Symmetry analysis of modified equal-width and nonlinear advection-diffusion equations

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by

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Declaration

I OKE DAVIES ADEYEMO, student number 29532892, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR. OKE DAVIES ADEYEMO

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF. C.M. KHALIQUE

Date: ....

Signed:..

DR. T. MOTSEPA

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 3

C.M. Khalique, O.D. Adeyemo, I. Simbanefayi, On optimal system, exact solutions and conservation laws of the modified equal-width equation, accepted and to appear in Applied Mathematics and Nonlinear Sciences

Chapter 4

O.D. Adeyemo, T. Motsepa, C.M. Khalique, Exact solutions and conservation laws of the generalized nonlinear advection-diffusion equation, submitted to Mathematical Methods in the Applied Sciences.

Dedication

To God, the creator of heaven and the earth from whom all blessings flow.

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Abstract

In this work we examine two nonlinear partial differential equations of fluid mechanics. The modified equal-width (MEW) equation, which is used in handling the simulation of a single dimensional wave propagation in nonlinear media with dispersion process, is studied first. We compute the optimal system of one-dimensional subalgebras and then use it to perform symmetry reductions and obtain group-invariant solutions. Also, we derive conservation laws of the MEW equation using the multiplier approach and the Noether theorem. Secondly we study the generalized nonlinear advection-diffusion equation, which describes the movement of a buoyancy-driven plume in an inclined porous medium. We consider three cases of n and in each case, we construct optimal system of one-dimensional subalgebras using the computed Lie point symmetries and then obtain symmetry reductions and group-invariant solutions based on these optimal systems of one-dimensional subalgebra. In addition, we determine the conservation laws of the equation by employing the multiplier approach and the new conservation theorem due to Ibragimov.

Introduction

For many years nonlinear partial differential equations (NLPDEs) have proven to be indispensable in modelling various nonlinear multidimensional systems which are present in numerous and varied natural phenomena. In recent years, many researchers have continuously explored the study of NLPDEs since they are fundamental to the understanding of the complex behaviours of these systems. Nonlinear equations are observed to be of great importance to this our contemporary world.

In view of the aforementioned, many methods of importance have been developed by notable scientists for generating exact solutions of NLPDEs. These include the ansatz method [1], the homogeneous balance method [2], the Bäcklund transformation [3], the inverse scattering transform method [4], the Darboux transformation [5], the simplest equation method [6], the Hirota bilinear method [7], the $(G'/G)-$expansion method [8], the Kudryashov method [9], the Jacobi elliptic function expansion method [10] and also the Lie symmetry method [11–16], to mention but a few.

Sophus Lie, a brilliant Norwegian mathematician in the late nineteenth century, developed an important revolutionary symmetry-based method for generating solutions to differential equations, popularly known today as Lie group analysis. This in turn has made it possible to obtain exact solutions to differential equations in a more systematic way. A highly robust amount of research which is based on Lie's work has been published by a number of researchers [11–19].

It is observed that conservation laws are established and entrenched laws of nature which have been experienced by many researchers in manifold scientific fields. Some of the conservation laws that are common in this respect are conservation of electric charge, conservation of linear momentum in an isolated system, the conservation of mechanical energy in the absence of dissipative forces, conservation of energy and many more. Moreover, in the field of applied mathematics, conservation laws are paramount in determining the extent to which the integrability of differential equations is ascertained, reduction and solutions of partial differential equations, development of numerical schemes, amongst others. For example, we can see that in [20–27] and the references contained therein.

The outline of this dissertation can simply be stated as follows:

In Chapter one, we give a brief presentation of some important preliminaries regarding Lie point symmetries, the extended Jacobi elliptic function expansion method, $(G'/G)-$expansion method, the Kudryashov method, Noether's theorem, multiplier approach and the conservation theorem due to Ibragimov.

In Chapter two, we study the potential Burgers equation as an illustrative example. Firstly, we compute Lie point symmetries and then use them to construct the commutator table. Furthermore, we obtain the one-parameter groups of the generated symmetries. We then compute the group-invariant solutions of the potential Burgers equation and thereafter find the travelling wave solution of the equation. Conclusively, we derive the conservation laws of the equation using the multiplier approach.

In Chapter three, we obtain infinitesimal generators of the modified equal-width (MEW) equation and then utilise the generators to construct an optimal system of one-dimensional subalgebras. Thereafter, we use the optimal system of one-dimensional subalgebras to obtain symmetry reductions as well as the group-invariant solutions of the modified equal-width equation which are the cnoidal and snoidal wave solutions of the equation. Penultimately, conservation laws are to be derived by the application of the multiplier approach and thereafter we use Noether's theorem also to obtain the conservation laws of the equation.

In Chapter four, we study the generalized nonlinear advection-diffusion equation with power law nonlinearity. The analysis of the equation prompts three different cases for n. For each case, we compute Lie point symmetries and then use them to construct optimal system of one-dimensional subalgebras. Thereafter, we obtain symmetry reductions and group-invariant solutions based on these optimal systems of one-dimensional subalgebras. Moreover, for each case we derive conservation laws by two differential approaches: the multiplier method and the conservation theorem due to Ibragimov.

In Chapter five, a summary of results discussed in the dissertation is provided and future work is proposed.

Bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter we present, very briefly, some basic concepts of Lie's theory which we will utilise in our dissertation. We also give some methods for finding exact solutions and deriving the conservation laws for the nonlinear partial differential equations to be studied in the dissertation.

1.1 Introduction

Marius Sophus Lie (1842-1899), a renowned and distinguished Norwegian mathematician in the later part of the nineteenth century developed a revolutionary symmetry-based method for obtaining solution to differential equations. This developed method is popularly known today as Lie group analysis which gives a more systematic and robust way of generating the exact solutions of differential equations. In recent times, several books based on Lie group analysis have been published, see [11–15]. Thus, many definitions and results that are presented in this Chapter are taken from the above-mentioned books.

1.2 Continuous one-parameter groups

Suppose $x = (x^1, \cdots, x^n)$ is the independent variable with coordinates x^i and $u = (u^1, \cdots, u^m)$ is the dependent variable with coordinates u^α (n and m finite). We consider the following change of the variables x and u:

$$T_a : \bar{x}^i = f^i(x, u, a), \ \bar{u}^\alpha = \phi^\alpha(x, u, a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A continuous one-parameter (local) Lie group of transformations in the space of variables x and u is a set G of transformations (1.1) which satisfies the following properties:

(i) If $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) There exists $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that from (i) the associativity property is satisfied. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if the group composition law is additive, i.e. $\phi(a, b) = a + b$.

Theorem 1.1 For any composition law $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)},$$

where

$$w(s) = \left. \frac{\partial \, \phi(s,b)}{\partial b} \right|_{b=0} .$$

1.3 Prolongation of point transformations and group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \cdots , \tag{1.3}$$

where the operator of total differentiation is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots , \quad i = 1, ..., n \tag{1.4}$$

The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Therefore the u_s^α are called differential variables.

Considering a pth-order PDE, namely

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

1.3.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^{\alpha}} = \phi^{\alpha}(x, u, a), \quad \phi^{\alpha}|_{a=0} = u^{\alpha}. \tag{1.6}$$

According to the Lie's theory, finding the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + a\,\xi^i(x, u), \quad \bar{u}^{\alpha} \approx u^{\alpha} + a\,\eta^{\alpha}(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^{α} into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^{\alpha}\big|_{a=0} = u^{\alpha}.$$

Consequently, we have

$$\xi^i(x, u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^{\alpha}(x, u) = \frac{\partial \phi^{\alpha}}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

We now introduce the symbol of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^{\alpha} \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x, u)\frac{\partial}{\partial u^{\alpha}}. \tag{1.9}$$

is known as the infinitesimal operator or generator of the group G. We say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5), if the group G is admitted by (1.5).

We now show how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}^{\alpha}_i = \bar{D}_j(u^{\alpha}), \qquad \bar{u}^{\alpha}_{ij} = \bar{D}_j(\bar{u}^{\alpha}_i) = \bar{D}_i(\bar{u}^{\alpha}_j), \cdots.$$

Let us now apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small a, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $(x, u, u_{(1)})$ space given by (1.6) and (1.13) form a one-parameter group called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We now let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14). Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

1.3.2 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we obtain

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j\text{)}. \end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Similarly, one can obtain the second prolongation

$$\zeta^{\alpha}_{ij} \;=\; D_j(\eta^{\alpha}_i) - u^{\alpha}_{ik} D_j(\xi^k), \quad \text{(sum on } k\text{)}. \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} \;=\; D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}\,j} D_{i_p}(\xi^j), \quad \text{(sum on } j\text{)}. \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i} \quad \text{(sum on } i, \alpha\text{)}, \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \end{aligned} \tag{1.18}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}. \tag{1.19}$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{1.20}$$

is a Lie point symmetry of the pth-order PDE (1.5), if

$$X^{[p]}\, E\big|_{E=0} = 0, \tag{1.21}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 An equation (1.21) that determines all the infinitesimal symmetries of (1.5) is called the determining equation.

Definition 1.4 A one-parameter group G of continuous transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{u}, \bar{u}_{(1)}, \cdots, \bar{u}_{(p)}) = 0, \tag{1.22}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \tag{1.23}$$

identically in x, u and a.

Theorem 1.2 A necessary and sufficient condition for a function $F(x, q)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.24}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n - 1$ functionally independent invariants. One can take, as basic invariants the left-hand side $n - 1$ first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x,) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.25}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The commutator of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \; [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Solution methods of differential equations

In this section, we give a brief presentation of some methods for obtaining exact solutions of differential equations.

1.7.1 The $(G'/G)-$expansion method

We give a compendium of the cogent features of the $(G'/G)-$expansion method. This method was introduced by Wang et al. [8] in the year 2008. In this method, the function G satisfies a second-order linear ordinary differential equation with constant coefficients. This method presents travelling wave solutions that are expressed as trigonometric, rational and hyperbolic functions. The $(G'/G)-$expansion method is a method that can simply be applied on nonlinear differential equations. Consider a NLPDE of the form

$$E(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0. \tag{1.26}$$

Firstly we transform the NLPDE (1.26) to a nonlinear ordinary differential equation (ODE) by making use of the substitution

$$u(t, x) = U(\phi), \quad \phi = x - ct, \tag{1.27}$$

where c is a constant. Using the above substitutions, equation (1.26) then transforms into the nonlinear ODE

$$E(U, -cU', U', c^2U'', -cU'', U'', \cdots) = 0. \tag{1.28}$$

Secondly we assume that

$$U(\phi) = \sum_{i}^{m} A_i \left(\frac{G'(\phi)}{G(\phi)}\right)^i \tag{1.29}$$

where $A_0, A_1, \cdots A_m$ are to be determined. The balancing procedure [2] is used to obtain the value of m, a positive integer. Here $G(\phi)$ satisfies the second-order linear homogeneous ODE

$$G''(\phi) + \lambda G'(\phi) + \mu G(\phi) = 0, \tag{1.30}$$

where λ and μ arbitrary constants. In equation (1.29) (G'/G) is given by

Case 1. When $M = \lambda^2 - 4\mu > 0$

$$\frac{G'(\phi)}{G(\phi)} = \Delta_1 \frac{A\cosh(\Delta_1\phi) + B\sinh(\Delta_1\phi)}{A\sinh(\Delta_1\phi) + B\cosh(\Delta_1\phi)} - \frac{\lambda}{2}, \tag{1.31}$$

where $\Delta_1 = \sqrt{M}/2$.

Case 2. When $M = \lambda^2 - 4\mu < 0$

$$\frac{G'(\phi)}{G(\phi)} = \Delta_2 \frac{-A\sin(\Delta_2\phi) + B\cos(\Delta_2\phi)}{A\cos(\Delta_2\phi) + B\sin(\Delta_2\phi)} - \frac{\lambda}{2}, \tag{1.32}$$

where $\Delta_2 = \sqrt{-M}/2$

Case 3. When $M = \lambda^2 - 4\mu = 0$

$$\frac{G'(\phi)}{G(\phi)} = \frac{B}{B\phi + A} - \frac{\lambda}{2} \tag{1.33}$$

In all the three cases mentioned above, A, B and C are arbitrary constants. The substitution of the value of $U(\phi)$ in equations (1.29) and (1.30) into (1.28) produces a polynomial in $(G'/G)^i$. Conclusively, equating the coefficients of $(G'/G)^i$ to zero and then solving the resultant system of algebraic equations yields the required values of A_i.

1.7.2 The extended Jacobi elliptic function expansion method

We give a synoptic outline of the Jacobi elliptic function expansion method which is another algorithm for deducing the exact solutions of differential equations. There has being an extensive literature, out of which some date back to several decades, covering various aspects of Jacobi elliptic functions [28–31] as to their interrelationships, derivation and applications. Many researchers [10, 32, 33] in recent times have utilised the properties of some of these elliptic functions in determining exact solutions of differential equations.

The procedure involved in applying the extended Jacobi elliptic function expansion method is explained as follows: Firstly the NLPDE (1.26) is transformed into the nonlinear ODE (1.28).

Secondly, we assume that our solutions can be expressed in the form

$$U(\phi) = \sum_{i=-M}^{M} A_i H(\phi)^i, \tag{1.34}$$

where a positive integer M is obtained by the balancing procedure and

$$H(\phi) = cn(\phi|\omega), \tag{1.35}$$

the cosine-amplitude function, is a solution to the first-order ODE [**?**, 28]

$$H'(\phi) = -\sqrt{(1 - H^2(\phi))(1 - \omega + \omega H^2(\phi))}, \tag{1.36}$$

and the sine amplitude function

$$H(z) = sn(\phi|w) \tag{1.37}$$

is a solution to the first-order ODE

$$H'(\phi) = \sqrt{(1 - H^2(\phi))(1 - \omega H^2(\phi))}. \tag{1.38}$$

The third step of our procedure includes substituting the value of $U(\phi)$ in equation (1.34), which can be subjected to either equation(1.36) or (1.38), into equation (1.28) in order to obtain a polynomial in powers of $H(\phi)$. Separating involved coefficients with respect to like powers of $H(\phi)$ produces an algebraic system of equations. These can then be solved to find the values of A_i, with $i = 0, \pm 1, \pm 2, \cdots \pm M$.

1.7.3 The Kudryashov method

This method is used in obtaining exact solutions of NLPDEs and it is as well described in [9]. It can also be seen, for example, in the papers [34, 35].

Consider the NLPDE

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{1.39}$$

We recall the algorithm involved in Kudryashov method as follows:

Step 1. The substitution $u(x,t) = U(\phi)$, $\phi = kx + \omega t$, with constants k and ω, transforms equation (1.39) to the ordinary differential equation

$$E_2(U, \omega U', kU', \omega^2 U'', k^2 U'', \cdots) = 0. \tag{1.40}$$

Step 2. Suppose that the exact solutions of equation (1.40) is presented as

$$U(\phi) = \sum_{n=0}^{N} a_n Q^n(\phi), \tag{1.41}$$

where a_n $(n = 0, 1, 2, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(\phi)$ becomes the solution of the first-order nonlinear ODE

$$Q_\phi(\phi) = Q^2(\phi) - Q(\phi). \tag{1.42}$$

Thus, equation (1.42) has the solution

$$Q(\phi) = \frac{1}{1 + e^\phi}. \tag{1.43}$$

Step 3. Next we substitute the value of $U(\phi)$ into equation (1.40) and then use equation (1.42) to generate an equation which involves the powers of Q.

Step 4.

Equating various powers of Q to zero yields the system of algebraic equations

$$P_n(a_N, a_{N-1}, \cdots, a_0, k, \omega, \cdots) = 0, \quad (n = 0, \cdots, N). \tag{1.44}$$

Step 5. Finally, the solution of the system of algebraic equations produces the values of coefficients $a_0, a_1, \cdots, a_{N-1}, a_N$ and relations for parameters of equation (1.40). Consequently, we obtain exact solutions of equation (1.40) in the form expressed in equation (1.41).

1.8 Conservation laws

Conservation laws originate from the field of Physics [36]. They are of immense importance because conservation laws give physical and conserved quantities for all solutions $u(t,x)$. It can be highlighted that they are useful in assessing the accuracy and stability of numerical methods for the solutions of partial differential equations. A local conservation law for a given PDE is a continuity equation

$$D_t T^t + D_x T^x = 0,$$

where T^t and T^x are respectively the conserved density and the spatial flux functions of t, x, u and the derivatives of u. D_t and D_x represent the total derivatives operators with respect to independent variables t and x respectively. Suppose that there exists a function $\Phi(t, x, u, u_t, u_x, ...)$ such that the conserved vector $(T^t, T^x) = (D_x\Phi, -D_t\Phi)$ holds for every solution $u(t,x)$, then this conservation law is said to be locally equivalent. A nontrivial conservation law can be expressed in a general form as

$$\frac{d}{dt}\int_{\Psi} T^t dx = -T^x|_{\partial\Psi},$$

with $\Psi \subseteq \mathbb{R}$ a fixed spatial domain.

Conservation laws are important in various fields of applications [37]. Precisely, they are highly essential in the study of solutions, integrability as well as in developing numerical solutions for PDEs. In view of this, several systematic approaches have been developed for calculating conservation laws such as the direct construction method popularly referred to as multiplier or variational derivatives approach, the symmetry or adjoint symmetry pair method.

Consider an rth-order system of partial differential equations which contain n independent variables $x = (x^1, x^2, \cdots, x^n)$ as well as m dependent variables $u = (u^1, u^2, \cdots, u^m)$, which are given by

$$E_\alpha(x, u, u_{(1)}, u_{(2)} \cdots, u_{(r)}) = 0, \quad \alpha = 1, \cdots, m, \tag{1.45}$$

with $u_{(i)}$ denoting the collection of all i-th-order partial derivatives of u. Moreover, n-tuple vector which is defined as $T = (T^1, T^2, \ldots, T^n)$, $T^j \epsilon\, \mathcal{A}$, $j = 1, \ldots, n$, ($\mathcal{A}$ is the space of differential functions) is a conserved vector of (1.45) if T^i satisfies equation

$$D_i T^i|_{(1.45)} = 0. \tag{1.46}$$

1.8.1 The multiplier approach

Multiplier approach has been employed by several researchers. See for example [16, 36, 38–43]. It is noteworthy that for a given differential system, a local conservation law arises from a linear combination formed by local multipliers or characteristics with each of the differential equations in the system, such that the multipliers Λ_α are functions of the independent and dependent variables and are of a finite number of derivatives with respect to the dependent variables of the said system of differential equations.

A multiplier $\Lambda_\alpha(x, u, u_1, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.47}$$

holds identically. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.48}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{ij_1\cdots j_s}^\alpha}, \alpha = 1, \ldots, m. \tag{1.49}$$

Once the multipliers have been obtained from (1.48), we can determine conserved vectors by invoking equation (1.47) as illustrated in [39].

1.8.2 The Noether theorem

The well-known theorem of an eminent and outstanding researcher and mathematician, Amalia 'Emmy' Noether [44] presented in 1918, plays an important fundamental role in various branches of theoretical physics due to the fact that it provides a highly straightforward connection between the conservation laws of a physical theory and the invariances of the variational integral whose Euler-Lagrange equations are the governing equations of that theory. In recent times, many researchers have applied Noether's theorem in various fields such as mechanics, and in finding conservation laws of PDEs. See, for example [45–47]. It may be said that Noether's theorem has placed the Lagrangian formulation in a position of primacy. Furthermore, the theorem brought into existence a situation whereby the search for conservation laws and selection rules have been reduced to a robust systematic study of the symmetries of a theory as well as the corresponding invariances of its Lagrangian. We begin with Euler-Langrange equations given as

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, ..., m, \tag{1.50}$$

where we have $\mathcal{L}(x, u, u_{(1)})$ as a first-order Lagrangian, that is, it involves the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ only, along with the independent variables $x = (x^1, ..., x^n)$ and the dependent variables $u = (u, ..., u^m)$.

Noether's theorem states that suppose that the variational integral with the Lagrangian $\mathcal{L}(x, u, u_{(1)})$ is invariant under a group G with a generator defined as

$$X = \xi^i(x, u, u_{(1)}, ...)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, ...)\frac{\partial}{\partial u^\alpha}, \tag{1.51}$$

then the vector field $C = (C^1, \cdots, C^n)$ defined by [45]

$$\begin{aligned} C^k &= \mathcal{L}\tau^k + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\Big(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^l x^k}}\Big)\right) \\ &+ \sum_{l=k}^{n}(\eta_l^\alpha - \psi^\alpha_{x^l x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} - B^k \end{aligned} \tag{1.52}$$

gives a conservation law for the Euler-Langrange equations (1.50), that is, obeys the equation $\text{div}C \equiv D_k(C^k) = 0$ for all solutions of system (1.45) that is

$$D_k(C^k)|_{(1.45)} = 0. \tag{1.53}$$

Any vector field C^k satisfying equation (1.53) is referred to as a conserved vector for equation (1.45).

1.8.3 The new conservation theorem due to Ibragimov

This relatively new method for generating the conservation laws states a general formula on conservation laws [48] for arbitrary partial differential equations by combining Lie symmetry operator and adjoint together with formal Lagrangians. Recently, this method has been put to use by several researchers, for instance [37,49–51]. The substance of the new conservation theorem due to Ibragimov is the fact that we can obtain a conservation law from every Lie generator, Lie-Bäcklund and non-local symmetry of a system of differential equations. We consider the system of nonlinear partial differential equations (1.45) and its adjoint equations given by

$$E^*_\alpha(x,u,v,\dots,v_{(s)},u_{(s)}) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta E_\beta) = 0, \tag{1.54}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator (1.49) and

m new dependent variables $v = (v^1,\dots,v^m)$.

Theorem 1.4 Consider a system of m equations (1.45). The adjoint system given by (1.54), inherits the symmetries of the system (1.45). Namely, if the system (1.45) admits a point transformation group with an operator (1.20), then the adjoint system (1.54) admits the generator (1.20) extended to the variable v^α by the formula

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta^\alpha_*\frac{\partial}{\partial v^\alpha} \tag{1.55}$$

with appropriately chosen $\eta^\alpha_* = \eta^\alpha_*(x,u,v)$.

