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On the study of an extended coupled KdV system: Analytical solutions and conservation laws

C. Mabenga [a], B. Muatjetjeja [a,b,*], T.G. Motsumi [a], A.R. Adem [c]

[a] Department of Mathematics, Faculty of Science, University of Botswana, Private Bag 22, Gaborone, Botswana
[b] Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, Republic of South Africa
[c] Department of Mathematical Sciences, University of South Africa, UNISA 0003, Pretoria, Republic of South Africa

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ABSTRACT

This paper aims to derive analytical solutions of an extended (2+1)-dimensional constant coefficients new coupled Korteweg–de Vries system. This will be achieved by implementing the classical symmetry method in conjunction with some simplest equation methods. The simplest equations that will be utilised includes among others the Bernoulli and Riccati equations. Furthermore, the conservation laws will be constructed through the multipliers approach, which subsequently reveals the conserved quantities. Moreover, a brief presentation of results obtained consisting of a variety of profile structures which include the kink type, bell and inverted bell shaped and singular wave solutions will be discussed.

1. Introduction

The research in constructing exact solutions of nonlinear partial differential equations (NLPDEs) has intensified in the recent years. The motive behind this is to enhance the understanding of the nonlinear physical wave phenomena appearing in mathematical problems described by these equations. Nonlinear partial differential equations have been widely used in describing many problems in the field of applied mathematics, engineering in areas such as fluid dynamics, solid state, plasma and quantum physics, optical fibre, ecology, chemistry, biology, oceanology and in many other areas of nonlinear science.[1,2] The importance of finding the exact solutions of NLPDEs together with the development of computational software packages such as Maple and Mathematica has led to the establishment of powerful and efficient methods to solve these equations. Some of the methods include the inverse scattering method,[3] the Darboux transformations method,[4] the homogeneous balance method,[5] the Hirota bilinear method,[6–12] the tanh-function methods,[13] the extended tanh-function method,[14,15] the Jacobi elliptic method[16] the Lie symmetry method[17–19] and many others. Recently, Qin[20] used a finite-dimensional integrable system to derive a new hierarchy of nonlinear evolution equations. An interesting system in this hierarchy is a new coupled Korteweg–de Vries system

$$
\begin{aligned}
u_t &= \beta u_{xxx} + \alpha(uv)_x + \gamma(vw)_x,\\
v_t &= \beta v_{xxx} + \lambda(uw)_x,\\
w_t &= \beta w_{xxx} + \lambda(uv)_x,
\end{aligned} \tag{1}
$$

where β, α, λ and γ are arbitrary constants. In general, coupled KdV systems are encountered in various physical phenomena, particularly in dynamical fluids.[21] Further work on this new coupled KdV system (1) was investigated by the authors in Refs. 22–24. Wu,[22] studied some exact travelling wave solutions of this new coupled equations using the extended tanh-function method with $\beta = -\frac{1}{4}$, $\alpha = \frac{3}{2}$, $\gamma = \frac{3}{4}$, $\lambda = \frac{3}{2}$. In Ref. 23, Wazwaz further extended system (1) into a new coupled Kadomtsev–Petviashvili system and studied the integrability of both systems with $\beta = -1$, $\alpha = \frac{3}{2}$, $\gamma = \frac{3}{2}$ and $\lambda = -3$. In addition, soliton solutions were also derived by employing Hirota's bilinear method. Inspired by the results in Ref. 23, Seadawy[24] extended system (1) to obtain a new coupled Zakharov–Kuznetsov system. Thereafter, explicit travelling wave solutions for both systems using the modified extended direct algebraic method were obtained.

Motivated by all the above extensions we propose a new extended (2+1)-dimensional NCKdV system of the form

$$
\begin{aligned}
u_t &= p_1 u_{xxx} + p_2(uv)_x + p_3(uv)_y + p_4(vw)_x + p_5(vw)_y,\\
v_t &= p_1 v_{xxx} + p_6(uw)_x + p_7(uw)_y,\\
w_t &= p_1 w_{xxx} + p_6(uv)_x + p_7(uv)_y,
\end{aligned} \tag{2}
$$

where p_1, p_2, p_3, p_4, p_5, p_6 and p_7 are arbitrary parameters.

The structure of this paper is to study an extended (2+1)-dimensional new coupled KdV system (2). Since there is no unified technique to solve nonlinear partial differential equations, we therefore aim to

* Corresponding author at: Department of Mathematics, Faculty of Science, University of Botswana, Private Bag 22, Gaborone, Botswana.
E-mail addresses: Mabengac@ub.ac.bw (C. Mabenga), Muatjetjejab@ub.ac.bw (B. Muatjetjeja), Motsumit@ub.ac.bw (T.G. Motsumi), ademar@unisa.ac.za (A.R. Adem).

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utilise the classical symmetry method together with the simplest equation method in order to construct exact solutions of system (2). We also employ the multiplier method to derive the conservation laws of the system (2).

2. Point symmetries of system (2)

Lie point symmetries of the extended (2+1)-dimensional system (2) are obtained using the symmetry generator

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w},$$

where the coefficients of the partial derivatives are functions of (t, x, y, u, v, w). Applying the third prolongation $\mathrm{pr}^{(3)} X$ to system (2) and splitting on the derivatives of u, v and w yields an overdetermined system of equations. Solving the resulting system of equations prompts the following two cases.

Case 1. $(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ arbitrary parameters but not in the form in Case 2 In this case, system (2) admits, respectively, three translations and a scaling symmetries spanned by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial y}, \\ &X_4 = -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w}. \end{aligned} \tag{3}$$

It is worth noting that these four symmetries are referred to as the *principal Lie algebra* of system (2).

Case 2. $p_2 p_7 - p_3 p_6 = 0$ and $p_4 p_7 - p_5 p_6 = 0$.

