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Set-Valued Contractions and q -Hyperconvex Spaces

by
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A dissertation prepared under the supervision of
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in fulfilment of the requirements for the degree of
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Abstract

In this MSc dissertation, we present fixed point theorems for a family of set-valued contraction maps in hyperconvex quasi-pseudometric spaces (called q -hyperconvex spaces). Our results generalise and extend the classical fixed point theorems of Espínola and Kirk. It is proved, for example, that if (X, d) is a bounded hyperconvex quasi-pseudometric space and f_α is a family of set-valued contractions indexed over a directed set Λ and taking values in the space of all nonempty externally q -hyperconvex subsets of X , then certain conditions imply that the fixed point set of the family $\{f_\alpha\}_{\alpha \in \Lambda}$ is nonempty and q -hyperconvex. Since quasi-pseudometric spaces and ultra-quasi-pseudometric spaces share many properties, in this MSc dissertation we show some of these properties. For example, we show that a contractive set-valued map T^* taking values in the space of all nonempty u -admissible subsets of a spherically complete ultra-quasi-pseudometric space (X, u) (called q -spherically complete space), has a point valued selection T which is also contractive. The remainder of the dissertation is devoted to fixed point theorems for a family of set-valued contractive maps in the setting of q -spherically complete ultra-quasi-pseudometric spaces.

Preface

The work described in this MSc dissertation was carried out under the supervision of Dr. Olivier Olela Otafudu, Department of Mathematical Sciences, North-West University, Mafikeng.

The MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:


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Dedication

I dedicate this work to my parents Fridah Kalomo and Leonard Sabao, my siblings Precious, Leonard, Kennedy, Mercy, Brian and Sharon, my friends Chrisper and Dennis, for the support they have rendered.

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Notation and Conventions

(X, m)	Pseudometric space
(X, d)	Quasi-pseudometric space
$B(x, \epsilon)$	Open ball of radius ϵ centred at x
$C(x, \epsilon)$	Closed ball of radius ϵ centred at x
M	Hausdorff pseudometric
D	Hausdorff quasi-pseudometric
U	Hausdorff ultra-quasi-pseudometric
$\mathcal{CB}(X)$	The collection of nonempty bounded closed subsets of the pseudometric space (X, m)
\mathcal{F}	Gauge
\mathcal{Q}	Quasi-gauge
\mathbb{E}	Complete gauge space
\mathbb{G}	Bicomplete quasi-gauge space
Λ	Directed set
$\mathcal{P}_0(X)$	The power set of X
$diam(X)$	Diameter of the set X
$\mathcal{A}(X)$	The collection of admissible subsets of a pseudometric space (X, m)
$\mathcal{A}_q(X)$	The collection of q -admissible subsets of a quasi-pseudometric space (X, d)
$\mathcal{A}_u(X)$	The collection of u -admissible subsets of an ultra-quasi-pseudometric space (X, u)
$\mathcal{E}(X)$	The collection of externally hyperconvex subsets of a pseudometric space (X, m)
$\mathcal{E}_q(X)$	The collection of externally q -hyperconvex subsets of a quasi-pseudometric space (X, d)
$\mathcal{H}(X)$	The collection of hyperconvex subsets of a pseudometric space (X, m)
$\mathcal{H}_q(X)$	The collection of q -hyperconvex subsets of a quasi-pseudometric space (X, d)
$Cl_{\tau(d)}(A)$	The closure of the set A with respect to the topology τ induced by the metric d .

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Introduction

The fundamental result in fixed point theory is the Banach contraction principle which states that if (X, m) is a complete metric space and $F : X \rightarrow X$ is a contraction map, that is,

$$m(F(x), F(y)) \leq k m(x, y) \text{ whenever } x, y \in X \text{ and } k \in [0, 1),$$

then F has a unique fixed point. Since most spaces studied in mathematical analysis share many properties, among others, topological, metric and algebraic properties, there is no clear line separating the metric fixed point theory from the topological or set theoretic branch of fixed point theory. For this reason, many authors have studied the problem of existence, uniqueness and approximation of fixed points for generalised contractions in metric spaces. However, many definitions and theorems in literature do not require that all properties of a metric hold. Hence, in the last decades, various concepts of generalised metrics such as quasi-pseudometrics and ultra-quasi-pseudometrics have been introduced.

Recently, Frigon [11] proved a theorem that serves as a basis for many other interesting fixed point theorem results. In particular, Frigon proved that if \mathbb{E} is a complete gauge space and $f : \mathbb{E} \rightarrow \mathbb{E}$ is a generalised contraction, then f has a unique fixed point. Frigon also remarked that it was an open question to define inductively a sequence converging to the fixed point of f . Espínola and Kirk [9] extended Frigon's Theorem to a family of set-valued contraction maps in complete gauge spaces and defined inductively a sequence converging to their fixed point. In particular, Espínola and Kirk proved that if \mathbb{E} is a complete gauge space and $f_\alpha : (\mathbb{E}, m_\alpha) \rightarrow (\mathcal{P}_0(\mathbb{E}), M_\alpha)$ is a family of contraction maps indexed over a directed set Λ satisfying the following:

- (i) $f_\alpha(x)$ is a nonempty closed subset of \mathbb{E} for every $\alpha \in \Lambda$,
- (ii) for every $x, y \in \mathbb{E}$ and $\alpha \in \Lambda$, we have $M_\alpha(f_\alpha(x), f_\alpha(y)) \leq k_\alpha m_\alpha(x, y)$, where $k_\alpha \in (0, 1)$ and M_α denotes the Hausdorff pseudometric on the nonempty closed subsets of \mathbb{E} ,
- (iii) for each $x \in \mathbb{E}$, $f_\beta(x) \subseteq f_\alpha(x)$ if $\beta \geq \alpha$,
- (iv) for every $\epsilon > 0$ and $\alpha \in \Lambda$, there exists $\beta \geq \alpha$ such that $\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon$ for every $x \in \mathbb{E}$.

Then there exists a unique point $z \in \mathbb{E}$ such that $z \in f_\alpha(z)$ for every $\alpha \in \Lambda$. Also, $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z for each $x \in \mathbb{E}$ where $\bar{f}(x)$ is a unique point of $\bigcap_\alpha f_\alpha(x)$.

Espínola and Kirk used the above result as a basis to construct fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces. However, condition (iv) was replaced by the weaker condition $\text{diam}(f_\alpha(x)) \leq \epsilon$ since fixed points in hyperconvex spaces are not unique. In particular, they proved that if (X, m) is a hyperconvex metric space and $f_\alpha : (X, m) \rightarrow (\mathcal{CB}(X), M)$ is a family of contraction maps indexed over a directed set Λ satisfying

- (i) for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,
- (ii) for every $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $\text{diam}(f_\alpha(x)) \leq \epsilon$ for every $x \in X$.

Then there exists at least one point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

Furthermore, it was shown that in the above result, if the collection $\mathcal{CB}(X)$ of bounded closed subsets of X is replaced by the collection $\mathcal{A}(X)$ of admissible subsets of X , then the fixed point set is hyperconvex.

The main goal of this MSc dissertation is to generalise and extend the above results of Espínola and Kirk [9] to **q -hyperconvex T_0 -quasi-metric spaces** (see Definition 3.2.1). We will also show that the extension of Frigon's Theorem in complete gauge spaces can be generalised to bicomplete quasi-gauge spaces and bicomplete quasi-metric spaces (see Theorem 3.1.1 and Corollary 3.1.2). Secondly, since quasi-pseudometric spaces and ultra-quasi-pseudometric spaces share many properties, in this MSc dissertation we show some of these properties. For example, among other properties, using the construction of Künzi and Otafudu [15], we will show that a set-valued map $T^* : H \rightarrow \mathcal{A}_u(X)$, where H is any set and (X, u) is a **q -spherically complete T_0 -ultra-quasi-metric space** (see Definition 4.1.4), admits a point valued selection $T : H \rightarrow X$ satisfying $u(T(x), T(y)) \leq U(T^*(x), T^*(y))$ whenever $x, y \in H$ (see Theorem 4.2.13). Finally, we will use the conditions of the results of Espínola and Kirk [9] to develop fixed point theorems for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces.

Outline of the MSc Dissertation

Chapter 1. In this chapter, we recall some of the important definitions to be used throughout the dissertation. In the first section, we first present a summary of pseudometric spaces, thereafter, we present a summary of quasi-pseudometric spaces. In the second section of this chapter, we use the concept of a pseudometric to recall the definition of a gauge space (see Definition 1.2.3) and discuss some of its examples. The last section of this chapter discusses the concept of a quasi-gauge space, an extension of the concept of a gauge space. We use the concept of a quasi-pseudometric to recall the definition of a quasi-gauge space (see Definition 1.3.2). We also recall an example of a quasi-gauge space and some of its properties.

Chapter 2. In this chapter, we present the results of Espínola and Kirk [9]. The first section of this chapter discusses Frigon's Theorem which guarantees the existence of a unique fixed point for a generalised contraction in gauge spaces. This result is then extended to fixed point theorems for a family of set-valued contraction maps in complete gauge spaces and complete metric spaces in the second section (see Theorem 2.2.4 and Corollary 2.2.5). The third section gives a brief summary of hyperconvex metric spaces and some of their properties. In the last section of this chapter, we present the results of Espínola and Kirk [9] on fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces. Theorem 2.4.1 shows that a family of set-valued contraction maps taking values in the space $\mathcal{CB}(X)$ of all nonempty bounded closed subsets of a hyperconvex metric space (X, m) has at least one fixed point. On the other hand, Theorem 2.4.4 shows that the fixed point set for a family of set-valued contraction maps taking values in the space $\mathcal{A}(X)$ of admissible subsets of a bounded hyperconvex metric space (X, m) is hyperconvex.

Chapter 3. In this chapter, we start our own investigations. We generalise the results of Espínola and Kirk [9] to q -hyperconvex T_0 -quasi-metric spaces. The first section of this chapter discusses the extension of Frigon's Theorem to a family of set-valued contraction maps in bicomplete quasi-gauge spaces (see Theorem 3.1.1). This result is then extended to bicomplete-quasi-metric spaces (see Corollary 3.1.2). The second section gives a brief summary of q -hyperconvex T_0 -quasi-metric spaces and some of their properties. The last section of this chapter is devoted to generalising Theorem 2.4.1 and Theorem 2.4.4 to q -hyperconvex T_0 -quasi-metric spaces. Theorem 3.3.1 shows that a family of set-valued contraction maps taking values in the collection $S_{cl} = \{A \subseteq X : A = cl_{\tau(d)}A \cap cl_{\tau(d^{-1})}A\}$ has at least one fixed point. On the other hand, Theorem 3.3.4 shows that the fixed point set for a family of set-valued contraction maps taking values in the space $\mathcal{E}_q(X)$ of externally q -hyperconvex subsets of a bounded q -hyperconvex T_0 -quasi-metric space (X, d) is q -hyperconvex.

Chapter 4. In the first section of this chapter, we recall some preliminaries on ultra-quasi-pseudometric spaces and give some of their examples. We also recall the definition of q -spherical completeness and recall an example of a q -spherically complete T_0 -ultra-quasi-metric space (see Definition 4.1.4). In the second section, using the construction of Künzi and Otafudu [15], we present some properties of q -spherically complete T_0 -ultra-quasi-metric spaces. For example, among other properties, we show that a set-valued map $T^* : H \rightarrow \mathcal{A}_u(X)$, where H is any set and (X, u) is a q -spherically complete T_0 -ultra-quasi-metric space, admits a point valued selection $T : H \rightarrow X$ satisfying $u(T(x), T(y)) \leq U(T^*(x), T^*(y))$ whenever $x, y \in H$ (see Theorem 4.2.13). In the third section, we use the conditions of the fixed point theorems of Espínola and Kirk [9] to develop fixed point theorems for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces (See Theorems 4.3.1 and 4.3.2).

Chapter 5. In this chapter, we summarise our investigations and present some open problems to be studied in future.

1

Preliminaries

In this chapter, we present the basic definitions to be used throughout the MSc dissertation. We first recall the definitions of a pseudometric space and quasi-pseudometric space and look at some of their examples. Thereafter, we use the definitions of a pseudometric and quasi-pseudometric to recall the concepts of a gauge space and quasi-gauge space respectively. For more details, we refer the reader to [7], [4], [15], [21] and [24].

1.1. Basic Definitions

Definition 1.1.1. ([4, Definition 1]) Let X be a set and let $m : X \times X \rightarrow [0, \infty)$ be a function mapping $X \times X$ into the set $[0, \infty)$ of nonnegative real numbers. Then m is called a pseudometric on X if

- (i) $m(x, x) = 0$ for all $x \in X$,
- (ii) $m(x, y) = m(y, x)$ for all $x, y \in X$,
- (iii) $m(x, y) \leq m(x, z) + m(z, y)$ for all $x, y, z \in X$.

The pair (X, m) is called a pseudometric space.

If in addition, for $x \neq y$ we have

$$m(x, y) > 0,$$

then m is a metric on X and the pair (X, m) is called a metric space.

It is clear from this definition that any metric space is a pseudometric space. Pseudometrics arise naturally in functional analysis, consider the following example:

Example 1.1.1. Let $F(X)$ be a space of real valued functions $f : X \rightarrow \mathbb{R}$, together with a special point $x_0 \in X$. Then x_0 induces a pseudometric on the space $F(X)$ with pseudometric

$$m(f, g) = |f(x_0) - g(x_0)| \quad \text{for all } f, g \in F(X).$$

Example 1.1.2. ([10, Example 2.4]) For a set X , define $m(x, y) = 0$ for all $x, y \in X$. We call m the trivial pseudometric on X . Thus the pseudometric space (X, m) is characterised by the fact that all distances are 0.

Let (X, m) be a pseudometric space. The *open ball* of radius $\epsilon > 0$ centred at $x \in X$ is the set $B_m(x, \epsilon) = \{y \in X : m(x, y) < \epsilon\}$. The collection of open balls yields a base topology $\tau(m)$, it is called the topology induced by m on X . Similarly, a *closed ball* of radius $\epsilon \geq 0$ centred at $x \in X$ is the set $C_m(x, \epsilon) = \{y \in X : m(x, y) \leq \epsilon\}$.

Definition 1.1.2. ([28]) Let (X, m) be a metric space.

- (i) A sequence $(x_n)_{n \in \mathbb{N}}$ in X is Cauchy if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n, p \geq N$, one has $m(x_n, x_p) \leq \epsilon$.
- (ii) A sequence $(x_n)_{n \in \mathbb{N}}$ converges to x if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m(x_n, x) < \epsilon$ for all $n \geq N$.
- (iii) A metric space (X, m) is complete if any Cauchy sequence is convergent.

If we delete the symmetry condition, that is, condition (ii) in Definition 1.1.1, we are led to the concept of a quasi-pseudometric space.

Definition 1.1.3. ([15, Definition 2.1]) Let X be a set and let $d : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of nonnegative real numbers. Then d is called a quasi-pseudometric on X if

- (i) $d(x, x) = 0$ for all $x \in X$,
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called a quasi-pseudometric space.

If in addition, for any $x, y \in X$,

$$d(x, y) = 0 = d(y, x) \implies x = y,$$

then d is called a T_0 -quasi-metric and the pair (X, d) is called a T_0 -quasi-metric space.

Remark 1.1.1. ([15]) If d is a quasi-pseudometric on a set X , then $d^{-1} : X \times X \rightarrow [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ for every $x, y \in X$, often called the conjugate quasi-pseudometric, is also quasi-pseudometric on X . The quasi-pseudometric on a set X such that $d = d^{-1}$ is a pseudometric. Note that if (X, d) is a T_0 -quasi-metric space, then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is also a pseudometric (Metric).

For any $x, y \in [0, \infty)$, we will set $x \dot{-} y = \max\{x - y, 0\}$.

Example 1.1.3. The space $([0, \infty), d)$, where $d(x, y) = x \dot{-} y$, is a T_0 -quasi-metric space.

Similar to pseudo-metric spaces, we get the following definitions and results in quasi-pseudometric spaces: Let (X, d) be a quasi-pseudometric space. The *open ball* of radius $\epsilon > 0$ centred at $x \in X$ is the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. The collection of *open balls* yields a *base topology* $\tau(d)$, it is called the topology induced by d on X . Similarly, the *closed ball* of radius $\epsilon \geq 0$ centred at $x \in X$ is the set $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that this set is $\tau(d^{-1})$ -closed, but not $\tau(d)$ -closed in general. As usual, in the theory of quasiuniformities, for a subset A of X and $\epsilon > 0$, we use notations like $B_d(A, \epsilon) = \bigcup_{a \in A} B_d(a, \epsilon)$ and $C_d(A, \epsilon) = \bigcup_{a \in A} C_d(a, \epsilon)$.

If (X, d) is a quasi-pseudometric space, then the pair $[C_d(x, r); C_{d^{-1}}(x, s)]$ where $x \in X$ and $r, s \in [0, \infty)$ is called a *double ball*. In general, $[(C_d(x_i, r_i))_{i \in I}; (C_{d^{-1}}(x_i, s_i))_{i \in I}]$, with $x_i \in X$ and $r_i, s_i \in [0, \infty)$, is called the *family of double balls*.

For any quasi-pseudometric space (X, d) , we denote the set of all *nonempty subsets* of X by $\mathcal{P}_0(X)$. Also, if $C \in \mathcal{P}_0(X)$, then $\text{dist}(x, C) = \inf\{d(x, c) : c \in C\}$ and $\text{dist}(C, x) = \inf\{d(c, x) : c \in C\}$ whenever $x \in X$.

For any $A, B \in \mathcal{P}_0(X)$, we set

$$D(A, B) = \max \left\{ \sup_{b \in B} \text{dist}(A, b), \sup_{a \in A} \text{dist}(a, B) \right\}.$$

Then D , is the so-called *extended* (as usual, a quasi-pseudometric that maps into $[0, \infty]$ (instead of $[0, \infty)$) will be called *extended Hausdorff (-Bourbaki) quasi-pseudometric* on $\mathcal{P}_0(X)$. It is known that D is an *extended T_0 -quasi-metric* when restricted to the set $S_{cl} = \{A \subset X : cl_{\tau(d)} A \cap cl_{\tau(d^{-1})} A\}$ (see [17, Page 164]).

A map $f : (X, d) \rightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called *isometry* or *isometric map* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$. Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a bijective isometry $f : (X, d) \rightarrow (Y, e)$. A map $f : (X, d) \rightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called *nonexpansive* provided that $e(f(x), f(y)) \leq d(x, y)$ whenever $x, y \in X$. A map $f : (X, d) \rightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) will be called *contracting* provided that $e(f(x), f(y)) \leq k d(x, y)$ whenever $x, y \in X$ and $k \in [0, 1)$.

Definition 1.1.4. Let (X, d) be a quasi-pseudometric space and $F : (X, d) \rightarrow (\mathcal{P}_0(X), D)$ be a set-valued map. Then F is said to be *contracting* if

$$D(F(x), F(y)) \leq k d(x, y) \text{ whenever } x, y \in X \text{ and } k \in [0, 1).$$

Definition 1.1.5. ([1]) If (X, d) is a quasi-pseudometric space, a sequence (x_n) is called:

(i) *Left-Cauchy* if for each $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m \geq n \geq k$.

(ii) Right-Cauchy if for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n \geq m \geq k$.

Definition 1.1.6. ([27]) Let (X, d) be a quasi-pseudometric space. The sequence (x_n) in X is a Cauchy sequence if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. A quasi-pseudometric space (X, d) is bicomplete if every Cauchy sequence (x_n) converges with respect to $\tau(d)$ and with respect $\tau(d^{-1})$ to a point $x_0 \in X$.

Remark 1.1.7. ([27]) A sequence (x_n) is a Cauchy sequence in (X, d) in this sense if and only if (x_n) is a Cauchy sequence in the pseudometric space (X, d^s) . A T_0 -quasi-metric space (X, d) is bicomplete if and only if the metric space (X, d^s) is complete.

We now turn our attention to the concept of a gauge space.

1.2. Concept of a Gauge Space

In this section, we recall some preliminaries on gauge spaces. A gauge space is simply a topological space whose topology is generated by a family of pseudometrics. We recall this definition as follows:

Definition 1.2.1. ([4]) Let X be a set. A family $\mathcal{F} = \{m_\lambda : \lambda \in \Lambda\}$ of pseudometrics on X is called a gauge.

Definition 1.2.2. ([4]) Let X be a nonempty set. A gauge $\mathcal{F} = \{m_\lambda | \lambda \in \Lambda\}$ is called separating if, for each pair $(x, y) \in X$ with $x \neq y$, there is a $m_\lambda \in \mathcal{F}$ such that $m_\lambda(x, y) \neq 0$.

Definition 1.2.3. ([4]) Let X be a nonempty set, and let $\mathcal{F} = \{m_\lambda | \lambda \in \Lambda\}$ be a separating family of pseudometrics on X . Then the topology $\mathcal{T}(\mathcal{F})$ having as subbase the family $\mathcal{B}(\mathcal{F})$ of all balls $B(x, m, \epsilon)$ with $m \in \mathcal{F}$, $x \in X$ and $\epsilon > 0$ is called the topology induced on X by the gauge \mathcal{F} , and the topological space (X, \mathcal{T}) is called a gauge space if there is a gauge \mathcal{F} with $\mathcal{T} = \mathcal{T}(\mathcal{F})$.

In order to fix our terminology, we shall denote a gauge space by $(\mathbb{E}, \mathcal{F})$.

Definition 1.2.4. ([4]) Let $(\mathbb{E}, \mathcal{F})$ be a gauge space with respect to the family $\mathcal{F} = \{m_\lambda | \lambda \in \Lambda\}$ of pseudometrics on \mathbb{E} . Let $\{x_n\}$ be a sequence in \mathbb{E} , and $x \in \mathbb{E}$. Then the following are considered:

- (i) The sequence $\{x_n\}$ converges to x if and only if $\forall \lambda \in \Lambda, \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $m_\lambda(x_n, x) < \epsilon, \forall n \geq N$.
- (ii) The sequence $\{x_n\}$ is Cauchy if and only if $\forall \lambda \in \Lambda, \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $m_\lambda(x_{n+p}, x_n) < \epsilon, \forall n \geq N, p \in \mathbb{N}$.
- (iii) $(\mathbb{E}, \mathcal{F})$ is complete if every Cauchy sequence in \mathbb{E} is convergent to an element of \mathbb{E} .
- (iv) A subset of \mathbb{E} is said to be closed if it contains the limit of any convergent sequence of its elements.

The following examples are well known.

Example 1.2.1. Any uniform space X is a gauge space. The pseudometrics are uniform continuous maps $f : X \times X \rightarrow \mathbb{R}$.

Example 1.2.2. A *completely regular* topological space X is a gauge space. The pseudometrics are continuous maps $f : X \times X \rightarrow \mathbb{R}$.

We now turn our attention to quasi-gauge spaces.

1.3. Concept of a Quasi-Gauge Space

In this section, we recall some preliminaries on quasi-gauge spaces. A quasi-gauge space is an extension of the concept of a gauge space, it is a topological space whose topology is generated by a family of quasi-pseudometrics. We recall this definition as follows:

Definition 1.3.1. ([24]) Let X be a nonempty set. A family $\mathcal{F} = \{d_\lambda : \lambda \in \Lambda\}$ of quasi-pseudometrics on X is called a quasi-gauge.

