

OPTIMAL PROVISIONING FOR DEPOSIT WITHDRAWALS AND LOAN LOSSES IN THE BANKING INDUSTRY

F. Gideon, M.Sc

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Figure 1: Overview of the Basel II Capital Accord

Supervisor: Prof. Mark A. Petersen

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Abstract

With the acceptance of the new Basel II banking regulation (implemented in South Africa in January 2008) the search for improved ways of modeling the most important banking activities has become very topical. Since the notion of Lévy-process was introduced, it has emerged as an important tool for modeling economic variables in a Basel II framework. In this study, we investigate the stochastic dynamics of banking items that are driven by such processes. In particular, we discuss bank provisioning for loan losses and deposit withdrawals.

The first type of provisioning is related to the earnings that the bank sets aside in order to cover loan defaults. In this case, we apply principles from robustness to a situation where the decision maker is a bank owner and the decision rule determines the optimal provisioning strategy for loan losses. In this regard, we formulate a dynamic banking loan loss model involving a provisioning portfolio consisting of provisions for expected losses and loan loss reserves for unexpected losses. Here, unexpected loan losses and provisioning for expected losses are modeled via a compound Poisson process and an exponential Lévy process, respectively. We use historical evidence from OECD (Organization for Economic Corporation and Development) countries to support the fact that the provisions for loan losses-to-total assets ratio is negatively correlated with aggregate asset prices and the private credit-to-GDP ratio.

Secondly, we construct models for provisioning for deposit withdrawals. In particular, we build stochastic dynamic models which enable us to analyze the interplay between deposit withdrawals and the provisioning for these withdrawals via Treasuries and reserves. Further insight is gained by considering a numerical problem and a simulation of the trajectory of the stochastic dynamics of the sum of the Treasuries and reserves. Since managing the risk that depositors will exercise their withdrawal option is an important aspect of this thesis, we consider the idea of a hedging provisioning strategy for deposit withdrawals in an incomplete market setting. In this spirit, we discuss an optimal risk management problem for a commercial bank whose main activity is to obtain funds through deposits from the public and use the Treasuries and reserves to cater for the resulting withdrawals. Finally, we provide a brief analysis of some of the issues arising from the dynamic models of the banking items derived.

KEYWORDS: Banks, Mixed Optimal/Robust Control; Expected and Unexpected Loan Losses; Loan Loss Provisioning; Loan Loss Reserves, Dynamics Modeling, Deposit Withdrawals, Lévy process .

2000 AMS SUBJECT CLASSIFICATION: 60G44, 90A09, 93B15.

Opsomming

Met die aanvaarding van die nuwe Basel II regulasies (vir implementering in Suid-Afrika in Januarie 2008) het die soeke na beter maniere om banke se belangrikste bedrywighede te modelleer meer belangrik geword. Lévy prosesse het ontpop as 'n belangrike wapen in die stryd om ekonomiese veranderlikes in 'n Basel II raamwerk te modelleer. In hierdie studie ondersoek ons die stogastiese dinamika van bankaktiwiteite wat deur hierdie prosesse gedryf word. In besonder, bespreek ons hoe banke voorsorg tref vir leningverliese en deposito ontrekkings.

Ons begin die studie deur te kyk na die verliese wat banke ly ten opsigte van lenings wat kliënte nie kan terugbetaal nie. Ons toets sekere afleidings oor die vraag na en die aanbod van lenings deur banke en kyk dan ook na die voorsorg wat banke tref om negatiewe gevolge te minimaliseer. Dit stel ons in staat om te kyk na hoe 'n mens waarde kan heg aan 'n sekere bank. Basel II gee sekere voorskrifte oor hierdie modelle en dit word hier in ag geneem.

Hierdie proefskrif beskou 'n manier om bank aktiwiteite soos byvoorbeeld die uitreik van lenings te modelleer deur te kyk na die sogenaamde Lévy proses. Hierdie proses word bestudeer omdat daar kritiek bestaan teen die algemeen gebruikte Brown se beweging wat beskou word as onvoldoende om realiteit te simuleer. Ons lei stogastiese differensiaal vergelykings af vir die bank se hoof balansstaat items om sodoende dan die kapitaal van die bank te simuleer. Basel II gee voorskrifte oor die vlak van kapitaal wat banke moet handhaaf vir tye waarin ekonomiese aktiwiteite afneem. Dit is dus vir ons belangrik om te kyk na die kapitaalberekenings proses siende dat dit ingevolge Basel II voorskrifte gebruik word om skokke te kan absorbeer.

Vervolgens kyk ons na die voorsorg wat getref word vir slegte skuld en die sikliese patroon van kapitaal van ontwikkelde lande sowel as die van Suid-Afrika. Ons vergelyk die verskil tussen die werklike produksie van lande soos gemeet deur die Bruto Binnelandse Produk (BBP) met dit wat hulle produksie potensiaal is. Hieruit kan ons belangrike gevolgtrekkings maak aangaande die siklusse wat kapitaal volg.

Ons beskou al die analise wat gedoen is in die tesis en kyk of dit aangepas kan word vir sekere uitsonderings.

Acknowledgements

In presenting this thesis, the result of study within the School of Computer, Mathematical and Statistical Sciences at the Potchefstroom Campus of the North-West University (NWU-PC), I wish to express my warmest thanks to those who have given me assistance and encouragement during my time here.

First of all, I am indebted to my supervisor, Prof M.A. Petersen from the Department of Mathematics and Applied Mathematics at NWU-PC for the guidance provided during the completion of this thesis. Such progress as I have made in the present enquiry owes much to the useful and regular discussions I have had with him. I have to thank him for his unfailing enthusiasm, encouragement and genuine interest in my academic work. He not only guided me through the thesis, but also equipped me with life-long research skills.

Above all, I would like to thank Almighty God for His grace in enabling me to complete this thesis. Thank you to my fiance, Hendrina Haidula for her encouragement and great support she has provided during this difficult time. Furthermore, I would like also to acknowledge the emotional support provided by my immediate family: Aina, Laimi and Gideon.

I am also grateful to the American-African Institute (AAI) for the financial assistance they have provided and the University of Namibia for recommending my studies for financial support.

Preface

One of the contributions made by the NWU-PC to the activities of the stochastic analysis community has been the establishment of an active research group that has an interest in institutional finance. In particular, this group has made contributions about modeling, optimization, regulation and risk management in insurance and banking. Students who have participated in projects in this programme under Prof. Petersen's supervision are listed below.

Level	Student	Graduation	Title
MSc	T Bosch	May 2003	Controllability of HJMM Interest Rate Models
MSc	CH Fouche	May 2006	Continuous-Time Stochastic Modelling of Capital Adequacy Ratios for Banks
MSc	MP Mulaudzi	May 2008	A Decision Making Problem in the Banking Industry
PhD	CH Fouche	May 2008	Dynamic Modeling of Banking Activities
PhD	F Gideon	Sept. 2008	Optimal Provisioning for Loan Losses and Deposit Withdrawals in the Banking Industry
PhD	T Bosch	May 2009	Optimal Auditing in the Banking Industry
MSc	MC Senosi	May 2009	Discrete Dynamics of Bank Credit and Capital and their Cyclicity
PhD	BA Tau	Current	Maximizing Banking Profit on a Fixed and Random Time Interval
PhD	MP Mulaudzi	Current	Lévy Process Based Models in Banking
PhD	MC Senosi	Current	Discrete-Time Modeling of the Basel II Capital Accord
Postdoc	J Mukuddem-Petersen	2006-8	Health Economics

Declaration

I declare that, apart from the assistance acknowledged, the research contained in the thesis is my own unaided work. It is being submitted in partial fulfilment of the requirements for the degree Philosophiae Doctor in Applied Mathematics at the Potchefstroom Campus of the North West University. It has not been submitted before for any degree or examination to any other University.

Nobody, including Prof. M.A. Petersen (Supervisor), but myself is responsible for the final version of this thesis.

Signature.....

Date.....

Index of Abbreviations

CAR - Capital Adequacy Ratio;
 OECD - Organization for Economic Corporation and Development;
 LLP - Loan Loss Provision;
 GDP - Gross Domestic Product;
 SDSTR - Stochastic Dynamics of the Sum of Treasuries and Reserves;
 PD - Probability Default;
 LGD - Loss Given Default;
 NPL - Non-Performing Loans;
 TA - Total Assets;
 NDISC - Non-Discretionary Component;
 DISC - Discretionary Component;
 VaR - Value-at-Risk;
 GKW - Galtchouck-Kunita-Watanabe;
 TCR - Total Capital Ratio;
 CIR - Cox, Ingersoll and Ross Process ;
 OU - Ornstein-Uhlenbeck Process.

Index of Symbols

$SIGN_{it}$ - One-Year-Ahead Changes of Earnings Before Taxes and Loan Loss Provisions;
 ER_{it} - Positive Correlation Between Earnings Before Taxes and Loan Loss Provisions;
 y'_{it} - Annual Growth Rate of GDP;
 P_{it} - Ratio of Loan Loss Provisions to Total Assets at the End of Year t for Bank i ;
 Λ - Loans;
 T - Treasuries;
 R - Reserves;
 K - Capital;
 L - Lévy Process;
 Δ - Deposits;
 ϕ - Characteristic Function of a Distribution;
 ψ - Lévy or Characteristic Exponent of L ;
 x - Variable;
 γ - Drift of a Stochastic Process;
 X - Value Process;
 Z - Standard Brownian Motion;

$Q(dt, dx)$ - Poisson Measure;
 ν - Lebesgue Measure;
 M - Martingale;
 ξ - Doléans-Dade Exponential;
 P - Total Provisioning for Loan Losses;
 A - Assets;
 r^Λ - Loan Rate;
 c^d - Default Premium;
 c^a - Administrative Cost;
 S^e - Expected Loan Losses;
 S^u - Unexpected Loan Losses;
 ν - Lévy Measure;
 \mathcal{B} - Borel Set;
 S - Aggregate Loan Losses;
 T - Terminal Time;
 P' - Nett Loan Loss Provisioning;
 ϱ - Nett Instantaneous Return of a Value Process;
 σ - Volatility of a Value Process;
 μ - Mean of a Value Process;
 π - Provisioning Strategy;
 k_d - Depository Value;
 D - Depository Contracts;
 L_{it} - Provisions for Loan Losses-to-Total Assets Ratio;
 n^T - Number of Treasuries;
 n^R - Number of Reserves;
 $\widehat{V}(\pi)$ - Provisioning Portfolio Value Process;
 c^o - Cost Process;
 Λ^c - Probability of Insolvency to Occur;
 C_T - Cost of Insolvency;
 N^L - Number of Loan Losses;
 l - Unexpected Loan Losses Sizes;
 r^R - Loan Loss Reserve Rate;
 P^π - Total Loan Loss Provisioning Under Strategy π ;
 R^l - Loan Loss Reserve;
 P'^π - Net Loan Loss Provisioning Under π ;
 r^R - Deterministic Rate of (Positive) Return on Reserves;
 f^R - Fraction of the Reserves Consumed by Deposit Withdrawals;
 σ^R - Volatility in the Level of Reserves;

G - Girsanov Parameter;
 \mathbb{Q}_g - Risk Neutral Martingale Measure related to the Kunita-Watanabe Measure;
 $M^{\mathbb{Q}}(dt, dx)$ - Compensated Jump Measure of L^R Under \mathbb{Q}_g ;
 W - Sum of Treasuries and Reserves;
 D^c - Sum of Cohort Deposits;
 w_{x+t} - Withdrawal Rate Function;
 N^I - Number of Withdrawals;
 M^I - Compensated Counting Process;
 w^{un} - Unanticipated Deposit Withdrawals;
 $f(w^{un})$ - Probability Density Function;
 c^l - Cost of Liquidation;
 r_t^P - Penalty Rate on Deposit Withdrawals;
 $c^{w^{un}}$ - Cost of Deposit Withdrawals;
 ρ^r - Relative Risk Ratio.

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Chapter 1

INTRODUCTION TO BANK PROVISIONING

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1.4 OUTLINE OF THE THESIS

1.1 INTRODUCTORY REMARKS

Bank provisioning for loan losses and deposit withdrawals are modern economic risk measuring techniques in the banking industry. In this regard, we discuss bank optimal provisioning strategies for both loan losses and deposit withdrawals.

1.1.1 Introduction to Robustness, Loan Losses and Lévy Processes

Principles from robust control theory have been used by economic decision makers to investigate the fragility of decision rules across a range of economic models. In line with this tendency, Chapter 3 and part of Chapter 6 applies principles from robustness to a situation where the decision maker is a bank owner and the decision rule determines the optimal provisioning strategy for loan losses.

In the sequel, we describe robust control theory in the following ways. Firstly, it can be regarded as a method to measure the *efficiency*¹ of models with changing parameters. Another characterization of robust control theory is that it is the control of models with uncertain dynamics subject to uncertain disturbances. In these cases, the common objective is often to explore the model design for alternatives that are insensitive to changes and that maintain stability and efficiency. As is evidenced by subsequent discussions, if we are given a robustness constraint on the uncertainty, then the model can deliver results (subject to an appropriate decision rule) that meet the requirements in all cases. It must be recognized that the overall efficiency may be compromised in order to guarantee that the model meets certain requirements. The next issue is how to choose a decision rule and to explain what it means for such a rule to *meet requirements* across models. In Bayesian analysis, the decision maker forms a prior over models and subsequently maximizes expected utility (or minimizes expected loss) by *averaging over models*². On the other hand, applications of robust control theory usually involve the minimization of a *worst case scenario*³ over the set of possible models. As a consequence, stochastic robust control problems either consider the cases where shocks are averaged over or where they are not as for worst case losses. The robust control approach thus accepts that decision makers are not

¹Efficiency is a state or quality of a model that performs at a level that is acceptable or desirable.

²Model averaging refers to the act of computing the average value of a parameter or a cost function over a set of possible models.

³The worst case scenario represents the worst possible environment or outcome out of the several possibilities in planning or simulation. For our situation, this may translate into a minimax problem in terms of losses or max-min expected utility.

able nor willing to form a prior over the forms of model specification. Despite this, decision makers must be able to specify the set of models which normally involves bounding the set of possibilities instead of specifying each alternative. The main objective of Chapter 3 of this thesis is to solve an optimal robust control problem with constraint for banks in a Lévy process framework. More specifically, we would like to obtain an optimal value for the process of provisioning for loan losses subject to a certain robustness constraint (involving risk) on the variance of this process. Our robust control approach represents a compromise between the aforementioned average and worst case paradigms in that it maximizes expected utility subject to a bound on the worst case scenario.

The economic health of a bank depends not only on its investment in loans, but also on how well it provisions for expected and unexpected losses from such loans. The need to create provisions arises because the loans are not recorded at market value, typically because imputed market values are either empirically difficult to obtain due to an absence of traded markets or a reliance on judgemental assumptions. Bad and doubtful bank loans defines two categories of loan loss provisions (LLP). *Specific provisions* are made for debts which have been identified as impaired or non-performing. *General provisions* are made for those doubtful debts which may turn out to be non-performing on the basis of historical performance or current economic conditions, although debt servicing is currently taking place. The distinction is that specific provisions are made for losses which have actually already occurred whereas general provisions are made for those loans which may occur in the future. Banks increase their provision for loan losses in order to write off loans closer to market values; the loan should be valued at the price it would command if traded in the open market. In this thesis, we are mainly interested in the latter type of debt.

Two problems are identified in the treatment of LLP under the Basel II Accord (see, for instance, [5] and [7]). It is suggested that a distinction should be made between expected and unexpected loan losses and that the former should be treated in the same way that identified losses are. That is, they should not be allowed to count as capital. However, capital should be maintained as a buffer stock available to absorb unexpected losses. Our argument is in line with Basel II in that provisions against expected losses should not count as capital and that they should be dynamically correlated with the rate at which loans are granted. On the other hand, loan loss reserves (Tier 2 Capital) should act as a first precautionary buffer against unexpected declines in asset values and should be a constituent of the provisioning portfolio along with provisions for unexpected loan losses. In particular, loan loss reserves fulfill the important role of being an estimate of future loan losses. The periodic provision is an outflow that supports the level of reserve. When loans are charged off, they deplete

the loan loss reserve. The periodic provision is an important discretionary managerial activity that is driven by sufficiency of the loan loss reserve.

We know that the current value of loans recorded on the balance sheet is equal to the bank's recorded investment (i.e., the amount outstanding or face value) less a provision for loan losses. It is an empirical fact that as macroeconomic conditions improve (deteriorate), the current value of loans increase (decrease) while there is a decrease (increase) in the provisions for expected loan losses. In other words, a strong negative correlation between the current value of loans and the provisions for expected losses exists (see, for instance, [16] for such evidence). This trend is reflected, for instance, in Figure 1.1, for the graphical representation of real aggregate asset prices and the total private credit-to-GDP ratio vs provisions for loan losses-to-total assets ratio for Spain.

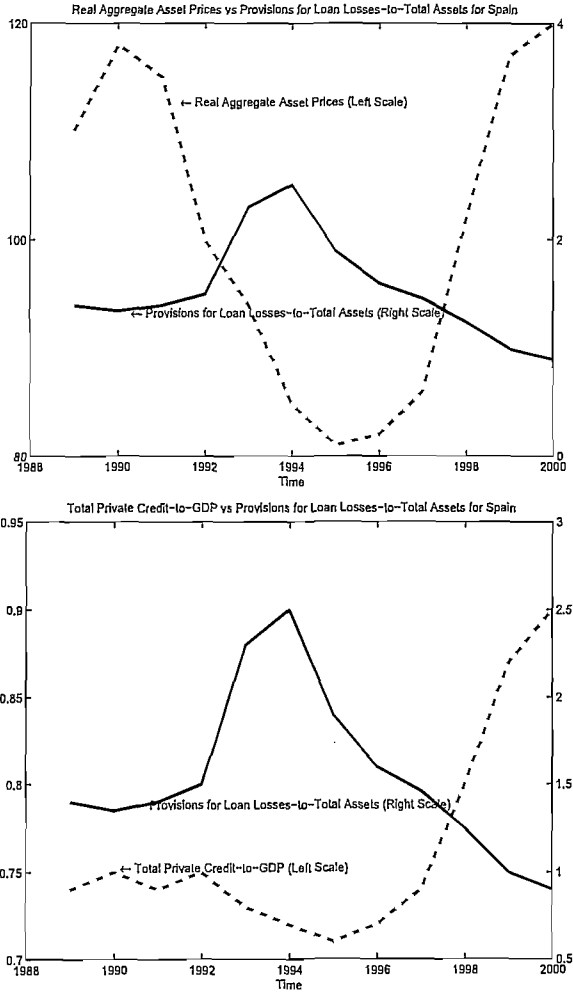


Figure 1.1: Real Aggregate Asset Prices and Total Private Credit-to-GDP Ratio vs Provisions for Loan Losses-to-Total Assets Ratio for Spain

Similar trends can also be confirmed in other OECD countries such as Australia, Norway, the United Kingdom and United States (see, for instance, Section 2.1.3 and the report [16]). As it is common to model the current value of loans via exponential Lévy processes, it is reasonable to assume that the dynamics of the provisions for expected loan losses can also be modeled in the same way (see, for instance, [26]).

In the recent past, more attention has been given to modeling procedures that deviate from those that rely on the seminal Black-Scholes financial model (see, for instance, [66] and [67]). Some of the most popular and tractable of these procedures are related to Lévy process-based models. In this regard, our thesis investigates the dynamics of banking items such as loans, reserves, capital and regulatory ratios that are driven by such processes. An advantage of Lévy-processes is that they are very flexible since for any time increment Δt any infinitely divisible distribution can be chosen as the increment distribution of periods of time Δt . In addition, they have a simple structure when compared with general semimartingales and are able to take different important stylized features of financial time series into account. A specific motivation for modeling banking items in terms of Lévy processes is that they have an advantage over the more traditional modeling tools such as Brownian motion (see, for instance, [29], [40], [63] and [84]), since they describe the non-continuous evolution of the value of economic and financial indicators more accurately. Our contention is that these models lead to analytically and numerically tractable formulas for banking items that are characterized by jumps. An important issue related to the dynamics of the provisioning and associated loan losses is whether traditional geometric Brownian motion appropriately describes the development of the provisions for expected loan losses and its dependence on the face value of the corresponding loans. Many empirical studies of bank loan portfolios indicate that the log returns of these loans exhibit a number of features which contradict the normality assumption, like skewness and heavy tails. In fact, the empirical distribution of real loan data is often *leptokurtic*. In other words, there are more values close to the mean than a normal law would suggest and that extremes indicating semi-heavy tails occur. This means that the face value of the loan portfolio may have sudden downward (or upward) jumps, which cannot be explained by continuous geometric Brownian motion. One method of solving this problem is to model the provisions for expected loan losses by using a more general exponential Lévy process with jumps. This leads to a generalization of the traditional class of Black-Scholes models by replacing the Wiener process in the classical geometric Brownian motion by a general Lévy process.

If there is a deviation from the Black-Scholes paradigm, one typically enters into the realm of incomplete market models. Most theoretical financial market models are incomplete, with academics and practitioners alike agreeing that "real-world"

markets are also not complete. The issue of completeness goes hand-in-hand with the uniqueness of the martingale measure (see, for instance, [30]). In incomplete markets, we have to choose an equivalent martingale measure that may emanate from the market. For the purposes of our investigation, for bank Treasuries and reserves, we choose a risk-neutral martingale measure, Q_g , that is related to the classical Kunita-Watanabe measure (see [60]). We observe that, in practice, it is quite acceptable to estimate the risk-neutral measure directly from market data via, for instance, the volatility surface. It is well-known that if the (discounted) underlying asset is a martingale under the original probability measure, P , the optimal hedging strategy is given by the Galtchouck-Kunita-Watanabe decomposition as observed in [39]. In the general case, the underlying asset has some drift under P , and the solution to the minimization problem is much more technical as it possesses a feedback component.

1.1.2 Introduction to Treasuries, Reserves and Deposit Withdrawals

We apply the quadratic hedging approach in part of Chapter 2 and Chapter 4 to a situation related to bank deposit withdrawals. In incomplete markets, this problem arises due to the fact that random obligations cannot be replicated with probability one by trading in available assets. For any hedging strategy, there is some residual risk. More specifically, in the quadratic hedging approach, the variance of the hedging error is minimized. With regard to this, our thesis addresses the problem of determining risk minimizing hedging strategies that may be employed when a bank faces deposit withdrawals with fixed maturities resulting from lump sum deposits.

Some banking activities that we wish to model dynamically are constituents of the assets and liabilities held by the bank. With regard to the former, it is important to be able to measure the volume of Treasuries and reserves that a bank holds. Treasuries are bonds issued by a national Treasury and may be modeled as a risk-free asset (bond) in the usual way. In the modern banking industry, it is appropriate to assign a price to reserves and to model it by means of a Lévy process because of the discontinuity associated with its evolution and because it provides a good fit to real-life data. Banks are interested in establishing the level of Treasuries and reserves on demand deposits that the bank must hold. By setting a bank's individual level of reserves, roleplayers assist in mitigating the costs of financial distress. For instance, if the minimum level of required reserves exceeds a bank's optimally determined level of reserves, this may lead to deadweight losses. While the academic literature on pricing bank assets is vast and well developed, little attention is given to pricing bank liabilities. Most bank deposits contain an embedded option which permits the depos-

itor to withdraw funds at will. Demand deposits generally allow costless withdrawal, while time deposits often require payment of an early withdrawal penalty. Managing the risk that depositors will exercise their withdrawal option is an important aspect of our contribution. The main thrust of Section 4.2 is the hedging of bank deposit withdrawals. In this spirit, we discuss an optimal risk management problem for commercial banks who use the Treasuries and reserves to cater for such withdrawals. In this regard, the main risks that can be identified are reserve, depository and intrinsic risk that are associated with the reserve process, the net cash flows from depository activity and cumulative costs of the bank's provisioning strategy, respectively.

1.2 PRELIMINARIES

In the ensuing discussion, for the sake of completeness, we firstly discuss a bank balance sheet, a general description of a Lévy process and an associated measure and then describe the Lévy decomposition that is appropriate for our analysis. We also provide a discussion on a mixed optimal/robust control problem, Ito's Formula and finally, we thrust through the general basic risk concepts. Throughout our contribution, we suppose for the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that $(\Omega, \mathbb{F}, \mathbb{P})$ is a filtered probability space. Subsequently, we use the notational convention "subscript t or s " to represent (possibly) random processes, while "bracket t or s " are used to denote deterministic processes.

