto$(isotropy subgroup of γ under $\{\breve{\alpha}_t : t \in \mathbb{R}\}$) is a Borel map. We let p_{II} be the projection in $\mathcal{Z}(H)$ that corresponds to P_{II}. Note that $s \neq t \implies s - t \neq 0 \implies \breve{\alpha}_{s-t} \neq \breve{\alpha}_0 = \mathbb{1}$, since $\breve{\alpha}$ is a group homomorphism. This shows that $\breve{\alpha}$ is free in the sense of [**Slu19**, p. 1992]; which also states that in this case we may identify an orbit with a copy of the real line.

Now if $\gamma' \in G_\gamma$, then there exists some $l \in G$ such that $l\gamma' = \gamma$. Then $G_{\gamma'} = \{g \in G : g\gamma' = \gamma'\} = \{g \in G : l^{-1}gl\gamma = \gamma\} = lG_\gamma l^{-1}$. This shows that if $\gamma \in P_{II}$ then $G(\gamma) \in P_{II}$. So we identify P_{II} with $(E \cap P_{II}) \times \mathbb{R}$. Then, as Sutherland insists, the action $\breve{\alpha}_s$ on P_{II} becomes $\breve{\alpha}_s(\gamma, t) = (\gamma, t - s)$ for $\gamma \in P_{II} \cap E$ and $s, t \in \mathbb{R}$.

As noted in Prop 4.5, $\{\check{\alpha}\}_{t\in\mathbb{R}}$ is a fixed point realization of $\{\bar{\alpha}_t\}_{t\in\mathbb{R}}$. But, by the preceding discussion, we have that $\{\check{\alpha}\}_{t\in\mathbb{R}}$ is equivalent translation on $\mathbb{R}$. Therefore, we must have that $\mathrm{VN}_l(H)p_{II} \rtimes_{\bar{\alpha}} \mathbb{R}$ ($\bar{\alpha}$ restricted to $\mathrm{VN}_l(H)p_{II}$) is identified with a direct integral of factors, each semifinite by Proposition 4.22. The crossed product must then be semifinite. We can also show that $\mathrm{VN}_l(H)p_{II} \rtimes_{\bar{\alpha}} \mathbb{R} \cong \mathrm{VN}_l(G)\kappa^{-1}(p_{II})$. Reasoning similarly as in the proof of (i), we find that $\kappa^{-1}(p_{II}) \leq q_{II}$.

Let σ^{II} denote the restriction of σ to $\mathrm{VN}_l(G)q_{II}$. As was the case in (i), we have that $\{\sigma_t^{II}\}_{t\in\mathbb{R}}$ is inner. As such, $\mathrm{VN}_l(G)q_{II} \rtimes_{\sigma^{II}} \mathbb{R} \cong \mathrm{VN}_l(G)\bar{\otimes}L^\infty(\mathbb{R})$. If we restrict θ the dual of σ to $\mathcal{Z}(\mathrm{VN}_l(G)_{II} \rtimes_{\sigma^{II}} \mathbb{R})$, it is possible to show that it is equivalent to translation on $L^\infty(\mathbb{R})$. We once again obtain access to Takesaki's type criterion (Proposition 4.22). By Proposition 4.5, Theorem 4.6 and reasoning similarly as in (i), we establish that $\mathrm{VN}_l(G)_{II} \rtimes_{\sigma^{II}} \mathbb{R} \cong \mathrm{VN}_l(H)\kappa(q_{II})\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$. Let P'_{II} be a Borel set that realises $\kappa(q_{II})$ in Γ_H. We can conclude that $\mu_H(P'_{II} - P_{II}) = 0$; furthermore, almost every $\gamma \in P'_{II}$, we find that G_γ is trivial. The arguments made therefore show that $\kappa(q_{II}) \leq p_{II}$.

(iii - iv) Let P_λ and q_λ be as in the hypothesis of the theorem. We let p_λ denote the projection which corresponds to P_λ in $\mathcal{Z}(H)$.

The prescription $\lambda_\lambda^H(h) := \lambda^H(h)p_\lambda$ defines a representation of H. This representation is a direct integral of representations

$$\lambda_\lambda^H = \int_{\Gamma_G}^{\oplus} \lambda_\omega^H d\mu_G(\omega),$$

that have following properties:

(1) λ_ω^H is $\{\check{\alpha}_t\}_{t\in\mathbb{R}}$-invariant; furthermore, the corresponding action of $\mathbb{R}$ on $\mathcal{Z}(\{\lambda_\omega^H(H)\}'')$ is periodic of period $\ln\lambda$.
(2) The representations $\mathrm{Ind}_H^G \lambda_\omega^H$ yield the central decomposition of the restriction of λ^G to $\kappa^{-1}(p_\lambda)$.

(See Proposition 4.5 and Theorem 4.13 for details.) Takesaki's type criterion then tells us that $\kappa^{-1}(p_\lambda) \leq q_\lambda$.

Let q_0 be the maximal central projection of pure type III_0 in $\mathcal{Z}(G)$. Similarly, we let q_{III} be the maximal central projection of type III in $\mathcal{Z}(G)$. We let Q_0 be an invariant Borel set that realises the projection $\kappa(q_0)$ and similarly Q_{III} realizes $\kappa(q_{III})$.

We list some facts. We let $\pi^H(h) := \lambda^H(h)\kappa\,(q_{III} - q_0)$. Let $\bar{\mu}$ be the restriction of μ_H to $Q_{III} - Q_0$ and $\bar{\mu} = \int_X^{\oplus} \mu_\omega d\mu(\omega)$ be the ergodic decomposition (see [**Hoc13**]) of $\bar{\mu}$. Then π^H has a corresponding decomposition given by $\pi^H = \int_X^{\oplus} \pi_\omega^H d\mu(\omega)$. For any $\omega \in X$ it can be shown that $\mathrm{Ind}_H^G \pi_\omega^H$ is a representation of G that generates a factor of type III_λ with $\lambda = \lambda(w) \neq 0$. Perusal of [**Sut77**] shows that the map $\omega \in X \to \lambda(\omega) \in (0,1]$ is Borel. The action of $\{\check{\alpha}_t : t \in \mathbb{R}\}$ on $\mathcal{Z}(\{\lambda_\omega^H(H)\}'')$ is equivalent to the canonical action of $\mathbb{R}$ on $L^\infty(\mathbb{R}/\lambda(\omega)\mathbb{Z})$.

Checking the facts listed above, we note we are allowed to use Proposition 4.24, from which we have $Q_{III} - Q_0 \subset B_H$. From this fact it is possible to show that $P_0 \subseteq Q_0$.

The initial arguments of (iii - iv) actually shows that $\Gamma_H - P_0 \cong \Gamma_H - Q_0$ so that we will have $P_0 = Q_0$. Equivalently $\kappa\left(q_0\right) = p_0$.

But then we also have that $\kappa(q_\lambda) = p_\lambda$ for any $\lambda \in (0,1]$; for we can show that if $\lambda \in (0,1]$, then $\kappa(q_\lambda) \geq p_\lambda$, and that the projections q_λ are disjoint. To prove this, suppose contrariwise that for some $\lambda_0 \in (0,1]$ we have $\kappa(q_{\lambda_0}) > p_{\lambda_0}$. Under this assumption, we have $\kappa\left(q_{\lambda_0}\right) \cap p_0 \neq 0$. This however forces $\kappa(q_{\lambda_0}) \cap \kappa(q_0) \neq 0$ to be true, which in turn implies that $q_{\lambda_0} \cap q_0 \neq 0$ for some $\lambda_0 \in (0,1]$. This is of course a contradiction.

We have shown that $\kappa(q_\lambda) = p_\lambda$ for all $\lambda \in [0,1]$. $\square$

CHAPTER 5

Sutherland's second structure theorem

In this chapter we give an overview of Sutherland's second structure theorem [**Sut78**, Theorem 4.4]. In this chapter we make the assumptions that G is a separable locally compact group and $\Delta_G = \{e^{nT}\}_{n\in\mathbb{Z}}$ for some $T > 0$, instead of $\mathbb{R}_+$. The method of our analysis mirrors that of the previous chapter. Once again, the author has made attempts to improve upon Sutherland's exposition. For example, instead of omitting the proof of [**Sut78**, Theorem 4.4], attempts are made to justify it, and Theorem 5.4 shows computations that were omitted.

There are some subtleties to consider in this case. We cannot claim, as we did in the hypothesis of Theorem 4.6, that $\mathrm{VN}_l(G) \rtimes_{\sigma_t} \mathbb{R}$ and $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}))$ are necessarily *-isomorphic. To remedy this, we make use of the theory of "induced covariant systems" as explored by Takesaki in [**Tak73**, §10].

