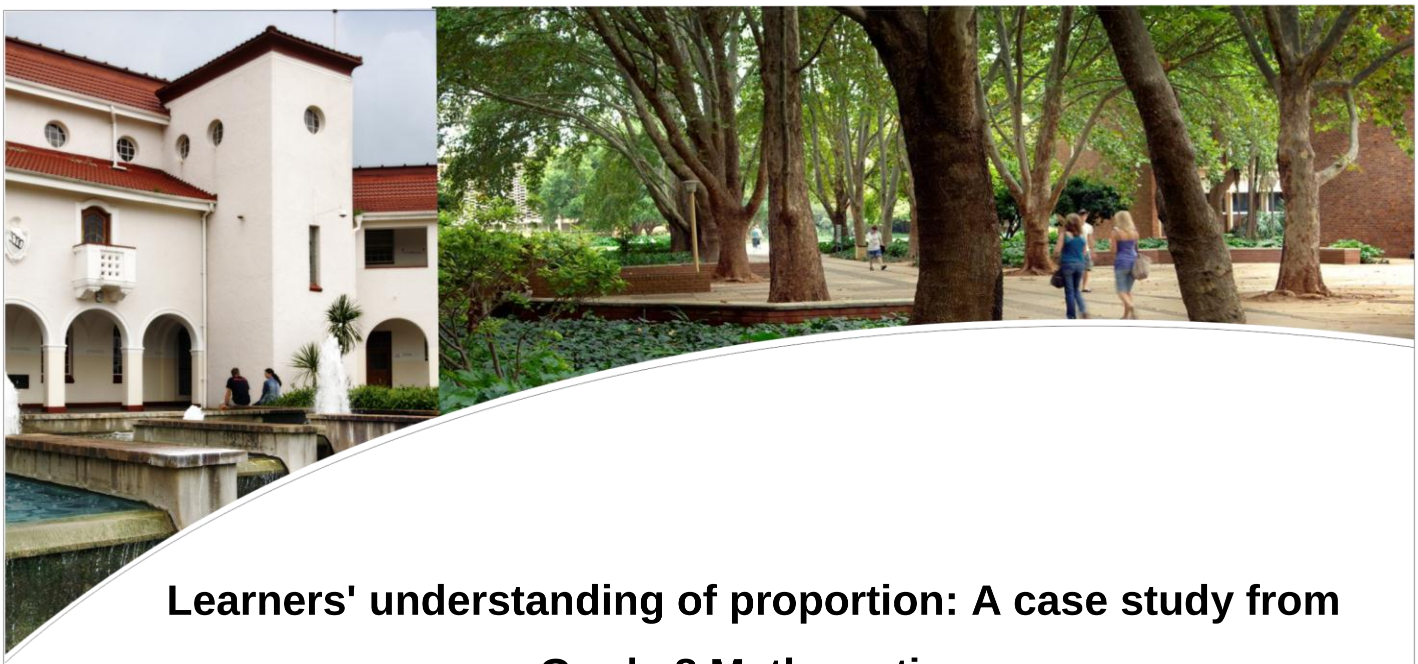

Learners' understanding of proportion: A case study from Grade 8 Mathematics

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Dissertation submitted in fulfillment of the requirements for the degree ***Magister Educationis*** in **Mathematics Education** at the Potchefstroom Campus of the North-West University

Supevisor: **Prof. HD Nieuwoudt**

Co- supervisor: **Dr. A Roux**

May 2014

It all starts here ™

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ACKNOWLEDGEMENTS

This study is dedicated to my parents, Rookeya and Mohamed Jeeva, my late mother-in-law, Mrs Fatima Bhoola, not forgetting the man behind the woman, my kind, loving husband, Farid Suliman. My proud parents are the sources of inspiration in my life. My husband was my pillar of strength during arduous days and nights.

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SHARIFA SULIMAN

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I herewith declare that I was responsible for the language editing of the dissertation:

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ABSTRACT

Underachievement in Mathematics hangs over South African Mathematics learners like a dark cloud. TIMSS studies over the past decade have confirmed that South African learners' results (Grades 8 and 9 in 2011) remained at a low ebb, denying them the opportunity to compete and excel globally in the field of Mathematics.

It is against this backdrop that the researcher investigated the meaningful understanding of the important yet challenging algebraic concept of Proportion. The theoretical as well as the empirical underpinnings of the fundamental idea of Proportion are highlighted. The meaningful learning of Algebra was explored and physical, effective and cognitive factors affecting meaningful learning of Algebra, views on Mathematics and learning theories were examined. The research narrowed down to the meaningful understanding of Proportion, misconceptions, and facilitation in developing Proportional reasoning.

This study was embedded in an interpretive paradigm and the research design was qualitative in nature. The qualitative data was collected via task sheets and interviews. The sample informing the central phenomenon in the study consisted of a heterogeneous group of learners and comprised a kaleidoscope of nationalities, both genders, a variety of home languages, differing socio-economic statuses and varying cognitive abilities. The findings cannot be generalised.

Triangulation of the literature review, the analysis of task sheets and interviews revealed that overall the participants have a meaningful understanding of the Proportion concept. However, a variety of misconceptions were observed in certain cases.

Finally, recommendations are made to address the meaningful learning of Proportion and its associated misconceptions. It is hoped that teachers read and act on the recommendations as it is the powerful mind and purposeful teaching of the teacher that can make a difference in uplifting the standard of Mathematics in South African classrooms!

KEY WORDS:

Mathematics education

Proportional reasoning

Learning

Algebra

Fractions

Decimals

Ratios

Percentages

Underachievement

Qualitative approach

Misconceptions

Understanding

Learners

Meaningful learning

Teacher knowledge

OPSOMMING

Onderprestering in Wiskunde hang soos ŉ donker wolk oor Suid-Afrikaanse Wiskunde-leerders. TIMSS-studies oor die afgelope dekade het bevestig dat Suid-Afrikaanse leerders se uitslae (Graad 8 en 9 in 2011) steeds op ŉ lae vlak was, wat hulle die geleentheid ontsê om op die gebied van Wiskunde globaal te wedywer en uit te blink.

Dit is teen hierdie agtergrond dat die navorser die betekenisvolle begrip van die belangrike, dog uitdagende algebraïese konsep van Proporsie ondersoek het. Die teoretiese sowel as die empiriese onderstutting van die fundamentele idee van Proporsie word uitgelig. Die betekenisvolle leer van Algebra is nagespeur en fisiese, doeltreffende en kognitiewe faktore wat die betekenisvolle leer van Algebra, menings oor Wiskunde en leerteorieë affekteer, is ondersoek. Die navorsing het neergekom op die betekenisvolle begrip van Proporsie, wanpersepsies, en fasilitering in die ontwikkeling van Proporsionele beredenering.

Hierdie studie is geanker in ŉ verklarende paradigma en die navorsingsontwerp was kwalitatief van aard. Die kwalitatiewe data is via werksvelle en onderhoude versamel. Die steekproef wat die sentrale fenomeen in die studie ingelig het, het bestaan uit 'n heterogene groep leerders wat ŉ kaleidoskoop nasionaliteite, beide geslagte, ŉ verskeidenheid moedertale, verskillende sosio-ekonomiese statusse en verskeie kognitiewe vermoëns ingesluit het. Die bevindinge kan nie veralgemeen word nie.

Triangulasie van die literatuur oorsig, die analise van werkskaarte en onderhoude het dit duidelik gemaak dat die deelnemers aan hierdie studie in die algemeen oor ŉ betekenisvolle begrip van die Proporsiekonsep beskik. ŉ Verskeidenheid wanpersepsies is egter in sekere gevalle waargeneem.

Ten laaste word aanbevelings gemaak om die betekenisvolle leer van Proporsie en die verwante wanpersepsies aan te spreek. Die hoop is dat onderwysers die aanbevelings sal lees en daarop reageer aangesien dit die onderwyser se kragtige denke en doelgerigte onderrig is wat 'n verskil kan maak in die opheffing van die standaard van Wiskunde in Suid-Afrikaanse klaskamers!

SLEUTELWOORDE:

Wiskunde-onderrig

Proporsionele denke/redenering

Leer

Algebra

Breuke

Desimale

Verhoudings

Persentasies

Onderprestering

Kwalitatiewe benadering

Wanpersepsies

Begrip

Leerders

Betekenisvolle leer

Onderwyserkennis

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STATEMENT OF THE PROBLEM AND MOTIVATION

1.1 STATEMENT OF THE PROBLEM

Mathematics is a key area of knowledge and competence for the development of the individual and the social and economic development of South Africa in a globalising world (Reddy, 2005:125). Following current insights about the nature of Mathematics and of school Mathematics education (NCTM, 2000), the definition of Mathematics provided in the New Revised National Curriculum Statement for Grades R-9 in South Africa (Department of Education, 2002:1) state that mathematical ideas and concepts build on one another to create a coherent structure. Such a view implies that learners need to understand certain key or fundamental mathematical concepts that contain the rudiments of more advanced mathematical concepts and provide a foundation for learners' further mathematical learning (Ma, 1999:124). The CAPS document on Mathematics in the Senior Phase, Grades 7-9 (DBE, 2013:8) defines Mathematics as follows:

"Mathematics is a language that makes use of symbols and notations to describe numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomenon and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem-solving that will continue in decision making" (DBE, 2013).

Understanding key ideas (e.g. number and spatial sense, proportion, function, problem-solving) is essential for learners to be able to build deep progressive conceptual understanding of the Mathematics they have to learn (Ma, 1999; Van den Heuvel-Panhuizen, 2008; Van Galen *et al*., 2008).

The reputed large scale international studies TIMSS (1998), its repeat TIMSS-R (1999) and TIMSS (2003) indicate that South African Grade 8 Mathematics learners do not understand key concepts of Mathematics and consequently lack Mathematics proficiency to such an extent that South Africa came last out of 50 participating countries, far below the international average (Reddy, 2006:12). In addition, in 2000 South Africa participated in the second study conducted by the Southern and Eastern African Consortium for Monitoring Education Quality (SACMEQ), a project popularly known as SACMEQ II, in which 15 countries from southern and eastern Africa participated. A random national sample of learners was tested in numeracy. Again, South African learners performed particularly poorly in Mathematics (Moloi, 2005:2).

Analyses of the findings of the above-mentioned high-profile studies highlight a major difference between South Africa's approach to the school Mathematics curriculum and that of other countries. Most countries place emphasis on understanding of mathematical concepts and principles while South Africa places emphasis on applying Mathematics to real-life situations and multicultural approaches (Reddy, 2006:82). South Africa evidently has a problem in school Mathematics education regarding the deep understanding of key mathematical concepts and the provision of learning programmes that provide effective opportunity for building conceptual understanding of those fundamental concepts. This is despite interventions such as the SYSTEM (Students and Youth in Science, Technology and Mathematics) Project and the Dinaledi Project that were introduced to improve the quality of the learning and teaching of Mathematics in South African schools.

Reddy (2005:127) posits that the ultimate indicator of the required success in school Mathematics will be the nature and quality of the learners' performance. She therefore accentuates the importance and necessity of finding meaningful ways to improve learner performance, particularly with regard to conceptual understanding.

The researcher undertook this study to find out how learners understand one such fundamental mathematical idea: the particularly important and difficult concept of proportion and how their understanding of it can be improved in classes. The researcher, therefore, focused her investigation on learners' conceptual understanding of proportion, misconceptions that may occur, grounds for those misconceptions, and what can be done to improve the learners' fundamental understanding of proportion.

1.2 REVIEW OF LITERATURE

Of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging and one of the most compelling research sites (Lamon, 2007:629). Proportional reasoning focuses on describing, predicting or evaluating the relationship between two relationships rather than a relationship between two concrete objects (Baxter & Junker, 2001:12). The development of learners' proportional reasoning can be regarded as the gateway to success in studying Algebra. Proportional reasoning is important as it is related to several key areas of school Mathematics curricula: fractions, long division, place value, percentage, ratio, rate, transformations and comparing costs (Baxter & Junker, 2001:5). From a Chinese perspective, for example, proportional reasoning is one of the most important forms of mathematical reasoning (Cai & Sun, 2002:195) as it entails the ability to mentally store and process several pieces of information.

One might learn the importance of understanding concepts in the Mathematics curriculum from Cangelosi (2003:177). He defines a concept as a category people mentally construct by creating a class of specifics possessing a common set of characteristics. He emphasises that concepts are the building blocks of knowledge and that constructing concepts in our mind enables us to extend what we understand beyond the specific situations we have experienced in the past. Conceptual knowledge is knowledge that is understood. This knowledge is equated with connected networks of ideas in the mind (long-term memory), and is required for mathematical expertise through its relationship with procedural knowledge (Hiebert & Carpenter, 1992:78). Conceptual knowledge is needed for problem- solving (Lesh & Zawojewski, 2007:782), therefore, for making sense of Mathematics and mathematical ideas to be learned and utilised in school and elsewhere.

Learners need to learn with understanding. According to Hiebert and Carpenter (1992:80), understanding promotes remembering and enhances transfer and use of knowledge that needs to be simultaneously held in short-term memory. The Carpenter and Lehrer model (1999, as quoted in Malloy, 2004:3) focuses on helping learners gain conceptual understanding. Learners understand when they can construct relationships, apply mathematical knowledge, reflect, articulate and make mathematical knowledge their own. This model explains that developing understanding requires more than connecting new and prior knowledge; it requires a structuring of knowledge so that new knowledge can be related to and incorporated into existing networks of knowledge. This leads to a reorganisation of the learner's mathematical knowledge. Teaching assists learners in this regard by providing meaningful "learning trajectories" that guide them through constructive engagement with relevant thinking, activities and reflection in their grappling with ideas (Van Galen *et al*., 2008; Van den Heuvel-Panhuizen, 2008).

Traditionally, learners think of doing (and learning) Mathematics as following set rules. If, however, they can be guided to see and form connections between different representational systems, they would learn to view Mathematics as a cohesive body of knowledge, and realise that information acquired in one setting will connect with information acquired in another setting. Such views or beliefs would, in turn, support meaningful learning and the formation of integrated mathematical knowledge (Hiebert & Carpenter, 1992:77).

As indicated above, proportional reasoning is a difficult and important conceptual leap for students (Baxter & Junker, 2001:5). Learners' experience with proportional reasoning begins in some elementary form in the early grades and extends through multiple topics in Mathematics. Before learning anything in our classrooms, learners engage in activities in the everyday world where they generate ideas about fractions, ratios and proportionality. They bring constructed prior knowledge into their Mathematics classrooms where it interacts with what the content curriculum and teaching offer. Mathematically successful learners manage to connect these two bodies of knowledge (Smith III, 2002:3), and teaching through proper trajectories needs to be instrumental toward accomplishing that (see above).

Proportional reasoning develops over a period and continues to be problematic for learners in middle school years and beyond. It is developed through trajectory activities that involve investigating and representing (for them) suitable realistic mathematical problem situations, comparing and determining relationships of uniqueness and sameness (equivalence) of fractions, decimals, percentages and ratios, and executing proportional tasks in a wide variety of problem-based contexts without recourse to rules or formulas (Van Galen *et al*., 2008). Lamon's (2007) research indicates that learners may need as much as three years worth of opportunities to reason in multiplicative situations in order to adequately develop proportional reasoning skills.

The ability to reason proportionally was a hallmark of Piaget's distinction between concrete levels of thought and formal operational thought. A learner has to begin to understand multiplicative relationships where most arithmetic concepts are additive in nature. Concepts and connections develop over a period of time, not in a day (Van de Walle *et al*., 2007:26). Through early counting experiences children begin to develop concepts of units and composite units. In turn, these conceptual structures provide a foundation for understanding mathematical topics that build on the concept of unit: place value, measurement, fractions and proportional reasoning (English, 2002:123). Different learners will use different ideas to give meaning to the same new idea. Every learner's construction of ideas is unique within the same environment or classroom (Van de Walle *et al*., 2004:22). In order to facilitate and support successful proportional reasoning and learning the teacher, therefore, cannot rely on a single-route trajectory, but has to support learners to negotiate through what Dolk and Fosnot (2001) fittingly identify as a "landscape of learning" events.

Misconceptions occur when learners' prior knowledge is incompatible with the notion of the new conceptualisation and where learners are prone to have systematic errors, suggesting that prior knowledge interferes with the acquisition of new concepts (Merenluoto, 2004:297). This kind of situation is typical when learners are struggling to learn the concept of rational numbers while their prior thinking of numbers is based on natural numbers. Learners are prone to have difficulties with decimals and fractions. Sometimes the reorganisation of new information requires the radical reorganisation of what is already known. It is likely that in the process of reorganisation learners will create misconceptions (Stafylidou & Vosniadou, 2004:504).

Various authors have written on learners' difficulty in understanding proportion. According to Hackenberg and Tillemma (2009:16), learners have difficulty with fractions because of whole number multiplicative concepts. Focusing on procedures encourages students to apply rules without thinking and thus the ability to reason proportionally often does not develop (Van de Walle *et al*., 2010:350). They also state that when it comes to connecting different representational systems, for children the world of fractions and the world of decimals are very distinct. Learners do not realise that decimal and percentage notation are simply two other methods of representing fractions. Without these connections, children learn each new piece of information they encounter as a separate unrelated idea. This is a result of failure to develop the percentage concept meaningfully.

From the above, it should be obvious that various factors influence the meaningful learning of Mathematics, particularly of proportionality. Researchers such as Maree *et al*. (1997), Hassan (2004) and Roux (2009) commonly assert that certain variables influence the meaningful learning of Mathematics.

Roux (2009:35) mentions some of these factors: learners' mathematical knowledge, problem-solving, learning environment, attitude towards Mathematics, their mathematical ability, Mathematics anxiety and instruction. She asserts that problem-solving requires a learner to interpret a situation mathematically, to describe and explain and not just use rules and procedures to solve a problem. Problem-solving strategies influence learning. She also mentions metacognition as another factor influencing the meaningful learning of Mathematics. Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data (Flavell, 1976:232). A metacognitively active learner will possess strategic knowledge, be self-regulated and plan before solving a problem.

Mathematical language proficiency also plays a vital role in conceptual understanding. International assessment studies have shown that in countries where a large proportion of learners are from homes where the language of teaching and learning in schools is not spoken at home, the Mathematics achievement scores are generally lower (Reddy, 2006:12). These learners need additional scaffolding to acquire mathematical linguistic competence. According to Barnes (2005:49), poor language skills in reading, writing and speaking is associated with low attainment in Mathematics. One aspect of language that is crucial to Mathematics learning pertains to the specialised terminology, symbols and syntax used to express mathematical ideas. The vocabulary and symbols of Mathematics make it similar to learning a foreign language (Clarke & Ramirez, 2004:57). Howie (2003:12) posits that the strength of the language component has strong effects on South African learners' performance in Mathematics. Learners who speak English at home tend to achieve higher scores in Mathematics.

Finally, cooperative and collaborative learning strategies could be used to enhance understanding of concepts, such as multifaceted ideas of proportion and proportional reasoning. When students work with each other, they share ideas, influence and build on the idea of others, justify their ideas to others and consequently create a deeper understanding of the concepts being explored (Dossey *et al*., 2002:502).

1.3 PURPOSE OF THE RESEARCH

The researcher intended to investigate how and why Grade 8 learners in the teaching-learning situation learn and understand proportion, what the grounds are for their understanding or misunderstanding and what can be done to support their meaningful learning of proportion and related concepts.

From the preceding arguments, the following anticipated research questions arose:

Research question 1

How do Grade 8 learners understand the concept proportion?

Research question 2

What are the grounds for their understanding?

Research question 3

What misconceptions occur?

Research question 4

What grounds exist for the observed misconceptions?

Research question 5

What can be done to support the learners to learn proportion with understanding?

1.4 RESEARCH DESIGN AND METHODOLOGY

1.4.1 Research paradigm

As this study's very nature was to understand how learners do or don't learn and understand the ideas relating to proportionality, the paradigm chosen to anchor and guide the theoretical and methodological positions of this study was interpretivism. The ultimate aim of interpretivist research is to offer a perspective of a situation and to analyse the situation under scrutiny to provide insight into the way in which a particular group of people, in this case Grade 8 Mathematics learners, make sense of their situation, particular in reference to the idea of proportionality (Nieuwenhuis, 2007a:60). The researcher became the instrument through which data was collected, analysed and interpreted in order to answer the research questions pertaining to the understanding, or not, of proportionality and to propose ways to support proper understanding through teaching.

1.4.2 The literature study

An intensive review of literature was undertaken regarding Grade 8 learners' conceptual understanding of proportion, whether misconceptions occur, and what grounds exist for such misconceptions, as well as what can be done to support learning with understanding. Included in the literature review are studies undertaken by other researchers to enrich knowledge and to update these studies with existing knowledge.

The researcher made extensive use of the North-West University library catalogue of a primary and secondary nature. Databases and search engines such as EBSCOhost, Eric, Academic Search Premier, JSTOR, Google Scholar and Journal Citation Report were exploited generously. Dissertations, books and journals were also consulted. Keywords for use include: Mathematics, Mathematics achievement/underachievement, fractions, percent, decimals, proportion, proportional reasoning, understanding, conceptual learning, teaching, misconception.

1.4.3 Research design

Qualitative research focuses on describing and understanding phenomena within their naturally occurring context with the intention of developing an understanding of the meaning(s) imparted by the respondents (Nieuwenhuis, 2007a:51). Qualitative research is typically used to answer questions about the complex nature of phenomena, often with the purpose of describing and understanding the phenomena from the participants' point of view (Leedy & Ormrod, 2005:94). The researcher therefore undertook a qualitative research with an interpretative, constructivist approach. A literature study formed the theoretical basis and referential framework for investigations that were undertaken in the classroom where the learners experienced the phenomena under discussion. The purpose of the study necessitated the use of an appropriate qualitative approach while the complexity of the matter required the approach to comprise multiple methods of investigation (see 4.6). A qualitative research design most suitable for the proposed research to understand how learners interpret and give meaning to their experiences in their Mathematics classes is a case study, this being an in-depth description and analysis of a bounded system (Merriam, 2009:40). From an interpretivist perspective, the typical characteristics of case studies is that they strive towards a comprehensive and deep understanding of how participants relate and interact with each other in a specific situation and how they make meaning of phenomena under scrutiny (Nieuwenhuis, 2007b:75). A case study was in this instance aimed at gaining greater insight into and understanding of the dynamics of a specific and complex situation in a particular context, namely of how Grade 8 Mathematics learners in class make sense of the ideas relating to proportionality and of identifying ways to support their efforts to effectively do so (Nieuwenhuis, 2007b:76).

1.4.4 Site or social network selection

The research was carried out at a secondary school in the Rustenburg area in the North West Province. In 2002 this school won an award for being the most racially integrated school. For many years this institution has produced excellent matric results, with distinctions in Mathematics topping the pyramid. This school of excellence was placed in the top 100 schools in a survey carried out by the Sunday Times newspaper. The language medium is English. Mathematics learners from Grade 8 with diverse cultural backgrounds, varying academic abilities and different economic, religious and social circumstances were selected as the study population: Three Grade 8 classes were selected to partake in the research. An assessment task on proportion learned in Grade 7 was given to the three classes. The marks attained gave the researcher an indication of the level of competence of each learner who completed this assessment task. A group of 18 learners (from the three classes), comprising six high achievers, six average achievers and six low achievers, was then selected to partake in individual task sheet completion and task-based interviews throughout that part of the learning programme in which the concept of proportionality was specifically taught.

1.4.5 Researcher's role

The researcher first sought ethical clearance from the Ethics Committee of the North-West University and carried out informed consent procedures with all relevant authorities and participants before commencing the research. The permission of the North West Province Education Department, the principal, educators, learners and their parents were therefore requested and obtained prior to the research commencing.

The role of the researcher was that of participant observer. She was the teacher of the participating class, the primary decision-maker, the monitor regarding the learning and other class events under scrutiny, the conductor of the task-based interviews, the collector of all data, the analyst and interpreter of the data.

1.4.6 Data generation

In order to find answers to the research questions 1 to 5 and to gain a deeper understanding of how Grade 8 learners understand the concept of proportion, multiple forms of data was utilised in this study. Task sheets and interviews were used to assist in finding a correspondence between the literature review and the empirical study.

Task-based interviews are becoming an increasingly important tool in qualitative research. The value of task-based interviews lies in the fact that they provide a structured mathematical environment that, to some extent, can be controlled. Interview contingencies can be decided explicitly and modified when appropriate (Goldin, 2000:520). Goldin further claims that task-based interviews make it possible to focus research attention more directly on the subjects' processes of addressing mathematical tasks rather than on the patterns of correct and incorrect answers in the results they produce.

Task-based interviews were used with the view to gaining greater insight and understanding of the dynamics of the situation and to gather more information on learners' conceptual understanding of proportion. All learners completed a worksheet composed of problems or activities, in writing. The focus was not on right or wrong answers. Instead the researcher focused on what and how learners understood or misunderstood the concept of proportion and its nuances. The researcher analysed responses according to indicators found in literature pertaining to conceptual understanding of proportion and proportional reasoning so as to gain deeper understanding of learners' understanding and misconceptions. Such indicators included the ability to: recognise/identify proportional situations as such; distinguish between and relate different nuances of proportionality; represent proportional situations; apply their knowledge and skill to solve problems involving proportions and proportional reasoning; use appropriate mathematical language and demonstrate positive study orientation elements.

The task-based interviews shed more light on whether learners envisage proportion as a connected network of ideas. The set tasks were the same for all participants and the interviews were conducted on an individual basis by the researcher. The written responses to tasks were analysed according to a rubric and related to the analyses of the verbal discourse between the researcher and the learner during the interviews. The researcher furthermore observed the specific learners in their engagement and participation in relevant tasks and exercises as it occurred in the natural class setting, and also monitored and analysed their written work in this regard.

To answer research question 5, the researcher related the findings of her analyses to recent relevant research literature to deduce and recommend critical measures to be taken to facilitate proper conceptual understanding of proportionality and proportional reasoning in classes, such as the Grade 8 class participating in this case study.

1.4.7 Data analyses

The initial task sheet assessment on proportion was monitored and task performance execution observed and monitored in relation to specific indicators of conceptual understanding of proportion and proportional reasoning (see above). The data analysis was inductive and comparative to develop common themes or patterns that cut across the data. The strategy of triangulation was used to ensure that the analyses were trustworthy. According to Merriam (2009:222), triangulation is a strategy in qualitative research that ensures consistency and dependability or reliability. In this case triangulation rested upon the use of multiple methods, data collection strategies and data sources to obtain data most congruent with reality as understood by the participants (Merriam, 2009:222). Triangulation ensured correlation between findings, the researcher's experience and the reality of the participants. With regard to the use of multiple methods of data collection (initial assessment task and task-based interviews) and analyses, what the participant reports and demonstrates can be used to check if it corresponds to what the literature yields.

In accordance with Creswell's (2013:115) argument concerning data analyses, the researcher read through the written scripts several times to obtain an overall feeling. The researcher identified significant phrases or sentences that pertained directly to the experiences of the participants. Clustering was the next phase, finally integrating the results into an in-depth description of the phenomenon.

1.4.8 Ethical aspects of the research

1.4.8.1 Informed consent (see Addendum A - C)

Permission was requested from the North West Education Department to conduct the empirical research in a secondary school in the Rustenburg region. The principal of the school needed to grant the researcher consent and learners' parents were also requested to grant consent for their children to participate in this research. These requests were all done in writing. The empirical research was carried out after permission was granted by the Ethics Committee of the NWU. As the research would interfere with the school timetable, arrangements were made to utilise available time.

1.4.8.2 Confidentiality and anonymity

Under no circumstances was any participant's name mentioned. They had a right to privacy and anonymity was assured. Following the principles of Resnick (2011:2), the researcher protected confidential communications, these being the task sheets and copies of the task sheets which contained information from the interviews. The learners' performance was not revealed to anyone and participants' names were not written down. The task sheets were labelled from A to R.

1.4.8.3 Use of volunteers

Participation was voluntary. Participants were not forced to partake in the research process. They needed to participate willingly and could withdraw at any time.

1.4.8.4 Honesty

Honesty being the top priority in ethical behaviour, the researcher was honest at all times. The participants were not deceived and relevant information concerning the research was expressed clearly and explicitly. The researcher was honest in reporting data, results, methods and procedures, in compliance with Resnik (2011:2).

1.4.8.5 Compliance with NWU Ethics Code

The researcher obtained ethical clearance from the NWU Ethics Committee for her study to ensure full compliance with the NWU Code of Research Ethics.

1.4.8.6 Objectivity

Resnik (2011:2) views objectivity as an important ethical principle in research with regard to avoiding bias in data analyses and interpretation. The researcher adhered to the principle of objectivity strictly by looking at the phenomenon under investigation and not at the face value of the respondents. It did not matter whose task sheet was under scrutiny.

