



Classes of unbounded Toeplitz operators

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Dissertation accepted in partial fulfilment of the requirements for the degree *Master of Science in Mathematics* at the North-West University

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Graduation: May 2025

Acknowledgements

I would like to express my sincere gratitude to my supervisor and mentor, Prof Sanne ter Horst, for his extensive support and insightful comments throughout the course of this dissertation, without whom this dissertation would not meet its completion. I would also like to thank Prof Sanne ter Horst for providing a systematic study of the various subjects that were necessary for this dissertation and, in particular, for organizing weekly seminars which exposed me to the various mathematical techniques used herein. I am immensely grateful to have Prof Sanne ter Horst as my supervisor, and I look forward to continuing as his student. I would like to thank my parents, Mohammed and Fazila for their monumental support and prayers. In particular, I would like to thank my Dad for inspiring me to study mathematics, in which I have found beauty, discipline, and humbleness.

I would like to thank the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa, for their generous financial support for the year 2024.

Abstract

The theory of bounded Toeplitz operators with symbols in $L^\infty(\mathbb{T})$ have been studied extensively in the last century with numerous sources describing their Fredholm and spectral theory, invertibility, their kernels and their adjoints. Once we consider symbols beyond $L^\infty(\mathbb{T})$, the respective Toeplitz operators become unbounded and their theory becomes complicated. In recent literature, various definitions have surfaced, which yield unbounded Toeplitz operators with more general symbols, which appear to be dichotomous and display faint correlation. In this dissertation, we study some of the recent approaches to unbounded Toeplitz operators and provide connections between them. Moreover, in some cases, we can describe their Fredholm and spectral properties and their adjoints, so as to give an extension of the results from the well-known bounded case.

Keywords: Toeplitz operators; unbounded operators; Fredholm theory; spectral theory.

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List of symbols

A table containing the list of symbols that will be used throughout text.

$\mathbb{Z}$	The integers
$\mathbb{N}$	The positive integers
$\mathbb{N}_0$	The non-negative integers
$\mathbb{T}$	The unit circle in the complex plane
$\mathbb{D}$	The open unit disk of the complex plane
L^p	The Lebesgue space of the unit circle $\mathbb{T}$
$\text{hol}(\mathbb{D})$	The space of all analytic functions on $\mathbb{D}$
H^p	The Hardy space of the open unit disk
K^p	The standard complement of H^p in L^p
P_+	The Riesz projection that maps L^p into the Hardy space H^p
$N(\mathbb{D})$	The Nevanlinna class of the open unit disk
$N^+(\mathbb{D})$	The Smirnov class of the open unit disk
$\mathcal{D}(T)$	The domain of a linear operator T
$\mathcal{R}(T)$	The range of a linear operator T
$\ker(T)$	The kernel of a linear operator T
$\Re(u)$	The real part of a complex number u
$\Im(u)$	The imaginary part of a complex number u
S_μ	The singular inner function defined by a positive measure μ
F_g	The outer spectral factor defined by a log-integrable function g
T_b	The Toeplitz operator defined by a function $b \in L^\infty$
T_ϕ^A	The Sarason-type Toeplitz operator
T_g^M	The Camara-Partington-Malheiro-type Toeplitz operator
T_w^R	The Groenewald-Ter Horst-Jaftha-Ran-type Toeplitz operator

Chapter 1

Introduction

The theory of Toeplitz operators begins with the classical paper of Otto Toeplitz [Toe10], where he constructed an infinite matrix from the Fourier coefficients of a continuous function ϕ defined on the unit circle $\mathbb{T}$ of the complex plane. These particular matrices have the property that the entries along the diagonals are constant, that is, of the form

$$T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

These matrices are called Toeplitz matrices. In particular, the sequence $(a_n)_{n \in \mathbb{Z}}$ are the Fourier coefficients of a function ϕ on $\mathbb{T}$. The classical Toeplitz operator is defined to be the compression of a multiplication operator by an essentially bounded function f defined on the Lebesgue space $L^p(\mathbb{T})$ to the Hardy space $H^p(\mathbb{T})$. It can be shown that the matrix representation of a classical Toeplitz operator with respect to the standard basis of $H^p(\mathbb{T})$ is a Toeplitz matrix. The correspondence between the bi-infinite sequence $(a_n)_{n \in \mathbb{Z}}$ and the Toeplitz matrix T brought forth an intimate interplay between complex analysis and operator theory. In particular, the notable paper by Brown and Halmos [BH64] sparked a cascading study into the theory of Toeplitz operators. In this regard, Toeplitz operators have caught the attention of many mathematicians leading to the publication of several books and thousands of research articles, [BH64; BS06; Dou72; HW54; NGG20] to name a few. Toeplitz operators can be defined on a variety of spaces, with recent focuses on reproducing kernel spaces such as Bergman, Dirichlet and Fock spaces. Toeplitz operators can also be defined on the Hardy spaces $H^p(\mathbb{T})$ and the concrete sequence spaces ℓ^p with each setting offering a rich theory in their own right [BS06].

In this dissertation, we will only consider Toeplitz operators acting on the Hardy space $H^p(\mathbb{T})$ for $1 < p < \infty$. One of the enticing properties of Hardy spaces is the fact that every function in $H^p(\mathbb{T})$ has an analytic extension to the open unit disc $\mathbb{D}$, which provides an analytic framework for Toeplitz operators. For $a \in L^\infty(\mathbb{T})$, we formally define the Toeplitz operator on $H^p(\mathbb{T})$ by $T_a = P_+ M_a$, where P_+ is the Riesz projection from $L^p(\mathbb{T})$ onto $H^p(\mathbb{T})$ and M_a is the multiplication operator defined by a . It can easily be shown that T_a is a bounded operator. The defining function a is called the symbol of T_a . A remarkable result by Brown and Halmos [BH64] states that every bounded operator on $H^p(\mathbb{T})$ with a Toeplitz matrix is a Toeplitz operator T_b for some b belonging to L^∞ . In particular, a Toeplitz operator T_b is bounded if and only if its symbol b belongs to $L^\infty(\mathbb{T})$. This exemplary relation permeates the theory of Toeplitz operators to their symbols and displays an elegant link between operator theory and complex function theory.

The symbols of Toeplitz operators reduces a significant portion of the questions concerning Toeplitz operators to the behavior of the symbol. Some particular instances are questions regarding commutativity, invertibility and Fredholmness. Regarding the invertibility of Toeplitz operators. A result by Gohberg and Krein [Dou72] shows that for a continuous symbol $a \in L^\infty(\mathbb{T})$, the bounded Toeplitz operator T_a is invertible if and only if a does not vanish on the unit circle $\mathbb{T}$ and the winding number of a with respect to the origin is zero. For a noncontinuous unimodular symbol $a \in L^\infty(\mathbb{T})$, the Devinatz and Widom invertibility criterion [Dev64; Wid60] shows that if a Toeplitz operator T_a is invertible, then the bounded inverse T_a^{-1} can be written as the product of two unbounded Toeplitz operators, $T_a = T_{1/\bar{O}} T_O$ with $\bar{O}, O$ belonging to the Hardy space $H^2(\mathbb{T})$. This is one instance where unbounded Toeplitz operators naturally arise.

In essence, unbounded Toeplitz operators arise by considering more general symbols, that is, by considering symbols beyond $L^\infty(\mathbb{T})$. Over the past decade, there have been a few research articles investigating unbounded Toeplitz operators [CMP21; Cas21; Gro+18; Gro+19a; Gro+19b; Sar08] with different domains, actions and different symbols. For bounded Toeplitz operators, the various definitions and actions which lead to bounded Toeplitz operators coincide. While for unbounded Toeplitz operators, the various definitions do not display any immediate correlation. This is due to the flexibility of choosing the symbol and the action of the operator. One of the aims of this dissertation is to try and classify, at least in part, the various unbounded Toeplitz operators.

In the broadest sense, the class of unbounded Toeplitz operators relies on two factors: The class of symbols and the domain of definition with the action of the operator. The various combinations that can be made from these factors produce assorted flavors of unbounded

Toeplitz operators. Hartman [Har63] considered unbounded Toeplitz operators with symbols belonging to $L^2(\mathbb{T})$. Sarason [Sar08] considered unbounded Toeplitz operators with symbols belonging to the Nevanlinna and Smirnov class. Câmara, et al [CMP21] considered unbounded Toeplitz operators with symbols that are measurable functions which are log-integrable. Groenewald, et al [Gro+18] considered unbounded Toeplitz operators with symbols that are rational functions with possible poles on $\mathbb{T}$. In particular, the various classes mentioned above have distinct domains and different actions.