The principal algebra of system (2) extends by one. Hence the Lie algebra is five dimensional. It is spanned by (3) and an additional scaling and space-dependent shift symmetry given by

$$X_5 = 3p_7 t\frac{\partial}{\partial t} + (2p_6 y + p_7 x)\frac{\partial}{\partial x} + 3p_7 y\frac{\partial}{\partial y}. \tag{4}$$

2.1. Symmetry reductions of system (2)

This subsection aims to transform system (2) into a nonlinear system of pdes or nonlinear system of odes by solving the associated characteristic equations, viz.,

$$\frac{\mathrm{d}t}{\xi^1} = \frac{\mathrm{d}x}{\xi^2} = \frac{\mathrm{d}y}{\xi^3} = \frac{\mathrm{d}u}{\eta^1} = \frac{\mathrm{d}v}{\eta^2} = \frac{\mathrm{d}w}{\eta^3}. \tag{5}$$

Firstly, we consider Case 1 and take the linear combination of the translation symmetries $a_1 X_1 + a_2 X_3 + a_3 X_2$ where (a_1, a_2, a_3) are arbitrary constants. Solving the associated Lagrange systems (5) we obtain the following five invariants

$$f = a_1 x - a_3 t, g = a_3 y - a_2 x, \theta = u, \phi = v, \psi = w. \tag{6}$$

Using these invariants, system (2) transforms into a nonlinear system of pdes in two new independent variables f and g given by

$$\begin{cases} a_3\theta_f + p_1 a_1^3\theta_{fff} + 3p_1 a_1 a_2(a_2\theta_{fgg} - a_1\theta_{ffg}) \\ -p_1 a_2^3\theta_{ggg} + p_2\phi(a_1\theta_f - a_2\theta_g) \\ +p_2\theta(a_1\phi_f - a_2\phi_g) + p_3 a_3(\theta\phi_g + \phi\theta_g) + p_4\psi(a_1\phi_f - a_2\phi_g) \\ +p_4\phi(a_1\psi_f - a_2\psi_g) + p_5 a_3(\psi\phi_g + \phi\psi_g) = 0, \\ a_3\phi_f + p_1 a_1^3\phi_{fff} + 3p_1 a_1 a_2(a_2\phi_{fgg} - a_1\phi_{fgg}) \\ -p_1 a_2^3\phi_{ggg} + p_6\psi(a_1\theta_f - a_2\theta_g) \\ +p_6\theta(a_1\psi_f - a_2\psi_g) + p_7 a_3(\psi\theta_g + \theta\psi_g) = 0, \\ a_3\psi_f + p_1 a_1^3\psi_{fff} + 3p_1 a_1 a_2(a_2\psi_{fgg} - a_1\psi_{ffg}) \\ -p_1 a_2^3\psi_{ggg} + p_6\phi(a_1\theta_f - a_2\theta_g) \\ +p_6\theta(a_1\phi_f - a_2\phi_g) + p_7 a_3(\phi\theta_g + \theta\phi_g) = 0. \end{cases} \tag{7}$$

System admits two translation symmetries, namely

$$Y_1 = \frac{\partial}{\partial f}, Y_2 = \frac{\partial}{\partial g}. \tag{8}$$

Taking the linear combination of $b_1 Y_1 + b_2 Y_2$, where b_1 and b_2 are arbitrary constants and solving the associated characteristic Eq. (5) yields the following four invariants

$$z = b_1 g - b_2 f, P = \theta, Q = \phi, R = \psi, \tag{9}$$

where P, Q and R are new dependent variables while z is a new independent variable. Making use of these invariants (9), system reduces to a third order nonlinear system of odes

$$\begin{cases} a_3 b_2 P'(z) + p_1\delta^3 P'''(z) + \Upsilon_1 (P(z)Q(z))' + \Upsilon_2 (Q(z)R(z))' = 0, \\ a_3 b_2 Q'(z) + p_1\delta^3 Q''(z) + \Upsilon_3 (P(z)R(z))' = 0, \\ a_3 b_2 R'(z) + p_1\delta^3 R'''(z) + \Upsilon_3 (P(z)Q(z))' = 0, \end{cases} \tag{10}$$

where $\delta = a_1 b_2 + a_2 b_1$, $z = -\delta x + a_3 b_1 y + a_3 b_2 t$, $\Upsilon_1 = \delta p_2 - a_3 b_1 p_3$, $\Upsilon_2 = \delta p_4 - a_3 b_1 p_5$ and $\Upsilon_3 = \delta p_6 - a_3 b_1 p_7$.

Secondly, by invoking symmetry X_4 in Case 1 and proceeding as above we obtain the following five invariants

$$f = \frac{y}{x}, g = \frac{t}{x^3}, \theta = ux^2, \phi = vx^2, \psi = wx^2. \tag{11}$$

Consequently, the group invariant solutions of the system (2) are

$$\begin{aligned} &u(x, y, t) = \frac{1}{x^2}\theta\left(\frac{1}{x}, \frac{t}{x^3}\right), v(x, y, t) = \frac{1}{x^2}\phi\left(\frac{1}{x}, \frac{t}{x^3}\right), \\ &\quad w(x, y, t) = \frac{1}{x^2}\psi\left(\frac{1}{x}, \frac{t}{x^3}\right), \end{aligned} \tag{12}$$

where $\theta(f, g)$, $\phi(f, g)$ and $\psi(f, g)$ are any solutions of the following system of nonlinear equations

$$\begin{cases} 27p_1 g^2(f\theta_{fgg} + g\theta_{ggg}) + p_1 f^2(f\theta_{fff} + 9g\theta_{ffg}) + 6p_1(2f^2\theta_{ff} \\ +15fg\theta_{fg} + 27g^2\theta_{gg}) + 3p_2 g(\phi\theta_g + \theta\phi_g) + 3p_4 g(\phi\psi_g + \psi\phi_g) \\ +(1 + 186p_1 g)\theta_g + (p_2 f - p_3)(\phi\theta_f + \theta\phi_f) + (p_4 f - p_5)(\phi\psi_f + \psi\phi_f) \\ +36p_1 f\theta_f + 4(p_2\theta + p_4\psi)\phi + 24p_1\theta = 0, \\ 27p_1 g^2(f\phi_{fgg} + g\phi_{ggg}) + p_1 f^2(f\phi_{fff} + 9g\phi_{ffg}) + 6p_1(2f^2\phi_{ff} + 15fg\phi fg \\ +27g^2\phi_{gg}) + 3p_6 g(\psi\theta_g + \theta\psi_g) + (1 + 186p_1 g)\phi_g + (p_6 f - p_7)(\psi\theta_f + \theta\psi_f) \\ +36p_1 f\phi_f + 4p_6\theta\psi + 24p_1\phi = 0, \\ 27p_1 g^2(f\psi_{fgg} + g\psi_{ggg}) + p_1 f^2(f\psi_{fff} + 9g\psi_{ffg}) + 6p_1(2f^2\psi_{ff} + 15fg\psi_{fg} \\ +27g^2\psi_{gg}) + 3p_6 g(\phi_g\theta + \theta\phi_g) + (1 + 186p_1 g)\phi_g + (p_6 f - p_7)(\phi_f\theta + \theta\phi_f) \\ +36p_1 f\psi_f + 4p_6\theta\phi + 24p_1\psi = 0. \end{cases} \tag{13}$$