Definition 1.3.2. ([24]) Let X be a nonempty set, and let $\mathcal{Q} = \{d_\lambda | \lambda \in \Lambda\}$ be a family of quasi-pseudometrics on X . Then the topology $\mathcal{T}(\mathcal{Q})$ having as subbase the family $\mathcal{B}(\mathcal{Q})$ of all balls $B(x, d, \epsilon)$ with $d \in \mathcal{Q}$, $x \in X$ and $\epsilon > 0$ is called the topology induced on X by the quasi-gauge \mathcal{Q} , and the topological space (X, \mathcal{T}) is called a quasi-gauge space if there is a quasi-gauge \mathcal{Q} with $\mathcal{T} = \mathcal{T}(\mathcal{Q})$.

Example 1.3.1. ([24, Theorem 2.6]) Any topological space is a quasi-gauge space. To see this, let (X, \mathcal{T}) be any topological space. For each $G \in \mathcal{T}$, define g on $X \times X$ by

$$g(x, y) = \begin{cases} 0 & \text{if } x \notin G \\ 0 & \text{if } x \in G, y \in G \\ 1 & \text{if } x \in G, y \notin G. \end{cases}$$

Then a simple discussion of cases shows that g satisfies the Triangle Inequality. Thus $\mathcal{Q} = \{g : G \in \mathcal{T}\}$ is a quasi-gauge on X . Furthermore,

$$B(x, g, \epsilon) = G \text{ if } x \in G \text{ and } \epsilon \leq 1, \text{ otherwise } B(x, g, \epsilon) = X.$$

Thus $\mathcal{B}(\mathcal{Q}) = \{G : G \in \mathcal{T}\}$, so that $\mathcal{T}(\mathcal{Q}) = \mathcal{T}$.

Not every topological space is a gauge space. A topological space is a gauge space if and only if it is completely regular(See Definition 1.3.3).

Definition 1.3.3. ([7]) Let X be a topological space. Then X is a completely regular space if given any closed set F and any point $x \in F$, then there is a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x) = 0$ and $\forall y \in F, f(y) = 1$.

In order to fix our terminology, we denote a quasi-gauge space by $(\mathbb{G}, \mathcal{Q})$.

Definition 1.3.4. ([21]) Let $(\mathbb{G}, \mathcal{Q})$ be a quasi-gauge space. A sequence $\{x_n\}$ is left \mathcal{Q} -Cauchy if and only if for each $d \in \mathcal{Q}$ and $\epsilon > 0$, there is a point $x \in \mathbb{G}$ and a positive integer k such that $d(x, x_m) < \epsilon$ for all $m \geq k$ (x and k depend on ϵ and d).

$(\mathbb{G}, \mathcal{Q})$ is left sequentially complete if every left \mathcal{Q} -Cauchy sequence in \mathbb{G} is convergent to a point $x_0 \in \mathbb{G}$.

Definition 1.3.5. ([21]) Let $(\mathbb{G}, \mathcal{Q})$ be a quasi-gauge space. A sequence $\{x_n\}$ is right \mathcal{Q} -Cauchy if and only if for each $d \in \mathcal{Q}$ and $\epsilon > 0$, there is a point $x \in \mathbb{G}$ and a positive integer k such that $d(x_m, x) < \epsilon$ for all $m \geq k$ (x and k depend on ϵ and d).

$(\mathbb{G}, \mathcal{Q})$ is right sequentially complete if and only if every right \mathcal{Q} -Cauchy sequence in \mathbb{G} is convergent to a point $x_0 \in \mathbb{G}$.

Definition 1.3.6. ([21]) A sequence $\{x_n\}$ in \mathbb{G} is \mathcal{Q} -Cauchy if and only if for each $d \in \mathcal{Q}$ and $\epsilon > 0$, there is a positive integer k such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$.

2

Set-Valued Contractions and Hyperconvex Metric Spaces

In this chapter, we present the results of Espínola and Kirk [9] on set-valued contractions and fixed points in hyperconvex spaces. We begin by discussing the result of Frigon [11] on the existence of a unique fixed point for generalised contractions in gauge spaces which Espínola and Kirk used as a basis to develop fixed point theorems for a family of set-valued contractions in hyperconvex spaces.

2.1. Frigon's Theorem

In this section, we discuss an important theorem on fixed points in gauge spaces that gives several applications. Following [11], we assume that $(\mathbb{E}, \mathcal{F})$ is a gauge space endowed with a complete gauge structure induced by a family $\mathcal{F} = \{m_\alpha : \alpha \in \Lambda\}$ of pseudometrics, where Λ is a directed set. Also, we assume that for every $x, y \in \mathbb{E}$ and every $\alpha \leq \beta$, the following condition is satisfied:

$$m_\alpha(x, y) \leq m_\beta(x, y).$$

Note that this means that any closed subset of (\mathbb{E}, m_α) is also closed in (\mathbb{E}, m_β) .

Let $x, y \in \mathbb{E}$. Suppose $x \sim_\alpha y$ if and only if $m_\alpha(x, y) = 0$. Then \sim_α defines an equivalence relation on \mathbb{E} and, in turn, a quotient metric space denoted by $\mathbb{E}_\alpha = ((\mathbb{E}, \sim_\alpha), m_\alpha)$.

Let $X \subset \mathbb{E}$ and $f : (X, m_\alpha) \rightarrow (\mathbb{E}, M_\alpha)$, set

$$f([x]_\alpha) = \{f(u) : u \sim_\alpha x\} = \{f(u) : m_\alpha(u, x) = 0\}.$$

Furthermore, we denote the α -diameter of the set $f([x]_\alpha)$ by $diam_\alpha$ whereas the generalised Hausdorff pseudometric induced by m_α on \mathbb{E} by M_α .

Definition 2.1.1. ([11]) Let $(\mathbb{E}, \mathcal{F})$ be a complete gauge space and $X \subset \mathbb{E}$. A map $f : (X, m_\alpha) \rightarrow (\mathbb{E}, M_\alpha)$ is said to be a generalised contraction if

(i) for every $\alpha \in \Lambda$, there exists $k_\alpha < 1$ such that $M_\alpha(f([x]_\alpha), f([y]_\alpha)) \leq k_\alpha m_\alpha(x, y)$ for all $x, y \in X$, where M_α denotes the Hausdorff metric on the sets $f([x]_\alpha)$.

(ii) for every $\epsilon > 0$ and every $\alpha \in \Lambda$, there exists $\beta \geq \alpha$ such that $\text{diam}_\beta(f([x])_\beta) < (1 - k_\beta)\epsilon$ for each $x \in X$, where $\text{diam}_\beta(f([x])_\beta)$ denotes the β -diameter of the set $f([x])_\beta$.

Frigon [11] gave the following result that guarantees the existence of a unique fixed point for a generalised contraction in a complete gauge space.

Theorem 2.1.2 (Frigon's Theorem). *Let \mathbb{E} be a complete gauge space. Then every generalised contraction $f : \mathbb{E} \rightarrow \mathbb{E}$ has a unique fixed point.*

Frigon [11] also remarked that it was an open question to define inductively a sequence converging to the fixed point of f . Therefore, in the next section, we outline the result of Espínola and Kirk [9] on the extension of Frigon's Theorem to a family of set-valued contraction maps in gauge spaces where this sequence is also defined.

2.2. An Extension of Frigon's Theorem in Gauge Spaces

We start this section by recalling a general observation which will be very useful in the outline of the proof of the extension of Frigon's Theorem due to Espínola and Kirk [9]. Before we do that, we recall the following definitions

Definition 2.2.1. *Let X be any set and $F : X \rightarrow \mathcal{P}_0(X)$ be a set-valued map. Then $f : X \rightarrow X$ is said to be a selection of F if $f(x) \in F(x)$ for any $x \in X$.*

Definition 2.2.2. *Let (X, m) be a metric space and for any $x \in X$, let $\{f^n(x)\}_{n \in \mathbb{N}}$ be a sequence. Then $\{f^n(x)\}_{n \in \mathbb{N}}$ is bounded if there exists a constant $K_n \geq 0$ such that $\forall x \in X$,*

$$m(f^n(x)) \leq K_n.$$

Lemma 2.2.3. *([9, Lemma 3]) Let (X, m) be a pseudometric space and $F : (X, m) \rightarrow (\mathcal{CB}(X), M)$ be a contraction map relative to the Hausdorff pseudometric on $\mathcal{CB}(M)$, where $\mathcal{CB}(X)$ are the nonempty bounded closed subsets of X . Suppose $f : (X, m) \rightarrow (X, m)$ is any selection of F . Then the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ is bounded for any $x \in X$.*

Proof. Define $\hat{F} : \mathcal{CB}(X) \rightarrow \mathcal{CB}(X)$ by

$$\hat{F}(A) = \overline{\bigcup_{a \in A} F(a)}.$$

We first show that the sequence $\{\hat{F}^n(\{x\})\}_{n \in \mathbb{N}}$ is bounded. Let $A, B \in \mathcal{CB}(X)$ and suppose $m = M(A, B)$. Let $u \in \bigcup_{a \in A} F(a)$. Then $\exists a \in A$ such that $u \in F(a)$. Let $\epsilon > 0$ and choose $b \in B$ such that $m(a, b) \leq m + \epsilon$.

Since F is a contraction map, $\exists \lambda \in (0, 1)$ such that

$$M(F(a), F(b)) < \lambda m(a, b).$$

Also, there exists $v \in F(b)$ such that

$$m(u, v) \leq \lambda(m(a, b) + \epsilon) \leq \lambda m + 2\lambda\epsilon.$$

Reversing the roles of u and v , we get

$$M(\hat{F}(A), \hat{F}(B)) \leq \lambda m + 2\lambda\epsilon,$$

and since $\epsilon > 0$ is arbitrary,

$$M(\hat{F}(A), \hat{F}(B)) \leq \lambda M(A, B).$$

This shows that \hat{F} is a contraction map from the pseudometric space $\mathcal{CB}(X)$ to itself.

Now we have that

$$M(\hat{F}^n(X), \hat{F}^{n+1}(X)) \leq \lambda M(\hat{F}^{n-1}(X), \hat{F}^n(X)) \leq \lambda^2 M(\hat{F}^{n-2}(X), \hat{F}^{n-1}(X)) \dots \leq \lambda^n M(\hat{F}^0(X), \hat{F}^1(X)).$$

Hence

$$\begin{aligned} M(\hat{F}^n(X), \hat{F}^{n+p}(X)) &\leq M(\hat{F}^n(X), \hat{F}^{n+1}(X)) + M(\hat{F}^{n+1}(X), \hat{F}^{n+2}(X)) + \dots + M(\hat{F}^{n+p-1}(X), \hat{F}^{n+p}(X)) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+p-1})M(\hat{F}^0(X), \hat{F}^1(X)) \\ &\leq \frac{\lambda^n}{1-\lambda}M(\hat{F}^0(X), \hat{F}^1(X)), \end{aligned}$$

and we see that this quantity goes to 0 as $n \rightarrow \infty$. It follows that $\{\hat{F}^n(X)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for any $X \in \mathcal{CB}(M)$. Therefore, $\{\hat{F}^n(\{x\})\}_{n \in \mathbb{N}}$ is bounded.

Now consider the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ where $x \in X$ with $f(x) \in F(x) = \hat{F}(\{x\})$. Then there exists $y \in F(x)$ such that $f(x) = y$. Thus

$$f^2(x) = f[f(x)] = f(y) \in F(y) \subseteq \hat{F}(\hat{F}(\{x\})) = \hat{F}^2(\{x\}).$$

From this, we see that

$$f^n(x) \in \hat{F}^n(\{x\}) \text{ for } n = 1, 2, \dots$$

Now we have already proved that $\{\hat{F}^n(\{x\})\}_{n \in \mathbb{N}}$ is bounded. It follows that $\{f^n(x)\}_{n \in \mathbb{N}}$ is also bounded. \square

The following result is an extension of Frigon's Theorem due to Espínola and Kirk [9]. This result will be generalised to bicomplete quasi-gauge spaces in Theorem 3.1.1.

Theorem 2.2.4. ([9, Theorem 4]) *Let $(\mathbb{E}, \mathcal{F})$ be a complete gauge space and $f_\alpha : (\mathbb{E}, m_\alpha) \rightarrow (\mathcal{P}_0(\mathbb{E}), M_\alpha)$ be a family of maps satisfying the following conditions:*

- (i) $f_\alpha(x)$ is a nonempty closed subset of \mathbb{E} for every $\alpha \in \Lambda$,
- (ii) for every $x, y \in \mathbb{E}$ and $\alpha \in \Lambda$, we have $M_\alpha(f_\alpha(x), f_\alpha(y)) \leq k_\alpha m_\alpha(x, y)$, where $k_\alpha \in (0, 1)$ and M_α denotes the Hausdorff pseudometric on the nonempty closed subsets of \mathbb{E} ,
- (iii) for each $x \in \mathbb{E}$, $f_\beta(x) \subseteq f_\alpha(x)$ if $\beta \geq \alpha$,
- (iv) for every $\epsilon > 0$ and $\alpha \in \Lambda$, there exists $\beta \geq \alpha$ such that $\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon$ for every $x \in \mathbb{E}$.

Then there exists a unique point $z \in \mathbb{E}$ such that $z \in f_\alpha(z)$ for every $\alpha \in \Lambda$. Also, $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z for each $x \in \mathbb{E}$ where $\bar{f}(x)$ is a unique point of $\bigcap_\alpha f_\alpha(x)$.

Proof. We start by proving the existence of a unique point $\bar{f}(x) \in \bigcap_\alpha f_\alpha(x)$ for which the sequence $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z for each $x \in \mathbb{E}$. Suppose $\alpha \in \Lambda$ and $x \in \mathbb{E}$. By condition (iii), we see that $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is a descending net of closed subsets of (\mathbb{E}, m_α) . Also, by condition (iv), for all $\alpha \in \Lambda$ and $\epsilon > 0$, there exists $\beta \geq \alpha$ such that

$$\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon \text{ for every } x \in \mathbb{E}.$$

This implies that the diameters of $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ tend to zero. Hence $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is Cauchy net in (\mathbb{E}, m_α) for each $\alpha \in \Lambda$. Therefore, $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is a Cauchy net in \mathbb{E} . Since \mathbb{E} is complete, there exists a unique point $\bar{f}(x)$ in \mathbb{E} such that $\lim_\alpha f_\alpha(x) = \bar{f}(x) \in \bigcap_\alpha f_\alpha(x)$.

We now recall [9] an important result which will be used in the next part of the proof.

Assertion. Suppose $S \subseteq \mathbb{E}$ satisfies $\text{diam}_\alpha(S) > \delta$ for $\alpha \in \Lambda$ and $\delta > 0$, and suppose $\beta \geq \alpha$ satisfies condition (iv) for $\epsilon = \frac{1}{4}\delta$. Then

$$\text{diam}_\beta(\bar{f}(S)) \leq \left(\frac{1 + k_\beta}{2}\right) \text{diam}_\beta(S).$$

Proof. From condition (iv), for each $\epsilon > 0$ and $\alpha \in \Lambda$, there exists $\beta \geq \alpha$ such that

$$\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon.$$

Now for any $u, v \in S$, the Triangle Inequality gives

$$m_\beta(\bar{f}(u), \bar{f}(v)) \leq m_\beta(\bar{f}(u), f_\beta(u)) + M_\beta(f_\beta(u), f_\beta(v)) + m_\beta(\bar{f}(v), f_\beta(v)).$$

Therefore,

$$\begin{aligned}
m_\beta(\bar{f}(u), \bar{f}(v)) &\leq M_\beta(f_\beta(u), f_\beta(v)) + \text{diam}_\beta(\bar{f}(u)) + \text{diam}_\beta(\bar{f}(v)) \\
&\leq k_\beta m_\beta(u, v) + 2(1 - k_\beta)\epsilon \\
&\leq k_\beta m_\beta(u, v) + \left(\frac{1 - k_\beta}{2}\right) \delta \quad [\text{since } \epsilon = \frac{1}{4}\delta] \\
&\leq k_\beta m_\beta(u, v) + \left(\frac{1 - k_\beta}{2}\right) \text{diam}_\beta(S) \quad [\text{since } \text{diam}_\beta(S) \geq \delta] \\
&\leq k_\beta \text{diam}_\beta(S) + \left(\frac{1 - k_\beta}{2}\right) \text{diam}_\beta(S) \\
&= \left(\frac{1 + k_\beta}{2}\right) \text{diam}_\beta(S).
\end{aligned}$$

Taking the sup over u and v , we get

$$\text{diam}_\beta(\bar{f}(S)) \leq \left(\frac{1 + k_\beta}{2}\right) \text{diam}_\beta(S).$$

□

We now show that the sequence $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z for each $x \in \mathbb{E}$. Suppose $x \in \mathbb{E}$ and let

$$S_n = \{\bar{f}^n(x), \bar{f}^{n+1}(x), \dots\}, \quad n = 1, 2, 3, \dots$$

Let $\alpha \in \Lambda$ be such that f_α takes α bounded values, that is, for every $\eta \in \Lambda$ and $\epsilon > 0$ there exists $\alpha \geq \eta$ such that

$$\text{diam}_\alpha(f_\alpha(x)) \leq (1 - k_\alpha)\epsilon.$$

Then by Lemma 2.2.3, the sets S_n are α -bounded. Now take β such that for any $\alpha \leq \beta$ and $\epsilon > 0$,

$$\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon.$$

Then f_β takes β -bounded values and hence by Lemma 2.2.3, the sets S_n are β -bounded as well. Now suppose to the contrary that $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ is not a Cauchy sequence in (\mathbb{E}, m_α) . Then

$$\lim_n \text{diam}_\alpha(S_n) = \delta > 0.$$

However, by the Assertion

$$\text{diam}_\beta(S_{n+1}) = \text{diam}_\beta(\bar{f}(S_n)) \leq \left(\frac{1 + k_\beta}{2}\right) \text{diam}_\beta(S_n).$$

This shows that the elements of S_{n+1} are arbitrarily closer to each other than the elements of S_n . It follows that $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (\mathbb{E}, m_β) . Since $m_\beta \geq m_\alpha$, then $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in (\mathbb{E}, m_α) . This contradicts our initial assumption. Hence, we conclude that $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (\mathbb{E}, m_α) for each $\alpha \in \Lambda$.

Since \mathbb{E} is complete, there exists $z \in \mathbb{E}$ such that for every α

$$\lim_n m_\alpha(\bar{f}^n(x), z) = 0.$$

This proves that the sequence $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z .

To show that z is the fixed point of \bar{f} , suppose to the contrary that $\bar{f}(z) \neq z$. Since Λ is a separating set, there exists $\alpha \in \Lambda$ such that for $\delta > 0$, $m_\alpha(z, \bar{f}(z)) = \delta$. Thus

$$\lim_n m_\alpha(\bar{f}^{n+1}(x), \bar{f}(z)) = \delta.$$

Now since $m_\alpha(z, \bar{f}(z)) = \delta$, then by the Assertion, there exists $\beta \in \Lambda$ satisfying $\beta \geq \alpha$ such that

$$m_\beta(\bar{f}^{n+1}(x), \bar{f}(z)) \leq \left(\frac{1+k_\beta}{2}\right) m_\beta(\bar{f}^n(x), z).$$

From this, we see that the right hand side goes to 0 as $n \rightarrow \infty$. This contradicts that $\bar{f}(z) \neq z$. Thus we conclude that $\bar{f}(z) = z$, that is, $z \in \bigcap_\alpha f_\alpha(z)$.

To prove that the fixed point is unique, suppose to the contrary that $\bar{f}(u) = u$ and $\bar{f}(v) = v$ with $u \neq v$. Since Λ is separating, there exists $\alpha \in \Lambda$ such that

$$m_\alpha(u, v) = \delta > 0.$$

Thus, by our Assertion there exists $\beta \geq \alpha$ such that

$$m_\beta(u, v) = m_\beta(\bar{f}(u), \bar{f}(v)) \leq \left(\frac{1+k_\beta}{2}\right) m_\beta(u, v),$$

which is not true since

$$\left(\frac{1+k_\beta}{2}\right) m_\beta(u, v) \leq m_\beta(u, v).$$

Hence the fixed point is unique and this concludes our proof. \square

Comparing Theorem 2.2.4 to Frigon's Theorem, let $f : \mathbb{E} \rightarrow \mathbb{E}$ be generalised contraction and for each $\alpha \in \Lambda$ and $x \in \mathbb{E}$, define

$$f_\alpha(x) = \overline{f([x]_\alpha)}.$$

Then, the family $\{f_\alpha\}_{\alpha \in \Lambda}$ satisfies all the assumptions of Theorem 2.2.4 in the following way:

- (i) Since $f_\alpha(x) = \overline{f([x]_\alpha)}$, then $f_\alpha(x)$ is a nonempty closed subset of \mathbb{E} for each $\alpha \in \Lambda$.
- (ii) Since $f : \mathbb{E} \rightarrow \mathbb{E}$ is a generalised contraction, then for every $\alpha \in \Lambda$, there exists $k_\alpha \in (0, 1)$ such that

$$M_\alpha(f([x]_\alpha), f([y]_\alpha)) \leq k_\alpha m_\alpha(x, y).$$

Then condition (ii) of Theorem 2.2.4 follows.

(iii) Since $\beta \geq \alpha \implies m_\beta(x, y) \geq m_\alpha(x, y)$ then $\overline{f([x]_\beta)} \subseteq \overline{f([x]_\alpha)}$, and therefore, condition (iii) of Theorem 2.2.4 follows.

(iv) Since f is a generalised contraction, then for every $\epsilon > 0$ and every $\alpha \in \Lambda$, there exists $\beta \geq \alpha$ such that

$$\text{diam}_\beta(f([x]_\beta)) < (1 - k_\beta)\epsilon \quad \forall x \in \mathbb{E}.$$

Hence condition (iv) of Theorem 2.2.4 follows.

Thus if \bar{f} is defined as in the proof of Theorem 2.2.4, then \bar{f} has a unique fixed point $z \in \mathbb{E}$ and for each $x \in \mathbb{E}$, the sequence $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z .

Taking $\mathbb{E} = X$ in Theorem 2.2.4, where X is a complete metric space, we get the following result that guarantees the existence of a unique fixed point for a family of set-valued contraction maps in complete metric spaces. This result will be generalised to bicomplete quasi-metric spaces in Corollary 3.1.2.

Corollary 2.2.5. ([9, Corollary 5]) *Let (X, m) be a complete metric space and $f_\alpha : (X, m) \rightarrow (\mathcal{P}_0(X), M)$ be a family of set-valued contraction maps indexed over the directed set Λ . Suppose*

- (i) $f_\alpha(x)$ is a closed subset of X for each $\alpha \in \Lambda$ and $x \in X$,
- (ii) $M(f_\alpha(x), f_\alpha(y)) \leq k_\alpha m(x, y)$ for each $x, y \in X$, where $k_\alpha \in (0, 1)$ and M denotes the Hausdorff metric over the nonempty closed subsets of X ,
- (iii) for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,
- (iv) for each $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $\text{diam}(f_\alpha(x)) \leq (1 - k_\alpha)\epsilon$ for each $x \in X$.