We fix an element $g_0 \in G$ such that $\Delta_G\,(g_0) = e^T$; furthermore, we denote by α_0 the automorphism of H defined by the prescription

$$\alpha_0(h) := g_0 h g_0^{-1}. \tag{5.1}$$

We may then identify G with $H \rtimes_{\alpha_0} \mathbb{Z}$, in which case $\mathrm{VN}_l(G) = \mathrm{VN}_l(H) \rtimes_{\bar{\alpha}_0} \mathbb{Z}$.

In the current context, we necessarily have that $t \mapsto \sigma_t$ is of period $2\pi/T$, since for all $g \in G$ we have $\sigma_t(\lambda^G(g)) = \Delta_G(g)^{it}\lambda^G(g)$. Thus the modular automorphism should be considered as an action of $\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ on $\mathrm{VN}_l(G)$. Let $\varepsilon : \mathbb{R} \to \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ be the quotient map. Since $(2\pi/T)\mathbb{Z}$ is normal, the canonical quotient map becomes a group homomorphism. This allows us to define an action of $\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ on $\mathrm{VN}_l(G)$ by the prescription

$$\psi_{\varepsilon(s)}(x) = \sigma_s(x); \quad x \in \mathrm{VN}_l(G).$$

The Pontryagin dual of $\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ is identified with $T\mathbb{Z}$ via the duality $\langle \varepsilon(s), nT\rangle := e^{isnT}$.

We also define the unitaries $p(\varepsilon(s))\,(s \in \mathbb{R})$ and $q(nT)\,(n \in \mathbb{Z})$ on $L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z})$ via the prescriptions

$$\begin{cases} (p(\varepsilon(s))\xi)(\varepsilon(t)) = \xi(\varepsilon(t-s)) \\ (q(nT)\xi)(\varepsilon(t)) = e^{intT}\xi(\varepsilon(t)), \end{cases}$$

where it is easily determined that $p(\varepsilon(s))\,(s \in \mathbb{R})$ is the left regular representation of $\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ on $L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z})$. On $L^2(T\mathbb{Z})$ we define the unitaries $\hat{p}(\varepsilon(s))\,(s \in \mathbb{R})$ and $\hat{q}(nT)\,(n \in \mathbb{Z})$ by

$$\begin{cases} (\hat{p}(\varepsilon(s))\xi)(mT) = e^{-ismT}\xi(mT) \\ (\hat{q}(nT)\xi)(mT) = \xi((m-n)T). \end{cases}$$

These unitaries will fulfil a role similar to that of $v(t)$ (see (4.2)) and $u(t)$ (see (4.3)) in the previous chapter. The generalised Fourier transform yields the following case-specific operator $\mathscr{F} : L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}) \to L^2(T\mathbb{Z})$ defined by $(\mathscr{F}\xi)(nT) = \int \overline{\langle \varepsilon(s), nT\rangle}\xi(\varepsilon(s))d(\varepsilon(s))$. Via direct computation one finds that $\mathscr{F}$ carries p to $\hat{p}$, more specifically:

$$\begin{aligned}(\mathscr{F}p(\varepsilon(r))\xi)(nT) &= \int \overline{\langle \varepsilon(s), nT\rangle}\xi(\varepsilon(s-r))\, d(\varepsilon(s)) \\ &= e^{-inTr}\int \overline{\langle \varepsilon(s), nT\rangle}\xi(\varepsilon(s))\, d(\varepsilon(s)) \\ &= (\hat{p}(\varepsilon(r))\mathscr{F}\xi)(nT).\end{aligned}$$

A similar computation shows that $\mathscr{F}q(nT)\mathscr{F}^* = \hat{q}(nT)$.

Theorem 5.1 ([**Sut78**, Proposition 4.1]). *There is an algebraic isomorphism of* $\mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ *with* $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(T\mathbb{Z}))$ *which carries the automorphism group dual to* $\{\psi_{t(s)}\}$ *to* $\{\bar{\alpha}_0^n \otimes \mathrm{Ad}\, q(nT)\}$. *Thus the restriction of the dual automorphism to the centre of the crossed product is equivalent to the action* $\{\tilde{\alpha}_0^n : n \in \mathbb{Z}\}$ *of* $\mathbb{Z}$ *on* $\mathcal{Z}(H)$.

Proof. If, in the proof of Theorem 4.6, we substitute $\{\sigma_t : t \in \mathbb{R}\}$ with $\{\psi_{\varepsilon(s)} : s \in \mathbb{R}\}$, $\mathbb{R}$ with $\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$, $\widehat{\mathbb{R}}$ with $T\mathbb{Z}$; and proceed verbatim, we then see that $\mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ is *-isomorphic with $[\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}))]^{\psi_{\varepsilon(\cdot)}\otimes \mathrm{Ad}\, p(\varepsilon(\cdot))^*}$.

In this case we find that the generators of $\mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ on $L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}, L^2(G))$ are given by

$$\begin{cases}[(A(g)\xi)(\varepsilon(t))](\cdot) := \delta(g)^{-ir}\lambda^G(g)[\xi(\varepsilon(r))](\cdot), & g \in G \\ [(B(r)\xi)(\varepsilon(t))](\cdot) := [\xi(\varepsilon(t-r))](\cdot), & r \in \mathbb{R}.\end{cases} \tag{5.2}$$

By substituting (4.7) in the proof of Theorem 4.6 with (5.2) and noting that p and q mimic the roles of v and u respectively; the generators of $\mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ as a subalgebra embedded into $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(\mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}))$, is given by

$$\begin{cases}\lambda^G(g) \otimes q(\chi(g))^*; & g \in G \\ \mathbb{1} \otimes p(\varepsilon(s)); & s \in \mathbb{R}.\end{cases} \tag{5.3}$$

where $\chi(g) = \ln \Delta_G(g) \in T\mathbb{Z}$.

Once again invoking Theorem 2.41, we see that the dual automorphism group is given by $\{\mathrm{Ad}(1 \otimes q(nT)); n \in \mathbb{Z}\}$. As was already determined, the bidual automorphism group is given by $\left\{\psi_{\varepsilon(s)} \otimes \mathrm{Ad}\, p(\varepsilon(s))^* : s \in \mathbb{R}\right\}$. We perform the Fourier transform in the second coordinate variable and obtain,

$$\begin{cases}\text{Generators} : \lambda^G(g) \otimes \hat{q}(\chi(g))^*; & g \in G \\ : \mathbb{1} \otimes \hat{p}(\varepsilon(s)); & s \in \mathbb{R} \\ \text{Dual automorphism} : \mathrm{Ad}(\mathbb{1} \otimes \hat{q}(nT)); & n \in \mathbb{Z} \\ \text{Bidual automorphism} : \psi_{\epsilon(s)} \otimes \mathrm{Ad}\, \hat{p}(\varepsilon(s))^*; & s \in \mathbb{R},\end{cases} \tag{5.4}$$

with respect to $\mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(T\mathbb{Z}))$.

We now attempt to define an operator that fulfils the role of W in Theorem 4.6. We define an operator W on $L^2(G \times T\mathbb{Z})$ by

$$(W\xi)(g, nT) = \xi(g_0^{-n}g, nT).$$

That W is an unitary and that $W^*\xi(g, nT) = \xi(g_0^n g, nT)$ follows mutatis mutandis from the proof of Theorem 4.6. As was done in Theorem 4.6 we compute

$$\begin{aligned}(W(\Delta_G^{it} \otimes \hat{p}(\varepsilon(t)^*)W^*\xi))(g, nT) &= (\Delta_G^{it} \otimes \hat{p}(\varepsilon(t)^*)W^*\xi)(g_0^{-n}g, nT)\\ &= \Delta_G(g_0^{-n}g)^{it}e^{itnT}\xi(g, nT)\\ &= ((\Delta_G^{it} \otimes \mathbb{1})\xi)(g, nT)\end{aligned}$$

and also

$$\begin{aligned}(W(\mathbb{1} \otimes \hat{q}(nT))W^*\xi)(g, mT) &= \xi(g_0^{m-n}g_0^{-\mathrm{m}}g, (m-n)T)\\ &= ((\lambda^G(g_0^n) \otimes \hat{q}(nT))\xi)(g, mT).\end{aligned}$$

Precisely as before, we may conclude that $W \in \mathrm{VN}_l(G)\bar{\otimes}\mathcal{B}(L^2(T\mathbb{Z}))$, that $\mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z} \cong \mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(T\mathbb{Z}))$, and that the dual automorphism is given by $\{\bar{\alpha}_0^n \otimes \operatorname{Ad}\hat{q}(nT) : n \in \mathbb{Z}\}$. $\square$

We let $\mathscr{N}_0 := \mathrm{VN}_l(G) \rtimes_{\psi_{\varepsilon(s)}} \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}$ and $\{\beta_{nT}\}_{n\in\mathbb{Z}}$ be the dual automorphism group on $\mathscr{N}_0$; let $\mathscr{N}_1 := \mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$ and $\{\theta_t\}_{t\in\mathbb{R}}$ is dual to $\{\sigma_t\}_{t\in\mathbb{R}}$. We now relate the covariant systems $(\mathcal{N}_0, T\mathbb{Z}, \beta)$ and $(\mathcal{N}_1, \mathbb{R}, \theta)$ to one another.