Lastly, we consider Case 2 and take the linear combination of $p_7 X_4 + X_5$. Solving the related characteristic Eq. (5), we get five invariants

$$f = t, g = y - \frac{p_7}{p_6}x, \theta = \frac{u}{p_6 y}, \phi = \frac{v}{p_6 y}, \psi = \frac{w}{p_6 y}. \tag{14}$$

Using these invariants, system (2) reduces to a system of nonlinear pdes in two independent variable f and g, namely,

$$\begin{cases} p_1 p_7^3\theta_{ggg} - p_6^3\theta_f - 2p_6^4\left(p_3\theta + p_5\psi\right)\phi = 0, \\ p_1 p_7^3\phi_{ggg} + p_6^3\phi_f - 2p_7 p_6^4\theta\psi = 0, \\ p_1 p_7^3\psi_{ggg} + p_6^3\psi_f - 2p_7 p_6^4\theta\phi = 0. \end{cases} \tag{15}$$

This system possess three point symmetries, viz.,

$$Y_1 = \frac{\partial}{\partial g}, Y_2 = \frac{\partial}{\partial f}, \quad Y_3 = 3f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} - 3\theta\frac{\partial}{\partial \theta} - 3\phi\frac{\partial}{\partial \phi} - 3\psi\frac{\partial}{\partial \psi}. \tag{16}$$

Invoking the linear combination of the translation symmetries $Y_1 + Y_2$ and solving the associated Lagrange system (5) gives four invariants, namely,

$$z = f - g, E = \theta, F = \phi, G = \psi, \tag{17}$$

which gives the group invariant solutions of the system (2) as

$$u(x,y,t)=p_6yE(z),\ v(x,y,t)=p_6yF(z),\ w(x,y,t)=p_6yG(z), \tag{18}$$

where $E(z)$, $F(z)$ and $G(z)$ are any solutions of the following system of third-order nonlinear equations

$$\begin{cases} p_1\,p_7^3E'''(z)-p_6^3E'(z)+2p_6^4(p_5G(z)+p_3E(z))F(z)=0,\\ p_1\,p_7^3F'''(z)-p_6^3F'(z)+2p_7p_6^4E(z)G(z)=0\\ p_1\,p_7^3G'''(z)-p_6^3G'(z)+2p_7p_6^4E(z)F(z)=0,\end{cases} \tag{19}$$

with $z=t-y+\dfrac{p_7}{p_6}x$.

Employing symmetry Y_3 gives the following invariants

$$z=\frac{f}{g^3},\ E=\theta g^3,\ F=\phi g^3,\ G=\psi g^3. \tag{20}$$

Thus, the group invariant solutions of the system (2) are obtained as

$$u(x,y,t)=\frac{p_6^4yE(z)}{(p_6y-p_7x)^3},\ v(x,y,t)=\frac{p_6^4yF(z)}{(p_6y-p_7x)^3},\ w(x,y,t)=\frac{p_6^4yG(z)}{(p_6y-p_6x)^3}, \tag{21}$$

where $E(z)$, $F(z)$ and $G(z)$ are any solutions of the following system of third-order nonlinear equations

$$\begin{cases} 60p_1p_7^3E(z)+2p_6^4(p_3E(z)+p_5G(z))F(z)-p_6^3E'(z)+276p_1p_7^3E'(z)z\\ +189p_1p_7^3E''(z)z^2+27p_1p_7^3E'''(z)z^3=0,\\ 60p_1p_7^3F(z)+2p_7p_6^4E(z)G(z)-p_6^3F'(z)+276p_1p_7^3F'(z)z\\ +189p_1p_7^3F''(z)z^2+27p_1p_7^3F'''(z)z^3=0,\\ 60p_1p_7^3G(z)+2p_7p_6^4E(z)F(z)-p_6^3G'(z)+276p_1p_7^3G'(z)z\\ +189p_1P_7^3G''(z)z^2+27p_1p_7^3G'''(z)z^3=0,\end{cases} \tag{22}$$

with $z=\dfrac{p_6^3t}{(p_6y-p_7x)^3}$.

2.2. Exact solutions using the simplest equation method

In this subsection, we derive exact solutions of system (2) by solving the system of nonlinear odes using the simplest equation method. The simplest equation method developed by Kudryashov[25,26] is a very powerful technique for computing exact solutions of NLPDEs. The method has also been successfully applied, extended and modified by the authors in Refs. 27–29 to solve a variety of NLPDEs. The outline of the simplest equation method is as follows.

The first step is to assume that the solutions of system take the form

$$P(z)=\sum_{k=0}^{L}E_kH(z)^k,\ Q(z)=\sum_{k=0}^{M}F_kH(z)^k,\ R(z)=\sum_{k=0}^{N}G_kH(z)^k, \tag{23}$$

where $H(z)$ satisfies the Bernoulli or Riccati equations, L, M and N are positive integers obtained by the balancing procedure while $E_0,\dots,E_L$, $F_0,\dots,F_M$ and $G_0,\dots,G_N$ are parameters to be determined. The Bernoulli equation has the form

$$H'(z)=aH(z)+bH(z)^2, \tag{24}$$

where a and b are constants and its solution is given by

$$H(z)=a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}. \tag{25}$$

The Riccati equation is defined by

$$H'(z)=aH(z)^2+bH(z)+c, \tag{26}$$

where a, b and c are constants. Its solutions are obtained as

$$H(z)=-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right], \tag{27}$$

and

$$H(z)=-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{1}{2}\sigma z\right)+\frac{\operatorname{sech}\left(\frac{1}{2}\sigma z\right)}{C\cosh\left(\frac{1}{2}\sigma z\right)-\frac{2a}{\sigma}\sinh\left(\frac{1}{2}\sigma z\right)}, \tag{28}$$

where $\sigma^2=b^2-4ac>0$ and C is a constant of integration.

Remark 1. It is observed that the cases $b^2-4ac=0$ and $b^2-4ac<0$ yield rational and periodic wave solutions, respectively.