1.5 CHAPTER FRAMEWORK

Chapter 1: Statement of the problem and motivation

The opening chapter provides an overview of the problem statement, the literature review and the anticipated research problems. The purpose of the research, research design and methodology is touched upon.

Chapter 2: The learning of algebra in school

Chapter 2 focuses on the teaching and learning of algebra in school. As reflected in the problem statement, according to international studies South African learners' performance is the lowest compared to other countries. With the displeasing results of Grade 8 learners in mind, the understanding of Mathematics, Algebra in particular, is investigated.

Chapter 3: The learning and teaching of proportion

The core of this chapter is proportional reasoning. The meaningful understanding of the concept of proportion, the multi-facets of proportion, and the misconceptions regarding proportion, the development of the concept and ultimately the facilitation of meaningful learning of proportion are discussed.

Chapter 4: Research design and methodology

Chapter 4 focuses on the research design and methodology. Data analysis, the role of the researcher, participants and site, data generation instruments and ethical principles are discussed. The trustworthiness of the research is touched upon.

Chapter 5: Analyses of the data

In this chapter the findings are discussed. The research questions are reflected upon from the results of data analysis. The data is presented in the form of written texts and interview dialogues.

Chapter 6: Summary and recommendations

The final chapter summarises the findings of the study. The researcher draws conclusions regarding the results that emanate from chapter 5. Possible recommendations based on this study are made for future research.

CHAPTER 2:
THE LEARNING OF ALGEBRA IN SCHOOL

2.1 INTRODUCTION

Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, Mathematics has facilitated the development of science, technology, engineering, business and government. Mathematics is also an intellectual achievement of great sophistication and beauty that epitomises the power of deductive reasoning (Kilpatrick *et al*., 2003:1).

The Grade 8 Mathematics curriculum comprises two components, Algebra and Geometry. The Curriculum and Assessment Policy statement (CAPS) (DBE, 2013:8) assigns more than sixty percent of the Mathematics curriculum weight to Algebra. The document also emphasises the importance of deep conceptual understanding to make sense of Mathematics (DBE, 2013:8).

Concepts and skills developed in Algebra lay the foundation for more advanced Mathematics. Krulik *et al.* (2003:26) view Algebra as a major component of Mathematics that permeates the entire middle school programme. Algebra is the bridge between the concrete Mathematics of primary school and the abstract Mathematics of senior high school and university (Krulik *et al*., 2003:26). Algebra is the language of Mathematics, the common denominator of all branches of Mathematics. As a way of thinking, Algebra helps learners to analyse, represent and solve problems. When dealing with arithmetic, learners focus on answers. In Algebra learners focus on relationships (Krulik *et al.,* 2003:26). Thus, Algebra, as an essential component of Mathematics, can be viewed as the pulsating heart of Grade 8 Mathematics.

A critical study of documented literature is presented on the learning of Algebra at school. The meaningful learning of Algebra will be gleaned at. A discussion of the dominant views of the teaching and learning of Mathematics follows. The researcher aimed to link these views to theories of learning in order to gain insight into the current approaches prevalent in the Mathematics classroom. Furthermore, how learners understand Mathematics will be investigated through what Cobb (1994:13) identifies as a socio-cultural perspective. The focus will be on the meaningful learning of Mathematics and will narrow down to a theoretical investigation of the various factors affecting the meaningful learning of Mathematics.

2.2 VIEWS OF SCHOOL MATHEMATICS

Various conceptions of Mathematics exist, resulting in different assumptions and implications. A teacher's view about Mathematics have a strong impact on the way in which Mathematics is approached in the classroom (Dossey, 1992). The view of Mathematics consequently has an influence on how the teaching and learning thereof is seen. Nieuwoudt (1998) further explicates this argument by stating that teaching is deeply rooted in the educators' views and educators do not discard their views easily. According to Ernest (1988), three views of Mathematics are prevalent, namely the Instrumentalist, the Platonist (or formalist) and the Problem-solving view.

2.2.1 Platonist (formalist) view of Mathematics

Discussions about the nature of Mathematics date back to the fourth century B.C. Among the first major contributors to the dialogue were Plato and his student, Aristotle. Plato took the position that the objects of Mathematics had an existence of their own, beyond the mind, in the external world (Dossey, 1992:40). This Platonic view of Mathematics is described as the formalist-static perspective. Mathematics is viewed as a fixed and static body of knowledge consisting of a logical and meaningful network of inter-related truths (facts, rules and algorithms), bound together by filaments of logic and meaning (Thompson, 1992:132).

In line with the Platonic view, the teacher is the giver of knowledge and the learner is a passive recipient of knowledge. According to Nieuwoudt and Golightly (2006:109), the teacher is the distributor of authority and content in class, while the role of the learner is to give responses in compliance with correct methods. They conclude that traditional positivist teaching limits meaningful development of integrated humans.

2.2.2 Instrumentalist view of Mathematics

The instrumentalist view of Mathematics sees Mathematics as a “toolbox” consisting of a set of unrelated but utilitarian rules, facts and skills (Thompson, 1992:132). Mathematics is taught through drill and practice, neglecting understanding. This view is criticised by Nieuwoudt (2006:33), who contends that the instrumentalists view Mathematics as a set of unrelated rules. The implications of this view in the Mathematics classroom are that learners learn procedures and algorithms. Tasks have fixed correct answers. When learners do not arrive at the correct answer, the result is failure. For example, when learners divide two fractions, they are taught to change the division sign to multiplication and invert the second fraction. Learners rote-learn the rules. When application is needed, they can conclude the correct response. On interrogation, the majority of the learners will not know why the *multiply and invert* rule works. Rules taught without understanding lead to misconceptions.

2.2.3 The problem-solving view

In contrast to the above rigid views, Mathematics can also be viewed from a relativist-dynamic perspective. In South Africa the problem-centred approach (PCA), which is a social-constructivist approach, is currently being proposed as the "best way" to approach Mathematics education at all school levels (Nieuwoudt, 2006:33). Niewoudt further claims that Mathematics is viewed from a ‘change and grow’ perspective. This view of Mathematics bears a strong resemblance to Aristotle’s experimental ideas about Mathematics. Dossey (1992:40) claims that in Aristotle’s view, the construction of a mathematical idea comes through idealisations performed by the mathematician (or mathematical person) as a result of experience with objects. Mathematics is not seen as a finished product with its origin outside the individual but it remains in the making in the individual's mind.

Successful problem-solving requires generating new representations from familiar representations using previous knowledge and experience (Kramarski, 2009:138). For the purpose of this study, the social constructivist view is prevalent as learners collaboratively and actively construct their own knowledge in a social setting. The teacher is the facilitator.

2.3 LEARNING THEORIES

Effective teaching-learning practices cannot be maintained without the support of grounded teaching and learning theories (Nieuwoudt, 2000:1). Teachers need to understand various learning theories in order to make choices in their teaching and to reinforce information from learner development in the context of instruction in the Mathematics classroom (Tipps *et al*., 2011:56). If teachers understand how learners learn, they can use the most effective teaching strategies. A learning theory is not a teaching strategy, but the theory informs teaching (Van de Walle *et al.*, 2010:20). Learning theories tend to fall into one of several perspectives or paradigms, including behaviourism, cognitive constructivism and social constructivism.

2.3.1 Behaviourism

Pavlov, Thorndike, Watson and Skinner (as cited by Pritchard, 2009:14-15) view learning as a relatively permanent, observable change in behaviour as a result of experience. This change is effected through a process of reward and reinforcement. The changes in behaviour occur as a response to a stimulus of one kind or another. The response leads to a consequence. When the consequence is pleasant and positive, the behaviour change is reinforced. With constant reinforcement, the behaviour pattern becomes conditioned (Pritchard, 2009:6).

Behaviourists call this method of learning ‘conditioning’ (Pritchard, 2009:6). The two types of conditioning are classical and operant conditioning. Classical conditioning occurs in four stages. The acquisition phase is the initial learning of the conditioned response. The second phase of extinction relates to the disappearance of the conditioned response. During the generalisation phase there is response to similar stimuli. The final phase is called the discrimination phase when a learner learns to produce a conditioned response to one stimulus but not to another similar one. Operant conditioning involves reinforcing a certain behaviour by reward or punishment (Pritchard, 2009:6).

In the classroom, repetition is seen in the drill and practice often associated with the learning of basic skills. Learners receive positive reinforcement (rewards) with each correct response. With each incorrect response, a learner receives negative reinforcement (punishment). All behaviour can be explained without the need to consider internal mental states or consciousness (Pritchard, 2009:6-15). Learners are viewed as passive recipients.

2.3.2 Cognitive Constructivism

Constructivism views learning as the result of mental construction, that is, learning takes place when new information is built into and added onto an individual’s current structure of knowledge, understanding and skills. We learn best when we construct our own understanding (Pritchard, 2009:17). Constructivism is rooted in the work of Jean Piaget, who introduced the notion of mental schema and developed a theory of cognitive development. At the heart of constructivism is the notion that children are not blank slates but rather creators of their own learning based on prior knowledge

(Van de Walle *et al*., 2010:20). Constructivism has become a driving force behind global educational reform (Nieuwoudt & Golightly, 2006:107).

In constructivist learning, individuals draw on their experience of the world around them in many different forms, and work to make sense of what they perceive in order to build an understanding of what is around them (Pritchard, 2009:20). Some essential features of constructivism are mentioned below (Pritchard, 2009:32-33):

1. The construction of knowledge and not the reproduction of knowledge is paramount. (Process is important and not the end-product).
2. Learning can lead to multiple representations of reality. (Multiple resources and alternative viewpoints lead to critical thinking skills).
3. Authentic tasks in a meaningful context are encouraged (such as problem-solving).
4. Reflection on prior experience is encouraged. (Integrating pre- and new knowledge).
5. Collaborative work for learning is encouraged.
6. Autonomy in learning is encouraged. (Learners are given responsibility for their own learning). (Pritchard, 2009:32-33).

Learners are viewed as information processors, information is received and processed. Children learn through being active, operating as lone scientists. The role of the teacher is to provide 'artefacts' needed for the child to work with and learn from. Cognitive growth has a biological, age-related, developmental basis. Children are unable to extend their cognitive capabilities beyond their stage of development. The implications are that it is no point in teaching a concept that is beyond a child's current stage of development. Piaget firmly believed that learning is inhibited when a child is shown how to do something rather than being encouraged to discover it for themselves.

2.3.3 Social Constructivism

The basic idea of constructivism is that learning is an active, constructive process. Learners are viewed as information constructors. New information is linked to prior knowledge.

Social interaction, being an important dimension, is added to the constructivist domain through social constructivism. According to Pritchard (2009:24), dialogue is important to share and develop ideas, as high priority is given to language in the process of intellectual development. He further elaborates on the importance of Vygotsky's notion of the zone of proximal development, which is a theoretical space of understanding just above the level of understanding of a given individual. It is the difference between a learner's assisted and unassisted performance (Van de Walle *et al*., 2010:21). It is the next area of understanding into which a learner will move. In the zone of proximal development a learner works effectively with support. A learner progresses to the next level of understanding as he or she moves into and across zones. The way information is internalised depends on whether it is within a learners' zone of proximal development (Van de Walle *et al*., 2010:21). In a nutshell, from the socio-cultural perspective, learning is dependent on the learners working in their zone of proximal development, the social interactions in the classroom and the culture within and beyond the classroom.

The teacher is the facilitator and scaffolder. Scaffolding is the process whereby support is given to learners at the appropriate time and at the appropriate level (Pritchard, 2009:25). Development is an internalisation of social experience. Learning is thus seen as a socially mediated activity. Watkins (2005:55) elaborates on the social dimension involved in enhancing learning. He iterates that classrooms as learning communities operate on the understanding that the growth of knowledge involves individual and social processes. It aims to enhance individual learning that is both a contribution to their own as well as the group's learning, and does this through supporting individual contributions to a communal effort (Watkins, 2005:55). He further attests that the agency of inquiry is not an individual, but a knowledge-building community.

The learning theory subscribing to this study is social constructivism. Learners actively and socially construct their own knowledge with the teacher facilitating and providing opportunities for scaffolding.

2.4 THE ACT OF LEARNING

2.4.1 What does the act of learning entail?

A basic understanding of the processes of learning is essential for teachers who intend to develop activities that will have the potential to lead to effective learning taking place in classrooms (Pritchard, 2009:1).

Pritchard (2009:17) defines learning as a result of mental construction from a constructivist's point of view. He asserts that learning takes place when new information is built and added onto one's existing knowledge and understanding. That is, for effective learning to take place, a person needs to construct his or her own understanding.

Van de Walle *et al*. (2010:20) refer to the theory of cognitive development embedded in the work of Piaget. The emphasis is on integrated networks, also known as cognitive schemas. Understanding is built onto a learner's prior knowledge, which is utilised to make sense of new information. As new information builds onto existing information, networks are formed and rearranged, whereby learning takes place. Piaget's two schemas are assimilation and accommodation. During assimilation new concepts fit with prior knowledge and the new information expands an existing network. Accommodation takes place when the new concept does not fit with the existing network, so the brain revamps or replaces the existing schema. Reflective thinking takes place as people sift through existing ideas (prior knowledge) to find ideas that seem to relate to the current thought or task. Hence, people construct their own knowledge based on their prior knowledge (Van de Walle *et al*., 2010). Van de Walle and Lovin (2006:2) stress that if minds are not actively engaged in thought, no effective learning occurs. Constructivists posit that for learning to take place, the learner's mind has to be active.

The Van Hieles made a significant impact on the understanding of levels of learning or levels of thought (Krulik *et al.,* 2003:7). They contested that learners pass through stages of geometric development. The geometric level at which the learner is operating need not necessarily be related to the learner's chronological age of the learner but rather reflects his or her experiences. The first stage is the visualisation phase, followed by analysis and deduction.

The Van Hieles' levels of thought reveal similar level structures in Algebraic thinking. Realistic Mathematics education (RME) also suggests that mathematical thinking develops through progressive levels of complexity .The concept of RME is explained by Nieuwoudt (2000:32-34): learners investigate mathematical problem situations in a realistic context which then leads to the formation of mathematical concepts, first on an intuitive level, gradually progressing towards more advanced levels (Nieuwoudt. 2000:32-34). The three levels are characterised as follows:

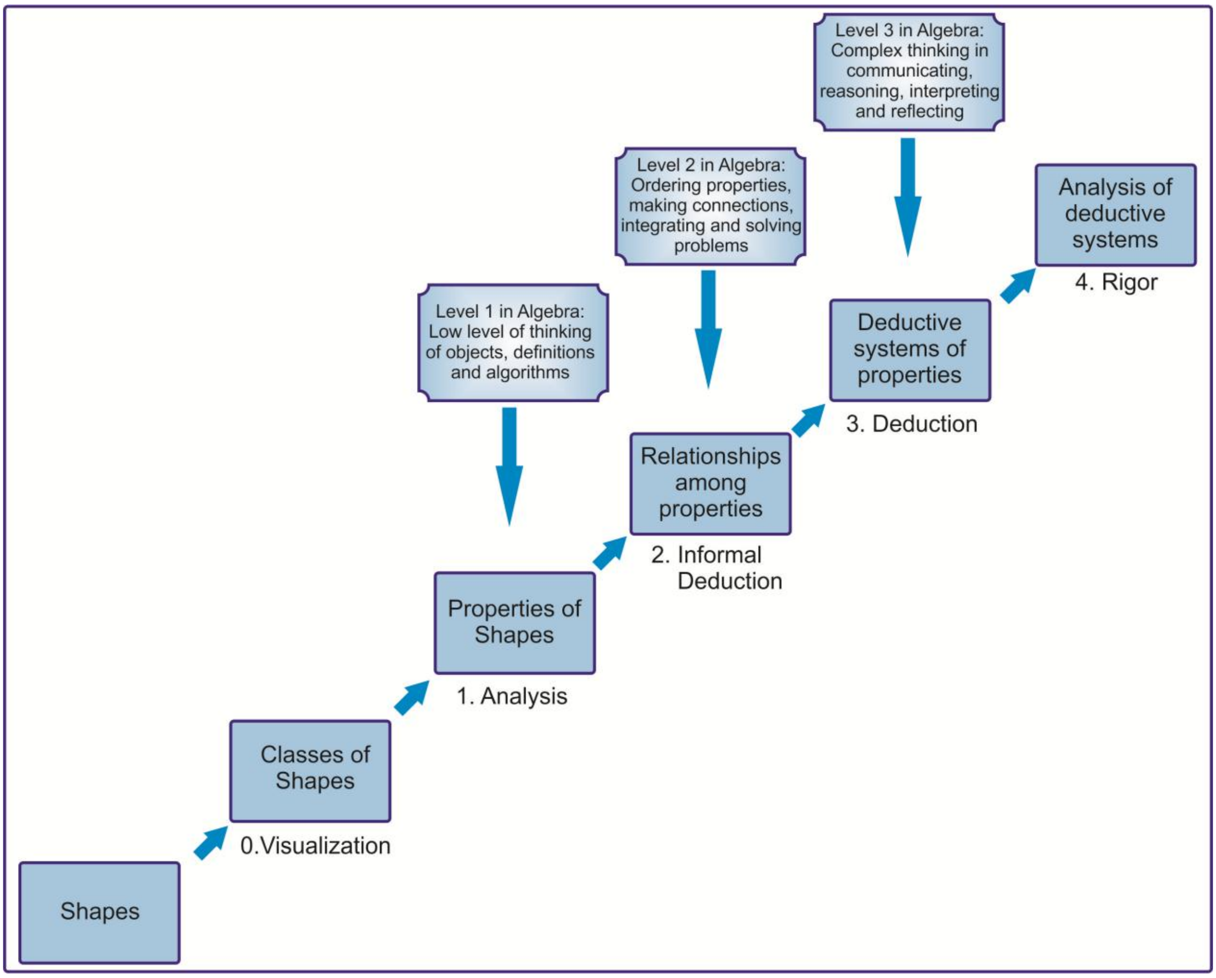


Figure 2.1: The van Hiele theory of geometric thought (adapted from Van de Walle *et al*. 2010:401).

In the Algebraic sense, learners at the lowest level (level 1) rely on low level thinking in terms of specific objects, definitions, techniques and standard algorithms. Level 2 (the middle or intermediate level) is characterised by learners ordering properties, making connections, integrating related issues and solving problems. Progression from level 1 to level 2 takes place during teaching and learning. Level 3 (high level) requires complex thinking in order to communicate reason, interpret and reflect (Nieuwoudt, 2000:34-35). In Algebraic learning, the learners therefore progress through levels at different rates. The teacher should take the various stages of development into consideration as a Mathematics classroom contains multiple intelligences. The cognitive levels of the learners vary.

2.4.2 Learning Algebra with understanding

Since most learning theories wrestle with the notion of understanding, the spotlight in this section will be on learning with understanding, or meaningful learning. Various researchers' notions on understanding are discussed below.

Understanding is a very complex phenomenon and a fundamental aspect of meaningful learning. Understanding should be the most fundamental goal of Mathematics instruction. Hiebert *et al.* (1997:2) highlight the importance of understanding in learning. They iterate that when learners understand what is being taught and learned, they feel satisfied, rewarded and have positive experiences. In contrast thereto, when learners do not understand what is being learned and taught, they become frustrated and have negative experiences. Learners who understand will remain engaged in learning, whilst learners who do not understand are likely to withdraw from learning as they feel frustrated and defeated. Learners who lack understanding will resort to memorising rules.

Hiebert and Carpenter (1992:67) define understanding in terms of the way information is represented and structured. They argue that a mathematical idea, procedure or fact is understood if it is part of a network of presentations. The degree of understanding is determined by the number of strengths of connections. The idea or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. This boils down to *understanding* as involving the recognition of relationships between pieces of information. Learners construct several kinds of connections to create mental networks (Hiebert & Carpenter, 1992:67).

The process of understanding is explained by the network theory (Hiebert & Carpenter, 1992:69). They refer to "webs" of interrelated ideas. Networks of mental representations are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information. Understanding grows as the networks become larger and more organised and therefore is not an all-or-none phenomenon. Understanding can be rather limited if only some of the mental representations of related ideas are connected or if the connections are weak. Understanding increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring. The implications are that each mathematical idea will be well understood if it is embedded in as rich a web of related mathematical ideas as possible (Hiebert & Carpenter,1992: 69).

Understanding in learning has been researched through decades. Dating back to 1986, Brownell's Meaning Theory suggested that children must understand what they are learning if learning is to be permanent (Tipps *et al*., 2011:59). When children create their own understanding they are demonstrating the Meaning Theory.

Skemp (1976:2) referred to the two contrasting views of understanding as instrumental understanding and relational understanding. Instrumental understanding is understanding 'rules without reason', and usually involves a multiplicity of disconnected maths rules. Hence, instrumental understanding refers to doing Mathematics without understanding. Teachers and learners still using the traditional syllabus promote instrumental understanding. The teacher-centered approach promotes instrumental understanding as rules are applied without reasoning. Skemp (1976:2) perceived a negative attitude towards Mathematics as the greatest measure of failure.

Relational understanding refers to the “what and why” of Mathematics. Learning Mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point (Skemp, 1976:2). Skemp’s sentiments seem to be supported by Van de Walle and Lovin (2006:3) as they explain the construction of ideas and understanding, using the diagram below:

Figure 2.2: A representation of relational and instrumental understanding (Van de Walle *et al*., 2010:25)

This figure demonstrates an individual’s understanding as it exists along a continuum. At one end of the continuum is a very rich set of connections. The understood idea is associated with many other existing ideas in a meaningful network of concepts and procedures. This is known as relational understanding. At the other end of the continuum is instrumental understanding. Ideas are completely isolated and have been rotely learned. Due to their isolation, poorly understood ideas are easily forgotten and are unlikely to be useful for constructing new ideas (Van de Walle & Lovin, 2006:3).

Understanding differs from learner to learner. For constructive and effective learning and teaching, learners should therefore be given ample opportunities to activate their existing prior knowledge. They should also be given opportunities to understand, apply and reflect on knowledge (Tipps *et al.,* 2011:60).

Piaget describes four developmental stages in learning, namely sensory motor, pre-operational, concrete operational and formal operational. Each stage has an approximate age of predominance in learning (Tipps *et al.*, 2011:56). During the concrete operational phase (ages 7-12) children begin the long process of developing mathematical meaning and constructing mathematical concepts using concrete objects.

Piaget's stages of mental development gives an indication of the cognitive ability of a learner. The average age of a Grade 8 learner is 13 years. This falls within the Formal Operational phase (Piaget). Learners at this stage should think abstractly and hypothetically, depending on language acquisition. Some have just moved from the Concrete Operational phase whilst some may be still at this stage. This level is mathematically important as many operations at this level include mathematical structure. This crucial transition stage has an impact on learning as learners glide from Arithmetic to Algebra.

According to Haylock (2010:26), understanding means to build up (cognitive) connections. To understand mathematical ideas we have to gradually build up these networks of connections, where each new experience is being connected with our existing understanding, and related in some way to other experiences. This is achieved through practical engagement with mathematical materials, through investigation and exploration, through talking about Mathematics – but above all, through the learner's own cognitive response, which has been shaped by prior successful learning to look for connections and relationships in order to learn in a meaningful way (Haylock, 2010:26).

Teachers must encourage learners to build networks of concepts. For example, when learners deal with fractions, networks of related concepts should form. The learners' minds should start thinking about *proper* and *improper* fractions. The networks should integrate, forming ideas about the relationships between fractions, decimals and percentages.

It is evident from the above that the essence of learning in the Algebra classroom boils down to *learning with understanding* or *meaningful learning*.

2.5 FACTORS AFFECTING MEANINGFUL LEARNING OF ALGEBRA

Physical, affective and cognitive factors inhibit the learning of Algebra in some cases. For the sake of this study, the researcher touched upon certain factors in the afore-mentioned three domains. A discussion follows from the literature review.

2.5.1 Maths anxiety

Mathematics anxiety refers to an emotional response when confronted with Mathematics (Maree *et al*., 1997:7). Panic, anxiety and concern are manifested in repetitive behaviour (like biting nails, scrapping of correct answers and an inability to speak clearly). Mathematics anxiety has been found to have an adverse effect on learner confidence, motivation and achievement (Hlalele, 2012:267). Some teachers treat Mathematics as an arbitrary collection of facts (see Section 2.2.2) and present

Mathematics as such. Learners' questions are ignored and classroom interaction is limited. Hlalele (2012:270) contends that such strategies perpetrate Mathematics anxiety in learners. Low quality teaching, inconsideration of diverse learning styles and language barriers cause Mathematics anxiety. When learners worry, they do not focus on mathematical tasks, they find it difficult to concentrate and think logically, having a tendency to make errors. Longer time is taken for task completion. Learners' negative attitude towards Mathematics leads to lowered self-esteem and eventually academic failure. Anxiety generally inhibits learning and achievement (Engelbrecht, 1996:233). High anxiety levels are associated with underachievement. Learners with learning difficulties often have a history of failure, anxiety and helplessness.

Learners need to develop a positive attitude towards Mathematics to reduce Maths anxiety. This necessitates a constructive approach to teaching. Creativity in lesson presentation is essential, keeping in mind modern technology and hands-on activities when introducing new concepts. By breaking the monotony in the Mathematics classroom, the learners will see Mathematics as fun, they will enjoy Mathematics and the joy will remain with them forever.

It is imperative to support these learners by means of creating a secure learning environment characterised by a pleasant teaching climate, as well as opportunities to succeed and to attribute success to themselves, thus enhancing a realistic positive self-concept (Engelbrecht, 1996:235).

2.5.2 Limited English proficiency

The South African Mathematics classroom consists of different races, cultures and nationalities. Most of these learners have limited English proficiency as English is not their mother-tongue. The home language of Limited English Proficiency (LEP) learners is tied to their cultural or ethnic background; it is the language spoken at home and in their social community (Tipps *et al*., 2011:37-43). At school, the learners have to communicate in English as English is the language of teaching and learning in most South African schools. Many of these learners speak English only at school. They argue that Mathematics has its own language as well as its own symbols. Words and terms used in a Mathematics class, such as denominator, numerator, quotient, composite, ratio, etc., make Mathematics more challenging and confusing for these learners. LEP learners can progress in Mathematics if they have a good command of English (Tipps *et al*., 2011:37-43).

Learners utilise language to communicate with peers and teachers, to understand the teachers' presentations and to articulate their own thinking (English, 2002:1010). This has specific ramifications for those learners for whom the language of instruction is not their home language.

In the Algebra classroom, communication between the teacher and the learner poses a problem for second- and third-language learners. The teacher must use mathematical vocabulary correctly so that the learners grasp the correct vocabulary. The learners need to articulate using appropriate mathematical language. The role of language therefore is crucial in understanding concepts in Algebra.

2.5.3 Learning difficulties

It is important for teachers to have an insight into learners' learning difficulties so as to understand how these difficulties manifest in unpleasant behaviours and attitudes, for appropriate strategies to be employed to deal with these problems (Pritchard, 2009:58). When learners experience difficulties in learning, the result is poor performance and low self-esteem. Pritchard (2009:59) further mentions four stages in learning: Input is the process of taking in and recording information, which is been received via the senses; Integration is the process whereby information is interpreted, categorised, placed in sequence, and linked with previous learning;

Memory is the third stage where information is placed into storage for later retrieval and use; Output is the final stage, that is, the time when actions are taken based on the processing of stored information. This can be in the form of language or action (for example, movements or gestures) (Pritchard, 2009:59). Pritchard further iterates that learning disabilities are classified by their effects at one or more of these stages and that each child is likely to have individual strengths and weaknesses at each of the stages. Difficulty in learning Mathematics denotes a wide range of impediments that learners must deal with (Tipps *et al*., 2009:43). Learning difficulties stem from various sources. Some learners may find it difficult to read or do Mathematics.

Tipps *et al*. (2011:43) emphasise that teachers work with learners having a variety of cognitive abilities, learning styles, social problems and physical challenges. The onus is on the teacher to find appropriate strategies so that all learners gain the maximum benefit from learning. All learners learn at a different rate and therefore should be given extra time to learn, practise and drill.

2.5.4 Multiple intelligences

Gardner (as cited in Tipps *et al.*, 2011:49), proposed The Theory of multiple intelligences.