1.2.1 Bank Balance Sheet

In this subsection, we provide some basic facts on bank balance sheet necessary for dealing with Section 2.2 in Chapter 2. In this regard, we describe a bank balance sheet equation at time t as

$$\Lambda_t + T(t) + R_t = \Delta_t + K_t$$

where Λ , T , R , Δ and K are loans, Treasuries, reserves, deposits, and bank capital, respectively. In this case, K_t consists of Tier 1 (bank's equity, E , plus retained earnings E_r) , Tier 2 and Tier 3 (subordinate debt, S , and loan-loss reserve, R^l) capital.

1.2.2 General Properties of Lévy Processes

This subsection on general properties of Lévy processes is useful for the dynamic bank models discussed in Chapters 2, 3 and 4. We start with a number of key definitions.

Definition 1.2.1 (Infinitely Divisible Distribution): *Assume that $\phi(\xi)$ is the characteristic function of a distribution, ξ . If for every positive integer n , $\phi(\xi)$ is also the n -th power of a characteristic function, we say that the distribution is infinitely divisible.*

Definition 1.2.2 (Lévy Process): *For each infinitely divisible distribution, a stochastic process $L = (L_t)_{0 \leq t}$ called a Lévy process exists. This process*

- *initiates at zero,*
- *has independent and stationary increments and*
- *has $(\phi(u))^t$ as a characteristic function for the distribution of an increment over $[s, s + t]$, $0 \leq s, t$, such that*

$$L_{t+s} - L_s.$$

Every Lévy process is a semi-martingale and has a càdlàg version (right continuous with left hand limits) which is itself a Lévy process. We will assume that the type of such processes that we work with are always càdlàg. As a result, sample paths of L are continuous a.e. from the right and have limits from the left.

Definition 1.2.3 (Jump of a Lévy Process): *The jump of L_t at $t \geq 0$ is defined by*

$$\Delta L_t = L_t - L_{t-}.$$

Definition 1.2.4 (Characteristic Exponent of a Lévy Process): *Since L has stationary independent increments its characteristic function must have the form*

$$\mathbb{E}[\exp\{-i\xi L_t\}] = \exp\{-t\Psi(\xi)\}$$

for some function Ψ called the Lévy or characteristic exponent of L .

Definition 1.2.5 (Lévy-Khintchine Formula): *The Lévy-Khintchine formula is given by*

$$\begin{aligned} \Psi(\xi) = & i\gamma\xi + \frac{c^2}{2}\xi^2 + \int_{|x|<1} \left[1 - \exp\{-i\xi x\} - i\xi x \right] \nu(dx) \\ & + \int_{|x|\geq 1} \left[1 - \exp\{-i\xi x\} \right] \nu(dx), \quad \gamma, c \in \mathbb{R} \end{aligned} \quad (1.1)$$

and for some σ -finite measure ν on $\mathbb{R} \setminus \{0\}$ with

$$\int \inf\{1, x^2\} \nu(dx) = \int (1 \wedge x^2) \nu(dx) < \infty.$$

Definition 1.2.6 (Lévy Triplet and Measure): *An infinitely divisible distribution has a Lévy triplet of the form*

$$[\gamma, c^2, \nu]$$

where the measure ν is called the Lévy measure.

The Lévy-Khintchine formula given by equation (1.1), is closely related to the Lévy process, L . This is particularly true for the *Lévy decomposition* of L . In particular, the description of the decomposition of the Lévy process, L , (see the Lévy-Khintchine representation (1.6) below) corresponds with that of [85, Chapter 4]. This decomposition is described in the rest of this paragraph. From equation (1.1) above, it is clear that L must be a linear combination of a Brownian motion and a quadratic jump process X which is independent of the Brownian motion.

Definition 1.2.7 (Quadratic Pure Jump Process): *A process is classified as quadratic pure jump if the continuous part of its quadratic variation $\langle X \rangle^c \equiv 0$, so that its quadratic variation becomes*

$$\langle X \rangle_t = \sum_{0 < s \leq t} (\Delta X_s)^2,$$

where $\Delta X_s = X_s - X_{s-}$ is the jump size at time s .

If we separate the Brownian component, Z , from the quadratic pure jump component X we obtain

$$L_t = X_t + cZ_t,$$

where X is quadratic pure jump and Z is standard Brownian motion on \mathbb{R} . Next, we describe the Lévy decomposition of Z . Let $Q(dt, dx)$ be the Poisson measure on $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$ with expectation (or intensity) measure $dt \times \nu$. Here dt is the Lebesgue measure and ν is the Lévy measure as before. The measure $dt \times \nu$ (or sometimes just ν) is called the compensator of Q . The Lévy decomposition of X specifies that

$$\begin{aligned} X_t &= \int_{|x|<1} x \left[Q((0, t], dx) - t\nu(dx) \right] + \int_{|x|\geq 1} x Q((0, t], dx) + t\mathbb{E} \left[X_1 - \int_{|x|\geq 1} x\nu(dx) \right] \\ &= \int_{|x|<1} x \left[Q((0, t], dx) - t\nu(dx) \right] + \int_{|x|\geq 1} x Q((0, t], dx) + \gamma t, \end{aligned} \quad (1.2)$$

where

$$\gamma = \mathbb{E} \left[X_1 - \int_{|x|\geq 1} x\nu(dx) \right].$$

The parameter γ is called the *drift of X* . In addition, in order to describe the Lévy decomposition of L , we specify more conditions that L must satisfy. The most important supposition that we make about L is that

$$\mathbb{E}[\exp\{-hL_1\}] < \infty, \quad \text{for all } h \in (-h_1, h_2), \quad (1.3)$$

where $0 < h_1, h_2 \leq \infty$. This implies that L_t has finite moments of all orders and in particular, $\mathbb{E}[X_1] < \infty$. In terms of the Lévy measure ν of X , we have, for all $h \in (-h_1, h_2)$, that

$$\begin{aligned} \int_{|x| \geq 1} \exp\{-hx\} \nu(dx) &< \infty; \\ \int_{|x| \geq 1} x^\alpha \exp\{-hx\} \nu(dx) &< \infty, \quad \forall \alpha > 0; \\ \int_{|x| \geq 1} x \nu(dx) &< \infty. \end{aligned}$$

The above assumptions lead to the fact that equation (1.2) can be rewritten as

$$X_t = \int_{\mathbb{R}} x \left[Q((0, t], dx) - t\nu(dx) \right] + t\mathbb{E}[X_1] = M_t + at,$$

where we have that

$$M_t = \int_{\mathbb{R}} x \left[Q((0, t], dx) - t\nu(dx) \right]$$

is a martingale and $a = \mathbb{E}[X_1]$.

In the specification of our model, we assume that the Lévy measure $\nu(dx)$ of L satisfies

$$\int_{|x| > 1} |x|^3 \nu(dx) < \infty. \quad (1.4)$$

As in the above, this allows a decomposition of L of the form

$$L_t = cZ_t + M_t + at, \quad 0 \leq t \leq T, \quad (1.5)$$

where $(cZ_t)_{0 \leq t \leq \tau}$ is a Brownian motion with standard deviation $c > 0$, $a = \mathbb{E}(L_1)$ and the martingale

$$M_t = \int_0^t \int_{\mathbb{R}} x M(ds, dx), \quad 0 \leq t \leq T,$$

is a square-integrable. Here, we denote the *compensated Poisson random measure* on $[0, \infty) \times \mathbb{R} \setminus \{0\}$ related to L by $M(dt, dx)$. Subsequently, if $\nu = 0$ then we will have that $L_t = Z_t$, where Z_t is appropriately defined Brownian motion.

Definition 1.2.8 (Stochastic Exponential): *For the Lévy process, L , that initiates at zero, the stochastic exponential of L (written as $\xi(L)$) is the (unique) Lévy process \widehat{L} that is a solution of*

$$\widehat{L}_t = 1 + \int_0^t \widehat{L}_{t-} dL_t.$$

The stochastic exponential is also known as the *Doléans-Dade exponential* (see, for instance, [81]). The contribution [47] indicates that if L is real-valued and has a characteristic triplet, (γ, σ^2, ν) , then it follows that the Doléans-Dade exponential, \widehat{L} , is a Lévy process with characteristic triplet $(\widehat{\gamma}, \widehat{\sigma}^2, \widehat{\nu})$. In this case, we have that

- $\widehat{\gamma} - \gamma = \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left[\left(\exp\{x\} - 1 \right) 1_{\{|\exp\{x\} - 1| < 1\}} - x 1_{\{|x| < 1\}} \right] \nu(dx);$
- $\widehat{\sigma}^2 = \sigma^2;$
- $\widehat{\nu}(\varphi) = \nu\left(\left\{x \in \mathbb{R} \mid \exp\{x\} - 1 \in \varphi\right\}\right)$ for any Borel set $\varphi \subset \mathbb{R}^*$.

In the book [85, Chapter 4], the following literature holds on the discussion of Lévy process theory. For all ω in the probability space denote by $\Delta L(t, \omega) = L(t, \omega) - L(t-, \omega)$ the jump of the process L at time $t > 0$. For all Borel sets $\mathcal{B} \subset [0, \infty) \times \mathbb{R} \setminus \{0\}$ set

$$M(\mathcal{B}, \omega) = \{(t, \Delta L(t, \omega)) \in \mathcal{B}\}.$$

Lévy theory dictates that M is a Poisson random measure with intensity

$$m(dt, dx) = dt\nu(dx),$$

where ν is the Lévy measure of the process L . Note that m is σ -finite and $M(\mathcal{B}, \cdot) = \infty$ a.s. when $m(\mathcal{B}) = \infty$. For $\mathcal{B} = [t_1, t_2] \times \mathcal{J}$, where $0 \leq t_1 < t_2 < \infty$ and a Borel set $\mathcal{J} \subset \mathbb{R} \setminus \{0\}$. Then

$$M(\mathcal{B}, \omega) = \{(t, \Delta L_t(\omega)) : t \in [t_1, t_2], \Delta L_t(\omega) \in \mathcal{J}\}$$

counts the jump size in \mathcal{J} which happens in the time interval $t_1 \leq t \leq t_2$. Therefore, $M(\mathcal{B}, \omega)$ is a Poisson random variable with mathematical expectation $(t_2 - t_1)\nu(\mathcal{J})$. Now, the Lévy-Khintchine representation in equation (1.1) corresponds to the representation

$$L_t = \gamma t + \sigma Z_t + \sum_{0 < s \leq t} \Delta L_s 1_{\{|\Delta L_s| > 1\}} + \int_0^t \int_{|x| \leq 1} x(M(ds, dx) - \nu(dx)ds), \quad t \geq 0. \quad (1.6)$$

In the case of finite variation of the jumps, i.e., when $\int_{\{|x| \leq 1\}} |x|\nu(dx) < \infty$, the last representation reduces to

$$L_t = \gamma_0 t + \sigma Z_t + \sum_{0 < s \leq t} \Delta L_s, \quad t \geq 0,$$

where

$$\gamma_0 = \gamma - \int_{\{|x| \leq 1\}} x\nu(dx).$$

This implies that L is the independent sum of a drift, a Gaussian component and a pure jump part represented by a process of finite variation. For instance, standard Brownian motion is obtained if we choose $\gamma = 0$ and $\nu = 0$ in equation (1.1). A homogeneous compound Poisson process

$$\sum_{j=1}^{N(t)} l_j, \quad t \geq 0, \quad \text{with intensity } \lambda > 0$$

of the Poisson process N has Lévy measure

$$\nu(dx) = \lambda F(dx),$$

where F is the common distribution function of an iid (independent and identically distributed) sequence of random variables $(v_j)_{j \in \mathbb{N}}$, and constants $\gamma = \sigma = 0$. The Lévy process L has finite mathematical expectation if

$$\int_{\{|x|>1\}} |x| \nu(dx) < \infty.$$

In this case, we have that

$$\mathbb{E}[L_t] = \gamma^1 t,$$

where

$$\gamma^1 = \gamma + \int_{\mathbb{R}} x(1 - 1_{\{|x| \leq 1\}}) \nu(dx)$$

(see, for example, [85]).

1.2.3 Mixed Optimal/Robust Control Problems

Our intention in this subsection is merely to provide a description of the class of mixed optimal/robust control problems that our main results in Chapters 2 and 3 may be broadly associated with. We have mentioned before that this thesis may be aligned with the interpretation of robust control in [90], where a compromise between the average and worst case approaches introduced in Subsection 1.3.1 is presented. This approach maximizes expected terminal provisioning subject to a bound on the worst case scenario. Below, we present a robust control problem with a constraint that may be solved in a mixed optimal/robust control framework (see, for example [90]).

Suppose that $(L_t)_{0 \leq t}$ is a Lévy process on the underlying filtered probability space $(\Omega, \mathbb{F}, \mathbb{F}, \mathbb{P})$ and \tilde{L} is a process such that

$$\exp\{L_t\} = \xi(\tilde{L}_t), \quad t \geq 0,$$

where ξ denotes the Doléans-Dade exponential. We represent the dynamics of the state variable, x_t , by the stochastic differential equation

$$dx_t = \mu(\kappa_t, x_{t-})dt + \sigma(\kappa_t, x_{t-})d\tilde{L}_t, \quad (1.7)$$

where the actions of the decisionmaker form a stochastic process $(\kappa_t)_{0 \leq t}$. In this case, a mixed optimal/robust control problem (that may be broadly associated with the problem solved in this thesis) may be formulated as follows.

Problem 1.2.9 (Statement of a Mixed Optimal/Robust Control Problem):

Suppose that the dynamics of the state variable, x_t , is given by equation (1.7) above. Then, for $t \geq 0$, we are able to formulate a mixed optimal/robust control problem on the period $[0, T]$ as

$$\max_{\kappa_t \in C} \mathbb{E}_P \left[f(\kappa_t, x_t) \right] \text{ subject to equation (1.7) and } \text{var}[f(\kappa_t, x_t)] \leq C, \quad (1.8)$$

where terminal time $T < \infty$, C is the admissible control set, f is some real-valued function and C is an upper bound that is measured against the variance, $\text{var}[f(\kappa_t, x_t)]$.

We note that Problem 1.2.9 can also be considered to be a finite horizon mean-variance problem. Furthermore, in the subsequent discussion (compare, for instance, the formulation of Problem 3.1.1 in Chapter 3), we consider the special case of statement (1.8) for which the state variable, x_t , itself is dependent on the control, κ_t , and C is a risk measure. In addition, statement (1.8) provides a link between the max-min expected utility theory (see, for instance, [46]) and applications of robust control.

1.2.4 Itô's Formula

In this subsection, we mention some theorems and lemmas from stochastic calculus and those will play a role in Chapter 2. In the book [12, Chapter 5], the following holds for the definition of X being semimartingale;

Definition 1.2.10 (Definition of a Semimartingale Process): *A semimartingale is a process X_t expressible as*

$$X_t = M_t + A_t^l$$

where M_t a local martingale and A_t^l locally of bounded variation.

Let X denote a semimartingale and H a predictable process. Furthermore, we use directly the notion of a stochastic integral of H with respect to X , of the form

$$(H.X)_t := \int_{0^+}^t H_s dX_s = \int_{[0,t]} H_s dX_s, \quad t \geq 0.$$

Definition 1.2.11 (Definition of Indistinguishable Processes): *Two processes Y and W^I are said to be indistinguishable if*

$$\mathbb{P}(\omega : t \rightarrow Y_t(\omega) \text{ and } t \rightarrow W_t^I(\omega) \text{ are the same functions}) = 1, \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega.$$

Next, we give the following result on the indistinguishability of the jump process that can be found in [81, Theorem 13 of Chapter 2].

Theorem 1.2.12 (Indistinguishability of the Jump Process): *The jump process $(\Delta(H.X))_{t \geq 0}$ is indistinguishable from the process $(H(t)(\Delta X(t)))_{t \geq 0}$.*

We need also the notion of quadratic (co)variation of a semimartingale.

Definition 1.2.13 (Definition of Indistinguishable of the Jump Process): *If X and Y are two semimartingales, the quadratic variation process of X , denoted by $[X, X] = ([X, X]_t)_{t \geq 0}$, is defined by*

$$[X, X]_t = X^2(t) - 2 \int_0^t X(s-) dX(s),$$

and the quadratic covariation of X and Y , denoted $[X, Y] = ([X, Y]_t)_{t \geq 0}$, is defined by

$$[X, Y]_t = X(t)Y(t) - \int_0^t X(s-) dY(s) - \int_0^t Y(s-) dX(s),$$

if they exist (see [81, Chapter 2] for details).

Moreover, we denote by $[X, X]^c$ the path continuous part of $[X, X]$. Then we can write

$$[X, X]_t = X^2(0) + [X, X]_t^c + \sum_{0 < s \leq t} (\Delta X_s)^2.$$

Furthermore, we need the following three theorems and a lemma (for proofs see [81, Section 6 of Chapter 2]).

Theorem 1.2.14 (Formula for Integration by Parts): *Let X and Y be two semimartingales. Then XY is a semimartingale and*

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + d[X, X]_t, \quad t > 0.$$

Theorem 1.2.15 (Quadratic Pure Jump Semimartingale): *Let X be a quadratic pure jump semimartingale. Then for every semimartingale Y we have*

$$[X, Y]_t = X_0 Y_0 + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s.$$

Theorem 1.2.16 (Semimartingales): *Let X and Y be two semimartingales, and let H and G be two predictable processes. Then*

$$[H.X, G.Y]_t = \int_{[0, t]} H_s G_s d[X, Y]_s, \quad t \geq 0,$$

and, in particular,

$$[H.X, H.X]_t = \int_{[0, t]} H_s^2 d[X, X]_s, \quad t \geq 0.$$

Lemma 1.2.17 (Itô's Formula for One-Dimensional Lévy Processes): *Let $(L_t)_{t \geq 0}$ be a one-dimensional Lévy process with characteristic triplet (γ, σ^2, ν) and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 function. Then*

$$\begin{aligned} g(L_t) &= g(0) + \frac{\sigma^2}{2} \int_{0+}^t g''(L_s) ds + \int_0^t g'(L_{s-}) dL_s \\ &\quad + \sum_{0 < s \leq t} [g(L_s) - g(L_{s-}) - \Delta L_s g'(L_{s-})]. \end{aligned}$$

1.2.5 Basic Risk Concepts

This subsection provide a background on basic risk concepts necessary for dealing with dicussions in Section 2.2 and Chapter 4. We assume that the actual provisions for deposit withdrawals are constituted by Treasuries and reserves with price processes $\mathbb{T} = (\mathbb{T}(t))_{0 \leq t \leq T}$ and $R = (R_t)_{0 \leq t \leq T}$, respectively. Suppose that $n_t^{\mathbb{T}}$ and n_t^R are the number of Treasuries and reserves held in the withdrawal provisioning portfolio, respectively. Let $L^2(\mathbb{Q}_{\widehat{R}})$ be the space of square-integrable predictable processes $n^R = (n_t^R)_{0 \leq t \leq T}$ satisfying

$$\mathbb{E}^{\mathbb{Q}} \left\{ \int_0^T (n_s^R)^2 d\langle \widehat{R} \rangle_s \right\} < \infty,$$

where $\widehat{R}_t = \mathbb{T}^{-1}(t)R_t$. For the discounted reserve price, \widehat{R}_t , we call $\pi_t = (n_t^R, n_t^{\mathbb{T}})$, $0 \leq t \leq T$, a *provisioning strategy* if

1. $n^R \in L^2(\mathbb{Q}_{\widehat{R}})$;
2. $n^{\mathbb{T}}$ is adapted;
3. the *discounted provisioning portfolio value process* is the value of the number of reserves and treasuries held in the withdrawals provisioning portfolio,

$$\widehat{V}_t(\pi) = V_t(\pi)\mathbb{T}^{-1}(t); \quad V_t(\pi) = n_t^R R_t + n_t^{\mathbb{T}} \mathbb{T}(t) \in L^2(\mathbb{Q}), \quad 0 \leq t \leq T;$$

4. $\widehat{V}_t(\pi)$ is càdlàg.

The (*cumulative*) *cost process* $c^o(\pi)$ associated with a provisioning strategy, π , is

$$c_t^o(\pi) = \widehat{V}_t(\pi) - \int_0^t n_s^R d\widehat{R}_s, \quad 0 \leq t \leq T.$$

The *intrinsic* or *remaining risk process*, $R(\pi)$, associated with a strategy is

$$R_t(\pi) = \mathbb{E}^{\mathbb{Q}}[(c_T^o(\pi) - c_t^o(\pi))^2 | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

It is clear that this concept is related to the conditioned expected square value of future costs. The strategy $\pi = (n_t^R, n_t^{\mathbb{T}})$, $0 \leq t \leq T$ is *mean self-financing* if its corresponding cost process $c^o = (c_t^o)_{0 \leq t \leq T}$ is a martingale. Furthermore, the strategy

π is *self-financing* if and only if

$$\widehat{V}_t(\pi) = \widehat{V}_0(\pi) + \int_0^t n_u^R d\widehat{R}_u, \quad 0 \leq t \leq T.$$

A strategy $\tilde{\pi}$ is called an *admissible time t continuation* of π if $\tilde{\pi}$ coincides with π at all times before t and $V_T(\tilde{\pi}) = D$ \mathbb{Q} -a.s. Moreover, a provisioning strategy is called *risk minimizing* if for any $t \in [0, T)$, π minimizes remaining risk. In other words, for any admissible continuation $\tilde{\pi}$ of π at t we have

$$R_t(\pi) \leq R_t(\tilde{\pi}), \quad \mathbb{P} - a.s.$$

The contribution [39] shows that a unique risk minimizing provisioning strategy π^D can be found using the generalized GKW decomposition of the *intrinsic value process*, $V^* = (V_t^*)_{0 \leq t \leq T}$, of a *contingent withdrawal*, D , given by

$$V_t^* = \mathbb{E}^{\mathbb{Q}}[D | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[D] = \int_0^t n_s^{RD} d\widehat{R}_s + K_t^D, \quad 0 \leq t \leq T,$$

where $K^D = (K_t^D)_{0 \leq t \leq T}$ is a zero-mean square-integrable martingale, orthogonal to the square-integrable martingale \widehat{R} and $n^{RD} \in L^2(\mathbb{Q}_{\widehat{R}})$. Furthermore, π_t^D is mean self-financing and given by

$$\pi_t^D = (n_t^{RD}, V_t^* - n_t^{RD} \widehat{R}_t), \quad 0 \leq t \leq T.$$

In this case, we have the intrinsic or remaining risk process

$$R_t(\pi^D) = \mathbb{E}^{\mathbb{Q}}[(K_T^D - K_t^D)^2 | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

1.3 RELATIONS TO THE PREVIOUS LITERATURE

Since the notion of Lévy process was introduced by Paul Lévy in the late 1980s (see, for instance [85]), there have been attempts to extend useful financial mathematics notions to a wider setting by replacing the famous continuous-time model (Black-Scholes) with the jump diffusion model (Lévy process-driven). An important development was the study of bank provisioning for loan losses and deposit withdrawals in the field of banking.

With regard to the above, we review some of the literature on Lévy processes, robust

control theory and its connection to loan losses and also Lévy processes with relation to loan losses and deposit withdrawals.

1.3.1 Literature Review on Lévy Processes

This thesis generalizes several aspects of the contribution [40] (see, also, [70], [71] and [73]) by extending the description of bank behavior in a continuous-time Brownian motion framework to one in which the dynamics of bank items may have jumps and be driven by Lévy processes. As far as information on these processes is concerned, Protter in [81, Chapter I, Section 4] and Jacod and Shiryaev in [56, Chapter II] are standard texts (see, also, [11] and [85]). Also, the connections between Lévy processes and finance are embellished upon in [86] (see, also, [58] and [61]).