The second step is to determine the values of L, M and N by taking the homogeneous balance between the highest-order nonlinear terms and the highest-order derivatives appearing in the system of odes . Lastly, we determine the values of the parameters $E_0,\dots,E_L$, $F_0,\dots,F_M$ and $G_0,\dots,G_N$ by substituting the values of L, M and N into Eq. (23). Making use of the Eqs. (24) and (26) and splitting on powers of the function $H(z)$ and thereafter, solving the resulting system of equations, we get the values of $E_0,\dots,E_L$, $F_0,\dots,F_M$ and $G_0,\dots,G_N$.

2.3. Bernoulli equation as the simplest equation method

Balancing the powers of the term $P'''(z)$ with $R'(z)Q(z)$, the term $Q'''(z)$ with $P'(z)R(z)$ and the term $R'''(z)$ with $P'(z)Q(z)$ in Eq. yields $L=2$, $M=2$ and $N=2$. Thus the solutions of system take the form

$$\begin{aligned}P(z)&=E_0+E_1H(z)+E_2H(z)^2,\\ Q(z)&=F_0+F_1H(z)+F_2H(z)^2,\\ R(z)&=G_0+G_1H(z)+G_2H(z)^2.\end{aligned} \tag{29}$$

Substituting (29) into the system and invoking the Bernoulli Eq. (24), thereafter, equating the coefficients of powers of the function $H(z)$ to zero yields an algebraic system of equations in terms of E_k, F_k and G_k for $(k=0,1,2)$. Solving the resulting system with the aid of Maple, prompts the following eight cases with $\omega=a_1b_2+a_2b_1,\Omega_1=a_3b_1p_3-p_2\omega,\Omega_2=a_3b_1p_7-p_6\omega$ and $\Omega_3=a_3b_1p_5-p_4\omega$.

Case 1.

$$p_4=\frac{a_3b_1p_5}{\omega},\ E_0=\frac{a^2p_1\omega^3-G_0\Omega_1+a_3b_2}{\Omega_2},\ E_1=\frac{6abp_1\omega^3}{\Omega_2},\ E_2=\frac{bE_1}{a},$$
$$F_0=G_0,\ F_1=\frac{aF_2}{b},\ F_2=\frac{6b^2p_1\omega^3}{\Omega_1},\ G_0=G_0,\ G_1=F_1,\ G_2=F_2. \tag{30}$$

Case 2.

$$p_4=\frac{a_3b_1p_5}{\omega},\ E_0=\frac{F_0\Omega_1-a^2p_1\omega^3-a_3b_2}{\Omega_2},\ E_1=\frac{-6abp_1\omega^3}{\Omega_2},\ E_2=\frac{bE_1}{a},$$
$$F_0=F_0,\ F_1=\frac{aF_2}{b},\ F_2=\frac{6b^2p_1\omega^3}{\Omega_1},\ G_0=-F_0,\ G_1=-F_1,\ G_2=-F_2. \tag{31}$$

Case 3.

$$p_4=\frac{3\Omega_1^2+16a_3b_1p_5\Omega_2}{16\omega\Omega_2},\ E_0=\frac{a^2p_1\omega^3+a_3b_2}{2\Omega_2},\ E_1=\frac{aE_2}{b},$$
$$E_2=\frac{6b^2p_1\omega^3}{\Omega_2},\ F_0=\frac{2(a^2p_1\omega^3+a_3b_2)}{3\,\Omega_1},\ F_1=\frac{aF_2}{b},$$
$$F_2=\frac{8b^2p_1\omega^3}{\Omega_1},\ G_0=F_0,\ G_1=F_1,\ G_2=F_2. \tag{32}$$

Case 4.

$$p_4=\frac{3\Omega_1^2+16a_3b_1p_5\Omega_2}{16\omega\Omega_2},\ E_0=-\frac{a^2p_1\omega^3+a_3b_2}{2\Omega_2},\ E_1=\frac{aE_2}{b},$$
$$E_2=\frac{-6b^2p_1\omega^3}{\Omega_2},\ F_0=\frac{2(a^2p_1\omega^3+a_3b_2)}{3\,\Omega_1},\ F_1=\frac{aF_2}{b},$$
$$F_2=\frac{8b^2p_1\omega^3}{\Omega_1},\ G_0=-F_0,\ G_1=-F_1,\ G_2=-F_2. \tag{33}$$

Case 5.

$$p_3=\frac{6p_1b^2\omega^4(6b^2p_1\omega^2+p_2F_2)-\Omega_2\Omega_3F_2^2}{6a_3b_1p_1b^2\omega^3F_2},\ E_0=\frac{a^2p_1\omega^3+a_3b_2}{2\Omega_2},$$
$$E_1=\frac{aE_2}{b},\ E_2=\frac{6b^2p_1\omega^3}{\Omega_2},\ F_0=\frac{(a^2p_1\omega^3+a_3b_2)F_2}{12b^2p_1\omega^3},$$
$$F_1=\frac{aF_2}{b},\ F_2=F_2,\ G_0=F_0,G_1=F_1,G_2=F_2. \tag{34}$$

Case 6.

$$p_3=\frac{6b^2p_1\omega^4(6b^2p_1\omega^2+p_2F_2)-\Omega_2\Omega_3F_2^2}{6b^2a_3b_1p_1\omega^3F_2},\ E_0=-\frac{(a^2p_1\omega^3+a_3b_2)}{2\Omega_2},$$
$$E_1=\frac{aE_2}{b},\ E_2=-\frac{6b^2p_1\omega^3}{\Omega_2},\ F_0=\frac{(a^2p_1\omega^3+a_3b_2)F_2}{12b^2p_1\omega^3},$$
$$F_1=\frac{aF_2}{b},\ F_2=F_2,G_0=-F_0,G_1=-F_1,G_2=-F_2. \tag{35}$$

Case 7.

$$a_3=\frac{a^2p_1\omega^3}{5b_2},\ p_4=\frac{p_1\omega^2(10b^2b_2E_2+a^2b_1p_5F_2^2)}{5b_2F_2^2},\ E_0=\frac{a^2E_2}{10b^2},$$
$$E_1=\frac{aE_2}{b},\ E_2=\frac{30b^2b_2p_1\omega^2}{a^2b_1p_1p_7\omega^2-5b_2p_6},\ F_0=\frac{a^2F_2}{10b^2},\ F_1=\frac{aF_2}{b},$$
$$F_2=\frac{40b^2b_2p_1\omega^2}{a^2b_1p_1p_3\omega^2-5b_2p_2},G_0=F_0,G_1=F_1,G_2=F_2. \tag{36}$$