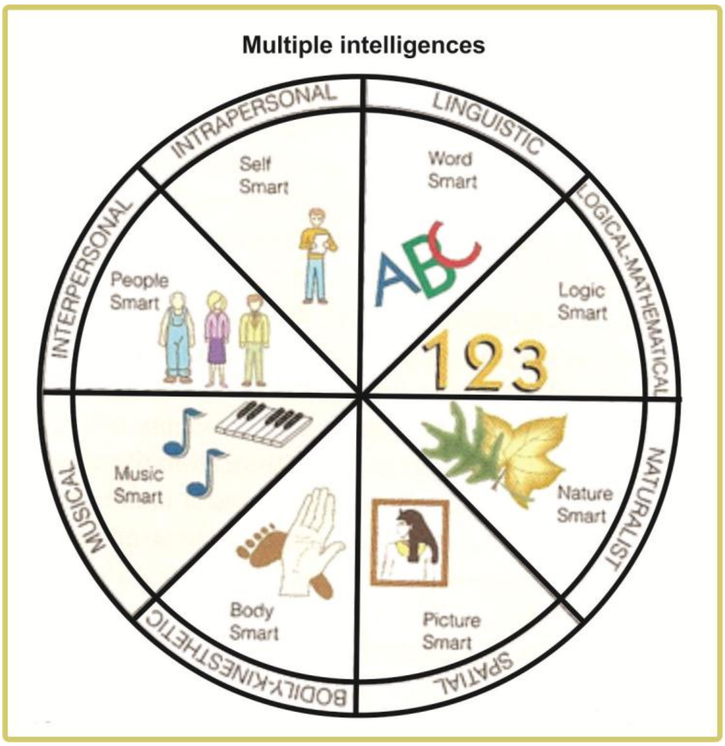


Figure 2.3: Gardener's 8 multiple intelligences (Tipps *et al.*, 2011:49)

The Theory of Multiple intelligences shows that people are capable, or even gifted, in different ways. Individuals may feel more comfortable with some of the intelligences and less comfortable with others. Some learners may lack logical-mathematical ability (Tipps *et al.*, 2011:49). Logical-mathematical intelligence, strongly related to problem-solving, reasoning, and mathematical thinking, receives little attention in classrooms where teacher-directed lessons are the primary means of instruction (Tipps *et al.*, 2011: 50). Hence, the lack of logical-mathematical ability may manifest in misunderstanding of Algebraic concepts.

2.5.5 Errors and misconceptions

Errors and misconceptions arise in Mathematics and should be viewed as 'necessary' stages in learners' mathematical development. An error could be made for many reasons, for example carelessness, misinterpretation of symbols or facts, lack of knowledge related to the concept or lack of ability to check the answer (Drews, 2006:14). A misconception could be the misapplication of a rule, an over- or under-generalisation or an alternative 'conception' of the situation (Drews, 2006:15). A social culture can be developed in which errors are not embarrassing signs of stupidity but are natural and constructive consequences of building improved methods of solving problems (Hiebert *et al*., 1997:168). Mistakes can therefore be viewed as learning sites.

If teachers examine errors to ascertain the roots, progress can be made towards assisting learners to overcome such barriers and enhancing understanding of the particular concepts in Algebra.

2.5.6 Learning styles

Pritchard (2009:42) defines a learning style as a mode of learning. Learners' learning styles include how learners approach a problem, how they strategise and their preferred means of acquiring knowledge and skills. Different contexts require different learning styles by learners. Synonymously, a learning style is also a learning preference.

There are three major types of learners: visual, auditory, and tactile. Learning styles is one way of describing how people receive, process and respond to their world (Tipps *et al*, 2011:51). Many teachers ignore the importance of learning styles. Teaching methods remain the same. Learners who do not adapt to the teachers' methods will not attain maximum benefit if their learning styles are not catered for.

2.5.7 Study attitudes

Feelings and attitudes affect learners' motivation and interests in Mathematics, including how they view the nature and learning of Mathematics. Attitudes include various factors, like enjoyment of Mathematics, self-confidence, usefulness of the subject and the challenge it offers (Maree *et al*., 1997:7). Learners' beliefs about their competence and their expectations for success in school have been directly linked to their levels of engagement, as well as to emotional states that promote or interfere with their ability to be academically successful (Akey, 2006:4). Akey further argues that learners who believe that they are academically incompetent are more nervous in the classroom and are scared of revealing their ignorance. They fear that educational interactions will result in embarrassment and humiliation, and this, in turn, inhibits them from behaving in ways that might help them, such as asking questions when they are confused or engaging in trial-and-error problem-solving. In addition, such learners are more likely to avoid putting much effort into a task so that they can offer a plausible alternative to low ability or lack of knowledge as an explanation for failure (Akey, 2006:4).

2.5.8 Study habits

Study attitude in Mathematics manifests in specific study habits, such as how learners plan their study time, how they prepare, how they learn theorems, rules and definitions. Other habits include how focused the learners are in completing assignments and working with familiar as well as unfamiliar problems (Maree *et al*., 1997:8).

2.5.9 Social interaction

Social interaction is a powerful catalyst in developing learners' mathematical thinking. Three principles underlying Vygotsky's theory are that learners construct their own knowledge, that their development cannot be separated from its social context, and that language takes a central role in this development (Hansen, 2006:9). Hansen explains that when two learners work together on a task, they initially have different understandings. By interacting with each other they share their understanding. The result is a richer experience than the experienced the learners gain by working individually.

Krulik *et al*. (2003:6) follows the same line of argument as Vygotsky, viewing learning as a social activity. They believe that children build and rebuild their knowledge by working together. Hiebert *et al.* (1997:9), view the classroom as a community of learners, where people relate to and interact with each other. This interaction is essential to build understanding of Mathematics.

In Mathematics classrooms, the teacher should initiate learners into a culture of mathematical inquiry where discussion and collaboration are valued in building a climate of intellectual challenge (Goos *et al*., 2007:30). Discussions allow learners to verbalise their thoughts. Not only do learners acquire understanding of mathematical content, they also acquire linguistic skills and vocabulary.

2.5.10 Teachers' mathematical knowledge

Teachers' mathematical knowledge is undoubtedly of paramount importance in creating mathematics-literate South African learners who can compete globally. In the past decades much research has been carried out on how teachers' mathematical knowledge shapes the way they teach. It is unlikely that teachers will be able to provide an adequate explanation of concepts they do not understand, and they can hardly engage their learners in productive conversations about multiple ways to solve a problem if they themselves can only solve it in a single way (Kilpatrick *et al*., 2003:377). They further contend that teachers with a relatively weak conceptual knowledge of Mathematics tend to demonstrate a procedure and then give learners opportunities to practise it. In this way learners receive little assistance in developing an understanding of what they are doing. In some cases inadequate conceptual knowledge results in incorrect procedures. Even teachers with strong conceptual knowledge do not necessarily use the knowledge to understand their learners' mathematical explanations, preferring instead to impose their own explanations. When learners try to solve problems relying on learned procedures, they will not see the logic in the steps. The learning will be devoid of conceptual understanding.

In a research study carried out by Plotz (2007:236-237) on teachers' current mathematical knowledge, the following conclusions were reached: Teachers do not have a proficient understanding of Proportion, hence lack proportional reasoning, and teachers use wrong procedures to solve problems, displaying unconnected mathematical knowledge.

The transition between different representations poses a problem. Misconceptions are evident in the written work of teachers. Difficulty arises when solving unfamiliar problems. These huge gaps in teachers' mathematical knowledge adversely impact on learners' meaningful learning of Mathematics (Plotz, 2007:236-237). Teachers cannot teach with understanding if they themselves lack conceptual understanding.

Research undertaken by Joint Educational Trust Educational Services (JET) as part of the National School Effectiveness Study (NSES) (Taylor, 2011:9) suggests that few primary school Mathematics teachers have the mathematical competency they need to adequately teach the subject. It was also noted that only between five percent and twenty percent of teachers covered the section on proportional reasoning. The study found clear links between teacher competence and learner performance. Teachers' mathematical competence is directly linked to learner performance. The results of these findings are similar to those drawn from a more comprehensive study undertaken as part of SACMEQ III. The results of many such studies boil down to the unanimous conclusion that for teachers to be effective they need to have a good conceptual understanding of the material they are teaching.

Ma (1999:124) indicates that profound understanding of fundamental Mathematics is more than a sound conceptual understanding of elementary Mathematics – it is the awareness of the conceptual structure and the ability to provide a foundation for that conceptual structure. A profound understanding of Mathematics has breadth, depth and thoroughness (Ma, 1999:124). She attests that breadth of understanding is the capacity to connect a topic with topics of less conceptual power. A depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness relates to the connection of all topics. Ma (1999:124) views a teacher with profound understanding as a teacher who uses multiple approaches, who reinforces ideas and helps learners to connect various mathematical concepts and procedures.

2.5.11 Study milieu (social, physical and experienced milieu)

Maree *et al*. (1997:9) elaborate on the study milieu experienced by learners. They contend that Mathematics learners come from different environments and different backgrounds. Learners emanating from non-stimulating environments often experience learning difficulties and learn at a slower rate because of non-exposure to experiences. Frustration, confusion and misunderstanding the language of teaching and learning lead to Maths anxiety and underachievement.

Many individuals manifest learning problems caused by environmental deprivation, such as poor nutrition, pedagogical neglect, lack of social interaction and intellectual stimulation. Engelbrecht *et al*. (1996:177) refer to Feuerstein's conceptual framework where it is stated that children from economically and psychologically impoverished homes perform poorly on intelligence tests and generally function at a lower level because they have been denied appropriate mediated learning experience.

A recent study carried out by Visser and Juan (2013) reveals that the socio-economic status of a particular household is positively associated with educational performance. Resources such as computers and internet access have a positive impact on learners' achievement scores.

2.5.12 Problem-solving behaviour

Effective problem-solving processes will enable teachers to prepare learners for the challenges they will face as adults (Maree *et al.*, 1997:8). Problem-solving is a process; it is the means by which individuals take the skills and understandings they have developed previously and apply them to unfamiliar situations (Krulik *et al*., 2003:92). The process begins with the initial confrontation of the problem and continues until an answer has been obtained and the learner has examined the solution process. Haylock (2010:56) views a problem as a situation in which something is given to reach a goal. Nieuwoudt (2000:20) refers to Polya's (four step) problem-solving model. Learners have to first understand the problem, a plan has to be made to solve the problem, and the plan has to be executed before the solution is assessed.

Haylock (2010:56) concludes that the central component of mathematical reasoning must be what is learned in problem-solving. This is due to the fact that the skills, concepts and principles of Mathematics that children master are used and applied to solve problems.

2.5.13 Metacognition

Metacognition refers to the idea of an individual's considering, being aware of and understanding his or her own mental (cognitive) processes and ways of learning (Pritchard, 2009:26). A metacognitive learner understands his learning, can reflect on and control his learning. The two components of metacognition are knowledge of cognition and regulation of cognition (Weimer, 2011). Learners possess knowledge of cognition when they are aware of knowing about things, knowing how to do things and knowing when and why to do things. Knowledge of cognition describes an individual's awareness of cognition at three different levels: declarative (knowing about things), procedural (knowing about how to do things), and conditional (knowing why and when to do things). Learners' control of their learning is referred to as regulation of cognition. Metacognitive learners plan their actions, monitor their activities, go back and evaluate progress.

As metacognition plays a critical role in successful learning it is important for both learners and teachers. The role of metacognitive knowledge in learning and teaching is accentuated by Pintrich (2002:220). He asserts that metacognitive knowledge has three interwoven strands: strategic knowledge, knowledge about cognitive tasks and self-knowledge. Strategic knowledge refers to the knowledge of general strategies for learning, thinking and problem-solving. This knowledge enables learners to memorise the content, extract meaning from text and to comprehend what they hear in the classroom (Pintrich, 2002:220). A learner possessing strategic knowledge will be able to set goals, check the answer, go back and fix the calculation mistake in a Mathematics problem.

Learners who are knowledgeable of cognitive tasks are aware that tasks vary in order of difficulty and that different tasks may require different cognitive strategies. Such learners have the knowledge of using strategies appropriately (Pintrich, 2002:221). He refers to self-knowledge as knowledge of one's strengths and weaknesses. Learners with self-knowledge develop self-awareness about their motivation.

Flavell (1979:906) concurs with Pintrich (2002:219), thereby further accentuating the important role of metacognitive knowledge in learning. According to Flavell (1979:908), metacognitive knowledge leads learners to select, evaluate and revise strategies.

Three sub-dimensions of metacognitive knowledge are: declarative, procedural and conditional (Schraw, Crippen & Hartley*,* 2006:114). Declarative knowledge is knowledge about the factors that can influence one's learning. For example, a learner will plan time effectively in completing a task as time can influence learning. When a learner is knowledgeable about procedures he or she possesses procedural knowledge. An example would be that a learner has procedural fluency in dividing fractions. Conditional knowledge refers to knowing when and why certain strategies are used. Learners having conditional knowledge are able to select the most appropriate strategies. Conditional knowledge enables a learner to use the cross-product algorithm in a direct proportion problem.

In view of the various researchers' arguments above, it is evident that metacognition plays a dynamic role in the learning of Algebra.

2.5.14 Mathematical proficiency

Mathematical proficiency is needed for learners to learn Mathematics successfully. Kilpatrick *et al.* (2003:5), contend that mathematical proficiency is made up of five interwoven strands. The first strand is *conceptual understanding*. Learners gain conceptual understanding when they understand mathematical concepts, operations and relations. *Procedural fluency* is achieved when learners have acquired skills in carrying out procedures flexibly, accurately, efficiently and appropriately. The third strand, *Strategic competence*, necessitates the ability to formulate, represent, and solve mathematical problems. *Adaptive reasoning* is the capacity for logical thinking, reflecting, explaining and justifying. The final strand is *Productive disposition,* being the inclination to see Mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Kilpatrick *et al.,* 2003:5).

From the above one can conclude that learners need the five strands in figure 2.4 (below) to learn Algebra successfully. Algebraic proficiency implies that learners need to understand concepts and procedures, possess problem-solving skills, display logical reasoning and view Mathematics as useful. Learners must also know what Algebraic symbols mean.

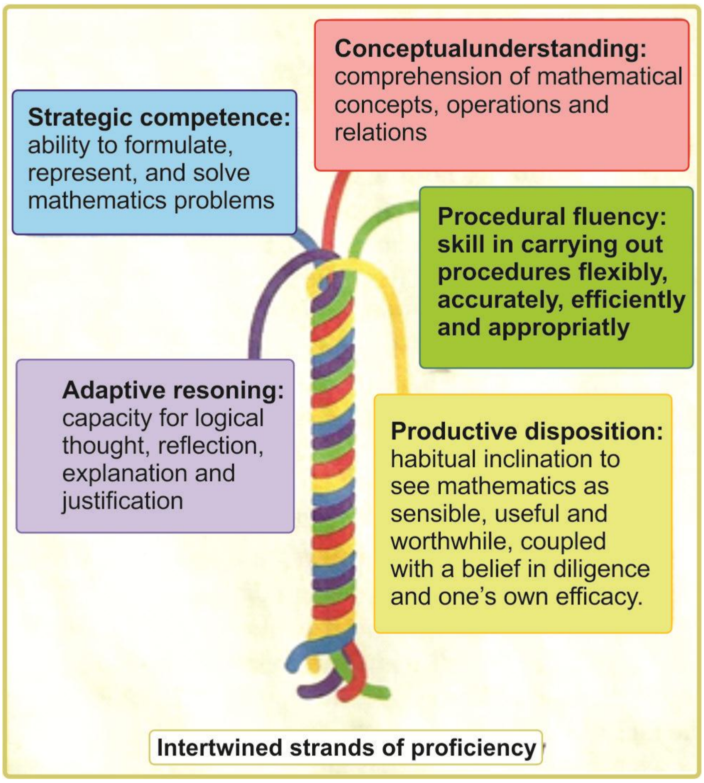


Figure 2.4: Intertwined strands of proficiency (Kilpatrick *et al*., 2003:5)

2.6 CONCLUSION

In this chapter the theoretical ideas about the learning of Algebra based on the literature review were examined. This was followed by the exploration of a melange of learning perspectives and theories. Finally, various factors affecting the meaningful learning of Algebra were discussed. In the next chapter the focus will be on the learning and teaching of Proportion.

CHAPTER 3:
THE LEARNING AND TEACHING OF PROPORTION

3.1 INTRODUCTION

Proportional reasoning is one of the most important algebraic abilities to be developed during primary and high school. Proportional reasoning provides opportunities for learners to consolidate their knowledge of primary school Mathematics and build a foundation for high school Mathematics and algebraic reasoning. Students who fail to develop proportional reasoning are likely to encounter obstacles in understanding higher-level Mathematics, particularly Algebra (Langrall & Swafford, 2000:254).

The focus in this chapter will be on proportional reasoning, as well as the meaningful understanding and the facilitation of proportion to enhance teaching and learning. The multi-faceted constructs of proportion will be discussed. Light will be shed on misconceptions occurring in the various constructs with a view to guide meaningful facilitation of the key mathematical concept in Algebra, namely, proportion.

3.2 PROPORTIONAL REASONING

The idea of proportional reasoning has been researched over many decades and a variety of literature documented on the dynamic mathematical idea. Proportional reasoning is rated one of the most important forms of mathematical reasoning. Hence, the development of learners' proportional reasoning can be viewed as the gateway to success in Algebra. Proportional reasoning involves a sense of covariation, multiple comparisons, the ability to mentally store and process several pieces of information (Post *et al.*, 1988:79). This view is supported by Cai and Sun

(2002:195), who also posit that proportional relationships provide a powerful means for learners to develop algebraic thinking and functional sense. Tipps *et al*. (2011:356) report that the same view is supported in the standards set by The National Council of Teachers of Mathematics (NCTM, 1989). The NCTM asserts that proportional reasoning is a key mathematical concept across all grades. Teachers should spend time and effort in aiding learners to develop this method of thinking.

3.2.1 The concept of proportional reasoning

Many researchers claim that it is difficult to define proportional reasoning. One way to describe proportional reasoning is to say it is the ability to think about and compare multiplicative relationships between quantities (Van de Walle *et al*., 2010:350). Proportional thinking is developed by learners when they compare, find equivalent ratios and solve proportions in different contexts without recourse to rules. According to Tipps *et al*. (2011:355-356), proportional reasoning involves understanding how quantities vary in relation to each other. While the actual numbers may vary, the relationship between them remains the same, for example, 2:4 = 10:20 = 50:100. They claim that learners reason proportionally when they compare percentages, work with decimal and common fractions and explore similar figures. Beyond the boundaries of school, learners need to reason proportionally when reading maps, doubling a recipe or are involved in rate transactions (Van de Walle *et al*., 2010:353).

Emanating from the above one cannot deny the important role played by proportional reasoning in laying the foundation for algebraic thinking. The meaningful understanding of Proportion is discussed below.

3.2.2 The meaningful understanding of proportion

It is estimated that more than half of the adult population cannot be viewed as proportional thinkers. Proportional reasoning is not dependent on age, which means that one does not acquire the habits and skills of proportional thinking simply by getting older (Van de Walle *et al*., 2010:350). Proportional reasoning develops over time. Thus instruction that focuses on proportional reasoning develops the concept of proportion (Lamon, 2007, as quoted by Van de Walle *et al*., 2010:350). Lamon further characterises proportional thinkers as understanding that ratios are distinct entities representing a relationship different from the quantities they compare. Meaningful understanding of proportion entails developing informal strategies in solving proportion problems, not just algorithms, that is, they understand relationships in which two quantities vary together and are able to see how the variation in one coincides with the variation in another. When learners reason proportionally, they recognise proportional relationships as distinct from non-proportional relationships in real-world contexts. Proportional reasoners therefore realise when a proportion is present. Although these learners use algorithms, conceptual understanding is present.

Learners must understand that a ratio is a multiplicative comparison of two quantities or that it is a joining of two quantities in a composed unit (for example: the number of boys : the number of girls). Ratios are often expressed in fraction notation , although ratios and fractions do not have identical meaning. Learners must be able to make "part to part" comparisons using ratios, but be knowledgeable that this cannot be done using fractions. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship. Proportional reasoning is complex and involves understanding that equivalent ratios can be created by iterating.

According to Lamon (2007:650), proportional reasoners have the ability to differentiate between additive and multiplicative situations. Learners associate addition with situations entailing adding, joining, subtracting and removing. They are familiar with addition because of their experience with counting and whole numbers. Multiplication is associated with processes such as enlarging, shrinking, scaling, duplicating and fair sharing. Learners need to interact with multiplicative situations and analyse the quantitative relationships to understand why additive transformations do not work. Lamon (2007) further contends that this takes time and experience and cannot happen until the learner can detect intensive quantities.

3.2.3 Ratios as a foundation for proportional reasoning

A ratio is a number that relates two quantities or measures within a given situation in a multiplicative relationship. A ratio can be expressed in several forms: as a fraction $\frac{2}{3}$, in colon notation (2:3) or as an expression (two to three) (Tipps *et al*., 2011:357). A ratio has to be expressed in simplified form.

Part of proportional reasoning is the ability to recognise ratios in various settings (Van de Walle *et al*., 2010:349). Types of ratios include part-to-whole ratios (for example, the number of girls in the class to the number of learners in the class), part-to-part ratios (comparing the number of boys to the number of girls) and rates (comparing the measurement of two quantities).

Learners need to realise the relationship between the two quantities in a ratio and that the relationship between them stays invariant. When two quantities in a ratio are multiplied by the same non-zero number, the value of the ratio remains the same (Charalombous & Pitta-Pantazi, 2007:291).

There appears to be a general consensus amongst Mathematics teachers that children and adults find ratio and proportion challenging (Lawton, 2006:50). The challenges may be due to the fact the problems are embedded in various sub-constructs of Proportion (fractions, decimals, percentages or measurements). Simple problems can be solved by learners intuitively, but solving complex problems needs more than intuition.

3.2.4 Proportions

A proportion is the equality between two or more ratios and a key part of solving problems that involve proportional reasoning (Tipps *et al*., 2011:363). In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change. Fractions indicate the proportion between a part and a whole. Percentages indicate the proportion of a specific total that is set to 100. Decimals are often measuring numbers that indicate the proportion with respect to a specific measurement (Van Galen *et al*., 2008:12). We speak of proportions if there is a linear relationship between the two (or more) numerical descriptions. The two types of Proportion are *direct proportion* and *indirect proportion.*

3.2.4.1 Direct Proportion

Direct Proportion means that if a certain number is increased or decreased by a certain factor, then the other number is also increased or decreased by a certain factor (Van Galen *et al*., 2008:29). Example: if the price of an item depends on the weight, as the weight increases, the price also increases and vice versa. This is a linear relationship. If you buy twice as much, you pay twice as much. When using a scale model, all the dimensions must be reduced or enlarged proportionally in order to retain the shape. If two figures are proportional (similar), any two linear dimensions measured will have the same ratio. For example, if a rectangle having length 6cm and breadth 4cm is enlarged by a factor of 3, the enlarged rectangle will have a length of 18cm and a breadth of 12cm.

3.2.4.2 Inverse Proportion

If one quantity is inversely proportional to another, it changes in the opposite way – as one quantity increases, the other quantity decreases.

Example

If 12 men take 5 days to build a wall, how long would it take 6 men to build the same wall (assuming they work at the same rate)?

First, we decide whether the problem is direct or inverse proportion. In this case, if fewer men are used, they will take longer, so it is inverse proportion.

Method

12 men take 5 days
6 men take *y* days
If the rate is the same, the products are equal.

Therefore, $12 \times 5 = 6 \times y$

$60 = 6y$

$10 = y$

6 men will take 10 days to build the wall.

Note

This process is the opposite of direct proportion

3.2.5 Fractions

Fraction notation is one of the ways to represent rational numbers. The mathematical notation for a fraction represents four different ways: to represent a part or a whole, to represent a part of a set, to model a division problem or as a ratio (Haylock, 2010:207).

Fractions are difficult in part because they actually represent three things: a number, a ratio and division. Vanhille and Baroody (2002:225) attribute learners' misunderstanding of fractions to two reasons. One possible reason is that learners lack concrete experiences that are necessary to construct conceptual understanding. The second reason could be that learners are not familiar with multiplicative reasoning.

Van Galen *et al*. (2008:63) argue that fractions originated from situations involving division and measurement. As fractions are the foundation for understanding percentages, decimals and proportions, learners need to be flexible in fraction computations. This will necessitate the knowledge of finding common denominators and equivalent fractions.

The concept of fraction does not comprise a single construct, but a set of five sub-constructs, that is, part-whole, ratio, operator, quotient, and measure (Pantziara & Phillipou, 2012:61).

Undoubtedly, learners must conceptualise the part-whole construct first. This is the basis for developing understanding of the other sub-constructs of Proportion.

The given example: $\frac{2}{5}$ can be conceived as part of a whole (two out of five equal parts), as a quotient (two divided by five), as an operator (two fifths of a quantity), as a ratio (two parts to five parts) and as a measure (as a point on the number line).

In order to understand how learners learn new mathematical ideas, two distinct though closely interrelated approaches have been identified by Pantziara and Phillipou (2012:61): *procedural* and *conceptual*. Many researchers are of the opinion that concept construction occurs through concrete situations, procedures and processes and moves on towards the abstraction of mathematical concepts, to the understanding of symbols and mental concepts (Pantziara & Phillipou, 2012:61).

Charalambous and Pitta-Pantazi (2007:299-300) refer to Lamon (2007): Learners understand the part-whole construct when they have the ability to portion a continuous quantity or a set of discrete objects into equal sized parts. Learners have to recognise that when an area or model is divided into parts of equal size, they have to be able to proceed with portioning and unitising, that is, to construct a fraction when a whole is given and to conceive of a whole when a fraction is given. Learners should realise that the relationship between the parts and the whole is conserved, regardless of the size, shape and arrangement of the equivalent parts and that the more parts of the whole are divided into, the smaller the produced parts become.

In the measure sub-construct, a measure is assigned to some interval on the number line, a unit fraction is defined and used repeatedly to determine a distance from a preset starting point (Charalambous & Pitta-Pantazi, 2007:299-300). Learners understand the measure sub-construct when they can perform partitions other than halving, find any number of fractions between two given fractions and use a given unit interval to measure any distance from the origin.

Ni (2001), as quoted by Pantziara and Phillipou (2012:65), ascribes the difficulty of understanding fraction equivalence to the multiplicative nature of the concept and also to the various sub-constructs related to the fraction concept. Ni (2001) asserts that this difficulty concerns two related aspects of operative thinking, the multiplicative thinking and the conservation of the whole and the parts. Pantziara and Phillipou (2012:65) give the example where multiplicative thinking is hierarchical; 4 x 3 is one "3", two "3"s, three "3"s and four "3"s at once. The second example concerns the conservation of the whole and the parts: $\frac{1}{4}$ and $\frac{3}{12}$. Learners need to understand that in equivalent fractions, the ratio remains constant while the number of counters increases or decreases.

Learners have to compare fractions. The comparison of fractions includes finding the order relation between two fractions. Learners should understand that the greater the number of parts into which the unit is partitioned, the smaller the fraction size (Pantziara & Phillipou, 2012:65). To be successful in ordering fractions, learners need to conceive that among fractions with the same denominator, the larger the numerator, the larger the fraction (example: $\frac{2}{4} < \frac{3}{4}$), while with the same numerator, the greater the denominator, the smaller the fraction (example, $\frac{3}{4} > \frac{3}{5}$) (Pantziara & Phillipou, 2012:65).

3.2.6 Percentages

The term percent is simply another name for hundredths and as such is a standardised ratio with a denominator of 100 (Van de Walle *et al*., 2010:337). In other words, percentages are numbers on a fixed scale that run from 0 to 100. Percentages offer a standardised way of describing proportions. Their kinship with fractions, proportions and decimals provides many possibilities to do arithmetic in a flexible fashion (Van Galen *et al*., 2008:29). Van Galen *et al*. (2008:95) claim that it is important for learners to convert percentages into fractions and fractions into percentages.

3.2.7 Decimals

Many learners find it difficult when computing with decimal fractions. Complete understanding requires multiplicative thinking, which does not come naturally to learners. Learners have to re-conceptualise the relationship of numbers. The mindset has to shift from additive thinking to multiplicative thinking. The study of decimal fractions is based upon expressions of the quantity with models of number lines, hundreds squares, place value charts, and currency in discussions of real life experiences (Sherman *et al*., 2005:196). Learners confuse decimal problems with whole number problems, applying procedures that do not exist. To gain conceptual understanding, learners can connect decimals to common fractions. For example, relating $\frac{3}{10}$ = 0,3 in drawings and on square paper are very effective strategies. The basis for learning decimals is established by calculating with simple fractions having the denominators 10, 100 and 1000 (Van Galen *et al*., 2008:119).

Sherman *et al.* (2005:196) contend that decimal number computation is often taught as a series of rules. The learners will apply these procedures incorrectly or even forget them because they do not understand the rules. It therefore becomes the responsibility of the teacher to gradually and developmentally introduce the algorithms in a sensible way. Place value understanding is important for learners to compute accurately and fluently when working with decimals (Sherman *et al.,* 2005:196).