1.3.2 Literature Review on Robust Control

In recent literature on Lévy processes, robust control theory and its connection with LLP in the banking system some contributions have been made on the aforementioned topic in [26], [81] and [85]. Literature dealing with the relationship between finance and optimal robust control, that we briefly consider, include [4], [52], [53], [54], [64] and [90] (see, also, [80]). A sample of contributions that cover LLP are [3], [65] and [79] (see, also, [31], [48], [49] and [78]).

Robust control started appearing in finance and economics literature in the late 1990's and has been applied very widely since then. For surveys of the development of this research area we refer to the contributions [4] and [52] (see, also, [53]). Furthermore, [54] offers an overview of the leading approach to robust control in economics. This approach is used, for instance, in [64], to present several results related to the application of robust control to the option pricing problem. Firstly, in these contributions, an application of robust control is discussed that obtains a simple option price independent of volatility. The authors then show that when robust control methods are applied to the standard stochastic model, the Black Scholes price arises in an easy way. Finally, the robust approach is used to derive a result demonstrating the validity of the Black Scholes price for systems with stochastic volatility. The paper [87] shows that a finite-horizon version of the robust control criterion appearing in such contributions as [53] and [54] can be described as a recursive utility, which in continuous time takes the form of the stochastic differential utility (SDU) of [35]. The paper [88] analyzes the monetary policy of a central bank in the context of a short run aggregate supply curve. In this regard, an interpretation of robust control

claims that the central bank follows a policy that takes its approximation as the true model, but maximizes the target function that reflects a risk sensitivity. Also, our thesis is broadly consistent with the interpretation of robust control in [90], where a compromise between the average and worst case approaches is made. This mixed optimal/robust control approach maximizes expected utility subject to a bound on the worst case scenario which in our case is related to risk.

1.3.3 Literature Review on Loan Losses

The economic health of a bank depends not only on its investment in loans, but also on how well it provisions for expected and unexpected losses from such loans. The need to create provisions arises because the loans are not recorded at market value, typically because imputed market values are either empirically difficult to obtain due to an absence of traded markets or a reliance on judgemental assumptions. Bad and doubtful bank loans defines two categories of loan loss provisions (LLP). The paper [2] (see, also, [19] and [72]) lays down a methodology for modeling loan loss provisions for banks. This paper exploits the 1990 change in capital adequacy regulations in order to construct more effective tests of capital and earnings management and its effect on bank loan loss provisions. They find support for the hypothesis that loan loss provisions are used for capital management but do not find evidence of earnings management via loan loss provisions. Furthermore, they document the reasons for the conflicting results on these effects observed in prior studies. Additionally, they find that loan loss provisions are negatively related to both future earnings changes and contemporaneous stock returns which contradicts the signaling results documented in prior work.

The contribution [3] expresses the concern that inefficient loan loss accounting may have a material impact on reported capital and earnings. Research prior to that in contribution [3], has examined banks' incentives to manipulate LLP and the resulting impact. However, most of this research has focussed on management incentives and other determinants of LLP decisions without addressing the relevant factors associated with best-practiced or efficient LLP decision making. In [3] a stochastic frontier model is identified that examines the efficiency of the LLP decisions of bank managers. Furthermore, the authors explore the relationship between efficient LLP decision-making and relevant factors that could potentially explain any efficiency. The evidence presented by the authors indicates that there is considerable inefficiency in loan loss decision-making among the sample institutions. The research is based on data from the Spanish banking industry, which is particularly relevant in light of the recent deregulatory initiatives in Spain. The finding in their study with regard to the

existence of inefficiency in loan loss decisions and the causes of such inefficiency have far-reaching implications for regulators throughout Europe.

The article [65] identifies two problems in the treatment of loan-loss provisions under the Basel Accord. The author distinguishes between expected and unexpected losses and asserts that the former should be treated in the same way as identified losses and not be counted as capital. However, capital should be maintained as a buffer stock available to absorb unexpected losses. The calculation of expected loan losses and the provisioning to cover these losses provides a deeper understanding of the economic structure of balance sheets. The authors claim that the calculation of expected losses is highly subjective but that the timing of the decision to identify a loss for balance sheet purposes and the level of provisioning against non-performing also involve highly subjective judgements. Attempts to calculate expected loan losses focus the attention of bank management on the factors which affect the creditworthiness of borrowers.

In [79], an accounting and behavioral framework is established from which the authors derive a reduced-form equation to test income smoothing and capital management practices (see, also, [72]) through loan loss provisions by Spanish banks (see, for instance, Figure 1.1). Spain offers a unique environment to perform tests because there are very detailed rules to set aside loan loss provisions and they are not counted as regulatory capital. Using panel data econometric techniques, we find evidence of income smoothing through LLP but not of capital management. The thesis draws some lessons for accounting rule setters and banking regulators regarding the current changes in the accounting framework (introduction of IFRS/IAS in Europe) as well as the new capital framework (see Basel II). In particular a very detailed set of rules to set aside loan loss provisions does not prevent managers from decreasing earnings volatility, similarly to what happens in a more principles-orientated accounting framework.

1.3.4 Literature Review on Treasuries, Reserves and Deposit Withdrawals

A vast literature exists on the properties of Treasuries and reserves and their interplay with deposit withdrawals. For instance, [10] (and the references contained therein) provides a neat discussion about Treasuries and loans and the interplay between them. Reserves are discussed in such contributions as [20], [25], [32], [36], [38], [95]. Firstly, [20] investigate the role of a central bank in preventing and avoiding financial contagion. Such a bank, by imposing reserve requirements on the banking industry, trades off the cost of reducing the resources available for long-term investment with the ben-

efit of raising liquidity to face an adverse shock that could cause contagious crises. We have that [25] presents a computational model for optimal reserve management policy in the banking industry. Also, [32] asserts that the standard view of the monetary transmission mechanism depends on the central bank's ability to manipulate the overnight interest rate by controlling reserve supply. They note that in the '90's, there was a marked decline in the level of reserve balances in the US accompanied at first by an increase in federal funds rate volatility. The article [36] examines how the risk of a bank run may affect the investment decisions made by a competitive bank. The basic premise is that when the probability of a run is small, the bank will offer a contract that admits a bank-run equilibrium. They show that in this case, the bank will hold an amount of liquid reserves exactly equal to what the withdrawal demand will be if a run does not occur; precautionary or excess liquidity will not be held. The paper [38] asserts that the payment of interest on bank reserves by the government assists in the implementation of monetary policy. In particular, it is demonstrated that paying interest on reserves financed by labor tax reduces welfare. Finally, [95] asserts that reserve requirements allow period-average smoothing of interest rates but are subject to reserve avoidance activities. A system of voluntary, period-average reserve commitments could offer equivalent rate-smoothing advantages. A common theme in the aforementioned contributions about reserves is the fact that they can be viewed as a proxy for general banking assets and that reserve dynamics are closely related to the dynamics of the deposits.

1.4 OUTLINE OF THE THESIS

In this section, we provide an outline of the thesis.

1.4.1 Outline of Chapter 1

In the current section, we provide preliminary information about Lévy processes, bank balance sheet, mixed optimal/robust control problems, Itô's formula and bank risk concepts and also, distinguish our thesis from the pre-existing literature. Under Section 1.1, the main problems addressed in the rest of our thesis are subsequently identified. Furthermore, in Section 1.4 we provide the outline of the thesis.

1.4.2 Outline of Chapter 2

Chapter 2 discusses the important features of models for loan losses and deposit withdrawals and their provisioning. This chapter is based on the research completed in [44] and [45]. In Section 2.1, we construct models for loan losses and their provisions (see Subsection 2.1.1). We assume that the bank provisions for loan losses by allocating funds to the provisions for expected losses and loan loss reserves for unexpected losses. In Subsection 2.1.2, the former is modeled by means of a general exponential Lévy process where the loan loss reserves increase by a constant interest rate. We assume that the bank has initial funds for provisioning for losses incurred by certain loans, and does not receive any external funds outside of the provisioning portfolio. Some historical evidence for our modeling choices is given in Subsection 2.1.3. Here particular emphasis is placed on provisioning data from Australia, Norway, the United Kingdom and United States. To our knowledge dynamic models for bank provisioning of the type mentioned above, have not appeared in the literature before.

Section 2.2 extends some of the modeling and optimization issues highlighted in [71] (see, also, [70], [40] and [73]) by presenting jump diffusion models for various bank items. Here we introduce a probability space that is the product of two spaces that models the uncertainty associated with the bank reserve portfolio and deposit withdrawals. As a consequence of this approach, the intrinsic risk of the bank arises now not only from the reserve portfolio but also from the deposit withdrawals. Throughout, we consider a depository contract that stipulates payment to the depositor on the contract's maturity date. We concentrate on the fact that deposit withdrawals are catered for by the Treasuries and reserves held by the bank. The stochastic dynamics of the latter mentioned items and their sum are presented in Subsections 2.2.1. In Subsection 2.2.2, our main focus is on depository contracts that permit a cohort of depositors to withdraw funds at will, with the stipulation that the payment of an early withdrawal is only settled at maturity. Furthermore, in Subsection 2.2.2.2, we suggest a way of counting deposit withdrawals by cohort depositors from which the bank has taken a single deposit at the initial time, $t = 0$.

1.4.3 Outline of Chapter 3

Chapter 3 provides information on optimal provisioning for loan losses. This chapter is based on the results determined in [45]. Section 3.1 provides a discussion on optimal provisioning in a robust control framework. Specifically, we present a statement of a mixed optimal/robust control problem. Furthermore, in Section 3.2, we give a solution of a mixed optimal/robust control problem in detailed. The solution is provided by

solving the nonlinear equation of the mixed optimal/robust control problem for bank provisioning stated in Problem 3.1.1 in Chapter 3.

1.4.4 Outline of Chapter 4

Chapter 4 deals with a discussion on optimal provisioning for deposit withdrawals. This chapter is based on the research completed in [44]. We consider basic risk concepts from Subsection 1.2.5 for application to Chapter 4 and Section 4.1 provides some risk *minimization* results that directly pertain to our studies. In Theorem 4.1.1, we derive a generalized GKW decomposition of the arbitrage-free value of the sum of cohort deposits depending on the reserve price. Theorem 4.2.1 provides a hedging strategy for bank reserve-dependent depository contracts in an incomplete reserve market setting. Intrinsic risk and the hedging strategies in Theorem 4.2.1 are derived with the (local) risk minimization theory contained in [39], assuming that bank deposits held accumulate interest on a risk free basis. In order to derive a hedging strategy for a bank reserve-dependent depository contract we require the generalized GKW decomposition for both its intrinsic value and the product of the inverse of Treasuries and the arbitrage free value of the sum of the cohort deposits. We accomplish this by assuming that the bank takes deposits (from a certain cohort of depositors with pre-specified characteristics) as a single lump sum at the beginning of a specified time interval and holds it until withdrawal some time later. More specifically, under these conditions, we show that the reserve risk (risk of losses from earning opportunity costs through bank and Federal government operations) is not diversifiable by raising the number of depository contracts within the portfolio. This is however the case with depository risk originating from the amount and timing of net cash flows from deposits and deposit withdrawals emanating from a cession of the depository contract.

1.4.5 Outline of Chapter 5

Chapter 5 deals with numerical simulations and examples for some historical data from member countries of the Organization for Economic Corporation and Development (OECD). In Section 5.1, the emphasis is placed on provisioning data from 9 OECD countries, viz. Australia, Finland, Italy, Norway, Spain, Sweden, the United Kingdom and the United States of America. In Section 5.2, we provide evidence that support the fact that the output gap and the provisions for loan losses/total assets are negatively correlated. Furthermore, we investigate the correlation between the output gap and provisions in relation to profitability. In Subsection 5.2.1 we look

at empirical evidence that provisions for loan losses is pro-cyclical. In this subsection we discuss provisioning for 9 OECD countries. In Subsection 5.2.2, we discuss correlations between profitability and provisions for loan losses. Here, we show that provisions typically do not increase until after economic growth has slowed down considerably and often not until the economy is in complete recession. Finally, Section 5.3 provides a simulation example for the stochastic dynamics of the sum of the Treasuries and reserves described in equation (2.23) in Chapter 2.

1.4.6 Outline of Chapter 6

In Chapter 6, we analyze the main risk management issues arising from the banking model we constructed previously. In Section 6.1, we do an analysis of the main robust control issues, while Subsection 6.1.3 outline a comparison with a discrete-time provisioning model. In Subsection 6.1.3, firstly we indicate that the non-discretionary component of total loan loss provisioning P , encapsulates expected loan losses. It follows in the second case that the discretionary component of the total loan loss provisioning P , results from three different management objectives. That is, the total loan loss provisioning P , is used to smooth earnings, manage regulatory capital, and signal the financial strength of the bank. In the final case, we articulate that the macroeconomic environment should affect the ability of borrowers to repay bank assets. In Section 6.2, we analyze the main risk management issues arising from the Lévy process-driven banking model constructed. Some of the highlights of this section are mentioned below. A description of the role that bank assets play is presented in Subsection 6.2.1. Furthermore, we provide more information about depository contracts and the stochastic counting process for deposit withdrawals in Subsection 6.2.2. Moreover, Subsection 6.2.3.3 provides a discussion on risk minimization and the hedging of deposit withdrawals.

1.4.7 Outline of Chapter 7

In Chapter 7, we provide concluding remarks in Section 7.1. In this section, we summarize all the chapters by outlining their objectives and main problems. We also highlight some topics for future research in Section 7.2 for both provisions for loan losses and deposit withdrawals.

1.4.8 Outline of Chapter 8

Chapter 8 consists of different references used in the thesis.

1.4.9 Outline of Chapter 9

Chapter 9 consists of appendices. In Section 9.1, we provide a table of parameters and values consider for the stochastic dynamics of the sum of the Treasuries and reserves (SDSTR) while Section 9.2 gives a lemma and remarks to support the proof of the Fubini Theorem. Lastly, in Section 9.3, we present characteristic triplets of one-dimensional Lévy process, L , and its linear transformation πL for $\pi \in \mathbb{R}$. Also, we present characteristic triplets for the natural logarithm of total provisioning

$$\ln \left(\frac{P_\pi}{p} \right).$$

Chapter 2

MODELING OF BANK PROVISIONING

2.1 MODEL FOR BANK LOAN LOSS PROVISIONING

2.1.1 Bank Loan Losses

2.1.2 Bank Loan Loss Provisioning

2.1.3 Historical Evidence

2.2 MODEL FOR BANK DEPOSIT WITHDRAWALS PROVISIONING

2.2.1 Assets

2.2.2 Liabilities

In this chapter, we describe bank models for provisioning for loan losses and deposit withdrawals.

2.1 MODEL FOR BANK LOAN LOSS PROVISIONING

This section is based on the paper [44]. In this context, a short description of the model is given below. The bank starts out with initial provisioning, p , and a *default premium rate*, $c^d > 0$. The rate, c^d , allows banks to load the repayments of creditors depending on the perceived credit risk the latter poses. The aggregate loan losses, S , are modeled by a compound Poisson process (see Section 2.1.1 below). Furthermore,

for the planning period $[0, T]$, we consider a characterization of the *nett loan loss provisioning*, P' , of the form

$$\begin{aligned} \text{Nett Loan Loss Provisioning } (P') = & \quad (2.1) \\ \text{Total Loan Loss Provisioning } (P) - \text{Aggregate Loan Losses } (S), \end{aligned}$$

where the *total loan loss provisions*, P , of the bank is the stochastic process $(P_t)_{t \geq 0}$, defined on the probability space, $(\Omega, \mathbb{F}, \mathbb{P})$, with *initial total loan loss provisions*, $P_0 = p \geq 0$. Furthermore, the *aggregate loan losses*, S , are defined as the sum of the unexpected and expected loan losses (see equation (2.4) in Section 2.1.1.3 below for an exact characterization).

2.1.1 Bank Loan Losses

In this section, we discuss expected and unexpected loan losses modeled via borrower asset values and an exponential Lévy process, respectively.

2.1.1.1 Expected Loan Losses

We consider the expected loan losses from the perspective of the borrower. We suppose that, after providing liquidity, the bank grants loans, Λ^0 , at the *interest rate on loans* or *loan rate*, r^Λ . This rate is assumed to be a random variable which is distributed according to some cumulative distribution function, F , and is the sum of risk-free Treasuries rate, r^T , and default premium rate, c^d , so that

$$r_t^\Lambda = r^T + c^d.$$

Due to the expenses related to monitoring and screening, we assume that these loans incur a constant *administrative cost*, c^a . This cost largely depends on the nature of the loan issued. For instance, for a loan with a guarantee the costs of monitoring and screening are generally higher than for a loan with no guarantee. Furthermore, we assume that the borrowers agree to repay, Λ^c , that is the product of the borrowed funds, Λ^0 , and an exponential growth factor, $\exp\{(r^T + c^d)T\}$, so that

$$\Lambda^c = \Lambda^0 \exp\left\{(r^T + c^d)T\right\}.$$

The borrowers would like to invest the borrowed funds in assets with value, V_t , at time t and V_T at the terminal time (of maturity of the loan contract), T . We assume that the value of these assets follow a geometric Lévy process given by

$$dV = \rho V dt + \sigma^v V dL_t,$$

where ρ is the nett instantaneous return and σ^v is an indicator of the volatility of the value of borrowers' assets. In the event of default, the bank will receive the value of the borrowers' assets minus the cost of bankruptcy. The expected cost of insolvency, $E[C_T]$, at T is assumed to equal the administrative costs, c^a , times the probability that insolvency will occur. In this case, symbolically we have that

$$E[C_T] = c^a \Lambda^c \int_0^{\Lambda^c} L'(v) dv,$$

where $L'(\cdot)$ is the density function of the log-normal distribution and C_T denotes the costs of insolvency at time T . Furthermore, we denote the expected loan losses by, S^e , and express it as

$$S^e = E[\min(V_T - \Lambda^c, 0)] - E[C_T] = \int_0^{\Lambda^c} \left\{ V_T - (1 + c^a)\Lambda^c \right\} L'(V_T) dV. \quad (2.2)$$

2.1.1.2 Unexpected Loan Losses

In the sequel, we assume that unexpected loan losses are recorded at the times

$$0 = T_0 < T_1 < T_2 < \dots,$$

where the corresponding unexpected loan loss amounts are described by the non-negative random variables

$$l_1 < l_2 < \dots,$$

called the *unexpected loan loss sizes*. Let

$$N_t^L = \sup\{n \geq 1 : T_n < t\}, \quad \sup(\emptyset) \equiv 0,$$

be the number of loan losses recorded during the interval $[0, t]$. We start with some initial loan loss provisioning, $p > 0$, which may, if necessary, be augmented by default premiums at a constant rate of $c^d > 0$. The *aggregate unexpected loan losses* is modeled by a compound Poisson process as

$$S_t^u = \begin{cases} \sum_{j=1}^{N_t^L} l_j, & N_t^L > 0; \\ 0, & N_t^L = 0. \end{cases} \quad (2.3)$$

Here $(l_n)_{n \in \mathbf{N}^L}$ is a sequence, independent of N^L , of positive i.i.d. random variables with a distribution function F and a mean $\mu = \mathbb{E}[l] < \infty$, modeling the values of the unexpected loan losses. Throughout our contribution, we will denote the value of the generic unexpected loan losses by S^u . The times between loan losses being recorded

$$T_n - T_{n-1}, \quad n \geq 1$$

are i.i.d. exponentially distributed random variables with parameter $\iota > 0$. The processes $(T_n)_{n \geq 1}$ and $(l_n)_{n \geq 1}$ are independent. It follows that the *unexpected loan loss number process*, $N^L = (N_t^L)_{t \geq 0}$, is a homogeneous Poisson process with an intensity $\iota > 0$, i.e.,

$$\mathbb{P}(N_t^L = k_1) = \exp\{-\iota t\} \frac{(\iota t)^{k_1}}{k_1!}, \quad k_1 = 0, 1, 2, \dots$$

so that $(S_t^u)_{t \geq 0}$ is a compound Poisson process, which is also a Lévy process.

2.1.1.3 Aggregate Loan Losses

We compute the *aggregate loan losses* as the sum of the unexpected and expected loan losses in the form

$$S_t = S_t^e + S_t^u, \quad (2.4)$$

where S^e and S^u are given by equations (2.2) and (2.3), respectively. As a result of this definition, we can write a straightforward expression for aggregate loan losses, S , in equation (2.4) as

$$S_t = \begin{cases} \int_0^{\Lambda^{c^d}} \left\{ V_T - (1 + c^a)\Lambda^{c^d} \right\} L'(V_T) dV + \sum_{j=1}^{N_t^L} l_j, & N_t^L > 0; \\ \int_0^{\Lambda^{c^d}} \left\{ V_T - (1 + c^a)\Lambda^{c^d} \right\} L'(V_T) dV, & N_t^L = 0. \end{cases} \quad (2.5)$$

2.1.2 Bank Loan Loss Provisioning

In this section, we discuss loan losses, loan loss reserves and provisioning. In particular, we construct models for these banking items.

2.1.2.1 Provisioning for Expected Losses

As was mentioned before, the current value of loans appearing on the balance sheet is equal to the face value of the loans less a provision for loan losses. A verifiable trend is that a strong negative correlation between the current value of loans and the provisions for expected losses exists (see, for instance, [16] for such evidence). Since the current value of loans may be modeled via exponential Lévy processes, it is reasonable to assume that the dynamics of the provisions for expected loan losses can also be modeled via an exponential Lévy process, L , as

$$X_t^{(1)} = x_0 \exp \left\{ c^d t + \sigma L_t \right\}, \quad t \geq 0, \quad X_0^{(1)} = x_0, \quad (2.6)$$

where $c^d \in \mathbb{R}$ is the constant default premium rate and can be chosen such that the provisions for expected loan losses has the desired appreciation rate. In essence, the provisions for expected loan losses given by equation (2.6) above should reflect the non-discretionary component of the total loan loss provisioning, P .

Lemma 2.1.1 (Stochastic Dynamics of $X_t^{(1)}$): Assume that $X_t^{(1)}$ is represented by equation (2.6) and \tilde{L} is the process such that

$$\exp\{L_t\} = \xi(\tilde{L}_t), \quad t \geq 0, \quad (2.7)$$

where ξ denotes the Doléans-Dade exponential and L is the Lévy process defined in Subsection 1.2.2 in Chapter 1. Then the stochastic differential equation (SDE) representing the dynamics of $X_t^{(1)}$ may be given by

$$\begin{aligned} dX_t^{(1)} &= X_{t-}^{(1)} \left(c^d dt + d\tilde{L}_t \right) \\ &= X_{t-}^{(1)} \left(\left[c^d + \frac{\sigma^2}{2} \right] dt + dL_t + \exp \left\{ \Delta L_t \right\} - 1 - \Delta L_t \right), \quad t > 0, \quad X_0^{(1)} = x_0. \end{aligned}$$

Proof. In order to verify the hypothesis of Lemma 2.1.1, we use Lemma 1.2.17 in Chapter 1 to present a form of Itô's formula that is suitable for our purposes. Furthermore, if we put $X_t^{(1)} = g_t(L_t)$, we have that

$$\begin{aligned} dg_t(L_t) &= \frac{\partial g_t(L_t)}{\partial t} dt + \frac{\partial g_t(L_t)}{\partial L_{t-}} dL_t + \frac{1}{2} \frac{\partial^2 g_t(L_t)}{\partial (L_t)^2} (dL_t)^2 \\ &\quad + d \sum_{0 < s \leq t} \left\{ g_t(L_t) - g_t(L_{t-}) - \Delta L_t \frac{\partial g_t(L_t)}{\partial L_{t-}} \right\}. \end{aligned}$$

In this situation, it follows that

$$\begin{aligned} \frac{\partial g_t(L_t)}{\partial t} &= c^d x_0 \exp\{c^d t + L_{t-}\}, \quad \frac{\partial g_t(L_t)}{\partial L_{t-}} = x_0 \exp\{c^d t + L_{t-}\}, \\ \frac{\partial^2 g_t(L_t)}{\partial (L_{t-})^2} &= x_0 \exp\{c^d t + L_{t-}\} \text{ and } (dL_t)^2 = \sigma^2 dt. \end{aligned}$$

Furthermore, for $X_0^{(1)} = x_0$, we have that

$$\begin{aligned}
dX_t^{(1)} &= c^d X_{t-}^{(1)} dt + X_{t-}^{(1)} dL_t + \frac{\sigma^2}{2} X_{t-}^{(1)} dt \\
&\quad + \left\{ x_0 \exp\{c^d t + L_t\} - x_0 \exp\{c^d t + L_{t-}\} - \Delta L_t x_0 \exp\{c^d t + L_{t-}\} \right\} \\
&= X_t^{(1)} \left\{ \left(c^d + \frac{\sigma^2}{2} \right) dt + dL_t \right\} + x_0 \exp\{c^d t + L_{t-}\} \left\{ \exp\{L_t - L_{t-}\} - 1 - \Delta L_t \right\} \\
&= X_{t-}^{(1)} \left\{ \left(c^d + \frac{\sigma^2}{2} \right) dt + dL_t \right\} + X_{t-}^{(1)} \left\{ \exp\{\Delta L_t\} - 1 - \Delta L_t \right\} \\
&= X_{t-}^{(1)} \left\{ \left(c^d + \frac{\sigma^2}{2} \right) dt + dL_t + \exp\{\Delta L_t\} - 1 - \Delta L_t \right\}, \quad t > 0.
\end{aligned}$$

□

2.1.2.2 Provisioning for Unexpected Losses

We suppose that the value process of the *loan loss reserve*, R^l , with constant loan loss reserve rate, r^R , follows

$$X^{(0)}(t) = \exp\{r^R t\}, \quad t \geq 0.$$

The corresponding SDE for the above value process becomes an ordinary differential equation (ODE) which is given by

$$dX^{(0)}(t) = r^R X^{(0)}(t) dt, \quad t > 0, \quad X^{(0)}(0) = 1, \quad (2.8)$$

where we use Lemma 1.2.17 in Chapter 1. This form is indicative of the fact that the value process for the loan loss reserve is riskless.