In [48], the coefficients η^α_* in (1.55) are given by

$$\eta^\alpha_* = -\left[\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)\right], \tag{1.56}$$

where λ^α_β can be computed by utilising the equation

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.57}$$

We can obtain a conserved vector, for instance, for a third-order Lagrangian by applying the formula

$$\begin{aligned} C^i = \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_i} - D_j\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}} + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}\right) + \cdots\right] \\ + D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}} - D_k\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}} + \dots\right] + D_jD_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_{ijk}} + \cdots, \end{aligned} \tag{1.58}$$

where $\mathcal{L}$ is the Lagrangian of the system E and E^* that is given by

$$\mathcal{L} = v^\alpha E_\alpha \tag{1.59}$$

and W^α is the Lie characteristic function defined as

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j,\ \alpha = 1, \ldots, m. \tag{1.60}$$

1.9 Concluding remarks

In this chapter, we gave a brief introduction to Lie group analysis. We presented three solution methods for obtaining the exact solutions of partial differential equations. Furthermore, a synopsis of three methods for deriving the conservation laws were discussed. The various techniques deliberated upon in this section will be utilised throughout this dissertation.

Chapter 2

Solutions and conservation laws for the potential Burgers equation: an illustrative example

In this chapter, we compute the Lie point symmetries of the potential Burgers equation. We then generate the commutator table for the Lie point symmetries, use Lie equations to obtain one-parameter group transformations for each of the symmetries and then utilise the symmetries to obtain the group-invariant solutions of the equation. Thereafter, we derive the conservation laws of the potential Burgers equation using the multiplier approach.

2.1 Introduction

The Burgers equation [52] is of considerable physical as well as mathematical interest. On one hand, it has a wide range of applications in hydrodynamics and other areas due to the fact that it is the simplest partial differential equation that combines a description of nonlinear effects with that of dissipative ones. Moreover, from a mathematical point of view, it is the prototype of an equation that is linearizable through a direct coordinate transformation. Indeed, the standard form of the Burgers equation is

$$v_t - 2vv_x - v_{xx} = 0. \tag{2.1}$$

Let

$$v = u_x.$$

Substituting the above value of v in equation (2.1) and integrating it with respect to x, we obtain the potential form of the Burgers equation, called the potential Burgers equation [14, 53]

$$u_t - u_x^2 - u_{xx} = 0, \tag{2.2}$$

where $u = u(t, x)$.

2.2 Solutions of the potential Burgers equation

In this section we obtain group-invariant solutions of the potential Burgers equation (2.2) by using its Lie point symmetries. Thus we start by first finding the Lie point symmetries of the potential Burgers equation (2.2).

2.2.1 Lie point symmetries

The potential Burgers equation (2.2) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.3}$$

if and only if

$$X^{[2]}(u_t - u_x^2 - u_{xx})\big|_{u_{xx}=u_t-u_x^2} = 0. \tag{2.4}$$

Here $X^{[2]}$ is the second prolongation of X and is defined as

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{12}\frac{\partial}{\partial u_{tx}} + \zeta_{22}\frac{\partial}{\partial u_{xx}}, \tag{2.5}$$

where ζ_1 and ζ_2 are given by equation (1.15), and ζ_{11}, ζ_{12} and ζ_{22} are given by equation (1.16) which can simply be expressed as

$$\zeta_1 \;\; = \;\; D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\zeta_2 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi)$$

and

$$\begin{aligned}
\zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).
\end{aligned}$$

where the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{2.6}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots. \tag{2.7}$$

Thus equation (2.4) becomes

$$\Big[\tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}}$$
$$+\zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}\Big]\Big(u_t - u_x^2 - u_{xx}\Big)\Big|_{u_{xx}=u_t-u_x^2} = 0,$$

which gives

$$\zeta_1 - 2u_x\zeta_2 - \zeta_{22}\big|_{u_{xx}=u_t-u_x^2} = 0. \tag{2.8}$$

Substituting the values of ζ_1, ζ_2 and ζ_{22} in equation (2.8), we obtain

$$\begin{aligned}
&\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x \xi_u - 2u_x\eta_x - 2u_x^2\eta_u + 2u_t u_x \tau_x \\
&+2u_t u_x^2 \tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - \eta_{xx} - 2u_x\eta_{xu} - u_{xx}\eta_u - u_x^2\eta_{uu} + 2u_{xx}\xi_x \\
&+u_x\xi_{xx} + 2u_x^2\xi_{xu} + 3u_x u_{xx}\xi_u + u_x^3\xi_{uu} + 2u_{tx}\tau_x + u_t\tau_{xx} + 2u_t u_x \tau_{xu} \\
&+u_t u_{xx}\tau_u + 2u_x u_{tx}\tau_u + u_t u_x^2 \tau_{uu}\big|_{u_{xx}=u_t-u_x^2} = 0.
\end{aligned} \tag{2.9}$$

By replacing u_{xx} by $u_t - u_x^2$ in the above equation, we obtain

$$\begin{aligned}
&\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x \xi_u - 2u_x\eta_x - 2u_x^2\eta_u + 2u_t u_x \tau_x \\
&+2u_t u_x^2 \tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - \eta_{xx} - 2u_x\eta_{xu} - u_t\eta_u + u_x^2\eta_u - u_x^2\eta_{uu} \\
&-2u_x^2\xi_x + 2u_t\xi_x + u_x\xi_{xx} + 2u_x^2\xi_{xu} + 3u_x u_t\xi_u - 3u_x^3\xi_u + u_x^3\xi_{uu} + 2u_{tx}\tau_x \\
&+u_t\tau_{xx} + 2u_t u_x \tau_{xu} + u_t^2\tau_u - u_t u_x^2\tau_u + 2u_x u_{tx}\tau_u + u_t u_x^2 \tau_{uu} = 0.
\end{aligned} \tag{2.10}$$

Since τ, ξ and η depend only on t, x and u, we can split on the derivatives of u to obtain an overdetermined system of linear homogeneous PDEs:

$$u_t u_{tx}: \quad \tau_u = 0, \tag{2.11}$$

$$u_{tx}: \quad \tau_x = 0, \tag{2.12}$$

$$u_t u_x: \quad \xi_u = 0, \tag{2.13}$$

$$u_x^2: \quad \eta_{uu} + \eta_u = 0, \tag{2.14}$$

$$u_x: \quad \xi_{xx} - 2\eta_{xu} - 2\eta_x - \xi_t = 0, \tag{2.15}$$

$$u_t: \quad 2\xi_x - \tau_t = 0, \tag{2.16}$$

$$\text{rest}: \quad \eta_t - \eta_{xx} = 0. \tag{2.17}$$

From equations (2.11) and (2.12), we obtain

$$\tau = a(t),$$

where $a(t)$ is an arbitrary function of t. From equation (2.13), we obtain

$$\xi = b(t, x),$$

where $b(t, x)$ is an arbitrary function of t and x. Now, by substituting these values of τ and ξ in (2.16) and integrating with respect to x yields

$$b(t, x) = \frac{1}{2}a'(t)x + c(t),$$

where $c(t)$ is an arbitrary function of t and so

$$\xi = \xi(t, x) = \frac{1}{2}a'(t)x + c(t). \tag{2.18}$$

Equation (2.14) is a second-order linear homogeneous PDE. Its solution is

$$\eta(t, x, u) = \alpha(t, x) + \beta(t, x)e^{-u}, \tag{2.19}$$

where $\alpha(t, x)$ and $\beta(t, x)$ are arbitrary functions of t and x. It follows by the substitution of the values of ξ and η from equations (2.18) and (2.19) into (2.15) that

$$\alpha_x(t, x) = -\frac{1}{4}a''(t)x - \frac{1}{2}c'(t).$$

By integrating the above equation with respect to x, we obtain

$$\alpha(t,x) = -\frac{1}{8}a''(t)x^2 - \frac{1}{2}c'(t)x + d(t),$$

where $d(t)$ is an arbitrary function of t and so

$$\eta = -\frac{1}{8}a''(t)x^2 - \frac{1}{2}c'(t)x + d(t) + \beta(t,x)e^{-u}.$$

Substituting the value of η from the above equation in (2.17), we obtain

$$-\frac{1}{8}a'''(t)x^2 - \frac{1}{2}c''(t)x + \frac{1}{4}a''(t) + d'(t) + \beta_t(t,x)e^{-u} - \beta_{xx}(t,x)e^{-u} = 0.$$

Splitting the above equation on e^{-u}, we obtain

$$e^{-u}: \quad \beta_t(t,x) - \beta_{xx}(t,x) = 0, \tag{2.20}$$

$$\text{rest}: \quad -\frac{1}{8}a'''(t)x^2 - \frac{1}{2}c''(t)x + \frac{1}{4}a''(t) + d'(t) = 0. \tag{2.21}$$

Splitting equation (2.21) on x, we obtain

$$x^2: \quad a'''(t) = 0, \tag{2.22}$$

$$x: \quad c''(t) = 0, \tag{2.23}$$

$$\text{rest}: \quad \frac{1}{4}a''(t) + d'(t) = 0. \tag{2.24}$$

Integrating (2.22) and (2.23) with respect to (t) yields, respectively,

$$a(t) = \frac{1}{2}C_1t^2 + C_2t + C_3 \tag{2.25}$$

and

$$c(t) = C_4t + C_5, \tag{2.26}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. It follows from (2.24) and (2.25) that

$$d(t) = -\frac{1}{4}C_1t + C_6, \tag{2.27}$$

where C_6 is an arbitrary constant. Hence the general solution of the system (2.11) to (2.17) is

$$\tau(t) = \frac{1}{2}C_1t^2 + C_2t + C_3,$$

$$
\begin{aligned}
\xi(t,x) &= \frac{1}{2}(C_1t + C_2)x + C_4t + C_5, \\
\eta(t,x,u) &= -\frac{1}{8}C_1x^2 - \frac{1}{2}C_4x - \frac{1}{4}C_1t + C_6 + \beta(t,x)e^{-u},
\end{aligned}
$$

where the Cs are constants and $\beta(t,x)$ satisfies the heat equation $\beta_t = \beta_{xx}$. Thus the Lie point symmetries of the potential Burgers equation are given by

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= \frac{\partial}{\partial u}, \\
X_4 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\
X_5 &= 2t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \\
X_6 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)\frac{\partial}{\partial u}, \\
X_\beta &= \beta(t,x)e^{-u}\frac{\partial}{\partial u},
\end{aligned}
$$

which generate a Lie algebra of infinite dimension, where $\beta(t,x)$ satisfies the heat equation $\beta_t = \beta_{xx}$. It is worthy of note that potential Burgers equation (2.2) has an infinite-dimensional Lie algebra of point symmetries and many higher symmetries [53].

2.2.2 Commutator table for the symmetries

We now calculate the commutation relations for all the symmetry generators. We first compute $[X_2, X_6]$. By the definition of the Lie bracket, we have

$$
\begin{aligned}
[X_2, X_6] &= X_2X_6 - X_6X_2 \\
&= \frac{\partial}{\partial t}\left(4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)\frac{\partial}{\partial u}\right) - \left(4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)\frac{\partial}{\partial u}\right)\frac{\partial}{\partial t} \\
&= 4\left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right) - 2\frac{\partial}{\partial u} \\
&= 4X_4 - 2X_3.
\end{aligned}
$$

Proceeding in a similar manner we compute other commutation relations. In a tabular form, these commutation relations are written as:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_β
X_1	0	0	0	X_1	$-X_3$	$2X_5$	X_{β_x}
X_2	0	0	0	$2X_2$	$2X_1$	$4X_4 - 2X_3$	X_{β_t}
X_3	0	0	0	0	0	0	$-X_\beta$
X_4	$-X_1$	$-2X_2$	0	0	X_5	$2X_6$	X_{β^1}
X_5	X_3	$-2X_1$	0	$-X_5$	0	0	X_{β^2}
X_6	$-2X_5$	$2X_3 - 4X_4$	0	$-2X_6$	0	0	X_{β^3}
X_β	$-X_{\beta_x}$	$-X_{\beta_t}$	X_β	$-X_{\beta^1}$	$-X_{\beta^2}$	$-X_{\beta^3}$	0

The values of β^1, β^2 and β^3 in the table above are given by,

$$\begin{aligned}\beta^1 &= x\beta_x + 2t\beta_t,\\ \beta^2 &= 2t\beta_x + x\beta,\\ \beta^3 &= 4tx\beta_x + 4t^2\beta_t + (x^2 + 2t)\beta.\end{aligned}$$

2.2.3 One-parameter groups of transformations

The corresponding one-parameter group of transformations can be obtained using the Lie equations

$$\begin{aligned}\frac{d\bar{t}}{da} &= \xi^1(t, x, u), \quad \bar{t}|_{a=0} = t,\\ \frac{d\bar{x}}{da} &= \xi^2(t, x, u), \quad \bar{x}|_{a=0} = x,\\ \frac{d\bar{u}}{da} &= \eta(t, x, u), \quad \bar{u}|_{a=0} = u.\end{aligned}$$

We now compute the one-parameter group of transformations for each Lie point symmetry of the potential Burgers equation. For each X_i, let T_{a_i} be the corresponding group. Let us first calculate the one-parameter group corresponding to infinitesimal generator X_4, namely

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 2\bar{t}, \quad \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \bar{x}, \quad \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = 0, \quad \bar{u}|_{a=0} = u.$$

Solving the above equations we get

$$\bar{t} = te^{2a}, \quad \bar{x} = xe^{a}, \quad \bar{u} = u.$$

Thus the one-parameter group T_{a_4} corresponding to the operator X_4 is given by

$$T_{a_4} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{2a_4}, xe^{a_4}, u).$$

If we continue in the same manner as above, we get the following one-parameter groups:

$$\begin{aligned}
T_{a_1} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_1, u), \\
T_{a_2} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_2, x, u), \\
T_{a_3} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, u + a_3), \\
T_{a_5} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + 2a_5 t, u - a_5^2 t - a_5 x), \\
T_{a_6} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \left(\frac{t}{1 - 4ta_6}, x(1 + 4ta_6), u + \frac{1}{2}\ln(1 - 4ta_6) - \frac{x^2 a_6}{1 - 4ta_6} \right), \\
T_{a_7} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, \ln[\beta(t, x)a_7 + e^u]).
\end{aligned}$$

2.2.4 Symmetry transformations

In this subsection we make use of the symmetries calculated in Section 2.2.1 to obtain special exact solutions for the potential Burgers equation. The Lie group analysis supplies us with two basic ways for constructing exact solutions of PDEs: group transformations of known solutions and construction of group-invariant solutions. These methods are described in detail by means of examples.

If $\bar{u} = h(\bar{t}, \bar{x})$ is a solution of equation (2.2), then so is

$$\phi(t, x, u, a) = h(f_1(t, x, u, a), f_2(t, x, u, a))$$

or in solved form with respect to $u : u = H_a(t, x)$ is a one-parameter family of solutions. For

$$T_{a_1} : \bar{t} = t, \quad \bar{x} = x + a_1, \quad \bar{u} = u,$$

if $\bar{u} = h(\bar{t}, \bar{x})$ is a solution, then

$$u = h(t, x + a_1).$$

We now write down the generated solutions for the other cases:

$$\begin{aligned}
T_{a_2} : u &= h(t + a_2, x), \\
T_{a_3} : u &= h(t, x) - a_3, \\
T_{a_4} : u &= h(te^{2a_4}, xe^{a_4}), \\
T_{a_5} : u &= h(t, x + 2a_5 t) + a_5^2 t + a_5 x, \\
T_{a_6} : u &= h\left(\frac{t}{1 - 4ta_6}, x(1 + 4ta_6)\right) - \frac{1}{2}\ln(1 - 4ta_6) + \frac{x^2 a_6}{1 - 4ta_6}, \\
T_\beta : u &= \ln[e^{h(t,x)} - \beta(t, x)a].
\end{aligned}$$

2.2.5 Construction of group-invariant solutions of (2.2)

Consider a Lie point symmetry

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.28}$$

of the potential Burgers equation (2.2). The group-invariant solutions under the one-parameter group generated by X are obtained as follows. We calculate two linearly independent invariants

$$J_1 = \phi(t, x) \text{ and } J_2 = \psi(t, x)$$

by solving the first-order quasi-linear PDE

$$X(J) \equiv \tau(t, x, u)\frac{\partial J}{\partial t} + \xi(t, x, u)\frac{\partial J}{\partial x} + \eta(t, x, u)\frac{\partial J}{\partial u} = 0$$

with characteristic equations otherwise known as the associated Lagrange system

$$\frac{dt}{\tau(t, x, u)} = \frac{dx}{\xi(t, x, u)} = \frac{du}{\eta(t, x, u)}.$$

Then we write one of the invariants as a function of the other, for example

$$J_2 = f(J_1), \tag{2.29}$$

where f is a function of J_1, and solve (2.29) for u. Finally, the expression of u is substituted in equation (2.2) and an ODE is obtained for the unknown function f. This procedure reduces the number of independent variables by one.

Let us now illustrate the above method by considering the five linearly independent Lie point symmetries X_1, X_2, X_4, X_5 and X_6 and construct group-invariant solutions under these operators.

Case 1. We first calculate the group-invariant solution under the symmetry operator X_1. The operator X_1 is given by

$$X_1 = \frac{\partial}{\partial x}.$$

The characteristics equations associated with X_1 are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which provide the two invariants $J_1 = t$ and $J_2 = u$. Thus the group-invariant solution is given by $J_2 = \psi(J_1)$, i.e.,

$$u = \psi(t).$$

Substituting this value of u in (2.2), we obtain

$$\psi'(t) = 0.$$

Thus the second-order potential Burgers PDE (2.2) reduces to first-order ODE

$$\frac{d\psi}{dt} = 0.$$

Solving the above equation, we obtain

$$\psi(t) = C,$$

where C is an arbitrary constant of integration. Hence the group-invariant solution of (2.2) under X_1 is given by

$$u(t, x) = C.$$

Case 2. We now obtain the group-invariant solution under the symmetry operator

$$X_2 = \frac{\partial}{\partial t}.$$

The Lagrangian system associated with X_2 is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which provide the two group-invariants $J_1 = x$ and $J_2 = u$. Thus the group-invariant solution is given by $J_2 = \psi(J_1)$, i.e.,

$$u = \psi(x).$$

Substituting this value of u in equation (2.2), we obtain

$$\psi''(x) + \psi'^2(x) = 0.$$

The solution to the above second-order nonlinear ODE is

$$\psi(x) = \ln(x + C_1) + C_2,$$

where C_1 and C_2 are arbitrary constants of integration. Hence the group-invariant solution of equation (2.2) under X_2 is given by

$$u(t, x) = \ln(x + C_1) + C_2.$$

Case 3. The Lie point symmetry X_3 defined by

$$X_3 = \frac{\partial}{\partial u}$$

does not have a group-invariant solution.

Case 4. Let us now construct the group-invariant solution under the symmetry generator

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

The characteristic equations associated with X_4 are

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{0}.$$

Thus, one invariant is $J_1 = u$. The other is obtained from the equation

$$\frac{dt}{2t} = \frac{dx}{x}$$

and is given by $J_2 = x/\sqrt{t}$.

Consequently, the group-invariant solution is $J_1 = \psi(J_2)$, i.e.,

$$u = \psi\left(\frac{x}{\sqrt{t}}\right).$$

Then

$$u_t = -\frac{1}{2}\frac{x}{t^{\frac{3}{2}}}\psi'\left(\frac{x}{\sqrt{t}}\right),$$

$$u_x = \frac{1}{\sqrt{t}}\psi'\left(\frac{x}{\sqrt{t}}\right),$$

$$u_{xx} = \frac{1}{t}\psi''\left(\frac{x}{\sqrt{t}}\right).$$

Substitution of the above values of u_t, u_x and u_{xx} in (2.2) gives us the second-order ODE

$$-\frac{1}{2}\frac{x}{t^{\frac{3}{2}}}\psi'\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{t}\psi'^2\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{t}\psi''\left(\frac{x}{\sqrt{t}}\right) = 0.$$

By letting $y = x/\sqrt{t}$ and $q = \psi'(y)$, we obtain

$$2\frac{dq}{dy} + yq = -2q^2.$$

The above is a Bernoulli equation with $n = 2$ whose solution is

$$q = \left(e^{\frac{y^2}{4}}\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{y}{2}\right) + C_1\right)\right)^{-1},$$

where C_1 is a constant of integration. Hence

$$\frac{d\psi}{dy} = \left(e^{\frac{y^2}{4}}\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{y}{2}\right) + C_1\right)\right)^{-1}.$$

Integrating the above equation, we obtain

$$\psi(y) = \ln\left(\left|\sqrt{\pi}\,\mathrm{erf}\left(\frac{y}{2}\right) + C_1\right|\right) + C_2,$$

where C_2 is a constant of integration. Reverting back to the original variables, we obtain

$$u(t,x) = \ln\left(\left|\sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) + C_1\right|\right) + C_2.$$

Case 5. We now calculate the group-invariant solution under the symmetry generator X_5 defined as

$$X_5 = 2t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}.$$

The associated Lagrangian system is

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{du}{-x}.$$

Therefore one of the invariants is $J_1 = t$. The other is obtained from the equation

$$\frac{dx}{2J_1} = \frac{du}{-x},$$

and is given by $J_2 = u + (x^2/4t)$.