Learners always think decimal fractions are actually whole numbers with arbitrarily placed decimal points. This misconception can lead to difficulties in both understanding and computational skill. Clearly, the concepts and procedures associated with decimal fractions require careful study and frequent reinforcement (Sherman *et al*., 2005:175).

Why do learners struggle with decimals? They do so because they do not fully understand the meaning of decimal number size and/or symbols; they cannot correctly order decimals with reference to size and have difficulty in choosing the correct operation to apply to a given situation (Sherman *et al.,* 2005:196-177). Learners must create meaning for written symbols and the connection between symbols and procedures. Teachers must promote meaningful understanding and experience through activities.

3.2.8 The relationship between decimals, fractions and percents

The fact that common fractions and decimal fractions are used to represent the same numbers is not understood by many learners (for example: $\frac{1}{2}$, 50% and 0,5). Van de Walle and Lovin (2006:402) attribute this lack of understanding or learners not seeing the connection to the following:

- When learners work with common fractions at one time and decimal fractions at another, connections are often not clear or not made at all.
- Teachers use the term fraction when referring to common fractions and the term decimal when referring to decimal fractions. Both terms refer to parts of units. Common fractions represent units separated into a number of parts, while decimal fractions represent units separated into 10 parts or parts that are powers of 10. Learners should understand that *percent* is a unit separated into 100 parts (Van de Walle & Lovin, 2006:402).

The knowledge that learners develop about the relationship between different types of proportion is called a network of relationships (Van Galen *et al*., 2008:70).

3.3 THE MEANINGFUL LEARNING AND THE DEVELOPMENT OF THE PROPORTION CONCEPT

Proportional reasoning is the main focus of number sense focus in the middle grades. It requires learners to transit from thinking additively to thinking multiplicatively. This change occurs gradually as learners actively engage in meaningful tasks and problems that involve ratios and proportions (Dacey & Collins, 2010:11).

Primary school learners need to add, subtract, multiply and divide, whereas high school learners need to reason about ratios and proportions. Teachers must help learners to develop the Proportion concept before resorting to algorithms. The importance of multiplicative thinking in proportional reasoning cannot be overemphasised. According to Singh (2000:271), mental operations, unitising and iterating play an important role in learners' use of multiplicative thinking in Proportion tasks. Unitising a composite unit and iterating it to its referent point enables one to preserve the invariance of a ratio. Proportions involve the coordination of two number sequences, keeping the ratio unit invariant under the iteration. In the iteration process, one needed to explicitly conceptualise the iteration action of the composite ratio unit to make sense of ratio problems and to have sufficient understanding of the meaning of multiplication and division and its relevance in the iteration process. One needs to construct multiplicative structures and iteration schemes in order to reason proportionally (Singh, 2000:271).

As learners move from additive to multiplicative reasoning with whole numbers, there are two significant related changes (Singh, 2000:273), namely changes in what the numbers are and changes in what the numbers are about. Singh argues that multiplicative reasoning is an entry point to the world of ratio and Proportion. Learners' thinking of composite unit schemes can develop into proportional reasoning.

Singh (2000:288) provides an example: To bake doughnuts, Mariah needs 8 cups of flour to bake 14 doughnuts. Using the same recipe, how many doughnuts can she bake with 12 cups of flour? A learner decomposed or unitised the composite ratio unit 8 cups to 14 doughnuts to find a ratio unit of 4 cups to 7 doughnuts and then iterated it to its referent point. She simultaneously coordinated two number sequences 4, 8, 12 with 7, 14, 21, and because of this she was able to preserve the relationship, not because of the pattern but because of the construction of the unit ratios 4 to 7. Iteration of ratios implies that there are two number sequences of 4, 8, 12 and 7, 14, 21 that are coordinated (Singh, 2000:288). Unitising appears to be an important development in proportional reasoning.

To gain meaningful insight into multiplicative schemes in proportional reasoning, one has to interpret the strategies. The ability to iterate composite units is a key foundation in meaningful understanding of proportion. The learner constructed iterable ratio units and was coordinating these units such that one ratio was distributed over the next ratio, which is basic to the construction of multiplication. The key foundation in meaningful dealing with proportional reasoning was the ability to iterate composite units. The learner unitised the composite unit to find a ratio unit and then iterated it to its referent point. She simultaneously coordinated two number sequences (example: 5, 10, 15, 20, 25 with 6, 12, 18, 24, 30) and was able to preserve the relationship in the iteration. She was able to unitise the units in a composite unit and to deal meaningfully with composite units, in which she was able to take a ratio as a composite unit while maintaining the ratio unit of its element (Singh, 2000:288).

3.4 MISCONCEPTIONS IN PROPORTIONAL REASONING

A misconception could be the misapplication of a rule, an over- or under-generalisation, or an incorrect conception of the situation. It is important to note that misconceptions are not limited to learners who need additional support: more able learners also make incorrect generalisations (Drews, 2006:15). Drews further explains that an error could be the result of carelessness; misinterpretation of symbols or text; lack of relevant experience or knowledge related to that topic or concept; a lack of awareness or inability to check the answer given; or the result of a misconception.

Researchers have recently contended and currently contend that teachers should adopt a constructivist mind-set and attitude to learners' mistakes. Learners must not be scared to make mistakes. If teachers accommodate learners' misconceptions and errors, learners' mathematical development will be enhanced (Drews, 2006:16).

Effective teaching of Mathematics involves planning to expose and discuss errors and misconceptions in such a way that learners are challenged to think, encouraged to ask questions and listen to explanations and helped to reflect upon these experiences. This suggests that the more teachers are aware of common errors and possible misconceptions associated with a topic, the more effective their planning will be to address and deal with learners' potential difficulties (Drews, 2006:20).

Swan (2001:155) attests that misconceptions are a natural stage of conceptual development. Consequently greater time in mathematical lessons should be given to encourage learners to make connections between aspects of mathematical learning and their own meanings.

Irwin (2001:402) conducted a study that investigated the role of learners' everyday knowledge of decimals in supporting the development of their knowledge of decimals. The study revealed the following commonly held misconceptions about decimal fractions:

- Longer decimal fractions are necessarily larger.
- Putting a zero at the end of a decimal number makes it ten times as large.
- Decimals act as "a decorative dot". When you do something to one side of the dot you also do it to the other side (for example: 2,5 + 1 = 3,6). Decimal fractions are "below zero" or negative numbers.
- Place-value columns include "ones" to the right of the decimal point. One hundredth is written 0,100; $\frac{1}{4}$ can be written either as 0,4 or as 0,25 (Irwin, 2001:402).

Learners look at the number after the decimal and apply whole number concept. They struggle with decimals when they lack the conceptual understanding of the value of each digit, as determined by its place (Sherman *et al*., 2005:29). The lack of understanding of place value manifests in learners being unable to order decimals.

Learners view the numerator and the denominator as separate values in a fraction. They find it difficult to view a fraction as a single value (Van de Walle *et al*., 2013:291). For example, it is difficult for them to see $\frac{2}{3}$ as a single number. Learners think that a fraction such as $\frac{1}{5}$ is smaller than a fraction $\frac{1}{10}$ because 5 is less than 10 (Van de Walle *et al*., 2013:292). Even and Tirosh (2008:203) follow the same line of argument, in that, when learners compare fractions like $\frac{1}{6}$ and $\frac{1}{8}$, they claim that $\frac{1}{8}$ is bigger because 8 is greater than 6.

Learners apply the rules applicable to whole numbers in fraction addition, adding the numerators to get a numerator and adding denominators to get an answer as a denominator. Overlapping of the whole number concepts leads to an answer such as $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ (Van de Walle *et al*., 2013:292). Chinn (2009:143) posits that the adding of the numerators and the denominators is a misconception that handicaps learners' understanding of fractions.

Multiplying fractions is difficult for learners as they associate multiplication with making something larger (Van Galen *et al*., 2008:83). It is difficult for learners to interpret a *certain fraction of something* as multiplication. As an example, learners find it strange that $\frac{1}{2}\times 4 = 2$, as 2 is less than 4.

Learners have the conception that *dividing makes smaller*. This is not always true. Consider the case of $2 \div \frac{1}{2}$. The answer is 4, which is greater than 2. Hence, learners tend to find the fractional division concept challenging (Van Galen *et al.,* 2008:84).

Learners do not understand why the denominators must be the same when subtracting fractions and why the numerators must be multiplied when multiplying fractions (Van de Walle *et al*., 2013:330). The reason is that when learners begin working with fraction multiplication, they have already internalised concepts of whole number multiplication. They get confused with fraction addition and subtraction.

Very few learners understand the measure sub-construct and that no matter what the unit of measurement is, you can break it into smaller and smaller subunits (Lamon, 2007:651). This lack of understanding makes it difficult for learners to locate a fraction on a number line.

3.5 FACILITATING MEANINGFUL LEARNING OF PROPORTION

Proportional reasoning can be a difficult concept for learners, even in the middle grades. A large percentage of learners work insightfully with fractions, percentages, decimals and proportions, but only at a concrete level – within meaningful contexts and with familiar numbers (Van Galen *et al*., 2008:12). Teachers are under pressure to "get through the text book", with the result that too little time is spent on conversations and classroom discussions. However, it is exactly these discussions – and not the lists of sums – that ensure in-depth understanding for the learners. Higher demands should be placed on learners' reasoning capacity. The teacher should link objectives to core insights and not to mastery of procedures.

Learners have the tendency to solve problems using an additive approach (Tipps *et al*., 2011:356). The essence of teaching should be to counteract additive thinking by building learners' understanding of multiplicative relationships. Teachers should not rush to introduce algorithms (for example, the cross-product algorithm). Algorithms should only be introduced when learners have fully developed and refined their understanding of multiplicative relationships as proportional relationships.

The understanding of proportional relationships is closely linked to the understanding of all facets of percent. Learners should be encouraged to use proportionally relationships to solve percent problems, rather than to rely on an algorithm applied by rote or a memorised rule to solve a problem.

Research findings suggest that it is crucial for teachers themselves to understand ratio and proportion and be able to recognise when these ideas are required to solve problems in other contexts (Lawton *et al.,* 2006:50). Lawton *et al*. demand that teachers need to proceed with caution when introducing ratio and Proportion as tension exists between teaching rules and teaching for understanding. The pendulum should swing from teaching rules to teaching for understanding. At a certain time a learner might not be cognitively mature to understand the concept of ratio and Proportion as understanding may be dependent on maturity. The suggestion is to sometimes delay the topic until the learner is cognitively mature enough to understand it. It is worth noting that these topics should be taught in context. Teachers should view language as an important factor in developing learners' understanding of ratio and proportion and emphasise it so that learners can draw concrete or mental images of proportional ideas. When learners devise incorrect algorithms, the result is a negative impact on future learning and understanding.

Useful approaches for teachers include emphasising the language of ratio and Proportion, asking learners to explain their reasoning when solving problems to assess their level of understanding and teaching proportion in a wide range of contexts (Lawton *et al*., 2006:50).

It is essential that conceptual understanding be established as a foundation for mastering algorithms and procedures. Concrete, hands-on materials and drawings to which symbols can be connected in the same lesson are critical components for lessons in which learners achieve both fractional number sense and then computational frequency (Sherman *et al*., 2005:166).

Teachers can help learners develop proportional thought processes by providing ratio and Proportion in a wide range of contexts, encouraging discussion and experimentation in predicting and comparing ratios, helping learners distinguish between proportional and non-proportional comparisons by providing examples of each, and discussing the differences (Van de Walle *et al*., 2010:350). Mechanical methods such as the cross-product algorithm should not be introduced until learners acquire conceptual understanding.

Fractions should be taught in a developmental, planned approach. This concept of part of a unit is more easily understood and should be introduced before the part/whole examples of sets and lines. As learners understand fraction symbolism, the operations are reinforced (Sherman *et al.,* 2005:166). As a suggestion, when introducing fraction addition and subtraction, begin with common denominator examples before touching on uncommon fractions. When introducing multiplication and division of fractions, begin with common fractions and gradually move to mixed fractions.

Van de Walle *et al*. (2010:316) posit that some teachers believe that they don't need to devote too much time to fraction operations and that algorithms are quicker to teach and lead to less confusion. This approach does not work as none of the algorithms help learners to think conceptually about the operations and what they mean. Van de Walle *et al*. further argue that when learners follow a procedure without understanding, they have no means of knowing when to use it and no way of assessing whether the answers make sense.

Percentages are introduced as a standardisation of the description of part-whole relationships with fractions. At the beginning, learners must therefore already have acquired the necessary elemental knowledge about fractions. Moreover, percentages are linked to "hundredths", which means that the learners must be specifically familiar with these fractions. It is therefore an obvious step to begin percentages only after decimals have been addressed, at least the common decimal fractions (Van Galen *et* *al*., 2008:103). In the learning-teaching trajectory for percentages, the emphasis should lie on flexible mental arithmetic.

Sherman *et al.* (2005:196) are of the opinion that learners mistake decimal problems for whole number computation, applying procedures that do not exist. Conceptual understanding can be developed by learners when they connect decimals to common fractions. For example, relating $\frac{3}{10}$ = 0.3 in drawings and on square paper are very effective strategies. Teachers must encourage learners to reflect on their answers and correct them. In the classroom, decimal computation is taught using rules. When learners do not understand the rules, the result is misapplication or forgotten rules.

Time is needed for making sense of algorithms and to develop fluency in learners' computations. Understanding place value is the stepping stone to success in decimal computations. Teachers can help clear misconceptions by giving counter examples (Sherman *et al.,* 2005:196).

A large percentage of learners work insightfully with fractions, percentages, decimals and proportions, but only at a concrete level – within meaningful contexts and with familiar numbers (Van Galen *et al*., 2008:11). Teachers should concentrate on learners' reasoning capacity and spend more time on discussions instead of racing through the text-book. The essence should be the linking of objectives to core insights and not to mastery of procedures.

Teachers can develop proportional reasoning in the Mathematics classroom through discussion, investigation and connection (Dacey & Collins, 2010:11-12). Learners must be shown ways of involving proportional reasoning in tasks. For algebraic success, teachers should discuss ways so that proportional reasoning permeates the study of fractions, decimals, percents. The role of proportional reasoning in converting between fractions and decimals should be evident. When teachers provide hands-on opportunities, learners make connections much easier.

Van Galen *et al*. (2008:12) brainstorm ways for the meaningful understanding of fractions as follows:

Developing language about fractions

When learners arrive from Grade 4, they already know about fractions in an informal way – as one-half, one-quarter and so on – but they have not yet gone deeply into fractions. The process begins with a systematic exploration of situations in which fractions occur. During this exploration, the language about fractions must be carefully developed. Teaching learners about sharing situations lead to all types of fractions. Sharing offers opportunity for developing fraction language. An important step in developing this language is the transformation of a “certain part of” into a measurement unit. Learners must wean themselves from making and counting parts. They must start to see fractions as descriptions of part-whole proportions (Van Galen *et al*., 2008:12).

Finding the common denominator by reasoning

Division is also used to prepare learners for finding the common denominator. For example, investigating the relationships between dividing into sixths, thirds and halves prepares learners for finding the common denominator of sixths, thirds and halves. While engaging in “reasoned sharing”, the learners are actually preparing for multiplication and division as well. This takes place when we ask how many times one-sixth fits into one-half, or what you get when you put two pieces of one-sixth together (Van Galen *et al*., 2008:12).

Number relationships

Computing activities on fair sharing help learners develop number relationships.

Performing reasoned operations

Reasoned addition and subtraction build upon finding the common denominator through reasoning. The objective is to have the learners reason time and again about how finding the common denominator actually works. Multiplication as repeated addition builds on reasoned addition, just like repeated subtraction is a logical expansion of reasoned subtraction (Van Galen *et al*., 2008:12).

"*Times*"

Relating "part of" to "times" is a difficult problem. Learners need to get accustomed to *part of* in proportion. The image of multiplication is expanded.

Encouraging the development of procedures

The more practice on procedure, the greater the fluency.

Unfortunately, many learners experience fractions only as parts and wholes and have no idea where they appear on the number line (Dacey & Collins, 2010:8). Teachers need to encourage learners to speak and experience different representations to reduce and clear misconceptions of the number line concept. For example: ask the learners to think about three-fourths. What first visual image do the learners get in their minds? Let them draw it, then describe it. As a group, discuss the images and words. Work together to create different visual images. Decide on the language you need to use when developing ideas about fractions. Think about the representations you use in the classroom. What are the limitations and challenges? Create problems in context, appropriate for your learners to solve, that involve comparing fractions. Encourage learners to draw, write or verbally explain their responses. Compare responses across the classroom. This will help you to draw conclusions about your learners' understanding (Dacey & Collins, 2010:8).

The relationship between fractions, percentages, decimals and proportions or ratios can be dealt with in a natural way if teachers make the context the central feature in teaching, affording the learners the opportunity to explore these contexts in various ways.

Drews (2006:17-18) firmly believes that educators should teach for cognitive conflict to accommodate errors and misconceptions. This describes learners presented with examples and problems which lead to illogical outcomes. For example, the adding of fractions $\frac{1}{2}+\frac{1}{4}$. If the strategy 'add across the top and bottom' is applied the result $\frac{2}{6}$ can be compared to a demonstration of a bar of chocolate where $\frac{1}{2}$ is given to pupil A and $\frac{1}{4}$ is given to pupil B – how much is given away? ($\frac{3}{4}$). The two different answers to the same example creates conflict between existing conceptual understanding (adding fractional values just across) and new information which challenges this existing framework. Teachers should resolve this conflict through peer discussion, sharing ideas, justifying responses, listening to others and teacher questioning. Accommodation can only occur when restructuring takes place within one's 'schema' to deal with the cognitive conflict. Learners need to accept that their response is not always quite right. The learning process and environment need to be of sufficient importance to the learners for them to make the effort to restructure and change their thinking. Teachers need to accept that explaining the misconception is not enough – the learners will also need help in the restructuring process (Drews, 2006:17-18).

Irwin (2001:491) claims that teachers should be aware of learners' misconceptions regarding decimals. Posing questions and mediating dialogue will help learners to reflect on their own thinking.

The problem of teaching multiplicative reasoning is compounded by the ways in which Proportions are taught in the school setting (Singh, 2000:271). According to Singh, teachers teach learners algorithms that are unfamiliar to them. Using these techniques, learners may arrive at the correct answer but are deprived of opportunities of sense making. If the concept is not well taught, learners will not learn well. Memorising rules such as "multiply the bottom and top", in fraction multiplication, poses a barrier when solving proportional algebraic expressions (Van de Walle *et al*., 2013:330).

Number lines are difficult for learners to manipulate. Teachers should help learners master other notions, such as the equivalence of fractions, before rushing to introduce this model in their teaching (Charalombous & Pitta-Pantazi, 2007: 311).

From the arguments above it is clear that addressing misconceptions is beneficial to the development of learners' meaningful understanding of Proportion. It therefore becomes incumbent upon teachers to address misconceptions and give learners adequate time to develop proportional reasoning.

3.6 CONCLUSION

Proportional reasoning is complex, both in terms of the underlying Mathematics and of the developmental experiences that it requires (Langrall & Swafford, 2000:261). Various factors affecting the meaningful learning of Proportion heighten the complexity of the concept. The arguments above seem to justify the notion that proportional reasoning is a difficult and important conceptual leap for learners. The facilitation of the meaningful learning of Proportion is discussed with a view to guide the teaching of this difficult concept. In the next chapter the research design and methodology inherent in the empirical study are discussed.

CHAPTER 4:
RESEARCH DESIGN AND METHODOLOGY

4.1 INTRODUCTION

The previous chapter explained the concept of proportional reasoning: its nature, the development of the multi-faceted constructs of proportion, misconceptions, the role and importance of proportional reasoning in the teaching and learning of Mathematics.

The researcher aims to provide a structure in this chapter for her investigation in answering the research questions regarding the Grade 8 learners' conceptual understanding of proportion. This chapter is devoted to an elaborate description of the research design and methodology used in the study. Data collection and data analyses strategies will be focused on in the latter part of the chapter.

4.2 AIM AND OBJECTIVES OF THE STUDY

The overall aim of this study was to investigate how Grade 8 learners learn and understand proportion and the grounds for their understanding or misunderstanding, culminating in the research question as to what can be done to support their meaningful learning of proportion and related concepts in the Mathematics classroom.

The overall aims were operationalised as follows:

By determining the meaning of the concept of proportion; the understanding and development thereof through a literature review; establishing the grounds for misunderstandings through a literature review; scrutinising how the concept of proportion is understood by learners through empirical research and determining what can be done to support the teaching and learning of proportional reasoning through a literature study based on empirical findings and in view of literature.

4.3 EMPIRICAL RESEARCH

This study is embedded in a qualitative research methodology. The rationale behind the chosen qualitative research methodology will be evident in the paragraphs that follow.

4.3.1 Research paradigm

A research paradigm refers to a pattern or model for research. To determine from which lens the research should be viewed, it has to be located within a research paradigm. Nieuwenhuis (2007a:52) refers to epistemology as "how can we know" and ontology as "what is the truth". This study is embedded in a paradigm that favours constructivist ontology and an interpretive epistemology.

This study is qualitative in nature. An understanding of qualitative research can be gained by looking at its philosophical foundation (Merriam, 2009:8). The philosophy that underlies the design and implementation of this research is interpretivism. Interpretivistic research assures that reality is socially constructed, that is, there are multiple interpretations of an event. Knowledge is not found but constructed. Constructivism is a term often used interchangeably with interpretivism (Merriam, 2009:9). According to Merriam (2009:9), in this world view individuals seek understanding of the world in which they live and work. They develop subjective meanings of their experiences. These subjective meanings are negotiated socially through interaction with others, hence social constructivism informs interpretive research. This is parallel to Denscombe's (2010:121) argument: "From the interpretivist's view, the social world does not have the tangible, material qualities that allow it to be measured in some literal way. It is a social creation constructed in the minds of people and reinforced through their interactions with each other" (Denscombe, 2010:121).

The researcher intended to investigate the deep knowledge emanating from the participants by personal interaction with each participant, spending extensive time with the participants and probing them to obtain detailed meaning of understanding the concept of proportion.

4.3.2 Research design

A qualitative approach is needed to answer the research questions. According to Merriam (2009:8), qualitative researchers are interested in understanding the meaning people have constructed, that is, how people make sense of their world or how they interpret their experiences in the world. The focus is on process, understanding and meaning. The researcher is the primary instrument of data collection and analysis, the process is inductive and the product is richly descriptive. The key concern is the understanding from the participants' perspectives, not the researcher's. Denscombe (2010:132) asserts that qualitative research is primarily concerned with the way in which people shape the world. It emphasises the ways in which human activity creates meaning and generates the social order that characterises the world in which we live.

The objectives of this qualitative research were to gain an in-depth understanding of how the participants from the Grade 8 classrooms interact with the concept under investigation. In a nutshell, the researcher intended to investigate the understanding of proportion with the participants being the human instruments through whom data would be generated and analysed.

4.3.3 Research methodology

Case study

Case study research is a qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) through in-depth data collection involving multiple sources of information (Creswell, 2013:97). This study was embedded in the qualitative research approach to interpret how the learners understand and what they understand.

The initial assessment was a written task-based assessment for all participants, conducted in the researcher's classroom, followed by face-to-face interviews with each learner. The researcher made photocopies of each learner's task sheet. The interviews took place within a few days. As this occurred during the post-exam period, a few learners were interviewed one after the other as they availed themselves. During the interview session, the learners were handed their original task sheets to review, while the researcher made notes on the photocopied task-sheets as the learners responded to the researcher's questions. The procedure corroborated with Singh (2000:274)'s opinion concerning interviewing: "Interviewing as a successful tool in research must be accompanied by appropriate learning tasks " (Singh, 2000:274).

4.3.4 Participants and setting

The population of the study involved learners of Mathematics in three Grade 8 classes. The study focused on purposive non-probability sampling. According to Creswell (2013:156), the concept of purposeful sampling used in qualitative research enables the inquirer to select individuals and sites for study because they can purposefully inform an understanding of the research problem and central phenomenon in the study. He further contends that decisions need to be made about whom or what should be sampled, what form the sampling will take and how many people need to be sampled (Creswell, 2013:156). McMillan and Schumacher (2006:126) further explicate this argument by stating that purposeful sampling requires judgement of the researcher to select subjects who will provide the best information to address the purpose of the research.

The sample was not randomly chosen. Eighteen learners were chosen from the three Grade 8 classes (8A, 8B and 8C). All three classes contain mix-ability learners. The participants were a heterogeneous group of 18 learners comprising different academic capabilities, genders, ethnic groups and home languages. The 18 learners were selected from low achievers to high achievers from all three classes.

The researcher teaches Mathematics to the three Grade 8 classes and is knowledgeable about the content related to Grade 8 Mathematics, namely proportional reasoning in Algebra. An initial assessment task on proportion learnt in Grade 7 was given to the learners of the three Grade 8 classes to guide the researcher as to the level of competence of each learner. Six learners were chosen from those obtaining over 70%, six learners scoring between 40% and 69% and the final six learners were the underachievers who were rated in the region of 0% to 39%. The selection of learners also took part in the task-based interviews.

The setting was the researcher's classroom. The physical environment was optimally geared with desks spaced out neatly, a notebook and signed consent forms.

4.3.5 The role of the researcher

The researcher is an educator in the field of Mathematics. She teaches Mathematics to three grade 8 classes, comprising 120 learners. The researcher submitted ethics forms to the NWU, the Department of Education in Rustenburg, the principal of this multicultural school of excellence and to the parents of the participants for their approval of this investigation.

The researcher collected data through the tasks sheets and interviews, analysed the data, reported and concluded with recommendations. Thus, the researcher was a participant observer and the primary instrument in data collection.

4.3.6 Data generation instruments

4.3.6.1 Task Sheet

The task-sheet (see Addendum D) was the first step and the core in the data collection process. The sheet was used as a means of determining the understanding of proportion or proportional reasoning. The task sheet intended to address the research questions:

Question 1: How do Grade 8 learners understand the concept proportion?

Question 2: What are the grounds for understanding?

Question 3: What misconceptions occur?

Question 4: What grounds exist for the observed misconceptions?

Question 5: What can be done to support the learners to learn proportion with understanding?

The task sheet was made as learner-friendly as possible. The aim was to capture and retain the participants' attention in order to improve concentration and motivation as these learners are at the age where they still think concretely. The task sheet consisted of 15 problems, covering all five facets of Proportion, at different cognitive levels, containing familiar and unfamiliar contexts. The task sheet did not have a heading as learners needed to recognise the tasks as proportional reasoning tasks. Nine out of the 15 tasks required the learners to explain their reasoning. The language used in the sheet was simple enough for the learners to understand.

In *Task 1* the learners needed to be able to iterate to calculate the ingredients needed for the chocolate recipe for six people, given the recipe for two people. This involved multiplicative reasoning. Learners had to motivate their reasoning.

Task 2 tested the scaling knowledge of the participants. The idea was to realise that the larger frame is the enlargement of the smaller frame and that the ratio of enlarging is fixed. The lengths and the breadths have to be in proportion, which means that the factor of enlargement must be a fixed ratio. Explanations of steps followed.

Task 3 was based on exchange rate. Conversion from Euros to Rands and vice-versa tested the idea of constant ratio as in task 2.

Tasks 4.1 and 4.2 required participants to order decimal numbers and fractions respectively, asking learners to explain their steps.

Task 5.1 entailed the extraction of the ratio concept from a statement. Task 5.2 was based on the idea of recognising that the same number can exist in three different forms, that is, fraction, decimal and percentage. In Task 5.3 the expectation was to convert a fraction to a ratio, finally simplifying the ratio. Task 5.4 investigated whether learners were aware of the fact that percentage runs up to 100 only.

Task 6 consisted of the concept of indirect proportion or inverse proportion. Learners needed to calculate and realise that this problem was embedded in indirect proportion.

Task 7 requested learners to locate a fraction on a number line.

Task 8 tested the procedural knowledge of fraction subtraction, multiplication and division.

Task 9 was of an unfamiliar context, where proportional or multiplicative reasoning was needed to solve the problem.

4.3.6.2 Task-based interviews

The interviews were conducted after the task sheets had been completed. All 18 learners were interviewed. The researcher did not make use of an interview schedule, but the participants were basically asked to elaborate on their answers and reasoning. The focus was not on right answers only. The researcher was searching for wrong answers where misconceptions were present. The zeal to answer the third and fourth research questions necessitated the search for misconceptions. The interviews were based on the tasks to establish correspondence between the written and the verbal answers.