2.1.2.3 Total and Nett Provisioning for Loan Losses

In the sequel, the *total provision* is made up of provisions for the expected and unexpected losses. In particular, the *provisioning portfolio* consists of the provisions for expected losses and the loan loss reserves (for unexpected losses) only. Under this regime, at each point in the planning period $[0, T]$ a fixed fraction, denoted by $\pi \in [0, 1]$, is assigned to the provisions for expected losses while the rest, $1 - \pi$, constitute the loan loss reserves (sometimes called a *constant mix strategy*). We call the fraction, π , the *provisioning strategy*, P^π the *total loan loss provisioning under*

strategy π and P'^π the *nett loan loss provisioning under π* . For $t > 0$ and S , $X^{(1)}$, \tilde{L} and $X^{(0)}$ given by equations (2.5), (2.6), (2.7) and (2.8), respectively, we use equation (2.1) above to represent the dynamics of the total and nett loan loss provisioning by

$$dP_t^\pi = P_{t-}^\pi \left(((1 - \pi)r^R + \pi c^d)dt + \pi d\tilde{L}_t \right) \quad (2.9)$$

and

$$dP_t'^\pi = P_{t-}^\pi \left(((1 - \pi)r^R + \pi c^d)dt + \pi d\tilde{L}_t \right) - dS_t, \quad P_0^\pi = p,$$

respectively. The solution of equation (2.9) is obtained via Itô's formula and, for $P_0 = p \in \mathbb{R}^+$, is found to be

$$P_t^\pi = p \exp \left\{ t(r^R + \pi(c^d - r^R)) \right\} \xi(\pi \tilde{L}_t) = p \exp \left\{ \gamma_X t + \pi \sigma \delta Z_t \right\} \tilde{P}_t^\pi, \quad t \geq 0, \quad (2.10)$$

where ξ is given by equation (2.7) above, γ_X is as in Lemma 9.3.3 of Appendix 9.3 in Chapter 9,

$$\Gamma(x) := \ln \left(1 + \pi(\exp\{\sigma x\} - 1) \right)$$

and

$$\begin{aligned} \tilde{P}_t^\pi &= \exp \left\{ \int_0^t \int_{\mathbb{R}} \Gamma(x) 1_{\{|\Gamma(x)| > 1\}} M(ds, dx) \right\} \\ &= \exp \left\{ \int_0^t \int_{\mathbb{R}} \Gamma(x) 1_{\{|\Gamma(x)| \leq 1\}} \left(M(ds, dx) - \nu(dx)ds \right) \right\}, \quad t \geq 0. \end{aligned} \quad (2.11)$$

We provided a solution for equation (2.9) by considering the following facts. The provisioning process may be modeled via an exponential Lévy process. In order to calculate the moments of this process we require the existence of the moment generating function in some neighborhood of 0. This corresponds to an analytic extension of the characteristic function. If this extension is possible (see [85, Theorem 25.17]

for exact specifications), for all $n \in \mathbb{N}$ and \tilde{P}_π given by equation (2.11), such that the n -th moment exists, then

$$\mathbb{E} \left[\left(P_t^\pi \right)^n \right] = p^n \exp \left\{ \left(n\gamma_X + \frac{n^2\sigma_X^2}{2} \right) t \right\} \mathbb{E} \left[\left(\tilde{P}_t^\pi \right)^n \right], \quad t \geq 0,$$

and

$$\mathbb{E} \left[\left(\tilde{P}_t^\pi \right)^n \right] = \exp \{ \tilde{\mu}_n t \}, \quad t \geq 0, \quad (2.12)$$

where

$$\tilde{\mu}_n = \int_{\mathbb{R}} \left(\left[1 + \pi \exp \{ \sigma x - 1 \} \right]^n - 1 - n\Gamma(x) 1_{\{|\Gamma(x)| \leq 1\}} \right) \nu(dx)$$

and ν is the Lévy measure of L . In particular,

$$\mathbb{E} \left[\tilde{P}_t^\pi \right] = \exp \left\{ t \int_{\mathbb{R}} \left(\pi \exp \{ \sigma x - 1 \} - \Gamma(x) 1_{\{|\Gamma(x)| \leq 1\}} \right) \nu(dx) \right\}, \quad t \geq 0.$$

The next result provides explicit formulae for the expectation and variance of P^π .

Proposition 2.1.2 (Explicit Formulae for $\mathbb{E}[P_t^\pi]$ and $\text{var}(P_t^\pi)$): *Suppose that equation (2.9) holds and, for equation (2.6), that L_1 has moment generating function*

$$\tilde{M}(s) = \mathbb{E} \left(\exp \{ sL_1 \} \right)$$

such that $\tilde{M}(\sigma) < \infty$. Then, for $t \geq 0$, we have that the following is true.

1. An explicit formula for $\mathbb{E}[P_t^\pi]$ is given by

$$\mathbb{E}[P_t^\pi] = p \exp \left\{ t(r^R + \pi(c^d - r^R + \ln \tilde{M}(\sigma))) \right\}.$$

2. An explicit formula for $\text{var}(P_t^\pi)$ is given by

$$\text{var}\left(P_t^\pi\right) = p^2 \exp\left\{2t(r^R + \pi(c^d - r^R + \ln \widetilde{M}(\sigma)))\right\} \left[\exp\{t\pi^2\rho\} - 1\right], \quad (2.13)$$

where $\rho = \ln \widetilde{M}(2\sigma) - 2 \ln \widetilde{M}(\sigma)$.

Proof. Since (γ, σ^2, ν) is the characteristic triplet of L , from Lemma 9.3.3 of Appendix 9.3 in Chapter 9 and equation (2.12) above we have, for $t \geq 0$, that

$$\begin{aligned} \mathbb{E}[P_t^\pi] &= p \exp\left\{t\left[r^R + \pi\left(c^d - r^R + \frac{(\sigma\delta)^2}{2} + \sigma\gamma + \int_{\mathbb{R}} \{\exp(\sigma x - 1) - \sigma x 1_{\{|x|<1\}}\} \nu(dx)\right]\right\} \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} \text{var}\left(P_t^\pi\right) &= p^2 \exp\left\{2t\left[r^R + \pi\left(c^d - r^R + \frac{(\sigma\delta)^2}{2} + \sigma\gamma\right) + \frac{(\pi\sigma\delta)^2}{2} \int_{\mathbb{R}} \frac{1}{2}\{1 + \pi \exp(\sigma x - 1)^2 - \frac{1}{2} - \pi \sigma x 1_{\{|x|<1\}}\} \nu(dx)\right]\right\} \\ &\quad \times p^2 \left[\exp\left\{t\left((r^R\pi(c^d - r^R + \frac{(\sigma\delta)^2}{2} + \sigma\gamma) + \int_{\mathbb{R}} \{(\exp(\sigma x - 1) - \sigma x 1_{\{|x|<1\}})\} \nu(dx))\right)\right\}^2\right]. \end{aligned} \quad (2.15)$$

On the other hand, we find that

$$\widetilde{M}(\sigma) = \mathbb{E}[\exp\{\sigma L_1\}] = \exp\left\{\sigma\gamma + \frac{(\sigma\delta)^2}{2} + \int_{\mathbb{R}} \left[\exp\{\sigma x\} - 1 - \sigma x 1_{\{|x|<1\}}\right] \nu(dx)\right\}$$

so that

$$\ln \widetilde{M}(\sigma) = \sigma\gamma + \frac{(\sigma\delta)^2}{2} + \int_{\mathbb{R}} \left[\exp\{\sigma x\} - 1 - \sigma x 1_{\{|x|<1\}}\right] \nu(dx) \quad (2.16)$$

and

$$\pi^2 \rho = (\pi \sigma \rho)^2 + \int_{\mathbb{R}} \left[\pi (\exp\{\sigma x\} - 1) \right]^2 \nu(dx). \quad (2.17)$$

Substituting equations (2.16) and (2.17) into equations (2.14) and (2.15), respectively, we have

$$\mathbf{E}[P_t^\pi] = p \exp \left\{ t \left(r^R + \pi \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] \right) \right\}$$

and

$$\text{var} \left(P_t^\pi \right) = p^2 \exp \left\{ 2t \left(r^R + \pi \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] \right) \right\} \left(\exp\{t\pi^2 \rho\} - 1 \right).$$

□

2.1.3 Historical Evidence

In this subsection, we consider historical evidence from OECD countries that will justify some of the modeling choices for the provisioning process made in Section 2.1. In this regard, we firstly provide evidence to support the fact that provisions for loan losses-to-total assets ratio is negatively correlated with aggregate asset prices and total private credit-to-GDP ratio. Throughout this section, we rely on historical data from member countries of the Organization for Economic Cooperation and Development (OECD) as supplied on the website [77]. The specific countries for which data was accessed are Australia, Spain, Sweden, Japan, Italy, Norway, the United Kingdom and United States. Figures 2.1, 2.2, 2.3 and 2.4 provides historical evidence that provisions for loan losses can generally be modeled as in Sections 2.1.1 and 2.1.2 for the period 1980 to 2000 in Australia, Norway, Spain and Sweden.

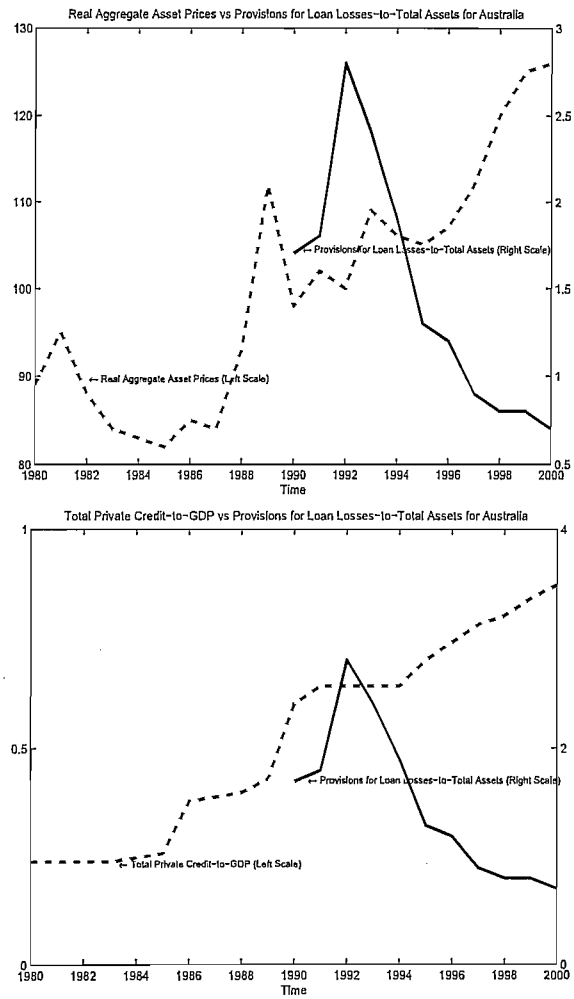


Figure 2.1: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Australia

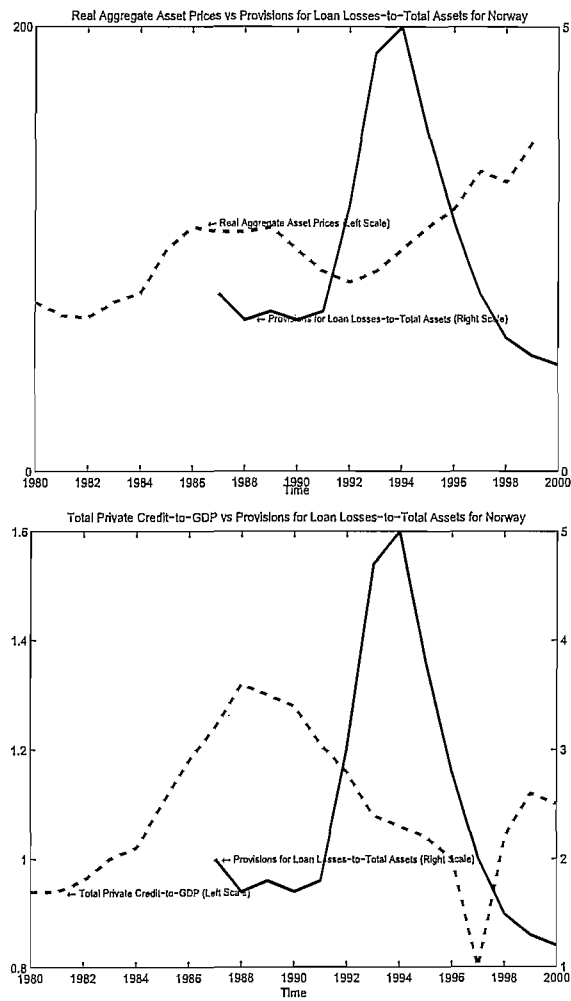


Figure 2.2: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Norway

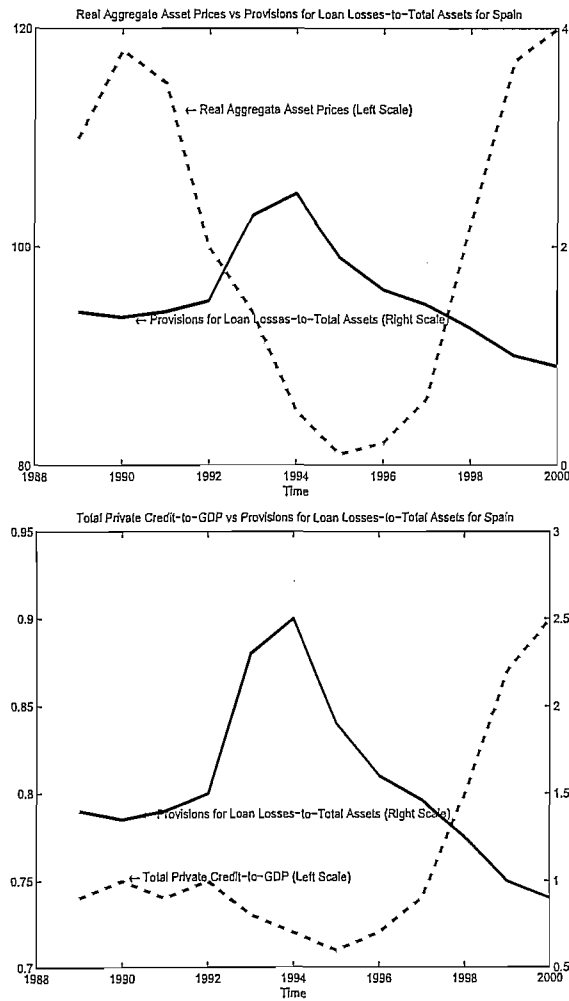


Figure 2.3: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Spain

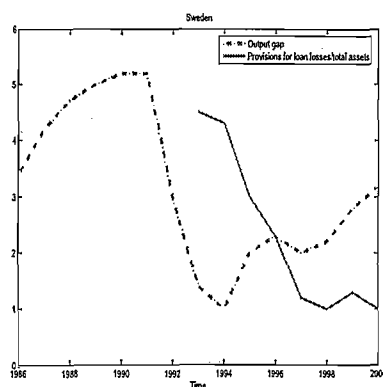


Figure 2.4: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Sweden

The data presented in Figures 2.5, 2.6, 2.7 and 2.8 provides evidence that provisions for loan losses can be modeled as in Sections 2.1.1 and 2.1.2 in the Finland, Italy, Japan and United Kingdom for most of the period 1980 to 2000. The USA, however, provides a few anomalies with the negative correlation between provisions and loan prices not being entirely clear.

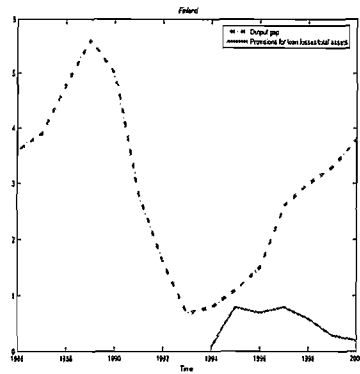


Figure 2.5: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Finland

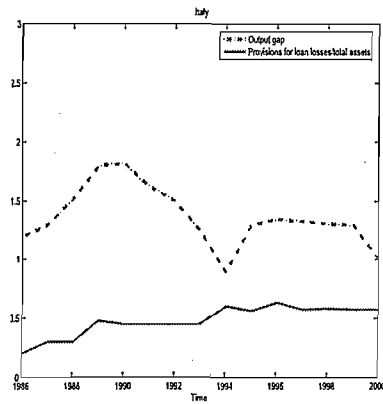


Figure 2.6: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Italy

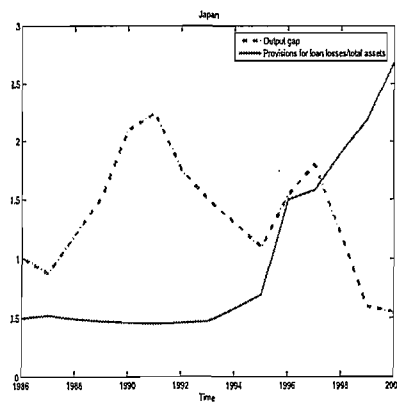


Figure 2.7: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for Japan

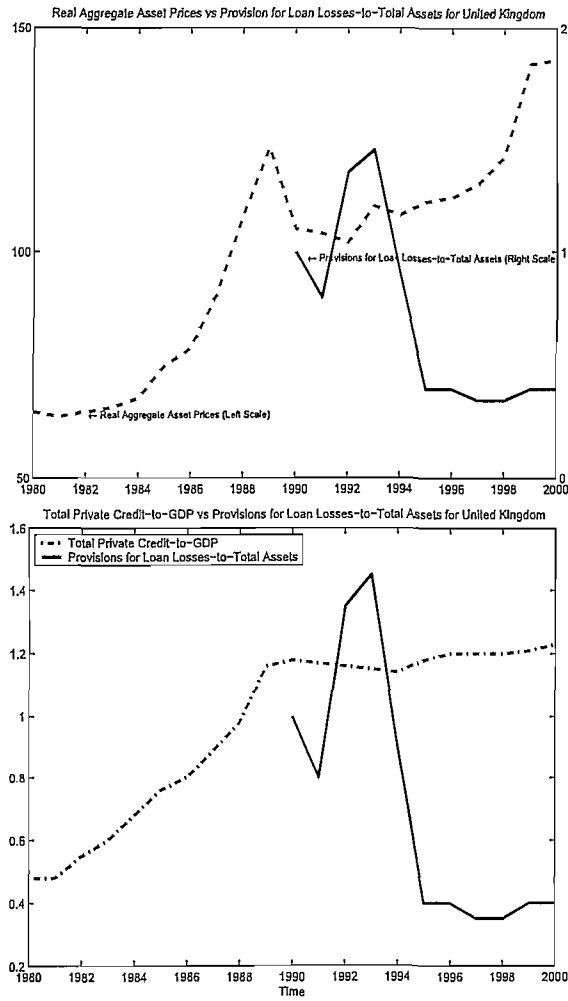


Figure 2.8: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for the United Kingdom

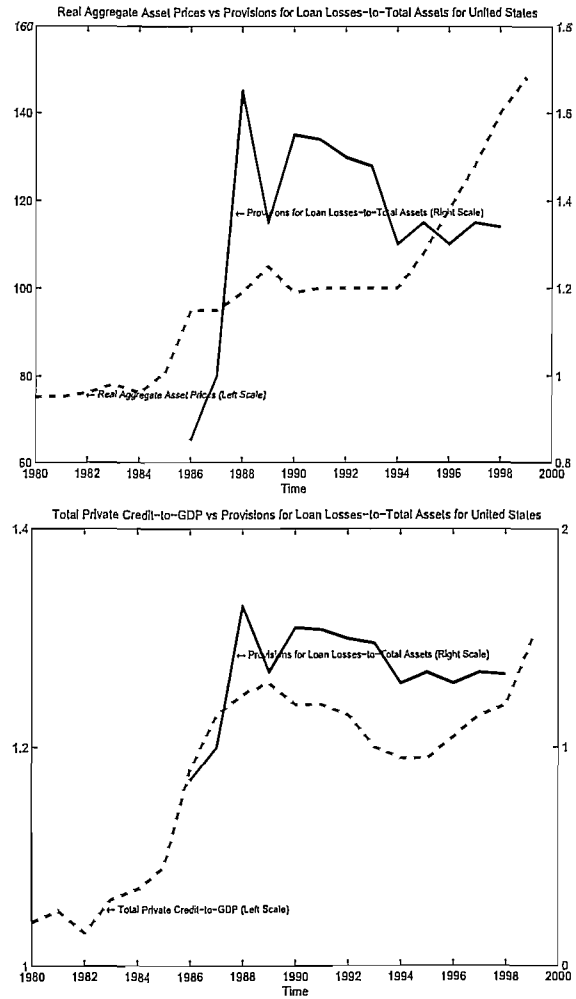


Figure 2.9: Output Gap vs Provisions for Loan Losses-to-Total Assets Ratio for the U.S.A.

2.2 MODEL FOR PROVISIONING FOR DEPOSIT WITHDRAWALS

This section is based on paper [44]. In this regard, our main objective is to construct a Lévy process-driven stochastic dynamic model that consists of assets, A , (uses of funds) and liabilities, Γ , (sources of funds and in connection with deposit withdrawals). In our contribution, these items can specifically be identified as

$$A_t = \Lambda_t + T(t) + R_t; \quad \Gamma_t = \Delta_t,$$

where Λ , T , R , and Δ are loans, Treasuries, reserves and outstanding debt, respectively.