Case 8.

$$a_3=\frac{a^2p_1\omega^3}{5b_2},\ p_4=\frac{p_1\omega^2(a^2b_1p_5F_2^2-10b^2b_2E_2)}{5b_2F_2^2},\ E_0=\frac{a^2E_2}{10b^2},$$
$$E_1=\frac{aE_2}{b},\ E_2=\frac{30b^2b_2p_1\omega^2}{5b_2p_6-a^2b_1p_1p_7\omega^2},\ F_0=\frac{a^2F_2}{10b^2},\ F_1=\frac{aF_2}{b},$$
$$F_2=\frac{40b^2b_2p_1\omega^2}{a^2b_1p_1p_3\omega^2-5b_2p_2},G_0=-F_0,G_1=-F_1,G_2=-F_2. \tag{37}$$

Thus the travelling wave solutions of an extended (2+1)-dimensional new coupled KdV system (2) using the Bernoulli equation as the simplest equation method are given by

$$u(x,y,t)=E_0+E_1a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}$$
$$+\ E_2a^2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2, \tag{38}$$
$$v(x,y,t)=F_0+F_1a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}$$
$$+\ F_2a^2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2, \tag{39}$$
$$w(x,y,t)=G_0+G_1a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}$$
$$+\ G_2a^2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2, \tag{40}$$

where $z=-(a_1b_2+a_2b_1)x+a_3b_1y+a_3b_2t$.

2.4. *Riccati equation as the simplest equation method*

By following a similar procedure as above, the balancing procedure yields $L=2$, $M=2$ and $N=2$. Consequently, the solutions of Eq. take the form

$$P(z)=E_0+E_1H(z)+E_2H(z)^2,$$
$$Q(z)=F_0+F_1H(z)+F_2H(z)^2,$$
$$R(z)=G_0+G_1H(z)+G_2H(z)^2. \tag{41}$$

Inserting Eq. (41) into Eq. and employing Eq. (26), thereafter, equating all coefficients of the exponents of $H(z)$ to zero, we obtain a system of algebraic equations in terms of (E_j,F_j,G_j) for $(j=0,1,2)$. Solving the resulting system with the aid of Maple, we obtain the following eight cases.

Case 1.

$$p_4=\frac{a_3b_1p_5}{\omega},\ E_0=\frac{p_1\omega^3(b^2+8ac)-G_0\Omega_1+a_3b_2}{\Omega_2},\ E_1=\frac{6abp_1\omega^3}{\Omega_2},$$
$$E_2=\frac{aE_1}{b},\ F_0=G_0,\ F_1=\frac{6abp_1\omega^3}{\Omega_1},\ F_2=\frac{aF_1}{b},\ G_0=G_0,\ G_1=F_1,$$
$$G_2=F_2. \tag{42}$$

Case 2.

$$p_4=\frac{a_3b_1p_5}{\omega},\ E_0=\frac{F_0\Omega_1-p_1\omega^3(b^2+8ac)-a_3b_2}{\Omega_2},\ E_1=\frac{-6abp_1\omega^3}{\Omega_2},$$
$$E_2=\frac{aE_1}{b},\ F_0=F_0,\ F_1=\frac{6abp_1\omega^3}{\Omega_1},\ F_2=\frac{aF_1}{b},\ G_0=-F_0,$$
$$G_1=-F_1,G_2=-F_2. \tag{43}$$

Case 3.

$$p_3=\frac{6a^2p_1\omega^4(6a^2p_1\omega^2+p_2F_2)-\Omega_2\Omega_3F_2^2}{6a^2a_3b_1p_1\omega^3F_2^2},\ E_0=\frac{p_1\omega^3(b^2+8ac)+a_3b_2}{2\Omega_2},$$
$$E_1=\frac{6abp_1\omega^3}{\Omega_2},\ E_2=\frac{aE_1}{b},\ F_0=\frac{(p_1\omega^3(b^2+8ac)+a_3b_2)F_2}{12a^2p_1\omega^3},$$
$$F_1=\frac{bF_2}{a},\ F_2=F_2,G_0=F_0,G_1=F_1,G_2=F_2. \tag{44}$$

Case 4.

$$p_3=\frac{6a^2p_1\omega^4(6a^2p_1\omega^2+p_2F_2)-\Omega_2\Omega_3F_2^2}{6a^2a_3b_1p_1\omega^3F_2},\ E_0=-\frac{p_1\omega^3(b^2+8ac)+a_3b_2}{2\Omega_2},$$
$$E_1=\frac{-6abp_1\omega^3}{\Omega_2},\ E_2=\frac{aE_1}{b},\ F_0=\frac{(p_1\omega^3(b^2+8ac)+a_3b_2)F_2}{12a^2p_1\omega^3},$$
$$F_1=\frac{bF_2}{a},\ F_2=F_2,G_0=-F_0,G_1=-F_1,G_2=-F_2. \tag{45}$$

Case 5.

$$a_3=\frac{p_1\omega^3(b^2-4ac)}{5b_2},\ p_4=\frac{p_1\omega^2(10a^2b_2E_2+b_1p_5(b^2-4ac)F_2^2)}{5b_2F_2^2},$$
$$E_0=\frac{(b^2+6ac)E_2}{10a^2},\ E_1=\frac{bE_2}{a},\ E_2=\frac{30a^2b_2p_1\omega^2}{b_1p_1p_7\omega^2(b^2-4ac)-5b_2p_6},$$
$$F_0=\frac{(b^2+6ac)F_2}{10a^2},\ F_1=\frac{bF_2}{a},\ F_2=\frac{40a^2b_2p_1\omega^2}{b_1p_1p_3\omega^2(b^2-4ac)-5b_2p_2},$$
$$G_0=F_0,G_1=F_1,G_2=F_2. \tag{46}$$

Case 6.