Since the Toeplitz operators considered in this dissertation are acting on the Hardy space of the open unit disk $H^p(\mathbb{D})$, Chapter 2 is dedicated to reviewing some of the basic properties of the Hardy spaces $H^p(\mathbb{D})$. Moreover, essential results concerning factorization of $H^p(\mathbb{D})$ functions are recalled to introduce more general analytic functions, which will be essential to characterize unbounded Toeplitz operators with analytic and rational symbols.

In Chapter 3, we review some of the theory of bounded Toeplitz operators, where the symbol belongs to $L^\infty(\mathbb{T})$. In particular, we briefly review their basic properties, kernels and spectral and Fredholm properties. The aim in this chapter is to gain insight into the interaction between bounded Toeplitz operators and their symbols. We will also review some of the basic theory of bounded analytic Toeplitz operators, which are Toeplitz operators with symbols in $H^\infty(\mathbb{T})$. The consideration of bounded analytic Toeplitz operators is conducive to the study of the class of unbounded Toeplitz operators studied by Sarason [Sar08]. The theory of bounded Toeplitz operators will provide a firm standpoint for the comparison of the various unbounded Toeplitz operators which will be discussed in Chapter 4.

In Chapter 4, we study some of the various types of unbounded Toeplitz operators which have appeared in the recent literature [CMP21; Gro+18; Gro+19a; Gro+19b; Sar08]. In particular, we provide some of their basic properties, spectral and Fredholm properties which allow us to compare the results of bounded Toeplitz operators discussed in Chapter 3. We also provide some connections between the various classes of unbounded Toeplitz operators. In particular, we show that the unbounded Toeplitz operator of Sarason [Sar08] arises when considering log-integrable symbols in the unbounded Toeplitz operator studied by Câmara et al [CMP21]. Moreover, without the log-integrable condition for the symbol, there is a slight indication that the unbounded Toeplitz operator considered by Câmara et al is more general than that of Sarason. Furthermore, the connection between the Toeplitz operators of Sarason and Câmara et al enables the study of minimal Toeplitz kernels for functions in the domain of the unbounded Toeplitz operator considered by Sarason. In the final section of Chapter 4, we study the unbounded Toeplitz operators considered by Groenewald, et al in [Gro+18; Gro+19a; Gro+19b]. One of the goals of this section is to establish a connection to the unbounded Toeplitz operator considered by Sarason [Sar08]. In particular, we show that the unbounded Toeplitz operator of Sarason coincides with

the Toeplitz operator of Groenewald et al with the assumption that the symbol is a proper rational function.

Chapter 2

Preliminaries

In this chapter, we discuss some of the preliminary material which will be used throughout this dissertation. We also take this opportunity to introduce some notation and basic definitions to produce in some sense a self-contained text. Since the Toeplitz operators in consideration act on analytic functions spaces, it is worthwhile to review some essential complex analysis results. We will also briefly recall some basic results of unbounded linear operators, as we will study unbounded Toeplitz operators in the last chapter.

2.1 Hardy Spaces

In this section we recall a special class of analytic functions with certain restricted growth estimates. These are the classical *Hardy spaces*. These classes commonly come in two flavours, the one considers functions analytic on the open unit disk and the other on the upper half plane. Both cases offer a rich theory in their own right, particularly concerning bounded and unbounded Toeplitz operators. Consequently, the matter of choice comes down purely to personal taste. In this regard, we have chosen the Hardy space of the open unit disk, which will form the underlying space of the operators forthcoming in the subsequent chapters. The results stated in this section come mostly from [Con95; Nik02; RC00]. Firstly, we begin with basic notation.

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. We let $\text{hol}(\mathbb{D})$ denote the vector space of all holomorphic functions on $\mathbb{D}$. We also denote by $L^p = L^p(\mathbb{T}, dm)$, the Lebesgue space of all functions measurable with respect to the normalized Lebesgue measure dm on the unit circle and p -integrable. In fact, the theory of overlap between $\text{hol}(\mathbb{D})$ and L^p is quite rich. In particular, there exists a subclass of $\text{hol}(\mathbb{D})$ whose functions admit boundary functions belonging to L^p obtained by *non-tangential limits* (defined below).

Definition 2.1.1. Let $f \in \text{hol}(\mathbb{D})$ and $t \in \mathbb{T}$. For $\alpha > 1$, the set defined by

$$\Gamma_\alpha(t) := \{z \in \mathbb{D} : |z - t| < \alpha(1 - |z|)\}$$

is called the *non-tangential approach region* at the point t with opening α . Note that this region is an open triangle in $\mathbb{D}$ with vertex t whose angle equals $2 \arccos 1/\alpha$. We then say that f has a non-tangential limit equal to $L \in \mathbb{C}$ at the point t if for all $\alpha > 1$ we have

$$f(z) \longrightarrow L \text{ as } z \longrightarrow t \text{ for } z \in \Gamma_\alpha(t).$$

We say that f has a radial limit equal to $M \in \mathbb{C}$ at a point $t \in \mathbb{T}$ if

$$f(rt) \longrightarrow M \text{ as } r \longrightarrow 1^-.$$

We will mostly be concerned with non-tangential limits, however it is worth noting its relationship to radial limits. Note that the radial limit path of a function f at a point $t \in \mathbb{T}$ falls in the region $\Gamma_\alpha(t)$ for any α . Thus if f has a non-tangential limit L at t , then f also has a radial limit L at t .

Let $f \in \text{hol}(\mathbb{D})$. Consider the following function of r ;

$$M_\infty(f, r) := \max\{|f(rt)| : t \in \mathbb{T}\}, \quad r \in (0, 1). \quad (2.1)$$

By using the maximum modulus principle, we see that (2.1) is an increasing function in r . In addition, Hardy [Har14] showed that for $0 < p < \infty$, the integral means

$$M_p(f, r) := \left[\int_{\mathbb{T}} |f(rt)|^p dt \right]^{1/p} \quad r \in (0, 1)$$

forms an increasing function in r . Summing up, we have

$$\sup_{0 < r < 1} M_p(f, r) = \lim_{r \rightarrow 1^-} M_p(f, r), \quad 0 < p \leq \infty. \quad (2.2)$$

Naturally, we are interested in those $f \in \text{hol}(\mathbb{D})$ for which the limit (2.2) is finite. In other words, we consider the class of analytic functions which have restricted growth near the boundary $\mathbb{T}$. This class is called the *Hardy space* of the open unit disk, which we denote by $H^p(\mathbb{D})$. The restraint on the growth estimates not only allows us to define a norm on $H^p(\mathbb{D})$, for $1 \leq p \leq \infty$ but also enables boundary functions, via non-tangential limits, belonging to the Lebesgue space of the unit circle L^p . This exposes an elegant overlap between analytic function theory and measure theory.

For $0 < p \leq \infty$, we formally define,

$$H^p(\mathbb{D}) := \left\{ f \in \text{hol}(\mathbb{D}) : \sup_{0 < r < 1} M_p(f, r) < \infty \right\}.$$

Note that the inequality [Dur00, Chapter 1, page 2]

$$(a + b)^p \leq 2^p(a^p + b^p), \quad a, b \geq 0 \quad 0 < p < \infty$$

implies that $H^p(\mathbb{D})$ is a complex vector space. Furthermore, by an application of Holder's inequality, we notice that for $0 < p \leq q \leq \infty$, we have

$$M_p(f, r) \leq M_q(f, r) \quad \text{for any } r \in (0, 1).$$

Thus $H^q(\mathbb{D}) \subset H^p(\mathbb{D})$. For $f \in \text{hol}(\mathbb{D})$ and $r \in (0, 1)$, define the function $f_r : \mathbb{T} \rightarrow \mathbb{C}$ by $f_r(t) = f(rt)$. Then $M_p(f, r)$ is just the L^p norm of the function f_r . From this observation, for $f \in H^p(\mathbb{D})$, $0 < p \leq \infty$, we can define $\|f\|_p := \sup_{0 < r < 1} M_p(f, r)$. It turns out that $\|\cdot\|_p$ is a norm on $H^p(\mathbb{D})$ for $1 \leq p \leq \infty$. In this regard, for the sequel we will only consider the Hardy spaces $H^p(\mathbb{D})$ for $1 \leq p \leq \infty$. We then have that $H^p(\mathbb{D})$ is a normed space. In fact, we will eventually see that $H^p(\mathbb{D})$ is a Banach space. However, we will first consider the boundary functions of $H^p(\mathbb{D})$.