Interviews are “technology of mind” tools of research (Henning, 2004:79). He declares that for interviews to be trustworthy and credible tools of data capturing, it is important that craftsmanship be built into their conceptualisation and their design as well as their implementation, recording and transcription. Henning (2004:70) further iterates that in essence, interviews are communicative events aimed at finding what participants think, know and feel. Task-based interviews are used in research for observing mathematical behaviour of children and drawing inferences from the observations (Goldin, 1997:40). Goldin further attests that task-based interviews have importance both as research instruments and as potential research-based tools for assessment and evaluation.

The interviews were planned with a clear overall design logic in mind. Task-based interviews were conducted at different times in the researcher’s classroom. Though all interviews were semi-structured and intentionally the same for all the participants, the probes depended on learners’ written responses in task completion and interview responses and varied from interview to interview.

The researcher assumed the role of a participant observer in the research process. According to Henning (2004:81), she becomes the instrument of observation and "sees for herself" how people act in a specific setting and what the setting comprises. What is observed (seen and heard) is the researcher's version of what is "there". He further argues that the interpretive researcher searches for the way in which the social actors make meaning on the stage of action that she is observing. The researcher therefore observed and recorded in such a way that the data could be used as building blocks. By then she had observed the data thrice – first in direct contact with the naturally occurring events, when the participants represented how they made meaning during task execution. This was evident in what they did, how they did it, what they used, and against what setting or back-drop they did it. Secondly, the interviews shed light on the task execution. Thirdly, the researcher observed through her notes after interviewing. She therefore interpreted thrice: firstly through the interpretation and presentation of the execution of the task sheet, secondly through the interviews and thirdly through the text she created from the interviews.

The researcher gathered purposeful data through task analyses and interviewing. A detailed recording of the data followed. Finally, the recorded data was prepared for scrutiny.

4.3.7 Data Analyses

For a strong research argument, the more methods of analysis used, the better the chances of the research crystallising into good craftsmanship (Henning, 2004:140). Henning (2004:138) mentions three phases of qualitative data analysis:

Table 4.1: Three phases of qualitative data analyses

PHASES OF ANALYSIS	DESCRIPTION OF PHASE	APPLICATION IN THIS STUDY
Orientation to the data	Reading or studying data sets to form overview and to apprehend the context (within the data text).	Studied task sheets and comments from interviews on copied task sheet thoroughly.
Working with data	Coding segments of meaning. Categorising related codes into groups. Seeking relationships between categories to form thematic patterns.	Searched for similarities and difference, forming categories. Looked for patterns.
Final composition of the analysed data text	Writing the final themes of the set of data. Presenting the pattern of related themes. The researcher will be able to make sense of the large quantity of data collected based on the construct developed during the literature review. The research will look for similarities.	After grouping similarities and differences, the final theme was presented, based on the literature review.

The column on the extreme right explains what was done by the researcher in accordance with the three phases of the analysis.

The scoring of the task sheet was based on a rubric .

According to Dossey *et al*. (2002:559), scoring guides for student-constructed items are known as rubrics, that is, the scoring codes in the rubric assess the learners' mathematical correctness or level of understanding, their communication of the response and the strategies employed.

The following rubric was used to assess understanding of the learners' tasks. The rubric was adapted and modified from the examples of rubrics as given by Dossey *et al.* (2002:556-564). This rubric's levels implicate various levels of understanding.

Table 4.2: Rubric to assess task performance

LEVEL		CRITERIA
4	Complete understanding	Correct answers. Complete reasoning. Coherently communicates answers/solutions. Evidence of deep understanding.
3	Moderate understanding	Correct answers. Reasoning valid, but not complete. Can communicate answers/solutions, but not coherently. Evidence of good understanding.
2	Poor understanding	Answers contain mathematical errors or are incomplete. Incomplete reasoning. Poor communication of answers/solutions. Evidence of partial understanding.
1	No understanding	Incorrect answers. Incorrect reasoning. Incoherent communication. No evidence of understanding.
0	No effort	No response or answer.

In a qualitative data analysis, the researcher aims to gain new understanding of the situations and processes being investigated (Creswell, 2009:183). In view of this the same rubric was used as a guide in assessing the learners' understanding of proportional reasoning during the interviews.

4.3.8 Ethical considerations

All aspects of the research process, from deciding upon the topic through to identifying a sample, conducting the research and disseminating the findings, have ethical implications, Creswell (2013:56-60) and Flick (2009:36-44) identified ethical principles to be observed. In the context of this study, the researcher complied with ethical principles as collated by the above authors in the following ways:

The researcher submitted an application to the Ethics Committee of the North-West University to obtain approval to conduct the research and permission was granted. Permission was obtained from the North-West Education Department of the Rustenburg region, the school principal, parents and learners to conduct the research.

The researcher explained the purpose of the research to the participants, verbally and by means of a letter, assuring them that their participation would be voluntary and could be terminated at any time at their request. The researcher assured the participants that their responses would be confidential. To conceal identities, tasks and interviews were completed anonymously. The social principles of social justice, equality and emancipation guided the researcher's ethical behaviour.

4.3.9 Trustworthiness, triangulation and crystallisation

Guion *et al*. (2011:1) explain validity in qualitative research. They posit that the findings of a study are true when the research findings accurately reflect the situation and are certain when these findings are supported by evidence.

In qualitative research the method of triangulation is used to check and establish validity in a study by analysing a research question from multiple perspectives (Guion *et al*., 2011:1). Data from the literature review, the tasks and task-based interviews served as basis for the triangulation employed in the analysis of the data. From comparison of the three sources of data, the researcher extracted common themes and patterns. Maree and Van der Westhuizen (2009:39) posit that triangulation is critical in facilitating validity and establishing data trustworthiness. The respondents' words were interpreted accurately, with no distortion of what was heard. Maree (2007:80) also argues that multiple methods of data collection, such as observation, interviews and document analyses, leads to trustworthiness.

Richardson (2000:394, as cited in Maree, 2007:81), proposes the concept of crystallisation.

Maree (2007:81) explains that crystallisation provides us with a deeper and complex understanding of the phenomenon and that the emergent reality is not a result of some form of measuring. The reality emerged from various data-gathering techniques and represented the researcher's own reinterpreted understanding of the phenomenon, that is, findings crystallised from the data. Seeing the same emerging pattern added trustworthiness to this research. Labanca (2010) is of the opinion that the trustworthiness of a qualitative study can be increased by maintaining high credibility and objectivity. The researcher had adequate contact with the participants, identifying and verifying recurrent patterns in the data, implying high credibility. To confirm the latter, the researcher used prominent research methods from the literature review and was not biased due to her being close to the participants as their teacher. Findings could be confirmed from the tasks and notes on the task-based interviews.

4.4 CONCLUSION

The main purpose of this chapter was to outline and motivate the choice of empirical design utilised in this study. The qualitative data was collected in two phases: phase one consisted of tasks and phase two consisted of interviews. The role of the researcher was discussed as well as the ethical connotations to this study. In the next chapter the researcher presents the data generated during the empirical study, followed by interpretation and reflection.

CHAPTER 5:
ANALYSIS OF THE DATA

5.1 INTRODUCTION

This chapter provides an interpretive synopsis of the data that was generated during the study. The four research questions are addressed through the discussion of empirical findings. The chapter firstly addresses the question: How do Grade 8 learners understand the concept of proportion? Then it focuses on the grounds for understanding, which addresses the second research question. The third and fourth research questions addressing misconceptions and the grounds for the observed misconceptions are also discussed. Finally, conclusions are drawn from the findings and discussions.

5.2 THE UNDERSTANDING OF THE PROPORTION CONCEPT

When learners are flexible in working with rational numbers, their understanding of the concept of proportion develops (Van De Walle *et al.*, 2010:363). They contend that equating two ratios to solve for a missing term is not an indication of understanding proportion. Learners need to recognise proportional quantities and their relationships.

The discussions that follow will focus on the understanding of the proportion concept, misunderstanding, as well as the grounds for understanding and misunderstanding, which emanated from the empirical study.

5.3 DISCUSSION OF THE LEARNERS' EXECUTION OF THE VARIOUS TASKS

Proportionality tasks in various contexts were posed in order to determine learners' development of proportional thought processes. The rubric in section 4.3.7 was used to assess the participants' understanding during the execution of the tasks as well as a guide to assess understanding during the interviews with the learners. A discussion of the learners' execution of the various tasks and findings from the interviews follows.

5.3.1 Task 1: Chocolate cake recipe

Proportional thinkers unitise a composite unit to find a ratio unit and then iterate it to its referent point, preserving the relationship in the iteration (Singh, 2000:282) (see section 3.3). The intent here was to investigate whether learners could iterate. According to Van de Walle *et al*. (2010:294), when learners count the fractional parts and then find multiples of the fractional parts to form a whole, iteration takes place. In Task 1, learners had to iterate:

Study the chocolate recipe for two people below:

4 tablespoons of flour

2 tablespoons of butter

2 tablespoons of sugar

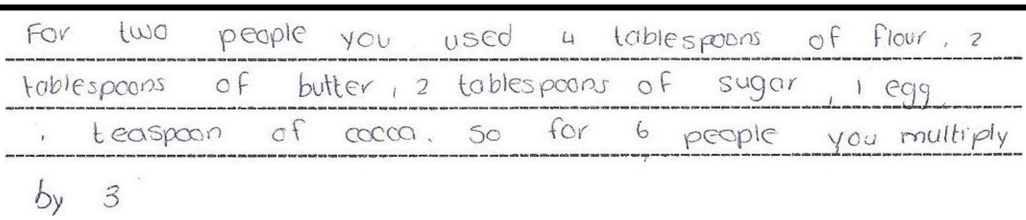
1 egg

1 teaspoon of cocoa

Calculate the ingredients needed to bake a cake for six people.

Figure 5.1: Task 1

Eleven of the 18 learners calculated the multiplication factor to be three, as the recipe was given for two people and ingredients for six people needed to be calculated. All 11 learners iterated the composite unit, in this case, two people (see section 3.3). Their answers were correct, their reasoning complete and coherent, indicating *complete understanding*, corresponding to rubric level 4. Evidence of iteration follows:



For two people you used 4 tablespoons of flour, 2 tablespoons of butter, 2 tablespoons of sugar, 1 egg, 1 teaspoon of cocoa. So for 6 people you multiply by 3

Figure 5.2: Response: Learner A

Learner B first found the recipe appropriate for one person and then multiplied by 6. She was able to unitise the units in a composite unit and deal meaningfully with composite units, in which she was able to take a ratio as a composite unit while maintaining the ratio unit of its element (see section 3.2.2). *Complete understanding* is evident (rubric level 4):

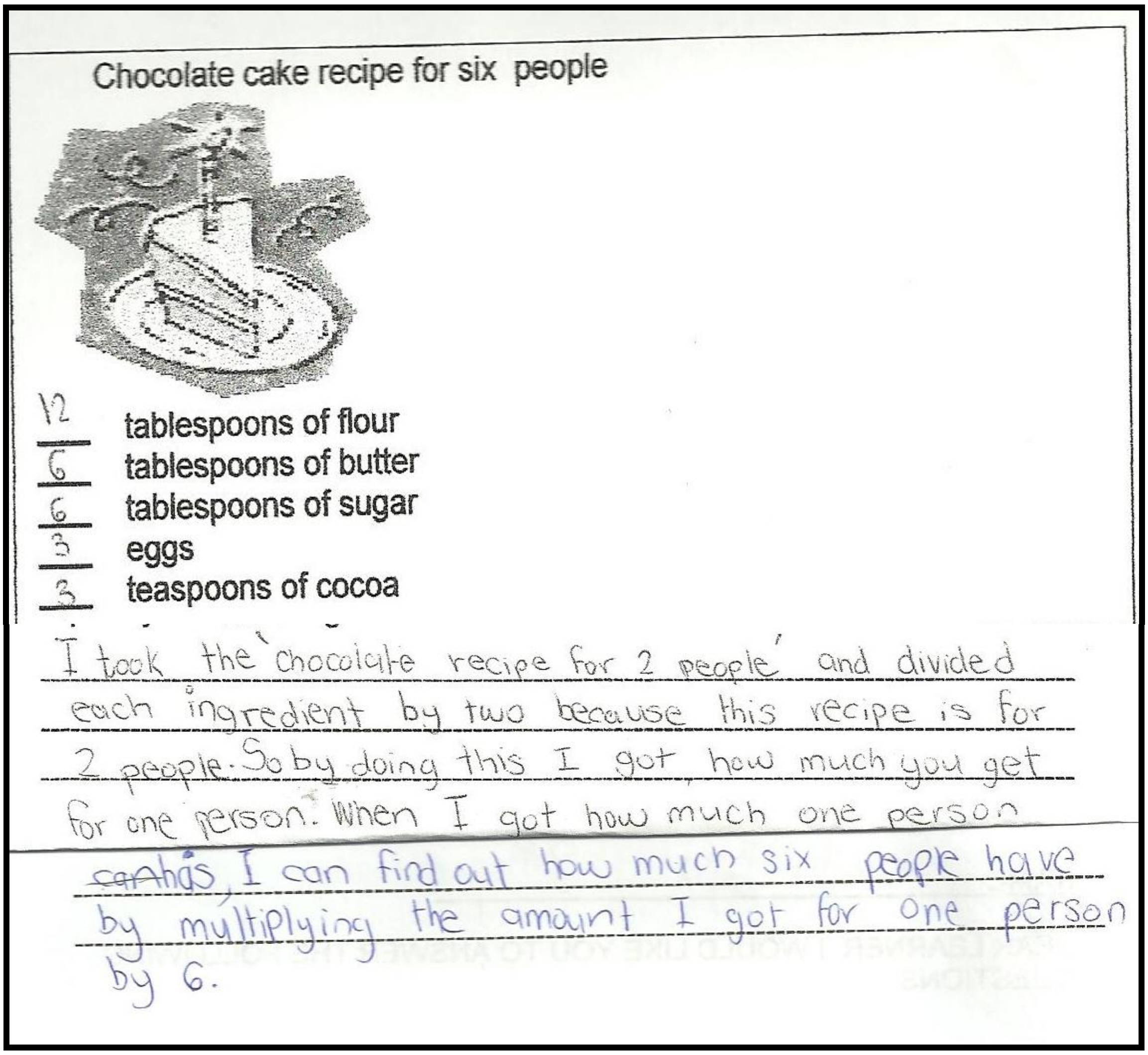


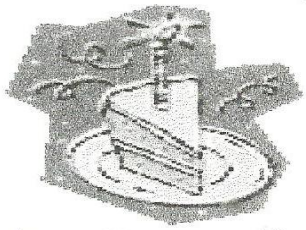
Figure 5.3: Response: Learner B

In total 12 of the 18 learners had *complete understanding* (rubric level 4) of direct proportion in the context of this task as revealed above. This level of understanding stems from the fact that these 12 learners had correct answers, their reasoning was complete and their communication of answers was coherent. This unitising ability parallels with Singh's (2000:288) comment that unitising is an important development in proportional reasoning (see section 3.3).

Six learners' responses were incorrect, their reasoning incomplete and their communication of solutions were not coherent, displaying rubric level 1 – *no understanding*. Two of the learners had a problem with computational accuracy when completing the task. When interviewed, they indicated that they made arithmetical errors, as shown below:

Learner K:

Chocolate cake recipe for six people



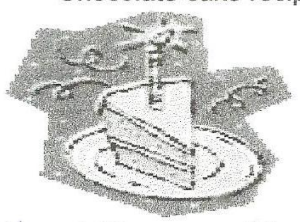
12 tablespoons of flour
10 tablespoons of butter
10 tablespoons of sugar
5 eggs
5 teaspoons of cocoa

Explain your reasoning:

I took the number of people and multiplied each one of them by six.

Learner O:

Chocolate cake recipe for six people



16 tablespoons of flour
4 tablespoons of butter
4 tablespoons of sugar
2 eggs
2 teaspoons of cocoa

Explain your reasoning:

multiply each number by itself to get the recipe for six people example ~~2 people multiply with the same 2~~.

Figure 5.4: Responses: Learner K and O

Though the response from Learners K and O were incorrect on the task sheets, interrogation during the interviews revealed that these two learners did not lack reflective practices in looking back and reflecting on the validity of the answer. In the words of Learner K: "…*I thought that the original recipe was for one person. That is why, I multiplied by six. 2 x 6 is wrong. It must be 2* x *3...".* When Learner O was asked to explain her answer, she smiled and quickly confirmed that each ingredient must be multiplied by 3. From the quick, corrective responses, one can deduce that these two learners did have the ability to reason multiplicatively. They probably did not fully attend to the problem. The incorrect reasoning during the task completion could be attributed to being in a hurry or careless (see section 3.4). Misconception is not evident, the two learners seemed to show *no understanding* during task completion but realised that their answers were wrong and verbally rectified their mistakes during the interviews, indicating some understanding.

Another incorrect response (out of the six) emanated from the script of Learner G, indicating a serious misconception in proportional reasoning. Learner G used an additive strategy rather than a multiplicative method to solve this problem. He argued that since the recipe was for two people, one needed to add 4 to each original quantity to obtain the recipe for six people. Evidence of Learner G's misconception follows:

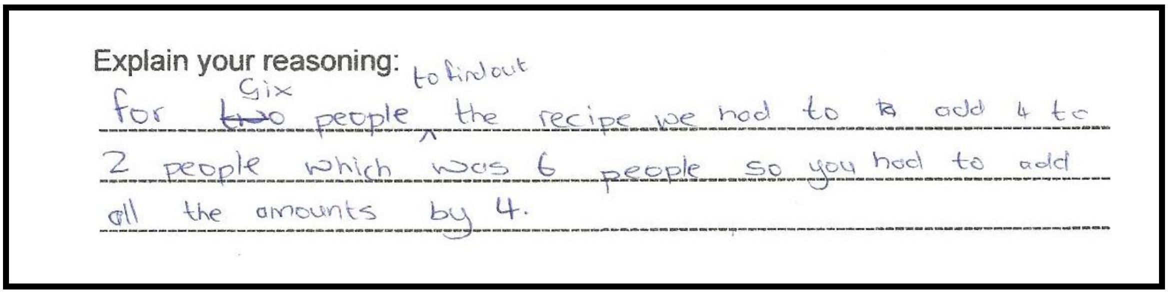
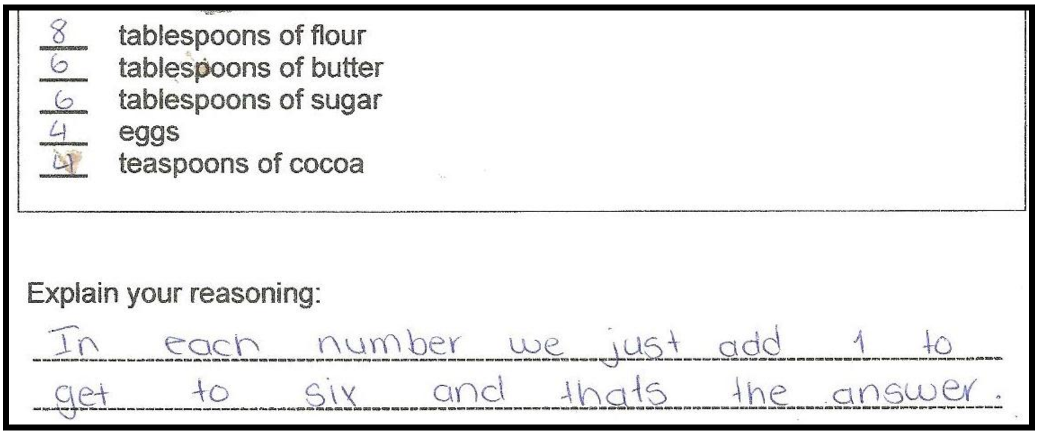

Explain your reasoning:
for ~~two~~ six people to find out the recipe we had to add 4 to 2 people which was 6 people so you had to add all the amounts by 4.

Figure 5.5: Response: Learner G

It is evident from the response above that the answer is incorrect as well as incoherent and displays *no understanding*. During the interview Learner G still demonstrated additive reasoning. Two more examples of *no understanding* are shown below:

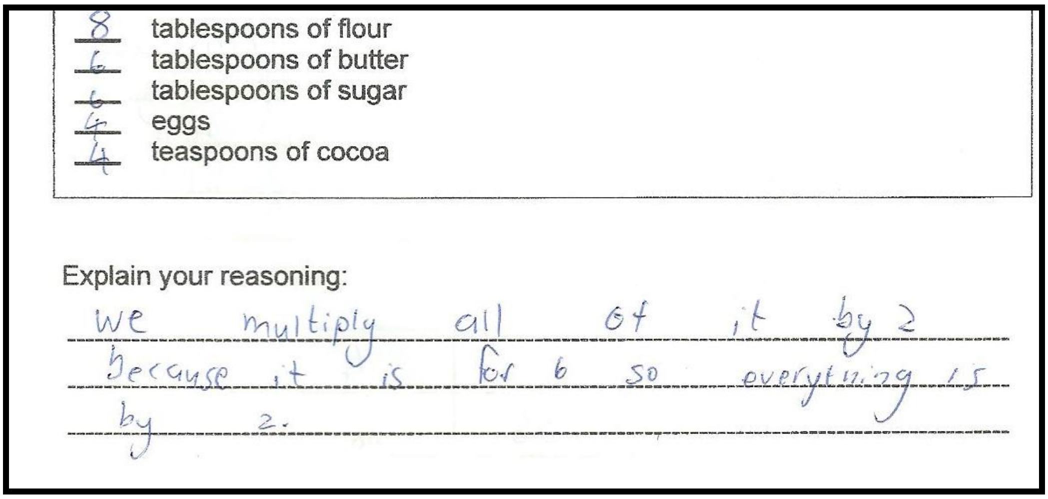
Learner M:



8 tablespoons of flour
6 tablespoons of butter
6 tablespoons of sugar
4 eggs
4 teaspoons of cocoa

Explain your reasoning:
In each number we just add 1 to get to six and thats the answer.

Learner I



8 tablespoons of flour
6 tablespoons of butter
6 tablespoons of sugar
4 eggs
4 teaspoons of cocoa

Explain your reasoning:
we multiply all of it by 2 because it is for 6 so everyting is by 2.

Figure 5.6: Responses: Learners M and I

In both these cases no correspondence exists between the reasoning and the mathematical computations. Learner M did not add 1 to the original recipe. She multiplied the 4 tablespoons of flour by 2, the 2 tablespoons of sugar and butter was multiplied by 3, the egg and teaspoon of cocoa was multiplied by 2. The ratio was not constant throughout for the required recipe. In fact, she did not add 1 at all. Learner I only multiplied the amount of flour by 2. The amounts for butter and sugar were multiplied by 3, while the amounts of egg and cocoa were multiplied by 4.

At the time of interviewing, both learners, Learner M and Learner I, were unable to give an explanation about the execution of the tasks and their answers. When Learner M was asked why she added 1, she replied: "…*I don't know*…" The result indicates a lack of understanding or *no understanding* (level 1) in the development of the concept of proportion during task execution and interviews.

The remaining learner (out of the six), also displaying *zero understanding* (rubric level 1), multiplied each ingredient by 4 and was unable to substantiate this during the interview session.

5.3.2 Geometric figures in understanding direct proportion.

Instead of using numbers, geometric figures are an alternative to test learners' understanding of proportion. When learners are requested to discuss proportional relationships using similar figures, they are forced to discuss the relative sizes without resorting to a rule (Slovin, 2000:58-60). The intent here was to give the participants the opportunity to acknowledge the development of their sense of proportion by the observation of the characteristics of proportional (that is, similar) figures. Task 2 below was aimed at learners' understanding of the factor of enlargement:

Study the picture below.

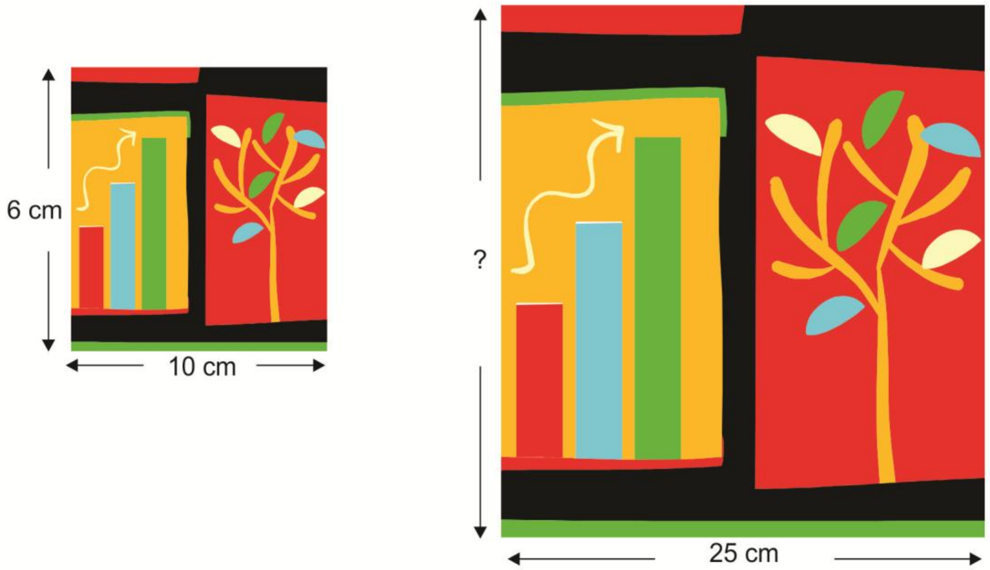


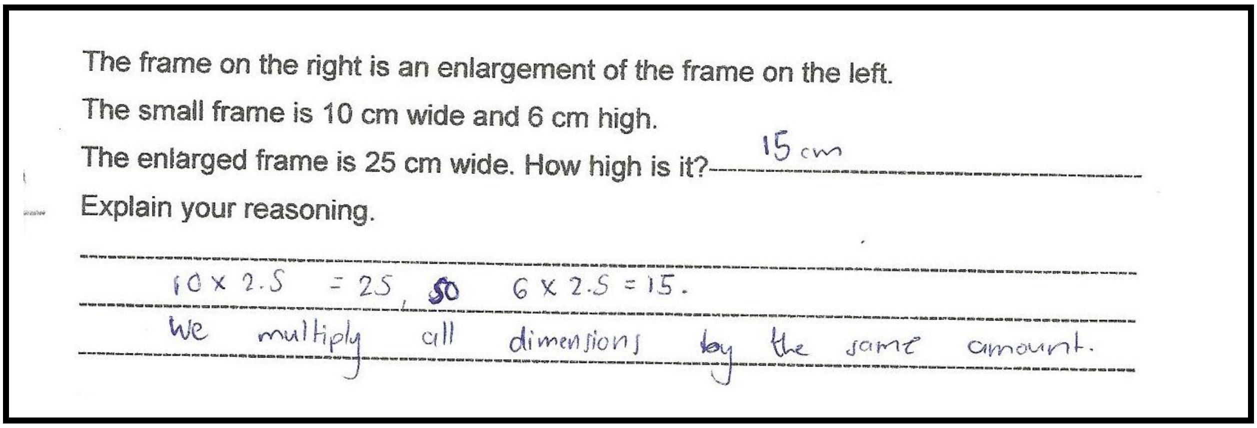
Figure 5.7: Task 2

The frame on the right is an enlargement of the frame on the left.

The small frame is 10cm wide and 6cm high.

The enlarged frame is 25cm wide. How high is it?

Eight of the 18 learners correctly displayed the scaling factor as 2.5, giving a coherent explanation of how they got the answer, as the following example shows:



The frame on the right is an enlargement of the frame on the left.
The small frame is 10 cm wide and 6 cm high.
The enlarged frame is 25 cm wide. How high is it? 15 cm
Explain your reasoning.
10 x 2.5 = 25, so 6 x 2.5 = 15.
We multiply all dimensions by the same amount.

Figure 5.8: Response: Learner C

The above-mentioned group of learners understood the concept of scaling up and down by a constant factor (that is, the ratio remains constant). The responses to the tasks and interviews revealed that learners were knowledgeable that scaling (enlargement in this task) has to be proportional to retain the original shape of the frame. The correct answers complete reasoning and coherent communication of answers yielded *complete understanding* (rubric level 4).