Definition 5.2 (Induced action, [**Tak73**, Definition 10.2]). We consider the von Neumann algebra $L^\infty(G)\bar{\otimes}\mathcal{M}$, where $\mathcal{M}$ is a von Neumann algebra acting on $\mathfrak{H}$. This von Neumann algebra may be identified with the algebra of all, σ-strong*-measurable, essentially bounded operator fields from G to $\mathcal{M}$. (See [**Tak03a**, Theorem 7.17] for more on this.) Let β be an action of a closed subgroup $H \subseteq G$ on $\mathcal{M}$. We define actions γ and χ of H and G on $L^\infty(G)\bar{\otimes}\mathcal{M}$ respectively, by the prescriptions.

$$\gamma_h(x)(g) = \beta_h(x(gh)), \qquad g \in G, h \in H \tag{5.5}$$

$$\chi_k(x)(g) = x(k^{-1}g), \qquad g, k \in G, x \in \mathcal{M} \otimes L^\infty(G). \tag{5.6}$$

Let $\mathcal{M}_0 := (L^\infty(G)\bar{\otimes}\mathcal{M})^\gamma$. It can be shown that χ_g and γ_h commute for all $g \in G$ and all $h \in H$, hence $\mathcal{M}_0$ is invariant under χ; we have obtained the action $\chi|_{\mathcal{M}_0}$ of G on $\mathcal{M}_0$, which we will also denote by χ.

The action χ of G on $\mathcal{M}_0$ is said to be **induced** up to G from the action β of H, and we write

$$(\mathcal{M}_0, G, \chi) = \operatorname{Ind}_H^G(\mathcal{M}, H, \beta).$$

Definition 5.3 (Equivalent covariant systems). We say that the covariant systems $(\mathcal{M}, G, \alpha)$ and $(\mathcal{N}, G, \beta)$ are **equivalent**, if there exists a normal *-isomorphism $\phi : \mathcal{M} \to \mathcal{N}$ such that $\phi(\alpha_g(x)) = \beta_g(\phi(x))$ for all $g \in G$ and $x \in \mathcal{M}$. By Theorem 2.39, such covariant systems generate the same crossed products.

THEOREM 5.4 ([**Sut78**, Proposition 4.2]). *Let* $(\mathscr{N}_0, T\mathbb{Z}, \beta)$ *be the covariant system described above. Then:*

- *(i) The covariant systems* $(\mathscr{N}_1, \mathbb{R}, \theta)$ *and* $\mathrm{Ind}_{T\mathbb{Z}}^{\mathbb{R}}(\mathscr{N}_0, T\mathbb{Z}, \beta)$ *are equivalent.*
- *(ii) The restriction* $\tilde{\theta}_t$ *of* θ_t *to the centre of* $\mathscr{N}_1$ *is the identity if and only if* $t = nT$ *for some* $n \in \mathbb{Z}$*, and, for this* n*, we have that* β_{nT} *is the identity on the centre of* $\mathscr{N}_0$*.*

PROOF. *(i)* From Definition 5.2 and the discussion preceding it, we ascertain the existence of the covariant system $(\mathcal{M}_0, \mathbb{R}, \chi) = \mathrm{Ind}_{T\mathbb{Z}}^{\mathbb{R}}(\mathscr{N}_0, T\mathbb{Z}, \beta)$. We note that $\mathcal{M}_0$ is the subalgebra of $L^\infty(\mathbb{R})\bar{\otimes}\mathscr{N}_0$ consisting of the operator fields $x(\cdot)$ in $L^\infty(\mathbb{R})\bar{\otimes}\mathscr{N}_0$ such that

$$x(t) = \beta_{nT}(x(t + nT)) \quad \mu_{\mathbb{R}} \text{ a.e.}, \tag{5.7}$$

and that the action χ of $\mathbb{R}$ on $\mathcal{M}_0$ is defined by

$$(\chi_s(x))\,(t) = x(t - s).$$

We proceed to identify $\mathcal{M}_0$ with $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$. Consider the map $\phi : L^\infty([0,T))\bar{\otimes}\mathscr{N}_0 \to \mathcal{M}_0$ defined by

$$[\phi(x(\cdot))](t) = \beta_{-nT}(x(r)) \quad \text{with } t = r + nT,\ n \in \mathbb{Z}, 0 \leq r < T.$$

Let $x(\cdot), y(\cdot) \in L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$ We see that

$$\begin{aligned} y(\cdot) = x(\cdot) &\iff \beta_{-nT}(x(r)) = \beta_{-nT}(y(r)) \text{ for all } 0 \leq r < T, n \in \mathbb{Z}. \\ &\iff [\phi(x(\cdot))](t) = [\phi(y(\cdot))](t) \text{ for all } t \in \mathbb{R}. \end{aligned}$$

So ϕ is well-defined, injective and certainly surjective by Equation (5.7). Under this identification, action χ_s is given by

$$(\chi_s(x))\,(t) = \begin{cases} \beta_{(n+1)T}(x(t - r + T)); & 0 \leq t < r \\ \beta_{nT}(x(t - r)); & r \leq t < T \end{cases} \tag{5.8}$$

where $s = r + nT$, $0 \leq r < T$. Recall from (5.4) that $\mathscr{N}_0$ may be regarded as a von Neumann algebra acting on $L^2(G \times T\mathbb{Z})$; therefore, $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$ may be considered as a von Neumann algebra acting on $L^2(G \times [0,T) \times T\mathbb{Z})$. The unitary $(U\xi)(g, r + nT) = \xi(r, g, nT)$ identifies the Hilbert spaces $L^2([0,T) \times G \times T\mathbb{Z})$ and $L^2(G \times \mathbb{R})$ with one another. We proceed to find generators of a suitable nature for $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$ on $L^2(G \times \mathbb{R})$. Adopting the notation used in (5.4), the generators of $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$ are operators on $L^2([0,T) \times G \times T\mathbb{Z})$ of the form

$$\begin{cases} f \otimes \lambda^G(g) \otimes \hat{q}(\chi(g))^*; & f \in L^\infty([0,T)),\ g \in G \\ f \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(s))^*; & f \in L^\infty([0,T)),\ s \in \mathbb{R}. \end{cases} \tag{5.9}$$

Let $\xi \in L^2(G \times \mathbb{R})$ and $h \in G$, then

$$\begin{aligned} [U(f \otimes \lambda^G(g) \otimes \hat{q}(\chi(g))^*)U^*\xi](h, r + nT) &= f(r)[U^*\xi](r, g^{-1}h, nT + \chi(g)) \\ &= f(r)\xi(g^{-1}h, r + nT + \chi(g)). \end{aligned}$$

Similarly $[U(f \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(s))^*)U^*\xi](h, r+nT) = f(r)e^{-isnT}\xi(h, r+nT)$. Evidently, the operators of the form

$$\begin{cases} \lambda^G(g) \otimes v(\chi(g)), & g \in G \\ \mathbb{1} \otimes u(s), & s \in \mathbb{R}. \end{cases}$$

generate $U(L^\infty([0,T))\bar{\otimes}\mathscr{N}_0)U^*$ on $L^2(G \times \mathbb{R})$. These operators however, are the Fourier transforms of the generators of the crossed product $\mathrm{VN}_l(G) \rtimes_\sigma \mathbb{R}$, as seen in (4.12).