Theorem 2.1.2. [RC00] Let $f \in H^p(\mathbb{D})$. Then $f = f_1/f_2$, for some $f_1, f_2 \in H^\infty(\mathbb{D})$. Moreover, the non-tangential limit

$$\lim_{z \rightarrow t} f(z) =: \tilde{f}(t)$$

exists for almost every $t \in \mathbb{T}$.

The above result is due to Fatou [Fat06]. In the sequel, we write $\tilde{f}$ for the boundary function of $f \in H^p(\mathbb{D})$ defined in Theorem 2.1.2. Moreover, Fatou showed that if the boundary function $\tilde{f} \in H^\infty$ vanishes on some measurable subset E of the circle $\mathbb{T}$, then f must necessarily be the zero function. This result was further improved by F. and M. Riesz [RR16] and is more commonly known as the Riesz brothers theorem.

Theorem 2.1.3. [RR16] Let $f \in H^p(\mathbb{D})$. If $f \not\equiv 0$, then $\log |\tilde{f}| \in L^1$. Moreover, if $\lim_{r \rightarrow 1^-} f(rt) = 0$ on a set $E \subset \mathbb{T}$ of positive measure, then $f \equiv 0$.

Definition 2.1.4. A measurable function $f : \mathbb{T} \rightarrow \mathbb{C}$ is said to be *log-integrable* if $\log |f| \in L^1$, i.e.

$$\int_{\mathbb{T}} |\log |f(t)|| dm(t) < \infty. \quad (2.3)$$

Theorem 2.1.5. [RC00] Let $f \in H^p(\mathbb{D})$ for $p \in [1, \infty]$. Then the boundary function $\tilde{f}$ belongs to L^p . Moreover

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{T}} |f_r(t) - \tilde{f}(t)|^p dm(t) = 0.$$

Thus every function in $H^p(\mathbb{D})$ has, via non-tangential limits, a boundary function belonging to L^p . A natural question arises. Which function in L^p appear in this manner?

Said differently, how can one recognize functions in L^p which are boundary functions of functions in $H^p(\mathbb{D})$? Before we answer this, we first recall some kernel functions.

For a fixed $t \in \mathbb{T}$, consider the function on $\mathbb{D}$ given by

$$z \mapsto \frac{t+z}{t-z}.$$

This function is called the *Herglotz kernel*. Moreover, it decomposes into two well-known functions given by

$$P_z(t) := \Re \left(\frac{t+z}{t-z} \right) = \frac{1-|z|^2}{|t-z|^2} \quad Q_z(t) := \Im \left(\frac{t+z}{t-z} \right) = \frac{2\Im(\bar{t}z)}{|t-z|^2}.$$

The functions P_z and Q_z are called the *Poisson kernel* and *conjugate Poisson kernel* respectively. For a function $f \in L^1$, the *Poisson integral* $\acute{f}$ and *conjugate Poisson integral* $\check{f}$ are defined respectively by

$$\acute{f}(z) := \int_{\mathbb{T}} P_z(t) f(t) dm(t), \quad \check{f}(z) := \int_{\mathbb{T}} Q_z(t) f(t) dm(t).$$

It can be shown that the Poisson integral of a function in L^1 defines an analytic function on $\mathbb{D}$ and likewise for the conjugate Poisson integral. Consequently, for any function $f \in L^1$, the Herglotz integral of f defined by

$$\vec{f}(z) := \int_{\mathbb{T}} \frac{t+z}{t-z} f(t) dm(t)$$

is analytic on $\mathbb{D}$ with $\vec{f}(z) = \acute{f}(z) + i\check{f}(z)$.

Let f be a function in L^p . The n -th Fourier coefficient of f is defined by

$$\hat{f}(n) := \int_{\mathbb{T}} f(t) \bar{t}^n dm(t).$$

It is known that $\{t^n : n \in \mathbb{Z}\}$ forms a Schauder basis for L^p , for $1 \leq p < \infty$ and every function $f \in L^p$ can be written as a Fourier series

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) t^n.$$

Theorem 2.1.6. [Con95, Theorem 1.3 page 270] If $1 \leq p \leq \infty$ and $f \in L^p$ such that $\hat{f}(n) = 0$ for $n < 0$, then the Poisson integral $\acute{f}$ belongs to H^p . Moreover:

- i) $\|f\|_p = \|\acute{f}\|_p$.
- ii) If $1 \leq p < \infty$, $\|\acute{f}_r - f\|_p \rightarrow 0$ as $r \rightarrow 1^-$.
- iii) If $p = \infty$, then $\acute{f}_r \rightarrow f$ in the weak* topology of L^∞ as $r \rightarrow 1^-$.

A natural question that arises is whether the converse of the preceding theorem holds too. That is, does the non-tangential limit of an H^p function have Fourier coefficients which vanish for negative indices? This is answered in the next theorem.

Theorem 2.1.7. If $1 < p \leq \infty$ and $f \in H^p(\mathbb{D})$, then the boundary function $g = \tilde{f}$ belongs to L^p and $\hat{g}(n) = 0$ for $n < 0$ and $\acute{g} = f$.

Proof. Suppose that $f \in H^p(\mathbb{D})$, $1 < p \leq \infty$. By Theorem 2.1.5, the boundary function $\tilde{f}$ exists almost everywhere on $\mathbb{T}$ and $\tilde{f} \in L^p$. We also have that $\|f_r - \tilde{f}\|_p \rightarrow 0$ as $r \rightarrow 1^-$. This implies that $\hat{g}(n) = \lim_{r \rightarrow 1^-} \hat{f}_r(n)$, since the functional $g \mapsto \hat{g}(n)$ is continuous on L^1 for each $n \in \mathbb{Z}$ [Nik02, page 25, lines 19-20]. Since $f \in H^p$, f has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ around 0 which converges uniformly on the $\mathbb{D}$. This implies that f_r has Taylor expansion $f_r(z) = \sum_{n=0}^{\infty} a r^n z^n$ around 0 and is analytic on the disc around 0 with radius $1/r > 1$. By Cauchy's theorem, it follows that $\hat{f}_r(n) = a_n r^n$ if $n \geq 0$ and $\hat{f}_r(n) = 0$ if $n < 0$. Hence $\hat{g}(n) = a_n$ if $n \geq 0$ and $\hat{g}(n) = 0$ if $n < 0$. Noting that g is a function in L^p , we have by Theorem 2.1.6 that $\acute{g} \in H^p(\mathbb{D})$. We also have for $z \in \mathbb{D}$ that

$$\acute{g}(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n = \sum_{n=0}^{\infty} a_n z^n = f(z).$$

Thus $\acute{g} = f$. □

The preceding results exposed an interesting subset of L^p during the identification of $H^p(\mathbb{D})$ functions. We formally define it by

$$H^p(\mathbb{T}) := \{f \in L^p : \hat{f}(n) = 0 \text{ for } n < 0\}.$$

It is easy to see that $H^p(\mathbb{T})$ is a linear subspace of L^p . To show that $H^p(\mathbb{T})$ is closed for $1 < p < \infty$, we first note that the functionals $f \mapsto \hat{f}(n)$, for $n \in \mathbb{Z}$, are continuous on L^p for each $1 < p < \infty$. Thus if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $H^p(\mathbb{T})$ such that $\|f_n - g\|_p \rightarrow 0$ for some $g \in L^p$. Then we have that $\hat{g}(k) = \lim_n \hat{f}_n(k)$. Thus we have that $\hat{g}(k) = 0$ for $k < 0$. Hence $g \in H^p(\mathbb{T})$ and so $H^p(\mathbb{T})$ contains all its limit points, hence it is closed. Likewise, the subset

$$H_0^p(\mathbb{T}) = \{f \in L^p : \hat{f}(n) = 0 \text{ for } n \leq 0\}$$

is also a closed subspace of L^p . Consequently, we can decompose L^p , for $1 < p < \infty$ in the following way

$$L^p = H^p(\mathbb{T}) \dot{+} \overline{H_0^p(\mathbb{T})},$$

where $\dot{+}$ indicates a direct sum. In the sequel, we will use the notation $K^p = \overline{H_0^p}$.

The preceding results can collectively be summed into the following theorem.

Theorem 2.1.8. If $1 < p \leq \infty$, then the map $\mathcal{J} : H^p(\mathbb{D}) \rightarrow H^p(\mathbb{T})$, which sends a function f in H^p to its non-tangential limit in $H^p(\mathbb{T})$ is a surjective isometry. Thus $H^p(\mathbb{D})$ is a Banach space.