In total five if the 18 learners were rated level 3 – *moderate understanding.* Their answers were correct, they could communicate answers (but not coherently), displaying good understanding. The two scenarios are discussed below:

Three learners (level 3), however, got the correct answer of 15cm as the above-mentioned ten learners. The three learners multiplied the height of 15cm by 2.5 to get the answer of 15cm. However, when explaining their reasoning, they wrote that they multiplied by 2.5cm to arrive at 15cm. Misconception is evident here as the scale factor is 2.5 only. One example follows:

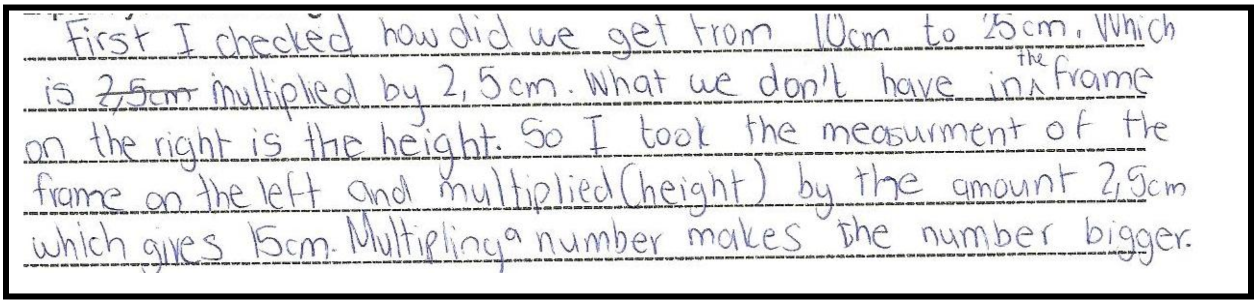

"First I checked how did we get from 10cm to 25cm. Which is ~~2,5cm~~ multiplied by 2,5cm. What we don't have in the frame on the right is the height. So I took the measurment of the frame on the left and multiplied (height) by the amount 2,5cm which gives 15cm. Multipling a number makes the number bigger.

Figure 5.9: Response: Learner B

Studying the last sentence above: "*Multiplying a number makes the number bigger*", shows that the learner had a misconception of the effect of multiplication (see section 3.4). It is however unclear what her thought processes were during the execution of this task. The correct answer of 15cm coupled with incorrect reasoning yields *moderate understanding* (rubric level 3) in the proportional reasoning of these three learners.

Level 3 (*moderate understanding*) was displayed by two other learners (out of the five) as well. Learner A wrote the correct answer but did not succeed in complete, coherent reasoning. Learner O explained his response coherently, had the correct answer but left out the unit of measurement (cm).

Two learners demonstrated the tendency to think additively in their comparisons of the lengths and breadths of the two figures:

Learner N:

10cm + 15cm = 25cm
~~4cm +~~ 6cm + 15cm = 21cm

Learner K:

I said 10cm + 10cm + 5 = 25cm, so then I said
6 cm + 6cm + 5 = 17 cm.

Figure 5.10: Responses: Learners N and K

Learner N added 15 cm to the height, reasoning that 15cm was added to the length. When interviewed, this learner stated: "…*the two figures will not be in proportion as the sides differ*…". Learner N had the misconception that whatever length is added to one side needs to be added to the other side as well. Multiplicative reasoning lacked in his response.

Learner K repeated addition of 10cm and added 5cm to get 25cm, followed the same procedure of repeated addition to 6cm, adding 5cm to get 17cm. Both Learners N and Learner K thought additively but in different ways.

Additive proportional reasoning of Learners N and K led to incorrect answers and incorrect reasoning, thereby displaying a misconception. According to the rubric *no understanding* exists (level 1).

Two learners answered haphazardly for the sake of answering, having no idea of how they arrived at the answers (which were wrong). These were Learners M and R. Learner M's response is shown below:

Learner M: 13cm high.

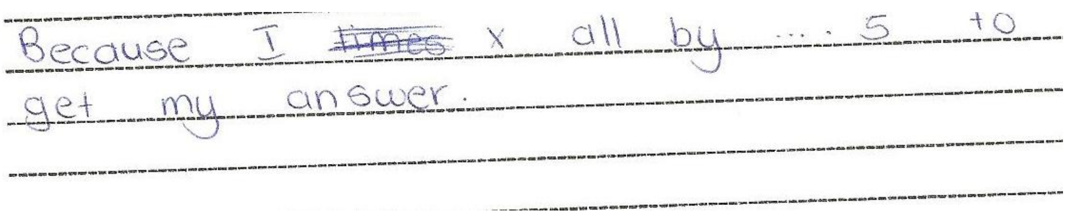


Figure 5.11: Response: Learner M

During the interview, Learner M was unable to motivate her response indicating *no understanding* (rubric level 1) of the concept of enlargement.

Learner R admitted to having guessed the answer as he did not understand the problem. The researcher appreciates his honesty and confirmation of *no understanding* (rubric level 1).

In total four learners revealed *no understanding* (level 1).

One of the remaining learners (out of 18), Learner Q, had an incorrect answer upon task completion (17cm). His response was: "…*you multiply the height by the number used to multiply the width*…". He explained (during the interview) that the length and the breadth had to be multiplied by the same factor. He recognised the constant factor needed to conserve proportion. This remark suffices to prove that the learner had partial conceptual understanding of the enlargement but had computation problems. This corresponds with rubric level 2 (*poor understanding*) as Learner Q's answer contained mathematical errors and the reasoning was poorly communicated.

5.3.3: Using tables to demonstrate the understanding of proportion.

The Ratio Table

To see the hidden multiplication factor in the ratio table, learners need to apply strategies of multiplication and division (Van Galen *et al.*, 2008:60).

Table 5.1: Task 3. Fill in the missing numbers:

Number of Euros	2	3		5	6
Number of Rands	24		48		

The table can be seen as a multiplication table in which the bottom row is made by multiplying the numbers in the top row by 12. It is always possible to write a proportion using two sets of numbers from a ratio table. The ratio table in this task is about exchange rate, relating the number of Euros to the number of Rands.

All the learners completed the table correctly. The interviews revealed that the participating learners have the ability to convert from Euros to Rands via multiplication and from Rands to Euros via division, using the constant factor of 12. The hidden multiplication factor was recognised by the learners in this case. The learners in this grade have already done exchange rates in class and are aware of different currencies of the world. Learner E's response follows:

Learner E

3. Fill in the missing numbers:

Number of Euros	2	3	4	5	6
Number of Rands	24	36	48	60	72

Figure 5.12: **Response: Learner E**

Learner E reasoned thus: "…2 x 12 = 24, when converting from Rands to Euros, I divided by 12 as the opposite of multiplication is division…". This indicates complete understanding based on correct answers, corresponding to rubric level 4.

5.3.4 The relationship between fractions, decimals and percentages

The relationship between fractions, percentages, decimals and proportions or ratios can be dealt with in a natural way if teachers make the context the central feature in teaching and give learners the opportunity to explore these contexts in many different ways (Van Galen *et al*., 2008:27). The same value is represented in three different presentations (as a fraction, as a decimal and as a percentage) in the following task.

Study the numbers in the box below:

$\frac{1}{2}$	0,5	50%

Does a relationship exist between the three? If so, explain.

Figure 5.13: Task 5.2

The concept of proportion in this case is flexible as all 18 learners realised as part of their fraction concept that a fraction can be represented as a decimal as well as a percentage. In other words, the participants unanimously agreed that the fraction $\frac{1}{2}$ is written as a decimal (0,5) and as a percentage (50%). The learners could see a number in different representations and not in isolation. The understanding that common fractions represent units separated into a number of parts, that decimal fractions represent units separated into 10 parts and that percentage is a unit separated into 100 parts, was evident from the interviews (see 3.2.8). Below is an example of a correct response:

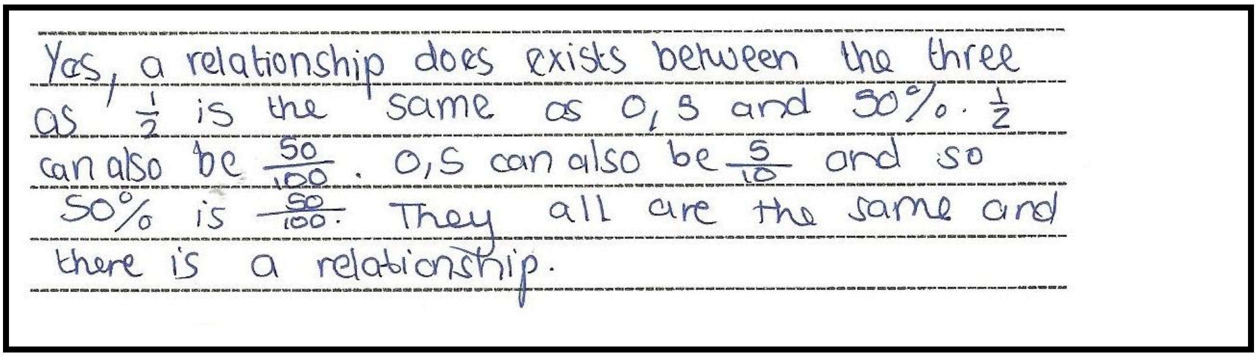


Figure 5.14: Response: Learner E

Complete understanding (rubric level 4) prevailed in this case ascribing to the fact that the correct answers were followed by coherent reasoning.

5.3.5 The ratio concept

Basic conversion of a fraction to a ratio was the essence of Task 5.3:

Table 5.2:

Fraction	Ratio	Simplified ratio
$\frac{14}{20}$	14:20	7:10

Eleven learners correctly converted $\frac{14}{20}$ to a ratio - 14:20, simplifying the ratio - 7:10. *Complete understanding* (level 4) was exhibited by these learners. The interviews confirmed the level of understanding.

Six learners converted the fraction to a ratio but three learners (Learners N, R and K) were unable to simplify 14:20 as 7:10. However, during the interviews, Learner N and Learner K succeeded in simplifying the ratio. The remaining three learners were able to convert the fraction to the correct ratio but did not simplify the ratio during task completion. All three were very accurate in simplification upon interviewing. According to the rubric, this indicates *partial understanding* (level 2), considering task analysis.

The remaining one learner (Learner P) had no idea of the problem, the analysis of the task revealed *no understanding* (level 1) as his answer was incorrect. Learner P rewrote the fraction, as shown below:

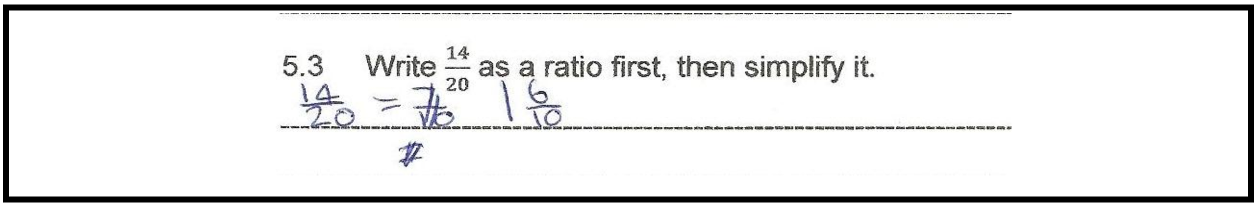


Figure 5.15: Response: Learner P

It is interesting to note that Learner P's first answer was correct. It suffices to prove that this learner was unable to convert a fraction to a ratio and to navigate between two forms of Proportion. The fact that she scratched out shows that she was confused. Surprisingly, when the researcher remarked: "Why did you rewrite the fraction? You were asked to convert the fraction into a ratio", she immediately said: "…*14:20 is equal to 7:10*…".

Another task relating to ratios is depicted in figure 5.17. The intent of the task below was to ascertain whether the learners grasped that ratios involve multiplicative rather than additive comparisons. To acquire sound understanding of the ratio concept, learners need to understand that in equivalent fractions, the ratio remains constant while the number of counters increases or decreases (Pantziara & Philippou, 2012:64).

Rusty High School has 16 grade 8 learners.

12 are basketball fans. The remaining learners are not.

Describe whatever relationships you can see between learners who are basketball fans and those who are not.

Figure 5.16: Task 5.1

(adapted from Van de Walle *et al*., 2010:352)

Van de Walle *et al*. (2010:352) further iterates that once learners determine that there are four non-fans, there are other possibilities:

- There are eight more fans than non-fans (additive relationship).
- There are three times as many fans as non-fans (multiplicative relationship).
- For every three students who like basketball, there is one who does not (multiplicative relationship).

Learners' understanding of ratio and proportion is evident when learners explicitly explain the type of relationships that exist between the two values (Van de Walle *et al*., 2010:352).

Eight learners had a sense of covariation. They saw the relationship between fans and non-fans as a relationship in which learners can solve problems or compare other ratios as well. A correct response is shown below:

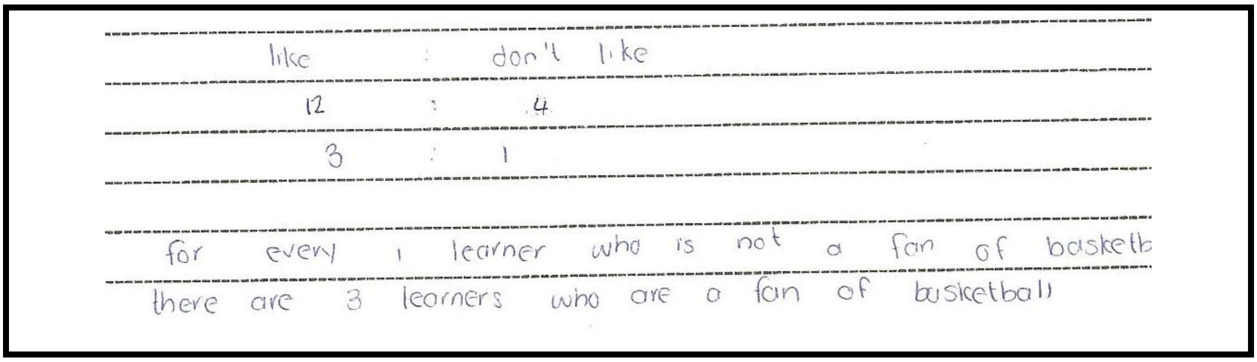


Figure 5.17: Response: Learner A

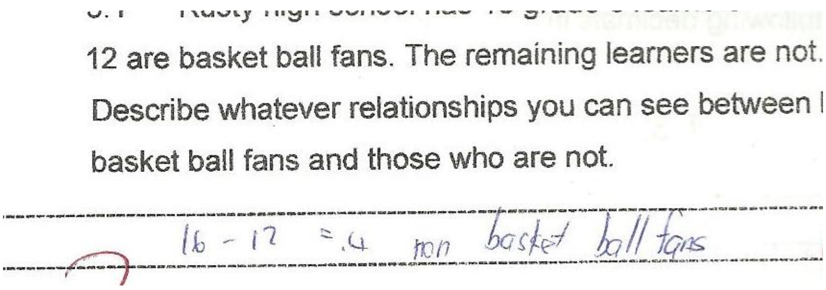
Complete understanding (level 4) was evident in the responses of the eight learners discussed above; the answers were correct, the reasoning complete, with coherent communication of answers.

The researcher posed the following questions to the learners during the interviews: Compare the ratio of fans to the number of learners in the class and compare the ratio of non-fans to the number of fans in class. The eight learners succeeded in writing the ratios as percentages and comparing percentages, as well as writing the ratios as fractions and comparing. Interview discussions also revealed the understanding that ratio is a multiplicative comparison between two quantities.

Learners G and K wrote down the correct ratio between fans and non-fans but did not simplify and thus could not compare fans and non-fans in basic ratio notation. Their reasoning was not complete, they could communicate answers but not coherently. The response of these two learners was rated as *moderate understanding* (rubric level 3).

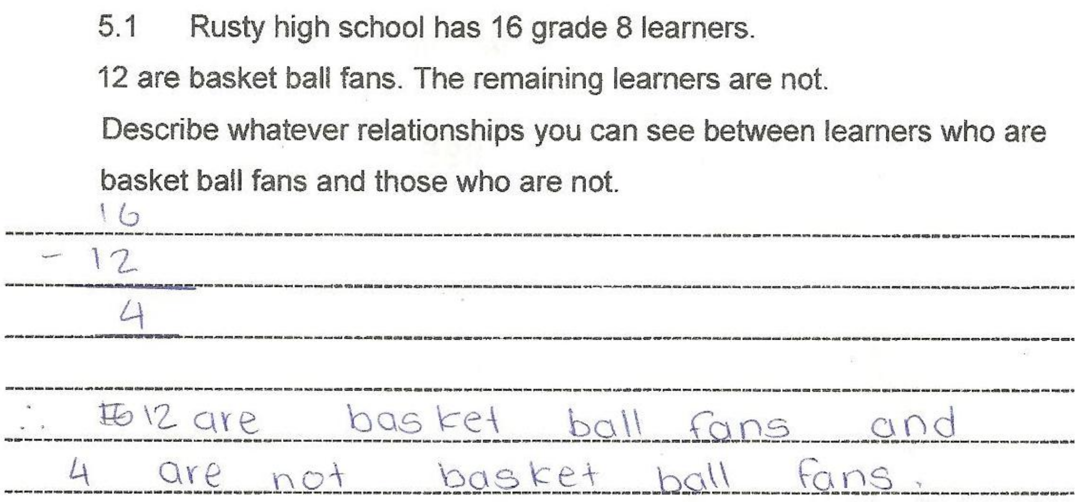
Seven learners attained *poor understanding* (level 2). The relationship between fans and non-fans was partially listed. Their communication of answers was poor and reasoning incomplete, resulting in some mathematical errors. The responses of Learner M and Learner R follow:

Learner R


12 are basket ball fans. The remaining learners are not.
Describe whatever relationships you can see between l
basket ball fans and those who are not.

16 - 12 = 4 non basket ball fans

Learner M


5.1 Rusty high school has 16 grade 8 learners.
12 are basket ball fans. The remaining learners are not.
Describe whatever relationships you can see between learners who are basket ball fans and those who are not.

16
− 12
4

∴ 12 are basket ball fans and 4 are not basket ball fans.

Figure 5.18: Response: Learner R and Learner M

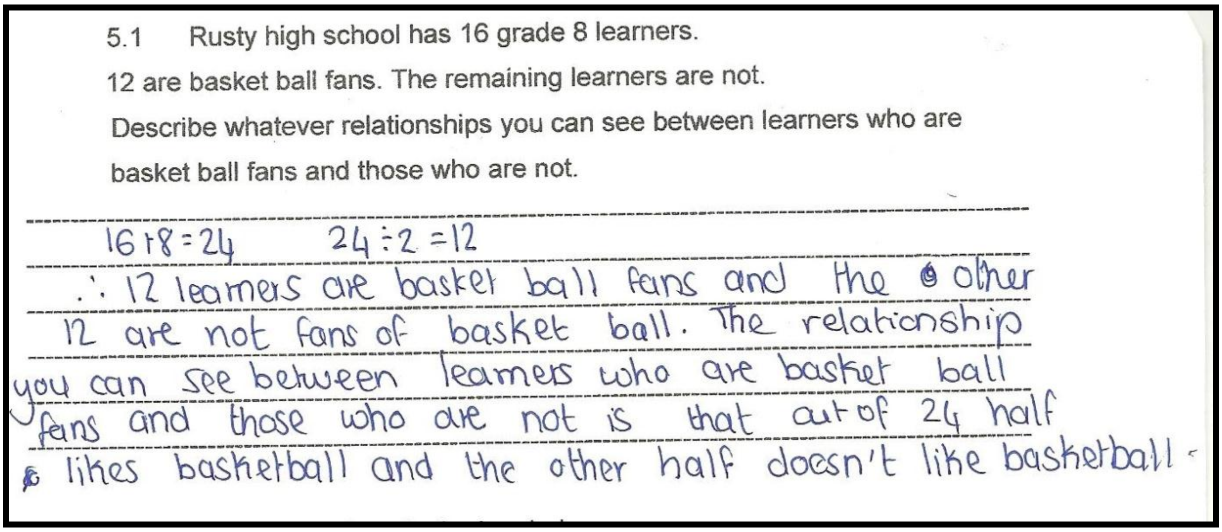
During the interview session, Learner R stated "… *Madam, I didn't understand what was asked*…". In this case, the question was not fully understood and therefore the learner guessed the answer as was evident, resulting in a partially correct response.

Learner M succeeded in calculating the number of non-fans but did not write the ratio between fans and non-fans during the task completion process. During the interview she verbally concluded the following ratios (correctly):

Table 5.3:

	Ratio	Simplified ratio
Fans : Non- fans	12 : 4	3 : 1
Learners in class : fans	16 : 12	4 : 3

One learner (Learner F) was unable to convert or simplify and revealed lack of understanding in the task. The result was *no understanding* (rubric level 1) as the response below indicates:



5.1 Rusty high school has 16 grade 8 learners.
12 are basket ball fans. The remaining learners are not.
Describe whatever relationships you can see between learners who are basket ball fans and those who are not.

16 + 8 = 24 24 ÷ 2 = 12
∴ 12 learners are basket ball fans and the other 12 are not fans of basket ball. The relationship you can see between learners who are basket ball fans and those who are not is that out of 24 half likes basketball and the other half doesn't like basketball.

Figure 5.19: Response: Learner F

When Learner F was interviewed, she realised that she had reasoned incorrectly at the time of completing the task sheet. The researcher rephrased the question: "If 12 learners liked basketball, how many out of the 16 did not like basketball?" She presented the correct ratio, concluding: "…*for every one non-fan there are three fans*…".

5.3.6 Comparing fractions

Below is an example of a conceptual item; the learners were asked to order the fractions. Although this item could be solved by implementing a procedure, the focus was to identify conceptual understanding. The task was included to ascertain whether learners understood that the greater the number of parts into which the unit is partitioned, the smaller the fraction size (see section 3.2.5).

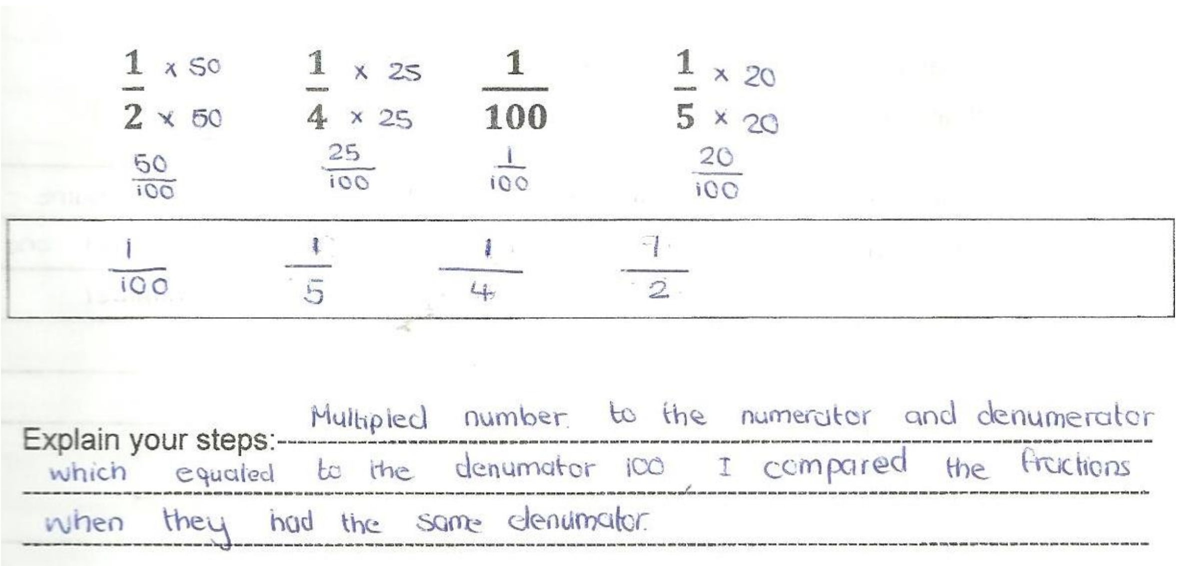
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{100}$	$\frac{1}{5}$

Figure 5.20: Task 4.2

(Write the fractions in ascending order (from smallest to biggest)

The Lowest Common Denominator concept was utilised by ten learners to compare and order the fractions according to size. Two responses of such a nature follow:

Learner D



Learner A

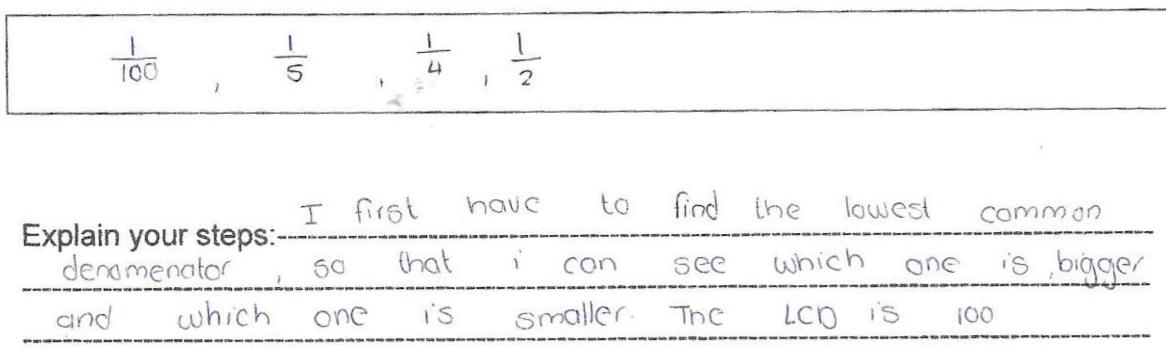


Figure 5.21: Response: Learners A and D.

One learner, Learner P, compared the fractions without finding the LCD. Therefore, a total of 11 learners attained *complete understanding* (level 4), as their answers were correct, their reasoning complete and solutions coherent. All 11 learners could order the fractions as required by just comparing the denominators as the numerators are constant. If so, they would have realised that the more parts the whole is divided into, the smaller the produced parts become (section 3.2.5).

The ten level 4 learners were prompted to find an alternative solution to the question during the interviews. They stated that if the numerators are the same as in this case, one has to look at the denominators: the greater the denominator, the smaller the fraction (see section 3.2.5).

Learner E was the only learner obtaining *moderate understanding* (level 3). He had the correct answer but his reasoning was incomplete and his solution was communicated incoherently.

One learner (Learner O) attained *poor understanding* (level 2). He showed evidence of partial understanding, with poor communication of answers and incorrect reasoning, though he did sequence the fractions as required.

Five learners showed lack of conceptual understanding, achieving level 1 and *no understanding.* From this group Learner M's reasoning manifested in a misconception (the smaller the denominator, the smaller the fraction) as she checked for the smallest denominator (see section 3.2.5). She confidently stated: "…*the smaller the denominator, the smaller the fraction*…", at the time of interviewing. Her incorrect response follows:

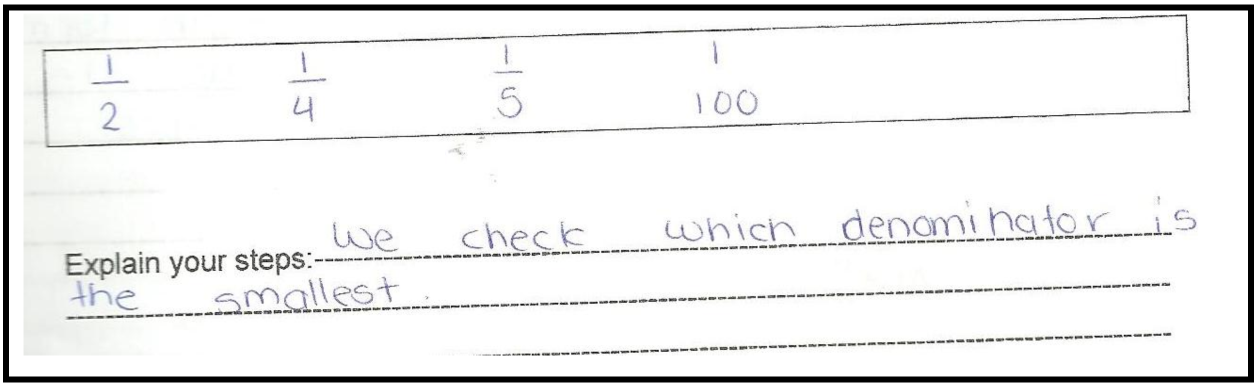


Figure 5.22: Response: Learner M

Besides Learner M, two other learners who possessed *no understanding* (Learner Q and Learner N) stated that $\frac{1}{100}$ is the smallest fraction but placed it at the end (right), at the time of completing the task. Upon interrogation, it was revealed that these learners could verbally explain the correct order but placed the fractions in descending order. A language barrier possibly caused misunderstanding and not a misconception. The instruction was clear as it stated: "…in ascending (from smallest to biggest)". Learner Q stated: "…$\frac{1}{100}$ *is the smallest because it has too many zero's*…". This response revealed a possible misconception. The learner is not clear about the "effect" of the *many zeros* on the size of the fraction.