2.2.1 Assets

In this subsection, the bank assets that we discuss are loans, provisions, Treasuries, reserves and unweighted and risk-weighted assets. In order to model the uncertainty associated with these items we consider the filtered probability space

$$(\Omega_1, \mathcal{G}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mathbb{P}_1).$$

2.2.1.1 Treasuries and Reserves

Treasuries are the debt financing instruments of the federal government. There are four types of Treasuries, viz., Treasury bills, Treasury notes, Treasury bonds and savings bonds. All of the Treasuries besides savings bonds are very liquid and are heavily traded on the secondary market. We denote the *interest rate on Treasuries* or *Treasury rate* by $r^T(t)$. In the sequel, the dynamics of the Treasuries will be described by

$$dT(t) = r^T(t)T(t)dt, \quad T(0) = t_0 > 0. \quad (2.18)$$

Bank reserves are the deposits held in accounts with a national agency (for instance, the Federal Reserve for banks) plus money that is physically held by banks (vault cash). Such reserves are constituted by money that is not lent out but is earmarked to

cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits will be needed as reserves. As a result of this description, we may introduce a reserve-deposit ratio, η , for which

$$R_t = \eta \Delta_t; \quad \Delta_t = \frac{1}{\eta} R_t, \quad 0 < \eta \leq 1. \quad (2.19)$$

The bank uses the remaining deposits to earn profit, either by issuing loans or by investing in assets such as Treasuries and stocks. The individual rationality constraint implies that reserves may implicitly earn at least their opportunity cost through certain bank operations and Federal government subsidies. For instance, members of the Federal Reserve in the United States may earn a return on required reserves through government debt trading, foreign exchange trading, other Federal Reserve payment systems and affinity relationships (outsourcing) between large and small banks. We note that vault cash in the Automated Teller Machines (ATMs) network also qualifies as required reserves. The conclusion is that banks may earn a positive return on reserves. In the sequel, we take the above discussion into account when assuming that the dynamics of the reserves are described by

$$dR_t = R_t \left\{ \left[r^R(t) - f^R(t) + a^R \sigma^R(t) \right] dt + \sigma^R(t) (c^R dZ_t^R + dM_t^R) \right\},$$

$$R_0 = r_0 > 0, \quad (2.20)$$

where r^R is the deterministic rate of (positive) return on reserves earned by the bank, f^R is the fraction of the reserves consumed by deposit withdrawals and σ^R is the volatility in the level of reserves. In order to have $R_t > 0$, we assume that $\sigma^R \Delta R_t > -1$ for all t a.s. Here, in a manner analogous to equation (1.5) of Subsection 1.2.2 in Chapter 1, we assume that L^R admits the decomposition

$$L_t^R = c^R Z_t^R + M_t^R + a^R t, \quad 0 \leq t \leq T,$$

where $(c^R Z_t^R)_{0 \leq t \leq \tau}$ is a Brownian motion with volatility $c^R > 0$, $a^R = E(L_1^R)$ and

$$M_t^R = \int_0^t \int_{\mathbb{R}} x M^R(ds, dx), \quad 0 \leq t \leq \tau,$$

is a square-integrable martingale. We know that the SDE from equation (2.20) above has the explicit solution

$$R_t = R_0 \exp \left\{ \int_0^t c^R \sigma^R(s) dZ_s^R + \int_0^t \sigma^R(s) dM_s^R + \int_0^t \left[a^R \sigma^R(s) + r^R(s) - f^R(s) - \frac{c^{R2} \sigma^{R2}(s)}{2} \right] ds \right\} \times \prod_{0 \leq s \leq t} (1 + \sigma^R(s) \Delta M_s^R) \exp(-\sigma^R(s) \Delta M_s^R).$$

We can use the notation $\widehat{R}_t = T^{-1}(t)R_t$ to denote the value of the discounted reserves. It is clear that \widehat{R}_t has a non-zero drift term so that it is only a semimartingale rather than a martingale. In order for \widehat{R} to be a martingale, under the approach of risk neutral valuation, a P_1 -equivalent martingale measure is required. There are infinitely many such measures in incomplete markets (see [24] for the incomplete information case). Girsanov parameter may be presented, for each $t \in [0, T]$, by

$$G_t = \frac{r^T(t) - r^R(t) + f^R(t) - a^R \sigma^R(t)}{\sigma^R(t)(c^{R2} + v)}, \quad v = \int_{\mathbb{R}} x^2 \nu(dx), \quad G_t \Delta R_t > -1.$$

In the sequel, the *compensated jump measure of L^R under \mathbb{Q}_g* is denoted by $M^{\mathbb{Q}}(dt, dx)$ and the Lévy measure $\nu(dx)$ under \mathbb{Q}_g has the form

$$\nu_t^{\mathbb{Q}}(dx) = (1 + G_t x) \nu(dx).$$

In addition, \widehat{R} is a square-integrable martingale under \mathbb{Q}_g (compare equation (1.4) of Subsection 1.2.2 in Chapter 1) that satisfies

$$d\widehat{R}_t = \sigma_t^R \widehat{R}_{t-} (c^R dZ_t^{\mathbb{Q}} + dM_t^{\mathbb{Q}}). \quad (2.21)$$

Here $Z^{\mathbb{Q}}$ is standard Brownian motion and

$$M_t^{\mathbb{Q}} = M_t - \int_0^t \int_{\mathbb{R}} G_s x^2 \nu(dx) ds = \int_0^t \int_{\mathbb{R}} x M^{\mathbb{Q}}(ds, dx) \quad (2.22)$$

is a square-integrable \mathbb{Q}_g -martingale. Under the above martingale, L^R may not be

a Lévy process since it may violate the fact that a semimartingale has stationary increments if and only if its characteristics are linear in time (cf. Jacod and Shiryaev in [56, Chapter II, Corollory 4.19]).

2.2.1.2 Provisions for Deposit Withdrawals

In the main, provisioning for deposit withdrawals involve decisions about the volume of Treasuries and reserves held by the bank. Without loss of generality, in the sequel, we suppose that the provisions for deposit withdrawals correspond with the sum of Treasuries and reserves as defined by equations (2.18) and (2.20) in the current chapter, respectively.

For withdrawal provisioning, we assume that the stochastic dynamics of the sum of Treasuries and reserves, W , is given by

$$\begin{aligned} dW_t &= W_{t-} \left[\left(r^T(t) + \pi_t^p(r^R(t) - f^R(t) - r^T(t) + a^R\sigma^R(t)) \right) dt + \right. \\ &\quad \left. \pi_t^p\sigma^R(t)(c^R dZ_t^R + dM_t^R) \right] - k_d(t)dt; \quad W_{0-} = \tau_0 + r_0 = w^o \geq 0, \\ &\quad W_t = W_t^u = T^u(t) + R_t^u \geq 0, \text{ for all} \\ t &\geq 0, \end{aligned} \tag{2.23}$$

where $\pi_t^p = \frac{R_t}{W_t}$ and the *depository value*, k_d , is the rate at which Treasuries and reserves are consumed by deposit withdrawals.

2.2.2 Liabilities

In the sequel, we assume that the bank deposit withdrawals are represented by the filtered probability space $(\Omega_2, \mathbf{H}, \mathbb{H}, \mathbf{P}_2)$. Here \mathbb{H} is the natural filtration generated by $I(T_i \leq t)$, $i = 1, \dots, n^x$, $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$, \mathcal{H}_0 is trivial and $\mathcal{H}_T = \mathbf{H}$. We suppose risk neutrality of the bank towards deposit withdrawals, which means that \mathbf{P}_2 is the risk neutral measure.

2.2.2.1 Depository Contracts

A *depository contract* is an agreement that stipulates the conditions for deposit taking and holding by the bank and withdrawal by the depositor. Depository contracts

typically specify the payment of some maturity amount that could be fixed or a function of some specified traded bank asset. Furthermore, we define the *deposit holding time* as the time between bank deposit taking and its withdrawal by the depositor. Our main supposition is that such times are mutually independent and identically distributed (i.i.d.). This assumption implies that depository contracts may be picked to form a cohort of individual contracts that have been held for an equal amount of time, x , with n^x denoting the number of such contracts. Ultimately, this situation leads to the description of the remaining deposit holding time by the i.i.d. non-negative random variables T_1, \dots, T_{n^x} . Under the assumption that the distribution of T_i is absolutely continuous, the *deposit survival conditional probability* may be represented by

$$P_2(T_i > x + t | T_i > x) = \exp \left\{ - \int_0^t \omega_{x+\tau} d\tau \right\},$$

where the *withdrawal rate function* is denoted by ω_{x+t} . Roughly speaking, for a deposit withdrawal at time instant T , $P_2(T > x + t | T > x)$ provides information about the probability that a deposit will still be held by a bank at $x + t$ conditional on a single deposit being taken by the bank at x .

In the sequel, reserves are related to outstanding debt (see, for instance, equation (2.19)) above and acts as a proxy for the assets held by the bank. This suggests that the *sum of cohort deposits*, D^c , may be dependent on the bank reserves, R_t , and as a consequence may be denoted by $D_t^c(R_t)$. For T and R from equations (2.18) and (2.20) in the current section, respectively, suppose that $D_t^c(R_t)$ is a \mathcal{G}_t -measurable function with

$$\sup_{u \in [0, T]} \mathbb{E}^Q[(T^{-1}(u)D_u^c(R_u))^2] < \infty. \quad (2.24)$$

We suppose that deposit withdrawals may take place at any time, $u \in [0, T]$, but that payment is deferred to the term of the contract. As a consequence, the contingent claim $D_u^c(R_u)$ must be time-dependent. From risk-neutral valuation, the *arbitrage-free value function*, $F_t(R_t, u)$, of the *sum of cohort deposits*, $D_t^c(R_t)$, is

$$F_t(R_t, u) = \begin{cases} \mathbb{E}^Q[T(t)T^{-1}(u)D_u^c(R_u) | \mathcal{G}_t], & 0 \leq t < u \leq T; \\ T(t)T^{-1}(u)D_u^c(R_u), & 0 \leq u \leq t \leq T. \end{cases} \quad (2.25)$$

From [81, Chapter I, Theorem 32], for $0 \leq t < u \leq T$ and $x \geq 0$, we have

$$F_t(x, u) = \mathbf{E}_{t,x}^{\mathbb{Q}}[\mathbb{T}(t)\mathbb{T}^{-1}(u)D_u^c(R_u)] = \mathbf{E}^{\mathbb{Q}}[\mathbb{T}(t)\mathbb{T}^{-1}(u)D_u^c(R_u)|R_t = x],$$

with $F(\cdot, u) \in C^{1,2}([0, T] \times [0, \infty))$ and $D_x F_t(x, u)$ bounded. Furthermore, we consider

$$j_t(x, u) = \mathbb{T}^{-1}(t) \left\{ F_t(R_{t-}(1 + \sigma^R(t)x), u) - F_t(R_{t-}, u) \right\} \quad (2.26)$$

to be the *value of the jump in the reserve process induced by a jump of the underlying Lévy process, L^R* .

In the case where equation (2.24) holds, the depository contract terminated at t receives the payout

$$D_{T_i}^c(R_{T_i})\mathbb{T}(T)\mathbb{T}^{-1}(T_i) \quad (2.27)$$

at time T . By way of consistency with our framework, the present value of the bank's depository obligation generated by the entire portfolio of depository contracts is considered to be \mathbb{Q} -a.s. of the form

$$\begin{aligned} D &= \mathbb{T}^{-1}(T) \sum_{i=1}^{n^x} D_{T_i}^c(R_{T_i})\mathbb{T}^{-1}(T_i)\mathbb{T}(T)\mathbf{I}(T_i \leq T) \\ &= \sum_{i=1}^{n^x} \int_0^T D_u^c(R_u)\mathbb{T}^{-1}(u)d\mathbf{I}(T_i \leq u) \\ &= \int_0^T D_u^c(R_u)\mathbb{T}^{-1}(u)dN_u^I. \end{aligned} \quad (2.28)$$

2.2.2.2 Stochastic Counting Process for Deposit Withdrawals

We assume that the bank takes a single deposit from each of n^x cohort depositors at $t = 0$. Furthermore, we model the *number of deposit withdrawals, N^I* , by

$$N_t^I = \sum_{i=1}^{n^x} \mathbf{I}(T_i \leq t), \quad l_t^x = n^x - N_t^I = \sum_{i=1}^{n^x} \mathbf{I}(T_i > t). \quad (2.29)$$

The *compensated counting process*, $M^I = (M_t^I)_{0 \leq t \leq T}$, expressible as

$$M_t^I = N_t^I - \int_0^t \iota_u du, \quad \text{where } \iota_t dt \equiv l_t^x \omega_{x+t} dt = \mathbb{E}[dN_t^I | \mathcal{H}_{t-}],$$

defines a H-martingale with

$$\langle M^I \rangle_t = \int_0^t \iota_u du, \quad 0 \leq t \leq T,$$

where ι is the (*stochastic*) *intensity of N^I* (compare with [56, Chapter II, Proposition 3.32]). In other words, ι is more or less the product of the withdrawal rate function, ω_{x+t} , and the remaining number of cohort depositors just before time instant t .

2.2.2.3 Cost of Deposit Withdrawals

Another modeling issue relates to the possibility that *unanticipated deposit withdrawals*, w^{un} , will occur. By way of making provision for these withdrawals, the bank is inclined to hold reserves, R , and Treasuries, T , that are very liquid. In our contribution, we propose that w^{un} may be associated with the *probability density function*, $f(w^{un})$, that is independent of time. In this regard, we may suppose that the unanticipated deposit withdrawals have a uniform distribution with support $[\underline{\Delta}, \overline{\Delta}]$ so that the *cost of liquidation*, c^l , or additional external funding is a quadratic function of the sum of Treasuries and reserves, $W = T + R$. In addition, for any t , if we have that

$$w^{un} > W_t,$$

then bank assets are liquidated at some *penalty rate*, r_t^p . In this case, the *cost of deposit withdrawals* is

$$c^{w^{un}}(W_t) = r_t^p \int_{W_t}^{\infty} [w^{un} - W_t] f(w^{un}) dw^{un} = \frac{r_t^p}{2\Delta} [\overline{\Delta} - W_t]^2. \quad (2.30)$$

Chapter 3

OPTIMAL PROVISIONING FOR LOAN LOSSES

3.1 STATEMENT OF A MIXED OPTIMAL/ROBUST CONTROL PROBLEM

3.2 SOLUTION OF A MIXED OPTIMAL/ROBUST CONTROL PROBLEM

In this short chapter, we state and prove a mixed optimal/robust control problem for bank provisioning. This chapter is based on the results determined in [45] (see, also, Section 2.1 for background). We note that the formulation of our control problem can be considered to be a special member of the class of constrained robust control problems stated in Problem 1.2.9 in Chapter 1. Below, we present a robust control problem with a constraint that may be solved in a mixed optimal/robust control framework (see, for example [90]).

Suppose that $(L_t)_{0 \leq t}$ is a Lévy process on the underlying filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and \tilde{L} is a process such that

$$\exp\{L_t\} = \xi(\tilde{L}_t), \quad t \geq 0,$$

where ξ denotes the Doléans-Dade exponential. We represent the dynamics of the state variable, x_t , by the stochastic differential equation

$$dx_t = \mu(\kappa_t, x_{t-})dt + \sigma(\kappa_t, x_{t-})d\tilde{L}_t, \quad (3.1)$$

where the actions of the decision-maker form a stochastic process $(\kappa_t)_{0 \leq t}$.

In the sequel, robust control theory investigates the fragility of decision rules across a range of economic indices. The decision rule determines the optimal provisioning strategy for loan losses. In this regard, historical evidence from Organization for Economic Corporation and Development (OECD) countries assists in confirming some of the modeling choices made in Chapter 5. This set-up naturally leads to a finite-horizon provisioning problem that may be solved via a mixed optimal/robust control approach involving a robustness constraint for risk stated above.

3.1 STATEMENT OF A MIXED OPTIMAL/ROBUST CONTROL PROBLEM

In this section, we consider a mixed optimal/robust control problem for bank provisioning in which we use the variance as a risk measure that is bounded by an appropriate robustness constraint. The problem can be formally stated as follows.

Problem 3.1.1 (Statement of a Mixed Optimal/Robust Control Problem):

Suppose that L is a Lévy process with a Lévy-Khintchine representation given in equation (1.6) of Subsection 1.2.2 in Chapter 1 and that P^π is given by equation (2.10) in Chapter 2. Then, for $t \geq 0$, we formulate the mixed optimal/robust control problem for bank provisioning as

$$\max_{\pi \in [0,1]} \mathbb{E}[P_T^\pi] \quad \text{subject to equation (2.10) in Chapter 2 and } \text{var}\left(P_T^\pi\right) \leq C, \quad (3.2)$$

where T is some given planning horizon and C is a given risk measure.

3.2 SOLUTION OF A MIXED OPTIMAL/ROBUST CONTROL PROBLEM

In this section, we solve the mixed optimal/robust control problem for bank provisioning as outlined in Problem 3.1.1 above.

Theorem 3.2.1 (Solution of a Mixed Optimal/Robust Control Problem):

Suppose that L is a Lévy process with a Lévy-Khintchine representation given in equation (1.6) of Subsection 1.2.2 in Chapter 1. Then the solution of the mixed optimal/robust control problem for bank provisioning, stated in Problem 3.1.1, is given by

$$\pi^* = \eta^* \rho^{-1/2}, \quad \pi^* \in [0, 1], \quad (3.3)$$

where $\rho = \ln \widetilde{M}(2\sigma) - 2 \ln \widetilde{M}(\sigma)$ and $\ln \widetilde{M}(\sigma)$ are defined as in Proposition 2.1.2 in Chapter 2. Also, η^* is the unique positive solution of

$$r^R T + \sqrt{\widehat{\gamma}} \eta T + \frac{1}{2} \ln \left(\frac{p^2}{C} \left[\exp \left\{ T \eta^2 \right\} - 1 \right] \right) = 0, \quad P_0 = p,$$

where

$$\widehat{\gamma} = \rho^{-1} \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right]^2.$$

The maximal expected terminal provisioning for loan losses under the variance constraint in equation (3.2) corresponding to equation (3.3) equals

$$\mathbb{E}[P_T^{\pi^*}] = p \exp \left\{ T \left[r^R + \eta^* \rho^{-1/2} \left(c^d - r^R + \ln \widetilde{M}(\sigma) \right) \right] \right\}. \quad (3.4)$$

Proof. Using equation (2.13) in Chapter 2, we can rewrite the variance constraint in Problem 3.1.1 as

$$p^2 \exp \left\{ 2T \left[r^R + \pi (c^d - r^R + \ln \widetilde{M}(\sigma)) \right] \right\} \left(\exp \{ T \eta^2 \} - 1 \right) \leq C, \quad \pi \sqrt{\rho} = \eta, \quad \forall \eta > 0.$$

As a consequence of this, we have that

$$T \pi \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] \leq \frac{1}{2} \ln \left(\frac{C}{p^2 \exp \{ T \eta^2 \} - p^2} \right) - r^R T. \quad (3.5)$$

In the sequel, let the right-hand side of equation (3.5) be denoted by $h(\eta)$. In this case, it follows that

$$T\pi \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] \leq h(\eta). \quad (3.6)$$

More precisely, if $\pi \in \mathbb{R}$ satisfies the constraints in equation (3.6) for $\eta > 0$, then it also satisfies the variance constraint in equation (3.2) and vice versa. We note that $h(\eta)$ is strictly decreasing in $\eta > 0$ with

$$\lim_{\eta \downarrow 0} h(\eta) = \infty, \quad \lim_{\eta \rightarrow \infty} h(\eta) = -\infty.$$

Now, if we have equality

$$T\pi_{\widehat{\eta}} \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] = h(\widehat{\eta}) \quad (3.7)$$

for the first time with increasing $\eta > 0$ then this determines the optimal $\widehat{\eta} > 0$. To see this, note that we have

$$\mathbb{E}[P_T^\pi] \leq \mathbb{E}[P_T^{\pi_{\widehat{\eta}}}], \quad \forall \pi, \quad \text{with } \pi\sqrt{\rho} \leq \widehat{\eta},$$

and for all admissible π with $\eta = \pi\sqrt{\rho} > \widehat{\eta}$ we obtain

$$T\pi_{\eta} \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right] \leq h(\eta) < h(\widehat{\eta}) = T\pi_{\widehat{\eta}} \left[c^d - r^R + \ln \widetilde{M}(\sigma) \right].$$

By solving the nonlinear equation given in equation (3.7) above for $\widehat{\eta}$, we have determined the solution of the mixed optimal/robust control problem for bank provisioning stated in Problem 3.1.1. \square

Chapter 4

OPTIMAL PROVISIONING FOR DEPOSIT WITHDRAWALS

4.1 GENERALIZED GKW DECOMPOSITION OF $T^{-1}(t)F_t(R_t, u)$

4.2 RISK-MINIMIZING STRATEGY FOR DEPOSITORY CONTRACTS

This chapter discusses the optimal provisioning for deposit withdrawals. This chapter is based on the research completed in [44] (see, also, Section 2.2 for background). Here, we focus on generalized GKW decomposition of $T^{-1}(t)F_t(R_t, u)$ and risk-minimizing strategy for depository contracts.

In this case, our model has far-reaching implications for risk management in the banking industry. For instance, we can apply the quadratic hedging theory developed in [17] and [39] to derive a risk minimizing strategy for deposit withdrawals. An approach that we adopt in this case, involves the introduction of a probability space that is the product of two spaces modeling the uncertainty associated with the bank's provision for deposit withdrawals via Treasuries and reserves and the withdrawals themselves given by

$$(\Omega_1, \mathcal{G}, \mathbb{G}, P_1) \text{ and } (\Omega_2, \mathcal{H}, \mathbb{H}, P_2),$$

respectively. In the sequel, we represent the product probability space by $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration, \mathbb{F} , is characterized by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

Here \mathcal{G}_t and \mathcal{H}_t are stochastically independent. As an equivalent martingale measure, \mathbb{Q} , we use the product measure of the generalized GKW measure \mathbb{Q}_g and of the risk-neutral deposit withdrawal law \mathbb{P}_2 . As a consequence of this approach, the intrinsic risk of the bank arises now not only from the Treasuries/reserves provisioning portfolio but also from the deposit withdrawals.

4.1 GENERALIZED GKW DECOMPOSITION OF

$$\mathbb{T}^{-1}(t)F_t(R_t, u)$$

Suppose that n_t^T and n_t^R are the number of treasuries and reserves in the provisioning strategy, $\pi = (n^R, n^T)$, respectively. Next, we produce the generalized GKW decomposition of $\mathbb{T}^{-1}(t)F_t(R_t, u)$, in order to eventually derive a hedging strategy for a reserve-dependent deposit withdrawal.

Theorem 4.1.1 (Generalized GKW Decomposition of $\mathbb{T}^{-1}(t)F_t(R_t, u)$): *Let $F_t(R_t, u)$ and j be defined by equations (2.25) and (2.26) of Subsection 2.2.2 in Chapter 2, respectively and assume that*

$$v_t^{\mathbb{Q}} = \int_{\mathbb{R}} x^2 \nu_t^{\mathbb{Q}}(dx) \text{ and } \zeta_t = c^2 + v_t^{\mathbb{Q}}, \quad t \in [0, T].$$

For $0 \leq t < u \leq T$, we define the predictable process

$$n_t^R(u) = \frac{c^2}{\zeta_t} D_x F_t(R_{t-}, u) + \frac{1}{\sigma(t) \widehat{R}_{t-} \zeta_t} \int_{\mathbb{R}} x j_t(x, u) \nu_t^{\mathbb{Q}}(dx) \quad (4.1)$$

and continuous and discontinuous terms by

$$\begin{aligned} \vartheta_t^{(1)}(u) &= c\sigma(t) \widehat{R}_{t-} \left\{ D_x F_t(\widehat{R}_{t-}, u) - n_t^R(u) \right\} \text{ and} \\ \vartheta_t^{(2)}(y, u) &= j_t(y, u) - y\sigma(t) \widehat{R}_{t-} n_t^R(u), \end{aligned} \quad (4.2)$$

respectively. In this situation, the generalized GKW decomposition of $T^{-1}(t)F_t(R_t, u)$ is given by

$$T^{-1}(t)F_t(R_t, u) = F_0(R_0, u) + \int_0^t n_s^R(u) d\widehat{R}_s + K_t(u),$$

where

$$K_t(u) = \int_0^t \vartheta_t^{(1)}(u) dZ_s^Q + \int_0^t \int_{\mathbb{R}} \vartheta_t^{(2)}(y, u) N^Q(ds, dy) \quad (4.3)$$

is orthogonal to \widehat{R} .