It can be shown that $U^*(\mathbb{1} \otimes u(t))U = e^{-it(\cdot)} \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(t))$. Let $\mathbb{R} \ni s = r + nT$ for some $0 \le r < T$ and $n \in \mathbb{Z}$, then by (5.4) and Equation (5.8); and using the tensor interpretation of $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$ we see that

$$\begin{aligned} \chi_s(e^{-it(\cdot)} \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(t))) &= \begin{cases} [e^{it(r-T)} \otimes \mathbb{1} \otimes \operatorname{Ad}\hat{q}(nT+T)](e^{-it(\cdot)} \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(t))); & 0 \le (\cdot) < r \\ [e^{itr} \otimes \mathbb{1} \otimes \operatorname{Ad}\hat{q}(nT)](e^{-it(\cdot)} \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(t))); & r \le (\cdot) < T \end{cases} \\ &= \begin{cases} e^{it(r-T)}e^{-it(\cdot)} \otimes \mathbb{1} \otimes e^{it(nT+T)}\hat{p}(\varepsilon(t)); & 0 \le (\cdot) < r \\ e^{itr}e^{-it(\cdot)} \otimes \mathbb{1} \otimes e^{itnT}\hat{p}(\varepsilon(t)); & r \le (\cdot) < T \end{cases} \\ &= e^{ist}[e^{-it(\cdot)} \otimes \mathbb{1} \otimes \hat{p}(\varepsilon(t))]. \end{aligned}$$

With slight abuse of notation, this shows that

$$\chi_s(\mathbb{1} \otimes u(t)) = e^{ist}\mathbb{1} \otimes u(t).$$

Similarly we also obtain

$$\chi_s(\lambda^G(g)) \otimes v(\chi(g)) = \lambda^G(g) \otimes v(\chi(g)).$$

But this is precisely the action of the dual automorphism by Theorem 2.41. Since we have identified $\{\chi_t\}_{t\in\mathbb{R}}$ with $\{\theta_t\}_{t\in\mathbb{R}}$ and $U(L^\infty([0,T))\bar{\otimes}\mathscr{N}_0)U^*$ is generated by the generators of $\mathscr{N}_1$, we conclude that $(\mathscr{N}_1, \mathbb{R}, \theta)$ and $\mathrm{Ind}_{T\mathbb{Z}}^{\mathbb{R}}(\mathscr{N}_0, T\mathbb{Z}, \beta)$ are equivalent.

(ii) We investigate the restriction $\tilde{\chi}_s$ of χ_s to the centre $L^\infty([0,T))\bar{\otimes}\mathcal{Z}(\mathscr{N}_0)$ of $L^\infty([0,T))\bar{\otimes}\mathscr{N}_0$. Let $s = r + nT$ for some $0 \le r < T$, $n \in \mathbb{Z}$ and suppose $\tilde{\chi}_s$ is the identity. Suppose $r \neq 0$. If $0 \le t < r$, then by Equation (5.8) we have $\beta_{-(n+1)T}(x(t)) = x(t - r + T)$ for all $x(\cdot) \in L^\infty([0,T))\bar{\otimes}\mathcal{Z}(\mathscr{N}_0)$. This is true iff $x(t+(n+1)T) = x(t-r+T)$ for all $x(\cdot) \in L^\infty([0,T))\bar{\otimes}\mathcal{Z}(\mathscr{N}_0)$, which is a contradiction. The converse is not difficult. □

We consider some additional results that help make up the prelude of Sutherland's second structure theorem.

PROPOSITION 5.5 ([**Sut78**, p. 242]). *Theorem 4.13 holds if the covariant system* $(\mathrm{VN}_l(G), \mathbb{R}, \sigma_t)$ *is replaced with* $(\mathrm{VN}_l(G), \mathbb{R}/(\frac{2\pi}{T})\mathbb{Z}, \psi)$.

Given (Γ_H, μ_H) which is the reduced quasi-dual of H, let $\check{\alpha}_0$ be the automorphism of (Γ_H, μ_H) corresponding to the automorphism α_0 of H, which is constructed as in the discussion preceding (4.4).

We also have the following result [**Sut78**, p. 242]. Careful perusal of [**Mac66**] and [**Amb41**] shows that the action $\{\check{\theta}_t : t \in \mathbb{R}\}$ of $\mathbb{R}$ on $\Gamma_H \times [0,T)$ induced by the action $\{\tilde{\theta}_t : t \in \mathbb{R}\}$ of $\mathbb{R}$ on $L^\infty([0,T))\bar{\otimes}\mathcal{Z}(\mathrm{VN}_l(H))$ is the flow built under the constant function $\Phi(\gamma) = T$ $(\gamma \in \Gamma_H)$ from $\check{\alpha}_0$. In essence, we are identifying $\mathscr{N}_0$ with $\mathrm{VN}_l(H)\bar{\otimes}\mathcal{B}(L^2(T\mathbb{Z}))$, from Theorem 5.1.

Let n denote the normalized Lebesgue measure on $[0,T)$. We wish to compare the ergodic decomposition of μ_H (with respect to $\check{\alpha}_0$) and $\mu_H \times n$ (with respect to θ_t). We also want to know how the "smooth parts" of these actions (see Theorem 4.20 and the discussion preceding it) compare. The following proposition provides the answers. We let π_H denote the projection of $\Gamma_H \times [0,T)$ on Γ_H.

PROPOSITION 5.6 ([**Sut78**, Lemma 4.3]). *Let $B_{\mathbb{Z}}, B_{\mathbb{R}}$ denote the smooth parts of the actions $\{\Gamma_H, \check{\alpha}_0\}$ and $\{\Gamma_H \times [0,T); \check{\theta}_t\}$ respectively. Then:*

(i) $\mu_H(\pi_H(B_{\mathbb{R}}) \triangle B_{\mathbb{Z}}) = 0.$

(ii) If $\mu_H = \int_x^{\oplus} \mu_z dm(x)$ is the ergodic decomposition of μ_H (with respect to $\check{\alpha}_0$), then $\mu_H \times n = \int_x^{\oplus}(\mu_x \times n) dm(x)$ is the ergodic decomposition of $\mu_H \times n$ with respect to $\{\check{\theta}_t\}$.

(iii) Let $\check{\alpha}_0 = \int_x^{\oplus} \check{\alpha}_{0,x} dm(x)$ be the decomposition of $\check{\alpha}_0$ corresponding to the decomposition in (ii), and $\check{\theta}_t = \int_x^{\oplus} \check{\theta}_{t,x} dm(x)$ the decomposition of $\{\check{\theta}_t : t \in \mathbb{R}\}$ corresponding to ergodic decomposition of $\mu_n \times n$. Then, for almost all x, $\{\check{\theta}_{t,x} : t \in \mathbb{R}\}$ on $\{\Gamma_H \times [0,T); \mu_x \times n\}$ is the flow built under the constant function $\Phi(\gamma) = T$ from the automorphism $\check{\alpha}_{0,x}$ on (Γ_H, μ_x).

Finally, our case-specific analogue of Theorem 4.26:

THEOREM 5.7. *Suppose G is separable, and $\Delta_G(G) = \left\{e^{nT} : n \in \mathbb{Z}\right\}$.*

(i) Let p_I and q_I denote the maximal central projections of type I in $\mathrm{VN}_l(H), \mathrm{VN}_l(G)$ respectively; then $\kappa(q_I) = p_I$.

(ii) Let q_{II} be the maximal central semifinite projection in $\mathrm{VN}_l(G)$ and p_{II} the central projection in $\mathrm{VN}_l(H)$ corresponding to $P_{II} = \{\gamma \in B_{\mathbb{Z}}\}$: the isotropy subgroup of γ under $\{\check{\alpha}_0^n : n \in \mathbb{Z}\}$ is trivial $\}$. Then $\kappa(q_{II}) = p_{II}$.

(iii) For each n, let p_n be the central projection in $\mathrm{VN}_l(H)$ corresponding to

$$P_n = \{\gamma \in B_{\mathbb{Z}} : \text{ isotropy subgroup of } \gamma \text{ under } \{\check{\alpha}_0^m : m \in \mathbb{Z}\} \text{ is } \{kn : k \in \mathbb{Z}\}\}.$$

Then $\kappa^{-1}(p_n)$ is the maximal central projection of $\mathrm{VN}_l(G)$ of pure type III_{λ^n} where $\lambda = e^{-T}$.

(iv) If $P_0 = \Gamma_H - B_{\mathbb{Z}}$, and p_0 is the corresponding central projection of $\mathrm{VN}_l(H)$, then $\kappa^{-1}(p_0)$ is the maximal central projection in $\mathrm{VN}_l(G)$ of pure type III_0.

We will not prove Theorem 5.7, but will justify its validity.

It is clear that Theorem 5.1 is the analogue of Theorem 4.6. So in the current context, we have that on the centre of $\mathrm{VN}_l(G)$ the induced dual action is identified with the action $\tilde{\alpha}_0^n$ of $\mathbb{Z}$ on

$\mathscr{Z}(H)$. We also see that Theorem 5.4 and Proposition 5.6, jointly serve as analogues of Theorem 4.13 and Proposition 4.5. No analogue of Theorem 4.20 is needed as its only role was to establish the existence of the smooth parts of any reduced quasi-dual under consideration.