This identification allows us to identify functions in $H^p(\mathbb{D})$ with their non-tangential limits in $H^p(\mathbb{T})$ and vice versa. In the subsequent chapters, we will not distinguish between them with clarity in a given context. Consequently, we will adopt the notation $H^p = H^p(\mathbb{D})$. Some care must be taken when considering the preceding theorem where we showed that a function $f \in L^p$ belongs to $H^p(\mathbb{T})$ if and only if $\hat{f}(k) = 0, k \leq 0$. Consider an arbitrary function $g \in L^p$ with Fourier expansion

$$g(t) = \sum_{j=-\infty}^{\infty} \hat{g}(j)t^j.$$

Let $P_+ : L^p \rightarrow \text{hol}(\mathbb{D})$ be the map defined by

$$(P_+g)(z) = \sum_{j=0}^{\infty} \hat{g}(j)z^j, \quad g \in L^p. \quad (2.4)$$

It is natural to think that $P_+g \in H^p(\mathbb{T})$. However, for $p = 1$ or $p = \infty$, this need not be the case. We demonstrate the case for $p = \infty$ in the following example.

Example 2.1.9. [RC00] Consider the function $f(e^{i\theta}) = \arg(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \sin(n\theta)/n$. It is clear that $f \in L^\infty$. Since $\sin(n\theta) = (e^{in\theta} - e^{-in\theta})/2i$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} &= \sum_{n=1}^{\infty} \frac{e^{in\theta} - e^{-in\theta}}{2in} \\ &= \frac{1}{2i} \left(\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} - \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n} \right) \\ &= \frac{1}{2i} \left(\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} - \sum_{n=-1}^{-\infty} \frac{e^{in\theta}}{n} \right). \end{aligned}$$

Then we have

$$(P_+f)(z) = \frac{1}{2i} \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) = \frac{1}{2i} \log(1 - z).$$

However, $\frac{1}{2i} \log(1 - z)$ does not belong to H^∞ .

The case for $p = 1$ is shown in [RC00, Example 3.3.4] and we make the observation that removing the Fourier coefficients with negative indices of a function in L^1 or L^∞ does not necessarily give a function in H^1 or H^∞ respectively. Fortunately, for the case where $1 < p < \infty$, we have a much more positive result due to F. Riesz.

Corollary 2.1.10. [Rie24] Let $1 < p < \infty$. The mapping P_+ , defined by (2.4), is a bounded linear operator from L^p into H^p .

Definition 2.1.11. For $1 < p < \infty$, the operator $P_+ : L^p \rightarrow H^p$ defined by

$$(P_+f)(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, \quad f \in L^p$$

is called the *Riesz projection*.

So far we have showed that H^p is Banach space for $1 \leq p \leq \infty$. In fact, more can be said about H^∞ and its relation to H^p for $1 \leq p < \infty$. It is easy to see that the product of two H^∞ functions is again in H^∞ and thus H^∞ is an algebra, in fact H^∞ is a Banach algebra. In the following lemma, we show that any H^∞ function defines a bounded multiplier on H^p .

Lemma 2.1.12. Let $1 \leq p \leq \infty$. If $f \in H^\infty$ and $g \in H^p$, then $fg \in H^p$. Moreover, $\|fg\|_p \leq \|f\|_\infty \|g\|_p$.

Proof. It is clear that $fg \in \text{hol}(\mathbb{D})$. On the other hand, for $1 < p < \infty$

$$\begin{aligned} \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rt)g(rt)|^p dm(t) &\leq \sup_{0 < r < 1} \left(\max_{t \in \mathbb{T}} |f(rt)|^p \int_{\mathbb{T}} |g(rt)|^p dm(t) \right) \\ &= \|f\|_\infty^p \|g\|_p^p. \end{aligned}$$

Thus $\|fg\|_p \leq \|f\|_\infty \|g\|_p < \infty$ by hypothesis and then we get $fg \in H^p$. For the case when $p = \infty$, the lemma follows from the observation that H^∞ is a Banach algebra. $\square$

Lemma 2.1.13. [RC00, Proposition 3.2.6 page 20] Let $0 < p < \infty$ and $f \in H^p$. Then for all $z \in \mathbb{D}$,

$$|f(z)| \leq 2^{1/p} \frac{\|f\|_p}{(1 - |z|)^{1/p}}. \quad (2.5)$$

Remark 2.1.14. [Vuk03] For any $z \in \mathbb{D}$, we have equality in (2.5) for the function

$$f_z(u) = \frac{(1 - |z|)^{1/p}}{(1 - \bar{z}u)^{2/p}} \quad u \in \mathbb{D}$$

and $\|f_z\|_p = 1$.

As an immediate consequence, we define for each $z \in \mathbb{D}$, the linear functional $E_z: H^p \rightarrow \mathbb{C}$ by $E_z(f) = f(z)$. By (2.5) we have

$$|E_z(f)| = |f(z)| \leq 2^{1/p} \frac{\|f\|_p}{(1 - |z|)^{1/p}}.$$

Which gives

$$\|E_z\| \leq 2^{1/p} (1 - |z|)^{-1/p}.$$

This discussion can be summarised by the following corollary.

Corollary 2.1.15. For $z \in \mathbb{D}$, the linear functional $E_z: H^p \rightarrow \mathbb{C}$ is continuous.

We point out in the case when $p = 2$, H^2 is a Hilbert space, and it is commonly called the *Hardy-Hilbert space*. The monograph [MR07] provides a thorough treatment of the Hardy-Hilbert space case. In particular, the arguments used therein are much simpler and elegant due to the rich geometric structure offered by Hilbert spaces in general, which is in contrast to the general Hardy spaces H^p . However, we note that H^p functions do offer a dual pairing.

Definition 2.1.16. Let $f \in H^p$ and $g \in H^q$, where $1 < p, q < \infty$ satisfies $1/p + 1/q = 1$. Then we define the *pairing*

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t) \overline{g(t)} dm(t).$$

2.2 Riesz-Smirnov factorization

In this section, we discuss a principle result of H^p function theory. This is the Riesz-Smirnov factorization theorem, more commonly known as the inner-outer factorization of H^p functions. It establishes that an H^p function can be decomposed into three factors: A *Blaschke product*, a *singular inner function* and an *outer function*. Moreover, these factors can be written entirely from the given function and are dependent on a few parameters. These factors inherently carry the various properties of the function which especially allows us to analyze the function from various directions. The results contained in this section come mostly from the sources [Con95; Nik02]. We will not dwell too far into their theory and cover only what is necessary and relevant to the subsequent chapters.

Definition 2.2.1. A function $f \in H^\infty$ such that $|\tilde{f}(t)| = 1$ a.e. on $\mathbb{T}$ is called an *inner function*.

Inner functions play an essential role in the factorization of H^p functions. There are fundamentally two inner functions, one with roots on the unit disc $\mathbb{D}$ and the other with no roots on the unit disc but possible roots on the unit circle $\mathbb{T}$. We first discuss the former.

Definition 2.2.2. Let $\{a_n\}_{n=1}^N$ be a sequence in $\mathbb{D} \setminus \{0\}$, where N can be finite or infinite. The sequence $\{a_n\}_{n=1}^N$ is said to satisfy the *Blaschke condition* if $\sum_{n=1}^N (1 - |a_n|) < \infty$. For $0 \neq a_j \in \{a_n\}_{n=1}^N$, the *Blaschke factor* of a_j is the function

$$B_{a_j}(z) = \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \overline{a_j}z},$$

while for $a_j = 0 \in \{a_n\}_{n=1}^N$, the corresponding Blaschke factor is

$$B_0(z) = z.$$

The *Blaschke product* for the sequence $\{a_n\}_{n=1}^N$ satisfying the Blaschke condition is defined to be the analytic function

$$b(z) = \prod_{n=1}^N B_{a_n}(z).$$

Lemma 2.2.3. [Con95, Proposition 2.6, pg 274] If b is a Blaschke product, then b is an inner function. Moreover, the set $\{a_n\}_{n=1}^N$ consists precisely the roots of b , multiplicities taken into account.

We recall that a positive measure ν on $\mathbb{T}$ is said to be *singular* relative to the Lebesgue measure m if there exists a measurable set A such that $\nu(A) = 0$ and $m(\mathbb{T} \setminus A) = 0$. In the subsequent text, we will only write “positive singular measure”, which is to be understood as a positive measure which is singular relative to the Lebesgue measure m on $\mathbb{T}$.

Definition 2.2.4. Let μ be a positive singular borel measure on $\mathbb{T}$. Then the analytic function

$$S_\mu(z) = \exp \left(- \int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t) \right)$$

is called a *singular inner function*.