Consequently, the group-invariant solution of (2.2) under X_5 is $J_2 = \psi(J_1)$, i.e.,

$$u = \psi(t) - \frac{x^2}{4t}, \tag{2.30}$$

where ψ is an arbitrary function. Substituting (2.30) into (2.2), gives the first-order ODE

$$\frac{d\psi}{dt} + \frac{1}{2t} = 0.$$

Solving for ψ, we obtain

$$\psi(t) = -\frac{1}{2}\ln t + C,$$

where C is an arbitrary constant of integration. Hence, the group-invariant solution of X_5 is

$$u(t,x) = -\left(\frac{1}{2}\ln t + \frac{x^2}{4t}\right) + C.$$

Case 6. Let us construct the group-invariant solution under operator X_6, namely

$$X_6 \;=\; 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2+2t)\frac{\partial}{\partial u}.$$

The characteristic equations associated with X_6 are

$$\frac{dt}{4t^2} = \frac{dx}{4tx} = \frac{du}{-(x^2+2t)}.$$

Therefore, one invariant is $J_1 = x/t$. The other is obtained from the equation

$$\frac{dx}{4tx} = \frac{du}{-(x^2+2t)}$$

and is given by

$$J_2 = u + \frac{1}{2}\ln x + \frac{x^2}{4t}. \tag{2.31}$$

As a result, the group-invariant solution is $J_2 = \psi(J_1)$, i.e.,

$$u = \psi\left(\frac{x}{t}\right) - \frac{x^2}{4t} - \frac{1}{2}\ln x.$$

Now

$$u_t = -\frac{x}{t^2}\psi'\left(\frac{x}{t}\right) + \frac{x^2}{4t^2},$$

$$u_x = \frac{1}{t}\psi'\left(\frac{x}{t}\right) - \frac{x}{2t} - \frac{1}{2x},$$

$$u_{xx} = \frac{1}{t^2}\psi''\left(\frac{x}{t}\right) - \frac{1}{2t} + \frac{1}{2x^2}.$$

Substitution of the above values of u_t, u_x and u_{xx} in (2.2) gives us the second-order ODE

$$\frac{1}{t^2}\psi''\left(\frac{x}{t}\right) + \frac{1}{t^2}\psi'^2\left(\frac{x}{t}\right) - \frac{1}{tx}\psi'\left(\frac{x}{t}\right) + \frac{3}{4x^2} = 0.$$

By letting $J_1 = z = x/t$ and $y = \psi'(z)$, we obtain the Riccati equation

$$z^2y' + z^2y^2 - zy + \frac{3}{4} = 0. \tag{2.32}$$

A particular solution of the above Riccati equation is $y_1 = 1/2z$. Thus letting $y = y_1 + 1/v$, we obtain

$$\frac{dy}{dz} = -\frac{1}{2z^2} - \frac{1}{v^2}\frac{dv}{dz}. \tag{2.33}$$

Now substituting the above value of y and dy/dz in (2.32), we obtain

$$-\frac{1}{2z^2} - \frac{1}{v^2}\frac{dv}{dz} = -\left(\frac{1}{2z} + \frac{1}{v}\right)^2 + \frac{1}{z}\left(\frac{1}{2z} + \frac{1}{v}\right) - \frac{3}{4z^2}, \tag{2.34}$$

which simplifies to

$$\frac{dv}{dz} = 1. \tag{2.35}$$

Integrating the above equation yields

$$v = z + C_1, \tag{2.36}$$

where C_1 is a constant of integration. Hence

$$y = \frac{1}{2z} + \frac{1}{z + C_1}$$

and so

$$\frac{d\psi}{dz} = \frac{1}{2z} + \frac{1}{z + C_1}.$$

Integrating the above equation, we obtain

$$\psi(z) = \frac{1}{2} \ln z + \ln(z + C_1) + C_2,$$

where C_2 is a constant of integration. Reverting back to the original variables, we obtain the group-invariant solution associated with X_6 as

$$u(t, x) = \ln \frac{1}{\sqrt{t}} \left(\frac{x}{t} + C_1 \right) - \frac{x^2}{4t} + C_2.$$

Travelling wave solutions

We can obtain travelling wave solutions of the potential Burgers equation by considering the linear combination X of the translation symmetries X_1 and X_2:

$$X = k \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad k \text{ a constant.} \tag{2.37}$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{0}. \tag{2.38}$$

Thus, one invariant is $J_1 = u$. The other is obtained from the equation

$$\frac{dx}{k} = \frac{dt}{1} \tag{2.39}$$

and is given by $J_2 = x - kt$. Thus, the group-invariant solution can be written as $J_1 = \psi(J_2)$, i.e.,

$$u = \psi(x - kt), \tag{2.40}$$

where ψ is an arbitrary function of its argument. Differentiation of u with respect to x and t, gives us

$$u_t = -k\,\psi', \quad u_x = \psi', \quad u_{xx} = \psi''.$$

Substituting these expressions into (2.1) we obtain the reduced ODE

$$\psi'' + \psi'^2 + k\psi' = 0, \tag{2.41}$$

which is a second-order ODE with constant coefficients. Suppose we let $z = x - kt$ and $\psi'(z) = y$, we obtain a simplified ODE of the form

$$y' + y^2 + ky = 0, \tag{2.42}$$

whose solution is

$$y(z) = \left(-\frac{1}{k} + C_1 e^{kz} \right)^{-1}, \tag{2.43}$$

where C_1 is a constant of integration. Hence

$$\psi'(z) = \left(-\frac{1}{k} + C_1 e^{kz} \right)^{-1}. \tag{2.44}$$

Integrating the above equation, we obtain

$$\psi(z) = \ln \left(\left| C_1 k e^{kz} - 1 \right| \right) - kz + C_2,$$

where C_2 is an arbitrary constant of integration. Thus reverting to the original variables, the travelling wave solution to the potential Burgers equation is of the form

$$u(t,x) = \ln \left(\left| C_1 k e^{kx - k^2 t} - 1 \right| \right) - kx + k^2 t + C_2.$$

2.3 Conservation laws of the potential Burgers equation

In this section, we derive the conservation laws for the potential Burgers equation (2.2) by employing the multiplier method. We recall the standard Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \cdots , \tag{2.45}$$

where the total derivatives D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + + u_{ttx} \frac{\partial}{\partial u_{tx}} + \cdots , \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{txx} \frac{\partial}{\partial u_{tx}} + \cdots . \end{aligned} \tag{2.46}$$

We achieve the aforementioned by using the zeroth-order multiplier $\Lambda(t, x, u)$. Following the procedure described in Section 1.8.1, the determining equation for the multiplier $\Lambda(t, x, u)$ is

$$\frac{\delta}{\delta u}[\Lambda(t, x, u)(u_t - u_x^2 - u_{xx})] = 0. \tag{2.47}$$

By expanding the above equation we obtain

$$u_t \Lambda_u - u_x^2 \Lambda_u - u_{xx} \Lambda_u - D_t \Lambda + 2D_x(\Lambda u_x) - D_x^2 \Lambda = 0. \tag{2.48}$$

Substituting for D_t and D_x from equation (2.46) into (2.48), gives

$$u_x^2 \Lambda_u - 2u_{xx} \Lambda_u - \Lambda_t + 2u_x \Lambda_x + 2u_{xx} \Lambda - \Lambda_{xx} - 2u_x \Lambda_{ux} - u_x^2 \Lambda_{uu} = 0. \tag{2.49}$$

Splitting the above equation on the derivatives of u, we obtain the following system of PDEs:

$$u_{xx} : \quad \Lambda - \Lambda_u = 0, \tag{2.50}$$

$$u_x : \quad \Lambda_x - \Lambda_{ux} = 0, \tag{2.51}$$

$$\text{rest} : \quad \Lambda_{xx} + \Lambda_t = 0. \tag{2.52}$$

Equation (2.50) yields

$$\Lambda(t, x, u) = C(t, x)e^u, \tag{2.53}$$

where $C(t, x)$ is an arbitrary function depending on t and x.

It is trivial to note that (2.53) satisfies equation (2.51). Substituting the value of Λ from equation (2.53) into (2.52), we obtain

$$C_{xx}e^u + C_t e^u = 0.$$

Thus becoming

$$C_{xx} + C_t = 0. \tag{2.54}$$

To solve equation (2.54), we assume that

$$C(t, x) = T(t)X(x). \tag{2.55}$$

Substituting the above value of $C(t, x)$ in equation (2.54) gives

$$X''T + T'X = 0.$$

Hence

$$\frac{X''}{X} = -\frac{T'}{T} = -a, \quad \text{(say)}$$

where a is an arbitrary constant. Thus we have

$$T' - aT = 0 \tag{2.56}$$

and

$$X'' + aX = 0. \tag{2.57}$$

Solving (2.56), we obtain

$$T(t) = C_1 e^{at},$$

where C_1 is a constant of integration. Solving equation (2.57) yields

$$X(x) = A_1 \sin \sqrt{a}x + B_1 \cos \sqrt{a}x,$$

where A_1 and B_1 are arbitrary constants. Thus

$$C(t, x) = C_1 e^{at} \left(A_1 \sin \sqrt{a}x + B_1 \cos \sqrt{a}x \right).$$

Putting $C_1A_1 = A$ and $C_1B_1 = B$, the above gives

$$C(t,x) = e^{at}\left(A\sin\sqrt{a}x + B\cos\sqrt{a}x\right).$$

Therefore equation (2.53) becomes

$$\Lambda(t,x,u) = e^{u+at}\left(A\sin\sqrt{a}x + B\cos\sqrt{a}x\right),$$

which yields the two multipliers

$$\Lambda_1(t,x,u) = e^{u+at}\sin\sqrt{a}x$$

and

$$\Lambda_2(t,x,u) = e^{u+at}\cos\sqrt{a}x$$

for equation (2.2).

We first compute conservation law of (2.2) associated with the multiplier $\Lambda_1(t,x,u)$. Therefore we have

$$\Lambda_1(t,x,u)\{u_t - u_x^2 - u_{xx}\} = D_tT^t + D_xT^x, \tag{2.58}$$

where the density T^t and flux T^x are respectively defined as

$$T^t = T^t(t,x,u) \ \text{ and } \ T^x = T^x(t,x,u,u_x). \tag{2.59}$$

Thus equation (2.58) becomes

$$e^{u+at}\sin\sqrt{a}x\left(u_t - u_x^2 - u_{xx}\right) = D_tT^t + D_xT^x.$$

The above yields

$$e^{u+at}\sin\sqrt{a}x\left(u_t - u_x^2 - u_{xx}\right) = T_t^t + u_tT_u^t + T_x^x + u_xT_u^x + u_{xx}T_{u_x}^x. \tag{2.60}$$

Splitting the above equation over the second derivatives of u, we obtain

$$u_{xx}: \quad T_{u_x}^x = -e^{u+at}\sin\sqrt{a}x, \tag{2.61}$$

$$\text{rest}: \quad T_t^t + u_tT_u^t + T_x^x + u_xT_u^x = \left(u_t - u_x^2\right)e^{u+at}\sin\sqrt{a}x. \tag{2.62}$$

Integration of equation (2.61) with respect to u_x gives

$$T^x = -u_x e^{u+at} \sin\sqrt{a}x + Q(t, x, u), \tag{2.63}$$

where $Q(t, x, u)$ is an arbitrary function of t, x and u. Substituting the above value of T^x into (2.62), we obtain

$$T^t_t + u_t T^t_u - \sqrt{a} u_x e^{u+at} \cos\sqrt{a}x + Q_x - u_x^2 e^{u+at} \sin\sqrt{a}x + u_x Q_u = e^{u+at} \sin\sqrt{a}x \left(u_t - u_x^2\right).$$

Simplification of the above equation and splitting it on the derivatives of u yields

$$u_t: \qquad T^t_u = e^{u+at} \sin\sqrt{a}x, \tag{2.64}$$

$$u_x: \qquad Q_u - \sqrt{a} e^{u+at} \cos\sqrt{a}x = 0, \tag{2.65}$$

$$\text{rest}: \qquad T^t_t + Q_x = 0. \tag{2.66}$$

Integrating equation (2.64) with respect to u produces

$$T^t = e^{u+at} \sin\sqrt{a}x + R(t, x), \tag{2.67}$$

where $R(t, x)$ is a function depending on t and x. Equation (2.65) then gives

$$Q = \sqrt{a} e^{u+at} \cos\sqrt{a}x + S(t, x),$$

where $S(t, x)$ is an arbitrary function of t and x. We set $R(t, x) = S(t, x) = 0$ since they both contribute to the trivial part of the conserved vector and consequently, equation (2.66) is satisfied. Therefore the density T^t and flux T^x defined above become

$$\begin{aligned} T^t_1 &= e^{u+at} \sin\sqrt{a}x, \\ T^x_1 &= -u_x e^{u+at} \sin\sqrt{a}x + \sqrt{a} e^{u+at} \cos\sqrt{a}x, \end{aligned}$$

which is the conserved vector of (2.2) associated with the multiplier $\Lambda_1(t, x, u)$.

We now compute conservation law of (2.2) associated with the multiplier $\Lambda_2(t, x, u)$. The determining equation gives

$$\Lambda_2(t, x, u)\{u_t - u_x^2 - u_{xx} = 0\} = D_t T^t + D_x T^x, \tag{2.68}$$

which implies

$$e^{u+at}\cos\sqrt{a}x\left(u_t-u_x^2-u_{xx}\right)=D_tT^t+D_xT^x,$$

where T^t and T^x are as defined in (2.59) . The above equation becomes

$$e^{u+at}\cos\sqrt{a}x\left(u_t-u_x^2-u_{xx}\right)=T_t^t+u_tT_u^t+T_x^x+u_xT_u^x+u_{xx}T_{u_x}^x. \tag{2.69}$$

Splitting the above equation on the second derivatives of u, we have

$$u_{xx}:\quad T_{u_x}^x=-e^{u+at}\cos\sqrt{a}x, \tag{2.70}$$

$$\text{rest}:\quad T_t^t+u_tT_u^t+T_x^x+u_xT_u^x=\left(u_t-u_x^2\right)e^{u+at}\cos\sqrt{a}x. \tag{2.71}$$

Integrating equation (2.70) with respect to u_x, we obtain

$$T^x=-u_xe^{u+at}\cos\sqrt{a}x+E(t,x,u), \tag{2.72}$$

where $E(t,x,u)$ is a function depending on t, x and u. Invoking the above value of T^x in (2.71), we obtain

$$T_t^t+u_tT_u^t+\sqrt{a}u_xe^{u+at}\sin\sqrt{a}x+E_x-u_x^2e^{u+at}\cos\sqrt{a}x+u_xE_u=e^{u+at}\cos\sqrt{a}x\left(u_t-u_x^2\right).$$

Simplifying and splitting the above equation on the derivatives of u yields

$$u_t:\quad T_u^t=e^{u+at}\cos\sqrt{a}x, \tag{2.73}$$

$$u_x:\quad E_u+\sqrt{a}e^{u+at}\sin\sqrt{a}x=0, \tag{2.74}$$

$$\text{rest}:\quad T_t^t+E_x=0. \tag{2.75}$$

Integrating equation (2.64) with respect to u gives

$$T^t=e^{u+at}\cos\sqrt{a}x+F(t,x),$$

where $F(t,x)$ is an arbitrary function depending on t and x. Solving equation (2.74), we obtain

$$E=-\sqrt{a}e^{u+at}\sin\sqrt{a}x+G(t,x).$$

where $G(t,x)$ is a function of t and x. We put $F(t,x)=G(t,x)=0$ since both functions contribute to the trivial part of the conservation law and so equation (2.75) is satisfied. Therefore density T^t and flux T^x are

$$T_2^t \;=\; e^{u+at}\cos\sqrt{a}x,$$

$$T_2^x = -u_x e^{u+at}\cos\sqrt{a}x - \sqrt{a}e^{u+at}\sin\sqrt{a}x,$$

which is the conserved vector of (2.2) associated with the multiplier $\Lambda_2(t, x, u)$.

2.4 Concluding remarks

In this chapter, we computed Lie point symmetries of the potential Burgers equation (2.2). We then constructed a commutator table for these Lie point symmetries. Sequel to that, we constructed the group-invariant solutions and also obtained the travelling wave solution of the equation (2.2). Furthermore, we proceeded to derive the conservation laws for the potential Burgers equation (2.2) using the multiplier approach.

Chapter 3

Solutions and conservation laws for the modified equal-width equation

3.1 Introduction

In this chapter we study the third-order modified equal-width (MEW) equation

$$u_t + 3\alpha u^2 u_x - \beta u_{txx} = 0, \tag{3.1}$$

where α and β are non-zero real numbers. Equation (3.1) is used in handling the simulation of a single dimensional wave propagation in nonlinear media with dispersion processes [54]. Researchers have used different techniques and methods to construct travelling wave solutions of the modified equal-width equation. In [55], a dynamical system technique for integer order was used to find travelling wave solutions of MEW equation which comprises solitary, periodic waves as well as kink and anti-kink wave solutions. Furthermore, the homotopy perturbation technique in [56] employed the use of analytical approach to find the numerical solutions of (3.1).

MEW equation (3.1) was investigated by Lu et al. in [54] where the extended simple equation method and the $\exp(-\varphi(\xi))$ expansion method were employed to generate the travelling wave solutions of the equation.

Here, we find the Lie point symmetries of the third-order MEW equation and then use the symmetries to construct an optimal system of one-dimensional subalgebras. Thereafter we use the optimal system of one-dimensional subalgebras to find system reductions, and present the new group-invariant solutions of the modified equal-width equation which are the cnoidal and snoidal solutions. Furthermore, we derive the conservation laws of the MEW equation by engaging two different methods which are the multiplier method and Noether approach.

3.2 Exact solutions of (3.1) constructed on optimal system

In this section, we first compute Lie point symmetries of equation (3.1) and then utilise the symmetries to construct an optimal system of one-dimensional subalgebras. Thereafter, we use this optimal system of one-dimensional subalgebras to find symmetry reductions and group-invariant solutions of (3.1).

3.2.1 Lie point symmetries of (3.1)

The vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of equation (3.1) if

$$X^{(3)}\Delta|_{\Delta=0} = 0, \tag{3.2}$$

where

$$\Delta \equiv u_t + 3\alpha u^2 u_x - \beta u_{txx}$$

and $X^{(3)}$ is the third prolongation [13] of X defined as

$$\mathrm{X}^{(3)} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}}. \tag{3.3}$$

Here ζ_t, ζ_x, ζ_{tx} and ζ_{txx} are determined by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{txx} &= D_x(\zeta_{tx}) - u_{ttx} D_x(\tau) - u_{txx} D_x(\xi),
\end{aligned} \tag{3.4}$$

where the total derivatives D_t and D_x are defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots , \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots .
\end{aligned}$$

Expanding (3.2) we obtain

$$\left(\zeta_t + 6\alpha u u_x \eta + 3\alpha u^2 \zeta_x - \beta \zeta_{txx}\right)\big|_{u_t+3\alpha u^2 u_x - \beta u_{txx}=0} = 0,$$

which gives

$$\begin{aligned}
&\beta\xi_{tuu}u_x^3 + \beta u_t \xi_{uuu} u_x^3 - 2u^2\alpha\xi_u u_x^2 + 3\beta u_{tx}\xi_{uu}u_x^2 + \beta u_{tt}\tau_{uu}u_x^2 - \beta\eta_{tuu}u_x^2 \\
&- \beta u_t \eta_{uuu} u_x^2 + 2\beta\xi_{txu}u_x^2 + 2\beta u_t \xi_{xuu} u_x^2 + \beta u_t \tau_{tuu} u_x^2 + \beta u_t^2 \tau_{uuu} u_x^2 + 4u\alpha\eta u_x \\
&+ 2u^2\alpha\eta_u u_x - \xi_t u_x - u_t \xi_u u_x - 2u^2\alpha\xi_x u_x - 2u^2\alpha u_t \tau_u u_x - 2\beta u_{tx}\eta_{uu}u_x \\
&+ 3\beta u_{xx}\xi_{tu}u_x + 3\beta u_t u_{xx}\xi_{uu}u_x + 4\beta u_{tx}\xi_{xu}u_x + 2\beta u_{tx}\tau_{tu}u_x + 4\beta u_t u_{tx}\tau_{uu}u_x \\
&+ 2\beta u_{tt}\tau_{xu}u_x + 2\beta\tau_u u_{ttx}u_x + 3\beta\xi_u u_{txx}u_x - 2\beta\eta_{txu}u_x - 2\beta u_t \eta_{xuu}u_x + \beta\xi_{txx}u_x \\
&+ \beta u_t \xi_{xxu}u_x + 2\beta u_t \tau_{txu}u_x + 2\beta u_t^2 \tau_{xuu}u_x + 2\beta\tau_u u_{tx}^2 + \eta_t + u_t\eta_u + 2u^2\alpha\eta_x \\
&- u_t\tau_t - u_t^2\tau_u - 2u^2\alpha u_t \tau_x + \beta\tau_u u_{tt}u_{xx} + 3\beta\xi_u u_{tx}u_{xx} - \beta u_{xx}\eta_{tu} - \beta u_t u_{xx}\eta_{uu} \\
&- 2\beta u_{tx}\eta_{xu} + 2\beta u_{xx}\xi_{tx} + 2\beta u_t u_{xx}\xi_{xu} + \beta u_{tx}\xi_{xx} + \beta u_t u_{xx}\tau_{tu} + 2\beta u_{tx}\tau_{tx} \\
&+ \beta u_t^2 u_{xx}\tau_{uu} + 4\beta u_t u_{tx}\tau_{xu} + \beta u_{tt}\tau_{xx} + 2\beta\tau_x u_{ttx} - \beta\eta_u u_{txx} + 2\beta\xi_x u_{txx} \\
&+ \beta\tau_t u_{txx} + 2\beta u_t \tau_u u_{txx} + \beta\xi_t u_{xxx} + \beta u_t \xi_u u_{xxx} - \beta\eta_{txx} - \beta u_t \eta_{xxu} \\
&+ \beta u_t \tau_{txx} + \beta u_t^2 \tau_{xxu}|_{u_t+3\alpha u^2 u_x - \beta u_{txx}=0} = 0.
\end{aligned}$$