Proof. We base our proof on the additivity of the projection in the GKW decomposition. From equation (1.4) of Subsection 1.2.2 in Chapter 1, equation (2.22) of Subsection 2.2.1 in Chapter 2 and the fact that $D_x F_t(x, u)$ is bounded, the integrals driven by \widehat{R} , $N^Q(\cdot, \cdot)$ and Z^Q are well-defined and square-integrable martingales. Furthermore, we note that [26, Proposition 10.5] determines n_t^R for the generalized GKW decomposition in the Lévy process case. Under the equivalent measure, \mathbb{Q} , this result extends quite naturally to the case of the additive process L . We are able to deduce from Ito's formula in [81, Chapter II, Theorem 33], that the discounted arbitrage-free value, $T^{-1}(t)F_t(R_t, u)$, admits the decomposition

$$T^{-1}(t)F_t(R_t, u) = F_0(R_0, u) + \int_0^t D_x F_s(R_{s-}, u) d\widehat{R}_s + \widetilde{K}_t(u), \quad 0 \leq t \leq T,$$

where

$$\widetilde{K}_t(u) = \int_0^t \int_{\mathbb{R}} \left\{ (J_s(y, u) - D_x F_s(R_{s-}, u) \sigma_s \widehat{R}_{s-} y) N^Q(ds, dy) \right\}.$$

This formula, along with the differential in equation (2.21) of Subsection 2.2.1 in Chapter 2, allows the orthogonal part of equation (4.3) in the hypothesis of Theorem 4.1.1 to be computed via

$$K_t(u) = \int_0^t \left\{ D_x F_s(R_{s-}, u) - n_s^R(u) \right\} d\widehat{R}_s + \widetilde{K}_t(u).$$

□

4.2 RISK-MINIMIZING STRATEGY FOR DEPOSITORY CONTRACTS

In the discussion thus far, bank obligations generated by depository contracts unfortunately do not correspond to a T -claim so that special assumptions are required. A way of transforming the aforementioned obligations into a T -claim is to suppose that deposit withdrawals are deferred to the term of the contract and are accumulated with a risk-free interest rate of r^f . In the wake of this specification, the depository contract terminated at time t would receive the payout $D_{T_i}^c(R_{T_i})\mathbb{T}(T)\mathbb{T}^{-1}(T_i)$ at time T , in the case where equation (2.24) of Subsection 2.2.1 in Chapter 2 holds. These contracts by deferment usually have short time horizons. By way of consistency with our framework, the present value of the bank obligation generated by the entire portfolio of depository contracts is considered to be \mathbb{Q} -a.s. of the form

$$\begin{aligned} D &= \mathbb{T}^{-1}(T) \sum_{i=1}^{n^x} D_{T_i}^c(R_{T_i})\mathbb{T}^{-1}(T_i)\mathbb{T}(T)\mathbb{I}(T_i \leq T) \\ &= \sum_{i=1}^{n^x} \int_0^T D_u^c(R_u)\mathbb{T}^{-1}(u)d\mathbb{I}(T_i \leq u) \\ &= \int_0^T D_u^c(R_u)\mathbb{T}^{-1}(u)dN_u^I. \end{aligned} \tag{4.4}$$

Also, we recall that the intrinsic risk process associated with D may be given by

$$R_t(\pi^*) = \mathbb{E}^{\mathbb{Q}} \left[\left(K_T^D - K_t^D \right)^2 \middle| \mathcal{F}_t \right], \tag{4.5}$$

where K_T^D is given in Subsection 1.2.5 in Chapter 1. The independence of the reserve market and deposit withdrawals, enables us to represent the *intrinsic value of the entire depository contract portfolio*, V_t^* , as

$$\begin{aligned}
 V_t^* &= \int_0^t T^{-1}(u) D_u^c(R_u) dN_u^I \\
 &\quad + \int_t^T T^{-1}(t) F_t(R_t, u) l_t^x P_2(T_i > u + x + t | T_i > x + t) \omega_{x+u} du,
 \end{aligned} \tag{4.6}$$

with initial value

$$V_0^* = \int_0^T F_0(R_0) n^x P_2(T_i > u + x | T_i > x) \omega_{x+u} du.$$

Under certain conditions, V_0 may be the single deposit taken by the bank at $t = 0$. The risk minimization approach adopted in the ensuing main result is dependent on the fact that the value of the optimal provisioning strategy, π^* , is exactly equal to the sum of cohort deposits that have already been withdrawn and expected possible future withdrawals as in

$$\widehat{V}_t(\pi^*) = V_t^* = \int_0^t T^{-1}(u) D_u^c(R_u) dN_u^I + \mathbb{E}^Q \left[\int_t^T T^{-1}(t) F_u(R_u) dN_u^I | \mathcal{F}_t \right].$$

Theorem 4.2.1 (GKW Decomposition of V_t^* & Risk-Minimizing Strategy): Suppose that n^R , $\vartheta_t^{(1)}$, $\vartheta_t^{(2)}$ and V_t^* are given by equation (4.1), equation (4.2) in Theorem 4.1.1 and equation (4.6), respectively.

1. For equation (4.4), with $0 \leq t \leq T$, the intrinsic value process, V^* , has the GKW decomposition

$$V_t^* = V_0^* + \int_0^t l_s^x n_s^R d\widehat{R}_s + K_t^H, \tag{4.7}$$

where

$$n_t^R = \int_t^T P_2(T_i > u + t + x | T_i > x + t) \omega_{x+u} n_t^R(u) du \tag{4.8}$$

and

$$K_t^D = \int_0^t l_{s-}^x \vartheta_s^{(1)} dZ_s^Q + \int_0^t \int_{\mathbb{R}} l_{s-}^x \vartheta_t^{(2)}(y) N^Q(ds, dy) + \int_0^t v_s dM_s^I, \quad (4.9)$$

is orthogonal to \widehat{R} . Here

$$\vartheta_t^{(1)} = \int_t^T \mathbb{P}_2(T_i > u + t + x | T_i > x + t) \omega_{x+u} \vartheta_t^{(1)}(u) du;$$

$$\vartheta_t^{(2)}(y) = \int_t^T \mathbb{P}_2(T_i > u + t + x | T_i > x + t) \omega_{x+u} \vartheta_t^{(2)}(y, u) du$$

and

$$v_t = \mathbb{T}^{-1}(T) D_t^c(R_t) - \int_t^T \mathbb{T}^{-1}(T) F_t(R_{t-}, u) \mathbb{P}_2(T_i > u + t + x | T_i > x + t) \omega_{x+u} du.$$

2. For $0 \leq t \leq T$, under \mathbb{Q} , the unique admissible risk-minimizing hedging strategy, $\pi^* = (n^{R*}, n^{T*})$, for the bank obligation in equation (4.4) is given by

$$\begin{aligned} n_t^{R*} &= l_{t-}^x \int_t^T \mathbb{P}_2(T_i > u + t + x | T_i > x + t) n_t^R(u) du, \\ n_t^{T*} &= \int_0^t \mathbb{T}^{-1}(u) D_u^c(R_u) dN_u^I \\ &\quad + l_{t-}^x \int_t^T \mathbb{T}^{-1}(t) F_t(R_t, u) \mathbb{P}_2(T_i > u + t + x | T_i > x + t) \omega_{x+u} du - n_t^{R*} \widehat{R}_t. \end{aligned} \quad (4.10)$$

For $0 \leq t \leq T$, the intrinsic risk process is

$$\begin{aligned}
 R_t(\pi^*) &= \int_t^T \left\{ \mathbb{E}^{\mathbb{Q}}[(l_t^x)^2 | \mathcal{F}_t] \right. & (4.11) \\
 &\quad \left. \mathbb{E}^{\mathbb{Q}} \left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} (\vartheta_s^{(2)}(y))^2 \nu_s^{\mathbb{Q}}(dy) | \mathcal{F}_t \right] \right\} ds \\
 &\quad + l_t^x \int_t^T \mathbb{E}^{\mathbb{Q}}[v_s^2(R_s) | \mathcal{F}_t] \mathbb{P}_2(T_i > s + t + x | T_i > x + t) \omega_{x+s} ds.
 \end{aligned}$$

Proof.

1. The proof of equation (4.7) in the first part of the theorem relies on the stochastic Fubini Theorem (see, for instance, Section 9.2 in Chapter 9).
2. In order to complete the proof, we make use of a combination of isometry results, Tonelli's theorem and orthogonality. In this regard, we bear in mind that

$$\begin{aligned}
 d\iota_s &= l_s^x \omega_{x+s} ds; \\
 \mathbb{E}^{\mathbb{Q}} \left[l_s^x | \mathcal{F}_t \right] &= l_s^x \mathbb{P}_2(T_i > s + t + x | T_i > x + t).
 \end{aligned}$$

□

Chapter 5

NUMERICAL SIMULATIONS AND EXAMPLES

5.1 HISTORICAL DATA

5.2 NUMERICAL ANALYSIS OF BANK PROVISIONING

5.2.1 Pro-Cyclicality of Provisions for Loan Losses

5.2.2 Correlations Between Profitability and Provisioning for Loan Losses

5.3 SIMULATIONS AND NUMERICAL EXAMPLES

5.3.1 Parameters and Values

5.3.2 Properties of the Trajectory

In this chapter, we justify the modeling in previous chapters by presenting historical data from OECD countries, a numerical analysis of bank provisioning for loan losses and then numerical simulations and examples of bank provisioning for deposit withdrawals.

5.1 HISTORICAL DATA

In this section, we consider historical data from OECD countries that will illuminate some features of the provisioning processes modeled in Chapter 2. In this regard, we firstly provide evidence to support the fact that the output gap (as calculated by the OECD) and the provisions for loan losses/total assets are negatively correlated. In essence this means that provisions for loan losses are pro-cyclical. The specific

countries for which data was accessed are Australia, Finland, Italy, Japan, Norway, Spain, Sweden, the United Kingdom and the United States of America.

The graphical representation of this data for the dynamic models are shown in Subsection 2.1.3 in Chapter 2. This empirical evidence for provisions for loan losses suggest that bank provisions are strongly procyclical with a negative correlation with the business cycle. More evidence is provided in the following table.

Countries	Provisions	Profitability
Australia	- 0,88	0,71
Finland	—	0,81
Germany	- 0,21	- 0,42
Italy	- 0,21	0,25
Japan	- 0,43	0,22
Norway	- 0,35	0,54
South Africa	- 0,85	0,74
Spain	- 0,41	0,84
Sweden	- 0,83	0,60
United Kingdom	- 0,38	0,12
United States	0,14	0,24

Table 5.1: Correlations between Output Gap and Provisions and Profitability

5.2 NUMERICAL ANALYSIS OF BANK PROVISIONING

In this section, we firstly review evidence from Subsection 2.1.3 from Chapter 2 to support the fact that the output gap (as calculated by the OECD) and the provisions for loan losses/total assets are negatively correlated. In essence this means that provisions for loan losses are pro-cyclical. Here the *output gap* is defined as the amount by which a country's output, or GDP, falls short of what it could be given its available resources. Of course, GDP is often used as a proxy for macroeconomic activity. In addition, we investigate the correlation between output gap and provisions in relation to profitability.

Throughout this section, we rely on historical data from member countries of the Organization for Economic Cooperation and Development (OECD) as supplied on the website [77]. The specific countries for which data was accessed are Australia, Finland, Italy, Japan, Norway, Spain, Sweden, the United Kingdom and the United States of America. The figures show that provisions typically do not increase until after economic growth has slowed considerably and often not until the economy is clearly in recession. This is best observed in the figures for Australia, Sweden, Norway and Spain illustrated in Subsection 2.1.3.

5.2.1 Discussion of Provisioning for the 9 OECD Countries

In this subsection, we provide a brief discussion of some of the outstanding features of the data for provisioning for loan losses provided in Subsection 2.1.3.

The data for Australia from Figure 2.1 in Subsection 2.1.3 shows that provisions failed to increase substantially in the late 1980's, when credit and asset prices were growing rapidly and the financial imbalances were developing. Moreover, the peak in provisions did not occur until at least one year after the economy had clearly slowed down.

The data for Norway from Figure 2.2 of Section 2.1.3, exhibits a similar behavior as the data for Finland from Figure 2.5 and the data for Spain from Figure 2.3 in Subsection 2.1.3. In these cases, provisions failed to increase substantially in the late 1980's, when credit and asset prices were growing rapidly and the financial imbalances were developing. In each of these figures the peak for provisions did not occur until the recession. However, one of the differences between these figures is the amount by which provisions increased when the economy had clearly slowed down.

In the data for Finland from Figure 2.5 of Subsection 2.1.3 we see a similar situation as in the data for Italy from Figure 2.6 where provisions failed to overlap the output gap during the recession. Again this can be linked with Japanese banking problems. Although in countries like the United States (see Figure 2.9 in Subsection 2.1.3) and Australia and Norway (see Figure 2.1 and 2.2 in Subsection 2.1.3), the provisions overlapped the output gap during the recessions, the situation in Italy (see Figure 2.6 in Subsection 2.1.3) is totally different. It seems that even after the banking problems that Japan experienced had been resolved, in Italy the situation changed slightly.

From the data for Japan from Figure 2.7 of Subsection 2.1.3, we can conclude that the level of provisioning only increased substantially during the second half of the 1990's, long after the problems in the Japanese banking system had been widely recognized.

The (low) positive correlation between provisions and the business cycle in the United

States (see Figure 2.9 in Subsection 2.1.3) appears to be driven by the surge in provisions in the second half of the 1980's. This phenomenon seems to reflect the delayed cleaning of the balance sheets following the developing countries' debt crisis of the early 1980's.

5.2.2 Correlations Between Profitability and Provisions for Loan Losses

As has been suggested in Subsection 5.2.1 above, bank provisions are strongly procyclical being negatively correlated with the business cycle. For instance, Figure 2.9 in Subsection 2.1.3 shows that provisions typically do not increase until after economic growth has slowed down considerably and often not until the economy is in complete recession. In the main, the behavior of provisions translates into a clear procyclical pattern in bank profitability, which further encourages procyclical lending practices. As is shown in Table 5.1, this pattern appears to be strongest in those countries that experienced banking system problems in the 1990's. In the main, the behavior of provisions translates into a procyclical pattern in bank profitability, which further encourages procyclical lending practices. This is borne out by the fact that we can express profit as

$$\begin{aligned} \Pi_t = & \left(r^T + k^d - c^A - r^d(M_t) \right) \Lambda_t + r_t^T W_t - \left(r_t^D + c^D \right) D_t \\ & - c^{w^{un}}(W_t) - (P(M_t) - \mathbb{E}(d)\Lambda_t) - r_t^T \gamma D_t. \end{aligned}$$

Our claim is thus that profit and provisions are negatively correlated. However, from Table 5.1, we also conclude that the profitability of German banks is not pro-cyclical. This may be due to their ability to smooth profits through hidden reserves. The procyclical nature of bank profits has arguably also contributed to the bank equity prices being positively correlated with the business cycle, although the correlation is typically weaker than that for profitability, reflecting the forward-looking nature of the equity market.

5.3 SIMULATIONS AND NUMERICAL EXAMPLES

In the sequel, further insight is gained by considering a simulation of a trajectory for the stochastic dynamics of the sum of the Treasuries and reserves, W (denoted by SDSTR), as given by equation (2.23) of Subsection 2.2.1 in Chapter 2.

5.3.1 Parameters and Values

We consider the SDSTR problem with the constant rates and variance functions. Figure 5.1 contains the trajectory for the simulated SDSTR given by (2.23) of Subsection 2.2.1 in Chapter 2 and is numerically simulated by using the parameter choices in Table 9.1 in Chapter 9.

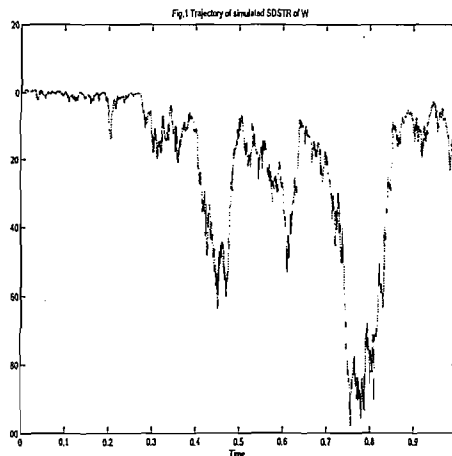


Figure 5.1: Trajectory for Simulated SDSTR

5.3.2 Properties of the Trajectory

Figure 5.1 shows the simulated trajectory for the CIR(Cox, Ingersoll and Ross) process also known as the square root process which is a Lévy process of the SDSTR problem with W being given by equation (2.23) of Subsection 2.2.1 in Chapter 2. This model give a much better fit to the data and lead to a significant improvement with respect to the Black-Scholes Model. Here different values for the banking parameters are collected in Table 9.1 in Chapter 9. The number of jumps of the trajectory was limited to 1000, with the initial values for T and R fixed at 1 and 20, respectively.

In the main, provisioning for the deposit withdrawals involve decisions about the volume of Treasuries and reserves held by the banks. For withdrawal provisioning, we assume that the stochastic dynamics of W is given by equation (2.23) of Subsection 2.2.1 in Chapter 2. This stochastic dynamic model enables us to analyze the interplay between deposit withdrawals and the provisioning for these withdrawals via Treasuries and reserves. In this spirit, we consider the stochastic dynamics of W that is driven by a *CIR* process with its trajectory given by Fig. 5.1. As has been noted of Subsection 2.2.1.2 in Chapter 2, deposits are subjected to the risk of early withdrawals. According to the trajectory in Fig.5.1 , we associate the latter assertion with the following trends: the trajectory initially depicted a steady path that shows the correspondence between the provisions of deposit withdrawals and W . Subsequently, a decrease in value occurs which may be due to the influence of an interest rate change. A model with such financial behavior forces depositors to become less inclined to withdraw their deposits. In this case, the volume of provisions for deposits withdrawals may decrease. Finally, we observe that the trajectory does not make an allowance for deposit withdrawals with $W < 0$. This is an indication that the bank may have inadequate securities and assets. With the higher interest rate volatility of 0.4, the adverse impact of the correlation between interest rate and withdrawals is diminished.

Chapter 6

ANALYSIS OF THE MAIN ISSUES

6.1 ANALYSIS OF PROVISIONING FOR LOAN LOSSES ISSUES

6.1.1 Banking Loan Loss Model

6.1.2 Optimal Provisioning in a Robust Control Framework

6.1.3 Comparison with a Discrete-Time Provisioning Model

6.1.4 Other Issues

6.2 ANALYSIS OF PROVISIONING FOR DEPOSIT WITHDRAWALS ISSUES

6.2.1 Assets

6.2.2 Liabilities

In this chapter, we provide an analysis of the main issues on modeling of bank provisioning given in Sections 2.1 and 2.2 in Chapter 2, and optimal provisioning for loan losses discussed in Chapter 3. We also analyze the main issues on optimal provisioning for deposit withdrawals and then on numerical simulations and examples discussed in Chapter 3 and 4, respectively.

6.1 ANALYSIS OF PROVISIONING FOR LOAN LOSSES ISSUES

In this section, we briefly analyze some of the issues emanating from Section 2.1 in Chapter 2.

6.1.1 Banking Loan Loss Model

In this subsection, we discuss the bank capital, loan losses and bank loan loss provisioning.

6.1.1.1 Bank Loan Losses

Statistical analysis of the bank loan losses mentioned in Section 2.1.1 in Chapter 2 has become a very interesting topic. In this regard, many applications of statistical modeling in the banking industry involve both summary and prediction of loan losses. However central interest often lies in prediction of quantities other than the mean of the distribution, and of variables that are not normally distributed, and so cannot be summarized by the mean. This makes our description of loan losses via Lévy processes an even more viable option. In this situation, when modeling features are measurable in terms of the quantiles of the distribution, techniques from *quantile regression* may allow prediction from information on covariates of how to predict the distributions of loan losses.

As we have mentioned in Subsection 1.1.1 in Chapter 1, the new Basel accord redefines the minimum capital requirements for depository financial institutions (see, for instance, [7]). In particular, the accord specifies new requirements for data, modeling, monitoring and documentation for retail loans. Understanding the effects of these requirements has the potential for providing a significant improvement in capital efficiency for secured loans. In short, the idea is to relate the amount of capital a particular loan requires to its riskiness. This is measured via the probability of default (PD), loss given default (LGD; potential percentage loss) and the exposure at default (EAD; monetary value of the loss). Under Basel II, institutions wanting to adopt the Internal Ratings Based Advanced approach are required to produce estimates for LGD as a percentage of the expected exposure. Modeling of LGD for secured loans, usually mortgages, is strongly dependent on the estimation of the likely realizable value of the security, such as the house and its mortgage. When this value is less than the loan amount a loss is realized. On the other hand, when there is an excess,

the benefit is returned to the original mortgage holder. Using mean house value estimates to estimate losses leads to unrealistic estimates of loss given default (LGD) as this ignores the asymmetric nature of the loss calculation.

6.1.1.2 Bank Loan Loss Provisioning

Provisions against expected loan losses should not count as capital since they are only marginally different from those that have been identified. Provision for unexpected loan losses should also be made, but these may count as capital since they satisfy the traditional role of regulatory bank capital as a precautionary buffer stock against unexpected declines in asset values. We note that the loan loss provisioning, P , fits into the paradigm of generalized Ornstein-Uhlenbeck (OU) processes driven by Lévy processes which have been thoroughly studied in the recent past. The OU process build the stochastic behaviour model for the volatility of the loan process, enabling it to give much better fit to the data for provisioning. Also, loan loss reserves may be viewed as the reserves left over after expected loan losses have been deducted from provisioning for loan losses.

The constant mix strategy used in this thesis is dynamic in the sense that it requires a re-balancing of the provisioning portfolio at any moment of time depending on the corresponding loan dynamics and losses suffered. From a practical point of view, the choice of a provisioning strategy of constant mix type is acceptable, as the bank operates under Basel II regulatory constraints (see, for instance, [5] and [7]), that suggests a portfolio weight of the provisions for expected losses that may have an upper risk bound. From a theoretical point of view, this condition is convenient since the provisioning process remains in the framework of exponential Lévy processes. While the formulae in Proposition 2.1.2 in Chapter 2 are explicit, they require the evaluation of a difficult numerical integral in equation (2.16) of Subsection 2.1.2 in Chapter 2. Lévy density doesn't always have a closed-form expression and finding its parameters from data is a non-trivial problem. Hence the integral is not easy to evaluate.

6.1.1.3 Historical Evidence

In Subsection 2.1.3 in Chapter 2, empirical data for Organization for Economic Corporation and Development (OECD) countries is used to establish the relationship between loan loss provisioning and aggregate asset prices. In this subsection, we provide a brief discussion of some of the outstanding features of the data for provisioning for loan losses provided in Subsection 2.1.3 in Chapter 2. In particular, the

figures show that provisions typically do not increase until after economic growth has slowed considerably and often not until the economy is clearly in recession. This is observed, for instance, in the figures for Australia, Norway, the UK and the USA. More discussion is provided below.

The data for Australia represented in Figure 2.1 of Subsection 2.1.3 in Chapter 2, shows that provisions failed to increase substantially in the late 1980's, when credit and asset prices were growing rapidly and the financial imbalances were developing. Moreover, the peak in provisions did not occur until at least one year after the economy had clearly slowed down. The data for Norway from Figure 2.2 of Subsection 2.1.3 in Chapter 2 exhibits similar behavior as the data for Spain from Figure 2.3 of Subsection 2.1.3 in Chapter 2. In these cases, provisions failed to increase substantially in the late 1980's, when credit and asset prices were growing rapidly and the financial imbalances were developing. In each of these figures the peak for provisions did not occur until the recession. However, one of the differences between these figures is the amount by which provisions increased when the economy had clearly slowed down.

In countries like the United States (see Figure 2.9 of Subsection 2.1.3 in Chapter 2), Australia and Norway (see Figure 2.2 of Subsection 2.1.3 in Chapter 2), the provisions overlapped the output gap during the recessions. It seems that even after the banking problems that Japan experienced had been resolved, in Italy the situation changed slightly. The (low) positive correlation between provisions and the business cycle in the United States (see Figure 2.9 of Subsection 2.1.3 in Chapter 2) appears to be driven by the surge in provisions in the second half of the 1980's. This phenomenon seems to reflect the delayed cleaning of the balance sheets following the developing countries' debt crisis of the early 1980's.

6.1.2 Optimal Provisioning in a Robust Control Framework

In this subsection, we provide a brief discussion of some of the robust control aspects of the statement and solution of the main result of this thesis.