Lemma 2.2.5. [Con95] Let S_μ be a singular inner function as defined above. Then S_μ is an inner function.

Singular inner functions are the inner functions which have no roots on $\mathbb{D}$. Singular inner functions are known candidates whose boundary functions have “pathological” behavior. Consider the function

$$u(z) = \exp \left(\frac{z+1}{z-1} \right).$$

Then u is an inner function [GMR16]. Consider the reciprocal of u , which is given by

$$v(z) = \exp \left(\frac{1+z}{1-z} \right).$$

Then v is analytic on $\mathbb{D}$ and $|v(e^{i\theta})| = |1/u(e^{i\theta})| = 1$ for $\theta \in (0, 2\pi)$. The latter implies that $v \in L^\infty$. Note that

$$|v(r)| = \left| \exp \left(\frac{1+r}{1-r} \right) \right| \longrightarrow \infty \quad \text{as } r \longrightarrow 1^-.$$

This implies that $v \notin H^\infty$ and thus not an inner function. More generally, the reciprocal of any nontrivial inner function is not an inner function. For more details look into [GMR16, page 45].

Inner functions enjoy a nice lattice structure as well. However, their use in this text is limited. We mention here a basic notion of inner functions that will be needed in the subsequent chapter. Let I_1, I_2 be two inner functions. Then I_1 is said to divide I_2 , there exists an inner function I_3 such that $I_1 = I_3 I_2$. More arithmetic can be done for inner functions and for further reading we refer to [FM16, Chapter 14].

Definition 2.2.6. Let I_1, I_2 be inner functions. Then I_1 and I_2 are called *relatively prime* if every inner function I_3 that divides both I_1 and I_2 is unimodular, and neither I_1 or I_2 divides the other.

We now turn to another class of analytic functions which may be viewed as “complimentary parts” to inner functions. In particular, these functions have no roots in $\mathbb{D}$. For a real-valued function $h \in L^1$ and $\lambda \in \mathbb{T}$, the Herglotz integral

$$F(z) = \lambda \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} |h(t)| dm(t) \right) \quad (2.6)$$

defines an analytic function on $\mathbb{D}$. Moreover, F admits a non-tangential extension to $\mathbb{T}$ with the property that $|F(t)| = e^{h(t)}$ a.e on $\mathbb{T}$, see [Hof14] for further details. The function F in (2.6) defined by h is called an *outer function*.

The following lemma is immediate from the above discussion.

Lemma 2.2.7. Let g be a measurable function on $\mathbb{T}$ with $\log |g| \in L^1$. Then

$$F_g(z) := \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right) \quad (2.7)$$

defines an outer function such that $|F_g(t)| = |g(t)|$ a.e on $\mathbb{T}$ and $F_g(0) > 0$. Furthermore, if $F \in \text{hol}(\mathbb{D})$ is an outer function with $|F(t)| = |g(t)|$ a.e on $\mathbb{T}$ and $F(0) > 0$, then $F = F_g$.

The function F_g given by (2.7) is called the *outer spectral factor* of g .

Lemma 2.2.8. Let g, h be log-integrable functions on $\mathbb{T}$. Then gh and g/h are log-integrable on $\mathbb{T}$. In that case, $F_{gh} = F_g F_h$ and $F_{g/h} = F_g / F_h$.

Proof. Suppose $\log |g|, \log |h| \in L^1$. Then

$$\begin{aligned} \int_{\mathbb{T}} |\log |g(t)h(t)|| dm(t) &= \int_{\mathbb{T}} |\log |g(t)| + \log |h(t)|| dm(t) \\ &\leq \int_{\mathbb{T}} |\log |g(t)|| dm(t) + \int_{\mathbb{T}} |\log |h(t)|| dm(t) \\ &< \infty. \end{aligned}$$

Hence, $\log |gh| \in L^1$. We also have

$$\begin{aligned} \int_{\mathbb{T}} \left| \log \left| \frac{g(t)}{h(t)} \right| \right| dm(t) &= \int_{\mathbb{T}} |\log |g(t)| - \log |h(t)|| dm(t) \\ &\leq \int_{\mathbb{T}} |\log |g(t)|| dm(t) + \int_{\mathbb{T}} |\log |h(t)|| dm(t) \\ &< \infty. \end{aligned}$$

Hence $\log |g/h| \in L^1$. In view of Corollary 2.2.7, the outer functions F_{gh} and $F_{g/h}$ exist.

More explicitly,

$$\begin{aligned}
 F_{gh}(z) &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)h(t)| dm(t) \right) \\
 &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} (\log |g(t)| + \log |h(t)|) dm(t) \right) \\
 &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) + \int_{\mathbb{T}} \frac{t+z}{t-z} \log |h(t)| dm(t) \right) \\
 &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right) \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |h(t)| dm(t) \right) \\
 &= F_g(z) F_h(z).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 F_{g/h}(z) &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log \left| \frac{g(t)}{h(t)} \right| dm(t) \right) \\
 &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} (\log |g(t)| - \log |h(t)|) dm(t) \right) \\
 &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right) \exp \left(- \int_{\mathbb{T}} \frac{t+z}{t-z} \log |h(t)| dm(t) \right) \\
 &= \frac{\exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right)}{\exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |h(t)| dm(t) \right)} \\
 &= \frac{F_g(z)}{F_h(z)}. \quad \square
 \end{aligned}$$

We have so far that a measurable function g which is log-integrable on $\mathbb{T}$ defines the outer function $F_g \in \text{hol}(\mathbb{D})$. We would like to know when $F_g \in H^p$. It turns out this is the case when $g \in L^p$, as stated by the following theorem.

Theorem 2.2.9. [Nik02, Theorem 3.9.1] Let $h \in L^p$, $1 \leq p \leq \infty$ with $\log |h| \in L^1$. Then the outer spectral factor

$$F_h(z) = \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |h(t)| dm(t) \right). \quad (2.8)$$

belongs to H^p .

Let $g \in H^p$. By the Riesz brothers theorem, Theorem 2.1.3, g is log-integrable and hence by Lemma 2.2.7, the outer spectral factor F_g exists and $|F_g(t)| = |g(t)|$ *a.e.* on $\mathbb{T}$. Furthermore, since $g \in H^p \subset L^p$, the preceding theorem gives that $F_g \in H^p$. This shows that every H^p function has an outer spectral factor belonging to H^p . Consequently, we say that g is an outer function in H^p if $g = \lambda F_g$ for some $\lambda \in \mathbb{T}$.

Outer functions in H^p arise naturally when studying certain invariant subspaces of the multiplication operator M_z on H^p . Let $f \in H^p$, then the closed linear subspace

$$E_f := \overline{\text{span}}\{z^n f : n \geq 0\}$$

is an invariant subspace of M_z . We naturally consider all functions f in H^p such that $E_f = H^p$. It turns out that these functions are precisely the outer functions in H^p .

Corollary 2.2.10. [Con95, Lemma 5.4, pg 291] Let $f \in H^p, 0 < p < \infty$. Then $E_f = H^p$ if and only if f is an outer function.

We now conclude with a result which encompasses the two types of functions we defined in this section. This result is due to F. Riesz and V. Smirnov and establishes that a given function $f \in H^p$ can be decomposed into a product of an inner function and an outer function.

Theorem 2.2.11. [Nik02] Let $f \in H^p \setminus \{0\}, 0 < p \leq \infty$. Then f admits a factorization $f = \lambda B S F$, where $\lambda \in \mathbb{T}$, B is a Blaschke product, S is a singular inner function and $F \in H^p$ is an outer function with $F(0) > 0$. In that case, $F = F_f$ is the outer spectral factor of f and B is the Blaschke product determined by the roots of f in $\mathbb{D}$, multiplicity taken into account. Moreover, the factors λ, B, S and F with the above mentioned properties are unique.

By the above theorem, the roots of a function $f \in H^p$ in $\mathbb{D}$ are contained in the Blaschke product B . By the Blaschke condition, the roots of B cannot accumulate within $\mathbb{D}$ and so must accumulate on the boundary. In particular, the roots of B must accumulate to a point on a set of Lebesgue measure zero.

In the above theorem, we note that the factor $\lambda B S$ is an inner function, so we denote it by $f_i := \lambda B S$. We also denote $f_o := F$. Consequently, we have $f = f_i f_o$. The following corollary summarizes this discussion.