Then there exists a unique point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

This corollary is a simple consequence of the classical extension of the Banach's contraction mapping principle due to Meir and Keeler [18]. Meir and Keeler [18] proved that if $f : (X, m) \rightarrow (X, m)$ satisfies the following condition:

Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq m(x, y) \leq \epsilon + \delta \implies m(f(x), f(y)) < \epsilon,$$

then f has a unique fixed point x_0 , and $\lim_n f^n(x) = x_0$ for each $x \in X$.

Now from the assumptions of Corollary 2.2.5, the map $f : X \rightarrow X$ defined by

$$\{f(x)\} = \bigcap_{\alpha \in \Lambda} f_\alpha(x),$$

satisfies this condition. To see this, let $\epsilon > 0$ and choose $\alpha \in \Lambda$ such that

$$\text{diam}_\alpha(f_\alpha(z)) \leq (1 - k_\alpha) \frac{\epsilon}{4} \quad \forall z \in X.$$

Now let $\delta > 0$ satisfy $(\frac{1+k_\alpha}{2})(\epsilon + \delta) < \epsilon$. If $\epsilon \leq d(x, y) \leq \epsilon + \delta$, then

$$\begin{aligned} m(f(x), f(y)) &\leq M(f_\alpha(x), f_\alpha(y)) + \text{diam}_\alpha(f_\alpha(x)) + \text{diam}_\alpha(f_\alpha(y)) \\ &\leq k_\alpha m(x, y) + 2(1 - k_\alpha) \frac{\epsilon}{4} \\ &\leq k_\alpha m(x, y) + \left(\frac{1 - k_\alpha}{2}\right) m(x, y) \\ &\leq \left(\frac{1 + k_\alpha}{2}\right) (\epsilon + \delta) \\ &< \epsilon. \end{aligned}$$

Hence, we have $m(f(x), f(y)) < \epsilon$. Therefore, we conclude that f has a unique fixed point say x , and it is clear that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

A question immediately arises whether we can replace condition (iv) of Corollary 2.2.5 with the following weaker condition:

(iv) for each $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $\text{diam}(f_\alpha(x)) \leq \epsilon$ for each $x \in X$.

In Corollary 2.2.5, condition (iv) is necessary for proving uniqueness of a fixed point (see Example 2.4.2 below). However, existence results can be obtained under the weaker condition. In Section 2.4, we recall fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces and we will use the weaker condition of Corollary 2.2.5. Before we do that, we first give a summary of hyperconvex metric spaces.

2.3. Hyperconvex Metric Spaces

The concept of hyperconvexity is due to Aronszajn and Panitchpakdi [2] who proved that a hyperconvex metric space is a nonexpansive absolute retract, that is, it is a nonexpansive retract of any metric space in which it is isometrically embedded. Recently, Sine [30] proved that some fixed point property for nonexpansive maps hold in bounded hyperconvex spaces. Since then, other authors have shown many other interesting results to hold in hyperconvex metric spaces. In this section, we will present a summary of hyperconvex spaces and some of their properties. For more results on hyperconvex metric spaces, we refer the reader to [2], [3], [14], [29] and [30].

We begin our discussion by recalling the definition of metric convexity.

Definition 2.3.1. ([8]) *A metric space (X, m) is said to be metrically convex if for any points $x, y \in X$ and positive real numbers r_1 and r_2 such that $m(x, y) \leq r_1 + r_2$, there exists $z \in X$ such that $m(x, z) \leq r_1$ and $m(z, y) \leq r_2$ or equivalently $z \in C_m(x, r_1) \cap C_m(x, r_2)$.*

Remark 2.3.2. ([8]) Let (X, m) be a metric space. Using the Triangle Inequality, we have $C_m(x_1, r_1) \cap C_m(x_2, r_2) \neq \emptyset$ implies $m(x_1, x_2) \leq r_1 + r_2$ for any $x_1, x_2 \in X$ and positive numbers r_1, r_2 . The converse is true on the real line and corresponds to Menger convexity in metric spaces.

We now recall the definition of an injective metric space.

Definition 2.3.3. ([14]) A metric space (X, m) is said to be injective if it has the following extension property: Whenever A is a subspace of a metric space Y and $f : A \rightarrow X$ is nonexpansive, then f has a nonexpansive extension $F : Y \rightarrow X$.

We now recall the definition of a hyperconvex metric space which was first introduced by Aronszajn and Panitchpakdi [2].

Definition 2.3.4. ([2]) A metric space (X, m) is called hyperconvex if for any indexed class of closed balls $C_m(x_i, r_i)$, $i \in I$, of X which satisfy

$$m(x_i, x_j) \leq r_i + r_j \text{ whenever } i, j \in I,$$

it is necessarily the case that

$$\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset.$$

According to Aronszajn and Panitchpakdi [2], hyperconvex spaces are characterised in the following way:

Theorem 2.3.5. ([2]) A metric space (X, m) is injective if and only if it is hyperconvex.

Hyperconvex metric spaces are metrically convex. To see this, let (X, m) be a hyperconvex metric space and $x, y \in X$ such that $x \neq y$. Let $r_1 = \alpha m(x, y)$ and $r_2 = (1 - \alpha)m(x, y)$, for any $\alpha \in [0, 1]$. Then $m(x, y) = r_1 + r_2$ and since (X, m) is hyperconvex, it follows that $C_m(x, r_1) \cap C_m(x, r_2) \neq \emptyset$. Let $z \in C_m(x, r_1) \cap C_m(x, r_2)$, then we have

$$m(x, z) \leq r_1 \text{ and } m(z, y) \leq r_2.$$

Moreover, the Triangle Inequality gives

$$m(x, z) = r_1 \text{ and } m(z, y) = r_2.$$

Therefore, (X, m) is metrically convex.

We now look at the concept of binary ball intersection property.

Definition 2.3.6. ([14]) Let (X, m) be a metric space. A family of balls $(C_m(x_i, r_i))_{i \in I}$, where each two intersect, is said to have a binary ball intersection property if for all $i \in I$,

$$\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset.$$

Suppose (X, m) is a metric space and $(C_m(x_i, r_i))_{i \in I}$ is a family of closed balls in X where each two of which intersect. Then if $z \in C_m(x_i, r_i) \cap C_m(x_j, r_j)$ whenever $i, j \in I$, then

$$m(x_i, x_j) \leq m(x_i, z) + m(z, x_j) \leq r_i + r_j.$$

Hence, if the space (X, m) is hyperconvex, then

$$\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset.$$

This shows that hyperconvex metric spaces have the binary ball intersection property.

Now suppose (X, m) is a complete metric space which has the binary ball intersection property, and suppose (X, m) is metrically convex. If $(C_m(x_i, r_i))_{i \in I}$ is a family of balls in X for which $m(x_i, x_j) \leq r_i + r_j$, then there is a line segment joining x_i and x_j and some point of this line segment lies in $C_m(x_i, r_i) \cap C_m(x_j, r_j)$. Therefore, $\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset$ by binary ball intersection property.

We can summarise these results in the following lemma:

Lemma 2.3.7. ([14]) *If (X, m) is a complete metric space, then the following are equivalent:*

(i) *X is hyperconvex.*

(ii) *X is metrically convex and has the binary ball intersection property.*

We now look at two well known examples of hyperconvex metric spaces.

Example 2.3.1. The set \mathbb{R} of real numbers equipped with the usual metric $m(x, y) = |x - y|$ is hyperconvex. Also, any closed interval in \mathbb{R} is also hyperconvex.

Example 2.3.2. The space l_∞ , whose elements consist of all bounded sequences (x_1, x_2, \dots) of real numbers, with distance $m_\infty(\mathbf{x}, \mathbf{y})$ between two such sequences $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ taken as

$$m_\infty(\mathbf{x}, \mathbf{y}) = \sup_{1 \leq i \leq \infty} |x_i - y_i|,$$

provides a nontrivial example of a metric space which is hyperconvex (see [14, Theorem 4.3]).

We now turn our attention to two important subsets of metric spaces which will be useful in our discussion: admissible subsets and externally hyperconvex subsets. We shall denote the collection of subsets of a metric space (X, m) which are hyperconvex by $\mathcal{H}(X)$.

Definition 2.3.8. ([8]) *Let (X, m) be a metric space. The cover of a subset A of X , denoted $\text{cov}(A)$, is the set*

$$\text{cov}(A) = \bigcap \{C_i : C_i \text{ is a closed ball and } A \subseteq C_i\}.$$

Definition 2.3.9. ([8]) Let (X, m) be a metric space. The set of admissible subsets of X , denoted $\mathcal{A}(X)$, is collection of all subsets of X which are intersections of closed balls, i.e $\mathcal{A}(X) = \{A \subset X : A = \text{cov}(A)\}$.

If (X, m) is hyperconvex, then $\mathcal{A}(X)$ contains all closed balls of X and the intersection of elements from $\mathcal{A}(X)$ is also in $\mathcal{A}(X)$.

We now look at externally hyperconvex subsets of a metric space. Note that the following definition strengthens the concept of a hyperconvex subset of a metric space (X, m) .

Definition 2.3.10. ([8]) A subset E of a metric space (X, m) is said to be externally hyperconvex (relative to X) if given any family $(x_\alpha)_{\alpha \in S}$ of points in X and any family $(r_\alpha)_{\alpha \in S}$ of nonnegative real numbers satisfying

$$m(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \quad \text{and} \quad \text{dist}(x_\alpha, E) \leq r_\alpha, \quad \text{whenever} \quad \alpha, \beta \in S,$$

it follows that

$$\bigcap_{\alpha \in S} C_m(x_\alpha, r_\alpha) \cap E \neq \emptyset.$$

We shall denote externally hyperconvex subsets of a metric space (X, m) by $\mathcal{E}(X)$.

We now look at the relationship among the sets $\mathcal{A}(X)$, $\mathcal{E}(X)$ and $\mathcal{H}(X)$. In order to do this, we need the concept of *proximality*.

Definition 2.3.11. ([8]) Let (X, m) be a metric space. A subset E of X is said to be proximal (with respect to X) if

$$E \cap C_m(x, \text{dist}(x, E)) \neq \emptyset \quad \text{for each} \quad x \in X.$$

Lemma 2.3.12. ([8, Lemma 3.8]) If E is either an admissible subset or externally hyperconvex subset of the hyperconvex metric space (X, m) , then E is proximal in X .

Proof. [8, Lemma 3.8]. □

We now recall and give an outline of the proof of a result that gives the relationship among the sets $\mathcal{A}(X)$, $\mathcal{E}(X)$ and $\mathcal{H}(X)$. This result will be generalised to q -hyperconvex T_0 -quasi-metric spaces in Theorem 3.2.14.

Theorem 2.3.13. ([8, Theorem 3.10]) Let (X, m) be a hyperconvex metric space, then

$$\mathcal{A}(X) \subseteq \mathcal{E}(X) \subseteq \mathcal{H}(X).$$

Proof. Firstly, we show that $\mathcal{A}(X) \subseteq \mathcal{E}(X)$. Let A be an admissible subset of X and let $(x_\alpha)_{\alpha \in S}$ be a family of points in X and $(r_\alpha)_{\alpha \in S}$ be a family of nonnegative real numbers satisfying $m(x_\alpha, x_\beta) \leq$

$r_\alpha + r_\beta$ and $\text{dist}(x_\alpha, A) \leq r_\alpha$ for any $\alpha, \beta \in S$. Since A is proximal, for any $\alpha \in S$, there exists $a_\alpha \in A$ such that $m(x_\alpha, a_\alpha) = \text{dist}(x_\alpha, A)$, which gives $A \cap C_m(x_\alpha, r_\alpha) \neq \emptyset$. Since X is hyperconvex, the conditions on both families imply $\bigcap_{\alpha \in S} C_m(x_\alpha, r_\alpha) \neq \emptyset$. Since A is admissible and $A \cap C_m(x_\alpha, r_\alpha) \neq \emptyset$, it follows that

$$A \cap \left(\bigcap_{\alpha \in S} C_m(x_\alpha, r_\alpha) \right) \neq \emptyset,$$

which proves the first inclusion.

The second inclusion $\mathcal{E}(X) \subseteq \mathcal{H}(X)$ follows directly from the definition of an externally hyperconvex subset of (X, m) . \square

In view of the relationship between hyperconvex spaces and nonexpansive maps via injectivity, it is not surprising that hyperconvex metric spaces admit interesting fixed point properties for nonexpansive maps. Consider the following theorem.

Theorem 2.3.14. ([14, Theorem 4.8]) *If (X, m) is a bounded hyperconvex metric space and $F : (X, m) \rightarrow (X, m)$ is nonexpansive, then the fixed point set $\text{Fix}(F)$ of F is nonempty and hyperconvex.*

Proof. [14, Theorem 4.8]. \square

We now look at the intersection property of hyperconvex metric spaces. In general, the intersection of two hyperconvex subsets of a given hyperconvex metric space need not be hyperconvex. Consider the following example in \mathbb{R}_∞^2 :

Example 2.3.3. ([14]) Let H_1 denote the line segment joining $(-1, 0)$ to $(1, 0)$ and let H_2 be the union of the line segment joining $(-1, 0)$ to $(0, 1)$ and the line segment joining $(0, 1)$ to $(1, 0)$. Then H_1 and H_2 are both hyperconvex subsets of \mathbb{R}_∞^2 but

$$H_1 \cap H_2 = \{(-1, 0), (1, 0)\}.$$

Since $H_1 \cap H_2$ consists of only two distinct points, it follows that $H_1 \cap H_2$ is not a hyperconvex subset of \mathbb{R}_∞^2 .

However, the intersection of two admissible subsets of a hyperconvex metric space is also admissible and hyperconvex. We now look at a very useful result of Baillon [3].

Theorem 2.3.15. ([3, Theorem 7]) *Let (X, m) be a bounded hyperconvex metric space and let $\{H_n\}$ be a descending sequence of nonempty hyperconvex subsets of X . Then $H = \bigcap_{n=1}^{\infty} H_n$ is nonempty and hyperconvex.*

Proof. [3, Theorem 7]. \square

We now recall the concept of approximate fixed points due to Sine [29].

Definition 2.3.16. ([14]) Let (X, m) be a metric space. Given a subset A of X , we define for $\epsilon > 0$ the ϵ -parallel set of A as

$$N_\epsilon(A) = \bigcup_{a \in A} C_m(a, \epsilon).$$

Note that $x \in N_\epsilon(A)$ if and only if there exists $a \in A$ such that $m(x, a) \leq \epsilon$. The following Lemma gives the characterization of $N_\epsilon(A)$ if $A \in \mathcal{A}(X)$.

Lemma 2.3.17. ([14, Lemma 4.2]) Suppose (X, m) is a hyperconvex metric space and let $A \in \mathcal{A}(X)$, say $A = \bigcap_{i \in I} C_m(x_i, r_i)$. Then for each $\epsilon > 0$,

$$N_\epsilon(A) = \bigcap_{i \in I} C_m(x_i, r_i + \epsilon).$$

Proof. [14, Lemma 4.2]. □

We now recall an important result that will be useful in the proof of Theorem 2.3.21.

Lemma 2.3.18. ([14, Lemma 4.3]) Suppose (X, m) is a hyperconvex metric space and let $A \in \mathcal{A}(X)$. Then for each $\epsilon > 0$ there is a nonexpansive retraction R of $N_\epsilon(A)$ into A which has the property that $m(x, R(x)) \leq \epsilon$ for each $x \in N_\epsilon(A)$.

Proof. [14, Lemma 4.3]. □

Definition 2.3.19. ([14]) Let (X, m) be a metric space. We say that the map $f : (X, m) \rightarrow (X, m)$ has approximate fixed points if

$$\inf_{x \in X} m(x, f(x)) = 0.$$

Definition 2.3.20. ([14]) Let (X, m) be a metric space. For the map $f : (X, m) \rightarrow (X, m)$ and any $\epsilon > 0$, we use $F_\epsilon(f)$ to denote the set of ϵ -approximate fixed points of f , that is,

$$F_\epsilon(f) = \{x \in X : m(x, f(x)) \leq \epsilon\}.$$

We now recall and outline the proof a very useful result.

Theorem 2.3.21. ([14, Theorem 4.1]) Suppose (X, m) is a hyperconvex metric space and $f : (X, m) \rightarrow (X, m)$ is nonexpansive. Then for any $\epsilon > 0$, the set $F_\epsilon(f) = \{x \in X : m(x, f(x)) \leq \epsilon\}$ is hyperconvex.

Proof. Clearly, we may suppose that $F_\epsilon(f)$ is nonempty. Let $x_i \in F_\epsilon(f)$ for each $i \in I$ and let $r_i \geq 0$ satisfy

$$m(x_i, x_j) \leq r_i + r_j.$$

Our goal is to show that

$$\left(\bigcap_{i \in I} C_m(x_i, r_i) \right) \cap F_\epsilon(f) \neq \emptyset.$$

Since $J = \bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset$ by the hyperconvexity of X , if $x \in J$, then for each $i \in I$

$$m(x_i, f(x)) \leq m(f(x), f(x_i)) + m(f(x_i), x_i) \leq m(x, x_i) + \epsilon \leq r_i + \epsilon.$$

This shows that $f(x) \in N_\epsilon(J)$. Now by Lemma 2.3.18, there is a nonexpansive retraction R of $N_\epsilon(J)$ onto J for which $m(R(x), x) \leq \epsilon$ for each $x \in N_\epsilon(J)$. Also, since $R \circ f$ is a nonexpansive map of J into J , it must have a fixed point by Theorem 2.3.14. Suppose $(R \circ f)(x_0) = x_0$ for each $x_0 \in J$. Then

$$m(x_0, f(x_0)) = m((R \circ f)(x_0), f(x_0)) \leq \epsilon.$$

From this, it follows that $x_0 \in J \cap F_\epsilon(f)$, and the proof is complete. \square

We now turn our attention to selection theorems for maps taking values in the spaces $\mathcal{A}(X)$ and $\mathcal{E}(X)$. We begin our discussion by recalling the following result that is used in the proofs of some of the selection theorems we are about to state.

Theorem 2.3.22. ([13, Theorem 1]) *Let H be a hyperconvex metric space, let S be any set, and let $F : S \rightarrow \mathcal{E}(H)$. Then there exists a mapping $f : S \rightarrow H$ for which $f(x) \in F(x)$ for each $x \in S$ such that $m(f(x), f(y)) \leq M(F(x), F(y))$ for each $x, y \in S$.*

Proof. [13, Theorem 1]. \square

The next result guarantees that if the set-valued map given in Theorem 2.3.22 is nonexpansive, then its selection is also nonexpansive.

Theorem 2.3.23. ([30, Theorem 1]) *Let (X, m) be an arbitrary metric space and (H, ρ) be a hyperconvex metric space. Suppose $F : (X, m) \rightarrow (\mathcal{A}(H), M)$ is nonexpansive in the sense:*

$$M(F(x), F(y)) \leq m(x, y) \text{ for all } x, y \in X.$$

Then F admits a nonexpansive point valued map $f : (X, m) \rightarrow (H, \rho)$ with $f(x) \in F(x)$ for any $x \in X$.

Proof. It is easy to see that if F is nonexpansive, then the selection f assumed in Theorem 2.3.22 is nonexpansive as well. \square

Since a nonexpansive point valued map from a bounded hyperconvex metric space to itself has at least one fixed point according to Theorem 2.3.14, it is also true for a set-valued map taking values in the space $\mathcal{A}(X)$. Consider the following result.

Corollary 2.3.24. ([30, Corollary 2]) Suppose (X, m) is a bounded hyperconvex metric space and $F : (X, m) \rightarrow (\mathcal{A}(X), M)$ is a nonexpansive map. Then F has fixed points.

Proof. By Theorem 2.3.23, there is a nonexpansive point valued map $f : (X, m) \rightarrow (X, m)$ such that $f(x) \in F(x)$ whenever $x \in X$. By Theorem 2.3.14, f has a fixed point say z . It is clear that z is also a fixed point of F . \square

Corollary 2.3.25. ([30, Corollary 3]) Let (X, m) be a hyperconvex metric space and $F : (X, m) \rightarrow (\mathcal{A}(X), M)$ be a nonexpansive map such that $\text{Fix}(F) \neq \emptyset$. Then $\text{Fix}(F)$ is hyperconvex.

Proof. The same conclusion holds for the nonexpansive selection $f : (X, m) \rightarrow (X, m)$. \square

In the following theorem, $\text{Fix}(F) = \{x \in X : x \in F(x)\}$. Corollary 2.3.25 implies that $\text{Fix}(F) \neq \emptyset$ if (X, m) is bounded.

Theorem 2.3.26. ([13, Theorem 2]) Let (X, m) be a hyperconvex metric space, let $F : (X, m) \rightarrow (\mathcal{E}(X), M)$ be nonexpansive and suppose $\text{Fix}(F) \neq \emptyset$. Then there is a nonexpansive map $f : (X, m) \rightarrow (X, m)$ with $f(x) \in F(x)$ for each $x \in X$ and for which $\text{Fix}(f) = \text{Fix}(F)$.

Proof. [13, Theorem 2]. \square

We now turn our attention to fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces.

2.4. Fixed Point Theorems in Hyperconvex Metric Spaces

In this section, we present the results of Espínola and Kirk [9] on fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces. We begin our discussion by recalling a result that guarantees the existence of at least one fixed point for a family of set-valued contraction maps taking values in the space $\mathcal{CB}(X)$ of all nonempty bounded closed subsets of a hyperconvex metric space (X, m) . This result will be generalised to a family of set-valued contraction maps in q -hyperconvex T_0 -quasi-metric spaces in Theorem 3.3.1. In addition to the proof of Theorem 6 in [9], we show that a map $f : (X, m) \rightarrow (X, m)$ defined by $\{f(x)\} = \bigcap_{\alpha \in \Lambda} f_\alpha(x)$, where $\{f_\alpha\}_{\alpha \in \Lambda}$ is a family of contraction maps indexed over a directed set Λ , is nonexpansive.