In the proof of Theorem 4.26 we relied on the fact that restrictions of the modular group on the semifinite parts were inner to, for example, identify the crossed product $\mathrm{VN}_l(G)_I \rtimes_{\bar{\alpha}^I} \mathbb{R}$ with $\mathrm{VN}_l(G)_I \bar{\otimes} L^\infty(\mathbb{R})$. We require something similar in this context.

The action $\psi_{\varepsilon(s)}$ of $\mathbb{R}/(\frac{2\pi}{T}\mathbb{Z})$ is built up from the modular automorphism group. But the modular automorphism group is trivial on the semifinite parts of $\mathrm{VN}_l(G)$, therefore the induced action $\psi_{\varepsilon(s)}$ will also be trivial on the semifinite parts. Let the type I and II parts of $\mathrm{VN}_l(G)$ be $\mathrm{VN}_l(G)_I$ and $\mathrm{VN}_l(G)_{II}$ respectively. By [**GL20**, Proposition 6.43] we have that $\mathrm{VN}_l(G) \cong \pi_\psi(\mathrm{VN}_l(G)_I) \bar{\otimes} \mathbb{C}$, since the left regular representation generates the algebra $\mathbb{C} \bar{\otimes} \mathrm{VN}_l(\mathbb{R}/(\frac{2\pi}{T}\mathbb{Z}))$, the crossed product $\mathrm{VN}_l(G)_I \rtimes_{\bar{\alpha}_0^I} \mathbb{Z}$ is of the form $\mathrm{VN}_l(G)_I \bar{\otimes} \mathrm{VN}_l(\mathbb{R}/(\frac{2\pi}{T}\mathbb{Z}))$. Hence, by application of the Fourier transform, we obtain equivalence with $VN(G_I) \bar{\otimes} L^\infty(T\mathbb{Z})$. Of course this result also holds in the case of $\mathrm{VN}_l(G)_{II}$.

With the facts established above, the proof of Theorem 5.7 develops similarly to that of Theorem 4.26.

CHAPTER 6

Constructing examples

Theorems 4.26 and 5.7 provide both necessary and sufficient conditions for the regular representation of a separable locally compact group to generate a von Neumann algebra with a central summand of pure type III_λ with $\lambda \in [0,1]$. We proceed, as in [**Sut78**, pp. 243-246], to construct such examples. As in the previous two chapters, the author has tried to make the exposition clearer. Definitions and background knowledge of the group measure space construction were incorporated from [**Tak03b**]. References to why the solvability of groups leads to crossed products that are hyperfinite were also given.

6.1. Hyperfinite examples

Definition 6.1 (Dual automorphism). Let α be an automorphism of G. We define an automorphism on $\widehat{G}$ by $\underline{\alpha}(\xi) := \xi \circ \alpha^{-1}$ for all $\xi \in \widehat{G}$. This automorphism is called the **dual automorphism** of α. In bracket notation, it is the unique automorphism such that $\langle \alpha(g), \xi \rangle = \langle g, \underline{\alpha}^{-1}(\xi) \rangle$.

Remark 6.2. This should not be mistaken for the dual action (see Definition 2.40).

Recall that given locally compact groups G and N and an action $\alpha : G \to \mathrm{Aut}(N)$, we can define a corresponding action $\bar{\alpha}$ of G on $\mathrm{VN}_l(N)$ by means of the unitary group $\{u_g\}_{g \in G}$ defined in Theorem 2.44.

The same prescription allows us to define an action of G on $L^\infty(G)$ also denoted $\bar{\alpha}$. That is, for all $g \in G$, we define an action $\bar{\alpha}$ of G on $L^\infty(G)$ by the prescription

$$\begin{aligned}(\bar{\alpha}_g(\eta)\xi)(s) &= (u_g \eta u_g^* \xi)(s) \\ &= \delta(\alpha)^{\frac{1}{2}} \eta u_g^* \xi(\alpha_{g^{-1}}(s)) \\ &= \eta(\alpha_{g^{-1}}(s))\xi(s) \\ &= [(\eta \circ \alpha_{g^{-1}})(\xi)](s).\end{aligned} \tag{6.1}$$

We will have need of the following theorem.

Theorem 6.3. *Let α be an automorphism of locally compact group G and $\underline{\alpha}$ the dual automorphism of α. Then $\mathscr{F}\,\bar{\alpha}(\mathrm{VN}_l(G))\mathscr{F}^{-1} = \bar{\underline{\alpha}}(L^\infty(\widehat{G}))$.*

Proof. By [**Tak03a**, VII, Theorem 3.14] $\mathscr{F}\,\mathrm{VN}_l(G)\mathscr{F}^{-1}$ is precisely the left multiplication algebra $L^\infty(\widehat{G})$ acting on $L^2(\widehat{G})$.

Let $g \in G$, $\xi \in L^2(G) \cap L^1(G)$, $\lambda(m) \in \mathrm{VN}_l(G)$ and $f \in \widehat{G}$. Then

$$\begin{aligned}(\mathscr{F}\lambda(m)\xi)(f) &= \int_G \overline{\langle s, f\rangle}\xi(m^{-1}s)\,ds \\ &= \int_G \overline{\langle ms, f\rangle}\xi(s)\,ds \\ &= \overline{\langle m, f\rangle}\int_G \overline{\langle s, f\rangle}\xi(s)\,ds \\ &= \overline{\langle m, f\rangle}\mathscr{F}\xi(f).\end{aligned}$$

We have shown that $\mathscr{F}\lambda(m) = M_m\mathscr{F}$, where M_m is the multiplication operator of $\overline{\langle m, \cdot\rangle}$. We see that $\overline{\langle \alpha(m), f\rangle}\xi(f) = \overline{\langle m, \underline{\alpha}^{-1}(f)\rangle}\xi(f)$, by Definition 6.1. That is to say, if $h \in L^\infty(\widehat{G})$ corresponds to M_m, then $h \circ \underline{\alpha}^{-1}$, corresponds to $M_{\alpha(m)}$. By Equation 6.1 we have $\bar{\underline{\alpha}}(h) = h \circ \underline{\alpha}^{-1}(h)$. The claim then holds for general elements of $\mathrm{VN}_l(G)$ by strong continuity. □

Suppose that A is a unimodular group such that $\mathrm{VN}_l(A)$ is a factor. For $\lambda \in [0,1]$ we let α_λ be a continuous automorphism of A such that $d\mu_A \circ \alpha_\lambda^{-1}/d\mu_A = \lambda$. Let $G_\lambda := A \ltimes_{\alpha_\lambda} \mathbb{Z}$. By Corollary 2.49 and Theorem 3.9, we have that $\phi^A \circ \bar{\alpha}_\lambda = \lambda$ and $\mathrm{VN}_l(G_\lambda) \cong \mathrm{VN}_l(A) \rtimes_{\bar{\alpha}_\lambda} \mathbb{Z}$. Therefore, by Proposition 3.17 we have that $\mathrm{VN}_l(G_\lambda)$ is a factor of type III_λ.

Our task will be to construct examples of such A and α_λ. We will need the following theorem.

Theorem 6.4. *Let A and α_λ be as above. Let $K = \{0,1\}^{\mathbb{Z}}$, regarded as a compact group, and ℓ the coordinate shift on K. Let $\widehat{K}$ be the dual group of K and $\underline{\ell}$ the automorphism of $\widehat{K}$ dual to ℓ on K. Define the automorphism β_λ of $A \times \widehat{K}$ by $\beta_\lambda(h,\xi) = (\alpha_\lambda(h), \underline{\ell}(\xi))$, and put $G_{0,\lambda} = (A \times \widehat{K}) \ltimes_{\beta_\lambda} \mathbb{Z}$. Then $\mathrm{VN}_l(G_{0,\lambda})$ is a factor of type III_0.*

Proof. We note that $\mathrm{VN}_l(A \times \widehat{K})$ is isomorphic with $\mathrm{VN}_l(A)\bar{\otimes}\mathrm{VN}_l(\widehat{K})$. Since $\mathrm{VN}_l(A)$ is a factor and $\widehat{K}$ is abelian, the centre of $\mathrm{VN}_l(A \times \widehat{K})$ is given by $\mathbb{C}\,\bar{\otimes}\,\mathrm{VN}_l(\widehat{K})$.