6.1.2.1 Statement of a Mixed Optimal/Robust Control Problem

We have seen in Section 3.1 in Chapter 3 that our robust control problem is a constrained one. Another type of control problem is the so-called *multiplier mixed optimal/robust control problem*. In our context, this problem can be reformulated as follows. For $t \geq 0$, we may formulate an optimal provisioning multiplier robust control problem as

$$\max_{\pi \in [0,1]} \mathbb{E}[P_{\pi}(T)] + \theta \text{var} \left(P_{\pi}(T) \right) \quad \text{subject to equation (2.10)} \quad (6.1)$$

where T is some given planning horizon. Problem 3.1.1 in Chapter 3 and the multiplier robust control problem formulated in equation (6.1) are related to each other. We can interpret the robustness parameter θ in equation (6.1) as an implied Lagrange multiplier on the constraint $\text{var} \left(P_{\pi}(T) \right) \leq C$.

6.1.2.2 Solution of a Mixed Optimal/Robust Control Problem

The solution of the mixed optimal/robust control problem for bank provisioning, π^* , given by equation (3.3) of Section 3.2 in Chapter 3 and presented in Theorem 3.2.1 of Section 3.2 has an interesting form. More specifically, the expression for π^* contains two important financial variables that are related to loan losses, S , namely, the loan loss reserves rate, r^R , and the default premium rate, c^d . If S increases (decreases) with an increased (decreased) probability of losses then r^R decreases (increases). Also, the form of the optimal solution of the robust control problem, π^* , seems to suggest that an increase (decrease) in r^R will result in an increase (decrease) in π^* . In other words, there seems to be a negative correlation between S and r^R and a positive correlation between r^R and π^* . On the other hand, if c^d increases (decreases) then r^R also increases (decreases). Although the connection between c^d and S is a complicated one, it is generally accepted that if S increases (decreases) then c^d will increase (decrease) subsequently. Also, the form of π^* given by equation (3.3) of Section 3.2 in Chapter 3 seems to suggest that an increase (decrease) in c^d will result in a decrease (increase) in π^* . Thus, there seems to be a positive correlation between both c^d and r^R and c^d and S but a negative correlation between c^d and π^* . In short, the above discussion supports the hypothesis that the level of reserves is increased with the provision for loan losses while the level is decreased by loans that are charged off. In this regard, we have to note that because recoveries are made on some bad loans, only nett loan losses deplete the level of the reserve.

The maximal expected terminal provisioning for loan losses, $\mathbb{E}[P_T^{\pi^*}]$ given by equation (3.4) of Section 3.2 in Chapter 3 under the variance constraint given by equation (3.2) in Theorem 3.2.1 of Section 3.2, was found to be dependent on both the default premium rate, c^d and the loan loss reserve rate, r^R . This means that both the optimal solution, π^* , and $\mathbb{E}[P_T^{\pi^*}]$ depends on c^d and r^R at the end of a planning period. As a consequence, $\mathbb{E}[P_T^{\pi^*}]$ is impacted on by both the loan loss reserves for unexpected

losses, $X^{(0)}$, and the provisioning for expected losses $X^{(1)}$. Economic intuition suggests that c^d and r^R are important instruments for manipulating provisions for loan loss at the end and throughout a planning period.

6.1.3 Comparison with a Discrete-Time Provisioning Model

Our contention is that the provisioning model developed in Section 2.1 in Chapter 2 and Section 3.1 in Chapter 3 is the first to utilize semi-martingale theory (via Lévy processes) to formalize some of the notions related to loan loss provisioning in the banking industry. From experience, we know that individual banks do not release complete information about provisioning to the public because of the sensitive nature of their individual provisioning practices. In addition, data that is freely available from central banks is often unsuitable for judging provisioning practice. As such it is difficult to access data that would help to verify good practice for our loan loss provisioning model. Instead, in the current subsection, we compare our model with a popular discrete-time model that has been made available in the public domain. Following [2], in order to determine a discrete-time model for total loan loss provisions, P , we may have to estimate the discretionary and the non-discretionary components of P . In the sequel, factors which may explain the form of P are grouped into three classes and compared with our provisioning model.

6.1.3.1 Non-Discretionary Component of Loan Loss Provisioning

Firstly, as is suggested in Section 6.1.2.2, the non-discretionary component of P involves expected loan losses. In the paradigm of [2], backward-looking rules based on identified credit losses give a strong cyclical to this component. Furthermore, the provisioning model includes three variables which represent the risk from a bank's portfolio. The *non-performing loans-to-gross loans ratio at the end of year t* denoted by NPL_{it} and the *first difference of NPL_{it}* given by

$$\Delta_{t/t+1}NPL_{it} = NPL_{it+1} - NPL_{it}$$

are indicative of the default risk associated with particular bank loans. As a result, [2] expects a positive correlation between these two variables and P . The said contribution also discusses the default risk for the overall credit portfolio which is measured by the *provisions for loan losses-to-total assets ratio*, L_{it} (recall Figure 1.1 from Section 1.1.1, and Figures 2.1, 2.2, 2.8 and 2.9 from Subsection 2.1.3 for real-world examples).

The coefficient associated with this variable should also be positive.

By comparison, in this thesis, the formula for expected loan losses, S^e , given by equation (2.2) of Section 2.1.1.1 in Chapter 2, is solely based on knowledge about the behavior of the borrower (related to NPL_{it} and $\Delta_{t/t+1}NPL_{it}$ in the previous paragraph; L_{it} reflects the behavior of the bank itself). Also, S^e and the provisioning for expected loan losses, $X^{(1)}$, presented in equation (2.6) of Section 2.1.2.1 in Chapter 2 are related via the default premium rate, c^d . Of course, the value of c^d is always a reasonable indication of the default risk of the specific loans. In particular, [2] predicts a positive correlation between c^d and P and suspects a strong connection with the credit cycle.

6.1.3.2 Discretionary Component of Loan Loss Provisioning

In the second place, the discretionary component of P results from three different management objectives. These management issues can be identified as the incoming smoothing hypothesis, regulatory capital management and the signalling hypothesis. The main features of each of these objectives are briefly described below.

Under the *income smoothing hypothesis*, banks understate (overstate) P when earnings are expected to be low (high) relative to that of other years (inter-temporal smoothing). If banks use P to smooth earnings, then a positive correlation between earnings before taxes and loan loss provisions, denoted by ER_{it} and P , respectively, should be expected (see, for instance, [72] and [74]). As the propensity to smooth income is higher for banks with good performance relative to banks with moderate current performance, [2] introduces a dummy variable, ERH_{it} , that may be realized as

$$ERH_{it} = \begin{cases} ER_{it}, & \text{Positive Earnings before Taxes and Loan Loss Provisions;} \\ 0, & \text{Otherwise.} \end{cases} \quad (6.2)$$

The coefficient for ERH_{it} from equation (6.2) above should be positive if there is non-linearity in the relation between P and earnings.

Poorly capitalized banks can also use P to *manage regulatory capital*. For the total capital ratio, TCR , the variable $TCRL_{it}$ takes the value

$$TCRL_{it} = \begin{cases} \frac{TCR - 8}{8}, & \text{Observations for Bank } i \text{ in the 1st Quartile of } TCR; \\ 0, & \text{Otherwise.} \end{cases} \quad (6.3)$$

A positive correlation between P and $TCRL_{it}$ could be expected if poorly capitalized banks are less willing to make provisions (see [89]). However, accounting relations could also influence the relation between bank capital and loan loss provisions. If regulatory capital variations are more related to retained earnings than loan loss allowances the correlation between P and $TCRL_{it}$ from equation (6.3) should be negative.

Finally, P can also be used to *signal the financial strength of the bank* or, equivalently, to verify the *signaling hypothesis*. In [9], it is suggested that loan loss provisions can indicate that management perceives the earnings capabilities of the bank to be sufficiently strong that it can withstand a hit to earnings in the form of additional loan loss provisions. If signaling is an important incentive in choosing P , then a positive correlation between P and changes in future earnings before taxes and P (see, for instance, [94]) should be observed. The variable, $SIGN_{it}$, defined as the *one-year-ahead changes of earnings before taxes and loan loss provisions* may be expressed as

$$SIGN_{it} = \frac{ER_{it+1} - ER_{it}}{0.5(TA_t + TA_{t+1})},$$

where TA is the total assets. In particular, $SIGN_{it}$ is computed in order to test the signaling hypothesis. A positive correlation with P is expected. In our provisioning model, the management objectives of income smoothing, managing regulatory capital and signaling the bank's financial strength has not been explored yet.

6.1.3.3 Macroeconomic Environment and Loan Loss Provisioning

Thirdly, the macroeconomic environment should affect the ability of borrowers to repay banks assets. The paper [2] introduces the annual growth rate of GDP, denoted by y'_{it} , in lieu of the fact that private sector wealth varies with the business cycle. A negative sign for the variable y'_{it} is usually found. In this case, we may model the relationship between loan loss provisions and the explanatory variables defined above as

$$\begin{aligned}
P_{it} = & \alpha_0 + \alpha_1 P_{it-1} + \alpha_2 NPL_{it} + \alpha_3 \Delta_{t/t+1} NPL_{it} + \alpha_4 L_{it} + \alpha_5 y'_{it} \\
& + \alpha_6 ER_{it} + \alpha_7 ERH_{it} + \alpha_8 TCRL_{it} + \alpha_9 SIGN_{it} + \zeta_{it},
\end{aligned} \tag{6.4}$$

where P_{it} is the ratio of loan loss provisions (specific provisions plus general provisions) to total assets at the end of year t for bank i . The contribution [2] introduces a lagged dependent variable as an explanatory variable to take into account a dynamic adjustment of P_{it} . If banks adjust their provisions slowly to recognize potential losses against loans following a default event, then provisions could be systematically related to each period. The model accounts for the possibility that the use of discretionary P for one purpose is conditional on the effects of the other two motivations; this is done by jointly estimating the relationships between loan loss provisions and income smoothing, capital management and signaling behaviors. Equation (6.4) is estimated to compute the non-discretionary component, $NDISC_{it}$, and the discretionary component, $DISC_{it}$, of P . One may assume that these two components are linear functions of the variables included in equation (6.4). Thus, the non-discretionary component of P is estimated as the sum of the products of its explanatory variable times the corresponding estimated coefficient from equation (6.4). The same method is used to compute the discretionary component. The affect of macroeconomic conditions on the ability of borrowers to repay loans has already been discussed in the technical report [41] where the provisioning model has some similarities with the one presented in this paper.

6.1.4 Other Issues

In this subsection, we speculate about a possible extension of our model to a Value-at-Risk (VaR) setting.

6.1.4.1 Value-at-Risk Approach

In this subsection, we provide a definition of Value-at-Risk (VaR) for our value risk process and consider certain aspects of a loan loss provisioning maximization problem with a VaR constraint. The Basel II capital accord permits an internal models approach to capital requirements, by which banks are allowed to use their internal VaR models to help determine their capital adequacy. Formally, VaR is specified as some high quantile of the corresponding loss distribution. For instance, in many countries, banks are required to report their VaR daily at a 99 % confidence

level over a one-day horizon as well as a two-week horizon (10 working days). The minimum capital requirement is then equal to the sum of a charge for credit risk and a charge for market risk. The credit risk charge is equal to 8 % of the risk-weighted assets and the charge for market risk is equal to a multiple of the average reported two-week VaRs in the last 60 trading days.

In the banking industry, VaR is a standard risk measure. The Basel II capital accord (see, for instance, [5] and [7]) permits an approach involving internal VaR models in order to determine the bank's capital requirements. This approach is not only allowed but in fact encouraged under Basel II. The distribution of the a.s. limit l_π^∞ in Proposition 2.1.2 of Subsection 2.1.2 in Chapter 2 allows us to obtain a measure of the risk in a stationary sense. This fact is borne out by the following definition.

Definition 6.1.1 (Value at Risk with Probability β): *Let $\pi \in \Pi \subseteq [0, 1]$ be a specific non-empty interval. In this case, for some probability $\beta \in (0, 1)$, we define*

$$\text{VaR}_\beta(l_\pi^\infty) = \inf\{x \in \mathbb{R} : \mathbb{P}(l_\pi^\infty > x) \leq \beta\}.$$

6.1.4.2 Value-at-Risk Maximization Problem

An ongoing problem is to solve the following maximization problem that is based on issues related to loan loss provisioning and regulatory constraints from the Basel II capital accord.

Problem 6.1.2 (Statement of Provisioning Maximization Problem): *Suppose that P_π is defined by equation (2.9) of Subsection 2.1.2 in Chapter 2. Then the loan loss provisioning maximization problem is to determine*

$$\max_{\pi \in \Pi} \mathbb{E}[P_\pi(t)] \text{ subject to } \text{VaR}_\beta(l_\pi^\infty) \leq C \quad (6.5)$$

for given constraint, $C > 0$, and probability, β , and some fixed time $t > 0$.

This set-up will enable us to consider several economic states of the bank involving, for instance, severe loan losses and risky provisioning portfolios. The use of $\text{VaR}_\beta(l_\pi^\infty)$ as a risk measure in the loan loss provisioning maximization problem may be justified by the fact that this quantity is equal to the loan loss provisioning required to prevent the loan loss provisioning slipping into negative territory for a long risk horizon with a

sufficiently high probability $1-\beta$. Note that in our definition the VaR does not depend on the initial loan loss provisioning and on the time t , but only on the selected π and the stochastic properties of the loan losses and provisioning processes. In reality, a bank may prefer a π that maximizes provisioning at the end of a stipulated period. Also, from Basel II, there are certain regulatory bounds (see, for instance, [5], [7] and [33]) imposed on banks related to the amount of risk exposure it may allow.

6.2 ANALYSIS OF PROVISIONING FOR DEPOSIT WITHDRAWALS ISSUES

The dynamic models of bank items constructed in this paper are compliant with the dictates of the Basel II capital accord. For instance, the properties of our models are positively correlated with the methods currently being used to assess the riskiness of bank provisioning portfolios and their minimum capital requirement (see [5] and [7]).

6.2.1 Assets

In this subsection, we analyze aspects of the bank assets such as provisions for deposit withdrawals, Treasuries and reserves.

6.2.1.1 Treasuries and Reserves

As was mentioned in Subsection 2.2.1.1 in Chapter 2, Treasuries are bonds issued by a national Treasury and may be modeled as a risk-free asset (bond) in the usual way. In modern times, it is possible to assign a price to reserves and to model them by means of Lévy processes. This is due to the discontinuity associated with their evolution and because they provide a good fit to real-life data. In this regard, several interesting contributions have led to the choice of representation with an expression give by equation (2.23) of Subsection 2.2.1 in Chapter 2 for the dynamics of the sum of the Treasuries and reserves. Amongst these is a paper by Chan (see [24]) that treats the case where the Lévy decomposition of general assets corresponds to our decomposition. The size of the depository value, k_d , from equation (2.23) of Subsection 2.2.1 in Chapter 2, can vary greatly.

Two economic aspects of required reserves on bank deposits are noteworthy. Firstly, their impact on bank-intermediated investment versus direct investment and, secondly, their opportunity cost. The main function of bank reserves is to serve as a

buffer to mitigate inefficient liquidation of a bank's assets in order to meet the demand for liquidity by investors. Due to some transaction costs or information costs, investors may prefer bank-intermediated investment to direct investment. Banks offer investors competitive deposit returns compared to the liquidation value of investment to attract funds from investors. If the Federal Reserve allows banks to set their individual optimal level of reserves, this might mitigate costs associated with required reserves. If banks implement the social optimum, this may introduce additional fragility into the banking system. We argue that required reserves might lead to deadweight loss if they are set above a bank's optimally determined reserves.

6.2.1.2 Provisions for Deposit Withdrawals

As has been noted in Subsection 2.2.1.2 in Chapter 2, deposits are subject to the risk of early withdrawal. This phenomenon can be associated with some interesting trends. For instance, as interest rates fall, depositors become less inclined to withdraw their deposits and the volume of provisions for deposit withdrawals may decrease. On the other hand, at higher rates of interest depositors have a greater propensity towards withdrawal and provisions for deposit withdrawals may increase. Furthermore, an increase in interest rate volatility diminishes the adverse impact of the correlation between interest rates and withdrawals. As a consequence, the optimal deposit rate may decrease which results in the widening of the optimal intermediation margin. An increase in the volatility of withdrawals exacerbates the impact of the correlation between interest rates and the propensity to withdraw. A negative correlation between interest rates and propensity to withdraw would be to the advantage of bank management because an increase in this correlation increases optimal deposit rates.

6.2.2 Liabilities

This subsection provides an analysis of aspects of the discussion about liabilities in Subsection 2.2.2 in Chapter 2.

6.2.2.1 Depository Contracts

In Subsection 2.2.2 in Chapter 2, we noted that equation (2.19) of Subsection 2.2.1 in Chapter 2 suggests an association between reserves and deposits that allows the sum of cohort deposits, D^c , to be dependent on the bank reserves, R_t , and denote this by $D_t^c(R_t)$. In reality, D^c will not only be dependent on R but on all the unweighted assets and several other banking items. The building of a model to incorporate this

dependence is the subject of much debate at present.

A simplifying modification of the depository contract can be considered where $T_i = T$ for all i . In this situation, equation (2.27) of Subsection 2.2.2 in Chapter 2 becomes

$$D_T^c(R_T) = D_T^c(R_T)\mathbb{T}(T)\mathbb{T}^{-1}(T). \quad (6.6)$$

6.2.2.2 Stochastic Counting Process for Deposit Withdrawals

In standard banking models, the volume of deposits and their withdrawals usually equates the input prices Lerner index with the inverse of the elasticity of supply to determine the optimal rate paid for deposits. These models have no explicit time dimension so that deposits have no meaningful term to maturity. Their implicit time to maturity is homogeneous across accounts and, without modeling the intertemporal behavior of interest rates, deposits must be assumed to be held until maturity. Clearly the models above ignore critical aspects of bank input pricing, viz., depository contracts have different times to maturity and the fact that depositors can withdraw funds before maturity. Both these issues are addressed in our contribution.

6.2.2.3 Cost of Deposit Withdrawals

In Subsection 2.2.2 in Chapter 2, the rate term for auxiliary profits, $\mu^a(s)$, may be generated from activities such as special screening, monitoring, liquidity provision and access to the payment system. Also, this additional profit may be arise from imperfect competition, barriers to entry, exclusive access to cheap deposits or tax benefits.

6.2.3 Risk and The Banking Model

This subsection provides some comments about the connection between our dynamic banking models and risk management.

6.2.3.1 Basic Risk Concepts

The cost process, $c^o(\pi)$, presented in Subsection 1.2.5 in Chapter 1, corresponds to the value of the provisioning portfolio, $\widehat{V}(\pi)$ less the accumulated return from reserves, R . Usually, the aggregate costs, $c_i^o(\pi)$, incurred on the time interval $[0, t]$ decompose easily into the cost incurred on $(0, t]$ and an initial cost $c_0^o(\pi) = V_0(\pi)$.

6.2.3.2 Generalized GKW Decomposition of $T^{-1}(t)F_t(R_t, u)$

From equations (4.1) and (4.2) in Theorem 4.1.1 in Chapter 4, we have two distinct parts associated with the continuous and discontinuous (jump) components of n^R given by $D_x F_t(x, R_{t-}, u)$ and $j_t(x, u)$, respectively. In the Brownian framework of the Black-Scholes asset market, a hedging strategy for reserves (assets) results from

$$n_t^R = D_x F_t(R_{t-}, u).$$

By contrast, Theorem 4.1.1 in Chapter 4 suggests that the hedging strategy for reserves with jumps is given by equation (4.1) of Section 4.1 in Chapter 4. This means that if L corresponds with Z then $\nu_t^Q(dx) = 0$ and $\kappa_t = c^2$.

6.2.3.3 Risk Minimizing Strategy for Depository Contracts

K^D from equation (4.9) in Theorem 4.2.1 has interesting ramifications for risk (minimization) management of the bank provisioning portfolio. For instance, K^D allows for the possibility that the credit risk can be reduced to intrinsic risk. In this regard, small changes in K^D can be represented as

$$dK_t^D = l_{s^-}^x \vartheta_s^{(1)} dZ_s^Q + \int_{\mathbb{R}} l_{s^-}^x \vartheta_t^{(2)}(y) N^Q(ds, dy) + v_s dM_s^I, \quad (6.7)$$

and interpreted as the losses incurred by the bank. If we analyze the first two terms of K^D and dK^D in equation (4.9) of Section 4.1 and equation (6.7) of Subsection 6.2.2, respectively, we may conclude that the integrals with respect to Z^Q and N^Q are the drivers of credit risk. In the context of incomplete loan markets in a Lévy-process setting, this demonstrates the influence of bank loan issuing on deposit risk. Here, the reliance of the provisioning strategy components, $\vartheta_t^{(1)}$ and $\vartheta_t^{(2)}$, on D_x and $j_t(x)$, results in a risk increase that originates from the continuous part of the bank loan process. Note that the credit risk driver is a function of the expected number of depository contracts surviving on the time interval $[t^-, T]$. Here a deposit withdrawal results in a decrease in dK_t^D . Also, the last term of dK^D in equation (6.7) of Subsection 6.2.2, that contains

$$dM_t^I = dN_t^I - \iota_t dt$$

can be interpreted as the source of risk for the entire bank provisioning portfolio.

From the first formula in equation (4.10) of Theorem 4.2.1, it is clear that the optimal investment in reserves, n_t^{R*} , is heavily dependent on the number of cohort deposits that are withdrawn during the time interval $[t^-, T]$. In particular, as the number of withdrawals, N^I , increases (l_t^x decreases) it is likely that n_t^{R*} will decrease. This trend is also possible for the optimal investment in the bank treasuries, n_t^{T*} , from the second formula in equation (4.10) above. The second part of Theorem 4.2.1 leads to an expression for the initial risk of the provisioning strategy, R_0 . In this regard, for $s > t$, and l_s^x defined by equation (2.29) of Subsection 2.2.2 in Chapter 2, the filtration \mathcal{F}_t follows a binomial distribution with the survival probability being $P_2(T_i > u + t + x | T_i > x + t)$. As a consequence, for $0 \leq t \leq T$, we have that

$$\begin{aligned} & \mathbb{E}^Q[(l_s^x)^2 | \mathcal{F}_t] \\ &= l_s^x P_2(T_i > s + t + x | T_i > x + t) (1 - P_2(T_i > s + t + x | T_i > x + t)) \\ & \quad + (l_t^x)^2 (P_2(T_i > s + t + x | T_i > x + t))^2. \end{aligned}$$

and

$$\begin{aligned} R_0(\pi^*) &= n^x \int_0^T \left\{ P_2(T_i > s + x | T_i > x) \left[1 - \right. \right. \\ & \quad \left. \left. P_2(T_i > s + x | T_i > x) + n^x P_2(T_i > s + x | T_i > x) \right] \right. \\ & \quad \left. \mathbb{E}^Q \left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} (\vartheta_s^{(2)}(y))^2 \right. \right. \\ & \quad \left. \left. \nu_s^Q(dy) \right] + \mathbb{E}^Q[\nu_s^2(R_s)] P_2(T_i > s + x | T_i > x) \omega_{x+s} \right\} ds. \quad (6.8) \end{aligned}$$

A first observation is that the reserve risk component of R_0 in equation (6.8) has the form

$$\begin{aligned} & n^x \int_0^T \left\{ P_2(T_i > s + x | T_i > x) \left[1 - P_2(T_i > s + x | T_i > x) + n^x P_2(T_i > s + x | T_i > x) \right] \right. \\ & \quad \left. \mathbb{E}^Q \left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} (\vartheta_s^{(2)}(y))^2 \nu_s^Q(dy) \right] \right\} ds. \quad (6.9) \end{aligned}$$

It is clear that for the reserve risk component given by equation (6.9) of Section 6.2

in this chapter, if n^x increases then division of the risk component by $(n^x)^2$ does not result in the value of the said component tending to 0. This means that in our incomplete information setting, by contrast to the findings in the Brownian motion framework, a portion of risk resulting from the holding of reserves cannot be hedged against by merely increasing the number of depository contracts, n^x . As far as bank deposit risk is concerned, R_0 and the *relative initial risk ratio* given by

$$\rho_0 = \frac{\sqrt{R_0}}{n^x}$$

may be used to measure the risk associated with the non-hedgeable part of the sum of cohort deposits. Also, the deposit risk component of R_0 given by equation (6.8) of Section 6.2 in this chapter behaves as in

$$n^x \rightarrow \infty \Rightarrow 1/n^{x2} \left\{ n^x \int_0^T \mathbb{E}^Q[v_s^2(R_s)] P_2(T_i > s + x | T_i > x) \omega_{x+s} ds \right\} \rightarrow 0.$$

It is not clear how a general risk analysis can be done for the *relative risk ratio* of the form

$$\rho_t = \frac{\sqrt{R_t}}{l_t^x}.$$

The simplifying modification of the depository contract in equation (6.6) above also has important ramifications for risk management. This scenario considers the cohort claim at terminal time, $D^c(R_T)$, rather than the more general $D_i^c(R_t, u)$ (see, for instance, equation (2.25) of Subsection 2.2.2 in Chapter 2 and equation (6.6) above) so that it becomes redundant to consider the variable "u". Henceforth, the bank obligation generated by the entire portfolio of depository contracts is considered to be the \mathcal{F}_T -measurable discounted deposit withdrawal

$$D^s = T^{-1}(T) D^c(R_T) \sum_{i=1}^{n^x} \mathbf{I}(T_i > T) = T^{-1}(T) D^c(R_T) l_T^x. \quad (6.10)$$

In the simplified case, the independence of the reserves and the bank withdrawals enables us to represent the intrinsic value of the entire depository contract portfolio,

V_t^{s*} , as

$$V_t^{s*} = l_t^x P_2(T_1 > T + t + x | T_1 > x + t) T_t^{-1} F_t(R_t), \quad 0 \leq t \leq T, \quad (6.11)$$

with initial value

$$V_0^{s*} = n^x P_2(T_1 > T + x | T_1 > x) F_0(R_0).$$

Under certain conditions, V_0^s may be considered to be the single deposit taken by the bank at $t = 0$.