Theorem 2.4.1. ([9, Theorem 6]) Suppose (X, m) is a hyperconvex metric space, and $\mathcal{CB}(X)$ is a collection of all nonempty bounded closed subsets of X endowed with the Hausdorff metric M . Let $f_\alpha : (X, m) \rightarrow (\mathcal{CB}(X), M)$ be a family of contraction maps indexed over a directed set Λ . Suppose

(i) for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,

(ii) for every $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $\text{diam}(f_\alpha(x)) \leq \epsilon$ for every $x \in X$,

Then there exists at least one point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

Proof. Define $f : (X, m) \rightarrow (X, m)$ by $\{f(x)\} = \bigcap_\alpha f_\alpha(x)$ for each $x \in X$. From the definition of the Hausdorff metric on $\mathcal{CB}(X)$, we see that for any $x, y \in X$,

$$M(f_\alpha(x), f_\alpha(y)) = \max \left\{ \sup \left\{ \text{dist}(f(x), f_\alpha(y)) \right\}_{f(x) \in f_\alpha(x)}, \sup \left\{ \text{dist}(f(y), f_\alpha(x)) \right\}_{f(y) \in f_\alpha(y)} \right\}.$$

Note that

$$\text{dist}(f(x), f_\alpha(y)) = \inf_{f(x) \in f_\alpha(x)} \{m(f(x), f(y)) : f(y) \in f_\alpha(y)\} \leq m(f(x), f(y)) \text{ for any } x, y \in X.$$

Therefore,

$$\sup_{f(x) \in f_\alpha(x)} \text{dist}(f(x), f_\alpha(y)) = m(f(x), f(y)).$$

Similarly,

$$\sup_{f(y) \in f_\alpha(y)} \text{dist}(f(y), f_\alpha(x)) = m(f(x), f(y)).$$

Hence,

$$M(f_\alpha(x), f_\alpha(y)) = m(f(x), f(y)) \text{ whenever } x, y \in X,$$

and since f_α is a contraction map for each $\alpha \in \Lambda$, we have

$$\begin{aligned} m(f(x), f(y)) &= M(f_\alpha(x), f_\alpha(y)) \\ &\leq k_\alpha m(x, y) \\ &\leq m(x, y). \end{aligned}$$

Therefore, f is nonexpansive. Now our goal is to show that f has a fixed point. Choose $\epsilon > 0$ so that the set

$$F_\epsilon(f) = \{x \in X : m(x, f(x)) \leq \epsilon\},$$

is nonempty. By Theorem 2.3.21, the set $F_\epsilon(f)$ is hyperconvex. Choose $\alpha \in \Lambda$ so that

$$\text{diam}(f_\alpha(z)) \leq \epsilon \text{ for all } z \in X.$$

If $u, v \in F_\epsilon(f)$, then the Triangle Inequality gives

$$\begin{aligned} m(u, v) &\leq m(u, f(u)) + m(f(u), f(v)) + m(v, f(v)) \\ &\leq \epsilon + m(f(u), f(v)) + \epsilon \\ &= m(f(u), f(v)) + 2\epsilon. \end{aligned}$$

Therefore, we have

$$\begin{aligned} m(u, v) &\leq m(f(u), f(v)) + 2\epsilon \\ &\leq M(f_\alpha(u), f_\alpha(v)) + \text{diam}(f_\alpha(u)) + \text{diam}(f_\alpha(v)) + 2\epsilon \\ &\leq k_\alpha m(u, v) + 4\epsilon. \end{aligned}$$

Hence $m(u, v) \leq 4(1 - k_\alpha)^{-1}\epsilon$. This shows that the distance between any two elements of $F_\epsilon(f)$ is bounded. Thus $F_\epsilon(f)$ is bounded. Since $\bar{f} : F_\epsilon(f) \rightarrow F_\epsilon(f)$, then \bar{f} has at least one fixed point by Theorem 2.3.14. Since \bar{f} is the restriction of f to $F_\epsilon(f)$, then it follows that f has at least one fixed point. \square

We saw that the fixed point in Corollary 2.2.5 was unique. However, under the assumptions of Theorem 2.4.1, the fixed point need not be unique. The next theorem involves the family of admissible subsets of the hyperconvex metric space (X, m) . Before we recall the theorem, we prove a useful result that guarantees the existence of at least one fixed point for a set-valued contraction map taking values in the space $\mathcal{A}(X)$ of admissible subsets of a hyperconvex metric space (X, m) . This result will be generalised to q -hyperconvex T_0 -quasi-metric spaces in Proposition 3.3.2.

Proposition 2.4.2. *Let (X, m) be a hyperconvex metric space and $f : (X, m) \rightarrow (\mathcal{A}(X), M)$ be a contraction map. Then f has at least one fixed point.*

Proof. Since f is a contraction map, then for any $k \in [0, 1)$ and $x, y \in X$,

$$M(f(x), f(y)) \leq k m(x, y) \leq d(x, y).$$

Therefore, f is nonexpansive. Thus by Theorem 2.3.23, there exists a nonexpansive map $\bar{f} : X \rightarrow X$ such that $\bar{f}(x) \in f(x)$ whenever $x \in X$.

Choose $\epsilon > 0$ so that the set

$$F_\epsilon(\bar{f}) = \{x \in X : d(x, \bar{f}(x)) \leq \epsilon\}$$

is nonempty. Using the argument in Theorem 2.4.1, we see that $F_\epsilon(\bar{f})$ is bounded and by Theorem 2.3.21, $F_\epsilon(\bar{f})$ is also hyperconvex. Thus by Theorem 2.3.14, $\bar{f}^* : F_\epsilon(\bar{f}) \rightarrow F_\epsilon(\bar{f})$ has at least one fixed point say z . Since \bar{f}^* is a restriction of \bar{f} to $F_\epsilon(\bar{f})$, then z is a fixed point of \bar{f} and f . \square

We next recall an important result which shows that the fixed point set for a set-valued contraction map is bounded. This result will be generalised to q -hyperconvex T_0 -quasi-metric spaces in Lemma 3.3.3.

Lemma 2.4.3. *([9, Lemma 7]) Let (X, m) be a hyperconvex metric space. If $T : (X, m) \rightarrow (\mathcal{CB}(X), M)$ is a contraction map, then the set of fixed points of T is bounded.*

Proof. Let $Fix(T)$ be the set of points fixed by T . Fix $x \in Fix(T)$ and let $\epsilon > 0$. If $y \in Fix(T)$, then since T is a contraction map, $\exists k \in [0, 1)$ such that

$$M(T(x), T(y)) \leq k m(x, y).$$

This implies that there exists $z \in T(x)$ such that

$$\begin{aligned} m(y, z) &\leq k m(x, y) + \epsilon \\ &\leq k[m(x, z) + m(z, y)] + \epsilon \\ &= k m(x, z) + k m(z, y) + \epsilon. \end{aligned}$$

Thus, we have

$$\begin{aligned} (1 - k)m(y, z) &\leq k m(x, z) + \epsilon \\ &\leq k \operatorname{diam}(T(x)) + \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} m(x, y) &\leq m(x, z) + m(z, y) \\ &\leq \operatorname{diam}(T(x)) + k \operatorname{diam}(T(x)) + \epsilon \\ &= (1 + k)\operatorname{diam}(T(x)) + \epsilon \\ &\leq (1 - k)^{-1}[(1 + k)\operatorname{diam}(T(x)) + \epsilon]. \end{aligned}$$

Hence,

$$m(x, y) \leq (1 - k)^{-1}((1 + k)\operatorname{diam}(T(x)) + \epsilon).$$

This shows that $m(x, y)$ is bounded for any $x, y \in \operatorname{Fix}(T)$. Therefore, $\operatorname{Fix}(T)$ is bounded. \square

If the family $\mathcal{CB}(X)$ in Theorem 2.4.1 is replaced with the family $\mathcal{A}(X)$ of admissible subsets of X , then assumption (ii) is no longer needed. The following result shows that the fixed point set for a family of set-valued contraction maps taking values in the space $\mathcal{A}(X)$ of admissible subsets of a hyperconvex metric space (X, m) is hyperconvex, this result will be generalised to q -hyperconvex T_0 -quasi-metric spaces in Theorem 3.3.4.

Theorem 2.4.4. ([9, Theorem 8]) *Suppose (X, m) is a bounded hyperconvex metric space and $f_\alpha : (X, m) \rightarrow (\mathcal{A}(X), M)$ is a family of contraction maps indexed over a directed set Λ , where $\mathcal{A}(X)$ is the collection of all nonempty admissible subsets of X endowed with the Hausdorff metric M . Suppose for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$. Then the set of points $x \in X$ such that $x \in \bigcap_{\alpha \in \Lambda} f_\alpha(x)$ is nonempty and hyperconvex.*

Proof. Since $f_\alpha : (X, m) \rightarrow (\mathcal{A}(X), M)$ is a contraction map for each $\alpha \in \Lambda$, then by Proposition 2.4.2, each f_α has a nonempty fixed point set H_α . Thus by Theorem 2.3.26, there is a nonexpansive map $\bar{f}_\alpha : X \rightarrow X$ with $\bar{f}_\alpha(x) \in f_\alpha(x)$ whenever $x \in X$ such that $\operatorname{Fix}(\bar{f}_\alpha) = H_\alpha$ for each $\alpha \in \Lambda$. Since fixed point sets of nonexpansive maps are hyperconvex by Theorem 2.3.14, then it follows that each H_α is a hyperconvex subset of X . Also, by Lemma 2.4.3, the sets H_α are bounded for each $\alpha \in \Lambda$. Hence $\{H_\alpha\}_{\alpha \in \Lambda}$ is a family of nonempty bounded hyperconvex subsets of X directed

by reverse set inclusion, that is, if $\beta \geq \alpha$, then $H_\beta \subseteq H_\alpha$. It follows from Theorem 2.3.15 that $H = \bigcap_{\alpha \in \Lambda} H_\alpha$ is nonempty and hyperconvex.

Now define $f : (X, m) \rightarrow (\mathcal{P}_0(X), M)$ for each $x \in X$ by

$$f(x) = \bigcap_{\alpha \in \Lambda} f_\alpha(x).$$

Then by Theorem 2.3.15, $f(x)$ is admissible and nonempty since $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is a family of nonempty admissible sets directed by reverse set inclusion (Note that admissible subsets are hyperconvex). Then f is a nonexpansive map of X into $\mathcal{A}(X)$. Therefore, by Corollary 2.3.24, f has at least one fixed point. Hence, it is clear that

$$x \in \bigcap_{\alpha \in \Lambda} f_\alpha(x) \text{ if and only if } x \in \text{Fix}(f) = H.$$

□

Remark 2.4.1. ([9]) The family $\mathcal{A}(X)$ in Theorem 2.4.4 can be replaced by the larger family $\mathcal{E}(X)$ of externally hyperconvex subsets of X .

We now apply Theorem 2.4.4 to the real line.

Example 2.4.1. ([9, Example 1]) Equip \mathbb{R} with the usual metric and let $\{f_n\}$ and $\{g_n\}$ be two sequences of mapping of $\mathbb{R} \rightarrow \mathbb{R}$ satisfying:

- (i) $f_n(x) \leq g_n(x)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$,
- (ii) $\{f_n\}$ is nondecreasing and $\{g_n\}$ is nonincreasing for each $x \in \mathbb{R}$,
- (iii) for each $n \in \mathbb{N}$, there exists $k_n \in (0, 1)$ such that for each $x, y \in \mathbb{R}$,

$$\max\{|f_n(x) - f_n(y)|, |g_n(x) - g_n(y)|\} \leq k_n|x - y|.$$

The conclusion under these assumptions is that there exists $a \leq b \in \mathbb{R}$ such that

$$\lim_n f_n(x) \leq x \leq \lim_n g_n(x) \text{ for each } x \in [a, b].$$

Proof. Let

$$F_n(x) = [f_n(x), g_n(x)] \text{ for each } x \in \mathbb{R} \text{ and } n \in \mathbb{N},$$

then $F_n(x)$ is admissible for each $n \in \mathbb{N}$. By condition (ii), we see that

$$F_{n+1}(x) \subseteq F_n(x).$$

Condition (iii) implies that F_n is a contraction map for each $n \in \mathbb{N}$. Then F_n satisfies all the conditions of Theorem 2.4.4 and therefore, the conclusion follows. □

The next example shows that the fixed point is not unique in Theorem 2.4.1.

Example 2.4.2. ([9, Example 2]) Equip $I = [0, 1]$ with the metric $m(x, y) = |x - y|$. Let $\mathcal{C}(I)$ denote the family of nonempty closed subintervals of I . Now for each $\alpha \in (0, 1)$, define

$$F_\alpha(x) = [f_\alpha(x), g_\alpha(x)] \cap I,$$

where

$$\begin{aligned} f_\alpha(x) &= \alpha(x + 1) - 1 \quad \text{and} \\ g_\alpha(x) &= \alpha(x - 2) + 2. \end{aligned}$$

Thus for any $\alpha \in (0, 1)$ and $x \in [0, 1]$,

$$f_\alpha(x) \leq g_\alpha(x).$$

Also, $\beta \geq \alpha$ implies

$$f_\alpha(x) \leq f_\beta(x) \quad \text{while} \quad g_\beta(x) \leq g_\alpha(x) \quad \text{for all } x \in I.$$

This implies that

$$F_\beta(x) \subseteq F_\alpha(x).$$

Also, we see that

$$\begin{aligned} M(F_\alpha(x), F_\alpha(y)) &\leq \max\{|f_\alpha(x) - f_\alpha(y)|, |g_\alpha(x) - g_\alpha(y)|\} \\ &= |\alpha(x + 1) - 1 - (\alpha(y + 1) - 1)| \\ &= \alpha|x - y|. \end{aligned}$$

and

$$\begin{aligned} \text{diam}(F_\alpha(x)) &\leq g_\alpha(x) - f_\alpha(x) \\ &= \alpha(x - 2) + 2 - [\alpha(x + 1) - 1] \\ &= 3(1 - \alpha). \end{aligned}$$

Hence

$$\text{diam}(F_\alpha(x)) \leq 3(1 - \alpha).$$

Now let the map $F : I \rightarrow I$ be defined by

$$F(x) = \bigcap_{\alpha \in (0, 1)} F_\alpha(x).$$

Then for every $x \in I$, $F(x) = x$. Therefore, this shows that the fixed point need not be unique.

3

Set-Valued Contractions and q -Hyperconvex T_0 -Quasi-Metric Spaces

In this chapter we start our own investigations. We generalise the results of Espínola and Kirk [9] presented in Chapter 2 to q -hyperconvex T_0 -quasi-metric spaces. Our investigations confirm the surprising fact that many classical results about fixed point theorems in hyperconvex metric spaces do not make essential use of the symmetry of the metric and, therefore, still hold in sometimes slightly modified form for our concept of fixed point theorems in q -hyperconvex T_0 -quasi-metric spaces. While many classical ideas about fixed point theorems in hyperconvex spaces can be generalised properly from the metric to quasi-pseudometric setting, these generalisations are not always easy and sometimes the asymmetry setting requires interesting new variations of old arguments. We begin by discussing fixed point theorems for a family of set-valued contraction maps in bicomplete quasi-gauge spaces and bicomplete quasi-metric spaces.

3.1. An Extension of Frigon's Theorem in Quasi-Gauge Spaces

In this section, we present an extension of Frigon's Theorem in quasi-gauge spaces. This result generalises the extension of Frigon's Theorem in gauge spaces due to Espínola and Kirk [9] presented in Section 2.2 (See Theorem 2.2.4).

We assume that \mathbb{G} is a quasi-gauge space endowed with a bicomplete quasi-gauge structure $\mathcal{Q} = \{d_\alpha : \alpha \in \Lambda\}$, where Λ is a directed set.

Theorem 3.1.1. *(Compare Theorem 2.2.4) Let \mathbb{G} be a bicomplete quasi-gauge space and let $\mathbb{G}_{cl} = \{A \subset \mathbb{G} : A = cl_{\tau(d_\alpha)}A \cap cl_{\tau(d_\alpha^{-1})}A\}$. Suppose $f_\alpha : (\mathbb{G}, d_\alpha) \rightarrow (\mathbb{G}_{cl}, D_\alpha)$ is a family of maps satisfying*

- (i) $f_\alpha(x)$ is a subset of \mathbb{G} for each $\alpha \in \Lambda$ and belongs to \mathbb{G}_{cl} ,
- (ii) $D_\alpha(f_\alpha(x), f_\alpha(y)) \leq k_\alpha d_\alpha(x, y) \forall x, y \in \mathbb{G}$ and $\alpha \in \Lambda$, where $k_\alpha \in (0, 1)$ and D_α denotes the Hausdorff T_0 -quasi-metric on \mathbb{G}_{cl} ,

(iii) $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(y)$ for each $x \in \mathbb{G}$,

(iv) $\forall \alpha \in \Lambda$ and $\epsilon > 0$, there exists $\beta \geq \alpha$ such that $\text{diam}_\beta(f_\beta(x)) \leq (1 - k_\beta)\epsilon$ for each $x \in \mathbb{G}$.

Then there exists a unique point $z \in \mathbb{G}$ such that $z \in f_\alpha(z)$ for each $\alpha \in \Lambda$. Moreover, $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z for each $x \in \mathbb{G}$ where $\bar{f}(x)$ is a unique point of $\bigcap_\alpha f_\alpha(x)$ whenever $x \in \mathbb{G}$.

Proof. Note that by assumption, $(\mathbb{G}, \mathcal{Q})$ is bicomplete. Since $f_\alpha(x) \in \mathbb{G}_{cl}$, then it follows that $f_\alpha(x)$ is a nonempty $\tau(d^s)$ -closed subset of \mathbb{G} for each $\alpha \in \Lambda$. Also, since f_α is a contraction map for each $\alpha \in \Lambda$, then

$$D_\alpha(f_\alpha(x), f_\alpha(y)) \leq k_\alpha d(x, y) \text{ whenever } k_\alpha \in (0, 1) \text{ and } x, y \in \mathbb{G},$$

and

$$D_\alpha^{-1}(f_\alpha(x), f_\alpha(y)) \leq k_\alpha d^{-1}(x, y) \text{ whenever } k_\alpha \in (0, 1) \text{ and } x, y \in \mathbb{G}.$$

Therefore,

$$D_\alpha^s(f_\alpha(x), f_\alpha(y)) \leq k_\alpha d^s(x, y) \text{ whenever } k_\alpha \in (0, 1) \text{ and } x, y \in \mathbb{G}.$$

Thus, $f_\alpha : (\mathbb{G}, d_\alpha^s) \rightarrow (\mathbb{G}_{cl}, D_\alpha^s)$ is a generalised contraction for each $\alpha \in \Lambda$. Hence by Theorem 2.2.4, there exists $z \in \mathbb{G}$ such that $z \in f_\alpha(x)$ for each $\alpha \in \Lambda$ and any sequence $\{\bar{f}^n(x)\}_{n \in \mathbb{N}}$ converges to z , where $\bar{f}(x)$ is a unique point of $\bigcap_\alpha f_\alpha(x)$. \square

If we let $\mathbb{G} = X$ in Theorem 3.1.1, where X is a bicomplete quasi-metric space, then we have the following general result that generalises Corollary 2.2.5 to bicomplete quasi-metric spaces.

Corollary 3.1.2. (Compare Corollary 2.2.5) Let (X, d) be a bicomplete quasi-metric space, $S_{cl} = \{A \subset X : A = cl_{\tau(d)} A \cap cl_{\tau(d^{-1})} A\}$ and $f_\alpha : (X, d) \rightarrow (S_{cl}, D)$ be a family of set-valued maps indexed over a directed set Λ satisfying

(i) $D(f_\alpha(x), f_\alpha(y)) \leq k_\alpha d(x, y) \forall x, y \in X, \alpha \in \Lambda$, where $k_\alpha \in (0, 1)$ and D denotes the Hausdorff T_0 -quasi-metric on S_{cl} ,

(ii) for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,

(iii) for each $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $\text{diam}(f_\alpha(x)) \leq (1 - k_\alpha)\epsilon$ for each $x \in X$.

Then there exists a unique point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

Proof. Using the arguments in the proof of Theorem 3.1.1, we see that Corollary 3.1.2 satisfies the conditions of Corollary 2.2.5. Therefore, there exists a unique point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$. \square

In Corollary 3.1.2, condition (iii) is necessary for proving uniqueness of a fixed point (see Example 3.3.2 below). However, existence results can be obtained under the weaker condition $\text{diam}(f_\alpha(x)) \leq \epsilon$. This leads us to the class of q -hyperconvex T_0 -quasi-metric spaces. In Section 3.3, we develop fixed point theorems for a family of set-valued contraction maps in q -hyperconvex T_0 -quasi-metric spaces and we will use the weaker condition of Corollary 3.1.2. Before we do that, we first give a summary of q -hyperconvex T_0 -quasi-metric spaces.

3.2. q -Hyperconvex T_0 -Quasi-Metric-Spaces

In this section, we present a summary of q -hyperconvex T_0 -quasi-metric spaces and some of their properties. We begin this section by recalling convexity in quasi-pseudometric spaces called q -hyperconvexity or Isbell convexity. For more information see [15], [12] and [26].

Definition 3.2.1. ([12]) *A quasi-pseudometric space (X, d) is said to be q -hyperconvex (Isbell-convex) provided that for each family $(x_i)_{i \in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ the following condition holds:*

$$\text{if } d(x_i, x_j) \leq r_i + s_j \text{ whenever } i, j \in I, \text{ then } \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

Remark 3.2.2. ([12]) *In the following, we are interested in working in T_0 -quasi-metric spaces, so we do not require that r_i or s_i (where $i \in I$) attain only positive values in Definition 3.2.1. We also note that we can assume without loss of generality that the points x_i ($i \in I$) are pairwise distinct in Definition 3.2.1: Indeed if this is not the case, then for each $x \in X$, set $T(x) = \{i \in I : x_i = x\}$ and consider only the points $x \in X$ which satisfy $T(x) \neq \emptyset$. Furthermore, set $r(x) = \inf\{r_i : i \in T(x)\}$ and $s(x) = \inf\{s_i : i \in T(x)\}$. Then we have $d(x, y) \leq r_i + s_i$ whenever $i \in T(x)$ and $j \in T(y)$. Thus $d(x, y) \leq r(x) + s_i$ whenever $j \in T(y)$, and consequently $d(x, y) \leq r(x) + s(y)$. Applying the definition of q -hyperconvexity to the family $(x)_{T(x) \neq \emptyset}$ of pairwise distinct points of X and families $(r(x))_{T(x) \neq \emptyset}$ and $(s(x))_{T(x) \neq \emptyset}$ of nonnegative real numbers, we find that $\emptyset \neq \bigcap_{T(x) \neq \emptyset} (C_d(x, r(x)) \cap C_{d^{-1}}(x, s(x))) \subseteq \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$. Hence the apparent weaker condition is indeed equivalent to our definition.*

We now turn our attention to some examples of q -hyperconvex T_0 -quasi-metric spaces.

Example 3.2.1. ([12, Example 1]) Let the set \mathbb{R} of real numbers be equipped with the T_0 -quasi-metric $d(x, y) = x - y = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$. Then (\mathbb{R}, d) is q -hyperconvex.

Proof. Note that $C_d(x, \epsilon) = [x - \epsilon, \infty)$ and $C_{d^{-1}}(x, \epsilon) = (-\infty, x + \epsilon]$ and $\epsilon \geq 0$.