$$a_3=\frac{p_1\omega^3(b^2-4ac)}{5b_2},\ p_4=\frac{p_1\omega^2(b_1p_5(b^2-4ac)F_2^2-10a^2b_2E_2)}{5b_2F_2^2},$$
$$E_0=\frac{(b^2+6ac)E_2}{10a^2},\ E_1=\frac{bE_2}{a},\ E_2=\frac{30a^2b_2p_1\omega^2}{b_1p_1p_7\omega^2(4ac-b^2)+5b_2p_6},$$
$$F_0=\frac{(b^2+6ac)F_2}{10a^2},\ F_1=\frac{bF_2}{a},\ F_2=\frac{40a^2b_2p_1\omega^2}{b_1p_1p_3\omega^2(b^2-4ac)-5b_2p_2},$$
$$G_0=-F_0,G_1=-F_1,G_2=-F_2. \tag{47}$$

Case 7.

$$p_4=\frac{3\Omega_1^2+16a_3b_1p_5\Omega_2}{16\omega\Omega_2},\ E_0=\frac{p_1\omega^3(b^2+8ac)+a_3\,b_2}{2\Omega_2},$$
$$E_1=\frac{6abp_1\omega^3}{\Omega_2},\ E_2=\frac{aE_1}{b},\ F_0=\frac{2(p_1\omega^3(b^2+8ac)+a_3b_2)}{3\Omega_1},$$
$$F_1=\frac{8abp_1\omega^3}{\Omega_1},\ F_2=\frac{aF_1}{b},\ G_0=F_0,G_1=F_1,G_2=F_2. \tag{48}$$

Case 8.

$$p_4=\frac{3\Omega_1^2+16a_3b_1p_5\Omega_2}{16\omega\Omega_2},\ E_0=-\frac{p_1\omega^3(b^2+8ac)+a_3b_2}{2\Omega_2},$$

$$E_1=\frac{-6abp_1\omega^3}{\Omega_2},E_2=\frac{aE_1}{b},F_0=\frac{2(p_1\omega^3(b^2+8ac)+a_3b_2)}{3\Omega_1},$$
$$F_1=\frac{8abp_1\omega^3}{\Omega_1},F_2=\frac{aF_1}{b},G_0=-F_0,G_1=-F_1,G_2=-F_2. \tag{49}$$

Therefore the solutions of system (2) using Riccati simplest equation method are

$$u(t,x,y)=E_0+E_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}+E_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}^2, \tag{50}$$
$$v(t,x,y)=F_0+F_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}+F_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}^2, \tag{51}$$
$$w(t,x,y)=G_0+G_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}+G_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left[\frac{1}{2}\sigma(z+C)\right]\right\}^2, \tag{52}$$

and

$$u(t,x,y)=E_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}^2+E_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}+E_0, \tag{53}$$
$$v(t,x,y)=F_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}^2+F_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}+F_0, \tag{54}$$
$$w(t,x,y)=G_2\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}^2+G_1\left\{-\frac{b}{2a}-\frac{\sigma}{2a}\tanh\left(\frac{\sigma z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\sigma z}{2}\right)}{C\cosh\left(\frac{\sigma z}{2}\right)-\frac{2a}{\sigma}\sinh\left(\frac{\sigma z}{2}\right)}\right\}+G_0, \tag{55}$$

where $\sigma^2=b^2-4ac>0,\ z=-(a_1b_2+a_2b_1)x+a_3b_1y+a_3b_2t$.

3. Analysis and discussions of solutions

In this section, a graphical demonstration in the form of 3-dimensional, 2-dimensional and density profiles of the solutions of the (2+1)-dimensional new coupled KdV system (2) is presented. We consider Case 1 solutions for both the Bernoulli and Riccati simplest equation methods. A careful observation resulted in the omission of profiles of the remaining solutions cases since they portray similar structures. Since nonlinear partial differential equations have found a great application in a variety of physical phenomena, it is therefore of paramount importance to derive and analyse their analytical solutions so as to have an in-depth comprehension of the physical dynamics of the problems they model. Thus, we determine the physical representations of the derived solutions. Firstly, we consider solutions derived using the Bernoulli equation.

In Fig. 1, we demonstrate 3D, 2D and contour graphs of solutions (38) describing bright soliton solution within the intervals $-15\le x\le 10$ and $-5\le y\le 5$.

The solution structures in Fig. 2 are 3D, 2D and contour graphs of solution (39) which portray dark soliton solution within the intervals $-15\le x\le 10$ and $-5\le y\le 5$.

In Fig. 3, we exhibit 3D, 2D and contour graphs of solution (40) representing a singular wave solution with the spatial variables ranging between $-15\le x\le 10$ and $-5\le y\le 5$.

Lastly, we present graphs of Case 1 solutions derived using the Riccati equation.

The graphs in Fig. 4 represent 3D, 2D and contour plots of solution (50) that resemble a kink type wave solution where the spatial variables range from $-15\le x\le 10$ and $-5\le y\le 5$.

In Fig. 5, we display 3D, 2D and contour structures of solution (51) which demonstrate an inverted singular wave solution within the intervals $-15\le x\le 10$ and $-5\le y\le 5$.

4. Conservation laws

This section aims to investigate the conservation laws for the extended (2+1)-dimensional new coupled KdV system (2). In order to compute conserved densities and fluxes, we resort to the invariance criteria and multiplier method due to the well-known result that the Euler–Lagrange operator annihilates a total divergence. A conserved vector (T^t,T^x,T^y) associated with a conservation law satisfies

$$D_tT^t+D_xT^x+D_yT^y\Big|_\varepsilon=0, \tag{56}$$

which vanishes of the solution space ε of the system (2). Furthermore, if nontrivial functions Λ_1, Λ_2 and Λ_3 called multipliers of system (2) exist, such that

$$\begin{aligned}&E_u\big[\Lambda_1\left(u_t-p_1u_{xxx}-p_2(uv)_x-p_3(uv)_y-p_4(vw)_x-p_5(vw)_y\right)\\&+\Lambda_2\left(v_t-p_1v_{xxx}-p_6(uw)_x-p_7(uw)_y\right)\\&+\Lambda_3\left(w_t-p_1w_{xxx}-p_6(uv)_x-p_7(uv)_y\right)\big]=0,\end{aligned} \tag{57}$$
$$\begin{aligned}&E_v\big[\Lambda_1\left(u_t-p_1u_{xxx}-p_2(uv)_x-p_3(uv)_y-p_4(vw)_x-p_5(vw)_y\right)\\&+\Lambda_2\left(v_t-p_1v_{xxx}-p_6(uw)_x-p_7(uw)_y\right)\\&+\Lambda_3\left(w_t-p_1w_{xxx}-p_6(uv)_x-p_7(uv)_y\right)\big]=0,\end{aligned} \tag{58}$$
$$\begin{aligned}&E_w\big[\Lambda_1\left(u_t-p_1u_{xxx}-p_2(uv)_x-p_3(uv)_y-p_4(vw)_x-p_5(vw)_y\right)\\&+\Lambda_2\left(v_t-p_1v_{xxx}-p_6(uw)_x-p_7(uw)_y\right)\\&+\Lambda_3\left(w_t-p_1w_{xxx}-p_6(uv)_x-p_7(uv)_y\right)\big]=0,\end{aligned} \tag{59}$$