Corollary 2.2.12. Let $f \in H^p, 0 < p \leq \infty$. Then f admits an inner-outer factorization $f = f_i f_o$, where f_i is an inner function and f_o is an outer function with $f_o(0) > 0$. Moreover, the functions f_i and f_o are uniquely determined by f .

In the subsequent section, outer functions of H^p for $1 \leq p \leq \infty$ will serve a pivotal role when considering more general classes of analytic functions. In this regard, it will be convenient to be able to recognize H^p outer functions.

Theorem 2.2.13. Let $1 \leq p \leq \infty$ and $f \in H^p$. Then the following are equivalent:

- (i) f is an outer function.
- (ii) If $g \in H^p$ and $g/f \in L^p$, then $g \in H^p$.

Proof. We first prove (i) $\implies$ (ii).

Suppose that f is an outer function. Let $g \in H^p$ such that $g/f \in L^p$. We need to show

that $g/f \in H^p$. Since f is an outer function, $f = \lambda F_f$, where $\lambda \in \mathbb{T}$ and F_f is the outer spectral factor of f . On the other hand, since $g \in H^p$. By Theorem 2.2.11 we have $g = \alpha B S F_g$, where $\alpha \in \mathbb{T}$, B is a Blaschke product, F_g is the outer spectral factor of g . We then have

$$\frac{g}{f} = \frac{\alpha B S F_g}{\lambda F_f} = (\alpha/\lambda) B S F_{g/f},$$

where the last equality follows from Corollary 2.2.8. Since $g/f \in L^p$ by assumption, we have by Theorem 2.2.9 that $F_{g/f} \in H^p$. Since $BS \in H^\infty$ and H^∞ is the multiplier algebra on H^p , we get $g/f \in H^p$.

We will now show that $(ii) \implies (i)$. Suppose for all $g \in H^p$ with $g/f \in L^p$ implies that $g/f \in H^p$. We need to show that f is an outer function or equivalently, f has no inner factor. Since $f \in H^p$, by Theorem 2.2.11 f has a RS-factorization $f = \lambda S_1 B_1 F_f$. Define the function $g := \min(|f|, 1)$. Since g is the minimum of two measurable function on $\mathbb{T}$, it follows that g is measurable. By definition, $|g(t)| \leq 1$ a.e. on $\mathbb{T}$. This gives us $g \in L^\infty$. Next we show that g is log-integrable. Let E be a measurable subset of $\mathbb{T}$ for which $|f(t)| < 1$ a.e. on E and let $\mathbb{T} \setminus E$ be the set where $|f(t)| \geq 1$ a.e. on $\mathbb{T} \setminus E$. We then have

$$\begin{aligned} \int_{\mathbb{T}} |\log(g(t))| dm(t) &= \int_{\mathbb{T}} |\log(\min(1, |f(t)|))| dm(t) \\ &= \int_E |\log(\min(1, |f(t)|))| dm(t) + \int_{\mathbb{T} \setminus E} |\log(\min(1, |f(t)|))| dm(t) \\ &= \int_E |\log |f(t)|| dm(t) + \int_{\mathbb{T} \setminus E} \log(1) dm(t) \\ &= \int_E |\log |f(t)|| dm(t) \\ &\leq \int_{\mathbb{T}} |\log |f(t)|| dm(t) \\ &< \infty. \end{aligned}$$

Thus $\log(g) \in L^1$ and consequently, the outer spectral factor F_g exists and belongs to H^∞ by Theorem 2.2.9. We then have

$$\left| \frac{F_g}{f} \right| = \frac{|F_g|}{|f|} = \frac{|g|}{|f|} \leq 1 \text{ a.e. } \mathbb{T}.$$

This shows that $F_g/f \in L^\infty$. Since $H^\infty \subset H^p$ and $L^\infty \subset L^p$, by hypothesis we have $F_g/f \in H^p$. Hence F_g/f also has a RS-factorization

$$\frac{F_g}{f} = S_2 B_2 F_h = S_2 B_2 F_{g/f} = S_2 B_2 \frac{F_g}{F_f}.$$

Multiplying both sides by f/F_g and substituting $f = S_1 B_1 F_f$ gives

$$1 = S_2 B_2 \frac{F_g}{F_f} \frac{f}{F_g} = S_1 S_2 B_1 B_2 := SB$$

with $S = S_1 S_2$ a singular inner function and $B = B_1 B_2$ a Blaschke product. Hence $1 = SB$ is the RS-factorization of the constant function with value 1. However, it is easy to see

that the RS-factorization of this function has both factors equal to the constant function with value 1. Hence, by uniqueness of the RS-factorization $S = 1$ and $B = 1$. This then implies that $S_1 = 1$ and $B_1 = 1$. We conclude that f is an outer function. $\square$

Corollary 2.2.14. Let $f \in H^p$ for $1 \leq p \leq \infty$. If $M = \inf_{z \in \mathbb{D}} |f(z)| > 0$, then f is an outer function in H^p .

Proof. First, we note that $1/f \in \text{hol}(\mathbb{D})$ since f has no roots in $\mathbb{D}$. Moreover, $|1/f(z)| \leq 1/M$ for all $z \in \mathbb{D}$, hence $1/f \in H^\infty$. Let $g \in H^p$. Then by Lemma 2.1.12, $g/f \in H^p$. By Theorem 2.2.13, f is an outer function in H^p . $\square$

Theorem 2.2.15. [Nik02, Theorem 4.2 pg 67] Let $f \in \text{hol}(\mathbb{D}) \setminus \{0\}$ such that $\Re f(z) \geq 0$ for all $z \in \mathbb{D}$. Then f is an outer function in H^1 .

Corollary 2.2.16. Let $f_1, f_2 \in H^\infty$. Then the product $f_1 f_2$ is an outer function in H^∞ if and only if f_1 and f_2 are outer functions.

Proof. Suppose the product $f_1 f_2$ is an outer function in H^∞ . Let

$$f_1 = \lambda_1 S_1 B_1 F_{f_1}, \quad f_2 = \lambda_2 S_2 B_2 F_{f_2}$$

be the RS-factorizations of f_1 and f_2 . Then

$$f_1 f_2 = \lambda_1 \lambda_2 S_1 S_2 B_1 B_2 F_{f_1} F_{f_2}.$$

Due to the uniqueness of the RS-factorization of H^∞ functions, the function $f_1 f_2$ being outer implies that $S_1 S_2 B_1 B_2$ is a unimodular constant. Since $|(S_1 S_2 B_1 B_2)(z)| \leq 1$ for all $z \in \mathbb{D}$, we must have that S_1, S_2, B_1, B_2 are unimodular constants. Hence $f_1 = \alpha_1 F_{f_1}$ and $f_2 = \alpha_2 F_{f_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{T}$ and thus f_1 and f_2 are outer functions in H^∞ .

Conversely, suppose that f_1 and f_2 are outer functions in H^∞ . Then by definition $f_1 = \lambda_1 F_{f_1}$ and $f_2 = \lambda_2 F_{f_2}$ for some $\lambda_1, \lambda_2 \in \mathbb{T}$. Then, by Corollary 2.2.8

$$f_1 f_2 = \lambda_1 \lambda_2 F_{f_1} F_{f_2} = \lambda F_{f_1 f_2},$$

where $\lambda = \lambda_1 \lambda_2 \in \mathbb{T}$. Hence, $f_1 f_2$ is an outer function in H^∞ . $\square$

Lemma 2.2.17. Let ρ be a polynomial with roots only on $\mathbb{T}$. Then ρ is an outer function in H^∞ .