Let $(x_i)_{i \in I}$ be a family of points in \mathbb{R} and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of nonnegative real numbers such that $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Suppose first that $\bigcap_{i \in F} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) = \emptyset$ for some finite subset F of I . We assume that F is nonempty. It follows that $\max\{x_i - r_i : i \in$

$F\} > \min\{x_i + s_i : i \in F\}$. Consequently there are $i_0, j_0 \in F$ such that $x_{i_0} - r_{i_0} > x_{j_0} + s_{j_0}$, that is, $C_d(x_{i_0}, r_{i_0}) \cap C_{d-1}(x_{j_0}, r_{j_0}) \neq \emptyset$. In particular, $x_{i_0} > x_{j_0}$. Thus $d(x_{i_0}, x_{j_0}) = x_{i_0} - x_{j_0} > r_{i_0} + s_{j_0}$. Hence we have a contradiction. Therefore, it must be the case that $\bigcap_{i \in F} (C_d(x_i, r_i) \cap C_{d-1}(x_i, s_i)) \neq \emptyset$ whenever F is a finite subset of I . Since for any $i \in I$, $C_d(x_i, r_i) \cap C_{d-1}(x_i, s_i)$ is compact with respect to the topology $\tau(d^s)$ on \mathbb{R} , we conclude that $\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d-1}(x_i, s_i)) \neq \emptyset$. Hence (\mathbb{R}, d) is q -hyperconvex. □

Corollary 3.2.3. ([12, Corollary 1]) *The subspace $[0, \infty)$ of (\mathbb{R}, d) is q -hyperconvex.*

Proof. In the proof of Example 3.2.1, we work with balls $C_d(x, \epsilon) \cap [0, \infty)$ and $C_{d-1}(x, \epsilon) \cap [0, \epsilon)$ where $x \in [0, \infty)$ and $\epsilon \geq 0$. □

Example 3.2.2. ([12, Example 2]) Let \mathbb{R} be equipped with the usual metric $d^s(x, y) = |x - y|$ whenever $x, y \in \mathbb{R}$. Then (\mathbb{R}, d^s) is not q -hyperconvex.

Proof. [12, Example 2]. □

Example 3.2.3. ([12, Example 3]) Consider \mathbb{R}^2 equipped with the T_0 -quasi-metric $d((\alpha, \beta), (\alpha', \beta')) = (\alpha - \alpha') \vee (\beta - \beta')$ whenever $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$. Then the diagonal $\{(\alpha, \alpha) : \alpha \in \mathbb{R}\}$ in this product T_0 -quasi-metric space is isometric to (\mathbb{R}, d^s) .

In Section 2.3, we saw that hyperconvex metric spaces are metrically convex (see Definition 2.3.1). This idea can be generalised to quasi-pseudometric spaces as follows:

Definition 3.2.4. ([12]) *Let (X, d) be a quasi-pseudometric space. We say that X is metrically convex if for any points $x, y \in X$ and nonnegative real numbers r and s such that $d(x, y) \leq r + s$, there exists $z \in X$ such that $d(x, z) \leq r$ and $d(z, y) \leq s$.*

Example 3.2.4. ([12, Example 4]) Consider the so-called Sorgenfrey T_0 -quasi-metric on \mathbb{R} which is defined for each $x, y \in \mathbb{R}$ as follows:

$$d(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 1 & \text{otherwise.} \end{cases}$$

Then (\mathbb{R}, d) is not metrically convex. Indeed, we have $d(\frac{1}{2}, 1) = 1 \leq \frac{1}{2} + \frac{1}{2}$. But there is no $z \in \mathbb{R}$ such that $d(\frac{1}{2}, z) \leq \frac{1}{2}$ and $d(z, 1) \leq \frac{1}{2}$, since such a z would satisfy $z \leq \frac{1}{2}$ and $z \geq 1$.

Definition 3.2.5. ([26, Definition 2]) *Let (X, d) be a quasi-pseudometric space. A family of balls $(C_d(x_i, r_i), C_{d-1}(x_i, s_i))_{i \in I}$ with $r_i, s_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$ is said to have a mixed binary intersection property if for all indices $i, j \in I$,*

$$\bigcap_{i \in I} C_d(x_i, r_i) \cap C_{d-1}(x_i, s_i) \neq \emptyset.$$

Definition 3.2.6. ([26, Definition 3]) A quasi-pseudometric space (X, d) is said to be q -hypercomplete or Isbell complete if every family of balls $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ where $r_i, s_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$, having a mixed binary intersection property satisfies,

$$\bigcap_{i \in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset.$$

Lemma 3.2.7. ([12, Proposition 1]) Let (X, d) be quasi-pseudometric space. Then the following are equivalent

- (i) (X, d) is q -hyperconvex.
- (ii) (X, d) is metrically convex and q -hypercomplete.

Proof. [12, Proposition 1] □

We now look at some properties of q -hyperconvex quasi-pseudometric spaces. We begin by looking at the following useful result.

Proposition 3.2.8. ([15, Proposition 3.1]) Let (X, d) be a q -hyperconvex quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a nonempty family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be two families of nonnegative real numbers such that $d(x_i, x_j) \leq r_i + s_j$. Set $D = \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$. Then D is nonempty and q -hyperconvex.

Proof. [15, Proposition 3.1] □

We now turn our attention to two important subsets of quasi-pseudometric spaces that will be useful in our discussion: q -admissible subsets and externally q -hyperconvex subsets. Note that we shall denote the collection of subsets of a quasi-pseudometric space which are q -hyperconvex by $\mathcal{H}_q(X)$.

We first recall q -admissible subsets of a quasi-pseudometric space (X, d) .

Let (X, d) be a quasi-pseudometric space. For a nonempty subset A of X , we set

$$\begin{aligned} \text{bicov}(A)_+ &= \bigcap \{C_d(x, r) : A \subseteq C_d(x, r), x \in X, r \geq 0\} \\ \text{bicov}(A)_- &= \bigcap \{C_{d^{-1}}(x, s) : A \subseteq C_{d^{-1}}(x, s), x \in X, s \geq 0\} \end{aligned}$$

Furthermore, we define the *bicover* of A by

$$\text{bicov}(A) := \text{bicov}(A)_+ \cap \text{bicov}(A)_-$$

Definition 3.2.9. ([15]) Let (X, d) be a quasi-pseudometric space. The set of q -admissible subsets of X , denoted $\mathcal{A}_q(X)$, is the collection of all subsets of X which can be written as the intersection of a nonempty family of sets of the form $C_d(x, \epsilon_1) \cap C_{d^{-1}}(x, \epsilon_2)$ where $\epsilon_1, \epsilon_2 \geq 0$ and $x \in X$, i.e. $\mathcal{A}_q(X) = \{A \subset X : A = \text{bicov}(A)\}$.

Therefore, if (X, d) is a q -hyperconvex quasi-pseudometric space, then every member of $\mathcal{A}_q(X)$ is also q -hyperconvex by Proposition 3.2.8. Let (X, d) be a quasi-pseudometric space and let A be a nonempty bounded subset in (X, d) . Then in accordance with Definition 2.3.8, we define the *cover* $\text{cov}(A)$ of A as follows:

$$\text{cov}(A) = \bigcap \{C_{d^s}(x) : A \subseteq C_{d^s}(x), x \in X\}.$$

It is clear that $A \subseteq \text{bicov}(A) \subseteq \text{cov}(A)$. The later inclusion can be strict as the next example shows.

Example 3.2.5. ([15, Example 3.2]) Let $X = [0, 1] \times [1/4, 3/4]$ be equipped with the T_0 -quasi-metric defined by

$$d((\alpha, \beta), (\alpha', \beta')) = (\alpha - \alpha') \vee (\beta - \beta') \text{ whenever } (\alpha, \beta), (\alpha', \beta') \in X.$$

Consider $A = \{(0, 1/2), (1, 1/2)\} \subseteq X$. Then, $\text{bicov}(A)$ is equal to the line segment in X from $x = (0, 1/2)$ to $y = (1, 1/2)$. This follows from the fact that, for each $\epsilon \in [0, 1/4]$, we have $x \in [0, 1] \times [1/4, 1/2 + \epsilon] = C_{d^{-1}}(y, \epsilon)$ and $y \in [0, 1] \times [1/2 - \epsilon, 3/4] = C_d(x, \epsilon)$ and that the line segment is a subset of any set of the form $C_d(a, r) \cap C_{d^{-1}}(b, s)$ for which $\{x, y\} \subseteq C_d(a, r) \cap C_{d^{-1}}(b, s)$. Indeed, assume that a point z belongs to this line segment. Then $d(z, y) = 0 = d(x, z)$ and therefore, $z \in C_d(a, r) \cap C_{d^{-1}}(b, s)$ by the Triangle Inequality.

On the other hand, $\text{cov}(A) = X$, since $\{x, y\} \subseteq C_{d^s}(z, \epsilon)$ with $z \in X$ implies that $\epsilon \geq 1/2$. Indeed, assume that $z = (a, b) \in X$. Then, $a \leq d^s((a, b), (0, 1/2)) \leq \epsilon$ and $1 - a \leq d^s((a, b), (1, 1/2)) \leq \epsilon$. Thus, $\epsilon \geq \max\{a, 1 - a\} \geq 1/2$ with $a \in [0, 1]$. In the light that the interval $[1/4, 3/4]$ has length $1/2$, it follows that $X \subseteq C_{d^s}(z, \epsilon)$. Therefore, $\text{cov}(A) = X$.

Let us now look at externally q -hyperconvex subsets of a quasi-pseudometric space (X, d) . Note that the following definition strengthens the concept of a q -hyperconvex subset of (X, d) .

Definition 3.2.10. ([15]) Let (X, d) be a quasi-pseudometric space. A subspace E of (X, d) is said to be externally q -hyperconvex (relative to X) if given any family $(x_i)_{i \in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ the following condition holds:

if $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, $\text{dist}(x_i, E) \leq r_i$ and $\text{dist}(E, x_i) \leq s_i$ whenever $i \in I$, then

$$\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap E \neq \emptyset.$$

We will denote the collection of externally q -hyperconvex subsets of a quasi-pseudometric space (X, d) by $\mathcal{E}_q(X)$.

The next theorem shows the behaviour of a descending family of externally q -hyperconvex subsets of a bounded q -hyperconvex T_0 -quasi-metric space (X, d) .

Theorem 3.2.11. ([15, Theorem 6.5]) *Let (X, d) be a bounded q -hyperconvex T_0 -quasi-metric space. Moreover, let $(X_i)_{i \in I}$ be a descending family of nonempty externally q -hyperconvex subsets of X , where I is assumed to be totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ if and only if $X_{i_2} \subseteq X_{i_1}$. Then $\bigcap_{i \in I} X_i$ is nonempty and externally q -hyperconvex relative to X .*

Proof. [15, Theorem 6.5]. □

We now look at the relationship that exists among the sets $\mathcal{A}_q(X)$, $\mathcal{E}_q(X)$ and $\mathcal{H}_q(X)$. Before we do that, we need the concept of a proximal subset of a quasi-pseudometric space (X, d) that generalises the concept given in Definition 2.3.11.

Definition 3.2.12. (Compare Definition 2.3.11) *A subset E of a quasi-pseudometric space (X, d) is said to be proximal (with respect to X) if*

$$E \cap C_d(x, \text{dist}(x, E)) \cap C_{d^{-1}}(x, \text{dist}(E, x)) \neq \emptyset \text{ for each } x \in X.$$

The following result generalises Lemma 2.3.12 to q -hyperconvex T_0 -quasi-metric spaces. This result will also be extended to the setting of q -spherically complete T_0 -ultra-quasi-metric spaces in Lemma 4.2.11.

Lemma 3.2.13. (Compare Lemma 2.3.12) *If E is either a q -admissible subset or an externally q -hyperconvex subset of a q -hyperconvex T_0 -quasi-metric space (X, d) , then E is proximal in X .*

Proof. See [15, Example 6.2 and Lemma 6.3]. □

The following result generalises Theorem 2.3.13 to q -hyperconvex T_0 -quasi-metric spaces. It gives the relationship that exists among the sets $\mathcal{A}_q(X)$, $\mathcal{E}_q(X)$ and $\mathcal{H}_q(X)$.

Theorem 3.2.14. (Compare Theorem 2.3.13) *Let (X, d) be a q -hyperconvex T_0 -quasi-metric space. Then*

$$\mathcal{A}_q(X) \subseteq \mathcal{E}_q(X) \subseteq \mathcal{H}_q(X).$$

Proof. We first show that $\mathcal{A}_q(X) \subseteq \mathcal{E}_q(X)$. Let A be a q -admissible subset of X and let $(x_\alpha)_{\alpha \in S}$ be a family of points in X and $(r_\alpha)_{\alpha \in S}$ and $(s_\alpha)_{\alpha \in S}$ be families of nonnegative real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$, $\text{dist}(x_\alpha, A) \leq r_\alpha$ and $\text{dist}(A, x_\beta) \leq s_\beta$ whenever $\alpha, \beta \in S$. Since A is proximal, for any $\alpha \in S$, there exists a $p_\alpha \in A$ such that $d(x_\alpha, p_\alpha) = \text{dist}(x_\alpha, A)$ and $d(p_\alpha, x_\alpha) = \text{dist}(A, x_\alpha)$ which gives $C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha) \cap A \neq \emptyset$. Since X is q -hyperconvex, the conditions on both families imply $\bigcap_{\alpha \in S} C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha) \neq \emptyset$. Since A is q -admissible and $C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha) \cap A \neq \emptyset$, it follows that

$$\bigcap_{\alpha \in S} (C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap A \neq \emptyset,$$

which proves the first inclusion.

The fact that $\mathcal{E}_q(X) \subseteq \mathcal{H}_q(X)$ follows from the definition of external q -hyperconvexity. \square

We now turn our attention to the intersection property of q -hyperconvex spaces. Recall that Theorem 2.3.15 tells us that a descending chain of hyperconvex subsets of a bounded hyperconvex metric space is nonempty and hyperconvex. The next theorem generalises this result to q -hyperconvex subsets of a bounded T_0 -quasi-metric space.

Theorem 3.2.15. (*[15, Theorem 4.1]*) *Let (X, d) be a bounded T_0 -quasimetric space and let $(H_i)_{i \in I}$ be a descending family of nonempty q -hyperconvex subsets of X , where one assumes that I is totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ hold if and only if $H_{i_2} \subseteq H_{i_1}$. Then $\bigcap_{i \in I} H_i$ is nonempty and q -hyperconvex.*

Proof. [15, Theorem 4.1]. \square

In view of the relationship between q -hyperconvex quasi-pseudometric spaces and nonexpansive maps via injectivity, it is not surprising that q -hyperconvex T_0 -quasi-metric spaces admit interesting fixed point theory for nonexpansive maps. In Theorem 2.3.14, we saw that a nonexpansive map of a bounded hyperconvex metric space to itself has at least one fixed point. The next theorem generalises this result to bounded q -hyperconvex T_0 -quasi-metric spaces.

Theorem 3.2.16. (*[15, Theorem 3.3]*) *If (X, d) is a bounded q -hyperconvex T_0 -quasi-metric space and $F : (X, d) \rightarrow (X, d)$ is a nonexpansive map, then the set $\text{Fix}(F)$ of fixed points of F in (X, d) is nonempty and q -hyperconvex.*

Proof. [15, Theorem 3.3]. \square

Let us now look at the approximation of fixed points in quasi-pseudometric spaces. We begin by recalling the definition an ϵ_1, ϵ_2 -parallel set of a subset in a quasi-pseudometric space that generalises Definition 2.3.16 to q -hyperconvex spaces.

Definition 3.2.17. (*[15, Definition 5.1]*) *Let (X, d) be a quasi-pseudometric space. Given a subset A of X , we define for $\epsilon_1, \epsilon_2 \geq 0$ the ϵ_1, ϵ_2 -parallel set of A as*

$$N_{\epsilon_1, \epsilon_2}(A) = \bigcup_{a \in A} C_d(a, \epsilon_2) \cap C_{d^{-1}}(a, \epsilon_1).$$

(Note that for each $\epsilon \geq 0$, in particular, $N_{\epsilon, \epsilon}(A) = \bigcup_{a \in A} C_{d^*}(a, \epsilon)$)

Note that from the above definition, $x \in N_{\epsilon_1, \epsilon_2}(A)$ if and only if there exists $a \in A$ such that $d(a, x) \leq \epsilon_2$ and $d^{-1}(a, x) \leq \epsilon_1$.

We now recall the characterisation of $N_{\epsilon_1, \epsilon_2}(A)$ if A is a q -admissible set in a q -hyperconvex quasi-pseudometric space that generalises Lemma 2.3.17.

Lemma 3.2.18. ([15, Lemma 5.2]) Let (X, d) be a q -hyperconvex quasi-pseudometric space and let A be a q -admissible subset of X , say $\emptyset \neq A = \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $x_i \in X$ and $r_i, s_i \in [0, \infty)$ whenever $i \in I \neq \emptyset$. Then, for each $\epsilon_1, \epsilon_2 \geq 0$,

$$N_{\epsilon_1, \epsilon_2}(A) = \bigcap_{i \in I} (C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1)).$$

Proof. [15, Lemma 5.2]. □

The following result generalises Lemma 2.3.18 to q -hyperconvex T_0 -quasi-metric spaces and will be very useful in the discussion below of approximate fixed points.

Lemma 3.2.19. ([15, Lemma 5.3]) Suppose that (X, d) is a q -hyperconvex T_0 -quasi-metric space and let A be a q -admissible subset of X . Then, for each $\epsilon_1, \epsilon_2 \geq 0$ there is a nonexpansive retraction R of $N_{\epsilon_1, \epsilon_2}(A)$ onto A which has the property that $d(x, R(x)) \leq \epsilon_1$ and $d(R(x), x) \leq \epsilon_2$ whenever $x \in N_{\epsilon_1, \epsilon_2}(A)$.

Proof. [15, Lemma 5.3]. □

Definition 3.2.20. ([15, Definition 5.4]) Let (X, d) be a T_0 -quasi-metric space. We say that a map $f : (X, d) \rightarrow (X, d)$ has approximate fixed points if $\inf_{x \in X} d^s(x, f(x)) = 0$.

Definition 3.2.21. ([15, Definition 5.5]) Let (X, d) be a T_0 -quasi-metric space. For a map $f : (X, d) \rightarrow (X, d)$ and for any $\epsilon_1, \epsilon_2 \geq 0$, we use $F_{\epsilon_1, \epsilon_2}(f)$ to denote the set of ϵ_1, ϵ_2 -approximate fixed points of f , that is, $F_{\epsilon_1, \epsilon_2}(f) = \{x \in X : d(x, f(x)) \leq \epsilon_2 \text{ and } d(f(x), x) \leq \epsilon_1\}$.

We now recall and give an outline of the proof of a very useful result that generalises Theorem 2.3.21 to q -hyperconvex T_0 -quasi-metric spaces.

Theorem 3.2.22. ([15, Theorem 5.6]) Suppose that (X, d) is a q -hyperconvex T_0 -quasi-metric space and that the map $f : (X, d) \rightarrow (X, d)$ is nonexpansive. Furthermore, suppose that for some $\epsilon_1, \epsilon_2 \geq 0$ one has that $F_{\epsilon_1, \epsilon_2}(f)$ is nonempty. Then, the set $F_{\epsilon_1, \epsilon_2}(f)$ is q -hyperconvex.

Proof. For each $i \in I$, where I is an indexing set, let $x_i \in F_{\epsilon_1, \epsilon_2}(f)$, and let $r_i \geq 0$ and $s_i \geq 0$ satisfy

$$d(x_i, x_j) \leq r_i + s_j.$$

We need to show that

$$\left[\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \right] \cap F_{\epsilon_1, \epsilon_2}(f) \neq \emptyset.$$

Since (X, d) is q -hyperconvex, then $\emptyset \neq J = \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ is q -hyperconvex by Proposition 3.2.8. Furthermore, J is obviously bounded in (X, d) .

Also, if $x \in J$, then for each $i \in I$,

$$\begin{aligned} d(x_i, f(x)) &\leq d(x_i, f(x_i)) + d(f(x_i), f(x)) \leq \epsilon_2 + d(x_i, x) \leq \epsilon_2 + r_i, \\ d(f(x), x_i) &\leq d(f(x), f(x_i)) + d(f(x_i), x_i) \leq d(x, x_i) + \epsilon_1 \leq s_i + \epsilon_1. \end{aligned}$$

This proves that $f(x) \in N_{\epsilon_1, \epsilon_2}(J)$ by Lemma 3.2.18. Now, by Lemma 3.2.19, there is a nonexpansive retraction R of $N_{\epsilon_1, \epsilon_2}(J)$ onto J for which $d(R(x), x) \leq \epsilon_2$ and $d(x, R(x)) \leq \epsilon_1$ whenever $x \in N_{\epsilon_1, \epsilon_2}(J)$. Also, since $R \circ f$ is a nonexpansive map of J into J , it must have a fixed point by Theorem 3.2.16.

Suppose that $(R \circ f)(x_0) = x_0$ for some $x_0 \in J$. Then,

$$\begin{aligned} d(x_0, f(x_0)) &= d((R \circ f)(x_0), f(x_0)) \leq \epsilon_2, \\ d(f(x_0), x_0) &= d(f(x_0), (R \circ f)(x_0)) \leq \epsilon_1. \end{aligned}$$

Thus $x_0 \in J \cap F_{\epsilon_1, \epsilon_2}(f)$, and the proof is complete. \square

We now turn our attention to selection theorems of maps taking values in the space $\mathcal{E}_q(X)$ of externally q -hyperconvex subsets of a q -hyperconvex T_0 -quasi-metric space (X, d) . We begin by stating a result that generalises Theorem 2.3.22 to q -hyperconvex T_0 -quasi-metric spaces. This result will be extended to the setting of q -spherically complete T_0 -ultra-quasi-metric spaces in Theorem 4.2.13.

Theorem 3.2.23. ([15, Theorem 6.6]) *Let (H, d) be a q -hyperconvex T_0 -quasi-metric space, let X be any set, and let a map $F : X \rightarrow \mathcal{E}_q(H)$ be given. Then, there exists a map $f : X \rightarrow H$ for which $f(x) \in F(x)$ whenever $x \in X$ and for which $d(f(x), f(y)) \leq D(F(x), F(y))$ whenever $x, y \in X$.*

Proof. [15, Theorem 6.6]. \square

Just as we saw in Theorem 2.3.23, if the set-valued map in Theorem 3.2.23 is nonexpansive, then its selection is also nonexpansive. In the next result, we recall the generalisation of Theorem 2.3.23 to q -hyperconvex T_0 -quasi-metric spaces, this result will be extended to contractive maps in the setting of q -spherically complete T_0 -ultra-quasi-metric spaces (see Corollary 4.2.14).

Corollary 3.2.24. ([15, Corollary 6.7]) *Let (H, d) be a q -hyperconvex T_0 -quasi-metric space. Moreover, let (X, ρ) be a T_0 -quasi-metric space, and suppose $F : X \rightarrow \mathcal{E}_q(H)$ is nonexpansive, that is, $D(F(x), F(y)) \leq \rho(x, y)$ whenever $x, y \in X$. Then, there is a nonexpansive map $f : (X, \rho) \rightarrow (H, d)$ for which $f(x) \in F(x)$ whenever $x \in X$.*

Proof. Since F is nonexpansive, then the selection obtained in Theorem 3.2.23 is also nonexpansive. \square

In view of Theorem 3.2.16, a point valued nonexpansive map of a bounded q -hyperconvex T_0 -quasi-metric space to itself has at least one fixed point. This result is extended to set-valued maps taking values in the space $\mathcal{E}_q(H)$ of externally q -hyperconvex subsets of a bounded T_0 -quasi-metric space in the following result that generalises Corollary 2.3.24. We will then extend this result to contractive maps in the setting of q -spherically complete T_0 -ultra-quasi-metric spaces in Corollary 4.2.15.