Replacing u_{xxt} by $(u_t + 2\alpha u^2 u_x)/\beta$ in the above equation, we obtain

$$\beta u_t u_x^3 \xi_{uuu} - \beta u_x^2 \eta_{tuu} - \beta u_t u_x^2 \eta_{uuu} - 2\beta u_x \eta_{uu} u_{tx} - 2\beta u_x \eta_{txu} - 2\beta u_t u_x \eta_{xuu}$$

$$
\begin{aligned}
&- \beta u_{xx}\eta_{tu} - \beta u_t\eta_{uu}u_{xx} - 2\beta u_{tx}\eta_{xu} - \beta u_t\eta_{xxu} + \beta u_x^3\xi_{tuu} + 3\beta u_x^2\xi_{uu}u_{tx} \\
&+ 2\beta u_x^2\xi_{txu} + 2\beta u_t u_x^2\xi_{xuu} + 3\beta u_x u_{xx}\xi_{tu} + 3\beta u_t u_x\xi_{uu}u_{xx} + 4\beta u_x u_{tx}\xi_{xu} \\
&+ \beta u_x\xi_{txx} + \beta u_t u_x\xi_{xxu} + 3\beta\xi_u u_{xx}u_{tx} + 2\beta u_{xx}\xi_{tx} + 2\beta u_t u_{xx}\xi_{xu} + \beta\xi_{xx}u_{tx} \\
&+ \beta\xi_t u_{xxx} + \beta u_t\xi_u u_{xxx} + \beta u_x^2 u_{tt}\tau_{uu} + \beta u_t u_x^2\tau_{tuu} + \beta u_t^2 u_x^2\tau_{uuu} + 2\beta u_x\tau_{tu}u_{tx} \\
&+ 4\beta u_t u_x\tau_{uu}u_{tx} + 2\beta u_x u_{tt}\tau_{xu} + 2\beta\tau_u u_x u_{ttx} + 2\beta u_t u_x\tau_{txu} + 2\beta u_t^2 u_x\tau_{xuu} \\
&+ 2\beta\tau_u u_{tx}^2 + \beta\tau_u u_{tt}u_{xx} + \beta u_t u_{xx}\tau_{tu} + 2\beta u_{tx}\tau_{tx} + \beta u_t^2\tau_{uu}u_{xx} + 4\beta u_t u_{tx}\tau_{xu} \\
&+ \beta u_{tt}\tau_{xx} + 2\beta\tau_x u_{ttx} + \beta u_t\tau_{txx} + \beta u_t^2\tau_{xxu} - \beta\eta_{txx} + \eta_t + 2\alpha u^2\tau_t u_x \\
&+ 2\alpha u^2 u_t\tau_u u_x - 2\alpha u^2 u_t\tau_x + u_t^2\tau_u - \xi_t u_x + 2u_t\xi_u u_x + 2u_t\xi_x + 2\alpha u^2\eta_x \\
&+ 4\alpha u^2\xi_u u_x^2 + 2\alpha u^2 u_x\xi_x + 4\alpha\eta u u_x = 0.
\end{aligned}
$$

Since the functions τ, ξ and η depend only on t, x and u we split the above determining equation on the derivatives of u and obtain the following overdetermined system of linear PDEs:

$$\tau_x = 0, \tag{3.5}$$

$$\tau_u = 0, \tag{3.6}$$

$$\xi_u = 0, \tag{3.7}$$

$$\eta_{uu} = 0, \tag{3.8}$$

$$\xi_t + \xi_x = 0, \tag{3.9}$$

$$2\xi_{tx} - \eta_{tu} = 0, \tag{3.10}$$

$$\xi_{xx} - 2\eta_{xu} = 0, \tag{3.11}$$

$$\xi_x - \beta\eta_{xxu} = 0, \tag{3.12}$$

$$\eta_t + 3\alpha u^2\eta_x - \beta\eta_{txx} = 0, \tag{3.13}$$

$$\beta\xi_{txx} - 2\beta\eta_{txu} + 3\alpha u^2\tau_t + 6\alpha u\eta - \xi_t = 0. \tag{3.14}$$

From equation (3.5) and (3.6), we have

$$\tau(t) = a(t),$$

where $a(t)$ is an arbitrary function of t. Solving equation (3.7) gives

$$\xi(t, x) = b(t, x),$$

where $b(t,x)$ is an arbitrary function of t and x. Equation (3.8) yields

$$\eta = c(t,x)u + d(t,x), \tag{3.15}$$

where $c(t,x)$ and $d(t,x)$ are arbitrary functions of t and x. Substituting the values of τ, ξ and η from the above equations into (3.14), we obtain

$$6\alpha u^2 c + 6\alpha u d - 2\beta c_{tx} + 3\alpha u^2 a' + \beta b_{txx} - b_t = 0.$$

Splitting the above equation on the powers of u, we obtain

$$u^2 : 2c(t,x) + a'(t) = 0, \tag{3.16}$$

$$u : d(t,x) = 0, \tag{3.17}$$

$$\text{rest} : \beta b_{txx} - 2\beta c_{tx} - b_t = 0. \tag{3.18}$$

From equation (3.16) we have

$$c(t,x) = -\frac{1}{2}a'(t), \tag{3.19}$$

and so equation (3.15) then becomes

$$\eta(t,x,u) = -\frac{1}{2}a'(t)u. \tag{3.20}$$

Substituting the above value of η into equation (3.12), we obtain

$$b(t,x) = e(t), \tag{3.21}$$

where $e(t)$ is an arbitrary function of t. Substituting the values of c and η in (3.19) and (3.21) respectively into equation (3.18), we have

$$e(t) = C_1,$$

where C_1 is a constant of integration. Trivially equations (3.9) and (3.11) are satisfied. Solving (3.10) we have

$$a(t) = C_2 t + C_3,$$

where C_2 and C_3 are constants of integration. Obviously from the above value of η, equation (3.13) is satisfied. Therefore the solution to the above system (3.5)–(3.14) is

$$\tau(t,x,u) = C_2 t + C_3,$$

$$\xi(t,x,u) = C_1,$$
$$\eta\,(t,x,u) = -\frac{1}{2}C_2 u.$$

Therefore the Lie point symmetries of (3.1) are given by

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

The infinitesimal generator X_3 represents scaling symmetry whereas the one-parameter groups generated by X_1 and X_2 demonstrate time and space-invariance of the MEW equation.

3.2.2 Optimal system of one-dimensional subalgebras

We now use the Lie point symmetries of (3.1) computed above to construct an optimal system of one-dimensional subalgebras in order to obtain symmetry reductions as well as group-invariant solutions of equation (3.1). Thereafter we generate adjoint representation using Lie series

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}(\mathrm{ad}X_i)^n(X_j) = X_j - \varepsilon[X_i,X_j] + \frac{\varepsilon^2}{2!}[X_i,[X_i,X_j]] - \cdots\,,$$

where ε is a real number and the commutator $[X_i, X_j]$ is defined by

$$[X_i,X_j] = X_iX_j - X_jX_i.$$

The table of commutators of Lie point symmetries of equation (3.1) and adjoint representations of the symmetry group of (3.1) on its Lie algebra are presented in Tables 1 and Table 2, respectively. Consequently, Table 1 and Table 2 are used to compute an optimal system of one-dimensional subalgebras for equation (3.1).

Table 1. Lie brackets for equation (3.1)

[,]	X_1	X_2	X_3
X_1	0	0	$2X_1$
X_2	0	0	0
X_3	$-2X_1$	0	0

Table 2. Adjoint representation of subalgebras

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$-2\varepsilon X_1 + X_3$
X_2	X_1	X_2	X_3
X_3	$e^{2\varepsilon}X_1$	X_2	X_3

Thus, following [13] and utilising Table 1 and Table 2, we obtain an optimal system of one-dimensional subalgebras which is given by

$$\{X_1 + cX_2, X_3 + aX_2\},$$

where c and a are arbitrary constants.

3.2.3 Symmetry reductions and solutions of (3.1)

We now employ the optimal system of one-dimensional subalgebras obtained in the previous subsection and find group-invariant solutions and symmetry reductions for equation (3.1).

Consider the first member of the optimal system of one-dimensional subalgebras, namely

$$X = X_1 + cX_2.$$

The symmetry X yields the two invariants

$$\xi = x - ct \quad \text{and} \quad U = u,$$

which give the group-invariant solution $U = U(\xi)$. Using ξ as our new independent variable, (3.1) is transformed into the nonlinear ordinary differential equation (NLODE)

$$c\beta U'''(\xi) + 3\alpha U^2(\xi)U'(\xi) - cU'(\xi) = 0. \tag{3.22}$$

We now use the extended Jacobi elliptic function expansion method [?] to obtain closed form solutions of (3.1). We assume that the solutions of the third-order

NLODE (3.22) can be expressed in the form

$$U(\xi) = \sum_{i=-M}^{M} A_i H(\xi)^i, \tag{3.23}$$

where M is a positive integer obtained by the balancing procedure. Here $H(\xi)$ satisfies the first-order ODE (1.36) or (1.38). We recall that

$$H(\xi) = \mathrm{cn}(\xi|\omega), \tag{3.24}$$

the Jacobi cosine-amplitude function, is a solution to (1.36), whereas the Jacobi sine-amplitude function

$$H(\xi) = \mathrm{sn}(\xi|w) \tag{3.25}$$

is a solution to (1.38). Here ω is a parameter such that $0 \leq \omega \leq 1$ [28, 29].

We note that when $\omega \to 1$, then $\mathrm{cn}(\xi|\omega) \to \mathrm{sech}(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \tanh(\xi)$. Also, when $\omega \to 0$, then $\mathrm{cn}(\xi|\omega) \to \cos(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \sin(\xi)$.

Cnoidal wave solution

Considering the NLODE (3.22), the balancing procedure yields $M = 1$, thus (3.23) becomes

$$U(\xi) = A_{-1}H^{-1}(\xi) + A_0 + A_1 H(\xi). \tag{3.26}$$

We now substitute the value of U from (3.26) into (3.22) and utilise (1.36) to obtain

$$\begin{aligned}
&H(\xi)^4\beta\, cA_{-1} - H(\xi)^6\beta\, cA_1 + H(\xi)^4\beta\, cA_1 - 7\, H(\xi)^2\beta\, cA_{-1} - 12\,\beta\, c\omega\, A_{-1} \\
&+ 6\,\beta\, c\omega^2 A_{-1} + 3\, H(\xi)^{10}\alpha\,\omega\, {A_1}^3 - 6\, H(\xi)^8\alpha\,\omega\, {A_1}^3 - H(\xi)^8 c\omega\, A_1 \\
&+ 6\, H(\xi)^7\alpha\, A_0 {A_1}^2 + 3\, H(\xi)^6\alpha\,\omega\, {A_1}^3 - 3\,\alpha\,\omega\, {A_{-1}}^3 - 3\, H(\xi)^6\alpha\, {A_1}^3 \\
&+ 3\, H(\xi)^8\alpha\, {A_1}^3 - H(\xi)^6 cA_1 + H(\xi)^4 cA_1 + H(\xi)^4 cA_{-1} - 3\, H(\xi)^2\alpha\, {A_{-1}}^3 \\
&- H(\xi)^2 cA_{-1} - 10\, H(\xi)^6\beta\, c\omega^2 A_1 - 2\, H(\xi)^6\beta\, c\omega^2 A_{-1} - 7\, H(\xi)^8\beta\, c\omega\, A_1 \\
&+ 14\, H(\xi)^8\beta\, c\omega^2 A_1 - 6\, H(\xi)^{10}\beta\, c\omega^2 A_1 + 21\, H(\xi)^2\beta\, c\omega\, A_{-1} \\
&- 14\, H(\xi)^2\beta\, c\omega^2 A_{-1} - 3\, H(\xi)^4\beta\, c\omega\, A_1 - 10\, H(\xi)^4\beta\, c\omega\, A_{-1} \\
&+ 2\, H(\xi)^4\beta\, c\omega^2 A_1 + 10\, H(\xi)^4\beta\, c\omega^2 A_{-1} + 10\, H(\xi)^6\beta\, c\omega\, A_1 + H(\xi)^6\beta\, c\omega\, A_{-1}
\end{aligned}$$

$$+ 3\,H(\xi)^8\alpha\,\omega\,{A_0}^2A_1 + 3\,H(\xi)^8\alpha\,\omega\,A_{-1}{A_1}^2 + 6\,H(\xi)^9\alpha\,\omega\,A_0{A_1}^2$$
$$- 6\,H(\xi)\alpha\,\omega\,{A_{-1}}^2A_0 - 3\,H(\xi)^2\alpha\,\omega\,A_{-1}{A_0}^2 - 3\,H(\xi)^2\alpha\,\omega\,{A_{-1}}^2A_1$$
$$+ 12\,H(\xi)^3\alpha\,\omega\,{A_{-1}}^2A_0 + 3\,H(\xi)^4\alpha\,\omega\,{A_0}^2A_1 + 3\,H(\xi)^4\alpha\,\omega\,A_{-1}{A_1}^2$$
$$+ 6\,H(\xi)^4\alpha\,\omega\,A_{-1}{A_0}^2 + 6\,H(\xi)^4\alpha\,\omega\,{A_{-1}}^2A_1 + 6\,H(\xi)^5\alpha\,\omega\,A_0{A_1}^2$$
$$- 6\,H(\xi)^5\alpha\,\omega\,{A_{-1}}^2A_0 - 6\,H(\xi)^6\alpha\,\omega\,{A_0}^2A_1 - 6\,H(\xi)^6\alpha\,\omega\,A_{-1}{A_1}^2$$
$$- 3\,H(\xi)^6\alpha\,\omega\,A_{-1}{A_0}^2 - 3\,H(\xi)^6\alpha\,\omega\,{A_{-1}}^2A_1 - 12\,H(\xi)^7\alpha\,\omega\,A_0{A_1}^2$$
$$+ 6\,H(\xi)\alpha\,{A_{-1}}^2A_0 + 3\,H(\xi)^2\alpha\,A_{-1}{A_0}^2 + H(\xi)^2c\omega\,A_{-1} + 3\,H(\xi)^2\alpha\,{A_{-1}}^2A_1$$
$$+ 6\,H(\xi)^2\alpha\,\omega\,{A_{-1}}^3 - H(\xi)^4c\omega\,A_1 - 6\,H(\xi)^3\alpha\,{A_{-1}}^2A_0 - 2\,H(\xi)^4c\omega\,A_{-1}$$
$$- 3\,H(\xi)^4\alpha\,A_{-1}{A_1}^2 - 3\,H(\xi)^4\alpha\,{A_0}^2A_1 - 3\,H(\xi)^4\alpha\,{A_{-1}}^2A_1 - 3\,H(\xi)^4\alpha\,A_{-1}{A_0}^2$$
$$+ 2\,H(\xi)^6c\omega\,A_1 - 6\,H(\xi)^5\alpha\,A_0{A_1}^2 - 3\,H(\xi)^4\alpha\,\omega\,{A_{-1}}^3 + 3\,H(\xi)^6\alpha\,A_{-1}{A_1}^2$$
$$+ 3\,H(\xi)^6\alpha\,{A_0}^2A_1 + H(\xi)^6c\omega\,A_{-1} + 6\,\beta\,cA_{-1} + 3\,\alpha\,{A_{-1}}^3 = 0.$$

The above equation can be separated on like powers of $H(\xi)$ to obtain an overdetermined system of eleven algebraic equations

$$A_0{A_1}^2 = 0,$$
$${A_{-1}}^2A_0 - \omega\,{A_{-1}}^2A_0 = 0,$$
$$2\,\omega\,{A_{-1}}^2A_0 - {A_{-1}}^2A_0 = 0,$$
$$A_0{A_1}^2 - 2\omega\,A_0{A_1}^2 = 0,$$
$$\alpha\,{A_1}^3 - 2\,\beta\,c\omega A_1 = 0,$$
$$\omega\,A_0{A_1}^2 - \omega\,{A_{-1}}^2A_0 - A_0{A_1}^2 = 0,$$
$$2\,\beta\,c\omega^2A_{-1} - \alpha\,\omega\,{A_{-1}}^3 + \alpha\,{A_{-1}}^3 - 4\,\beta\,c\omega\,A_{-1} + 2\,\beta\,cA_{-1} = 0,$$
$$3\,\alpha\,\omega\,A_{-1}{A_1}^2 + 3\,\alpha\,\omega\,{A_0}^2A_1 - 6\,\alpha\,\omega\,{A_1}^3 + 14\,\beta\,c\omega^2A_1 + 3\,\alpha\,{A_1}^3 - 7\,\beta\,c\omega\,A_1$$
$$- c\omega\,A_1 = 0,$$
$$6\,\alpha\,\omega\,{A_{-1}}^3 - 3\,\alpha\,\omega\,{A_{-1}}^2A_1 - 3\,\alpha\,\omega\,A_{-1}{A_0}^2 - 14\,\beta\,c\omega^2A_{-1} - 3\,\alpha\,{A_{-1}}^3$$
$$+ 3\,\alpha\,{A_{-1}}^2A_1 + 3\,\alpha\,A_{-1}{A_0}^2 + 21\,\beta\,c\omega\,A_{-1} - 7\,\beta\,cA_{-1} + c\omega\,A_{-1} - cA_{-1} = 0,$$
$$3\,\alpha\,\omega\,{A_1}^3 - 3\,\alpha\,\omega\,{A_{-1}}^2A_1 - 3\,\alpha\,\omega\,A_{-1}{A_0}^2 - 6\,\alpha\,\omega\,A_{-1}{A_1}^2 - 6\,\alpha\,\omega\,{A_0}^2A_1$$
$$- 2\,\beta\,c\omega^2A_{-1} + 3\,\alpha\,A_{-1}{A_1}^2 - 10\,\beta\,c\omega^2A_1 + 3\,\alpha\,{A_0}^2A_1 - 3\,\alpha\,{A_1}^3 + \beta\,c\omega\,A_{-1}$$

$$+ 10\,\beta\, c\omega\, A_1 - \beta\, cA_1 + c\omega\, A_{-1} + 2\, c\omega\, A_1 - cA_1 = 0,$$

$$6\,\alpha\,\omega\, A_{-1}{}^2 A_1 - 3\,\alpha\,\omega\, A_{-1}{}^3 + 6\,\alpha\,\omega\, A_{-1} A_0{}^2 + 3\,\alpha\,\omega\, A_{-1} A_1{}^2 + 3\,\alpha\,\omega\, A_0{}^2 A_1$$

$$+ 10\,\beta\, c\omega^2 A_{-1} + 2\,\beta\, c\omega^2 A_1 - 3\,\alpha\, A_{-1}{}^2 A_1 - 3\,\alpha\, A_{-1} A_0{}^2 - 3\,\alpha\, A_{-1} A_1{}^2 - 3\,\alpha\, A_0{}^2 A_1$$

$$- 10\,\beta\, c\omega\, A_{-1} - 3\,\beta\, c\omega\, A_1 + \beta\, cA_{-1} + \beta\, cA_1 - 2\, c\omega\, A_{-1} - c\omega\, A_1 + cA_{-1} + cA_1 = 0.$$

Solving the above system we obtain

$$\omega = \frac{8\beta + 3k - 1}{16\beta}, \quad A_0 = 0, \quad A_1 = \pm\sqrt{\frac{c(3k + 8\beta - 1)}{8\alpha}}, \quad A_{-1} = -\frac{3\beta \pm k}{\beta + 1} A_1$$

with $k = \sqrt{8\beta^2 + 1}$.