then the following equation holds

$$\begin{aligned}D_tT^t+D_xT^x+D_yT^y=&\left(v_t-p_1v_{xxx}-p_6(uw)_x-p_7(uw)_y\right)\Lambda_2\\&+\left(w_t-p_1w_{xxx}-p_6(uv)_x-p_7(uv)_y\right)\Lambda_3\\&+\left(u_t-p_1u_{xxx}-p_2(uv)_x-p_3(uv)_y-p_4(vw)_x-p_5(vw)_y\right)\Lambda_1,\end{aligned} \tag{60}$$

where E_u, E_v, and E_w are the standard Euler–Lagrange operators.[18,19,30–32] The analysis of Eqs. (57)–(59) prompts the following theorem.

Theorem 1. *Let $(\Lambda_1,\Lambda_2,\Lambda_3)$ being nontrivial differential functions then the extended (2+1)-dimensional new coupled KdV system (2) admits zeroth order multipliers of the form*

$$\Lambda_1(t,x,y,u,v,w)=A(t,x,y),\Lambda_2(t,x,y,u,v,w)=B(t,x,y),$$
$$\Lambda_3(t,x,y,u,v,w)=C(t,x,y), \tag{61}$$

if and only if linear coupled third-order partial differential equations of the form

$$\begin{cases}(p_2p_5-p_3p_4)A_x+p_5p_6C_x+p_5p_7C_y=0,\\ p_4A_x+p_5A_y=0,\\ p_6B_x+p_7B_y=0,\\ p_1A_{xxx}-A_t=0,\\ p_1B_{xxx}-B_t=0,\\ p_1C_{xxx}-C_t=0,\end{cases} \tag{62}$$

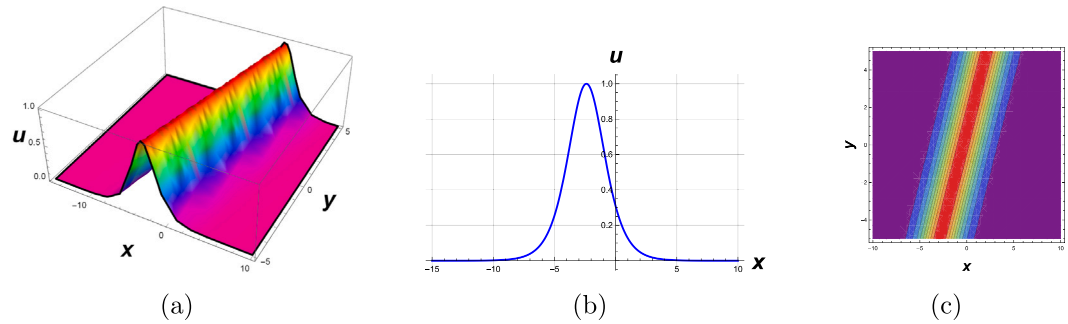

Fig. 1. Wave profile of solution (38) for $a = 0.5$, $b = -0.25$, $p_1 = -1$ and $p_6 = -1$ with other parameters set to unity. (a) 3D plot. (b) 2D plot. (c) Contour plot.

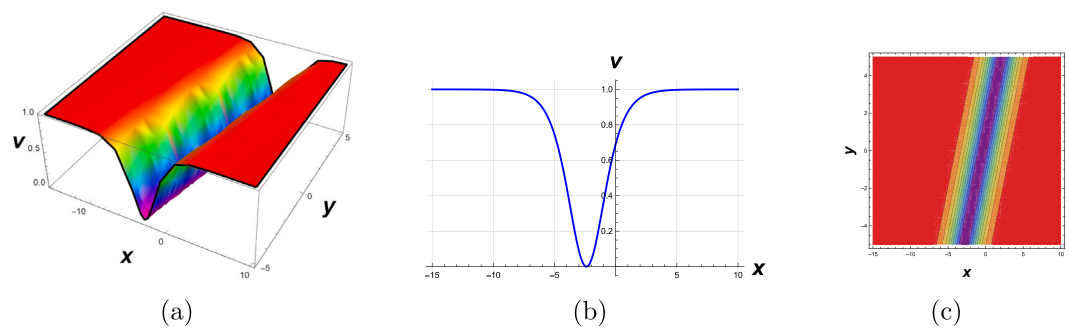

Fig. 2. Solution (39) profile for $a = 0.5$, $b = -0.25$ and $p_2 = -1$ with other parameters set to one. (a) 3D plot. (b) 2D plot. (c) Contour plot.

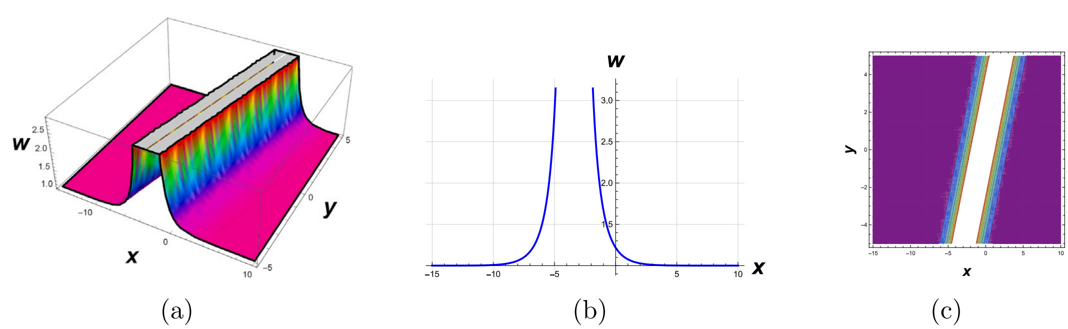

Fig. 3. Wave structure of solution (40) for $a = 0.5$, $b = 0.25$ and $p_1 = -0.5$ where the remaining parameters are chosen to be one. (a) 3D plot. (b) 2D plot. (c) Contour plot.