Corollary 3.2.25. (*[15, Corollary 6.8]*) *Let H be a bounded q -hyperconvex T_0 -quasi-metric space and suppose that $F : H \rightarrow \mathcal{E}_q(H)$ is nonexpansive. Then F has a fixed point, that is, there exists $x \in H$ such that $x \in F(x)$.*

Proof. Corollary 3.2.24 implies that F has a nonexpansive selection $f : H \rightarrow H$. By Theorem 3.2.16, f has a fixed point. \square

The following result recalls the generalisation of Theorem 2.3.26 to q -hyperconvex T_0 -quasi-metric spaces. We set $Fix(F) = \{x \in H : x \in F(x)\}$. By Corollary 3.2.25, $Fix(F) \neq \emptyset$ if H is bounded and q -hyperconvex, and F is nonexpansive.

Theorem 3.2.26. (*[15, Theorem 6.9]*) *Let (H, d) be a q -hyperconvex T_0 -quasi-metric space, let $F : H \rightarrow \mathcal{E}_q(H)$ be a nonexpansive map and suppose that $Fix(F) \neq \emptyset$. Then, there exists a nonexpansive map $f : H \rightarrow H$ with $f(x) \in F(x)$ whenever $x \in H$ such that $Fix(f) = Fix(F)$.*

Proof. [15, Theorem 6.9]. \square

We now turn our attention to fixed point theorems for a family of set-valued contraction maps in q -hyperconvex spaces.

3.3. Fixed Point Theorems in q -Hyperconvex T_0 -Quasi-Metric Spaces

In this section, we generalise the results of Espínola and Kirk [9] presented in Section 2.4 to q -hyperconvex T_0 -quasi-metric spaces. We begin our discussion by proving a theorem that guarantees the existence of at least one fixed point for a family of set-valued contraction maps taking values in the collection $S_{cl} = \{A \subseteq X : A = cl_{\tau(d)}A \cap cl_{\tau(d^{-1})}A\}$ of subsets of a q -hyperconvex T_0 -quasi-metric space (X, d) . This result generalises Theorem 2.4.1 to q -hyperconvex T_0 -quasi-metric spaces.

Theorem 3.3.1. (*Compare Theorem 2.4.1*) *Let (X, d) be a q -hyperconvex T_0 -quasi-metric space and let $S_{cl} = \{A \subseteq X : A = cl_{\tau(d)}A \cap cl_{\tau(d^{-1})}A\}$ be endowed with the Hausdorff T_0 -quasi-metric D . Suppose $f_\alpha : (X, d) \rightarrow (S_{cl}, D)$ is a family of contraction maps indexed over a directed set Λ satisfying*

(i) *for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,*

(ii) *for each $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $diam(f_\alpha(x)) \leq \epsilon$ for each $x \in X$.*

Then there exists at least one point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

Proof. Define $f : (X, d) \rightarrow (X, d)$ by $\{f(x)\} = \bigcap_\alpha f_\alpha(x)$ for each $x \in X$. From the definition of the Hausdorff T_0 -quasi-metric on S_{cl} , we see that for any $x, y \in X$,

$$D(f_\alpha(x), f_\alpha(y)) = \max \left\{ \sup \left\{ \text{dist}(f_\alpha(x), f_\alpha(y)) \right\}_{f(y) \in f_\alpha(y)}, \sup \left\{ \text{dist}(f_\alpha(x), f_\alpha(y)) \right\}_{f(x) \in f_\alpha(x)} \right\}.$$

Note that

$$\text{dist}(f_\alpha(x), f_\alpha(y)) = \inf_{f(y) \in f_\alpha(y)} \{d(f(x), f(y)) : f(x) \in f_\alpha(x)\} \leq d(f(x), f(y)) \text{ for any } x, y \in X.$$

Taking the sup over all $x, y \in X$ we get

$$\sup_{f(y) \in f_\alpha(y)} \text{dist}(f_\alpha(x), f_\alpha(y)) = d(f(x), f(y)).$$

Similarly,

$$\sup_{f(x) \in f_\alpha(x)} \text{dist}(f_\alpha(x), f_\alpha(y)) = d(f(x), f(y)).$$

Therefore, we get

$$D(f_\alpha(x), f_\alpha(y)) = d(f(x), f(y)) \text{ whenever } x, y \in X,$$

and since f_α is a contraction map for each $\alpha \in \Lambda$, we have

$$\begin{aligned} d(f(x), f(y)) &= D(f_\alpha(x), f_\alpha(y)) \\ &\leq k_\alpha d(x, y) \\ &\leq d(x, y). \end{aligned}$$

Therefore, f is nonexpansive. Our goal is to show that f has a fixed point. Choose $\epsilon_1, \epsilon_2 > 0$ such that the set

$$F_{\epsilon_1, \epsilon_2}(f) = \{x \in X : d(x, f(x)) \leq \epsilon_2 \text{ and } d(f(x), x) \leq \epsilon_1\}$$

is nonempty. Then $F_{\epsilon_1, \epsilon_2}(f)$ is q -hyperconvex by Theorem 3.2.22. We now show that $F_{\epsilon_1, \epsilon_2}(f)$ is also bounded.

Let $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and choose $\alpha \in \Lambda$ so that $\text{diam}(f_\alpha(x)) \leq \epsilon$. If $u, v \in F_{\epsilon_1, \epsilon_2}(f)$, then

$$\begin{aligned} d(u, v) &\leq d(u, f(u)) + d(f(u), f(v)) + d(f(v), v) \\ &\leq \epsilon_2 + d(f(u), f(v)) + \epsilon_1 \\ &\leq d(f(u), f(v)) + \epsilon_1 + \epsilon_2 \\ &\leq d(f(u), f(v)) + 2\epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} d(u, v) &\leq d(f(u), f(v)) + 2\epsilon \\ &\leq D(f_\alpha(u), f_\alpha(v)) + \text{diam}(f_\alpha(u)) + \text{diam}(f_\alpha(v)) + 2\epsilon \\ &\leq k_\alpha d(u, v) + 4\epsilon. \end{aligned}$$

Therefore, $d(u, v) \leq (1 - k_\alpha)^{-1}4\epsilon$. This means that $F_{\epsilon_1, \epsilon_2}(f)$ is bounded. By Theorem 3.2.16, $\bar{f} : F_{\epsilon_1, \epsilon_2}(f) \rightarrow F_{\epsilon_1, \epsilon_2}(f)$ has a fixed point since $F_{\epsilon_1, \epsilon_2}(f)$ is bounded and q -hyperconvex. Therefore, f has a fixed point since \bar{f} is a restriction of f to $F_{\epsilon_1, \epsilon_2}(f)$. \square

The next theorem involves the family of externally q -hyperconvex subsets of a bounded q -hyperconvex T_0 -quasi-metric space (X, d) . Before we give an outline of the theorem, we prove a useful result that guarantees the existence of at least one fixed point for a set-valued contraction map taking values in the space $\mathcal{E}_q(X)$ of externally q -hyperconvex subsets of a q -hyperconvex T_0 -quasi-metric space (X, d) . This result generalises Proposition 2.4.2 to q -hyperconvex T_0 -quasi-metric spaces.

Proposition 3.3.2. (Compare Proposition 2.4.2) *Let (X, d) be a q -hyperconvex T_0 -quasi-metric space and let $T : (X, d) \rightarrow (\mathcal{E}_q(X), D)$ be a contraction map. Then T has at least one fixed point.*

Proof. Since T is a contraction map, then for any $k \in [0, 1)$ and $x, y \in X$,

$$D(T(x), T(y)) \leq k d(x, y) \leq d(x, y).$$

Therefore, T is nonexpansive. Thus by Corollary 3.2.24, there exists a nonexpansive map $f : (X, d) \rightarrow (X, d)$ such that $f(x) \in T(x)$ whenever $x \in X$.

Let $\epsilon_1, \epsilon_2 > 0$ so that the set

$$F_{\epsilon_1, \epsilon_2}(f) = \{x \in X : d(x, f(x)) \leq \epsilon_2 \text{ and } d(f(x), x) \leq \epsilon_1\}$$

is nonempty. Using the argument in Theorem 3.3.1, we see that $F_{\epsilon_1, \epsilon_2}(f)$ is bounded and by Theorem 3.2.22, this set is also q -hyperconvex. Thus by Theorem 3.2.16, $\bar{f} : F_{\epsilon_1, \epsilon_2}(f) \rightarrow F_{\epsilon_1, \epsilon_2}(f)$ has at least one fixed point say z . Since \bar{f} is a restriction of f to $F_{\epsilon_1, \epsilon_2}(f)$, then z is also a fixed point of f . It is clear that z is also a fixed point of T . \square

The next result shows that the fixed point set for a set-valued contraction map is bounded, it generalises Lemma 2.4.3 to q -hyperconvex T_0 -quasi-metric spaces.

Lemma 3.3.3. (Compare Lemma 2.4.3) *If (X, d) is a q -hyperconvex T_0 -quasi-metric space and $T : (X, d) \rightarrow (S_{cl}, D)$ is a contraction map, then the set of fixed points of T is bounded.*

Proof. Fix $x \in \text{Fix}(T)$ and let $\epsilon_1, \epsilon_2 > 0$. Since T is a contraction map, if $y \in \text{Fix}(T)$, then

$$D(T(x), T(y)) \leq k d(x, y) \text{ where } k \in (0, 1).$$

This implies that $\exists z \in T(x)$ such that

$$\begin{aligned} d(z, y) &\leq k d(x, y) + \epsilon_1 \\ &\leq k[d(x, z) + d(z, y)] + \epsilon_1. \end{aligned}$$

Hence,

$$\begin{aligned}(1 - k)d(z, y) &\leq k d(x, z) + \epsilon_1 \\ &\leq k \operatorname{diam}(T(x)) + \epsilon_1.\end{aligned}$$

Similarly, since T is a contraction map, if $y \in \operatorname{Fix}(T)$, then

$$D(T(y), T(x)) \leq k d(y, x) \text{ where } k \in (0, 1).$$

This implies that

$$\begin{aligned}d(y, z) &\leq k d(y, x) + \epsilon_2 \\ &\leq k[d(y, z) + d(z, x)] + \epsilon_2.\end{aligned}$$

Hence,

$$\begin{aligned}(1 - k)d(y, z) &\leq k d(z, x) + \epsilon_2 \\ &\leq k \operatorname{diam}(T(x)) + \epsilon_2.\end{aligned}$$

Choose $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Then

$$\begin{aligned}d(x, y) &\leq d(x, z) + d(z, y) \\ &\leq \operatorname{diam}(T(x)) + k \operatorname{diam}(T(x)) + \epsilon \\ &\leq (1 - k)^{-1}[(1 + k)\operatorname{diam}(T(x)) + \epsilon].\end{aligned}$$

This means that the distance between any points of $\operatorname{Fix}(T)$ is bounded. Therefore, $\operatorname{Fix}(T)$ is bounded. \square

Following Remark 2.4.1, we prove the existence of a q -hyperconvex fixed point set for a family of set-valued contraction maps taking values in the space $\mathcal{E}_q(X)$ of externally q -hyperconvex subsets of a bounded q -hyperconvex T_0 -quasi-metric space (X, d) . This result generalises Theorem 2.4.4 to q -hyperconvex T_0 -quasi-metric spaces.

Theorem 3.3.4. (Compare Theorem 2.4.4) *Let (X, d) be a bounded q -hyperconvex T_0 -quasi-metric space, let $\mathcal{E}_q(X)$ denote the collection of all nonempty externally q -hyperconvex subsets of X endowed with the Hausdorff quasi-pseudometric D . Let $f_\alpha : (X, d) \rightarrow (\mathcal{E}_q(X), D)$ be a family of contraction maps indexed over a directed set Λ such that for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$. Then the set of points $x \in X$ such that $x \in \bigcap_{\alpha \in \Lambda} f_\alpha(x)$ is nonempty and q -hyperconvex.*

Proof. By Proposition 3.3.2, each f_α has a nonempty fixed point set H_α . By Theorem 3.2.26, there is a nonexpansive map $\bar{f}_\alpha : X \rightarrow X$ with $\bar{f}_\alpha(x) \in f_\alpha(x)$ whenever $x \in X$ such that $\operatorname{Fix}(\bar{f}_\alpha) = \operatorname{Fix}(f_\alpha)$ for each $\alpha \in \Lambda$. Thus $\operatorname{Fix}(\bar{f}_\alpha) = H_\alpha$ for each $\alpha \in \Lambda$. Since fixed point sets for nonexpansive maps in bounded q -hyperconvex T_0 -quasi-metric spaces are q -hyperconvex by Theorem 3.2.16, then H_α

is a q -hyperconvex subset of X for each $\alpha \in \Lambda$. Also, Lemma 3.3.3 implies that H_α is bounded for each $\alpha \in \Lambda$. Thus $\{H_\alpha\}_{\alpha \in \Lambda}$ is a family of nonempty bounded q -hyperconvex subsets of X satisfying the following condition:

$$\beta \geq \alpha \implies H_\beta \subseteq H_\alpha.$$

Therefore, by Theorem 3.2.15, $H = \bigcap_{\alpha \in \Lambda} H_\alpha$ is a nonempty and q -hyperconvex.

Now define $f : (X, d) \rightarrow (\mathcal{P}_0(X), D)$ by

$$f(x) = \bigcap_{\alpha \in \Lambda} f_\alpha(x) \text{ whenever } x \in X.$$

Note that $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is a family of nonempty externally q -hyperconvex subsets of X satisfying the following condition:

$$\beta \geq \alpha \implies f_\beta(x) \subseteq f_\alpha(x).$$

Thus by Theorem 3.2.11, $f(x)$ is nonempty and externally q -hyperconvex relative to X . Therefore, f is a nonexpansive map of X into $\mathcal{E}_q(X)$. By Corollary 3.2.25, f has at least one fixed point and it is clear that

$$x \in \bigcap_{\alpha \in \Lambda} f_\alpha(x) \text{ if and only if } x \in \text{Fix}(f) = H.$$

□

Corollary 3.3.5. *Let (X, d) be a bounded q -hyperconvex T_0 -quasi-metric space, let $\mathcal{A}_q(X)$ denote the collection of all nonempty q -admissible subsets of X endowed with the Hausdorff quasi-pseudometric D . Let $f_\alpha : (X, d) \rightarrow (\mathcal{A}_q(X), D)$ be a family of contraction maps indexed over a directed set Λ such that for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$. Then the set of points $x \in X$ such that $x \in \bigcap_{\alpha \in \Lambda} f_\alpha(x)$ is nonempty and q -hyperconvex.*

Proof. It follows from the proof of Theorem 3.3.4 since any q -admissible subset of a q -hyperconvex T_0 -quasi-metric space (X, d) is externally q -hyperconvex relative to X (see Theorem 3.2.14). □

We now see what Corollary 3.3.5 says when applied to the real line.

Example 3.3.1. (Compare Example 2.4.1) Equip \mathbb{R} with the T_0 -quasi-metric $d(x, y) = x - y = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of mapping of $\mathbb{R} \rightarrow \mathbb{R}$ satisfying:

- (i) $f_n(x) \leq g_n(x)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$,
- (ii) $\{f_n\}$ is nondecreasing and $\{g_n\}$ is nonincreasing for each $x \in \mathbb{R}$,

(iii) for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that for each $x, y \in \mathbb{R}$,

$$\max\{f_n(x) \dot{-} f_n(y), g_n(x) \dot{-} g_n(y)\} \leq k_n(x \dot{-} y),$$

The conclusion under these assumptions is that there exists $a \leq b \in \mathbb{R}$ such that

$$\lim_n f_n(x) \leq x \leq \lim_n g_n(x) \text{ for each } x \in [a, b].$$

Proof. Let

$$F_n(x) = [f_n(x), g_n(x)] \text{ for each } x \in \mathbb{R} \text{ and } n \in \mathbb{N},$$

then $F_n(x)$ is q -admissible for each $n \in \mathbb{N}$. By condition (ii), we see that

$$F_{n+1}(x) \subseteq F_n(x).$$

Condition (iii) implies that F_n is a contraction map for each $n \in \mathbb{N}$. Then F_n satisfies all the conditions of Corollary 3.3.5 and therefore, the conclusion follows. \square

Example 3.3.2. (Compare Example 2.4.2) Let $I = [0, 1]$ and $\mathcal{C}(I)$ denote the closed subintervals of I . Consider the T_0 -quasi-metric $d(x, y) = x \dot{-} y = \max\{x - y, 0\}$ whenever $x, y \in I$. For each $\alpha \in (0, 1)$, define

$$F_\alpha(x) = [f_\alpha(x), g_\alpha(x)] \cap I,$$

where $f_\alpha(x) = \alpha(x + 1) - 1$ and $g_\alpha(x) = \alpha(x - 2) + 2$. Then $f_\alpha(x) \leq g_\alpha(x)$ for any $x \in I$. Moreover, $\beta \geq \alpha$ implies

$$f_\alpha(x) \leq f_\beta(x) \text{ and } g_\beta(x) \leq g_\alpha(x).$$

Thus for any $x \in I$,

$$F_\beta(x) \subseteq F_\alpha(x).$$

Also,

$$D(F_\alpha(x), F_\alpha(y)) = \max \left\{ \sup \left\{ \text{dist}(F_\alpha(x), t) \right\}_{t \in F_\alpha(y)}, \sup \left\{ \text{dist}(t, F_\alpha(y)) \right\}_{t \in F_\alpha(x)} \right\}.$$

Note that

$$\begin{aligned} \text{dist}_{F_\alpha(y)}(F_\alpha(x), t) &= \inf_{t \in F_\alpha(y)} \{d(p, t) : p \in F_\alpha(x)\} = \inf_{t \in F_\alpha(y)} \{\max\{p - t, 0\} : p \in F_\alpha(x)\} \\ &\leq \max\{g_\alpha(x) - t, 0\} \text{ whenever } x \in I \text{ and } t \in F_\alpha(y). \end{aligned}$$

Taking the sup over $t \in F_\alpha(y)$, we have

$$\begin{aligned} \max\{g_\alpha(x) \dot{-} t, 0\} &\leq \max\{g_\alpha(x) - g_\alpha(y), 0\} \\ &= \alpha(\max\{x - y, 0\}) \text{ whenever } x, y \in I. \end{aligned}$$

Therefore,

$$\sup \left\{ \text{dist}_{F_\alpha(y)}(F_\alpha(x), t) \right\} \leq \alpha(\max\{x - y, 0\}) = \alpha(x \dot{-} y).$$

using the same argument, we see that

$$\sup \left\{ \operatorname{dist} (r, F_\alpha(y)) \right\}_{r \in F_\alpha(x)} \leq \alpha(\max\{x - y, 0\}) = \alpha(x - y).$$

Hence, we conclude that

$$D(F_\alpha(x), F_\alpha(y)) \leq \alpha(\max\{x - y, 0\}) = \alpha(x - y).$$

Therefore, F_α is contracting for each α . Also,

$$\begin{aligned} \operatorname{diam}(F_\alpha(x)) &= \sup\{d(x, y) : x, y \in F_\alpha(x)\} \\ &= \sup\{\max\{x - y, 0\} : x, y \in F_\alpha(x)\} \\ &\leq \max\{g_\alpha(x) - f_\alpha(x), 0\} \\ &= \max\{\alpha(x - 2) + 2 - \alpha(x + 1) + 1, 0\} \\ &= \max\{3(1 - \alpha), 0\} = 3(1 - \alpha). \end{aligned}$$

Now define $F : I \rightarrow I$ by the map

$$F(x) = \bigcap_{\alpha \in (0,1)} F_\alpha(x) \text{ for each } x \in (0, 1).$$

Then $F(x) = x$ for each $x \in I$. This proves that the fixed point need not be unique in Theorem 3.3.1.

4

Set-Valued Contractive Maps and Ultra-Quasi-Pseudometric Spaces

Ultra-quasi-pseudometric spaces and quasi-pseudometric spaces share many common properties, yet they are quite different in very distinctive ways. The most striking similarity has to do with injective extension property; the most striking difference is likely the fact that while quasi-pseudometric spaces are metrically convex, ultra-quasi-pseudometric spaces are not. It is important to mention that the ultra-quasi-pseudometric spaces should not be confused with *quasi-ultra-metric spaces* as they are discussed in the theory of dissimilarities (see [6]). In this chapter, we present some interesting properties of q -spherically complete T_0 -ultra-quasi-metric spaces and some fixed point theorems for set-valued contractive maps. For more information on ultra-quasi-pseudometric spaces, we refer the reader to [19] and [16].

4.1. Ultra-Quasi-Pseudometric Spaces

We start this section by recalling the definition of an ultra-quasi-pseudometric space and some of its examples, thereafter, recall the concept of q -spherical completeness.

Definition 4.1.1. ([19, Definition 1.1]) *Let X be a set and u be a nonnegative real valued function on $X \times X$. Then u is an ultra-quasi-pseudometric on X if*

- (i) $u(x, x) = 0$ for all $x \in X$, and
- (ii) $u(x, z) \leq \max\{u(x, y), u(y, z)\}$ whenever $x, y, z \in X$.

The above inequality is called the strong Triangle Inequality.

If u satisfies the following additional condition:

- (iii) $u(x, y) = 0 = u(y, x)$ implies that $x = y$ for any $x, y \in X$, then u is called T_0 -ultra-quasi-metric and the pair (X, u) is called T_0 -ultra-quasi-metric space.

Remark 4.1.2. ([19]) We remark that

- (i) If u is an ultra-quasi-metric, then the so called conjugate u^{-1} of u , where $u^{-1}(x, y) = u(y, x)$ whenever $x, y \in X$, is also an ultra-quasi-metric.
- (ii) The set of open balls $\{y \in X : u(x, y) < \epsilon; x \in X, \epsilon > 0\}$ yields a base topology $\tau(u)$ induced by u on X .
- (iii) $u^s = \max\{u, u^{-1}\}$ is an ultra-metric on X .

We now look at some examples.

Example 4.1.1. ([25]) Let (X, T) be a topological space and $G \in T$. Then the function u defined by

$$u(x, y) = \begin{cases} 1 & \text{if } x \in G \text{ and } y \notin G \\ 0 & \text{otherwise} \end{cases}$$

is an ultra-quasi-pseudometric.

Example 4.1.2. ([19, Example 2.2]) Let $X = \mathbb{R}$. Define u by

$$u(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ x - y & \text{if } x > y. \end{cases}$$

Then the two topologies $\tau(u)$ and $\tau(u^{-1})$ are distinct.

Example 4.1.3. ([16, Example 1]) Let $X = [0, \infty)$ be equipped with $u(x, y) = x$ if $x, y \in X$ and $x > y$, and $u(x, y) = 0$ if $x, y \in X$ and $x \leq y$. It is easy to check that (X, u) is a T_0 -ultra-quasi-metric space. Also, for $x, y \in [0, \infty)$, the map u^s defined by $u^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $u^s(x, y) = 0$ if $x = y$ is an ultra metric on $[0, \infty)$.

Let us now turn our attention to the concept of q -spherical completeness.

Let (X, u) be an ultra-quasi-pseudometric space and for each $x \in X$ and $r \in [0, \infty)$, let $C_u(x, r) = \{y \in X : u(x, y) \leq r\}$ be the $\tau(u^{-1})$ -closed ball of radius r at x .