Thus, the solution of equation (3.1) is

$$u(t, x) = \pm\sqrt{\frac{c(3k + 8\beta - 1)}{8\alpha}} \left\{ \mathrm{cn}\,(\xi\,|\omega\,) - \left(\frac{3\beta \pm k}{\beta + 1}\right) \mathrm{nc}\,(\xi\,|\omega\,) \right\}.$$

Snoidal wave solutions

We now obtain snoidal wave solutions for the equation (3.1). We recall that the balancing procedure yields $M = 1$, thus substituting the value of U from (3.26) into (3.22) and making use of (1.37) we obtain the determining equation

$$3\, H(\xi)^6 \alpha\, A_1{}^3 - H(\xi)^4 cA_1 + H(\xi)^2 cA_{-1} + 3\, H(\xi)^2 \alpha\, A_{-1}{}^3 + 3\, H(\xi)^4 \alpha\, A_{-1}{}^2 A_1$$

$$+ 3\, H(\xi)^4 \alpha\, A_{-1} A_0{}^2 + H(\xi)^6 \beta\, cA_1 - H(\xi)^4 \beta\, cA_{-1} - H(\xi)^4 \beta\, cA_1$$

$$+ 7\, H(\xi)^2 \beta\, cA_{-1} + 3\, H(\xi)^4 \alpha\, A_{-1} A_1{}^2 + 3\, H(\xi)^4 \alpha\, A_0{}^2 A_1 + 3\, H(\xi)^{10} \alpha\,\omega\, A_1{}^3$$

$$- 3\, H(\xi)^8 \alpha\,\omega\, A_1{}^3 - H(\xi)^8 c\omega\, A_1 - 6\, H(\xi)^7 \alpha\, A_0 A_1{}^2 - 3\, H(\xi)^6 \alpha\, A_{-1} A_1{}^2$$

$$- 3\, H(\xi)^6 \alpha\, A_0{}^2 A_1 + H(\xi)^6 c\omega\, A_{-1} + H(\xi)^6 c\omega\, A_1 + 6\, H(\xi)^5 \alpha\, A_0 A_1{}^2$$

$$- 3\, H(\xi)^4 \alpha\,\omega\, A_{-1}{}^3 - 3\,\alpha\, A_{-1}{}^3 - 6\, H(\xi)^7 \alpha\,\omega\, A_0 A_1{}^2 - 3\, H(\xi)^6 \alpha\,\omega\, A_{-1}{}^2 A_1$$

$$- 3\, H(\xi)^6 \alpha\,\omega\, A_{-1} A_0{}^2 - 3\, H(\xi)^6 \alpha\,\omega\, A_{-1} A_1{}^2 - 3\, H(\xi)^6 \alpha\,\omega\, A_0{}^2 A_1 - 6\, H(\xi)^5 \alpha\,\omega\, A_{-1}{}^2 A_0$$

$$+ 3\, H(\xi)^4 \alpha\,\omega\, A_{-1}{}^2 A_1 + 3\, H(\xi)^4 \alpha\,\omega\, A_{-1} A_0{}^2 + 6\, H(\xi)^9 \alpha\,\omega\, A_0 A_1{}^2 + 3\, H(\xi)^8 \alpha\,\omega\, A_{-1} A_1{}^2$$

$$+ 3\, H(\xi)^8 \alpha\,\omega\, A_0{}^2 A_1 - 3\, H(\xi)^8 \alpha\, A_1{}^3 + H(\xi)^6 cA_1 - H(\xi)^4 cA_{-1}$$

$$- 6\,\beta\, cA_{-1} + 7\, H(\xi)^2 \beta\, c\omega\, A_{-1} - H(\xi)^4 \beta\, c\omega\, A_1 - 8\, H\,(\xi)^4\, \beta\, c\omega\, A_{-1}$$

$$- H(\xi)^4 \beta\, c\omega^2 A_{-1} + 8\, H(\xi)^6 \beta\, c\omega\, A_1 + H(\xi)^6 \beta\, c\omega\, A_{-1} + H(\xi)^6 \beta\, c\omega^2 A_1$$

$$+ H(\xi)^6\beta\, c\omega^2 A_{-1} - 7\, H(\xi)^8\beta\, c\omega\, A_1 - 7\, H(\xi)^8\beta\, c\omega^2 A_1 - 3\, H(\xi)^2\alpha\, A_{-1}{A_0}^2$$

$$- 6\, H\left(\xi\right)\alpha\, {A_{-1}}^2 A_0 + 3\, H(\xi)^2\alpha\,\omega\, {A_{-1}}^3 - 3\, H(\xi)^2\alpha\, {A_{-1}}^2 A_1 - H(\xi)^4 c\omega\, A_{-1}$$

$$+ 6\, H(\xi)^3\alpha\, {A_{-1}}^2 A_0 + 6\, H(\xi)^{10}\beta\, c\omega^2 A_1 + 6\, H(\xi)^3\alpha\,\omega\, {A_{-1}}^2 A_0 = 0.$$

Splitting on powers of $H(\xi)$ yields the following overdetermined system of algebraic equations:

$${A_{-1}}^2 A_0 = 0,$$

$$A_0 {A_1}^2 = 0,$$

$$\alpha\, {A_{-1}}^3 + 2\,\beta\, cA_{-1} = 0,$$

$$\omega\, {A_{-1}}^2 A_0 + {A_{-1}}^2 A_0 = 0,$$

$$\omega\, A_0 {A_1}^2 + A_0 {A_1}^2 = 0,$$

$$\alpha\, {A_1}^3 + 2\,\beta\, c\omega A_1 = 0,$$

$$\omega\, {A_{-1}}^2 A_0 - A_0 {A_1}^2 = 0,$$

$$3\,\alpha\,\omega\, A_{-1}{A_1}^2 + 3\,\alpha\,\omega\, {A_0}^2 A_1 - 3\,\alpha\,\omega\, {A_1}^3 - 7\,\beta\, c\omega^2 A_1 - 3\,\alpha\, {A_1}^3$$

$$- 7\,\beta\, c\omega\, A_1 - c\omega\, A_1 = 0,$$

$$3\,\alpha\,\omega\, {A_{-1}}^3 + 3\,\alpha\, {A_{-1}}^3 - 3\,\alpha\, {A_{-1}}^2 A_1 - 3\,\alpha\, A_{-1}{A_0}^2 + 7\,\beta\, c\omega\, A_{-1}$$

$$+ 7\,\beta\, cA_{-1} + cA_{-1} = 0,$$

$$\beta\, c\omega^2 A_{-1} - 3\,\alpha\,\omega\, {A_{-1}}^2 A_1 - 3\,\alpha\,\omega\, A_{-1}{A_0}^2 - 3\,\alpha\,\omega\, A_{-1}{A_1}^2 - 3\,\alpha\,\omega\, {A_0}^2 A_1$$

$$+ \beta\, c\omega^2 A_1 + 3\,\alpha\, {A_1}^3 - 3\,\alpha\, A_{-1}{A_1}^2 - 3\,\alpha\, {A_0}^2 A_1 + \beta\, c\omega\, A_{-1} + 8\,\beta\, c\omega\, A_1$$

$$+ \beta\, cA_1 + c\omega\, A_{-1} + c\omega\, A_1 + cA_1 = 0,$$

$$3\,\alpha\,\omega\, {A_{-1}}^2 A_1 - 3\,\alpha\,\omega\, {A_{-1}}^3 + 3\,\alpha\,\omega\, A_{-1}{A_0}^2 - \beta\, c\omega^2 A_{-1}$$

$$+ 3\,\alpha\, {A_{-1}}^2 A_1 + 3\,\alpha\, A_{-1}{A_0}^2 + 3\,\alpha\, A_{-1}{A_1}^2 + 3\,\alpha\, {A_0}^2 A_1 - 8\,\beta\, c\omega\, A_{-1}$$

$$- \beta\, c\omega\, A_1 - \beta\, cA_{-1} - \beta\, cA_1 - c\omega\, A_{-1} - cA_{-1} - cA_1 = 0.$$

Solving the above equations, the values of β, A_{-1}, A_0 and A_1 are

$$\beta = -\frac{1}{1+\omega}, \quad A_{-1} = A_0 = 0, \quad A_1 = \pm\sqrt{\frac{2c(\beta+1)}{\alpha}}. \tag{3.27}$$

Reverting to original variables we obtain

$$u(t,x) = \pm\sqrt{\frac{2c(\beta+1)}{\alpha}}\,\mathrm{sn}\left(\xi\,|\omega\right). \tag{3.28}$$

We now take into consideration the second operator $X_3 + aX_2$ of the optimal system. This symmetry operator produces two invariants $J_1 = e^x t^{-a/2}$ and $J_2 = ut^{1/2}$. Thus $J_2 = f(J_1)$ gives a group-invariant solution to (3.1). That is

$$u = t^{-1/2} f(e^x t^{-a/2}). \tag{3.29}$$

Substituting the above value of u in (3.1), we obtain the third-order nonlinear ODE

$$a\beta z^3 f'''(z) + \beta(3a+1)z^2 f''(z) + (a\beta - a + \beta)zf'(z) + 6azf^2(z)f'(z) - f(z) = 0,$$

where $z = e^x t^{-a/2}$.

3.3 Conservation laws of the MEW equation

In the section we derive conservation laws for (3.1) by employing two different methods, namely the multiplier method and Noether's approach.

3.3.1 Conservation laws of (3.1) using the multiplier approach

We look for the zeroth-order multiplier $\Lambda = \Lambda(t, x, u)$. Thus, the determining equation for this multiplier is

$$\frac{\delta}{\delta u}\Big\{\Lambda(t,x,u)\Big(u_t + 3\alpha u^2 u_x - \beta u_{txx}\Big)\Big\} = 0, \tag{3.30}$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_t D_x^2 \frac{\partial}{\partial u_{txx}} \tag{3.31}$$

and the total derivatives D_t and D_x are as defined in (1.4). The above equation yields

$$u_t\Lambda_u + 3\alpha u^2 u_x \Lambda_u - \beta u_{txx}\Lambda_u + 6\alpha u u_x \Lambda - D_t(\Lambda) - D_x(3\alpha u^2 \Lambda) + \beta D_x^2 D_t(\Lambda) = 0,$$

which on expanding gives

$$u_t\Lambda_u + 3\alpha u^2 u_x \Lambda_u - \beta u_{xxt}\Lambda_u + 6\alpha u u_x \Lambda - \Lambda_t - u_t\Lambda_u - 3\alpha u^2 \Lambda_x - 6\alpha u u_x \Lambda$$

$$- 3\alpha u^2 u_x \Lambda_u + \beta \Lambda_{txx} + \beta u_t \Lambda_{uxx} + \beta u_x \Lambda_{tux} + \beta u_t u_x \Lambda_{uux} + \beta u_{tx} \Lambda_{ux} + \beta u_x \Lambda_{tux}$$
$$+ \beta u_t u_x \Lambda_{uux} + \beta u_x^2 \Lambda_{tuu} + \beta u_t u_x^2 \Lambda_{uuu} + \beta u_{tx} u_x \Lambda_{uu} + \beta u_{tx} \Lambda_{ux} + \beta u_{tx} u_x \Lambda_{uu}$$
$$+ \beta u_{xx} \Lambda_{tu} + \beta u_{xx} u_t \Lambda_{uu} + \beta u_{xxt} \Lambda_u = 0.$$

Splitting the above equation on the derivatives of u, we have

$$\Lambda_{uu} = 0, \tag{3.32}$$
$$\Lambda_{ux} = 0, \tag{3.33}$$
$$\Lambda_{tu} = 0, \tag{3.34}$$
$$\beta \Lambda_{txx} - 3\alpha u^2 \Lambda_x - \Lambda_t = 0. \tag{3.35}$$

Equation (3.32) yields

$$\Lambda = a(t,x)u + b(t,x),$$

where $a(t,x)$ and $b(t,x)$ are arbitrary functions of t and x. From equation (3.33), we obtain

$$a(t,x) = c(t),$$

where $c(t)$ is an arbitrary function of t. Equation (3.34) gives

$$c(t) = C_1,$$

where C_1 is an arbitrary constant of integration. Thus

$$\Lambda = C_1 u + b(t,x).$$

Invoking the above value of Λ into (3.35), we have

$$\beta b_{txx} - 3\alpha u^2 b_x - b_t = 0.$$

Splitting the above equation on u, we get

$$u^2 : b_x = 0, \tag{3.36}$$
$$\text{rest} : \beta b_{txx} - b_t = 0. \tag{3.37}$$

Solving (3.36) we have

$$b(t,x) = d(t),$$

where $d(t)$ is a function of t. From (3.37) we obtain

$$d(t) = C_2,$$

where C_2 is an arbitrary constant of integration. Thus the zeroth-order multiplier Λ becomes

$$\Lambda(t, x, u) = C_1 u + C_2.$$

Therefore we have two multipliers for the MEW equation (3.1) which are

$$\Lambda_1(t, x, u) = u \tag{3.38}$$

and

$$\Lambda_2(t, x, u) = 1. \tag{3.39}$$

We first generate conservation laws of (3.1) associated with the multiplier $\Lambda_1(t, x, u)$. Therefore we have

$$\Lambda_1(t, x, u)\{u_t + 3\alpha u^2 u_x - \beta u_{txx}\} = D_t T^t + D_x T^x \tag{3.40}$$

in which the density and flux are defined respectively as

$$T^t = T^t(t, x, u, u_x) \text{ and } T^x = T^x(t, x, u, u_x, u_{tx}). \tag{3.41}$$

Thus equation (3.40) becomes

$$u(u_t + 3\alpha u^2 u_x - \beta u_{txx}) = D_t T^t + D_x T^x.$$

The above equation yields

$$u\{u_t + 3\alpha u^2 u_x - \beta u_{txx}\} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{txx} T^x_{u_{tx}}.$$

Splitting the above equation on the third derivatives of u, we have

$$u_{txx} : T^x_{u_{tx}} = -\beta u, \tag{3.42}$$

$$\text{rest} : T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} = u(u_t + 3\alpha u^2 u_x). \tag{3.43}$$

Integrating (3.42) with respect to u_{tx} gives

$$T^x = -\beta u u_{tx} + G(t, x, u, u_x),$$

where $G(t,x,u,u_x)$ is a function depending on t, x, u and u_x. Substituting the above value of T^x in (3.43), we have

$$T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + G_x - \beta u_x u_{tx} + u_x G_u + u_{xx} G_{u_x} = u(u_t + 3\alpha u^2 u_x). \quad (3.44)$$

Splitting the above equation on the second derivatives of u, we have

$$u_{tx} : T^t_{u_x} - \beta u_x = 0, \quad (3.45)$$

$$u_{xx} : G_{u_x} = 0, \quad (3.46)$$

$$\text{rest} : T^t_t + u_t T^t_u + G_x + u_x G_u = u(u_t + 3\alpha u^2 u_x). \quad (3.47)$$

Integration of equation (3.45) with respect to u_x yields

$$T^t = \frac{1}{2}\beta u_x^2 + H(t,x,u),$$

where $H(t,x,u)$ is a function depending on t, x and u and equation (3.46) produces

$$G = L(t,x,u),$$

where $L(t,x,u)$ is an arbitrary function of t, x and u. Equation (3.47), now becomes

$$H_t + u_t H_u + L_x + u_x L_u = uu_t + 3\alpha u^3 u_x.$$

Splitting the above equation over the derivatives of u, we obtain

$$u_t : H_u = u, \quad (3.48)$$

$$u_x : L_u = 3\alpha u^3, \quad (3.49)$$

$$\text{rest} : H_t + L_x = 0. \quad (3.50)$$

Equation (3.48) gives

$$H = \frac{1}{2}u^2 + M(t,x), \quad (3.51)$$

where $M(t,x)$ is an arbitrary function of depending on t and x. Solving equation (3.49), we obtain

$$L = \frac{3}{4}\alpha u^4 + K(t,x), \quad (3.52)$$

where $K(t,x)$ is a function dependent of t and x. We set $K(t,x) = M(t,x) = 0$ since they contribute to the trivial part of the conserved vector, values of H and L respectively become

$$H = \frac{1}{2}u^2 \quad \text{and} \quad L = \frac{3}{4}\alpha u^4.$$

Thus, the conservation law of (3.1) associated with $\Lambda_1(t,x,u)$ is

$$\begin{aligned} T_1^t &= \frac{1}{2}u^2 + \frac{1}{2}\beta u_x^2, \\ T_1^x &= \frac{3}{4}\alpha u^4 - \beta u u_{tx}. \end{aligned}$$

Next, we compute the conservation law of (3.1) related to the second multiplier $\Lambda_2(t,x,u)$. Thus we consider

$$\Lambda_2(t,x,u)\{u_t + 3\alpha u^2 u_x - \beta u_{txx}\} = D_t T^t + D_x T^x, \tag{3.53}$$

where the density T^t and the flux T^x are as defined in (3.41). Therefore, equation (3.53) becomes

$$u_t + 3\alpha u^2 u_x - \beta u_{txx} = D_t T^t + D_x T^x.$$

Expanding the above equation produces

$$u_t + 3\alpha u^2 u_x - \beta u_{txx} = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x + u_{txx} T_{u_{tx}}^x.$$

Splitting the above equation on the third derivatives of u, we obtain

$$u_{txx} : T_{u_{tx}}^x = -\beta, \tag{3.54}$$

$$\text{rest} : T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x = u_t + 3\alpha u^2 u_x. \tag{3.55}$$

Integration of equation (3.54) with respect to u_{tx} gives

$$T^x = -\beta u_{tx} + A(t,x,u,u_x),$$

where $A(t,x,u,u_x)$ is an arbitrary function depending on t, x, u and u_x. Thus equation (3.55) now becomes

$$T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + A_x + u_x A_u + u_{xx} A_{u_x} = u_t + 3\alpha u^2 u_x.$$

Splitting the above equation over the second derivatives of u gives

$$u_{tx} : T^t_{u_x} = 0, \tag{3.56}$$

$$u_{xx} : A_{u_x} = 0, \tag{3.57}$$

$$\text{rest} : T^t_t + u_t T^t_u + A_x + u_x A_u = u_t + 3\alpha u^2 u_x. \tag{3.58}$$

Solving equations (3.56) gives the density

$$T^t = B(t, x, u),$$

where $B(t, x, u)$ is an arbitrary function of t, x and u. Equation (3.57) then becomes

$$A = A(t, x, u),$$

where $A(t, x, u)$ is an arbitrary function depending on t, x and u. Invoking the above values of T^t and A in equation (3.58), we have

$$B_t + u_t B_u + A_x + u_x A_u = u_t + 3\alpha u^2 u_x.$$

Splitting the above equation over the derivatives of u gives

$$u_t : B_u = 1, \tag{3.59}$$

$$u_x : A_u = 3\alpha u^2, \tag{3.60}$$

$$\text{rest} : B_t + A_x = 0. \tag{3.61}$$

Integrating equation (3.59) with respect to u, we have

$$B = u + Q(t, x),$$

where $Q(t, x)$ is a function of t and x. Equation (3.60) then yields

$$A = \alpha u^3 + R(t, x),$$

where $R(t, x)$ is an arbitrary function of t and x. We take $Q(t, x) = R(t, x) = 0$ as they both contribute to the trivial part of the conserved vector and so B and A become respectively

$$B = u \quad \text{and} \quad A = \alpha u^3.$$

Substituting back the above values of A and B, we have the conservation law of equation (3.1) related to $\Lambda_2(t,x,u)$ as stated below

$$\begin{aligned} T_2^t &= u, \\ T_2^x &= \alpha u^3 - \beta u_{tx}. \end{aligned}$$

Remark: It should be noted that the multiplier $\Lambda_2 = 1$ tells us that the MEW equation is itself a conservation law.

3.3.2 Conservation laws of (3.1) using the Noether theorem

In this subsection we derive the conservation laws for the modified equal-width equation (3.1) using the Noether theorem. This equation as it is, does not have a Lagrangian. In order to apply Noether's theorem we transform equation (3.1) into a fourth-order equation which has a Lagrangian. Thus using the transformation $u = V_x$, equation (3.1) becomes

$$V_{tx} + 3\alpha V_x^2 V_{xx} - \beta V_{txxx} = 0. \qquad (3.62)$$

It can readily be verified that the second-order Lagrangian for equation (3.62) is given by

$$\mathcal{L} = -\frac{1}{2}V_x V_t - \frac{1}{4}\alpha V_x^4 - \frac{1}{2}\beta V_{xx} V_{tx} \qquad (3.63)$$

because $\delta\mathcal{L}/\delta V = 0$ on (3.62). Here $\delta/\delta V$ is the Euler-Lagrange operator defined as

$$\frac{\delta\mathcal{L}}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_t^2\frac{\partial}{\partial V_{tt}} + D_x^2\frac{\partial}{\partial V_{xx}} + D_t D_x\frac{\partial}{\partial V_{tx}} - \cdots, \qquad (3.64)$$

where the total derivatives D_t, D_x are as defined in (1.4).

Consider the vector field

$$X = \tau(t,x,V)\frac{\partial}{\partial t} + \xi(t,x,V)\frac{\partial}{\partial x} + \eta(t,x,V)\frac{\partial}{\partial V}, \qquad (3.65)$$

where τ, ξ and η depend on t, x and V. To determine the Noether symmetries X of (3.62) we insert the value of $\mathcal{L}$ from (3.63) in

$$X^{[2]}(\mathcal{L}) + \mathcal{L}[D_t(\tau) + D_x(\xi)] = D_t(B^t) + D_x(B^x), \qquad (3.66)$$

where $B^t = B^t(t,x,V)$ and $B^x = B^x(t,x,V)$ are the gauge terms and $X^{[2]}$ is the second prolongation of X defined as

$$X^{[2]} = X + \zeta_t \frac{\partial}{\partial V_t} + \zeta_x \frac{\partial}{\partial V_x} + \zeta_{tt} \frac{\partial}{\partial V_{tt}} + \zeta_{xx} \frac{\partial}{\partial V_{xx}} + \zeta_{tx} \frac{\partial}{\partial V_{tx}} \tag{3.67}$$

with ζ_t and ζ_x defined in (1.15) as well as ζ_{tt}, ζ_{xx} and ζ_{tx} given in (1.16). Equation (3.66) becomes

$$-\frac{V_x}{2}\zeta_t - \frac{V_t}{2}\zeta_x - \alpha V_x^3 \zeta_x - \frac{\beta}{2} V_{xx}\zeta_{tx} - \frac{\beta}{2} V_{tx}\zeta_{xx} = B_t^t + B_x^x + V_t B_u^t + V_x B_u^x. \tag{3.68}$$

Expansion of the above equation gives

$$\begin{aligned}
&-\frac{1}{2}\eta_t V_x - \frac{1}{2}\eta_V V_t V_x + \frac{1}{2}\tau_t V_t V_x + \frac{1}{2}\tau_V V_t^2 V_x + \frac{1}{2}\xi_t V_x^2 + \frac{1}{2}\xi_V V_t V_x^2 - \frac{1}{2}\eta_x V_t \\
&-\frac{1}{2}\eta_V V_t V_x + \frac{1}{2}\tau_x V_t^2 + \frac{1}{2}\tau_V V_t^2 V_x + \frac{1}{2}\xi_x V_t V_x + \frac{1}{2}\xi_V V_t V_x^2 - \alpha\eta_x V_x^3 - \alpha\eta_V V_x^4 \\
&+\alpha\tau_x V_t V_x^3 + \alpha\tau_V V_t V_x^4 + \alpha\xi_x V_x^4 + \alpha\xi_V V_x^5 - \frac{1}{2}\beta\eta_{tx} V_{xx} - \frac{1}{2}\beta\eta_{tV} V_x V_{xx} \\
&-\frac{1}{2}\beta\eta_{xV} V_t V_{xx} - \frac{1}{2}\beta\eta_V V_{tx} V_{xx} - \frac{1}{2}\beta\eta_{VV} V_t V_x V_{xx} + \frac{1}{2}\beta\tau_t V_{tx} V_{xx} + \frac{1}{2}\beta\xi_x V_{tx} V_{xx} \\
&+\frac{1}{2}\beta\tau_{xt} V_t V_{xx} + \frac{1}{2}\beta\tau_x V_{tt} V_{xx} + \frac{1}{2}\beta\tau_{tV} V_t V_x V_{xx} + \frac{1}{2}\beta\xi_{xV} V_t V_x V_{xx} + \frac{1}{2}\beta\tau_{xV} V_t^2 V_{xx} \\
&+\beta\tau_V V_t V_{tx} V_{xx} + \frac{1}{2}\beta\tau_V V_x V_{tt} V_{xx} + \frac{1}{2}\beta\tau_{VV} V_t^2 V_x V_{xx} + \frac{1}{2}\beta\xi_{xt} V_x V_{xx} + \frac{1}{2}\beta\xi_t V_{xx}^2 \\
&+\frac{1}{2}\beta\xi_{tV} V_x^2 V_{xx} + \beta\xi_V V_x V_{xt} V_{xx} + \frac{1}{2}\beta\xi_V V_t V_{xx}^2 + \frac{1}{2}\beta\xi_{VV} V_t V_x^2 V_{xx} - \frac{1}{2}\beta\eta_{xx} V_{tx} \\
&-\beta\eta_{xV} V_x V_{tx} - \frac{1}{2}\beta\eta_V V_{tx} V_{xx} - \frac{1}{2}\beta\eta_{VV} V_x^2 V_{tx} + \beta\xi_x V_{tx} V_{xx} + \frac{1}{2}\beta\xi_{xx}^1 V_x V_{tx} \\
&+\beta\xi_{xV} V_x^2 V_{tx} + \frac{3}{2}\beta\xi_V V_x V_{tx} V_{xx} + \frac{1}{2}\beta\xi_{VV} V_x^3 V_{tx} + \beta\tau_x V_{tx}^2 + \frac{1}{2}\beta\tau_{xx} V_t V_{tx} \\
&+\beta\tau_{xV} V_t V_x V_{tx} + \frac{1}{2}\beta\tau_V V_t V_{tx} V_{xx} + \beta\tau_V V_x V_{tx}^2 + \frac{1}{2}\beta\tau_{VV} V_t V_x^2 V_{tx} - \frac{1}{2}\tau_t V_t V_x \\
&-\frac{1}{2}\tau_V V_t^2 V_x - \frac{1}{2}\xi_x V_t V_x - \frac{1}{2}\xi_V V_t V_x^2 - \frac{1}{4}\alpha\tau_t V_x^4 - \frac{1}{4}\alpha\tau_V V_t V_x^4 - \frac{1}{4}\alpha\xi_x V_x^4 - \frac{1}{4}\alpha\xi_V V_x^5 \\
&-\frac{1}{2}\beta\tau_t V_{tx} V_{xx} - \frac{1}{2}\beta\tau_V V_t V_{tx} V_{xx} - \frac{1}{2}\beta\xi_x V_{tx} V_{xx} - \frac{1}{2}\beta\xi_V V_x V_{tx} V_{xx} \\
&= B_t^t + B_x^x + V_t B_V^t + V_x B_V^x.
\end{aligned} \tag{3.69}$$