Therefore the centre can also be given by $(\mathbb{1} \otimes \mathscr{F}^{-1})\mathbb{C}\,\bar{\otimes}\,\mathrm{VN}_l(\widehat{K})(\mathbb{1} \otimes \mathscr{F}) \equiv L^\infty(K)$, see [**GL20**, Theorem 6.45]. Since $L^\infty(K)$ is the left multiplication algebra on $L^2(K)$, the projections of $L^\infty(K)$ are the characteristic functions of the Borel measurable sets. For more on this see [**KR83**, 2.5.12. Example]. Hence, the nonatomic definition of a (tracial) von Neumann algebra simplifies to that of the measure theoretic case. The measure space $([0,1], m)$ where m is the Lebesgue measure, is nonatomic. By Example 1.20 we can identify $L^\infty(K, \mu_K)$ with $L^\infty([0,1]^2, m \times m)$. Consequently, the centre of $\mathrm{VN}_l(A \times \widehat{K})$ is nonatomic.

Note that $\bar{\beta}_\lambda$ restricted to the centre simplifies to the automorphism $\bar{\underline{\ell}}$ on $\mathrm{VN}_l(\widehat{K})$, so that it may be viewed as $\bar{\ell}$ acting on $L^\infty(K)$ by Theorem 6.3. The automorphism β_λ induces an action of $\mathbb{Z}$ on $A \times \widehat{K}$ by $\mathbb{Z} \ni z \mapsto \beta_\lambda^z$.

The fixed point subalgebra $\mathcal{Z}(L^\infty(K))^{\bar{\ell}}$ is isomorphic to the complex plane, as only the constant functions can remain fixed under coordinate shifts. Hence we have shown that $\bar{\ell}$ is centrally ergodic.

Let τ denote the Plancherel weight on $\mathrm{VN}_l(A)\bar{\otimes}\,\mathrm{VN}_l(\widehat{K})$. Theorem 3.8 and the identification of $L^2(A)\otimes L^2(\widehat{K})$ with $L^2(A\times\widehat{K})$ (as was done in Theorem 4.6), ensures that τ is the unique trace such that $\tau(x\otimes y) = \phi^A(x)\phi^{\widehat{K}}(y)$, where $x\otimes y \in \mathrm{VN}_l(A\times\widehat{K})$. Theorem 3.9 then shows that $\delta_{\beta_\lambda}(1) = \delta_{\alpha_\lambda}(1)\delta_{\underline{\ell}}(1)$.

By [**Fol16**, 4.25 Proposition], we have that the Haar measure $\mu_{\widehat{K}}$ is the counting measure (up to multiplication by some scalar). Since $\underline{\ell}$ is a $1-1$ mapping, $\mu_{\widehat{K}}\circ\underline{\ell}^{-1} = \mu_{\widehat{K}}$, hence we must have $\delta(\underline{\ell}) = 1$.

By assumption $\frac{d\mu_A\circ\alpha_\lambda^{-1}}{d\mu_A} = \lambda$, therefore $\delta(\alpha_\lambda) = \lambda$. So all the requirements of [**Tak03a**, XII, Theorem 3.7 (ii)] are satisfied and $\mathrm{VN}_l(A\times\widehat{K})\rtimes_{\bar{\beta}_\lambda}\mathbb{Z}$ is a factor of type III_0. Since $\mathrm{VN}_l(A\times\widehat{K})\rtimes_{\bar{\beta}_\lambda}\mathbb{Z}\cong\mathrm{VN}_l(G_{0,\lambda})$ by Corollary 2.49, we have obtained the desired result. □

6.1.1. Hyperfinite example.

Definition 6.5. We say a von Neumann algebra $\mathcal{M}$ is **hyperfinite** or **approximately finite dimensional** if there exists an increasing chain of finite dimensional von Neumann subalgebras $\mathcal{M}_n$ of $\mathcal{M}$, such that their union is weak dense in $\mathcal{M}$.

We will have need of some theory from the group measure space construction of factors.

Definition 6.6 ([**Tak03b**]). Let G be a separable locally compact group, and $\{\Omega,\mu\}$ a σ-finite standard measure space. By an action of G on $\{\Omega,\mu\}$, we mean a Borel map $T:(s,\omega)\in G\times\Omega\mapsto T_s\omega\in\Omega$ such that

(i) for each fixed $s\in G$, the map: $\omega\in\Omega\mapsto T_s\omega\in\Omega$ is a non-singular bijection of $\{\Omega,\mu\}$;
(ii) $T_s(T_t\omega) = T_{st}\omega, s,t\in G,\omega\in\Omega$;
(iii) $T_e\omega = \omega$, where e is the unit of G.

In most cases, we assume that $T_s\omega=\omega$ for every $\omega\in\Omega$ implies $s=e$ to avoid a triviality. We also say that $\{\Omega,\mu\}$ is a G-measure space.

With $\{G,\Omega,\mu\}$, we consider an action α of G on the abelian von Neumann algebra $\mathcal{A}=L^\infty(\Omega,\mu)$ given by the following:

$$[\alpha_s(a)](\omega) = a\left(T_s^{-1}\omega\right),\quad s\in G,\quad a\in\mathcal{A},\quad \omega\in\Omega.$$

Definition 6.7 ([**Tak03b**, Definition 1.3]). We say that an action T of G on $\{\Omega,\mu\}$ is free if for any compact subset K of G such that $e\notin K$ and any Borel subset E of Ω with $\mu(E)>0$, there exists a Borel subset $F\subset E$ such that $\mu(F)>0$ and $T_s(F)\cap F=\emptyset$ for every $s\in K$. The action T is said to be ergodic if $T_sE=E$ for every $s\in G$ implies $\mu(E)=0$ or $\mu(\Omega\backslash E)=0$.

From [**Tak03b**, p. 4] we have that for $\mathcal{A}:=L^\infty(\Omega,\mu)$ and the action α of G on $\mathcal{A}$ as above, the freeness and the ergodicity of T may be rephrased as follows. The action T is ergodic if and only if $\{\mathcal{A},G,\alpha\}$ is ergodic in the sense that $\mathcal{A}^\alpha=\mathbb{C}$; T is free if and only if for any compact subset K of $G \setminus \{e\}$ and a nonzero projection $p \in \mathcal{A}$ there exists a nonzero projection $f \in \mathcal{A}$ such that $f \leq p$ and $f\alpha_s(f) = 0$ for every $s \in K$.

Definition 6.8 ([**Tak03b**]). Given $\{G, \Omega, \mu, T\}$ as above, the crossed product $\mathcal{A} \rtimes_\alpha G$ is called the von Neumann algebra associated with $\{G, \Omega, \mu, T\}$.

We now proceed to the examples.

Let $U := U(2, \mathbb{Q})$ denote the square upper triangular matrices with rational entries, and nonzero determinant. Let $N := N(2, \mathbb{Q})$ be the normal subgroup consisting of elements from U having determinant one. Let $N_\lambda := N_\lambda(2, \mathbb{Q})$ be the subgroup of $GL(2, \mathbb{R})$ (the general linear group) generated by $N(2, \mathbb{Q})$ and the matrices $\begin{pmatrix} \lambda^{n/2} & 0 \\ 0 & \lambda^{n/2} \end{pmatrix}$, $\lambda \in (0, 1)$. Note that $N_\lambda(2, \mathbb{Q})$ may be regarded as the direct product of $N(2, \mathbb{Q})$ and $\mathbb{Z}$. We consider the groups P_λ ($\lambda \in (0, 1]$) defined by

$$P_1 = \mathbb{R}^2 \ltimes_d U$$
$$P_\lambda = \mathbb{R}^2 \ltimes_d N_\lambda$$

where $d_T(\mathbf{x}) := T\mathbf{x}$ where T is a square matrix and $\mathbf{x} \in \mathbb{R}^2$.

Consider $(\mathbb{R}^2 \ltimes_d N) \ltimes_{z(\lambda)} \mathbb{Z}$ where the action $z(\lambda)$ of $\mathbb{Z}$ on $\mathbb{R}^2 \ltimes_d N$ is given by

$$[z(\lambda)_n](\mathbf{x}, T) := (\lambda^n \mathbf{x}, T)\,.$$

We identify $(\mathbb{R}^2 \ltimes_d N) \ltimes_{z(\lambda)} \mathbb{Z}$ with P_λ in the obvious way.

Theorem 6.9. *Let P_λ be as above, then $\mathrm{VN}_l(P_\lambda)$ is a hyperfinite factor of type III_λ for each $\lambda \in (0, 1]$.*

Proof. We start by obtaining the type of $\mathrm{VN}_l(P_\lambda)$. Firstly, let $A := \mathbb{R}^2 \ltimes_d N$, we proceed to show that $\mathrm{VN}_l(A)$ is a hyperfinite factor of type II_∞.