Lemma 4.1.3. ([16, Lemma 9]) Let (X, u) be an ultra-quasi-pseudometric space. Moreover, let $x, y \in X$ and $r, s \geq 0$. Then $C_u(x, r) \cap C_{u^{-1}}(y, s) \neq \emptyset$ if and only if $u(x, y) \leq \max\{r, s\}$.

Definition 4.1.4. ([16, Definition 2]) Let (X, u) be an ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative real numbers. We say that $(C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i))_{i \in I}$ has a mixed binary intersection property provided that $u(x_i, x_j) \leq \max\{r_i, s_j\}$ whenever $i, j \in I$. We say that (X, u) is q -spherically complete provided that each family $(C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i))_{i \in I}$ possessing a mixed binary intersection property satisfies

$$\bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset.$$

Remark 4.1.5. *Following the argument in Remark 3.2.2, we assume without loss of generality that the points x_i ($i \in I$) in Definition 4.1.4 are pairwise distinct.*

Example 4.1.4. ([16, Example 2]) The T_0 -ultra-quasi-metric space $([0, \infty), u)$, where u is defined as in Example 4.1.3, is q -spherically complete.

Proof. see [16, Example 2]. □

We also recall the following definition.

Definition 4.1.6. ([16]) *An ultra-metric space (X, m) is called spherically complete if for any family $(x_i)_{i \in I}$ of points in X and any family of positive real numbers $(r_i)_{i \in I}$ such that $m(x_i, x_j) \leq \max\{r_i, r_j\}$ whenever $i, j \in J$, we have that $\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset$.*

Remark 4.1.7. *Spherical completeness is an intersection property. In other words, if a metric space is spherically complete, then the family of admissible sets has some kind of compactness behaviour. Indeed, let (X, m) be a metric space and consider the family of admissible sets $\mathcal{A}(X)$. We say that $\mathcal{A}(X)$ is compact if for any family $\{A_\alpha\}$ of elements in $\mathcal{A}(X)$, we have $\bigcap_\alpha A_\alpha \neq \emptyset$ provided finite intersections are not empty. Spherical completeness obviously implies that $\mathcal{A}(X)$ is compact.*

Proposition 4.1.8. ([19, Proposition 2])

- (i) *Let (X, u) be an ultra-quasi-pseudometric space. Then (X, u) is q -spherically complete if and only if (X, u^{-1}) is q -spherically complete.*
- (ii) *Let (X, u) be a T_0 -ultra-quasi-metric space. If (X, u) is q -spherically complete, then (X, u^q) is spherically complete.*

Similar to q -admissible subsets of a quasi-pseudometric space, we define u -admissible subsets of an ultra-quasi-pseudometric space as follows:

Definition 4.1.9. (Compare Definition 3.2.9) *Let (X, u) be an ultra-quasi-pseudometric space. The set of u -admissible subsets of X , denoted $\mathcal{A}_u(X)$, is the collection of all subsets of X which can be written as the intersection of a nonempty family of sets of the form $C_u(x, \epsilon_1) \cap C_{u^{-1}}(x, \epsilon_2)$ where $\epsilon_1, \epsilon_2 \geq 0$ and $x \in X$, i.e. $\mathcal{A}_u(X) = \{A \subset X : A = \text{bicov}(A)\}$.*

Lemma 4.1.10. *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and let $\mathcal{A}_u(X)$ be a collection of u -admissible subsets of X . If $\{A_\alpha\}$ is a family of elements in $\mathcal{A}_u(X)$, then*

$$\bigcap_\alpha A_\alpha \neq \emptyset.$$

Proof. Since (X, u) is q -spherically complete, then Proposition 4.1.8 implies that (X, u^s) is spherically complete. Thus Remark 4.1.7 implies that $\mathcal{A}(X)$ is compact with respect to u^s . Therefore, any collection $\{A_\alpha\}$ of elements in $\mathcal{A}_u(X) \subseteq \mathcal{A}(X)$ satisfies

$$\bigcap_{\alpha} A_\alpha \neq \emptyset.$$

□

4.2. Properties of q -Spherically Complete T_0 -Ultra-Quasi-Metric Spaces

In this section, we present some interesting properties of q -spherically complete T_0 -ultra-quasi-metric spaces. We begin our discussion by recalling two important fixed point theorems that will be useful throughout this chapter. Before we do that, we recall the following definition.

Definition 4.2.1. ([23]) *Let (X, u) be a metric space and $T : X \rightarrow X$ a self map. The map T is said to be contractive if $u(T(x), T(y)) < u(x, y)$ for all $x, y \in X$ with $x \neq y$ and nonexpansive if $u(T(x), T(y)) \leq u(x, y)$ for all $x, y \in X$.*

Unlike q -hyperconvex T_0 -quasi-metric spaces, fixed points for contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces are unique. Consider the following theorem:

Theorem 4.2.2. ([19, Theorem 4.1]) *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space. If $T : (X, u) \rightarrow (X, u)$ is a contractive map, then T has a unique fixed point.*

Proof. Let $C_a = C_u(a, u(a, T(a))) \cap C_{u^{-1}}(a, u(T(a), a))$ such that $a \in X$ and let \mathcal{A} be the collection of C_a for all $a \in X$. Define a partial order on \mathcal{A} by

$$C_a \preceq C_b \Leftrightarrow C_b \subseteq C_a.$$

Consider the totally ordered subfamily \mathcal{A}_1 of \mathcal{A} . By q -spherical completeness of X , we have

$$\emptyset \neq \bigcap_{C_a \in \mathcal{A}_1} C_a = C.$$

Let $b \in C$ and $C_a \in \mathcal{A}_1$. Let $x \in C_b$. Then our goal is to show that $x \in C_a$.

By the strong Triangle Inequality, we have

$$u(a, x) \leq \max\{u(a, b), u(b, x)\}. \quad (4.1)$$

Since $b \in C$, we have

$$u(a, b) \leq u(a, T(a)) \text{ and } u(b, a) \leq u(T(a), a). \quad (4.2)$$

Then,

$$u(b, x) \leq u(b, T(b)) \leq \max\{u(b, a), u(a, T(a)), u(T(a), T(b))\},$$

and by the contractiveness of T , we have

$$u(b, x) \leq u(b, T(b)) < \max\{u(b, a), u(a, T(a)), u(a, b)\}.$$

Therefore, by inequality (4.2)

$$u(b, x) \leq u(b, T(b)) \leq \max\{u(T(a), a), u(a, T(a)), u(a, T(a))\}. \quad (4.3)$$

Furthermore, we have

$$u(T(a), a) \leq \max\{u(T(a), T(b)), u(T(b), T(a)), u(T(a), a)\},$$

which in view of (4.2) and the contractiveness of T , we have

$$u(T(a), a) < \max\{u(a, T(a)), u(T(b), b)\}. \quad (4.4)$$

Equations (4.3) and (4.4) give

$$u(b, x) < \max\{u(a, T(a)), u(T(b), b)\}. \quad (4.5)$$

Now by inequalities (4.1) and (4.5), we get

$$u(a, x) \leq \max\{u(a, b), u(b, x)\} \leq \max\{u(a, T(a)), u(T(b), b)\}.$$

We claim that $u(T(b), b) \leq u(a, T(a))$. Suppose to the contrary that $u(a, T(a)) < u(b, T(b))$. Now, using the strong Triangle Inequality and the fact that T is contractive, we obtain

$$\begin{aligned} u(a, b) &\leq \max\{u(a, T(a)), u(T(a), T(b)), u(T(b), b)\} \\ &< \max\{u(a, T(a)), u(a, b), u(T(b), b)\} \leq u(T(b), b). \end{aligned}$$

Now combining the above inequality and $u(a, T(a)) < u(T(b), b)$, we get $u(a, T(a)) < u(a, b)$, that is, $b \notin C_u(a, u(a, T(a)))$. This contradicts that $b \in C_u(a, u(a, T(a))) \cap C_{u^{-1}}(a, u(T(a), a))$. Therefore, $x \in C_u(a, u(a, T(a)))$.

Using the similar argument, one can show that $u(x, a) \leq u(T(a), a)$, and then $x \in C_{u^{-1}}(a, u(T(a), a))$. We have that $x \in C_u(a, u(a, T(a))) \cap C_{u^{-1}}(a, u(T(a), a)) = C_a$ and $C_b \subseteq C_a$ whenever $C_a \in \mathcal{A}$. Therefore, C_b is an upper bound in \mathcal{A} for the family \mathcal{A}_1 . By Zorn's lemma, \mathcal{A} has a maximal element, say C_z , for some $z \in X$.

We claim that $T(z) = z$. Suppose to the contrary that $z \neq T(z)$. Since

$$u(T(z), T^2(z)) < u(z, T(z)) \quad \text{and} \quad u(T^2(z), T(z)) < u(T(z), z) \quad (4.6)$$

and

$$T(z) \in C_u(T(z), u(T(z), T^2(z))) \cap C_{u^{-1}}(T(z), u(T^2(z), T(z))),$$

we have

$$T(z) \in C_u(z, u(z, T(z))) \cap C_{u^{-1}}(z, u(T(z), z)).$$

Hence $T(z) \in C_{T(z)} \cap C_z$.

We have that $C_{T(z)} \subseteq C_z$. However, by inequality (4.6), $z \notin C_{T(z)}$, so $C_{T(z)} \subset C_z$, and this contradicts the maximality of C_z . Therefore, T has a fixed point, obviously unique. \square

Unlike q -hyperconvex T_0 -quasi-metric spaces, the existence of fixed points for nonexpansive maps in q -spherically complete T_0 -ultra-quasi-metric spaces is not guaranteed. However, such maps leave invariant a specific ball, say C . Consider the following theorem.

Theorem 4.2.3. ([19, Theorem 4.2]) *Suppose (X, u) is a q -spherically complete T_0 -ultra-quasi-metric space and suppose $T : (X, u) \rightarrow (X, u)$ is nonexpansive. Then either T has a fixed point or there exists a ball of the form $C = C_u(x, r) \cap C_{u^{-1}}(x, s)$ where $r, s \in [0, \infty)$ such that*

$$(i) \quad T : C \rightarrow C,$$

$$(ii) \quad u(z, T(z)) = r \text{ and } u(T(z), z) = s \text{ for any } z \in C.$$

Proof. By defining C_a and \mathcal{A} as in the proof of Theorem 4.2.2, we can find a maximal element C_z of \mathcal{A} . For any $b \in C_z$, we have

$$u(b, T(b)) \leq \max\{u(b, z), u(z, T(z)), u(T(z), T(b))\}.$$

Since $u(z, b) \leq u(z, T(z))$ and T is nonexpansive, we have

$$u(b, T(b)) \leq \max\{u(T(z), z), u(z, T(z))\}. \quad (4.7)$$

Thus, $u(T(z), z) \leq \max\{u(T(z), T(b)), u(T(b), b), u(b, z)\}$ by the strong Triangle Inequality. Again, since $u(b, z) \leq u(T(z), z)$, $u(z, b) \leq u(z, T(z))$ and T is nonexpansive, we have

$$u(T(z), z) \leq \max\{u(z, T(z)), u(T(b), b)\}. \quad (4.8)$$

From inequalities (4.7) and (4.8), we have

$$u(b, T(b)) \leq \max\{u(z, T(z)), u(T(b), b)\}. \quad (4.9)$$

Similarly, we have

$$u(T(b), b) \leq \max\{u(T(z), z), u(b, T(b))\}. \quad (4.10)$$

We claim that $u(T(b), b) \leq u(z, T(z))$ and $u(b, T(b)) \leq u(T(z), z)$. Suppose to the contrary that $u(T(b), b) > u(z, T(z))$. Then,

$$u(z, b) \leq \max\{u(z, T(z)), u(T(z), T(b)), u(T(b), b)\}.$$

By the nonexpansiveness of T , we have

$$u(z, b) \leq \max\{u(z, T(z)), u(T(b), b)\} \leq u(T(b), b) > u(z, T(z)).$$

Hence, we conclude that $b \notin C_u(z, T(z))$, which is a contradiction. By similar arguments, one can show that $u(b, T(b)) \leq u(T(z), z)$.

Thus, from inequality (4.9), $u(b, T(b)) \leq u(z, T(z))$. Since b is a common point of $C_u(b, u(b, T(b)))$ and $C_u(z, u(z, T(z)))$ and $u(z, T(b)) \leq \max\{u(z, T(z), u(T(z), T(b)))\} \leq u(z, T(z))$ which implies that $T(b) \in C_u(z, u(z, T(z)))$, we have

$$C_u(b, u(b, T(b))) \subseteq C_u(z, u(z, T(z))).$$

By similar arguments, one can show that

$$C_{u^{-1}}(b, u(T(b), b)) \subseteq C_{u^{-1}}(z, u(T(z), z)).$$

Hence

$$C_u(b, u(b, T(b))) \cap C_{u^{-1}}(b, u(T(b), b)) \subseteq C_u(z, u(z, T(z))) \cap C_{u^{-1}}(z, u(T(z), z)).$$

Therefore, $C_b \subseteq C_z$.

If $u(b, T(b)) < u(z, T(z))$ and $u(T(b), b) < u(T(z), z)$, then

$$u(b, z) \leq u(T(z), z) > u(T(b), b),$$

which implies that $z \in C_u(z, u(z, T(z)))$ but $z \notin C_u(b, u(b, T(b)))$. Similarly, $z \in C_{u^{-1}}(z, u(T(z), z))$ but $z \notin C_{u^{-1}}(b, u(T(b), b))$.

Moreover, $z \in C_z$ but $z \notin C_b$, which is not possible by the maximality of C_z , thus we have $u(b, T(b)) = u(z, T(z)) = r$ and $u(T(b), b) = u(T(z), z) = s$ whenever $b \in C_z$. \square

Using the construction of Künzi and Otafudu [15], we recall and present some interesting properties of q -spherically complete T_0 -ultra-quasi-metric spaces. We begin our discussion by presenting a result that extends Proposition 3.2.8 to q -spherically complete T_0 -ultra-quasi-metric spaces.

Proposition 4.2.4. (Compare [15, Proposition 3.1]) *Let (X, u) be a q -spherically complete ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a nonempty family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be two families of nonnegative real numbers such that $u(x_i, x_j) \leq \max\{r_i, s_i\}$. Set $D = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$. Then D is nonempty and q -spherically complete.*

Proof. Since X is q -spherically complete, then $D \neq \emptyset$. For each $\alpha \in S$, let $x_\alpha \in D$ and $(r_\alpha)_{\alpha \in S}$ and $(s_\alpha)_{\alpha \in S}$ be families of nonnegative real numbers such that $u(x_\alpha, x_\beta) \leq \max\{r_\alpha, s_\beta\}$ whenever $\alpha, \beta \in S$. We show that the family

$$[(C_u(x_\alpha, r_\alpha))_{\alpha \in S}; (C_u(x_i, r_i))_{i \in I}; (C_{u^{-1}}(x_\alpha, s_\alpha))_{\alpha \in S}; (C_{u^{-1}}(x_i, s_i))_{i \in I}]$$

of double balls satisfies the hypothesis of q -spherical completeness. We see that for each $\alpha \in S$ and $i \in I$, we have $u(x_\alpha, x_i) \leq s_i \leq \max\{r_\alpha, s_i\}$ and $u(x_i, x_\alpha) \leq r_i \leq \max\{r_i, s_\alpha\}$. Hence by q -spherical completeness of X , we have

$$\emptyset \neq \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \cap \bigcap_{\alpha \in S} (C_u(x_\alpha, r_\alpha) \cap C_{u^{-1}}(x_\alpha, s_\alpha)).$$

Therefore,

$$\emptyset \neq D \cap \bigcap_{\alpha \in S} (C_u(x_\alpha, r_\alpha) \cap C_{u^{-1}}(x_\alpha, s_\alpha)).$$

Hence, the subspace D of X is q -spherically complete. \square

Note that Proposition 4.2.4 implies that each member of $\mathcal{A}_u(X)$ is q -spherically complete.

Let us now recall the concept of the approximation of fixed points in q -spherically complete ultra-quasi-pseudometric spaces. We begin recalling the definition of an ϵ_1, ϵ_2 -parallel set of a subset in an ultra-quasi-pseudometric space which extends Definition 3.2.17 to ultra-quasi-pseudometric spaces.

Definition 4.2.5. ([19, Definition 3.6]) Let (X, u) be an ultra-quasi-pseudometric space. Given a subset A of X , we define for $\epsilon_1, \epsilon_2 \geq 0$ the ϵ_1, ϵ_2 -parallel set of A as

$$N_{\epsilon_1, \epsilon_2}(A) = \bigcup_{a \in A} C_u(a, \epsilon_2) \cap C_{u^{-1}}(a, \epsilon_1).$$

(Note that for each $\epsilon \geq 0$, in particular, $N_{\epsilon, \epsilon}(A) = \bigcup_{a \in A} C_{u^*}(a, \epsilon)$)

From the above definition, $x \in N_{\epsilon_1, \epsilon_2}(A)$ if and only if there exists $a \in A$ such that $u(a, x) \leq \epsilon_2$ and $u^{-1}(a, x) \leq \epsilon_1$.

We now recall the characterisation of $N_{\epsilon_1, \epsilon_2}(A)$ if A is a u -admissible set in a q -spherically complete ultra-quasi-pseudometric space which extends Lemma 3.2.18 to q -spherically complete T_0 -ultra-quasi-metric spaces.

Lemma 4.2.6. ([19, Lemma 3.7]) Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and let A be a u -admissible subset of X , say $\emptyset \neq A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$ with $x_i \in X$ and r_i, s_i nonnegative real numbers whenever $i \in I \neq \emptyset$. Then, for each $\epsilon_1, \epsilon_2 \geq 0$,

$$N_{\epsilon_1, \epsilon_2}(A) = \bigcap_{i \in I} (C_u(x_i, \max\{r_i, \epsilon_2\}) \cap C_{u^{-1}}(x_i, \max\{s_i, \epsilon_1\})).$$

Proof. Suppose $y \in N_{\epsilon_1, \epsilon_2}(A)$. Then $u(a, y) \leq \epsilon_2$ and $u(y, a) \leq \epsilon_1$ for some $a \in A$. But for each $i \in I$,

$$u(x_i, y) \leq \max\{u(x_i, a), u(a, y)\} \leq \max\{r_i, \epsilon_2\}$$

and

$$u(y, x_i) \leq \max\{u(y, a), u(a, x_i)\} \leq \max\{\epsilon_1, s_i\}.$$

Then for each $i \in I$, we have $y \in C_u(x_i, \max\{r_i, \epsilon_2\})$ and $y \in C_{u^{-1}}(x_i, \max\{s_i, \epsilon_1\})$ which implies that

$$N_{\epsilon_1, \epsilon_2}(A) \subseteq \bigcap_{i \in I} (C_u(x_i, \max\{r_i, \epsilon_2\}) \cap C_{u^{-1}}(x_i, \max\{s_i, \epsilon_1\})).$$

Now, let us consider $y \in \bigcap_{i \in I} (C_u(x_i, \max\{r_i, \epsilon_2\}) \cap C_{u^{-1}}(x_i, \max\{s_i, \epsilon_1\}))$ and let $i \in I$. We have that

$$u(x_i, y) \leq \max\{r_i, \epsilon_2\}$$

and

$$u(y, x_i) \leq \max\{\epsilon_1, s_i\}.$$

Since A is nonempty and by the definition of A , we must have for any $i, j \in I$,

$$u(x_i, x_j) \leq \max\{u(x_i, a), u(a, x_i)\} \leq \max\{r_i, s_j\}.$$

Therefore, by the q -spherical completeness of X ,

$$\begin{aligned} \emptyset &\neq \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_u(y, \epsilon_2)) \cap \bigcap_{i \in I} (C_{u^{-1}}(x_i, s_i) \cap C_{u^{-1}}(y, \epsilon_1)) \\ &= \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \cap C_u(y, \epsilon_1) \cap C_{u^{-1}}(y, \epsilon_2) \\ &= A \cap C_u(y, \epsilon_1) \cap C_{u^{-1}}(y, \epsilon_2). \end{aligned}$$

Hence, there is $a \in A$ such that $u(y, a) \leq \epsilon_1$ and $u(a, y) \leq \epsilon_2$. Therefore, $y \in N_{\epsilon_1, \epsilon_2}(A)$ and this completes our proof. \square

We now recall a result that extends Lemma 3.2.19 to the setting of q -spherically complete T_0 -ultra-quasi-metric spaces that will be useful in the discussion below of approximate fixed point sets.

Lemma 4.2.7. (*[19, Lemma 3.8]*) *Suppose that (X, u) is a q -spherically complete T_0 -ultra-quasi-metric space and let A be a u -admissible subset of X . Then, for each $\epsilon_1, \epsilon_2 \geq 0$ there is a nonexpansive retraction R of $N_{\epsilon_1, \epsilon_2}(A)$ onto A which has the property that $u(x, R(x)) \leq \epsilon_1$ and $u(R(x), x) \leq \epsilon_2$ whenever $x \in N_{\epsilon_1, \epsilon_2}(A)$.*

Proof. Assume that $\emptyset \neq A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$ with $I \neq \emptyset$. By Lemma 4.2.6, we know that $N_{\epsilon_1, \epsilon_2}(A)$ is q -spherically complete. Consider the family $\mathcal{F} = \{(D, R_D) : A \subseteq D \subseteq N_{\epsilon_1, \epsilon_2}(A) \text{ and } R_D : D \rightarrow A \text{ is a nonexpansive retraction such that } u(x, R(x)) \leq \epsilon_1 \text{ and } u(R(x), x) \leq \epsilon_2 \text{ for each } x \in D\}$. Note that $(A, I_A) \in \mathcal{F}$, where I_A is the identity map on A . So $\mathcal{F} \neq \emptyset$. If one orders \mathcal{F} in the usual way $((D, R_D) \preceq (H, R_H)$ if and only if $D \subseteq H$ and R_H is an extension of R_D), then each chain in (\mathcal{F}, \preceq) is bounded above, so by Zorn's Lemma, \mathcal{F} has a maximal element which we denote by (D, R_D) .

We need to show that $D = N_{\epsilon_1, \epsilon_2}(A)$. Suppose there exists $x \in N_{\epsilon_1, \epsilon_2}(A)$ such that $x \notin D$, and consider the set

$$C = \left[\bigcap_{w \in D} (C_u(R_D(w), u(w, x)) \cap C_{u^{-1}}(R_D(w), u(x, w))) \right] \\ \cap \left[\bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \cap (C_u(x, \epsilon_1) \cap C_{u^{-1}}(x, \epsilon_2)) \right].$$

We first show that $C \neq \emptyset$, and in order to do this, we need to show that C has the mixed binary intersection property. If $w_1, w_2 \in D$, then

$$u(R_D(w_1), R_D(w_2)) \leq u(w_1, w_2) \leq \max\{u(w_1, x), u(x, w_2)\}.$$

This proves that

$$C_u(R_D(w_1), u(w_1, x)) \cap C_{u^{-1}}(R_D(w_2), u(x, w_2)) \neq \emptyset$$

by Lemma 4.1.3, and so C has a mixed binary intersection property for the first family. Also, for each $w \in D$, $R_D(w) \in A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$. So the mixed binary intersection property is satisfied for the second family.