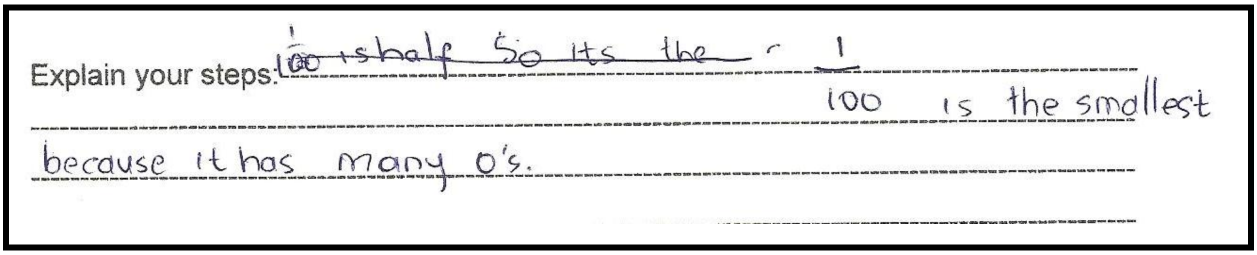


Figure 5.23: Response: Learner Q

No understanding is evident as the reasoning and answers were incorrect and the communication incoherent, as revealed by the tasks completed by the five learners.

5.3.7 Subtracting, multiplying and dividing fractions

Procedural knowledge of the learners was assessed by the subtraction, multiplication and division computation items in the tasks below:

a) $\frac{3}{5} - \frac{1}{2}$

b) $\frac{2}{3} \times \frac{1}{3}$

c) $\frac{2}{3} \div \frac{4}{5}$

Figure 5.24: Task 8

If learners possess procedural understanding it does not necessarily mean that they have conceptual understanding. Learners may memorise the rules of procedures and arrive at the right answers without knowing the logic behind the steps in an algorithm. When learners solve problems correctly using algorithms, there is no indication that they understand the multiplicative relationships embedded in proportions (Slovin, 2000:58).

The participants were interrogated on conceptual understanding during interviewing, even though there was evidence of procedural knowledge on task completion. Learners were not asked to explain their reasoning in Task 8.

5.3.7.1 Adding and subtracting fractions

Adding fractions with different denominators seems to have required the same type of knowledge that children use to compare fractions and decide whether they are equivalent or different (Hallet *et al*., 2010:396). Therefore, to be able to add and subtract fractions with different denominators, learners need to understand equivalent fractions as a common denominator is needed.

Task (8a): $\frac{3}{5} - \frac{1}{2}$, is based on the subtraction of fractions.

Twelve learners who displayed procedural knowledge on task completion also showed conceptual understanding during the interviews. They realised that the reason for finding the Lowest Common Denominator (LCD), was a necessary condition for obtaining equivalent fractions. This condition resulted in the denominators being the same in both fractions (see section 3.2.5), allowing subtraction of the numerators. One correct response follows:

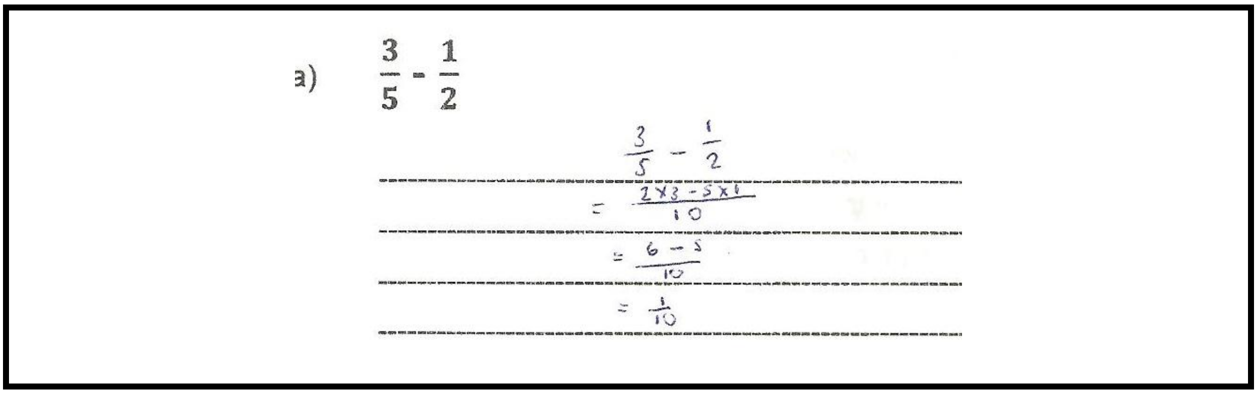


Figure 5.25: Response: Learner C

The **12** learners attained *complete understanding* (level 4) based on task analysis. Misconceptions were however evident, resulting in two learners (Learners I and K) obtaining *poor understanding* (level 2). All three arrived at the correct answer of $\frac{1}{10}$ but immediately after this fraction wrote: = 10. Misconception was evident in their last steps (see section 3.4) . No motivation for the incorrect responses was given during interviewing.

Four out of 18 learners attained level 1 – *no understanding* of this task. Their answers were incorrect. Two of these learners' responses are shown and discussed below:

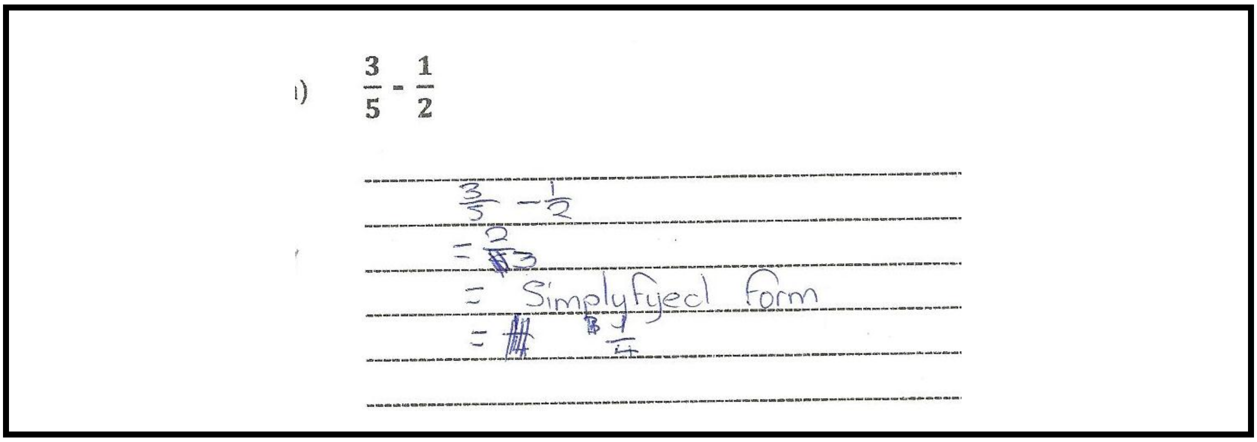


Figure 5.26: Response: Learner N

Learner N subtracted the numerators and the denominators, resulting in an incorrect "simplified" form. This confirms that learner N lacked the ability to form equivalent fractions. During the interview Learner N remarked: "…*3 – 1 = 2 and 5 – 2 = 3. I wrote 5 instead of 3. I don't know what happened*…"

Learner R found the correct Lowest Common Denominator (LCD), but subtracted the numerators, as shown below:

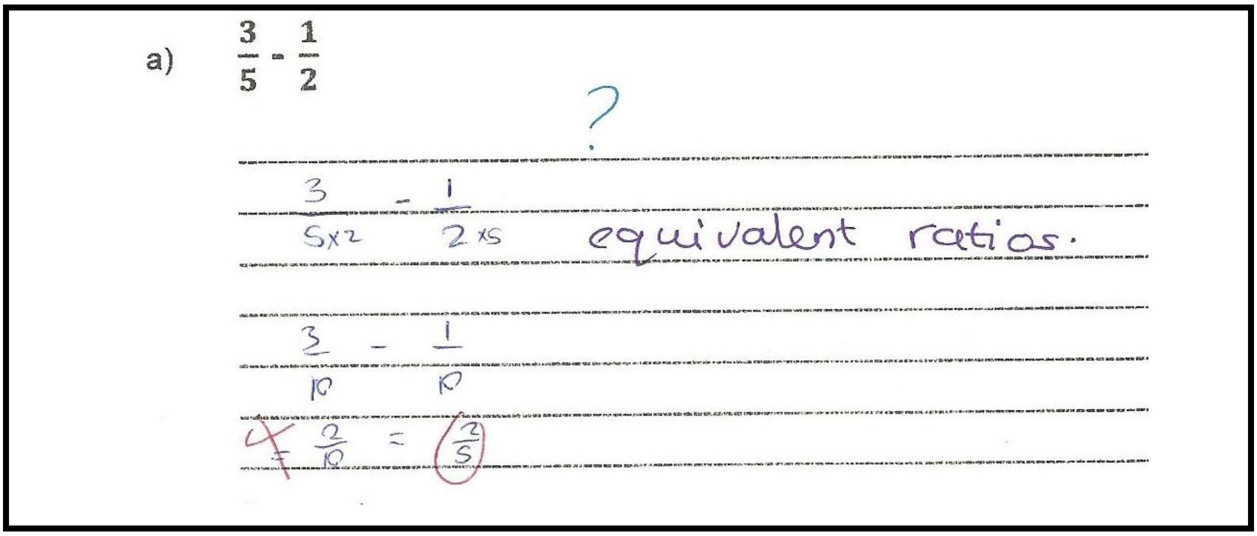


Figure 5.27: Response: Learner R

The resistance to accept fractions as numbers led Learner N and Learner R to conceptualise fractions as two different whole numbers, a misconception that results in computational bugs (Charalombous & Pantazi, 2007:300) as discussed in section 3.4.

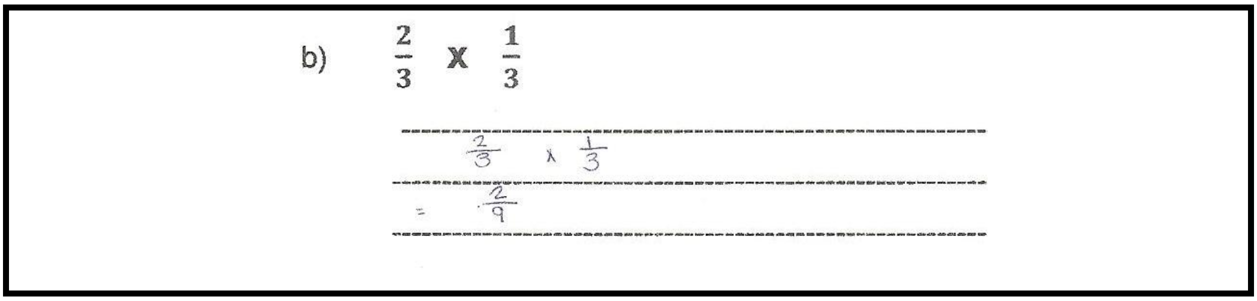
Learner R, displaying no understanding of fraction addition, subtracted the numerators, leaving the denominators the same. He was unable to find equivalent ratios for $\frac{3}{5}$ and $\frac{1}{2}$ although he was aware that the LCD is 10. During the interview, he stated that: "… $\frac{3}{10}$ *was an equivalent ratio for* $\frac{3}{5}$ *and that* $\frac{1}{10}$ *was an equivalent ratio for* $\frac{1}{2}$.."

Two learners, Learner I and Learner K, attained *poor understanding* (level 2). Although both learners found the LCD correctly and computed $\frac{1}{10}$ as the answer, the solution did not stop there. They had an additional final step: $\frac{1}{10}$ = 10. During the interviews both learners were unable to explain how they arrived at 10, from $\frac{1}{10}$.

5.3.7.2 Multiplying fractions

During fraction multiplication, meaning must be sought for interpreting such calculations (Anghileri, 2007:126). Anghileri furthermore refers to the rule as multiplying the 'tops' and multiplying the ' bottoms', meaning 'numerators' and 'denominators' respectively.

Ten learners presented correct answers by applying the rule as discussed above, achieving *complete understanding* (level 3). One example follows:


b) $\frac{2}{3}$ x $\frac{1}{3}$

$\frac{2}{3}$ x $\frac{1}{3}$

= $\frac{2}{9}$

Figure 5.28: Response: Learner A

The incorrect answers of the eight learners resulted in achievement level 1 (*no understanding*) as their tasks and interviews revealed misconceptions. Three out of the eight learners inverted the second fraction, displaying a lack of understanding of the concept of multiplication of fractions (see section 3.4). One example is shown:

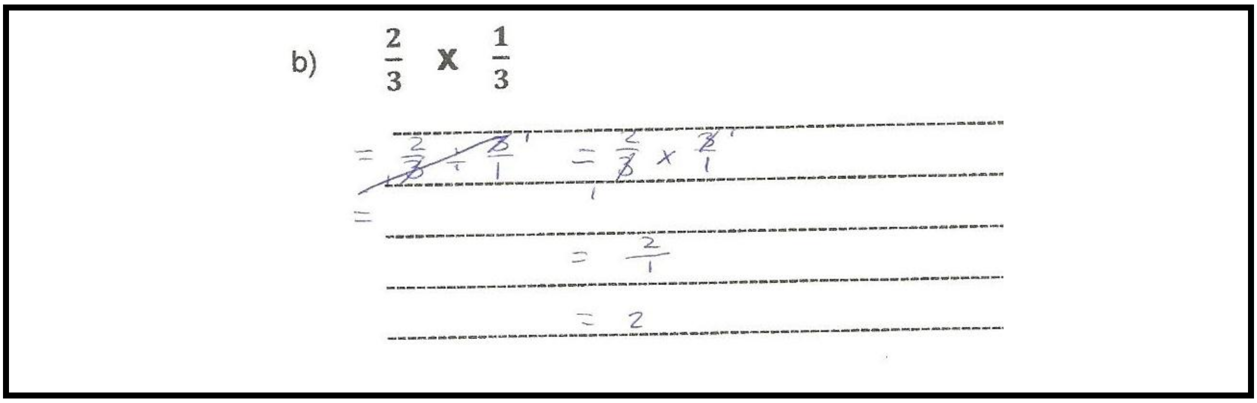


Figure 5.29: Response: Learner K

This learner was in doubt as to what should be done, as is evident from her scratching out. She initially changed the multiplication sign into a division sign, inverted the second fraction and cancelled the 3's. For some reason, she realised that this step was incorrect and scratched out the step. In her second attempt, she left the multiplication sign as is, inverted the original second fraction and cancelled. She was aware that cancelling is part of procedure in fraction multiplication. Misconception is evident as the second fraction is inverted as in the case of division of fractions.

The interview revealed the misapplication of a rule. "… *because I multiplied fractions, I thought this rule exists*…" was her response when the researcher asked her as to her reason for inverting.

The interviewing session revealed that Learner C made a mathematical error: *2 x1 = 3*. She was quite shocked when she realised her carelessness. Her words: "… *2 x 1 = 2, not 3*...", indicated that she did not have a conception regarding the multiplication of fractions.

5.3.7.3 Dividing fractions

In order to explain the division involving two fractions, the idea of ratios can be used (Anghileri, 2007:127). She explains the idea of multiplicative inverses. Applying the rule to the expression $\frac{2}{3} \div \frac{4}{5}$, rewrite the expression as $\frac{\frac{2}{3}}{\frac{4}{5}}$. Change this ratio into an equivalent fraction by multiplying the top and bottom by the same number. The number is chosen so that the bottom number becomes 1. That is, multiply the top and the bottom by $\frac{5}{4}$, the multiplicative inverse of $\frac{4}{5}$.

Eleven learners had the procedural fluency in simplifying the above. Learners correctly applied the invert-and-multiply procedure during division of fractions. One example of a correct response follows:

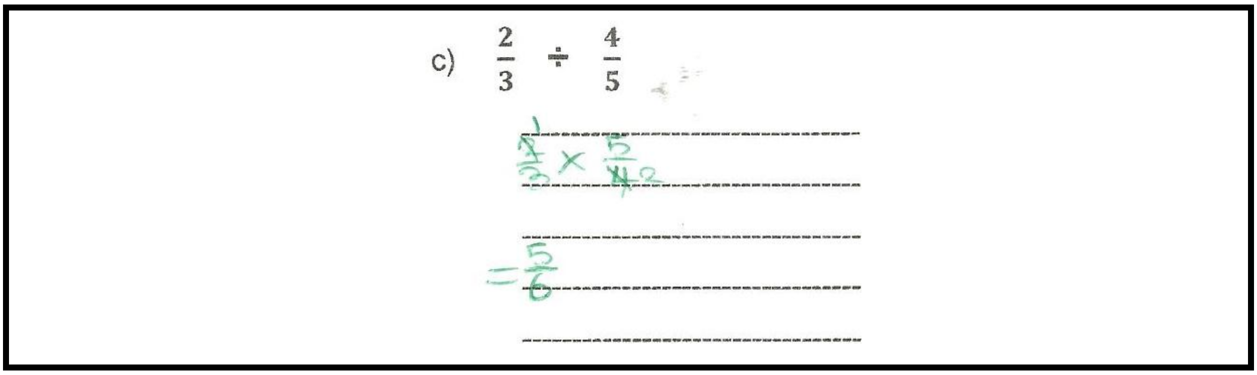


Figure 5.30: Response: Learner E

The correct procedure was followed in the response above; the division was converted into multiplication, cancelling procedure was carried out to arrive at the simplified answer. The score level of the rubric was level 4 (*complete understanding*) in the case of the 11 learners above. Interviews revealed conceptual understanding (see section 3.5.3).

Three learners scored level 2 (*poor understanding*). They succeeded in converting the division into multiplication and inverting the second fraction. Two of the three learners managed to cancel, the same as Learner E above. The remaining learner did not cancel, concluding with an incorrect response. Learner M's response follows:

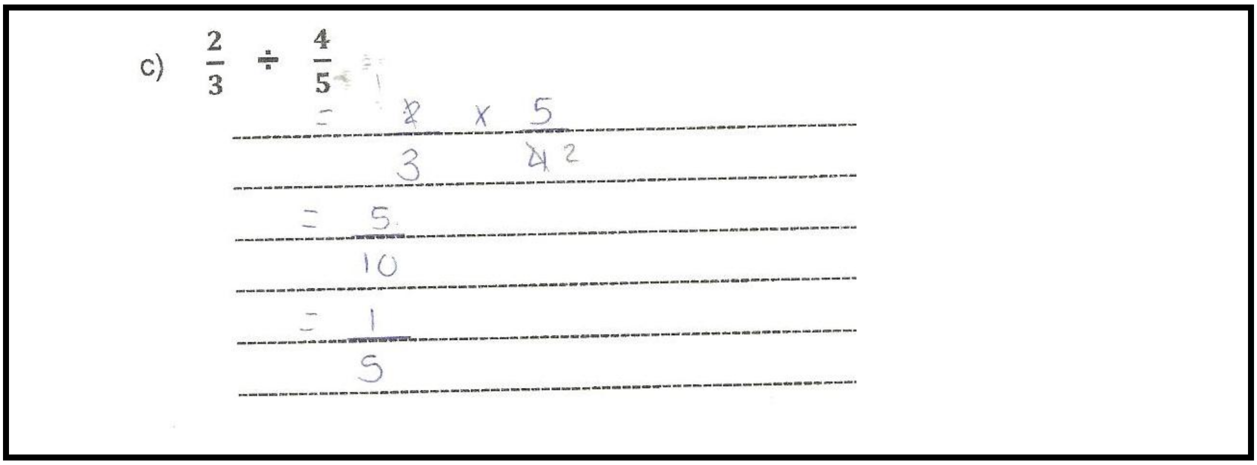


Figure 5.31: Response: Learner M

Learner M applied the algorithm correctly in step 1. Step 2 contained a mathematical error. The multiplication of the numerators is correct, but mathematical error resulted in the denominator. This learner multiplied 3 by 2 and got 10 instead of 6. This is not a misconception but a careless error.

Four of the 18 learners revealed *no understanding* (rubric level 1) as their answers were totally incorrect. Two of the learners at this level found the LCD, mistaking the algorithms for addition or subtraction of fractions with multiplication of fractions. One level 1 learner found a remainder while the remaining level 1 learner also tried to find the LCD without success:

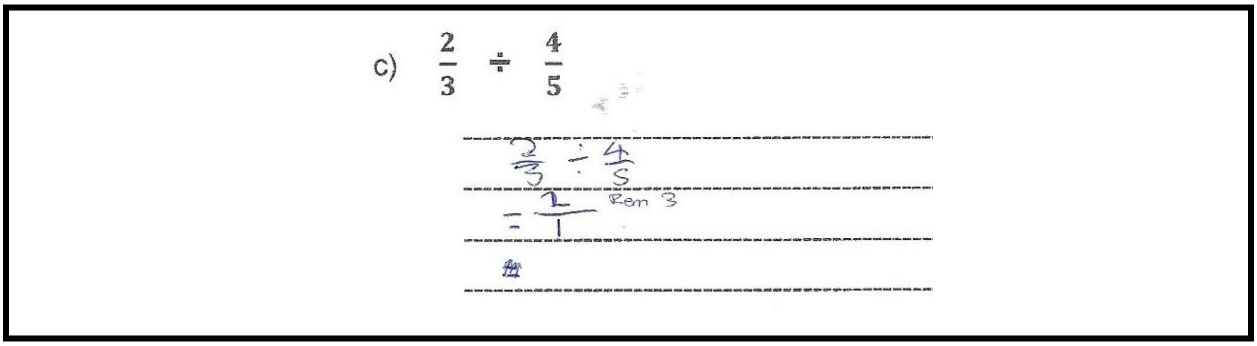


Figure 5.32: Response: Learner N

When the researcher asked Learner N to explain his last step, he exclaimed: "…*I did it wrong. Supposed to give me one…"*. This remark also revealed *no understanding* (level 1) during the interview.

In all components of Task 8, there is no evidence of moderate understanding in the task sheets. This implies that learners had either *complete understanding, poor understanding* or *no understanding* of the procedure. Misapplication of the algorithms in all components of Task 8 is evident.

5.3.8 Decimals

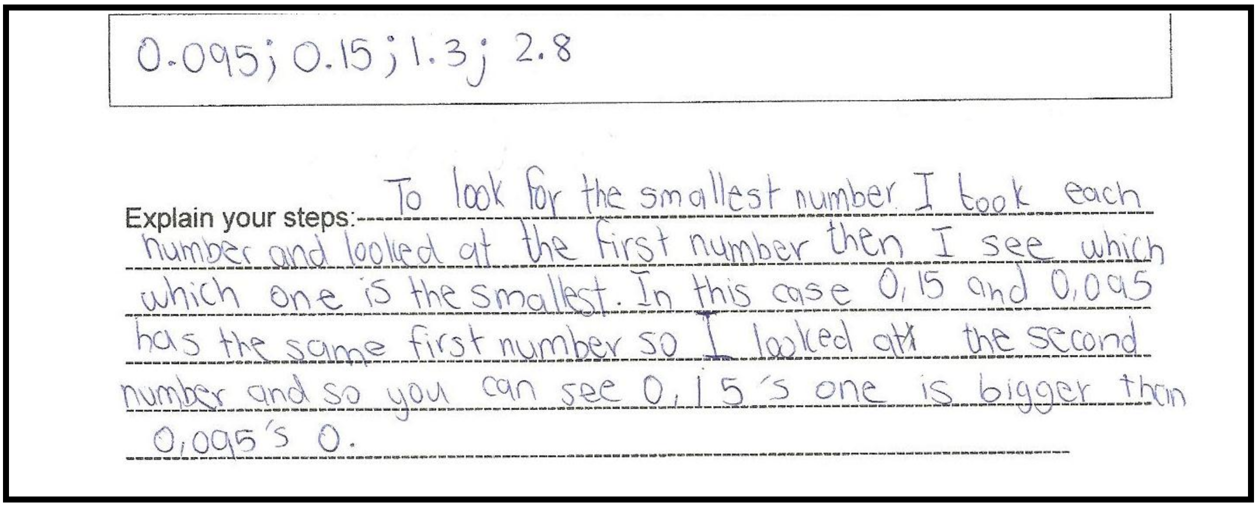
In this task of ordering decimals, the aim of the researcher was to assess decimal understanding and not skill development:

4.1 Place the following decimals in order from smallest to biggest:

0.15, 1.3, 0.095, 2.8

Figure 5.33: Task 4.1

Eleven learners succeeded in ordering the decimals in ascending order. Seven learners showed *complete understanding* (level 4), their reasoning was complete and their communication of answers was coherent. Learner B was one of the five learners demonstrating *complete understanding:*


0.095; 0.15; 1.3; 2.8

Explain your steps: To look for the smallest number I took each number and looked at the first number then I see which which one is the smallest. In this case 0,15 and 0,095 has the same first number so I looked at the second number and so you can see 0,15's one is bigger than 0,095's 0.

Figure 5.34: Response: Learner B

Four learners who placed the decimals in the correct order attained level 3 (*moderate understanding*) as their reasoning was incomplete and their communications of answers not coherent.

Five learners were rated at level 2 (*poor understanding*). Although their ordering of the decimals was correct, they were unable to explain how they arrived at the answer. Learner R, being one of the five, remarked during the interview "… *the more the zero's, the smaller the decimal*…". This portrayed a misconception (see section 3.4).

However, two other cases of misconception were evident. One incorrect response is shown below:

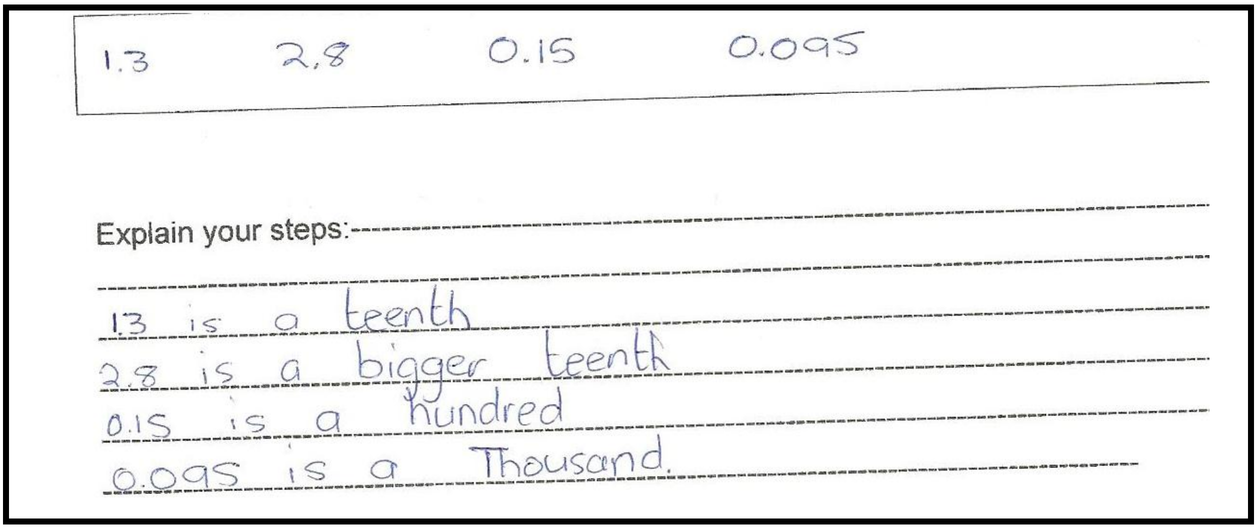


Figure 5.35: Response: Learner N

The above learner explained during the interview: *"…the longer the decimal, the bigger the number…".* That explains why he placed 0.095 last as the "biggest decimal". Words like "teenth, bigger teenth, hundred and thousand" reveal that this learner does not understand place value of each digit in a decimal.

5.4 CONCLUSION

This chapter provided the analysis and interpretation of the data generated for the inquiry. The data analyses gave rise to many observations. The data analyses indicate different stages in the learners' proportional reasoning or proportional understanding. To summarise the findings of the investigation, learners' understanding ranged from complete to no understanding, with moderate understanding and poor understanding in between. Misconceptions did occur. Triangulation exists as the literature review, tasks and the interviews triangulated the data. In the final chapter the research questions will be addressed based on the findings and the necessary recommendations are put forward.

CHAPTER 6:
SUMMARY AND RECOMMENDATIONS

6.1 INTRODUCTION

A synopsis of the investigation is presented in this chapter. Findings from the literature review and empirical research pave the way to discussions of important insights gained from this investigation. Most apparent findings are discussed. The chapter concludes with a discussion of the limitations of the research, recommendations for further research, the value of this research and general concluding remarks.

6.2 SUMMARY OF CHAPTERS

In **Chapter 1** the low achievement of South African learners in Mathematics is the concern. The problem statement leads the discussion to the posing of the research questions. The research focuses on a particular algebraic concept: Proportion. The first research question aims at investigating the learners' understanding of the concept of Proportion, linking to the grounds for understanding the concept as the second research question. The third research question is based on misconceptions, linking to the fourth research question, being the grounds for misconceptions. The fifth research question centres on what can be done to support the meaningful learning of Proportion. A brief outline of the study follows.