In order to hedge deposit withdrawals, it is appropriate to adopt a (local) risk minimization approach, since

$$V_t^s(\pi^{s*}) = V_t^{s*}, \quad \text{for all } t \in (0, T).$$

Suppose that the intrinsic value, V_t^{s*} , is given by equation (6.11) above and the variables $\vartheta_t^{(1)s}$ and $\vartheta_t^{(2)s}$ are analogous to those given by equation (4.2) in Chapter 4. For the depository contract in equation (6.10) above, with $0 \leq t \leq T$, the intrinsic value process, V^* , has the generalized GKW decomposition

$$V_t^{s*} = V_0^{s*} + \int_0^t l_s^x P_2(T_1 > T + s + x | T_1 > x + s) n_s^R d\widehat{R}_s + K_t^{D^s},$$

where

$$\begin{aligned} K_t^{D^s} &= \int_0^t l_s^x P_2(T_1 > T + s + x | T_1 > x + s) \vartheta_t^{(1)s} dZ_s^{\mathbb{Q}} \\ &\quad + \int_0^t \int_{\mathbb{R}} l_s^x P_2(T_1 > T + s + x | T_1 > x + s) \vartheta_t^{(2)s}(y) N^{\mathbb{Q}}(ds, dy) \\ &\quad + \int_0^t -T_s^{-1} F_s(R_{s-}) P_2(T_1 > T + s + x | T_1 > x + s) dM_s^I, \end{aligned}$$

is orthogonal to \widehat{R} . For $0 \leq t \leq T$, under the equivalent probability measure \mathbb{Q} , the unique admissible risk-minimizing provisioning strategy, $\pi^{s*} = (n^{Rs*}, n^{Ts*})$, for the

deposit withdrawal in equation (6.10) has the form

$$\begin{aligned} n_t^{Rs*} &= l_t^x P_2(T_1 > T + t + x | T_1 > x + t) n_t^R, \\ n_t^{Ts*} &= l_t^x P_2(T_1 > T + t + x | T_1 > x + t) \Gamma^{-1}(t) F_t(R_t) - n_t^{Rs*} \widehat{R}_t. \end{aligned}$$

For $0 \leq t \leq T$, the intrinsic or remaining risk process may be expressed as

$$\begin{aligned} R_t^s(\pi^{s*}) &= \int_t^T \left\{ P_2^2(T_1 > T - s + x | T_1 > x + s) \mathbb{E}^Q[(l_t^x)^2 | \mathcal{F}_t] \right. \\ &\quad \times \mathbb{E}^Q \left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} (\vartheta_s^{(2)s}(y))^2 \nu_s^Q(dy) | \mathcal{F}_t \right] \Big\} ds \\ &\quad + l_t^x P_2(T_1 > T + t + x | T_1 > x + t) \\ &\quad \int_t^T P_2(T_1 > T + s + x | T_1 > x + s) \omega_{x+s} \Gamma^{-2}(s) \mathbb{E}^Q[F_s^2(R_s) | \mathcal{F}_t] ds. \end{aligned}$$

Chapter 7

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 CONCLUSIONS

7.2 FUTURE INVESTIGATIONS

In this chapter, we provide concluding remarks and comment about possible topics for future research.

7.1 CONCLUSIONS

In this section, we consider concluding remarks about all the Chapters.

7.1.1 Concluding Remarks About Chapter 1

Chapter 1 established the preliminary information about Lévy processes and distinguished the thesis from the pre-existing literature. Subsequently, the main problems addressed in the thesis were also identified in this chapter.

7.1.2 Concluding Remarks About Chapter 2

In Chapter 2, we discussed important features for a model for loan losses and its provisioning. Banks provision loan losses by allocating funds to the provisioning for

expected losses and loan loss reserves for unexpected losses. In Subsection 2.1.3, we provided historical evidence for our modeling choices and a particular emphasis was placed on provisioning data from Australia, Norway, the United Kingdom and United States.

Section 2.2 extended some of the modeling and optimization issues highlighted in [71] (see, also, [70], [40] and [73]) by presenting jump diffusion models for various bank items. Here we introduced a probability space that was the product of two spaces that models the uncertainty associated with the bank reserve portfolio and deposit withdrawals. As a consequence of that approach, the intrinsic risk of the bank arose then not only from the reserve portfolio but also from the deposit withdrawals. Throughout we considered a depository contract that stipulated payment to the depositor on the contract's maturity date. We concentrated on the fact that deposit withdrawals were catered for by the Treasuries and reserves held by the bank. The stochastic dynamics of the latter mentioned items and their sum were presented in Subsections 2.2.1.1. In Subsection 2.2.2, our main focus was on depository contracts that permitted a cohort of depositors to withdraw funds at will, with the stipulation that the payment of an early withdrawal was only settled at maturity. This issue was outlined in more detail in Subsection 2.2.2.1. Furthermore, in Subsection 2.2.2.2, we suggested a way of counting deposit withdrawals by cohort depositors from which the bank has taken a single deposit at the initial time, $t = 0$.

7.1.3 Concluding Remarks About Chapter 3

In Chapter 3, we discussed optimal provisioning for loan losses. In this regard, we applied principles from robustness to a situation where the decision maker was a bank owner and the decision rule determined the optimal provisioning strategy for loan losses. The aforementioned principles were well articulated in Section 3.1. The solution of a mixed optimal/robust control problem was detailed out in Section 3.2.

7.1.4 Concluding Remarks About Chapter 4

Chapter 4 dealt with a discussion on optimal provisioning for deposit withdrawals. For incomplete bank reserves, we derived a (locally) risk-minimizing hedging strategy for deposit withdrawals. This context provided a fertile environment for the derivation of general Lévy-driven models for reserves. Furthermore, we investigated the generalized GKW decomposition of the intrinsic value of the sum of cohort deposits contingent on a reserve process. This led naturally to a solution of a risk minimization problem for banks that provided a hedging strategy for deposit withdrawals. The spe-

cific risk types related to our study were intrinsic, reserve and depository risk those were associated with the cumulative cost of the bank provisioning strategy, reserve processes and the amount and timing of net cash flows from deposits and deposit withdrawals emanating from a cession of the depository contract, respectively.

7.1.5 Concluding Remarks About Chapter 5

Chapter 5 dealt with numerical simulations and examples for some historical data from member countries of the Organization for Economic Corporation and Development (OECD). In Section 5.1, the emphasis was placed on provisioning data from 9 OECD countries, viz. Australia, Finland, Italy, Norway, Spain, Sweden, the United Kingdom and the United States of America. In Section 5.2, we provided evidence that supported the fact that the output gap and the provisions for loan losses/total assets were negatively correlated. Furthermore, we investigated the correlation between the output gap and provisions in relation to profitability. In Subsection 5.2.1 we looked at empirical evidence that provisioning for loan losses in 9 OECD countries is pro-cyclical. In Subsection 5.2.2, we discussed correlations between profitability and provisions for loan losses. Here, we showed that provisions typically do not increase until after conomic growth has slowed down considerably and often not until the economy was in complete recession. Moreover, Subsection 5.3 provided a numerical simulation of provisioning via the sum of Treasuries and reserves.

7.1.6 Concluding Remarks About Chapter 6

In Chapter 6, we analyzed the main risk management issues arising from the banking model we constructed previously. In Subsection 6.1, we did an analysis of the main robust control issues, while Subsection 6.1.3 outline a comparison with a discrete-time provisioning model. In this case, firstly we indicated that the non-discretionary component of the total loan loss provisioning P , encapsulated expected loan losses. In the second case, the discretionary component of the total loan loss provisioning P , resulted from three different management objectives. That is, the total loan loss provisioning P , was used to smooth earnings, manage regulatory capital, and signal the financial strength of the bank. In the final case, we articulated that the macroeconomic environment should affect the ability of borrowers to repay bank assets. In Section 6.2, we analyzed the main risk management issues arising from the Lévy process-driven banking model that were constructed in the foregoing sections. A description of the role that bank assets played was presented in Subsection 6.2.1. Furthermore, we provided more information about depository contracts and the stochastic count-

ing process for deposit withdrawals in Subsection 6.2.2. Risk minimization and the hedging of withdrawals was discussed in Subsection 6.2.3.

7.1.7 Concluding Remarks About Chapter 7

In the current chapter, we provide concluding remarks in Subsection 7.1 and highlight some topics for future research in Subection 7.2.

7.1.8 Concluding Remarks About Chapter 8

In Chapter 8, we have shown different references used in the thesis.

7.1.9 Concluding Remarks About Chapter 9

In Chapter 9, we provided appendices. In Section 9.1, we provided a table of parameters and values consider for SDSTR problem. Section 9.2 establishes useful results from measure theory and states a Fubini Theorem. Finally, in Section 9.3, we presented characteristic triplets of one-dimensional Lévy process, L , and its linear transformation πL which is given for $\pi \in \mathbb{R}$. We also presented characteristic triplets for the natural logarithm of total provisioning

$$\ln \left(\frac{P_\pi}{p} \right).$$

7.2 FUTURE INVESTIGATIONS

In this section, we establish possible topics for future research for provisioning for loan losses and deposit withdrawals, respectively.

An ongoing debate revolves around an investigation into the implications for provisioning (see, for instance, [65]) of the

- implicit subsidy enjoyed by borrowers in the form of limited liability;
- stable assessment of the provisioning needed to prevent a negative outcome of the loan loss process with a high probability;
- expected costs of administering insolvency in the event of borrower default.

Also, we should strive to generalize our provisioning model in the following respects.

- dynamics of the Treasuries, risk premium and expected loss rates;
- incorporation of banking business lines for operational risk;
- liquidity of the provisioning portfolio;
- stochastic banking rates;
- optimization within a provisioning portfolio with a large number of loan positions;
- optimization within a dynamic provisioning portfolio;
- dependence between the provisioning portfolio and the total loan loss process;
- inter-temporal dependence in the provisioning portfolio.

An open problem is to directly determine the hedging strategies for loan processes that are semimartingales and are not subjected to transformation by an equivalent measure into a martingale. Further issues that have not been resolved yet relate to the generalization from 1 to n (multidimensional) reserve types (compare, for instance, equation (2.20)) of Section 2.2 in Chapter 2 and deterministic to stochastic interest rates (compare, for instance, equation (2.18) of Subsection 2.2.1 in Chapter 2).

More generally, the important open problems in banking are related to the Basel II capital accord and several perceived defects in its specifications. In the main, these concerns involve the computation of capital charges via the internal risk forecasting models proposed by banks themselves. These have a heavy reliance on Value-at-Risk (VaR) and related methodologies which have proven to have major defects for these purposes. In the sequel, we identify some elements of the context in which the problems with the Basel II capital accord has arisen. More specifically, these relate to the accord's failure to acknowledge the endogenous nature (or externality) of risk and liquidity, the use of VaR as a risk measure, the implementation of the Standard approach to credit risk, the implementation of the Internal Ratings Based (IRB) approach to credit risk, the definition of and charges against operational risk, the flexibility of regulator behavior under Pillar 2: Supervisory Review and exacerbation of procyclicality.

- Firstly, existing banking risk models consider risk as a fixed exogenous process. This, however, does not hold true in general. Market volatility is, in part at least, the outcome of interaction between market players and is thus endogenous. This endogeneity may be of considerable consequence during times of crisis.
- In the second place, VaR is a misleading risk measure when the returns are not normally (elliptically) distributed as is the case with credit, market and, more particularly, operational risk. In addition, VaR-based measurement does not determine the distribution or extent of risk in the tail, but only provides an estimate of a specific data point in the distribution.

- A third point is that the Standard approach to credit risk differentiates assets not only according to the borrower but according to riskiness proxied by credit rating agencies' assessment of the borrower. This approach represents an improvement in the sense that corporations are rated. However, it is deficient with respect to the viewpoint that ratings properly reflect risk and the inducing of procyclical capital charges which will lead to overlending during booms and underlending during recessions.
- Fourthly, the capital formula for the Internal Ratings Based (IRB) approach to credit risk is problematic. In short, the IRB capital formula is deduced by considering the large-portfolio asymptotic dynamics of a Merton model with a single common risk-factor. Many empirical studies have confirmed that this formula gives rise to procyclicality.
- In the fifth place, the justification for and feasibility of holding regulatory capital against operational risk is debatable. By contrast to market and credit risk, operational risk is mainly idiosyncratic which makes the regulation to prevent contagion a moot point. Any estimation of operational risk is hampered by an absence of data and a lack of a precise definition of such risk.
- Sixthly, the increased flexibility that regulators have under "Pillar 2: Supervisory Review" may create incentives for inconsistent regulatory review if its implementation is not subject to careful international monitoring which in all likelihood may not occur.
- A consequence of the aforementioned six issues is that Basel II will exacerbate procyclicality of regulation and the susceptibility of the financial system to financial instability. This will negate the central purpose of the new regulation drafted in this accord.

Chapter 8

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Chapter 9

APPENDIX

9.1 APPENDIX A: PARAMETERS USED IN SIMULATING THE SD-STR

9.2 APPENDIX B: MEASURE THEORY AND FUBINI THEOREM

9.3 APPENDIX C: CHARACTERISTIC TRIPLETS

In the current chapter we provide a table of parameters and values considered for the SDSTR problem given in equation (2.23) of Subsection 2.2.1 in Chapter 2. We also provide a table with values for the correlations between output gap and provisioning and profitability. Here, we also establish some useful results from measure theory and then state a Fubini Theorem. Furthermore, we discuss some results on characteristic triplets of one-dimensional Lévy process, L , and its linear transformation πL given for $\pi \in \mathbb{R}$. We also present a characteristic triplets for the natural logarithm of total provisioning

$$\ln \left(\frac{P_\pi}{p} \right).$$

Most of the contents of Section 9.3 may be found in the standard textbooks [81] and [26].

9.1 APPENDIX A: PARAMETERS USED IN SIMULATING THE SDSTR

In this section, we provide a table of parameters and values used for simulating the SDSTR given in (2.23) of Subsection 2.2.1 in Chapter 2.

Parameters	Values
r^T	0.4
π	2
r^R	0.8
f^R	5
a^R	1.25
σ^R	0.4
c^R	0.0006
κ	1.5
γ	1.9992
λ	2

Table 9.1: Parameters Used in Simulating the SDSTR

9.2 APPENDIX B: MEASURE THEORY AND FUBINI THEOREM

In this section, we reproduce some useful results from measure theory and state Fubini's Theorem.

Now consider the two measure spaces (X_1, K_σ^1, P'_1) and (X_2, K_σ^2, Q'_1) . The following lemma holds.

Lemma 9.2.1 *Let P'_1 and Q'_1 be σ -finite measures and C a subset of $X_1 \times X_2$ that is measurable with respect to the measure $(P'_1 \times Q'_1)$. Then for almost all $x_1 \in X_1$ (with respect to the measure P'_1) the set C_{x_1} is measurable with respect to the measure Q'_1 . The function $f_C(x_1) = Q'_1(C_{x_1})$ is measurable with respect to the measure Q'_1 , and*

$$(P'_1 \times Q'_1)(C) = \int_x f_C dP'_1.$$

(here both sides may be simultaneously infinite)

Remark 1: By the virtue of the fact that the measures P'_1 and Q'_1 , as well as the sets X_1 and X_2 , occur symmetrically throughout, we can write

$$(P'_1 \times Q'_1)(C) = \int_{x_2} P'_1(C'_{x_2}) dQ'_1,$$

where $C'_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in C\}$. It follows also from this that

$$\int_{x_1} (C_{x_1}) dP'_1 = \int_{x_2} P'_1(C'_{x_2}) dQ'_1.$$

Remark 2: For the product of three measure spaces

$$(X_1, K^1_\sigma(X_1), P'_1), (X_2, K^2_\sigma(X_2), Q'_1), (X_3, K^3_\sigma(X_3), Q''),$$

we can write similarly,

$$(P'_1 \times Q'_1 \times Q'')(C) = \int_{x_1 \times x_2} Q''(C_{x_1, x_2}) d(P'_1 \times Q'_1) = \int_{x_2} (P'_1 \times Q'_1)(C_{x_2}) dQ'',$$

where

$$\begin{aligned} C_{x_1, x_2} &= \{x_3 \in X_3 : (x_1, x_2, x_3) \in C\}, \\ C_{x_3} &= \{(x_1, x_2) \in X_1 \times X_2 : (x_1, x_2, x_3) \in C\}. \end{aligned}$$

Theorem 9.2.2 (Fubini Theorem:) *Let $f(x_1, x_2)$ be a function on the product of the space $(X_1, K^1_\sigma(X_1), P'_1)$ and $(X_2, K^2_\sigma(X_2), Q'_1)$ that is integrable with respect to $(P'_1 \times Q'_1)$. Then*

1. *for almost all $x_1 \in X_1$ (with respect to the measure P'_1) the function $f(x_1 \times x_2)$ is integrable on X_2 (with respect to the measure Q'_1) and its integral over X_2 is an integrable function on X_1 ;*
2. *for almost all $x_2 \in X_2$ (with respect to the measure Q'_1) the function $f(x_1, x_2)$*

is integrable on X_1 (with respect to the measure \mathbf{Q}'_1) and its integral over X_1 is an integrable function X_2 ;

3. the following equality hold:

$$\int_{x_1 \times x_2} f(x_1, x_2) d(\mathbf{P}'_1 \times \mathbf{Q}'_1) = \int_{x_1} \left(\int_{x_2} f(x_1, x_2) d\mathbf{Q}'_1 \right) d\mathbf{P}'_1 = \int_{x_2} \left(\int_{x_1} f(x_1, x_2) d\mathbf{P}'_1 \right) d\mathbf{Q}'_1;$$

4. for non-negative functions that are measurable with respect to the measure $(\mathbf{P}'_1 \times \mathbf{Q}'_1)$ the existence of either of the iterated integrals implies the existence of the double integral, i.e., the integrability of $f(x_1, x_2)$ over $X_1 \times X_2$.

Proof. Consider the case of non-negative function f . Let the set C be contained in $X_1 \times X_2 \times X_3$, where $X_3 = \mathbb{R}^1$ is the real axis with usual Lebesgue measure $\mathbf{Q}'' = d_{x_3}$

$$C = \{(x_1, x_2, x_3) \in (X_1 \times X_2 \times \mathbb{R}^1) : 0 \leq x_3 \leq f(x_1, x_2)\}$$

We apply the relations written out in Remark 2 above this case. We obtain

$$\begin{aligned} C_{x_2, x_2} &= \{x_2 \in \mathbb{R}^1 : 0 \leq x_3 \leq f(x_1, x_2)\} \mathbf{Q}''(C_{x_1, x_2}) = f(x_1, x_2); \\ C_{x_1} &= \{(x_1, x_2) \in X_2 \times \mathbb{R}^1 : 0 \leq x_3 \leq f(x_1, x_2)\} \\ (\mathbf{P}'_1 \times \mathbf{Q}'_1)(C_{x_1}) &= \int_{x_2} f(x_1, x_2) d\mathbf{Q}'_1. \end{aligned}$$

All the assertions of the theorem follow from this for a non-negative function. The decomposition $f = f^+ - f^-$ finishes the proof in the general case. \square

9.3 APPENDIX C: CHARACTERISTIC TRIPLETS

In this section, we present characteristic triplets for

$$\exp L = \xi(\widehat{L}), \pi L \text{ and } \ln \left(\frac{P_\pi}{p} \right).$$

Lemma 9.3.1 (Characteristic Triplet for $\exp L = \xi(\widehat{L})$): *If L is a real-valued Lévy process with characteristic triplet (γ, σ^2, ν) , then also \widehat{L} defined by*

$$\exp L = \xi(\widehat{L})$$

is a Lévy process with characteristic triplet $(\widehat{\gamma}, \widehat{\sigma}^2, \widehat{\nu})$ given by

$$\widehat{\gamma} - \gamma = \frac{\sigma^2}{2} + \int \left((\exp x - 1)1_{\{|\exp x - 1| < 1\}} - x1_{\{|x| < 1\}} \right) \nu(dx)$$

$$\widehat{\sigma}^2 = \sigma^2, \quad \widehat{\nu}(J) = \nu(\{x \in \mathbb{R} \mid \exp x - 1 \in J\}),$$

for any Borel set $J \subset \mathbb{R} - \{0\}$.

In the following lemma, the relation between the characteristic triplet of one-dimensional Lévy process, L , and its linear transformation πL is given for $\pi \in \mathbb{R}$.

Lemma 9.3.2 (Characteristic Triplet for πL): *If L is a Lévy process with characteristic triplet (γ, σ^2, ν) , then πL is for $\pi \in \mathbb{R}$ a Lévy process with characteristic triplet $(\gamma_\pi, \sigma_\pi^2, \nu_\pi)$ given by*

$$\begin{aligned} \gamma_\pi &= \pi\gamma + \int \pi x \{1_{\{|\pi x| < 1\}} - 1_{\{|x| < 1\}}\} \nu(dx), & \sigma_\pi^2 &= (\pi\sigma)^2, \\ \nu_\pi(J) &= \nu(\{x \in \mathbb{R} \mid \pi x \in J\}) \end{aligned}$$

for any Borel set $J \subset \mathbb{R} - \{0\}$.

In the next result, we are able to calculate the characteristic triplet for the natural logarithm of total provisioning by means of Lemma 9.3.1 and 9.3.2.

Lemma 9.3.3 (Characteristic Triplet for $\ln\left(\frac{P_\pi}{p}\right)$):

Consider model in equation (2.6) of Subsection 2.1.2 in Chapter 2 with Lévy process, L , and characteristic triplet, (γ, σ^2, ν) . The process $\ln\left(\frac{P_\pi}{p}\right)$ is a Lévy process with characteristic triplet $(\gamma_X, \sigma_X^2, \nu_X)$ given by

$$\begin{aligned} \gamma_X &= r^R \pi (c^d + \frac{\sigma^2 \delta^2}{2} - r^R + \sigma \gamma) - \frac{(\pi \sigma \delta)^2}{2} \\ &\quad + \int_{\mathbb{R}} \left(\Gamma(x) 1_{\{|\Gamma(x)| \leq 1\}} - \pi \sigma x 1_{|x| \leq 1} \right) \nu(dx), \\ \sigma_X^2 &= (\pi \sigma \delta)^2, \quad \nu_X(\mathcal{J}) = \nu(\{x \in \mathbb{R} | \Gamma(x) \in \mathcal{J}\}), \end{aligned}$$

for any Borel set $\mathcal{J} \subset \mathbb{R} - \{0\}$ and $\Gamma(x) = \ln(1 + \pi(\exp \sigma x - 1))$.