Since

$$x \in N_{\epsilon_1, \epsilon_2}(A) = \bigcap_{i \in I} (C_u(x_i, \max\{r_i, \epsilon_2\}) \cap C_{u^{-1}}(x_i, \max\{s_i, \epsilon_1\})),$$

we know that $(C_u(x, \epsilon_1) \cap C_{u^{-1}}(x, \epsilon_2)) \cap (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset$ for each $i \in I$.

Finally, if $w \in D$, then

$$u(R_D(w), x) \leq \max\{u(R_D(w), w), u(w, x)\} \leq \max\{\epsilon_2, u(w, x)\}$$

and

$$u(x, R_D(w)) \leq \max\{u(x, w), u(w, R_D(w))\} \leq \max\{u(x, w), \epsilon_1\}.$$

Thus, by Lemma 4.1.3, we have

$$C_u(R_D(w), u(w, x)) \cap C_{u^{-1}}(x, \epsilon_2) \neq \emptyset$$

as well as

$$C_{u^{-1}}(R_D(w), u(x, w)) \cap C_u(x, \epsilon_1) \neq \emptyset.$$

Of course, $C_u(x, \epsilon_1)$ and $C_{u^{-1}}(x, \epsilon_2)$ intersect.

Thus we have shown that the family

$$[(C_u(R_D(w)), u(w, x))_{w \in D}, (C_u(x_i, r_i))_{i \in I}, C_u(x, \epsilon_1), \\ (C_{u^{-1}}(R_D(w)), u(x, w))_{w \in D}, (C_{u^{-1}}(x_i, s_i))_{i \in I}, C_{u^{-1}}(x, \epsilon_2)]$$

of double balls has a mixed binary intersection property. Therefore, we conclude that $\emptyset \neq C \subseteq A$. Now let $p \in C$ and define $R' : D \cup \{x\} \rightarrow A$ by setting $R'(w) = R_D(w)$ if $w \in D$ and $R'(x) = p$. Then for $w \in D$,

$$u(R'(x), R'(w)) = u(p, R(w)) \leq u(x, w)$$

and

$$u(R'(w), R'(x)) = u(R(w), p) \leq u(w, x).$$

So R' is nonexpansive. Also, $u(R'(x), x) = u(p, x) \leq \epsilon_2$ and $u(x, R'(x)) = u(x, p) \leq \epsilon_1$. Hence, we conclude that the pair $(D \cup \{x\}, R')$ contradicts the maximality of (D, R_D) in (\mathcal{F}, \preceq) . Therefore, $D = N_{\epsilon_1, \epsilon_2}(A)$ and the proof is complete. \square

Definition 4.2.8. (Compare Definition 3.2.20) Let (X, u) be a T_0 -ultra-quasi-metric space. We say that a map $f : (X, u) \rightarrow (X, u)$ has approximate fixed points if $\inf_{x \in X} u^s(x, f(x)) = 0$.

Definition 4.2.9. (Compare Definition 3.2.21) Let (X, u) be a T_0 -ultra-quasi-metric space. For a map $f : (X, u) \rightarrow (X, u)$ and for any $\epsilon_1, \epsilon_2 \geq 0$, we use $F_{\epsilon_1, \epsilon_2}(f)$ to denote the set of ϵ_1, ϵ_2 -approximate fixed points of f , that is,

$$F_{\epsilon_1, \epsilon_2}(f) = \{x \in X : u(x, f(x)) \leq \epsilon_2 \text{ and } u(f(x), x) \leq \epsilon_1\}.$$

Recall that Theorem 3.2.22 tells us that the set $F_{\epsilon_1, \epsilon_2}(f)$ is q -hyperconvex. This concept is extended to q -spherically complete T_0 -ultra-quasi-metric spaces in the following theorem.

Theorem 4.2.10. ([19, Theorem 4.3]) Suppose that (X, u) is a q -spherically complete T_0 -ultra-quasi-metric space and that the map $f : (X, u) \rightarrow (X, u)$ is nonexpansive. Furthermore, suppose that for some $\epsilon_1, \epsilon_2 \geq 0$, one has $F_{\epsilon_1, \epsilon_2}(f)$ is nonempty. Then, the set $F_{\epsilon_1, \epsilon_2}(f)$ is q -spherically complete.

Proof. For each $i \in I$, where I is an indexing set, let $x_i \in F_{\epsilon_1, \epsilon_2}(f)$ and let $r_i \geq 0$ and $s_i \geq 0$ satisfy

$$u(x_i, x_j) \leq \max\{r_i, s_j\}.$$

We need to show that

$$\left[\bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \right] \cap F_{\epsilon_1, \epsilon_2}(f) \neq \emptyset.$$

Since (X, u) is q -spherically complete, then $\emptyset \neq J = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$ is q -spherically complete by Proposition 4.2.4. Furthermore, J is obviously bounded in (X, u) .

Also, if $x \in J$, then for each $i \in I$,

$$\begin{aligned} u(x_i, f(x)) &\leq \max\{u(x_i, f(x_i)), u(f(x_i), f(x))\} \leq \max\{\epsilon_2, u(x_i, x)\} \leq \max\{\epsilon_2, r_i\}, \\ u(f(x), x_i) &\leq \max\{u(f(x), f(x_i)), u(f(x_i), x_i)\} \leq \max\{u(x, x_i), \epsilon_1\} \leq \max\{s_i, \epsilon_1\}. \end{aligned}$$

This proves that $f(x) \in N_{\epsilon_1, \epsilon_2}(J)$ by Lemma 4.2.6. Now, by Lemma 4.2.7, there is a nonexpansive retraction R of $N_{\epsilon_1, \epsilon_2}(J)$ onto J for which $u(R(x), x) \leq \epsilon_2$ and $u(x, R(x)) \leq \epsilon_1$ whenever $x \in N_{\epsilon_1, \epsilon_2}(J)$. Also, since $R \circ f$ is a nonexpansive map of J into J , it must have a fixed point by Theorem 4.2.3.

Suppose that $(R \circ f)(x_0) = x_0$ for some $x_0 \in J$. Then,

$$\begin{aligned} u(x_0, f(x_0)) &= u((R \circ f)(x_0), f(x_0)) \leq \epsilon_2, \\ u(f(x_0), x_0) &= u(f(x_0), (R \circ f)(x_0)) \leq \epsilon_1. \end{aligned}$$

Thus $x_0 \in J \cap F_{\epsilon_1, \epsilon_2}(f)$, and the proof is complete. \square

We now show that a u -admissible subset of a q -spherically complete T_0 -ultra-quasi-metric space (X, u) is proximal in X (See Definition 3.2.12).

Lemma 4.2.11. *(Compare [15, Lemma 6.3]) Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and let $x \in X$. Furthermore, let $\emptyset \neq A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap (C_{u^{-1}}(x_i, s_i)))$ where $(x_i)_{i \in I}$ is a nonempty family of points in X and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ are families of nonnegative real numbers. Then there is $p \in A$ such that $\text{dist}(x, A) = u(x, p)$ and $\text{dist}(A, x) = u(p, x)$.*

Proof. Evidently, the family of double balls

$$[(C_u(x_i, r_i))_{i \in I}; (C_u(x, \text{dist}(x, A) + \epsilon))_{\epsilon > 0}; (C_{u^{-1}}(x_i, s_i))_{i \in I}; (C_{u^{-1}}(x, \text{dist}(A, x) + \epsilon))_{\epsilon > 0}]$$

has a mixed binary intersection property. Thus, there is

$$p \in A \cap C_u(x, \text{dist}(x, A)) \cap C_{u^{-1}}(x, \text{dist}(A, x))$$

since X is q -spherically complete. And it is clear that p satisfies the stated conditions. \square

The following result is useful in the discussion of selection theorems in q -spherically complete T_0 -ultra-quasi-metric spaces.

Lemma 4.2.12. *(Compare [15, Lemma 6.4]) Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and suppose A is u -admissible, that is, $A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$ where $(x_i)_{i \in I}$ is a nonempty family of points in X and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ are families of nonnegative real numbers. Let $(x_\alpha)_{\alpha \in S}$ be a nonempty family of points in X and $(r_\alpha)_{\alpha \in S}$ and $(s_\alpha)_{\alpha \in S}$ be two families of nonnegative real numbers such that $u(x_\alpha, x_\beta) \leq \max\{r_\alpha, s_\beta\}$ whenever $\alpha, \beta \in S$, $\text{dist}(x_\alpha, A) \leq r_\alpha$ and $\text{dist}(A, x_\alpha) \leq s_\alpha$ whenever $\alpha \in S$. Then*

$$\left[\bigcap_{\alpha \in S} (C_u(x_\alpha, r_\alpha) \cap C_{u^{-1}}(x_\alpha, s_\alpha)) \right] \cap A \neq \emptyset.$$

Proof. Let $(s_\alpha)_{\alpha \in S}$ be a family of points in X and let $(r_\alpha)_{\alpha \in S}$ and $(s_\alpha)_{\alpha \in S}$ be two families of nonnegative real numbers such that $u(x_\alpha, x_\beta) \leq \max\{r_\alpha, s_\beta\}$, $\text{dist}(x_\alpha, A) \leq r_\alpha$ and $\text{dist}(A, x_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$.

Since $A = \bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i))$, $\text{dist}(x_\alpha, A) \leq r_\alpha$ and $\text{dist}(A, x_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$, choose $p \in A$ according to Lemma 4.2.11, then we have

$$\begin{aligned} u(x_\alpha, x_i) &\leq \max\{u(x_\alpha, p), u(p, x_i)\} \leq \max\{r_\alpha, s_i\}, \\ u(x_i, x_\alpha) &\leq \max\{u(x_i, p), u(p, x_\alpha)\} \leq \max\{r_i, s_\alpha\} \end{aligned}$$

and it is clear that $u(x_i, x_j) \leq \max\{r_i, s_j\}$. Therefore, the family of double balls

$$[(C_u(x_\alpha, r_\alpha))_{\alpha \in S}; (C_u(x_i, r_i))_{i \in I}; (C_{u^{-1}}(x_\alpha, s_\alpha))_{\alpha \in S}; (C_{u^{-1}}(x_i, s_i))_{i \in I}]$$

satisfies the mixed binary intersection property. Thus, by the q -spherical completeness of X , we have

$$\begin{aligned} \emptyset &\neq \left[\bigcap_{\alpha \in S} (C_u(x_\alpha, r_\alpha) \cap C_{u^{-1}}(x_\alpha, s_\alpha)) \right] \cap \left[\bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \right] \\ &= \left[\bigcap_{\alpha \in S} (C_u(x_\alpha, r_\alpha) \cap C_{u^{-1}}(x_\alpha, s_\alpha)) \right] \cap A. \end{aligned}$$

□

We now look at selection theorems in q -spherically complete T_0 -ultra-quasi-metric spaces. We begin our discussion by extending Theorem 4.2.13 to q -spherically complete T_0 -ultra-quasi-metric spaces.

Theorem 4.2.13. (Compare [15, Theorem 6.6]) *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and let H be any set. Let $T^* : H \rightarrow \mathcal{A}_u(X)$ be given. Then, there exists a map $T : H \rightarrow X$ for which $T(x) \in T^*(x)$ whenever $x \in H$ such that $u(T(x), T(y)) \leq U(T^*(x), T^*(y))$ whenever $x, y \in H$.*

Proof. Let \mathcal{F} denote the set $\{(D, T) : D \subseteq H, T : D \rightarrow X, T(d) \in T^*(d) \text{ whenever } d \in D \text{ and } u(T(x), T(y)) \leq U(T^*(x), T^*(y)) \text{ whenever } x, y \in D\}$. Note that $\mathcal{F} \neq \emptyset$ since $(\{x_0\}, T) \in \mathcal{F}$ for any choice of $x_0 \in H$ and $T(x_0) \in T^*(x_0)$.

Define a partial order relation on \mathcal{F} by setting $(D_1, T_1) \preceq (D_2, T_2)$ if and only if $D_1 \subseteq D_2$ and $T_2|_{D_1} = T_1$. Let $((D_\alpha, T_\alpha))_{\alpha \in S}$ be an increasing chain in (\mathcal{F}, \preceq) . Then it follows that $(\bigcup_{\alpha \in S} D_\alpha, T) \in \mathcal{F}$ where $T|_{D_\alpha} = T_\alpha$. By Zorn's Lemma, (\mathcal{F}, \preceq) has a maximal element say (D, T) . Suppose to the contrary that $D \neq H$ and select $x_0 \in H \setminus D$. Set $\bar{D} = D \cup \{x_0\}$ and consider the set

$$J = \bigcap_{x \in D} (C_u(T(x), U(T^*(x), T^*(x_0))) \cap C_{u^{-1}}(T(x), U(T^*(x_0), T^*(x)))) \cap T^*(x_0).$$

Since $T^*(x_0) \in \mathcal{A}_u(X)$, then by Lemma 4.2.12, $J \neq \emptyset$ if $\text{dist}(T^*(x_0), T(x)) \leq U(T^*(x_0), T^*(x))$, $\text{dist}(T(x), T^*(x_0)) \leq U(T^*(x), T^*(x_0))$ and $u(T(x), T(y)) \leq \max\{U(T^*(x), T^*(x_0)), U(T^*(x_0), T^*(y))\}$ whenever $x, y \in D$. Therefore, we need to check if these conditions hold.

Let $x \in D$. For each $\epsilon > 0$, we have

$$T^*(x) \subseteq B_u(T^*(x_0), U(T^*(x_0), T^*(x)) + \epsilon)$$

and

$$T^*(x) \subseteq B_{u^{-1}}(T^*(x_0), U(T^*(x), T^*(x_0)) + \epsilon)$$

by definition of the Hausdorff ultra-quasi-pseudometric. Since $T(x) \in T^*(x)$, then for each $\epsilon > 0$, there exists $a \in T^*(x_0)$ such that $u(T(x), a) \leq U(T^*(x), T^*(x_0)) + \epsilon$ and $b \in T^*(x_0)$ such that $u(b, T(x)) \leq U(T^*(x_0), T^*(x)) + \epsilon$. Therefore, $\text{dist}(T(x), T^*(x_0)) \leq U(T^*(x), T^*(x_0))$ and $\text{dist}(T^*(x_0), T(x)) \leq U(T^*(x_0), T^*(x))$. Also, from the hypothesis of T , for each $x, y \in D$, $u(T(x), T(y)) \leq U(T^*(x), T^*(y)) \leq \max\{U(T^*(x), T^*(x_0)), U(T^*(x_0), T^*(y))\}$. Hence $J \neq \emptyset$.

Now choose $y_0 \in J$ and define $\bar{T}(x) = y_0$ if $x = x_0$ and $\bar{T}(x) = T(x)$ if $x \in D$. Since for each $x \in D$ we have

$$u(\bar{T}(x_0), \bar{T}(x)) = u(y_0, T(x)) \leq U(T^*(x_0), T^*(x))$$

and

$$u(\bar{T}(x), \bar{T}(x_0)) = u(T(x), y_0) \leq U(T^*(x), T^*(x_0)),$$

we conclude that $(D \cup \{x_0\}, \bar{T}) \in \mathcal{F}$ and this contradicts the maximality of (D, T) . Hence $D = H$. □

In Theorem 4.2.13, if the set-valued map is contractive, then its selection is also contractive. The next result shows this and it is an extension of Corollary 3.2.24 to contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces.

Corollary 4.2.14. *(Compare [15, Corollary 6.7]) Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space. Moreover, Let (H, ρ) be a T_0 -ultra-quasi-metric space and suppose $T^* : (H, \rho) \rightarrow (\mathcal{A}_u(X), U)$ is a contractive map, that is, $U(T^*(x), T^*(y)) < u(x, y)$ whenever $x, y \in H$. Then, there exists a contractive map $T : (H, \rho) \rightarrow (X, u)$ for which $T(x) \in T^*(x)$ whenever $x \in H$.*

Proof. Because T^* is a contractive map, then the selection obtained in Theorem 4.2.13 is contractive. □

Note that the above result also holds for a nonexpansive map.

Theorem 4.2.2 guarantees the existence of a unique fixed point for a point valued contractive map of a q -spherically complete T_0 -ultra-quasi-metric space to itself. The next corollary generalises this result to set-valued contractive maps taking values in the space $\mathcal{A}_u(X)$. This result extends Corollary 3.2.25 to q -spherically complete T_0 -ultra-quasi-metric spaces.

Corollary 4.2.15. ([15, Corollary 6.8]) *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and suppose that $T^* : (X, u) \rightarrow (\mathcal{A}_u(X), U)$ is contractive. Then T^* has unique fixed point, that is, there exists a unique $x \in X$ such that $x \in T^*(x)$.*

Proof. The existence of a contractive selection $T : X \rightarrow X$ of T^* follows from Corollary 4.2.14 and has a unique fixed point by Theorem 4.2.2. \square

We now turn our attention to fixed point theorems for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces.

4.3. Fixed Point Theorems in q -Spherically Complete T_0 -Ultra-Quasi-Metric Spaces

In this section, we use the conditions of the fixed point theorems of Espínola and Kirk [9] presented in Section 2.4 to develop fixed point theorems for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces.

Unlike q -hyperconvex T_0 -quasi-metric spaces, fixed points for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces are unique. Thus in the following theorem, we use the strong condition of Corollary 2.2.5.

Theorem 4.3.1. (Compare Theorem 2.4.1) *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space and let $S_{cl} = \{A \subseteq X : A = cl_{\tau(u)}A \cap cl_{\tau(u^{-1})}A\}$ be endowed with the Hausdorff T_0 -ultra-quasi metric U . Let $f_\alpha : (X, u) \rightarrow (S_{cl}, U)$ be a family of contractive maps indexed over a directed set Λ satisfying*

(i) *for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$,*

(ii) *for each $\epsilon > 0$, there exists $\alpha \in \Lambda$ such that $diam(f_\alpha(x)) \leq (1 - k_\alpha)\epsilon$.*

Then there exists a unique point $x \in X$ such that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

Proof. Define $f : (X, u) \rightarrow (X, u)$ by $\{f(x)\} = \bigcap_\alpha f_\alpha(x)$ for each $x \in X$. From the definition of the Hausdorff T_0 -ultra-quasi-metric on S_{cl} , we see that for any $x, y \in X$,

$$U(f_\alpha(x), f_\alpha(y)) = \max \left\{ \sup \left\{ \text{dist}_{f(y) \in f_\alpha(y)}(f_\alpha(x), f(y)) \right\}, \sup \left\{ \text{dist}_{f(x) \in f_\alpha(x)}(f(x), f_\alpha(y)) \right\} \right\}.$$

Note that

$$\text{dist}_{f(y) \in f_\alpha(y)}(f_\alpha(x), f(y)) = \inf_{f(y) \in f_\alpha(y)} \{u(f(x), f(y)) : f(x) \in f_\alpha(x)\} \leq u(f(x), f(y)) \text{ for any } x, y \in X.$$

Taking the sup over all $x, y \in X$, we get

$$\sup_{f(y) \in f_\alpha(y)} \text{dist}(f_\alpha(x), f(y)) = u(f(x), f(y)).$$

Similarly,

$$\sup_{f(x) \in f_\alpha(x)} \text{dist}(f(x), f_\alpha(y)) = u(f(x), f(y)).$$

Therefore, we get

$$U(f_\alpha(x), f_\alpha(y)) = u(f(x), f(y)) \quad \text{whenever } x, y \in X,$$

and since f_α is contractive for each $\alpha \in \Lambda$, we have

$$\begin{aligned} u(f(x), f(y)) &= U(f_\alpha(x), f_\alpha(y)) \\ &< u(x, y). \end{aligned}$$

Therefore, $f : (X, u) \rightarrow (X, u)$ is a contractive map. Thus by Theorem 4.2.2, f has a unique fixed say x . It is clear that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$.

□

In Theorem 4.3.1, if the family S_α is replaced with the family $\mathcal{A}_u(X)$ of u -admissible subsets of a q -spherically complete T_0 -ultra-quasi-metric space (X, u) , then condition (ii) is no longer needed. Hence we have the following result.

Theorem 4.3.2. (Compare Theorem 2.4.4) *Let (X, u) be a q -spherically complete T_0 -ultra-quasi-metric space, let $\mathcal{A}_u(X)$ denote the collection of all u -admissible subsets of X endowed with the Hausdorff ultra-quasi-pseudometric U , and let $f_\alpha : (X, u) \rightarrow (\mathcal{A}_u(X), U)$ be a family of contractive maps indexed over a directed set Λ . Suppose for each $x \in X$, $\beta \geq \alpha$ implies $f_\beta(x) \subseteq f_\alpha(x)$. Then there exists a unique point $x \in X$ such that $x \in f_\alpha(x)$ for every $\alpha \in \Lambda$.*

Proof. Define $f : (X, u) \rightarrow (\mathcal{P}_0(X), U)$ by $f(x) = \bigcap_{\alpha \in \Lambda} f_\alpha(x)$ for each $x \in X$. Since $\{f_\alpha(x)\}_{\alpha \in \Lambda}$ is a family of nonempty u -admissible sets directed by reverse set inclusion, then by Lemma 4.1.10, $f(x)$ is nonempty and u -admissible. Hence f is a contractive map of X into $\mathcal{A}_u(X)$ and by Corollary 4.2.15, f has a unique fixed point say x . It is clear that $x \in f_\alpha(x)$ for each $\alpha \in \Lambda$. □

5

Conclusion

In this MSc dissertation, we have successfully generalised the results of Espínola and Kirk [9] on fixed point theorems for a family of set-valued contraction maps in hyperconvex metric spaces to q -hyperconvex T_0 -quasi-metric spaces. In this last part of our work, we present a summary of our investigations.

In the first part of our investigations, we first generalised the extension of Frigon's Theorem in gauge spaces to quasi-gauge spaces. Then, we generalised the fixed point theorem on the existence of a unique fixed point for a family of set-valued contractions in complete metric spaces to bicomplete quasi-metric spaces. After that, we generalised the fixed point theorem on the existence of at least one fixed point for a family of set-valued contraction maps in hyperconvex metric spaces to q -hyperconvex T_0 -quasi-metric spaces. Thereafter, we generalised the result on the existence of a hyperconvex fixed point set for a family of set-valued contractions in hyperconvex metric spaces to q -hyperconvex T_0 -quasi-metric spaces.

In the second part of our work, we first provided some interesting properties of q -spherically complete T_0 -ultra-quasi-metric spaces. Then we used the conditions of the fixed point theorems of Espínola and Kirk [9] to develop fixed point theorems for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces.

Our conclusion leads us to list some open problems encountered throughout the present investigations. We hope to study these problems in future work.

Problem 1. *Under what conditions does the best approximation of a set-valued nonexpansive map in q -hyperconvex spaces exist?*

Problem 2. *Is it possible to define inductively a sequence converging to the unique fixed point for a family of set-valued contractive maps in q -spherically complete T_0 -ultra-quasi-metric spaces?*

Problem 3. *Under what assumptions can we get a q -spherically complete fixed point set for a family of set-valued maps in the setting of q -spherically complete T_0 -ultra-metric spaces?*

Problem 4. *Is it possible to extend the classical result of Baillon [3] to bounded q -spherically complete T_0 -ultra-quasi-metric spaces; i.e to prove that a descending chain of q -spherically complete spaces has a nonempty intersection and is q -spherically complete?*

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