Proof. Suppose ρ is a polynomial with roots only on $\mathbb{T}$. Let $\alpha_1, \dots, \alpha_n$ be the roots of ρ , with multiplicity taken into account. We can then write

$$\rho(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n) = (-1)^n \alpha_1 \alpha_2 \dots \alpha_n \left(1 - \frac{z}{\alpha_1}\right) \left(1 - \frac{z}{\alpha_2}\right) \dots \left(1 - \frac{z}{\alpha_n}\right).$$

Fix $1 \leq j \leq n$. Consider the factor $(1 - z/\alpha_j) \in \text{hol}(\mathbb{D})$, since $\alpha_j \in \mathbb{T}$. Write $z = x + iy$ and $\alpha_j = a + ib$. We then have

$$1 - \frac{z}{\alpha_j} = 1 - \frac{x + iy}{a + ib}.$$

So that

$$\Re\left(1 - \frac{z}{\alpha_j}\right) = 1 - \frac{xa + yb}{a^2 + b^2} = 1 - (xa + yb).$$

Since $z \in \mathbb{D}$ and $\alpha_j \in \mathbb{T}$, $xa + yb + \Re(z\bar{\alpha}_j) \leq |z\bar{\alpha}| = |z||\bar{\alpha}| = |z| < 1$. Hence $\Re(1 - z/\alpha_j) \geq 0$ and thus by Theorem 2.2.15, $z \mapsto (1 - z/\alpha_j)$ is an outer function in H^1 . Since $(1 - z/\alpha_j)$ is also in H^∞ , we then have that it is an outer function in H^∞ . Since $1 \leq j \leq n$ was arbitrary, we have that $(1 - z/\alpha_j)$ is an H^∞ outer function for all $1 \leq j \leq n$. By inductively applying Corollary 2.2.16 and using the fact that $(-1)^n \alpha_1 \alpha_2 \dots \alpha_n$ is a unimodular constant, we see that the product

$$(-1)^n \alpha_1 \alpha_2 \dots \alpha_n \left(1 - \frac{z}{\alpha_1}\right) \left(1 - \frac{z}{\alpha_2}\right) \dots \left(1 - \frac{z}{\alpha_n}\right) = \rho(z)$$

is an outer function in H^∞ . □

2.3 Nevanlinna and Smirnov class

In this section we will discuss a much broader class of analytic functions which, in particular, contain the Hardy spaces H^p , for $1 \leq p \leq \infty$. We recall from Theorem 2.2.11 that any function $f \in H^p \setminus \{0\}$ admits a factorization $f = \lambda SBF$, where $\lambda \in \mathbb{T}$, $S = S_\mu$, is a singular inner function with a singular inner function μ , B is a Blaschke factor and $F = F_f$ is the outer spectral factor of f . Conversely, consider the product

$$\alpha S_\nu B F_g, \tag{2.9}$$

where $\alpha \in \mathbb{T}$, S_ν is a singular inner function with singular positive measure ν , B is a Blaschke product and F_g is the outer spectral factor of $g \in L^p$ with $\log|g| \in L^1$. By Theorem 2.2.9, $F_g \in H^p$. Since $\alpha S_\nu B \in H^\infty$, we have $\alpha S_\nu B F_g \in H^p$. In a “loose” sense, the widest class of functions one can form so that the factors in (2.9) are still defined is by considering just a singular measure in S_ν (with ν not necessarily positive), and the function $\log|g| \in L^1$ (with g not necessarily in L^p). This class in fact turns out to be precisely the *Nevanlinna class* which we will define below. Note that we can write the singular measure ν as a difference of two positive singular measures $\nu_1 - \nu_2$ and then (2.9) can also be written as

$$\frac{S_{\nu_1}}{S_{\nu_2}} B F_g.$$

The above discussion gave a heuristic definition of the Nevanlinna class. There are many ways of defining the Nevanlinna class and some of them can be found in [Con95; Nik02; RC00] and can be shown to be equivalent. For the relevance of this text, we adopt the definition of [Nik02]. Formally, we define the Nevanlinna class as follows.

Definition 2.3.1. The *Nevanlinna class* of $\mathbb{D}$ is defined to be the set

$$N(\mathbb{D}) := \left\{ f \in \text{hol}(\mathbb{D}) : f = \frac{f_1}{f_2}, f_1, f_2 \in \bigcup_{p \geq 1} H^p \right\}.$$

It is clear from the definition that $N(\mathbb{D})$ contains all the Hardy spaces H^p for $1 \leq p \leq \infty$. Moreover, since H^p functions are log-integrable, by Lemma 2.2.8, we see that each $f \in N(\mathbb{D})$ is log-integrable. We will eventually see that $N(\mathbb{D})$ is a vector space and an algebra. However, we will first give an equivalent definition of $N(\mathbb{D})$ in terms of the algebra H^∞ .

Lemma 2.3.2. Let $f \in \text{hol}(\mathbb{D})$. Then $f \in N(\mathbb{D})$ if and only if $f = f_1/f_2$, where $f_1, f_2 \in H^\infty$.

Proof. Suppose that $f \in N(\mathbb{D})$. Then $f \in \text{hol}(\mathbb{D})$ and by definition $f = f_1/f_2$, with $f_1, f_2 \in H^1$ (since $H^p \subset H^1, p \geq 1$). By Theorem 2.2.11, we can write $f_1 = \lambda_1 S_1 B_1 F_{f_1}$ and similarly $f_2 = \lambda_2 S_2 B_2 F_{f_2}$, where $\lambda_1, \lambda_2 \in \mathbb{T}$, S_1, S_2 are singular inner functions, B_1, B_2 are Blaschke products and $F_{f_1}, F_{f_2} \in H^1$ are the outer spectral factors of f_1 and f_2 , respectively. We then have

$$f = \frac{f_1}{f_2} = \frac{\lambda_1 S_1 B_1 F_{f_1}}{\lambda_2 S_2 B_2 F_{f_2}}.$$

Since $f \in \text{hol}(\mathbb{D})$, f does not have any poles on $\mathbb{D}$, hence the roots of B_2 must be the roots of B_1 , counting multiplicities. Hence, we may write $B_1 = B_2 B$ for some Blaschke product B . We then have

$$f = \frac{\lambda_1 S_1 B_1 F_{f_1}}{\lambda_2 S_2 B_2 F_{f_2}} = \frac{\lambda_1 S_1 B F_{f_1}}{\lambda_2 S_2 F_{f_2}}$$

Consider the function $g = \min(1, |f|)$. Since g is the minimum of two measurable functions, it is again measurable. By definition, $g(t) \leq 1$ a.e. on $\mathbb{T}$. This gives us $g \in L^\infty$. We also need to show that $\log(g) \in L^1$. Let E be a measurable subset of $\mathbb{T}$ for which $|f(t)| < 1$ a.e. on E and $|f(t)| \geq 1$ a.e. on $\mathbb{T} \setminus E$. Then

$$\begin{aligned} \int_{\mathbb{T}} |\log(g(t))| dm(t) &= \int_{\mathbb{T}} |\log(\min(1, |f(t)|))| dm(t) \\ &= \int_E |\log(\min(1, |f(t)|))| dm(t) + \int_{\mathbb{T} \setminus E} |\log(\min(1, |f(t)|))| dm(t) \\ &= \int_E |\log |f(t)|| dm(t) + \int_{\mathbb{T} \setminus E} \log(1) dm(t) \\ &= \int_E |\log |f(t)|| dm(t) \\ &\leq \int_{\mathbb{T}} |\log |f(t)|| dm(t) \\ &< \infty. \end{aligned}$$

The last inequality follows since $f = f_1/f_2$ which is a ratio of two log-integrable functions. Since $\log(g) \in L^1$, we can define the outer function $F_{\min(|f|,1)}$. Since $g \in L^\infty$, by Theorem 2.2.9 we have $F_{\min(|f|,1)} \in H^\infty$. Similarly, define the function $g' = \min(|f|^{-1}, 1)$, which also belongs to L^∞ . Likewise, we can define the outer function $F_{\min(|f|^{-1},1)}$ and by the same argument as above we get $F_{\min(|f|^{-1},1)} \in H^\infty$. Notice that

$$F_g = F_{\min(|f|,1)} = F_{|f| \min(|f|^{-1},1)} = F_{|f|} F_{\min(|f|^{-1},1)} = F_{|f|} F_{g'} = F_f F_{g'}$$

We define the functions $G_1 := \lambda_1 S_1 B F_g \in H^\infty$ and $G_2 := \lambda_2 S_2 F_{g'} \in H^\infty$. Then

$$\begin{aligned} \frac{G_1}{G_2} &= \frac{\lambda_1 S_1 B F_g}{\lambda_2 S_2 F_{g'}} \\ &= \frac{\lambda_1 S_1 B F_f F_{g'}}{\lambda_2 S_2 F_{g'}} \\ &= \frac{\lambda_1 S_1 B F_{f_1/f_2}}{\lambda_2 S_2} \\ &= \frac{\lambda_1 S_1 B F_{f_1}}{\lambda_2 S_2 F_{f_2}} \\ &= f. \end{aligned}$$

The converse is trivial. □

We recall that for a log-integrable function g , the outer function F_g is defined by

$$F_g(z) = \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right).$$

Lemma 2.3.3. [Con95, Proposition 3.6 pg 279] Let g be a measurable function on $\mathbb{T}$ and $\log |g| \in L^1$. Then the outer function F_g belongs to $N(\mathbb{D})$.

There is an interesting subclass of $N(\mathbb{D})$ which is also a vector space and in particular also contains the Hardy spaces H^p for $1 \leq p \leq \infty$. This is the *Smirnov class* which can be viewed as the natural limit of H^p as $p \rightarrow 0$.