From Theorem 6.3 and Corollary 2.49, we obtain $\mathrm{VN}_l(A) \cong \mathrm{VN}_l(\mathbb{R}^2) \rtimes_{\bar{d}} N \cong L^\infty(\widehat{\mathbb{R}^2}) \rtimes_{\bar{d}} N$. It is an exercise to use Definition 6.1 and the fact that $\widehat{\mathbb{R}^2} \cong \mathbb{R}^2$, to show that: $L^\infty(\widehat{\mathbb{R}^2}) \rtimes_{\bar{d}} N \cong L^\infty(\mathbb{R}^2) \rtimes_{\bar{d}} N$. But $L^\infty(\mathbb{R}^2) \rtimes_{\bar{d}} N$ is the von Neumann algebra associated with $\{N, \mathbb{R}^2, m \times m, d\}$ in the sense of Definition 6.8.

So we have access to results obtained from the group measure space construction. We wish to make use of the classical criteria for the type of factors arising in the group measure space construction, i.e., [**Tak03b**, XIII, Theorem 1.7]; to which end we show that d is ergodic and free.

We see that $L^\infty(\mathbb{R}^2)^{\bar{d}} = \{f \in L^\infty(\mathbb{R}^2) : \bar{d}_T(f) = f \text{ for all } T \in N\} \cong \mathbb{C}$, since $f \circ d_T = f$ for all $T \in N$, implies that f must be a scalar multiple of the unit of $L^\infty(\mathbb{R}^2)$. By the remarks after Definition 6.7, the preceding fact shows that d acts ergodically.

Since N is a countably infinite discrete group, we obtain access to an another definition of "free" from [**KR86**, p. 553]. In our case it reads: The action d of N acts freely on $\mathbb{R}^2$, if for all $T \in N$ such that T is not the identity, we have $\{\mathbf{x} \in \mathbb{R}^2 : T\mathbf{x} = \mathbf{x}\}$ is a null set. Let $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in N$. Then

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cy \end{pmatrix}$$

We are interested in $(x, y) \in \mathbb{R}^2$ such that $ax + by = x$ and $cy = y$, with $ac = 1$. This is always true for $(0, 0)$. Clearly, $cy = y$ forces $c = 1$, except if $y = 0$. Let us assume $y \neq 0$, then $a = 1$ as $ac = 1$. In this case, for $x \in \mathbb{R}$ we have that $x + by = x$ holds iff $b = 0$. If $y = 0$ and $x \neq 0$, then $ax = x$ forces $a = 1$, so $c = 1$, but b may take on any value.

The facts above show that $\{\mathbf{x} \in \mathbb{R}^2 : T\mathbf{x} = \mathbf{x}\} \subseteq \mathbb{R} \times \{0\}$ for any $T \in N$ not equal to the identity. As $\mathbb{R} \times \{0\}$ is a null set, we have shown that d acts freely on $\mathbb{R}^2$.

By [**Tak03b**, XIII, Theorem 1.7] (or [**Tak01**, Theorem 7.12]) we obtain that $L^\infty(\widehat{\mathbb{R}^2}) \rtimes_{\bar{\underline{d}}} N \cong \mathrm{VN}_l(A)$ is a factor of type II_∞.

We know from [**Con75a**, p. 1091] that any representation of a solvable group generates a hyperfinite von Neumann algebra. Any abelian group is solvable. In particular, we have that $\mathrm{VN}_l(\mathbb{R}^2)$ is hyperfinite. Furthermore, [**Con75a**, Corollary 4] then tells us that since N is solvable (see [**Tak03b**, p. 64]) and $\mathrm{VN}_l(\mathbb{R}^2)$ is hyperfinite, we must have that $\mathrm{VN}_l(\mathbb{R}^2) \rtimes_{\bar{\underline{d}}} N$ is hyperfinite. So we have argued that $\mathrm{VN}_l(A)$ is a hyperfinite factor of type II_∞.

We verify that $\mathrm{VN}_l(P_\lambda)$ is of type III_λ for $\lambda \in (0, 1)$. For $P_\lambda = A \ltimes_{z(\lambda)} \mathbb{Z}$, the action of the generator of $\mathbb{Z}$ on A scales the Haar measure of A by a factor of λ. The discussion preceding Theorem 6.4 shows that $\mathrm{VN}_l(P_\lambda)$ is of type III_λ $(\lambda \in (0, 1))$. Similarly since $\mathrm{VN}_l(A)$ is hyperfinite and $\mathbb{Z}$ is solvable, we have that $\mathrm{VN}_l(P_\lambda)$ is hyperfinite.

We turn our attention to $\mathrm{VN}_l(P_1)$. Some additional theory is required.

Definition 6.10 (Almost freely acting group, [**Con73**, Définition 1.4.7]). Let μ be a positive σ-finite measure on a measurable space Ω and G a subgroup of the group of bimeasurable bijections $s : \Omega \to \Omega$, such that the measure $s.\mu$ (or $\mu \circ s^{-1}$) is equivalent to μ.

We say that G acts almost freely in Ω if for every non-negligible measurable subset A of Ω and all $s \neq e_G, s \in G$ there exists a measurable subset B which is non-negligible and such that $s(B) \cap B = \emptyset$.

Definition 6.11 (Ratio set, [**Con73**, Définition 3.3.3]). We call the **ratio set** of G, the set $r(G)$ of $\lambda \geq 0$ having the following property: For all $\epsilon > 0$ and any non-negligible measurable subset $A \subset \Omega$, there exists a non-negligible measurable set $B \subset A$ and a $s \in G$ such that $s(B) \subset A$ and that $|\frac{d\mu \circ s^{-1}}{d\mu}(\omega) - \lambda| < \epsilon$ for all $\omega \in B$.

Proposition 6.12 ([**Con73**, Corollaire 3.3.4]). *Let G be a group acting ergodically and almost freely on a measurable space Ω provided with a positive, σ-finite measure μ, that is quasi-invariant* *with respect to G. Then we have $S(L^\infty(\Omega,\mu)\rtimes_s G) = r(G)$, where $S(\cdot)$ is as in Definition 3.14.*

Arguing identically as in the case of $\mathrm{VN}_l(A)$, we see that $\mathrm{VN}_l(P_1) \cong L^\infty(\mathbb{R}^2)\rtimes_{\bar{d}} U$. It can be shown that U complies with the assumptions made in Definition 6.10, so that it makes sense to show it is indeed a freely acting group. Let $V \subset \mathbb{R}^2$ such that $m(V) > 0$.

Let $T \in U$ and $V \subset \mathbb{R}^2$ be measurable such that $m(V) > 0$. Suppose that for any measurable set $B \subset V$ such that $m(B) > 0$, we have $TB \cap B \neq \varnothing$. We may suppose that $m(TB \triangle B) = 0$ if $m(B) > 0$; for if we suppose contrariwise, then (using the properties of measures) we obtain a contradiction with our assumption that $TB \cap B \neq \varnothing$ for all measurable $B \subset V$. Set $V_1 = \{\mathbf{x} \in V : T\mathbf{x} \neq \mathbf{x}\}$. Since $V_1 \subseteq V \triangle TV$, it is a null set. As T is linear, it must be the identity when restricted to the vector space difference $V - V_1 = \{\mathbf{x} \in V : T\mathbf{x} = \mathbf{x}\}$. We see that $m(V - V_1) = m(V) > 0$. This shows that $V - V_1$ contains a neighbourhood of $\mathbf{0}$, and so T must be the identity.

The preceding paragraph shows that, in accordance with Definition 6.10, U is a freely acting group.

We calculate Krieger's ratio set $r := r(U)$ as in our current context. Our aim is to show that $p \in r$ for any $p \in \mathbb{Q}_+$. Let $T \in U$ such that $|T| = p$, and $V \subseteq \mathbb{R}^2$ such that $m(V) > 0$. Using the ergodicity of N, we find a matrix $S \in N$ such that $m(V \cap STV) > 0$. We set $B = S^{-1}T^{-1}V \cap V$. Then, it is true that $B \subseteq V$ and $m(B) > 0$. Since $|ST| = |S||T| = p$, we have $\frac{dm\circ(ST)^{-1}}{dm} = p$ on B (as $ST.m$ is equivalent to m, see Definition 6.10.) This shows that $p \in r$. In general, Krieger's ratio set is in fact closed, implying that $r = [0, +\infty)$.