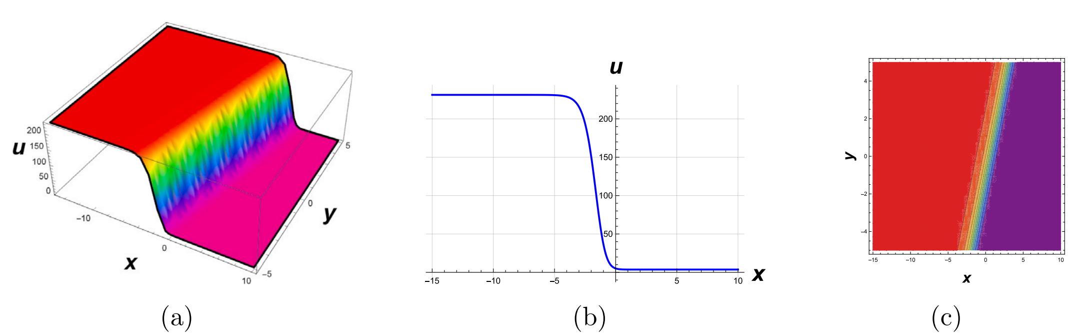

Fig. 4. Profile of solution (50) for $a = 0.25$, $c = 0.1$ and $p_6 = -1$ while the values of the other parameters are set to unity. (a) 3D plot. (b) 2D plot. (c) Contour plot.

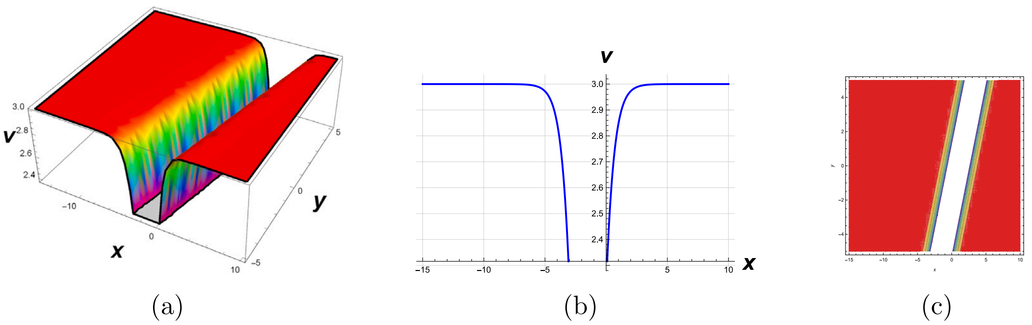

Fig. 5. Wave structure of solution (51) for $a = 0.25$, $b = 0.5$, $c = -0.5$ and $p_2 = -1$ with other parameters set to unity. (a) 3D plot. (b) 2D plot. (c) Contour plot.

exist.

Proof. A tedious but straightforward computations from Eqs. (57)–(59).

Lemma 1. *Suppose the extended (2+1)-dimensional new coupled KdV system (2) admits zeroth-order multiplier functions of the form*

$$\Lambda_1(t,x,y,u,v,w) = 0,\ \Lambda_2(t,x,y,u,v,w) = 0,\ \Lambda_3(t,x,y,u,v,w) = C(t,x,y), \tag{63}$$

then there exists a linear third-order system of equations given by

$$\begin{cases} p_6 C_x + p_7 C_y = 0, \\ p_1 C_{xxx} - C_t = 0, \end{cases} \tag{64}$$

such that Eq. (2) possesses an infinite local conservation law of the form

$$\begin{cases} T_1^t = wC, \\ T_1^x = -(p_6 uv + p_1 w_{xx})C + p_1 w_x C_x - p_1 w C_{xx}, \\ T_1^y = -p_7 uvC. \end{cases} \tag{65}$$

Lemma 2. *Let the extended (2+1)-dimensional new coupled KdV system (2) establish a nontrivial differential functions of zeroth-order, namely*

$$\Lambda_1(t,x,y,u,v,w) = A(t,x,y),\ \Lambda_2(t,x,y,u,v,w) = 0,\quad \Lambda_3(t,x,y,u,v,w) = 0, \tag{66}$$

then the associated infinite conserved vectors are

$$\begin{cases} T_2^t = uA, \\ T_2^x = -(p_2 uv + p_1 u_{xx} + p_4 vw)A + p_1 u_x A_x - p_1 u A_{xx}, \\ T_2^y = -(p_3 u + p_5 w)vA. \end{cases} \tag{67}$$

provided the linear system of equations

$$\begin{cases} p_4 A_x + p_5 A_y = 0, \\ (p_2 p_5 - p_3 p_4)A_x = 0, \\ p_1 A_{xxx} - A_t = 0, \end{cases} \tag{68}$$

holds.

Lemma 3. *Assume that the extended (2+1)-dimensional new coupled KdV system (2) formally admits zeroth-order multiplier functions of the form*

$$\Lambda_1(t,x,y,u,v,w) = 0,\ \Lambda_2(t,x,y,u,v,w) = B(t,x,y),\ \Lambda_3(t,x,y,u,v,w) = 0, \tag{69}$$

then there exists a linear third-order system of equations given by

$$\begin{cases} p_6 B_x + p_7 B_y = 0, \\ p_1 B_{xxx} - B_t = 0, \end{cases} \tag{70}$$

and the corresponding infinite conserved vectors are

$$\begin{cases} T_3^t = vB, \\ T_3^x = -(p_6 uw + p_1 v_{xx})B + p_1 v_x B_x - p_1 v B_{xx}, \\ T_3^y = -p_7 uwB. \end{cases} \tag{71}$$

Proof. The proofs of the above Lemmas are straightforward, but a bit lengthy. They consist of the application of the divergence equation $D_t T_i^t + D_x T_i^x + D_y T_i^y = 0$ for $(i = 1,2,3)$ which holds for all solutions of system (2).

5. Concluding remarks

In this work exact solutions of the extended (2+1)-dimensional new coupled KdV system (2) were derived using the symmetry method. Other analytical solutions were obtained with the aid of Bernoulli and Riccati equations as the simplest equation method. In order to have an in-depth conception of the derived results a variety of profile structures which includes among others the bell shaped, inverted bell shaped, kink and singular wave solutions were obtained. It is observed that the aforementioned system admits infinitely many conservation laws. The correctness of the obtained results have been verified with the aid of Maple software package.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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