Splitting (3.69) on the derivatives of V we obtain

$$V_{tt}V_{xx} : \tau_x = 0, \tag{3.70}$$

$$V_t^2 V_x : \tau_u = 0, \tag{3.71}$$

$$V_{xx}^2 : \xi_t = 0, \tag{3.72}$$

$$V_{tx}V_{xx} : \xi_x = 0, \tag{3.73}$$

$$V_tV_x^2 : \xi_u = 0, \tag{3.74}$$

$$V_tV_x : \eta_u = 0, \tag{3.75}$$

$$V_x^3 : \eta_x = 0, \tag{3.76}$$

$$V_x^4 : 3\xi_x - \tau_t = 0, \tag{3.77}$$

$$V_x : -\eta_t = 2B_V^x, \tag{3.78}$$

$$V_t : -\eta_x = 2B_V^t, \tag{3.79}$$

$$\text{rest} : B_t^t + B_x^x = 0. \tag{3.80}$$

From (3.70) and (3.71) we have

$$\tau = a(t),$$

where $a(t)$ is an arbitrary function of t. Equations (3.72-3.74) yield

$$\xi = C_1,$$

where C_2 is an arbitrary constant. Solving equations (3.75) and (3.76), we have

$$\eta = f(t),$$

where $f(t)$ is an arbitrary function of t. From (3.77), we obtain

$$a(t) = C_2,$$

where C_1 is an arbitrary constant of integration. Equation (3.78) equally yields

$$B^x = -\frac{1}{2}f'(t)V + g(t,x),$$

where $g(t,x)$ is an arbitrary function of t and x. From (3.76) and (3.79), we have

$$B^t = h(t,x),$$

where $h(t,x)$ is an arbitrary function of t and x. Here we can choose $h(t,x) = 0$ and $g(t,x) = 0$ as they contribute to the trivial part of the conserved vector, thus satisfying equation (3.80). Therefore we have the following solution:

$$\tau = C_2,$$

$$\begin{aligned}
\xi &= C_1, \\
\eta &= f(t), \\
B^t &= 0, \\
B^x &= -\frac{1}{2}f'(t)V.
\end{aligned}$$

Thus we obtain the following Noether point symmetries and their corresponding gauge functions:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad B^t = 0, \quad B^x = 0, \\
X_2 &= \frac{\partial}{\partial x}, \quad B^t = 0, \quad B^x = 0, \\
X_f &= f(t)\frac{\partial}{\partial V}, \quad B^t = 0, \quad B^x = -\frac{1}{2}f'(t)V.
\end{aligned}$$

Next, we use the above results to compute the conserved vectors of the fourth-order equation (3.62). Using the formulae for the conserved vector (T^t, T^x) in chapter 1 [45], we obtain the following three conserved vectors associated with the above Noether symmetries X_1, X_2 and X_f as

$$\begin{aligned}
T_1^t &= -\frac{1}{4}\alpha V_x^4 - \frac{1}{2}\beta V_{xx}V_{tx} - \frac{1}{2}\beta V_t V_{xxx}, \\
T_1^x &= \frac{1}{2}V_t^2 + \alpha V_t V_x^3 - \frac{1}{2}\beta V_t V_{txx} + \frac{1}{2}\beta V_{tx}^2 + \frac{1}{2}\beta V_{xx}V_{tt};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}V_x^2 - \frac{1}{2}\beta V_x V_{xxx}, \\
T_2^x &= \frac{3}{4}\alpha V_x^4 - \frac{1}{2}\beta V_x V_{txx} + \frac{1}{2}\beta V_{xx}V_{tx};
\end{aligned}$$

$$\begin{aligned}
T_f^t &= -\frac{1}{2}fV_x + \frac{1}{2}\beta f(t)V_{xxx}, \\
T_f^x &= -\frac{1}{2}f(t)V_t - \alpha f(t)V_x^3 + \frac{1}{2}\beta f(t)V_{txx} - \frac{1}{2}\beta f'(t)V_{xx} + \frac{1}{2}f'(t)V,
\end{aligned}$$

respectively. Reverting to the original variables we obtain one local and two non-local conserved vectors of (3.1) given by

$$\begin{aligned}
T_1^t &= -\frac{1}{4}\alpha u^4 - \frac{1}{2}\beta u_x u_t - \frac{1}{2}\beta u_{xx}\int u_t dx, \\
T_1^x &= \frac{1}{2}\left(\int u_t dx\right)^2 + \alpha u^3 \int u_t dx - \frac{1}{2}\beta u_{tx}\int u_t dx + \frac{1}{2}\beta u_t^2 + \frac{1}{2}\beta u_x \int u_{tt} dx;
\end{aligned}$$

$$
\begin{aligned}
T_2^t &= \frac{1}{2}u^2 - \frac{1}{2}\beta u u_{xx}, \\
T_2^x &= \frac{3}{4}\alpha u^4 - \frac{1}{2}\beta u u_{tx} + \frac{1}{2}\beta u_x u_t;
\end{aligned}
$$

$$
\begin{aligned}
T_f^t &= -\frac{1}{2}u f(t) + \frac{1}{2}\beta u_{xx} f(t), \\
T_f^x &= -\frac{1}{2}f(t)\int u_t dx - \alpha u^3 f(t) + \frac{1}{2}\beta u_{tx} f(t) - \frac{1}{2}\beta u_x f'(t) + \frac{1}{2}f'(t)\int u dx.
\end{aligned}
$$

Remark: It should be noted that due to the presence of arbitrary function $f(t)$ we have infinitely many nonlocal conservation laws.

3.4 Concluding remarks

In this chapter we studied the modified equal-width equation (3.1). For the first time, Lie point symmetries of (3.1) were computed and used to construct an optimal system of one-dimensional subalgebras. Thereafter utilising this optimal system of one-dimensional subalgebras, symmetry reductions and new group-invariant solutions of MEW equation (3.1) were presented. The solutions obtained were cnoidal and snoidal waves. Again for the first time, we computed the conservation laws for (3.1) by employing two different methods; the multiplier method and the Noether approach.

Chapter 4

Exact solutions and conservation laws of a generalized nonlinear advection-diffusion equation

4.1 Introduction

The nonlinear advection-diffusion equation [57]

$$u_t - 2\alpha u u_x - u_x^2 - u u_{xx} = 0, \tag{4.1}$$

that recounts the movement of a buoyancy-driven plume in an inclined porous medium, with constant α has a specific physical significance connected to the bed inclination. The solution is characterized by two moving boundaries for compactly supported initial data, propagated with finite speed and spanning a distance of $\mathcal{O}(\sqrt{t})$. In [57] the authors generated late-time asymptotic solutions to a nonlinear advective-diffusion equation that has various applications in porous media flow.

In this chapter, we study the generalized nonlinear advection-diffusion equation

$$u_t - \omega u^n u_x - u_x^2 - u u_{xx} = 0. \tag{4.2}$$

We compute the Lie point symmetries of this equation for three different values of n. For each case we obtain group-invariant solutions under their Lie point symmetries.

We then proceed to find the travelling wave solutions for each case. Furthermore, we derive conservation laws associated with each case by utilising the multiplier approach and the new conservation theorem due to Ibragimov.

This work has been submitted for publication, See [58]

4.2 Exact solutions of (4.2)

In this section we present exact solutions of the generalized nonlinear advection-diffusion equation (4.2) by applying Lie symmetry method.

4.2.1 Lie point symmetries of (4.2)

We begin by determining the Lie point symmetries of (4.2). Let the symmetry group be generated by the vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{4.3}$$

Then X is a Lie point symmetry of (4.2) if

$$X^{[2]}\{u_t - \omega u^n u_x - u_x^2 - u u_{xx}\}\big|_{u_t-\omega u^n u_x - u_x^2 - u u_{xx}=0} = 0, \tag{4.4}$$

where $X^{[2]}$ is the second prolongation of X as defined in (2.5). Expanding equation (4.4), we obtain

$$\zeta_1 - \omega n u^{n-1} u_x \eta - \omega u^n \zeta_2 - 2u_x \zeta_2 - u_{xx}\eta - u\zeta_{22}\big|_{u_t-\omega u^n u_x - u_x^2 - u u_{xx}=0} = 0, \tag{4.5}$$

where ζ_1, ζ_2 and ζ_{22} are as defined in chapter 1. Expansion of equation (4.5) gives

$$\begin{aligned}
&\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x \xi_u - \omega n u^{n-1} u_x \eta - u_{xx}\eta \\
&-\omega u^n \eta_x - \omega u^n u_x \eta_u + \omega u^n u_t \tau_x + \omega u^n u_t u_x \tau_u + \omega u^n u_x \xi_x + \omega u^n u_x^2 \xi_u \\
&-2u_x\eta_x - 2u_x^2\eta_u + 2u_t u_x \tau_x + 2u_t u_x^2 \tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - u\eta_{xx} \\
&-2uu_x\eta_{xu} - uu_{xx}\eta_u - uu_x^2\eta_{uu} + 2uu_{xx}\xi_x + uu_x\xi_{xx} + 2uu_x^2\xi_{xu} \\
&+3uu_x u_{xx}\xi_u + uu_x^3\xi_{uu} + 2uu_{tx}\tau_x + uu_t\tau_{xx} + 2uu_t u_x\tau_{xu} + uu_t u_{xx}\tau_u
\end{aligned}$$

$$+2uu_xu_{tx}\tau_u + uu_tu_x^2\tau_{uu}\big|_{u_t-\omega u^n u_x - u_x^2 - uu_{xx}=0} = 0. \tag{4.6}$$

Substituting $\omega u^n u_x + u_x^2 + uu_{xx}$ for u_t in equation (4.6), we obtain

$$\begin{aligned}
&\eta_t + \omega u^n u_x\eta_u + u_x^2\eta_u + uu_{xx}\eta_u - \omega u^n u_x\tau_t - u_x^2\tau_t - uu_{xx}\tau_t - \omega^2 u^{2n}u_x^2\tau_u \\
&-2\omega u^n u_x^3\tau_u - 2\omega u^{n+1}u_xu_{xx}\tau_u - u_x^4\tau_u - 2uu_x^2u_{xx}\tau_u - u^2u_{xx}^2\tau_u - u_x\xi_t \\
&-\omega u^n u_x^2\xi_u - u_x^3\xi_u - uu_xu_{xx}\xi_u - \omega nu^{n-1}u_x\eta - u_{xx}\eta - \omega u^n\eta_x - \omega u^n u_x\eta_u \\
&+\omega^2u^{2n}u_x\tau_x + \omega u^n u_x^2\tau_x + \omega u^{n+1}u_{xx}\tau_x + \omega^2u^{2n}u_x^2\tau_u + \omega u^n u_x^3\tau_u + \omega u^{n+1}u_xu_{xx}\tau_u \\
&+\omega u^n u_x\xi_x + \omega u^n u_x^2\xi_u - 2u_x\eta_x - 2u_x^2\eta_u + 2\omega u^n u_x^2\tau_x + 2u_x^3\tau_x + 2uu_xu_{xx}\tau_x \\
&+2\omega u^n u_x^3\tau_u + 2u_x^4\tau_u + 2uu_x^2u_{xx}\tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - u\eta_{xx} - 2uu_x\eta_{xu} - uu_{xx}\eta_u \\
&-uu_x^2\eta_{uu} + 2uu_{xx}\xi_x + uu_x\xi_{xx} + 2uu_x^2\xi_{xu} + 3uu_xu_{xx}\xi_u + uu_x^3\xi_{uu} + 2uu_{tx}\tau_x \\
&+\omega u^{n+1}u_x\tau_{xx} + uu_x^2\tau_{xx} + u^2u_{xx}\tau_{xx} + 2\omega u^{n+1}u_x^2\tau_{xu} + 2uu_x^3\tau_{xu} \\
&+2u^2u_xu_{xx}\tau_{xu} + \omega u^{n+1}u_xu_{xx}\tau_u + uu_x^2u_{xx}\tau_u + u^2u_{xx}^2\tau_u + 2uu_xu_{tx}\tau_u \\
&+\omega u^{n+1}u_x^3\tau_{uu} + uu_x^4\tau_{uu} + u^2u_x^2u_{xx}\tau_{uu} = 0.
\end{aligned}$$

Now splitting the above equation on the derivatives of u, we obtain

$$\tau_u = 0, \tag{4.7}$$

$$\tau_x = 0, \tag{4.8}$$

$$\xi_u = 0, \tag{4.9}$$

$$2u\xi_x - u\tau_t - \eta = 0, \tag{4.10}$$

$$2\xi_x - u\eta_{uu} - \eta_u - \tau_t = 0, \tag{4.11}$$

$$u\xi_{xx} + \omega u^n\xi_x - \omega u^n\tau_t - \xi_t - \omega nu^{n-1}\eta - 2\eta_x - 2u\eta_{xu} = 0, \tag{4.12}$$

$$\eta_t - \omega u^n\eta_x - u\eta_{xx} = 0. \tag{4.13}$$

Equations (4.7) and (4.8) yield

$$\tau = A(t), \tag{4.14}$$

where $A(t)$ is an arbitrary function depending on t. Solving (4.9) gives

$$\xi = B(t, x), \tag{4.15}$$

where $B(t,x)$ is a function of t and x. Substituting these values of τ and ξ into (4.10), we obtain

$$\eta = 2B_x(t,x)u - A'(t)u.$$

We note that equation (4.11) is satisfied for the above values of τ, ξ and η. Now substituting the values of τ, ξ and η in equation (4.12) we have

$$\omega n A'(t)u^n + \omega B_x u^n - 7B_{xx}(t,x)u - \omega A'(t)u^n - 2\omega n B_x(t,x)u^n - B_t(t,x) = 0.$$

Splitting the above equation on the powers of u, provided $n \neq 1$, we have

$$u^n : nA'(t) + B_x(t,x) - A'(t) - 2nB_x(t,x) = 0, \tag{4.16}$$

$$u : B_{xx}(t,x) = 0, \tag{4.17}$$

$$\text{rest} : B_t(t,x) = 0. \tag{4.18}$$

Equation (4.18) gives

$$B = B(x),$$

where $B(x)$ is an arbitrary function of x. Equation (4.17) now yields

$$B(t,x) = C_1 x + C_2,$$

where C_1 and C_2 are constants of integration. Substituting the value of B into equation (4.16) and solving for $A(t)$, provided $2n \neq 1$, $n \neq 1$, we obtain

$$A(t) = \frac{2n-1}{n-1}C_1 t + C_3.$$

We also note that equation (4.13) is satisfied for the above values of τ, ξ and η. Hence, the solution to the system of (4.7)-(4.13) is

$$\begin{aligned} \tau &= \frac{2n-1}{n-1}\,C_1 t + C_3, \\ \xi &= C_1 x + C_2, \\ \eta &= 2C_1 u - \frac{2n-1}{n-1}\,C_1 u. \end{aligned}$$

Thus, we obtain the following Lie point symmetries of (4.2) for $n \neq 1/2$ and $n \neq 1$:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u}. \end{aligned} \tag{4.19}$$

Case 2. n=1/2

We first consider the generalized nonlinear advection-diffusion equation (4.2) when $n = 1/2$ which becomes

$$u_t - \omega u^{1/2} u_x - u_x^2 - u u_{xx} = 0. \qquad (4.20)$$

The determining equations for (4.20) can be obtained from (4.7)-(4.13) with $n = 1/2$. These are

$$\tau_u = 0, \qquad (4.21)$$

$$\tau_x = 0, \qquad (4.22)$$

$$\xi_u = 0, \qquad (4.23)$$

$$2u\xi_x - u\tau_t - \eta = 0, \qquad (4.24)$$

$$2\xi_x - u\eta_{uu} - \eta_u - \tau_t = 0, \qquad (4.25)$$

$$u\xi_{xx} + \omega u^{1/2}\xi_x - \omega u^{1/2}\tau_t - \xi_t - \frac{1}{2}\omega u^{-1/2}\eta - 2\eta_x - 2u\eta_{xu} = 0, \qquad (4.26)$$

$$\eta_t - \omega u^{1/2}\eta_x - u\eta_{xx} = 0. \qquad (4.27)$$

Equations (4.21) and (4.22) yield

$$\tau = P(t),$$

where $P(t)$ is an arbitrary function t. Solving (4.23), we have

$$\xi = Q(t, x),$$

where $Q(t, x)$ is an arbitrary function depending on t and x. Invoking the values of τ and ξ in (4.24) above , the η gives

$$\eta = 2Q_x(t, x)u - P'(t)u. \qquad (4.28)$$

Obviously, equation (4.25) is satisfied. Substituting the value of η above in equation (4.26) produces

$$7Q_{xx}(t, x)u + \frac{1}{2}\omega P'(t)u^{1/2} + Q_t(t, x) = 0. \qquad (4.29)$$

Splitting (4.29) above over the derivatives of u, we have

$$u : Q_{xx}(t, x) = 0, \qquad (4.30)$$

$$u^{1/2} : P'(t) = 0, \tag{4.31}$$

$$\text{rest} : Q_t(t,x) = 0. \tag{4.32}$$

Solving equation (4.32) gives

$$Q = Q(x),$$

where $Q(x)$ is a function of x and subsequently (4.30) yields

$$Q = C_1 x + C_2,$$

where C_1 and C_2 are constants. Equation (4.31) now becomes

$$P(t) = C_3,$$

where C_3 is a constant of integration and we note that equation (4.27) is satisfied. Therefore τ, ξ and η become

$$\tau = C_3,$$

$$\xi = C_1 x + C_2,$$

$$\eta = 2C_1 u.$$

Thus, the Lie point symmetries of (4.20) are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{4.33}$$

Case 3. n=1

For the case when $n = 1$, the generalized nonlinear advection-diffusion equation (4.2) becomes

$$u_t - \omega u u_x - u_x^2 - u u_{xx} = 0. \tag{4.34}$$

The determining equations for (4.34) can be obtained from (4.7)-(4.13) with $n = 1$. These are

$$\tau_u = 0, \tag{4.35}$$

$$\tau_x = 0, \tag{4.36}$$

$$\xi_u = 0, \tag{4.37}$$

$$2u\xi_x - u\tau_t - \eta = 0, \tag{4.38}$$

$$\eta - u\eta_u - u^2\eta_{uu} = 0, \tag{4.39}$$

$$u\xi_{xx} - 2u\eta_{xu} - \omega u\xi_x - 2\eta_x - \xi_t = 0, \tag{4.40}$$

$$\eta_t - \omega u\eta_x - u\eta_{xx} = 0. \tag{4.41}$$

Equations (4.35) and (4.36) yields

$$\tau = G(t),$$

where $G(t)$ is a function with argument t and solving (4.37), we obtain

$$\xi = H(t, x),$$

where $H(t, x)$ is a function of t and x. Substituting the values of τ and ξ in (4.38), then

$$\eta = 2H_x(t, x)u - G'(t)u.$$

We note that (4.39) is satisfied. Equation (4.40) now becomes

$$7H_{xx}(t, x)u + \omega H_x(t, x)u + H_t(t, x) = 0. \tag{4.42}$$

Splitting the above equation on u, we obtain

$$u : 7H_{xx}(t, x) + \omega H_x(t, x) = 0, \tag{4.43}$$

$$\text{rest} : H_t(t, x) = 0. \tag{4.44}$$

Solving equation (4.44) gives

$$H = H(x),$$

where $H(x)$ is a function dependent on x. Solution to equation (4.43) yields

$$H = C_1 + C_2 e^{-\frac{\omega}{7}x},$$

where C_1 and C_2 are constants. Therefore ξ and η become respectively

$$\xi = C_1 + C_2 e^{-\frac{\omega}{7}x} \quad \text{and} \quad \eta = -\frac{2\omega}{7}uC_2 e^{-\frac{\omega}{7}x} - uG'(t).$$

Substituting the above values of η and ξ in (4.41), we obtain

$$uG''(t) + \frac{48}{343}\omega^2 u^2 C_2 e^{-\frac{\omega}{7}x} = 0.$$

Splitting the above equation over u, we have

$$u^2 : C_2 e^{-\frac{\omega}{7}x} = 0, \tag{4.45}$$

$$u : G''(t) = 0. \tag{4.46}$$

Equation (4.45) obviously produces

$$C_2 = 0 \quad \text{since} \quad e^{-\frac{\omega}{7}x} \neq 0.$$

Solving equation (4.46) gives

$$G(t) = C_3 t + C_4,$$

where C_3 and C_4 are constants. Thus, variables τ, ξ and η now become

$$\tau = C_3 t + C_4,$$
$$\xi = C_1,$$
$$\eta = -C_3 u.$$

Thus, the infinitesimal generators of (4.34) are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{4.47}$$

4.2.2 Optimal system of one-dimensional subalgebras

We now use the Lie point symmetries of the three cases of n for equation (4.2) computed above to construct an optimal system of one-dimensional subalgebras in order to obtain symmetry reductions as well as group-invariant solutions.