Chapter 2 provides insight into the learning and teaching of Algebra in school. Views of school mathematics and learning theories are explored. The literature review delves into the meaningful learning of Algebra and the various factors affecting learning are examined.

In **Chapter 3** proportional reasoning is discussed. The concept is highlighted through a literature review. Meaningful understanding of Proportion, the development of the idea of Proportion, resulting misconceptions and ways to facilitate meaningful learning of Proportion dominate this chapter.

Chapter 4 broadly outlines the research design and methodology. A qualitative research method validates this study. The aim is to identify philosophical aspects of the investigation and motivation for the choice of method. A discussion of the measuring instruments, being tasks sheets and one-to-one interviews, ensues. The procedures for the data collection, data analyses and the trustworthiness of the data are also outlined.

Chapter 5 provides a lens to gain insight into the results of the investigation whereby the research questions are addressed. The first step is to analyse the task sheets that were completed by the participants, followed by a discussion of the results of the interviews, alluding to the fact that the interviews aimed at authenticating the data. Examples of learners' responses are given. The results of the empirical study are contextualised with regard to the theoretical findings.

In **Chapter 6** recommendations that emanate from the results in Chapter 5 are discussed. Light is shed on the limitations and contributions of the study. Final analysis concludes the chapter. The qualitative findings initiate discussions through which the insight gained can be used to answer the research questions.

6.3 SUMMARY OF FINDINGS

6.3.1 Addressing the research questions

6.3.1.1 Research question 1

How did Grade 8 learners understand the concept proportion?

In answering this research question, the researcher reflected on proper as well as improper understanding with regard to the empirical study as both types of understanding were evident.

Multiplicative reasoning, required in proportional reasoning, was evident. Learners understood the concept of enlargement in scaling. The scaling factor's constancy was noted. The place value in decimals was understood, enabling learners to order decimals. The same applied to ordering fractions. Many learners were aware that the greater the denominator, the smaller the fraction if the numerator is kept constant. Learners realised that a fraction can be written as a decimal and as a percentage. The conversion between the three different presentations was understood. Understanding of Indirect proportion as well as Direct proportion was displayed. The ability to convert a fraction to a ratio and then simplify the ratio was present. Learners had procedural and conceptual competency in fraction subtraction, division and multiplication. A small number of learners had conceptual understanding of unfamiliar Proportion problems. This is a good indication of an advanced form of proportional reasoning. Ratio tables were viewed as a strategy in solving Proportion. The percentage concept was thoroughly understood.

However, misunderstanding or improper understanding of Proportion was also evident. The place value in decimals posed a problem. Many numerical errors were present in the tasks sheets. Learners could not calculate the factor of enlargement on the tasks sheets but displayed understanding during the interviews. Errors were verbally rectified after some probes during the interview sessions. Algorithms for fraction subtraction, division and multiplications were muddled up, being used in the wrong places. This can be attributed to the fact that learners did not understand the procedures that they had learnt. Although most learners could differentiate between Direct and Indirect Proportion, there was uncertainty when the cross-product algorithm should be used in problem–solving. Some learners failed to realise that the maximum percentage attainable in a test is hundred. Creativity and critical thinking lacked with regard to unfamiliar, seemingly complex problems. Iterating required multiplicative thinking. Some learners still displayed concrete thinking and additive reasoning.

6.3.1.2 Research question 2

What were the grounds for their understanding?

Relational understanding was the basis for understanding Proportion, while the reason for not understanding proportion can be attributed to instrumental understanding. The learners who had relational understanding associated Proportion with many other existing ideas in a meaningful network of concepts and procedures. Instrumental understanding was evident, indicating that learners could have rote-learnt procedures without understanding them. Poorly understood ideas were easily forgotten and not useful for constructing new ideas.

Learners understood that a ratio is a number relating two quantities in a multiplicative relationship. Ratios were recognised in various settings. Multiplicative reasoning promoted identification of multiplicative relationships. Most learners could distinguish between multiplicative and additive approaches. A network of relationships is built between the various sub-constructs of Proportion. Abstract levels of understanding existed; the transition from the concrete phase was evident as learners thought multiplicatively. The conceptual understanding was also evident as learners could distinguish between proportional and non-proportional situations. The relationship between the numerator and the denominator in a fraction was understood. Knowledge of why procedures work was based on conceptual understanding. Similarity of figures was seen as a multiplicative relationship.

In Figure 6.1, below, the *web-like* connections of Proportional concepts show that proportional reasoning was complex and not linear, therefore myriad possibilities were at play as learners viewed the same concept differently.

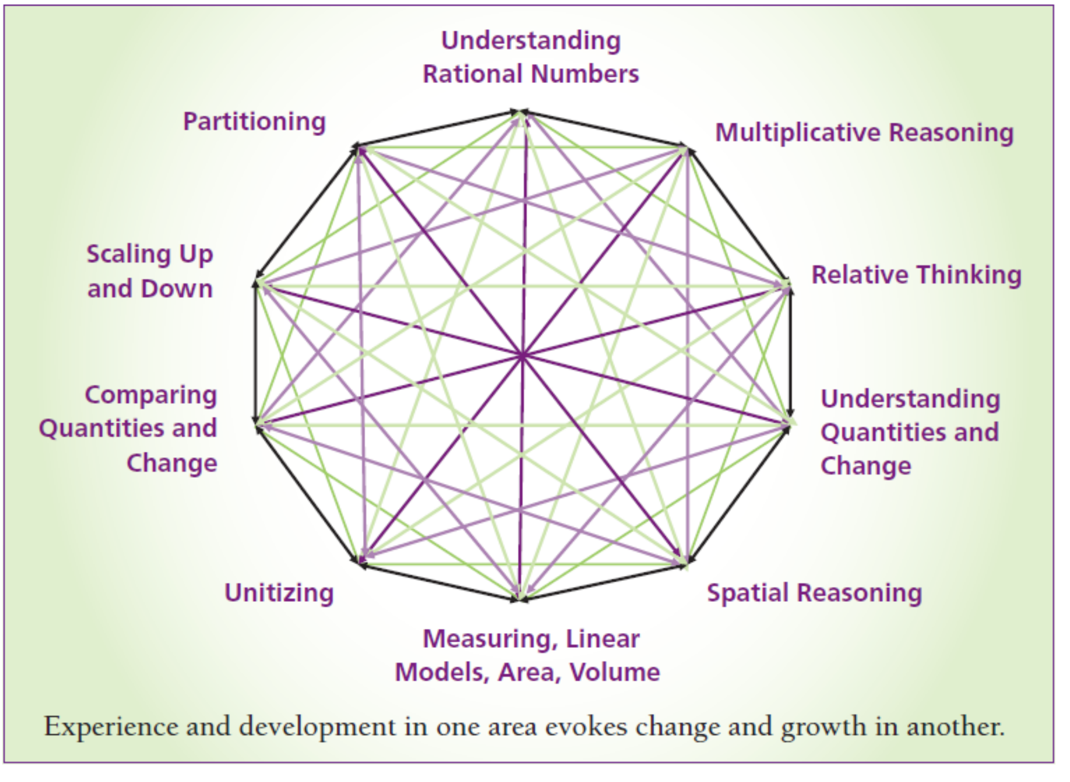


Figure 6.1 Some interconnecting proportional reasoning concepts. (K-12, Ontario: 4)

According to the above model, experience and development in one area evoke change and growth in another. The interconnected Proportional reasoning concepts include scaling, unitising, comparing quantities, partitioning, measuring and multiplicative reasoning (K-12, Ontario:4). The learners who had a good conceptual understanding, developed web-like thinking. Learners who did not acquire the interrelated web-like connections lacked understanding.

6.3.1.3 Research question 3

What misconceptions occurred?

In some cases, learners still had the tendency to reason additively where multiplicative reasoning was required. As an example, in the case of the chocolate cake recipe, the answer was found by multiplying the original recipe by 3. Instead of multiplying 2 by 3 to get 6, some learners added 4 to the original recipe, reasoning that 2 + 4 = 6.

The geometric figures exhibited scaling by a factor, that is, enlargement by a factor of 2.5. It was evident that some learners added 15cm to 10cm to get the given height of 25cm. They did not realise that 10cm x 2.5 = 25cm was the proper response to resolve the matter. From their additive reasoning, the idea of calculating the height as 6cm + 15cm = 21cm, was a misconception. "*Multiplying a number makes the number bigger*", was a serious misconception. (See section 3.4).

Learners thought that the numerator and the denominators were separate unrelated numbers. They did not conceive a fraction as a single value. This was evident when subtracting two fractions, the perception that the numerators and denominators should be subtracted, led to a misconception.

The comparison of fraction size posed misconceptions. One response indicated that the more zeros in the denominator, the smaller the fraction, given that the numerators are the same. The fraction multiplication algorithm was misconstrued. The fraction after the multiplication sign was inverted. Inversion as such is supposed to be the case in fraction division problems. Common misconceptions regarding decimal notation, such as: "…*the more the zeros the smaller the decimal…and* "…*the longer the decimal, the bigger the number*…", were evident.

When comparing heights some learners used additive reasoning. Learners counted the division lines and not partitions on the number lines. They were inclined to reason additively. The fact that six paper clips and six matchsticks were used for comparison, was misconceived by one learner who stated that: "…Mr Short and Mr Tall exchanged places…".

The misconceptions mentioned above were gleaned from the empirical study and supported by the theoretical findings.

6.3.1.4 Research question 4

What grounds exist for the observed misconceptions?

The reason for misconceptions is a lack of meaningful understanding of the concept of Proportion. Learners treated fractions as whole numbers when solving problems. That is, they overlapped whole number concepts and fraction concepts. Algorithm rules were memorised, at times, without understanding. The result was muddling up the signs, inverting fractions when not necessary. Hence, the wrong rule was used when a learner did not know when, where and how to apply a rule. This displays a lack of conditional knowledge. Knowledge about the content, that is, declarative knowledge, does not imply that the learner possesses conditional knowledge.

Learners also tended to muddle up the rules for addition, multiplication and division. The reflection of rote learning without conceptual understanding of rules is manifested in misconceptions. When learners follow a procedure they do not understand, they have no means for knowing when to use it and no way of assessing when answers make sense (Van de Walle *et al*., 2013:316). In the Mathematics classroom, learners are taught that common denominators are required for adding and subtracting fractions. Some learners are perplexed by "why?". They are not aware that finding common denominators enables them to compare equal-sized parts. At times the denominators are ignored, whilst the numerators are added. Additive thinking caused major misconceptions.

Misconceptions occurred in cases where the learners' mathematical knowledge did not exist as a web of interrelated connections. Inflexible strategies are the result of lack of understanding.

6.3.1.4 Research question 5

What can be done to support the learners to learn proportion with understanding?

The insights gained from the empirical findings formed the basis in attempting to answer this question. In view of the literature review, some solutions follow.

Teachers should transform their classrooms into *a community of learners* (Van de Walle & Lovin, 2006:9). Learners need to interact with each other and the teacher to share ideas, compare and evaluate strategies. Rich interaction will produce reflective thinking. All learners should participate. Learners will learn from each other, enhancing their understanding of the concept of Proportion. One cannot deny the role played by misconceptions and errors. These should be highlighted to clear any misunderstandings that are prevalent. When learners and teachers discuss misconceptions and errors, other learners who are harbouring the same misconceptions and errors also benefit. In this way, these errors and misconceptions become handy tools for constructive learning. Teachers should create an environment where learners are not afraid to make mistakes. Van de Walle *et al*. (2010:43) stress the value of classroom discussion. They contend that as learners describe and evaluate solutions to tasks, share approaches and make conjectures, learning will occur in ways that are otherwise unlikely to occur. In this way learners take ownership of ideas and make sense of Mathematics (Van de Walle *et al*., 2010:43).

In viewing the classroom as learning communities, language is a crucial aspect. Mathematics has its own language. Learners must translate their thinking into words. Algebraic terms should be discussed, for example: ratio, numerator, denominator, direct proportion, indirect proportion, etc. Encourage learners to explain their steps in solving algebraic problems. *Think Mathematics, speak Mathematics and explain Mathematics* should be the logo of the Mathematics classroom. The idea of learning as a social activity is mirrored by Vygotsky’s notion that children are best capable of internalising complex knowledge when they are involved in a discussion (Krulik *et al*., 2003:6). He further argues that children must think aloud and learn new ideas from others.

Learners need time to think and conceptualise. Teachers need to give learners time to learn. Lessons should not be rushed in order to meet deadlines of work schedules. The most important aspect of lessons is the introduction of a concept. Rushing the introduction will result in learners not understanding the concept. More time spent at this stage will lessen the time in rectifying misunderstanding or re-teaching the same concept.

Scaffolding is very important. Tipps *et al*. (2011:58) define scaffolding as the social support that learners receive while constructing meaning from their experience. The social constructivist domain impounds on teachers to use meaningful language for children's learning (Tipps *et al*., 2011:58). It was evident from the interviews that merely rephrasing the questions made a difference to the responses of the learners.

Models provide a testing ground for emerging ideas. Concrete models give learners something to think about, explore with, talk about and reason with (Van de Walle & Lovin, 2006:9). Most of the learners in this study were not exposed to models in primary school. When learners are exposed to models, conceptual knowledge becomes enhanced. Models add colour and excitement to the Mathematics lesson that could otherwise be viewed as *boring*. A teacher can improvise if models are not available. For example, an orange can be cut up into four pieces. Learners see each piece as one out of four or a quarter. Grouping two such pieces, shows learners that $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, and also that $\frac{2}{4}=\frac{1}{2}$. Learners hereby note how equivalent fractions are formed.

These are some suggestions in developing deep progressive understanding of the concept of Proportion. By facilitating the meaningful learning of Proportion, not only is Mathematics achievement enhanced, but Mathematics anxiety could also decrease.

6.4 CONTRIBUTIONS OF THIS STUDY

Research showed how learners understood the concept of Proportion, whether misconceptions occurred, where the misconceptions originated and what could be done to support learning with understanding. This study contributes to a better understanding of the complexity of teaching and learning Proportional concepts. The study will make a significant contribution to mathematical teaching and learning. The intention is not to generalise the findings of this study. The researcher intends to learn or find out how learners understand Proportion or what issues are relevant in learners' understanding of Proportion.

6.5 LIMITATIONS OF STUDY

6.5.1 The language of the data collection

Language posed a challenge for some learners as the language of the data collection was English, which was these learners' second or third language. Interpreting and comprehending the questions were problematic. The researcher had to rephrase some of the questions during the interviews so that the learners could comprehend what was being asked.

6.5.2 Time factor

The passage of time that elapsed between the task execution and the interviews may have affected the quality of responses and reflections at the time of conducting the interviews. Learners completed the tasks individually after the final exam session. The interviews were conducted over a few days.

6.5.3 Researcher-Teacher

This limitation stems from the fact that the researcher was also the participants' Mathematics teacher for the current year. The researcher assured the participants on various occasions during the investigation that their names would not be mentioned and that every step of the investigation process would be confidential. There is still a possibility that these learners might have been ambivalent in completing the tasks and interacting honestly and openly in the interview sessions.

6.5.4 Generalisations

The results cannot be generalised as it was a case study involving only 18 learners from one school with one teacher (the researcher). The study was intended to come to an understanding of how the learners understood, or misunderstood, the idea of Proportion and why that was the case and not to generalise. Empirical results were compared with findings from literature to identify different or similar patterns of understanding or misunderstanding.

6.6 RECOMMENDATIONS

6.6.1 Recommendations with regard to the understanding of Proportion

To develop understanding of Proportion, learners must explore proportional situations in a wide variety of contexts. Concrete models should be used to enhance understanding. Dialogue should be the vehicle for understanding. Learners should verbalise their thinking so that they can learn from each other in a learning community. The idea is not only to improve understanding but also to clear misconceptions. Learners need to form web-like connections between the different proportional concepts (as in Figure 6.1) to develop understanding. The researcher highly recommends that learners question teachers on the logic behind procedures to gain a deeper understanding of Proportion.

6.6.2 Recommendations with regard to addressing misconceptions

Teachers should create an environment where mistakes are treated as opportunities for learning and rectifying misunderstandings. Exposing learners to multiple problems and activities and allowing them to discuss their thought processes will help to further develop understanding. Learners need to work with non-examples as well to differentiate between proportional and non-proportional relationships.

6.6.3 Recommendations with regard to teaching for understanding

Teachers concentrate on procedural understanding. If more time is spent introducing a concept, the longer the concept will be retained in the learners' long-term memory. The Department of Education should consider training teachers to view the importance of teaching for understanding. Enough effort and ample time should be spent on the introduction of the concept of Proportion. Rules and algorithms should be delayed until learners have gained conceptual understanding.

6.6.4 Recommendations with regard to further research

Literature shows that proportion is a difficult idea for learners to conceive properly. The learners in this study overall understand the ideas under scrutiny quite well. A logical recommendation would therefore also be to investigate the understanding of Proportion in other (less favourable) conditions.

6.7 FINAL ANALYSIS

Proportional reasoning is sometimes perceived as only being the study of ratios, rates and rational numbers such as fractions, decimals and percents, but it actually permeates all strands of mathematics (K-12, Ontario:4). Proportional reasoning is a very important dimension in the development of a learner's mathematical understanding. According to Lamon (2005:10), ninety percent of learners that enter high school lack reasoning in Mathematics and Science.

It consequently becomes apparent that the teacher is the driving force behind most of the knowledge acquired by the learner. Formal learning takes place in the classroom. It therefore is imperative that the educator facilitates this knowledge effectively and accurately through a variety of teaching methods. It has been concluded through a variety of research and studies that learners need a solid and stable foundation, especially within the subject Mathematics. The Mathematics teacher must therefore be strong, surefooted in the classroom and also have the necessary knowledge and skills to assist learners and provide support in this subject. A good Mathematics teacher who teaches for conceptual understanding will empower the learners with skills to tackle Proportional problems. Education goes beyond teaching and learning. It is thus the teacher, who is the backbone of the education system, who can facilitate effectively to make education last a lifetime!

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ADDENDUM A:
LETTER TO THE NORTH-WEST DEPARTMENT OF EDUCATION

P.O. Box 29

Derby

0347

6 March 2013

The Area Manager-Mrs Paledi

The North West Department of Education

Bojanala

Private bag

Sir/Madam

Request: Learners to actively participate in the M.ED research

I, Mrs Sharifa Suliman (university number: 20984758) who is registered for the M. Ed. Degree in Mathematics Education at the Potchefstroom Campus of the North-West University, and is working under supervision of Prof HD Nieuwoudt and Dr A. Roux, hereby wish to conduct empirical work. My study focuses on *"*Learners' understanding of proportion: A case study from Grade 8 Mathematics*".*

I will do the investigation strictly according to the proposal that has been approved by the Faculty's Master and Doctoral Programme Committee. A copy of the proposal is available to you on your request. In order to finish the dissertation in the allocated period, I need to complete the planned empirical work in 2013 involving learners in the state school where I work, Zinniaville Secondary School. I have selected this school, to become part of this study. The learners will be requested to complete a task sheet and be interviewed. The purpose of the tasks and interviews is to identify where the learner will need help to improve on performance. The tasks and interviews will be done anonymously, and the scores would not be used against the school, teachers or learners. I do have consent to approach you to request permission to do so, and Prof H. Nieuwoudt and Dr A. Roux strongly support my request as the outcomes of this study will be beneficial not only to the schools but also to learners and teachers in a much wider context and in the critical field of learning and teaching of Mathematics.

Please do not hesitate to contact me should there be any further queries.

You may contact my supervisors:

Prof HD Nieuwoudt (018 299 1875)

Dr A. Roux (018 299 1895)

Thanking you in anticipation.

Sincerely,

....

Sharifa Suliman

082 303 9184

sulimansharifa@gmail.com

ADDENDUM B:

LETTER TO PRINCIPAL

P.O. Box 29

Derby

0347

Mr Abdull [Principal]

Zinniaville Secondary School

42 Hollis Street

Zinniaville

0299

2 May2013

Sir

Request: To conduct m.eD research at school

I, Mrs Sharifa Suliman (university number: 20984758) who is registered for the M.ED degree in Mathematics Education at the Potchefstroom Campus of the North-West University, and working under supervision of Prof HD Nieuwoudt and Dr A. Roux, hereby wish to request your school to conduct empirical work. My study focuses on *"*Learners' understanding of proportion: A case study from Grade 8 Mathematics*".*

I will do the investigation strictly according to the proposal that has been approved by the Faculty's Master and Doctoral Programme Committee. A copy of the proposal is available to you on your request. In order to finish the dissertation in the allocated period, I need to complete the planned empirical work in 2013 involving learners in the state school where I work. I have selected this school to become part of this study. I do have consent to approach you to request permission to do so. Prof H. Nieuwoudt and Dr A. Roux strongly support my request as the outcomes of this study will be beneficial not only to the school, but also to learners and teachers in a much wider context and in the critical field of learning and teaching of Mathematics.

Please do not hesitate to contact me should there be any further queries.

You may contact my supervisors:

Prof HD Nieuwoudt (018 299 1875)

Dr A. Roux (018 299 1895)

Thanking you in anticipation.

Sincerely,

....

Sharifa Suliman

082 303 9184

sulimansharifa1@gmail.com

ADDENDUM C:
LETTER TO PARENT

P.O. Box 29

Derby

0347

4 March 2013

MS .... [Parent/Guardian]

Zinniaville Secondary School

42 Hollis Street

Zinniaville

Sir/Madam

Request: Learner to actively participate in the M.ED research

I, Mrs Sharifa Suliman (university number: 20984758) who is registered for the M. Ed. degree in Mathematics Education at the Potchefstroom Campus of the North-West University, and is working under supervision of Prof HD Nieuwoudt and Dr A. Roux, hereby wish to conduct empirical work. My study focuses on "Learners' understanding of proportion: A case study from Grade 8 Mathematics".

I will do the investigation strictly according to the proposal that has been approved by the Faculty's Master and Doctoral Programme Committee. A copy of the proposal is available to you on your request. In order to finish the dissertation in the allocated period, I need to complete the planned empirical work in 2013 involving learners in this state school where I work. I have selected this school, to become part this study, and would need your approval for your child to participate in the research. The learners will be requested to complete a task-sheet and be part of an interview. The purpose of the tasks and interviews is to identify where the learner will need help to improve on performance. The questionnaire and interview will be done anonymously, and the scores would not be used against the school, teacher or learners. I do have consent to approach you to request permission to do so, and Prof H. Nieuwoudt and Dr A. Roux strongly support my request as the outcomes of this study will be beneficial not only to the schools but also to learners and teachers in a much wider context and in the critical field of learning and teaching of Mathematics.

Please do not hesitate to contact me should there be any further queries.

You may contact my supervisors:

Prof HD Nieuwoudt (018 299 1875)

Dr A. Roux (018 299 1895)

Thanking you in anticipation.

Sincerely,

....

Sharifa Suliman

082 303 9184

sharifasuliman1@telkomsa.net

ADDENDUM D:

TASK SHEET

NAME________________________________

DEAR LEARNER. I WOULD LIKE YOU TO ANSWER THE FOLLOWING QUESTIONS.

1. Study the chocolate recipe for 2 people below.

Chocolate cake recipe for two people

4 tablespoons of flour

2 tablespoons of butter

2 tablespoons of sugar

1 egg

1 teaspoon of cocoa

Calculate the ingredients needed to bake a cake for six people:

Chocolate cake recipe for six people:

___ tablespoons of flour

___ tablespoons of butter

___ tablespoons of sugar

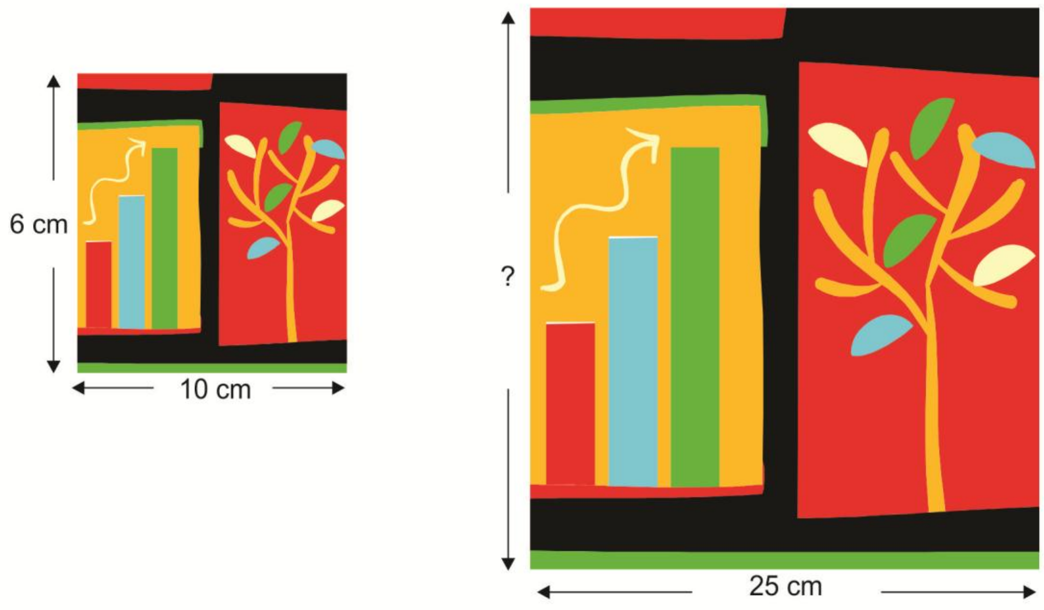
___ eggs

___ teaspoons of cocoa

Explain your reasoning:

----

2 Study the picture below.



The frame on the right is an enlargement of the frame on the left.

The small frame is 10 cm wide and 6 cm high.

The enlarged frame is 25 cm wide. How high is it?--

Explain your reasoning.

----

3. Fill in the missing numbers:

Number of Euros	2	3		5	6
Number of Rands	24		48		

4.

4.1 Place the following decimals in order from smallest to biggest:

0.15, 1.3, 0.095, 2.8

Explain your steps:---

4.2 Write the fractions below in ascending order (from smallest to biggest):

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{100}$ $\frac{1}{5}$

Explain your steps:---

5.

5.1 Rusty high school has 16 grade 8 learners.

12 are basket ball fans. The remaining learners are not.

Describe whatever relationships you can see between learners who are basket ball fans and those who are not.

5.2 Study the numbers in the box below:

$\frac{1}{2}$	0,5	50%

Does a relationship exist between the three?

If so, explain.

5.3 Write $\frac{14}{20}$ as a ratio first, then simplify it.

5.4 A learner studied for 1 hour and earned 90% for her test. If she had studied for two hours, she would have scored 180% for the test. Does this make sense? Explain.

----

6. It takes 6 builders 12 hours to build a wall. How long will it take 9 workers to build the same wall if they all work at a constant rate. Explain your reasoning. Also explain the type of proportional relationship involved.

----

7. On which number line can the point marked by x be named by the fraction

$\frac{1}{3}$?

a I----------I----------I----------I----------I----------I

x

b. I----------I----------I----------I----------I----------I
X

c. I----------I----------I----------I----------I----------I
X

d. I----------I----------I----------I----------I----------I
X

e. None of the above.

Answer: ---------------------------------------

8. Simplify:

a) $\frac{3}{5} - \frac{1}{2}$

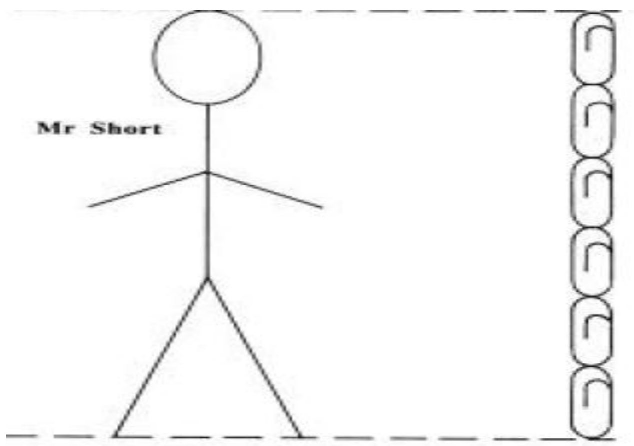
----

b) $\frac{2}{3} \times \frac{1}{3}$

----

c) $\frac{2}{3} \div \frac{4}{5}$

9.



Mr Short has a friend Mr Tall.

Mr Short is six paper clips in height.

When we measure their heights with matchsticks:

Mr Short's height is four matchsticks.

Mr Tall's height is six matchsticks.

How many paper clips are needed for Mr Tall’s height? Explain.