By Theorem 6.12, we know that $S(L^\infty(\mathbb{R}^2)\rtimes_{\bar{d}} U) = [0, +\infty)$. Connes' classification of type *III* factors, in this case Proposition 3.15, asserts that $\mathrm{VN}_l(P_1) \cong L^\infty(\mathbb{R}^2)\rtimes_{\bar{d}} U$ is a factor of type III_1.

From the solvability of U ([**Tak03b**, p. 64]), we conclude that $\mathrm{VN}_l(P_1)$ is hyperfinite. ☐

Corollary 6.13. *Let P_λ ($\lambda \in (0,1]$) be as in Theorem 6.9, then $\mathrm{VN}_l(P_{0,\lambda})$ as defined in Theorem 6.4, is a hyperfinite factor of type III_0.*

Proof. Invoke Theorem 6.4. ☐

6.2. Nonhyperfinite examples

Let $G_1 = \mathbb{R}^2 \ltimes_s \mathrm{GL}_2(\mathbb{Q})$, (where $\mathrm{GL}_2(\mathbb{Q})$ is the general linear group over $\mathbb{Q}$). Let $\lambda \in (0,1)$. We denote by SL_λ the group generated by the special linear group of degree 2 over $\mathbb{Q}$ and $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}$. Set $G_\lambda = \mathbb{R}^2 \ltimes_s \mathrm{SL}_\lambda$. It follows mutatis mutandis from the proof of Theorem 6.9 that $\mathrm{VN}_l(G_\lambda)$ is a factor of type III_λ for any $\lambda \in (0,1]$.

Bibliography

[AM66] L Auslander and CC Moore, *Unitary representations of solvable Lie groups*, no. 1-62, American Mathematical Soc., 1966.

[Amb41] W Ambrose, *Representation of ergodic flows*, Annals of Mathematics (1941), 723–739.

[Bla06] B Blackadar, *Operator algebras: theory of C*-algebras and von Neumann algebras*, vol. 122, Springer Science & Business Media, 2006.

[CLvS10] G Cornelissen, K Landsman, and WD van Suijlekom, *The flashes of insight never came for free: interview with Alain Connes.*

[Coh13] DL Cohn, *Measure theory*, vol. 5, Springer, 2013.

[Con73] A Connes, *Une classification des facteurs de type III*, Annales Scientifiques de l'École Normale Supérieure, vol. 6, 1973, pp. 133–252.

[Con75a] ———, *Classification of automorphisms of hyperfinite factors of type* II_2 *and* II_∞ *and application to type III factors*, Bulletin of the American Mathematical Society **81** (1975), no. 6, 1090–1092.

[Con75b] ———, *Structure theory for type III factors*, Proc. Internat. Congr. Math, vol. 2, 1975, pp. 84–91.

[Dix77] J Dixmier, *C*-algebras*, North-Holland mathematical library, vol. 15, North-Holland Publishing Company, 1977.

[erg] Strong ergodicity, *Encyclopedia of mathematics*, URL: `http://encyclopediaofmath.org/index.php?title=Strong_ergodicity&oldid=33303`, Accessed: 2024-05-17.

[Fol16] GB Folland, *A course in abstract harmonic analysis*, vol. 29, CRC press, 2016.

[Fre01] DH Fremlin, *Broad foundations, measure theory, vol. 2*, Torres Fremlin, Colchester (2001).

[GL20] S Goldstein and LE Labuschagne, *Notes on noncommutative* L^p *and Orlicz spaces*, Wydawnictwo Uniwersytetu Łódzkiego, 2020.

[Haa33] A Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Annals of mathematics (1933), 147–169.

[Haa75] U Haagerup, *The standard form of von Neumann algebras*, Mathematica Scandinavica **37** (1975), no. 2, 271–283.

[Hoc13] M Hochman, *Notes on ergodic theory*, URL: `http://math.huji.ac.il/~mhochman/courses/ergodic-theory-2012/notes.final.pdf`, 2013, Accessed: 2024-03-20.

[Kel17] JL Kelley, *General Topology*, Dover Books on Mathematics, Courier Dover Publications, 2017.

[KR83] RV Kadison and JR Ringrose, *Fundamentals of the theory of operator algebras. volume I: Elementary theory*, 1st ed., Pure and applied mathematics 100-100, 2, vol. Volume 1, Academic Press, 1983.

[KR86] ———, *Fundamentals of the theory of operator algebras. volume II: Advanced theory*, Pure and applied mathematics 100-100, 2, vol. Volume 2, Academic Press, 1986.

[KT13] E Kaniuth and KF Taylor, *Induced representations of locally compact groups*, 1 ed., Cambridge Tracts in Mathematics 197, no. 197, Cambridge University Press, 2013.

[Loo52] LH Loomis, *Note on a theorem of Mackey*, Duke Mathematical Journal **19** (1952), no. 4.

[LT04] PA Loeb and E Talvila, *Lusin's theorem and bochner integration*, arXiv preprint math/0406370 [math.CA] (2004).

[M70] Takesaki M, *Tomita's theory of Modular Hilbert algebras and its applications*, Springer Lecture Notes in Mathematics **128** (1970).

[Mac49] George W Mackey, *A theorem of Stone and von Neumann*, Duke Mathematical Journal **16** (1949), no. 2, 313.

[Mac57] GW Mackey, *Borel structure in groups and their duals*, Transactions of the American Mathematical Society **85** (1957), no. 1, 134–165.

[Mac62] ______, *Point realizations of transformation groups*, Illinois Journal of Mathematics **6** (1962), no. 2, 327–335.

[Mac66] ______, *Ergodic theory and virtual groups*, Mathematische Annalen **166** (1966), 187–207.

[MZ55] D Montgomery and L Zippin, *Topological transformation groups*, Interscience Tracts in Pure and Applied Mathematics No 1, Interscience, 1955.

[NU61] M Nakamura and H Umegaki, *Heisenberg's commutation relation and the planeherel theorem*, Proceedings of the Japan Academy **37** (1961), no. 5, 239–242.

[Ped00] GK Pedersen, *The existence and uniqueness of the Haar integral on a locally compact topological group*, Preprint, University of Copenhagen, November (2000), 2004–2.

[Pet13] J Peterson, *Notes on von Neumann algebras*, URL: `https://math.vanderbilt.edu/peters10/teaching.html`, 2013, Accessed: 2024-03-20.

[Pow67] RT Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Annals of Mathematics (1967), 138–171.

[Rou77] R Rousseau, *The left Hilbert algebra associated to a semi-direct product*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 82, Cambridge University Press, 1977, pp. 411–418.

[Rou81] ______, *Crossed products of von Neumann algebras and semi-direct products of groups*, Indagationes Mathematicae (Proceedings), vol. 84, Elsevier, 1981, pp. 105–116.

[Rud87] W Rudin, *Real and complex analysis*, McGraw-Hill International Editions: Mathematics Series, McGraw-Hill, 1987.

[Rud90] ______, *Fourier analysis on groups*, Wiley Classics Library, Wiley-Interscience, 1990.

[Sal16] D Salamon, *Measure and integration*, European Mathematical Society London, 2016.

[Slu19] K Slutsky, *Regular cross sections of Borel flows*, Journal of the European Mathematical Society **21** (2019), no. 7, 1985–2050.

[Sut74] CE Sutherland, *Direct integral theory for weights, and the plancherel formula*, Bulletin of the American Mathematical Society **80** (1974), no. 3, 456–461.

[Sut77] ______, *Crossed products, direct integrals and connes' classifcation of type III factors*, Mathematica Scandinavica **40** (1977), no. 2, 209–214.

[Sut78] ______, *Type analysis of the regular representation of a nonunimodular group*, Pacific Journal of Mathematics **79** (1978), no. 1, 225–250.

[Tak73] M Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Mathematica **131** (1973), 249–310.

[Tak01] ______, *Theory of operator algebras I*, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2001.

[Tak03a] ______, *Theory of operator algebras II*, vol. 125, Springer, 2003.

[Tak03b] ______, *Theory of operator algebras III*, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2003.

[Tat72] N Tatsuuma, *Plancherel formula for non-unimodular locally compact groups*, Journal of Mathematics of Kyoto University **12** (1972), no. 1, 179–261.

[VD74] A Van Daele, *A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras*, Journal of Functional Analysis **15** (1974), no. 4, 378–393.

[VD78] ______, *Continuous crossed products and type III von Neumann algebras*, vol. 31, Cambridge University Press, 1978.

[vN49] J von Neumann, *On rings of operators. Reduction theory*, Annals of Mathematics (1949), 401–485.

[Wat88] K Watanabe, *Dual of non-commutative L^p-spaces with* $0 < p < 1$, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 103, Cambridge University Press, 1988, pp. 503–509.

Notation Index