Case 1. n$\neq$ 1/2, 1

We first compute all the commutators of the Lie symmetries (4.19). This is then presented in Table 1. Using the commutator Table 1 we construct the adjoint representations of the symmetry group of (4.2) and present it in Table 2.

Table 1. Commutator table of the Lie algebra of equation (4.2)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$\frac{n-1}{2n-1}X_2$
X_3	$-X_1$	$-\frac{n-1}{2n-1}X_2$	0

Table 2. Adjoint table of the Lie algebra of equation (4.2)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \varepsilon X_1$
X_2	X_1	X_2	$X_3 - \frac{n-1}{2n-1}\varepsilon X_2$
X_3	$e^{\varepsilon}X_1$	$e^{\varepsilon(n-1)/(2n-1)}X_2$	X_3

Following the procedure given in [13], it turns out that an optimal system of one-dimensional subalgebras is given by

$$\{X_2, X_1 + cX_2, X_3\},$$

where $c \in \mathbb{R}$.

Case 2. n=1/2

The commutators of the Lie symmetries (4.33) are presented in Table 3. Using this commutator table we construct the adjoint representations of the symmetry group of (4.20) and present it in Table 4.

Table 3. Commutator table of the Lie algebra of equation (4.20)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	X_2
X_3	0	$-X_2$	0

Table 4. Adjoint table of the Lie algebra of equation (4.20)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - \varepsilon X_2$
X_3	X_1	$e^{\varepsilon} X_2$	X_3

As before, following [13], an optimal system of one-dimensional subalgebras is given by

$$\{X_2, X_1 + cX_2, bX_1 + X_3\},$$

where $b, c \in \mathbb{R}$.

Case 3. n=1

Lastly, we compute all the commutators of the Lie symmetries (4.47) and present them in Table 5. The adjoint representations of the symmetry group of (4.34) is presented in Table 6.

Table 5. Commutator table of the Lie algebra of equation (4.34)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	0
X_3	$-X_1$	0	0

Table 6. Adjoint table of the Lie algebra of equation (4.34)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \varepsilon X_1$
X_2	X_1	X_2	X_3
X_3	$e^{\varepsilon} X_1$	X_2	X_3

Thus, in this case an optimal system of one-dimensional subalgebras is given by

$$\{X_2, X_1 + cX_2, bX_2 + X_3\},$$

where $b, c \in \mathbb{R}$.

4.2.3 Symmetry reductions and solutions

We now engage the optimal system of one-dimensional subalgebras obtained earlier in the previous subsection and find group-invariant solutions and symmetry reductions for each of the three cases of n for equation (4.2).

Case 1. n $\neq$ 1/2, 1

In this case the optimal system of one-dimensional subalgebras is $\{X_2, X_1 + cX_2, X_3\}$. We find the group-invariant solutions under each of these three members of the optimal system.

(i) X_2

The associated Lagrangian system for X_2 is given by

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{4.48}$$

which produces two group-invariant solutions $J_1 = t$ and $J_2 = u$. Therefore $J_2 = f(J_1)$. Obviously

$$u = f(t). \tag{4.49}$$

Substituting the above value of u in (4.2), we obtain

$$f'(t) = 0. \tag{4.50}$$

Solving the above equation yields

$$f(t) = \text{C}, \tag{4.51}$$

where C is a constant. Therefore

$$u(t, x) = \text{C}.$$

(ii) $X_1 + cX_2$

This symmetry operator will provide the travelling wave solutions. Following the procedure adopted in chapter 2, we have the invariants $J_1 = u$ and $J_2 = x - ct$, so that from $J_1 = f(J_2)$, we obtain a group-invariant solution to (4.2) under $X_1 + cX_2$. Thus

$$u = f(x - ct). \tag{4.52}$$

Substituting the above value of u in (4.2) with $\lambda = x - ct$, we have

$$cf'(\lambda) + \omega f'(\lambda) f^n(\lambda) + (f(\lambda)f'(\lambda))' = 0.$$

(iii) X_3

We now consider the group-invariant solution of scaling symmetry

$$X_3 = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u}. \tag{4.53}$$

The Lagrangian system associated with X_3 is

$$\frac{dt}{t} = \frac{(2n-1)}{n-1}\frac{dx}{x} = \frac{(2n-1)du}{-u}. \tag{4.54}$$

The above system (4.54) generates two invariants $J_1 = x^{\frac{2n-1}{n-1}}/t$ and $J_2 = u/x^{\frac{1}{n-1}}$. Hence, the group-invariant solution of (4.34) is $J_2 = f(J_1)$ and so

$$u = x^{-1/n-1} f\left(\frac{x^{(2n-1)/(n-1)}}{t}\right).$$

Substituting the above value of u into (4.2) and simplifying it, we obtain the ODE

$$\begin{aligned} &f(z)\left(\left(2n^2 - 9n + 4\right)zf' + (1-2n)^2 z^2 f'' + (1-n)\omega f(z)^n\right) \\ &+ zf'\left((1-2n)^2 zf' + (n-1)\left((2n-1)\omega f(z)^n + (n-1)z\right)\right) + (n+1)f(z)^2 = 0. \end{aligned}$$

Case 2. n=1/2

The optimal system of one-dimensional subalgebras in this case is

$$\{X_2, X_1 + cX_2, bX_1 + X_3\}.$$

(i) X_2

The characteristic equations of $X_2 = \partial/\partial x$ are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{4.55}$$

which give rise to two group-invariant solutions $J_1 = t$ and $J_2 = u$. Thus $J_2 = f(J_1)$. Obviously

$$u = f(t). \tag{4.56}$$

Substituting the above value of u in (4.34), we obtain

$$f'(t) = 0.$$

Therefore,

$$f(t) = C_1,$$

where C_1 is a constant of integration. Therefore,

$$u(t, x) = C_1.$$

(ii) $X_1 + cX_2$

Following the procedural steps taken in chapter 2, we obtain

$$u = f(x - ct).$$

Therefore

$$u_t = -cf'(\epsilon), \quad u_x = f'(\epsilon) \quad \text{and} \quad u_{xx} = f''(\epsilon), \quad \text{where} \quad \epsilon = x - ct.$$

Substituting the above values of u_t, u_x and u_{xx} in (4.34) and simplifying it, we obtain the ordinary differential equation

$$cf'(\epsilon) + \omega f^{1/2}(\epsilon)f'(\epsilon) + (f(\epsilon)f'(\epsilon))' = 0.$$

(iii) $bX_1 + X_3$

We subsequently compute the group-invariant solution of the scaling symmetry X_4 stated as

$$X_4 = b\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}.$$

The associated Lagrangian system with X_4 is

$$\frac{dt}{b} = \frac{dx}{x} = \frac{du}{2u}.$$

The above system generates two invariants $J_1 = b\ln -t$ and $J_2 = u/x^2$. Hence, the group-invariant solution of (4.34) is $J_2 = f(J_1)$ and eventually

$$u = x^2 f\left(b\ln x - t\right).$$

Therefore, substituting the above values of u in (4.34), we have the first order ODE

$$b^2\left(ff'\right)' + \left(b\omega\sqrt{f} + 7bf + 1\right) f' + 2\omega f^{3/2} + 6f^2 = 0.$$

Case 3. n=1

The optimal system of one-dimensional subalgebras in this case is

$$\{X_2, X_1 + cX_2, bX_2 + X_3\}.$$

(i) X_2

The characteristic equations associated with $X_2 = \partial/\partial x$ are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which give rise to two invariants $J_1 = t$ and $J_2 = u$. Therefore $J_2 = f(J_1)$ and then

$$u(t, x) = f(t).$$

Substituting the above value of u into (4.34), we obtain

$$f(t) = C_1,$$

where C_1 is a constant. Hence, the group-invariant solution to X_2 is

$$u(t, x) = C_1.$$

(ii) $X_1 + cX_2$

Following the steps taken above we have

$$u = f(\lambda), \quad \text{where} \quad \lambda = x - ct. \tag{4.57}$$

Substituting the above value of u in (4.34), we have the second order nonlinear ODE

$$cf'(\lambda) + \omega f(\lambda)f'(\lambda) + (f(\lambda)f'(\lambda))' = 0$$

whose solution is given by

$$f(z) = c_1 e^{-\frac{\omega z}{2}} - \frac{2c}{\omega},$$

where c_1 is a constant of integration. Hence, the solution of (4.34) is

$$u(t,x) = c_1 e^{-\frac{\omega}{2}(x-ct)} - \frac{2c}{\omega}.$$

(iii) $bX_2 + X_3$

Lastly, we generate the group-invariant solution under scaling symmetry X_4 defined by

$$X_4 = b\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{4.58}$$

The associated Lagrangian system to X_4 is

$$\frac{dt}{t} = \frac{dx}{b} = \frac{du}{-u}. \tag{4.59}$$

System (4.59) yields two invariants $J_1 = x - b\ln t$ and $J_2 = tu$. Thus, the group-invariant solution of (4.34) is $J_2 = f(J_1)$ and so

$$u = \frac{1}{t} f\left(x - b\ln t\right).$$

Substituting the above values of u in (4.34) yields

$$bf' + \omega f f' + (f f')' + f = 0.$$

4.3 Conservation laws of (4.2)

In this section we derive conservation laws of the generalized nonlinear advection-diffusion equation (4.2) by considering its three cases. We use two different methods of finding conservation laws; the multiplier method and the new conservation theorem due to Ibragimov.

4.3.1 Conservation laws of (4.2) using multiplier method

In this subsection we employ the multiplier method to construct conservation laws for the generalized nonlinear advection-diffusion equation (4.2).

Case 1. n$\neq$ 1/2, 1

We look for the zeroth-order multiplier $\Lambda = \Lambda(t, x, u)$ and hence solve the determining equation

$$\frac{\delta}{\delta u}\Big\{\Lambda(t,x,u)\left(u_t - \omega u^n u_x - u_x^2 - u u_{xx}\right)\Big\} = 0, \tag{4.60}$$

which gives

$$u_t\Lambda_u - \omega u^n u_x \Lambda_u - u_x^2\Lambda_u - u u_{xx}\Lambda_u - \omega n u^{n-1} u_x \Lambda - u_{xx}\Lambda - D_t(\Lambda)$$
$$+D_x\left(\omega u^n \Lambda\right) + D_x(2u_x\Lambda) - D_x^2(\Lambda u) = 0.$$

Equation above expands to

$$u_t\Lambda_u - \omega u^n u_x\Lambda_u - u_x^2\Lambda_u - u u_{xx}\Lambda_u - \omega n u^{n-1} u_x\Lambda - u_{xx}\Lambda - \Lambda_t - u_t\Lambda_u$$
$$+\omega u^n\Lambda_x + \omega n u^{n-1}u_x\Lambda + \omega u^n u_x\Lambda_u + 2u_x\Lambda_x + 2u_x^2\Lambda_u + 2u_{xx}\Lambda - u\Lambda_{xx}$$
$$-u_x\Lambda_x - u u_x\Lambda_{ux} - u_x\Lambda_x - u_x^2\Lambda_u - u_{xx}\Lambda - u u_x\Lambda_{ux} - u_x^2\Lambda_u - u u_x^2\Lambda_{uu}$$
$$-u u_{xx}\Lambda_u = 0.$$

Simplifying and splitting the above equation on the derivatives of u, we obtain

$$u_{xx} : \Lambda_u = 0, \tag{4.61}$$

$$\text{rest} : \omega u^n\Lambda_x - \Lambda_t - u\Lambda_{xx} = 0. \tag{4.62}$$

Equation (4.61) yields

$$\Lambda = Q(t, x),$$

where $Q(t, x)$ is a function whose arguments are t and x. Substituting the above value of Λ in (4.62), we have

$$\omega u^n Q_x - u Q_{xx} - Q_t = 0.$$

Splitting the above equation over u yields

$$u^n : Q_x(t, x) = 0,$$
$$u : Q_{xx}(t, x) = 0,$$
$$\text{rest} : Q_t(t, x) = 0.$$

Solving the above equations, we obtain

$$Q = C_1,$$

where C_1 is a constant. Therefore $\Lambda = 1$ is the multiplier for (4.2). We obtain the conservation law for the above multiplier by considering

$$\Lambda(t,x,u)\{u_t - \omega u^n u_x - u_x^2 - uu_{xx}\} = D_t T^t + D_x T^x, \tag{4.63}$$

where density T^t and flux T^x are defined respectively as

$$T^t = T^t(t,x,u) \quad \text{and} \quad T^x = T^x(t,x,u,u_x). \tag{4.64}$$

and so equation (4.63) becomes

$$u_t - \omega u^n u_x - u_x^2 - uu_{xx} = D_t T^t + D_x T^x,$$

which after simplification gives

$$u_t - \omega u^n u_x - u_x^2 - uu_{xx} = T_t^t + u_t T_u^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x.$$

Now splitting the above equation on the second derivatives of u, we obtain

$$u_{xx} : T_{u_x}^x = -u, \tag{4.65}$$

$$\text{rest} : T_t^t + u_t T_u^t + T_x^x + u_x T_u^x = u_t - \omega u^n u_x - u_x^2. \tag{4.66}$$

Integration of equation (4.65) with respect to u_x, we obtain the flux

$$T^x = -uu_x + P(t,x,u),$$

where $P(t,x,u)$ is a function of t, x and u. Consequently, equation (4.66) gives

$$T_t^t + u_t T_u^t + P_x + u_x P_u - u_x^2 = u_t - \omega u^n u_x - u_x^2.$$

Simplifying and splitting the above equation on the derivatives of u, we have

$$u_t : T_u^t = 1, \tag{4.67}$$

$$u_x : P_u = -\omega u^n, \tag{4.68}$$

$$\text{rest}: P_x + T^t_t = 0. \tag{4.69}$$

Solving equation (4.67) gives

$$T^t = u + Q(t,x),$$

where $Q(t,x)$ is an arbitrary function of t and x. Equation (4.68) then produces

$$P = -\frac{u^{n+1}}{n+1}\omega + R(t,x),$$

where $R(t,x)$ is a function of t and x. We now take $R(t,x) = Q(t,x) = 0$ since both functions contribute to the trivial part of the conserved vectors. Obviously, equation (4.69) is satisfied. Therefore the density and flux (T^t, T^x) for the (4.2) is

$$T^t = u,$$
$$T^x = -uu_x - \frac{\omega}{n+1}u^{n+1}.$$

Remark: It is worthy to note that the multiplier $\Lambda = 1$ gives us an inclination that the generalized nonlinear advection-diffusion equation (4.2) is itself a conservation law.

Case 2. n=1/2.

We look for the zeroth-order multiplier $\Lambda = \Lambda(t,x,u)$. The determining equation for the multiplier is

$$\frac{\delta}{\delta u}\left\{\Lambda(t,x,u)\left(u_t - \omega u^{1/2}u_x - u_x^2 - uu_{xx}\right)\right\} = 0, \tag{4.70}$$

where $\delta/\delta u$ is as defined in chapter 1. The above equation then becomes

$$\begin{aligned} &u_t\Lambda_u - \omega u^{\frac{1}{2}}u_x\Lambda_u - u_x^2\Lambda_u - uu_{xx}\Lambda_u - \frac{1}{2}\omega u^{-\frac{1}{2}}u_x\Lambda - u_{xx}\Lambda - D_t(\Lambda) \\ &+D_x(\omega u^{\frac{1}{2}}\Lambda) + D_x(2u_x\Lambda) - D_x^2(\Lambda u) = 0, \end{aligned} \tag{4.71}$$

which expands to

$$\begin{aligned} &u_t\Lambda_u - \omega u^{\frac{1}{2}}u_x\Lambda_u - u_x^2\Lambda_u - uu_{xx}\Lambda_u - \frac{1}{2}\omega u^{-\frac{1}{2}}u_x\Lambda - u_{xx}\Lambda - \Lambda_t - u_t\Lambda_u \\ &+\omega u^{\frac{1}{2}}\Lambda_x + \frac{1}{2}\omega u^{-\frac{1}{2}}u_x\Lambda + \omega u^{\frac{1}{2}}u_x\Lambda_u + 2u_x\Lambda_x + 2u_x^2\Lambda_u + 2u_{xx}\Lambda - u\Lambda_{xx} \end{aligned}$$

$$-u_x\Lambda_x - uu_x\Lambda_{ux} - u_x\Lambda_x - u_x^2\Lambda_u - u_{xx}\Lambda - uu_x\Lambda_{ux} - u_x^2\Lambda_u - uu_x^2\Lambda_{uu}$$
$$-uu_{xx}\Lambda_u = 0. \tag{4.72}$$

Splitting the above equation on the derivatives of u after further simplification, we have

$$u_{xx} : \Lambda_u = 0, \tag{4.73}$$
$$\text{rest} : \omega u^{\frac{1}{2}}\Lambda_x - u\Lambda_{xx} - \Lambda_t = 0. \tag{4.74}$$

From equation (4.73), we obtain

$$\Lambda = Q(t, x), \tag{4.75}$$

where $Q(t, x)$ is an arbitrary function depending on t and x. Substituting the above value of Λ in (4.74), we obtain

$$\omega Q_x(t, x)u^{\frac{1}{2}} - Q_t(t, x) - Q_{xx}(t, x)u = 0. \tag{4.76}$$

Splitting (4.76) on u, we have

$$u^{\frac{1}{2}} : Q_x(t, x) = 0, \tag{4.77}$$
$$u : Q_{xx}(t, x) = 0, \tag{4.78}$$
$$\text{rest} : Q_t(t, x) = 0. \tag{4.79}$$

Equation (4.77) and (4.79) yields

$$Q = C_1,$$

where C_1 is a constant. Thus, the multiplier of (4.34) yields

$$\Lambda = C_1.$$

Therefore, $\Lambda = 1$. Obviously, (4.78) is satisfied.

Next, we derive the conservation law for the constant-valued multiplier. Therefore we consider

$$\Lambda(t, x, u)\{u_t - \omega u^{1/2}u_x - u_x^2 - uu_{xx}\} = D_tT^t + D_xT^x, \tag{4.80}$$

where density T^t and flux T^x are as defined in (4.64). Equation (4.80) becomes

$$1\{u_t - \omega u^{1/2} u_x - u_x^2 - u u_{xx}\} = D_t T^t + D_x T^x.$$

which simplifies to

$$u_t - \omega u^{1/2} u_x - u_x^2 - u u_{xx} = T_t^t + u_t T_u^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x.$$

Splitting the above equation on second derivatives of u, we obtain

$$u_{xx} : T_{u_x}^x = -u, \tag{4.81}$$

$$\text{rest} : T_t^t + u_t T_u^t + T_x^x + u_x T_u^x = u_t - \omega u^{1/2} u_x - u_x^2. \tag{4.82}$$

Integrating equation (4.81) with respect to u_x, we obtain

$$T^x = -u u_x + A(t, x, u),$$

where $A(t, x, u)$ is an arbitrary function of t, x and u. Equation (4.82) then becomes

$$T_t^t + u_t T_u^t + A_x + u_x A_u = u_t - \omega u^{1/2} u_x.$$

Splitting the above equation on the derivatives of u, we obtain

$$u_t : T_u^t = 1, \tag{4.83}$$

$$u_x : A_u = -\omega u^{1/2}, \tag{4.84}$$

$$\text{rest} : T_t^t + A_x = 0. \tag{4.85}$$

Equation (4.83) produces

$$T^t = u + B(t, u).$$

where $B(t, x)$ is a function depending on t and x. Solving equation (4.84) yields

$$A = -\frac{2}{3}\omega u^{\frac{3}{2}} + C(t, x),$$

where $C(t, x)$ is a function of t and x. We set $B(t, x) = C(t, x) = 0$ since they both contribute to the trivial part of the conservation law and consequently equation (4.85) is satisfied. Thus the local conserved vector related to multiplier $\Lambda = 1$ is

$$T^t = u,$$

$$T^x = -uu_x - \frac{2}{3}\omega u^{\frac{3}{2}}.$$

Remark: It is worthy of note that the multiplier $\Lambda = 1$ tells us that the nonlinear advection-diffusion equation (4.20) is itself a conservation law.

Case 3. n=1

We now compute the conservation laws for (4.2) with $n = 1$ which is (4.34). Looking for the zeroth-order multiplier $\Lambda = \Lambda(t, x, u)$ we obtain the determining equation for (4.34) as

$$\frac{\delta}{\delta u}\left\{\Lambda(t,x,u)\left(u_t - \omega u u_x - u_x^2 - u u_{xx}\right)\right\} = 0, \tag{4.86}$$

which simplifies to

$$\begin{aligned} &u_t\Lambda_u - \omega u u_x\Lambda_u - u_x^2\Lambda_u - u u_{xx}\Lambda_u - \omega u_x\Lambda - u_{xx}\Lambda - D_t(\Lambda) \\ &+D_x(\omega u\Lambda) + D_x(2u_x\Lambda) - D_x^2(\Lambda u) = 0. \end{aligned} \tag{4.87}$$

Expanding the above equation, we obtain

$$\begin{aligned} &u_t\Lambda_u - \omega u u_x\Lambda_u - u_x^2\Lambda_u - u u_{xx}\Lambda_u - \omega u_x\Lambda - u_{xx}\Lambda - \Lambda_t - u_t\Lambda_u \\ &+\omega u\Lambda_x + 2u_x\Lambda_x + \omega u_x\Lambda + \omega u u_x\Lambda_u + 2u_x^2\Lambda_u + 2u_{xx}\Lambda - u\Lambda_{xx} \\ &-u_x\Lambda_x - u u_x\Lambda_{ux} - u_x\Lambda_x - u u_x\Lambda_{ux} - u_x^2\Lambda_u - u_x^2\Lambda_u - u u_x^2\Lambda_{uu} \\ &-u_{xx}\Lambda - u u_{xx}\Lambda_u = 0. \end{aligned} \tag{4.88}$$

Simplification of the above equation and splitting on the various derivatives of u, we obtain

$$\Lambda_u = 0, \tag{4.89}$$

$$\omega u\Lambda_x - \Lambda_t - u\Lambda_{xx} = 0. \tag{4.90}$$

Solving equation (4.89), we obtain

$$\Lambda = Q(t, x), \tag{4.91}$$

where $Q(t, x)$ is a function of t and x. Substituting the value of Λ into (4.90) gives

$$\omega u Q(t,x) - Q_t(t,x) - uQ_{xx} = 0. \tag{4.92}$$

Splitting the above equation on u, we have

$$Q_t(t,x) = 0, \tag{4.93}$$

$$\omega Q_x(t,x) - Q_{xx}(t,x) = 0. \tag{4.94}$$

Equation (4.93) yields

$$Q = Q(x), \tag{4.95}$$

where Q is a function of x. Substituting this value of Q in (4.94) and solving it produces

$$Q(t,x) = C_1 e^{\omega x} + C_2, \tag{4.96}$$

where C_1 and C_2 are constant. Thus, the two multipliers for (4.34) are given as

$$\Lambda_1 = 1 \quad \text{and} \quad \Lambda_2 = e^{\omega x}.$$

We first generate conservation laws of (4.34) associated with the multiplier $\Lambda_1(t,x,u)$. Therefore we consider

$$\Lambda_1(t,x,u)\{u_t - \omega u u_x - u_x^2 - u u_{xx}\} = D_t T^t + D_x T^x, \tag{4.97}$$

where the density T^t and flux T^x are as defined in (4.64). Thus the above equation (4.97) then expands to

$$u_t - \omega u u_x - u_x^2 - u u_{xx} = T_t^t + u_t T_u^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x.$$

Splitting the above equation on second derivatives of u

$$u_{xx} : T_{u_x}^x = -u, \tag{4.98}$$

$$\text{rest} : T_t^t + u_t T_u^t + T_x^x + u_x T_u^x = u_t - \omega u u_x - u_x^2. \tag{4.99}$$

Integrating equation (4.98) yields

$$T^x = -u u_x + A(t,x,u),$$

where $A(t,x,u)$ is an arbitrary function of t, x and u. Substituting the above value of T^x in equation (4.99), we have

$$T_t^t + u_t T_u^t + A_x + u_x A_u = u_t - \omega u u_x.$$

Splitting the above equation on the derivatives of u, we obtain

$$u_t : T^t_u = 1, \tag{4.100}$$

$$u_x : A_u = -\omega u, \tag{4.101}$$

$$\text{rest} : T^t_t + A_x = 0. \tag{4.102}$$

Equation (4.100) produces

$$T^t = u + B(t, x),$$

where $B(t, x)$ is a function of t and x. Solving equation (4.101) then gives

$$A = -\frac{1}{2}\omega u^2 + C(t, x),$$

where $C(t, x)$ is a function depending on t and x. We take functions $B(t, x)$ and $C(t, x)$ to be zero since they both contribute to the trivial part of the conservation law so that equation (4.102) is satisfied. Therefore, the conserved vector associated with multiplier Λ_1 is

$$T^t = u,$$
$$T^x = -uu_x - \frac{1}{2}\omega u^2.$$

We now compute the conservation law for the second multiplier $\Lambda_2 = e^{\omega x}$. Thus we consider

$$\Lambda_2(t, x, u)\{u_t - \omega u u_x - u_x^2 - u u_{xx}\} = D_t T^t + D_x T^x \tag{4.103}$$

and so we have

$$e^{\omega x}\{u_t - \omega u u_x - u_x^2 - u u_{xx}\} = D_t T^t + D_x T^x.$$

Expanding the above equation, we have

$$e^{\omega x}\{u_t - \omega u u_x - u_x^2 - u u_{xx}\} = T^t_t + u_t T^t_u + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x}.$$

Splitting the above equation on the second derivatives of u, we have

$$u_{xx} : T^x_{u_x} = -u e^{\omega x}, \tag{4.104}$$

$$\text{rest}: T^t_t + u_t T^t_u + T^x_x + u_x T^x_u = e^{\omega x}\{u_t - \omega u u_x - u_x^2\}. \tag{4.105}$$

Integrating equation (4.104) with respect to u_x gives flux

$$T^x = -u u_x e^{\omega x} + E(t, x, u),$$

where $E(t, x, u)$ is a function of t, x and u. Consequently, equation (4.105) becomes

$$T^t_t + u_t T^t_u - u u_x \omega e^{\omega x} + E_x + u_x E_u - u_x^2 e^{\omega x} = e^{\omega x}\{u_t - \omega u u_x - u_x^2\}.$$

Simplifying the above equation further and splitting it on the derivative of u yields

$$u_t : T^t_u = e^{\omega x}, \tag{4.106}$$

$$u_x : E_u = 0, \tag{4.107}$$

$$\text{rest} : T^t_t + E_x = 0. \tag{4.108}$$

Solving equation (4.106) produces

$$T^t = u e^{\omega x} + F(t, x),$$

where $F(t, x)$ is an arbitrary function of t and x. Equation (4.107) then gives

$$E = E(t, x),$$

where $E(t, x)$ is a function depending on t and x. We set $E(t, x) = F(t, x) = 0$ due to the fact that they contribute to the trivial part of the conserved vectors and so equation (4.108) is satisfied. Thus, the conservation law for the second multiplier Λ_2 is

$$T^t = u e^{\omega x},$$

$$T^x = -u u_x e^{\omega x}.$$

Remark: It should be noted that the multiplier $\Lambda_1 = 1$ tells us that the nonlinear advection-diffusion equation (4.34) is itself a conservation law.

4.3.2 Conservation laws of (4.2) using Ibragimov's method

In this subsection we construct conservation laws of the three cases of the generalized nonlinear advection-diffusion equation using the new conservation theorem due to Ibragimov [48].

Case 1. n≠ 1/2, 1

We first derive the conservation laws of equation (4.2) when $n \neq 1/2, 1$ by determining its adjoint equation using the formula

$$F^* \equiv \frac{\delta}{\delta u}\left\{v\left(u_t - \omega u^n u_x - u_x^2 - u u_{xx}\right)\right\} = 0, \tag{4.109}$$

where $v = v(t,x)$ and the Euler-Lagrange operator $\delta/\delta u$ is defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_x D_t\frac{\partial}{\partial u_{tx}} - \cdots,$$

with the total derivatives D_t and D_x being given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \cdots. \end{aligned}$$

Simplification of equation (4.109) gives the adjoint equation

$$F^* \equiv -v_t + \omega u^n v_x - u v_{xx} = 0. \tag{4.110}$$

Considering equation (4.2) and its adjoint (4.110), we obtain the Lagrangian of the equation as

$$\mathcal{L} = v\left(u_t - \omega u^n u_x - u_x^2 - u u_{xx}\right). \tag{4.111}$$

According to the Lagrangian (4.111) and the formula in equation (1.58), then the expanded formulae for computing the conserved vectors corresponding to an infinitesimal generator are

$$C^t = \tau\mathcal{L} + W^1\frac{\partial\mathcal{L}}{\partial u_t} + W^2\frac{\partial\mathcal{L}}{\partial v_t}, \tag{4.112}$$

$$C^x = \xi\mathcal{L} + W^1\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_x\frac{\partial\mathcal{L}}{\partial u_{xx}}\right] + D_x\left(W^1\right)\frac{\partial\mathcal{L}}{\partial u_{xx}}. \tag{4.113}$$

We recall that (4.2) has three symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \quad \text{and} \quad X_3 = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u}.$$

Let us first take into consideration the infinitesimal generator X_1. We easily note that the extension of the generator X_1 is the same translation symmetry $X_1 = \partial/\partial t$. In order to obtain the value of λ we apply the equation

$$X(F) = \lambda(F),$$

which produces $\lambda = 0$. Again, we observe that

$$D_t(\tau) = 0.$$

Applying (1.56), we obtain $\eta^* = 0$. Consequently, the operator admitted by the adjoint equation (4.110) is

$$Y = \frac{\partial}{\partial t}.$$

We now use the above value of generator Y together with (1.60) to compute the Lie characteristic functions which are

$$W^1 = -u_t \quad \text{and} \quad W^2 = -v_t.$$

Therefore, the conserved vectors for equations (4.2) and (4.110) corresponding to the infinitesimal generator $X_1 = \partial/\partial t$ can be obtained by using the values of C^t and C^x in (4.112) and (4.113). Thus the following conserved vectors for the generator are obtained:

$$C^t = -\omega u^n v u_x - v u_x^2 - u v u_{xx},$$

$$C^x = \omega u^n v u_t + v u_x u_t - u v_x u_t + u v u_{tx}.$$

Similarly, we generate the conserved vector corresponding to the infinitesimal generator $X_1 = \partial/\partial x$. In this case we have the Lie characteristics $W^1 = -u_x$ and $W^2 = -v_x$ and so the conserved vector stipulated by applying (4.112) and (4.113) gives

$$C^t = -v u_x,$$

$$C^x = vu_t - uv_xu_x.$$

We now consider the third infinitesimal generator

$$X_3 = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u}, \tag{4.114}$$

admitted by the generalized nonlinear advection-diffusion equation (4.2). The prolongation of X to the derivatives involved in the advection-diffusion equation is of the form

$$X_3 = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}, \tag{4.115}$$

where ζ_t, ζ_x are as defined in (1.15) and ζ_{xx} in (1.16). Hence their values are given as

$$\zeta_t = \frac{-2n}{2n-1}u_t, \quad \zeta_x = \frac{-n}{2n-1}u_x \quad \text{and} \quad \zeta_{xx} = -u_{xx}, \tag{4.116}$$

which makes the equation (4.115) become

$$X_3 = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u} - \frac{2n}{2n-1}u_t\frac{\partial}{\partial u_t} - \frac{n}{2n-1}u_x\frac{\partial}{\partial u_x} - u_{xx}\frac{\partial}{\partial u_{xx}}.$$

Thus, application of equation (1.57) produces

$$\lambda = -\frac{2n}{2n-1}. \tag{4.117}$$

We also note that by applying equation (1.56), we obtain

$$\eta^* = -\left(\frac{n-2}{2n-1}\right)v. \tag{4.118}$$

The values of D_t and D_x are

$$D_t(\tau) = 1 \quad \text{and} \quad D_x = \frac{n-1}{2n-1}.$$

Therefore the extension (1.55) of the operator (4.114) to v put up the form

$$Y = t\frac{\partial}{\partial t} + \frac{n-1}{2n-1}x\frac{\partial}{\partial x} - \frac{1}{2n-1}u\frac{\partial}{\partial u} - \frac{n-2}{2n-1}v\frac{\partial}{\partial v}. \tag{4.119}$$

One can obviously verify that it is admitted by (4.2) and (4.110). Subsequently, from (1.60) the Lie characteristic functions become

$$W^1 = -\frac{1}{2n-1}u - tu_t - \frac{n-1}{2n-1}xu_x \quad \text{and} \quad W^2 = -\frac{n-2}{2n-1}v - tv_t - \frac{n-1}{2n-1}xv_x.$$

Thus, the conserved vector for the equation (4.2) and (4.110) corresponding to the Lie operator X_3 can then be obtained by using (4.112) and (4.113). Hence the conserved vectors are

$$\begin{aligned} C^t &= -\omega t v u^n u_x - t v u_x^2 - t u v u_{xx} - \frac{1}{2n-1} u v - \frac{n-1}{2n-1} x v u_x, \\ C^x &= \frac{n-1}{2n-1} x v u_t + \frac{1}{2n-1} \omega u^{n+1} v + \omega t u^n v u_t + \frac{1}{2n-1} u v u_x \\ &+ t v u_x u_t - \frac{1}{2n-1} u^2 v_x - t u v_x u_t - \frac{n-1}{2n-1} x u v_x u_x + \frac{1}{2n-1} u v u_x \\ &+ t u v u_{tx} + \frac{n-1}{2n-1} u v u_x. \end{aligned}$$

Case 2. n$\neq$ 1/2

We now compute the conservation laws of equation (4.20) by determining its adjoint equation using the well-known formula

$$F^* \equiv \frac{\delta}{\delta u}\left\{ v\left(u_t - \omega u^{1/2} u_x - u_x^2 - u u_{xx}\right)\right\} = 0, \tag{4.120}$$

which yields the adjoint equation of (4.20) as

$$F^* \equiv -v_t + \omega u^{1/2} v_x - u v_{xx} = 0. \tag{4.121}$$

Now considering equation (4.20) as well as its adjoint (4.121), we then obtain the Lagrangian of (4.20) as

$$\mathcal{L} = v\left(u_t - \omega u^{1/2} u_x - u_x^2 - u u_{xx}\right). \tag{4.122}$$

We recall that equation (4.20) has the following Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \quad \text{and} \quad X_3 = x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

We first consider the symmetry generator $X_1 = \partial/\partial t$. We note that the extension of the generator X_1 to the new variable v produces the same translation symmetry $X_1 = \partial/\partial t$. In order to obtain the value of λ we apply the equation (1.57) which gives $\lambda = 0$. We observe that $D_t(\tau) = 0$ and so $\eta^* = 0$. Eventually, the operator admitted by the adjoint equation (4.121) is

$$Y = \frac{\partial}{\partial t}.$$

Using the above value of operator Y together with (1.60) to obtain the Lie characteristic functions we have

$$W^1 = -u_t \quad \text{and} \quad W^2 = -v_t.$$

Hence, the conserved vectors for equations (4.20) related to the Lie symmetry $X_1 = \partial/\partial t$ can be established using the values of C^t and C^x. Thus, the conserved vectors for the operator are

$$C^t = -\omega u^{1/2} v u_x - v u_x^2 - u v u_{xx},$$
$$C^x = \omega u^{1/2} v u_t + v u_x u_t - u v_x u_t + u v u_{tx}.$$

Likewise, we compute the conserved vector related to the Lie operator $X_1 = \partial/\partial x$. In this case we obtain the Lie characteristics $W^1 = -u_x$ and $W^2 = -v_x$ and so the conserved vectors acquired by applying equations (4.112) and (4.113) are

$$C^t = -v u_x,$$
$$C^x = v u_t - u v_x u_x.$$

We next take the scaling Lie symmetry generator

$$X_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \tag{4.123}$$

admitted by the equation (4.20) and subsequently extend it to another variable v. The prolongation of the above symmetry to the derivatives involved in the generalized nonlinear advection-diffusion equation with $n = 1/2$ is of the form

$$X_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + 2u_t\frac{\partial}{\partial u_t} + u_x\frac{\partial}{\partial u_x}.$$

We now use the equations (1.57) and (1.56) to obtain the values of λ and η^* respectively, and so $\lambda = 2$ and $\eta^* = -3v$. We equally note that $D_t(\tau) = 0$ and $D_x(\xi) = 1$. Hence, the extension (1.55) of the generator to v is of the form

$$Y = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} - 3v\frac{\partial}{\partial v}. \tag{4.124}$$

Using equation (1.60), the Lie characteristics functions become

$$W^1 = 2u - x u_x \quad \text{and} \quad W^2 = -3v - x v_x.$$

Finally, from equations (4.112) and (4.113) we obtain the conserved vectors C^t and C^x which corresponds to the above scaling symmetry and satisfies the conservation equation $D_t(C^t) + D_x(C^x) = 0$. Therefore the required conserved vectors are

$$C^t = 2uv - xvu_x,$$
$$C^x = xvu_t - 2\omega u^{3/2}v - 3uvu_x + 2u^2v_x - xuu_xv_x.$$

Case 3. n=1

Conclusively, we generate the conservation laws of equation (4.34) by first obtaining its adjoint equation using

$$F^* \equiv \frac{\delta}{\delta u}\left\{v\left(u_t - \omega uu_x - u_x^2 - uu_{xx}\right)\right\} = 0. \tag{4.125}$$

The above equation produces the adjoint equation of (4.34) which is

$$F^* \equiv -v_t + \omega uv_x - uv_{xx} = 0. \tag{4.126}$$

We note that considering equation (4.34) and its adjoint (4.126), we have the Lagrangian of (4.34) to be

$$\mathcal{L} = v\left(u_t - \omega uu_x - u_x^2 - uu_{xx}\right). \tag{4.127}$$

We equally note that equation (4.34) has the following translational and scaling symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \quad \text{and} \quad X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

In the first place, we consider the infinitesimal generator $X_1 = \partial/\partial t$ of (4.34). We see that extending X_1 to the new variable v gives the same translation symmetry $X_1 = \partial/\partial t$. In a bid to determine the value of λ we use the equation (1.57) so that we obtain $\lambda = 0$. We observe that $D_t(\tau) = 0$ and so $\eta^* = 0$. Consequently, the operator that is being admitted by the adjoint equation (4.126) is

$$Y = \frac{\partial}{\partial t}.$$

We now use the above equation together with (1.60) to obtain the values of W^1 and W^2 which are

$$W^1 = -u_t \quad \text{and} \quad W^2 = -v_t.$$

Therefore, the conservation law for (4.34) correlating to the translational symmetry $X_1 = \partial/\partial t$ using the formula of C^t and C^x in equations (4.112) and (4.113) respectively yields

$$C^t = -\omega uvu_x - vu_x^2 - uvu_{xx},$$

$$C^x = \omega uvu_t + vu_xu_t - uv_xu_t + uvu_{tx}.$$

Correspondingly, we generate the conserved vector rendered to the Lie point symmetry $X_1 = \partial/\partial x$. In this case we have the Lie characteristics $W^1 = -u_x$ and $W^2 = -v_x$, having realized that $D_x(\xi) = 0$. Therefore, the conserved vectors obtained by applying equations (4.112) and (4.113) produce

$$C^t = -vu_x,$$

$$C^x = vu_t - uv_xu_x.$$

Finally, we consider the scaling symmetry generator

$$X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \tag{4.128}$$

admitted by the nonlinear advection-diffusion equation (4.34) and then extend it to a new variable v. The prolongation of X_3 in (4.128) to the derivatives included in the equation (4.34) is of the form

$$X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2u_t\frac{\partial}{\partial u_t} - u_x\frac{\partial}{\partial u_x} - u_{xx}\frac{\partial}{\partial u_{xx}}.$$

Using equations (1.57) and (1.56) in order to obtain the values of λ and η^* respectively, we have $\lambda = -2$ and $\eta^* = v$. We note that $D_t(\tau) = 1$ and $D_x(\xi) = 0$ and the extension (1.55) of the generator to v has the form

$$Y = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

Applying equation (1.60), the Lie characteristics functions W^1 and W^2 then become

$$W^1 = -u - tu_t \quad \text{and} \quad W^2 = v - tv_t.$$

Therefore, the conserved vector for the equations (4.34) and (4.126) relating to the infinitesimal generator X_3 of the nonlinear advection-diffusion equation (4.34) can now be obtained by the application of the formulae in (4.112) and (4.113). Hence the conserved vectors found are

$$C^t = -uv - \omega tuvu_x - tvu_x^2 - tuvu_{xx},$$
$$C^x = \omega u^2 v + 2uvu_x - u^2 v_x + \omega tuvu_t + tvu_t u_x - tuv_x u_t + tuvu_{tx}.$$

4.4 Concluding remarks

In this chapter we studied the generalized nonlinear advection-diffusion with power law nonlinearity (4.2). The analysis of the equation prompted three different cases for n. For each case Lie point symmetries were computed and then used to construct an optimal system of one-dimensional subalgebras. Thereafter symmetry reductions and group-invariant solutions were obtained based on these optimal systems of one-dimensional subalgebras. Moreover, for each case conservation laws were derived by two differential approaches: the multiplier method and the conservation theorem due to Ibragimov.

Chapter 5

Concluding remarks

Many of the real-world physical systems of fluid mechanics, material science, elasticity, thermodynamics, biology, gas dynamics and so on are modelled by nonlinear partial differential equations. It is therefore of importance to study such systems with a view to finding their exact solutions and conservation laws. Thus, in this dissertation we have studied two nonlinear partial differential equations namely the modified equal-width and the nonlinear generalized advection-diffusion equations.

In Chapter one we presented the relevant literature which was used in this dissertation. Various methods for finding the exact solutions of nonlinear partial differential equations were described and different methods for deriving conservation laws were discussed.

Chapter two studied the potential Burgers equation as an illustrative example. We obtained the Lie point symmetries of the equation, computed the commutator table for the symmetries and generated the one-parameter groups for them. We later found group-invariant solutions including the travelling wave solution of the potential Burgers equation. Moreover, we derived the conservation laws for the equation using the multiplier approach.

In Chapter three we examined the modified equal-width equation by first computing the infinitesimal generators of the equation. We proceeded into using the obtained generators to construct an optimal system of one-dimensional subalgebras. Moreover, the utilisation of the optimal system of one dimensional subalgebras, symmetry reductions and new group-invariant solutions of the modified equal-width equation (3.1) were presented. The solutions obtained were cnoidal and snoidal waves. We also derived the conservation laws of the modified equal-width equation using the multiplier approach, which resulted in two multipliers and consequently, two conserved vectors. Furthermore, we employed Noether's theorem to obtain the conservation laws of the equation which made it possible to generate one local and two nonlocal conserved vectors of the equation.

In Chapter four, the generalized nonlinear advection-diffusion equation was investigated, in which we had to consider three cases depending on the values of the parameter n. Lie symmetry analysis for each case was performed in which the Lie symmetries found were utilised for constructing the optimal system of one-dimensional subalgebras in each case. Furthermore, we engaged the optimal system of one-dimensional subalgebras to obtain the symmetry reductions, group-invariant solutions as well as the travelling wave solutions in all the cases. In addition, we then derived the conservation laws of the cases by applying the multipliers method as well as the new conservation theorem due to Ibragimov.

In future work, we plan to use the conservation laws obtained here to construct exact solutions for the partial differential equations studied in this dissertation.

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