Definition 2.3.4. The *Smirnov class* of $\mathbb{D}$ is defined to be the set

$$N^+(\mathbb{D}) := \left\{ f \in \text{hol}(\mathbb{D}) : f = \frac{f_1}{f_2}, f_1, f_2 \in \bigcup_{p \geq 1} H^p, f_2 \text{ outer} \right\}.$$

Analogous to the Nevanlinna class, every function in the Smirnov class can be written as a ratio of H^∞ functions with the denominator being outer.

Lemma 2.3.5. A function $f \in N^+(\mathbb{D})$ if and only if $f = f_1/f_2$ where $f_1, f_2 \in H^\infty$ and f_2 is outer.

Proof. The proof of this result is similar to the proof of Lemma 2.3.2. Suppose $f \in N^+(\mathbb{D})$ with f_1/f_2 with $f_1, f_2 \in H^1$ and f_2 is outer. Using the same inner-outer factorization in the preceding proof with $S_2 = 1$, we have the desired result. The converse is again trivial. $\square$

By Lemma 2.3.3, for a measurable function g on $\mathbb{T}$ with $\log |g| \in L^1$, the outer function F_g belongs to the Nevanlinna class $N(\mathbb{D})$. In the following lemma, we show that F_g in fact belongs to the Smirnov class.

Lemma 2.3.6. Let g be a measurable function on $\mathbb{T}$ such that $\log |g| \in L^1$. Then the outer spectral factor F_g belongs to $N^+(\mathbb{D})$.

Proof. Suppose g is measurable on $\mathbb{T}$ and $\log |g| \in L^1$. Then the outer function F_g exists and belongs to the Nevanlinna class by the above lemma. Suppose $F_g = f_1/f_2$, where $f_1, f_2 \in H^1$. Then we have on $\mathbb{T}$, $|g| = |F_g| = \frac{|f_1|}{|f_2|}$. Consequently,

$$\begin{aligned} F_g(z) &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |g(t)| dm(t) \right) \\ &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log \frac{|f_1(t)|}{|f_2(t)|} dm(t) \right) \\ &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} (\log |f_1(t)| - \log |f_2(t)|) dm(t) \right) \\ &= \frac{\exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |f_1(t)| dm(t) \right)}{\exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \log |f_2(t)| dm(t) \right)} \\ &= \frac{F_{f_1}(z)}{F_{f_2}(z)}. \end{aligned}$$

Since $f_1, f_2 \in H^1$, by Theorem 2.2.9 we have F_{f_1}, F_{f_2} are outer functions in H^1 . Hence $F_g \in N^+(\mathbb{D})$. $\square$

By Lemma 2.3.2 and Lemma 2.3.5 and using the fact that H^∞ is an algebra, it is easy to see that $N(\mathbb{D})$ and $N^+(\mathbb{D})$ are vector spaces and algebras. Moreover, the representation of functions in $N(\mathbb{D})$ allows us to deduce their properties from their corresponding bounded analytic functions.

Corollary 2.3.7. Let $f = f_1/f_2 \in N(\mathbb{D}) \setminus \{0\}$. Then the non-tangential limit $\tilde{f}$ of f exists almost everywhere on $\mathbb{T}$. Moreover, $\log(|\tilde{f}(t)|) \in L^1$ and thus the limit $\tilde{f}$ does not vanish on a set of positive measure.

Proof. By Lemma 2.3.2, we can write $f = f_1/f_2$ where $f_1, f_2 \in H^\infty \setminus \{0\}$. By Theorem 2.1.2, f_1 and f_2 have non-tangential limits $\tilde{f}_1, \tilde{f}_2$ a.e. on $\mathbb{T}$ and by Lemma 2.1.3, $\tilde{f}_2$ cannot vanish on a set $E \subset \mathbb{T}$ of positive measure. Therefore, $f = f_1/f_2$ has non-tangential limits

almost everywhere on $\mathbb{T}$. By the same Lemma, we also have that $\tilde{f}_1, \tilde{f}_2$ are log-integrable on $\mathbb{T}$. Consequently, the non-tangential limit of f is also log-integrable, which implies that f cannot vanish on a subset of $\mathbb{T}$ of positive measure. $\square$

One of the enticing properties of the Smirnov class is the fact that one can recover H^p functions from L^p functions, as demonstrated by the following lemma.

Lemma 2.3.8. For $1 \leq p \leq \infty$, $L^p \cap N^+(\mathbb{D}) = H^p$.

Proof. Suppose $f \in L^p \cap N^+(\mathbb{D})$. By Lemma 2.3.5, we have $f = f_1/f_2$, where $f_1, f_2 \in H^\infty$ and f_2 is outer. On the other hand we also have $f_1/f_2 \in L^p$. Since $f_1 \in H^\infty \subset H^p$ and f_2 is outer, by Theorem 2.2.13 we have $f = f_1/f_2 \in H^p$. So we have the inclusion $L^p \cap N^+(\mathbb{D}) \subset H^p$. The reverse inclusion easily follows from the fact that $H^p \subset N^+(\mathbb{D})$ and $H^p \subset L^p$. $\square$

We have showed that every function in the Smirnov class can be written as quotients of H^∞ functions with their denominators being outer. However such a representation is not unique. The following result, from [Sar08], demonstrates that every function in $N^+(\mathbb{D})$ has a unique representation.

Lemma 2.3.9. [Sar08] A function $f \in N^+(\mathbb{D})$ can be written uniquely as $f = b/a$, where a and b belong to the unit ball of H^∞ , a is an outer function with $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$.

Proof. Let $f \in N^+(\mathbb{D})$. By Lemma 2.3.5 we can write $f = f_1/f_2$ with $f_1, f_2 \in H^\infty$ and f_2 is an outer function. We first show that $\log(|f_1|^2 + |f_2|^2) \in L^1$. Since $f_1 \in H^\infty$, $\log |f_1| \in L^1$ and by the monotone property of log, we have

$$\log |f_1(t)|^2 \leq \log(|f_1(t)|^2 + |f_2(t)|^2) \text{ a.e. on } \mathbb{T}$$

Hence,

$$\int_{\mathbb{T}} \log(|f_1(t)|^2 + |f_2(t)|^2) dm(t) \geq \int_{\mathbb{T}} \log |f_1(t)|^2 dm(t) = \int_{\mathbb{T}} 2 \log |f_1(t)| dm(t) > -\infty.$$

On the other hand, since $f_1, f_2 \in H^\infty$, there exists constants M, N such that $|f_1(t)| \leq M$ a.e on $\mathbb{T}$ and $|f_2(t)| \leq N$ a.e. on $\mathbb{T}$. Consequently,

$$\int_{\mathbb{T}} \log(|f_1(t)|^2 + |f_2(t)|^2) dm(t) \leq \int_{\mathbb{T}} \log(M^2 + N^2) dm(t) = \log(M^2 + N^2) \int_{\mathbb{T}} dm(t) = \log(M^2 + N^2).$$

Therefore, $\log(|f_1|^2 + |f_2|^2) \in L^1$. Next, we consider the function

$$h(t) := \frac{|f_2(t)|^2}{|f_1(t)|^2 + |f_2(t)|^2}.$$

Since $f_2 \in H^\infty$, $\log |f_2| \in L^1$. Moreover, as h is a ratio of two log-integrable functions, h is also log-integrable. Thus the outer function $F_h =: a$ exists with $a(0) > 0$ and $|a(t)| = |F_h(t)| = |h(t)|$ a.e. on $\mathbb{T}$. Moreover, $|h(t)| \leq 1$ a.e. on $\mathbb{T}$ hence $a \in L^\infty$. By Theorem 2.2.9, a is an outer function in H^∞ . Define the function $b := fa$. Note that $b \in N^+(\mathbb{D})$. We also have

$$|a|^2 + |b|^2 = |h|^2 + |b|^2 = \frac{|f_2|^2}{|f_2|^2 + |f_1|^2} + \frac{|f_2|^2}{|f_2|^2 + |f_1|^2} \frac{|f_1|^2}{|f_2|^2} = 1 \text{ a.e. on } \mathbb{T}.$$

The above identity shows that $b \in L^\infty$. Summing up, we have $b \in N^+(\mathbb{D}) \cap L^\infty = H^\infty$. Thus we have $f = b/a$ with a and b having the desired properties. Note that the representation is unique since the outer function a is uniquely determined. $\square$

The unique representation $f = b/a$ in the above lemma is called the *canonical representation* of f . For the sequel, when a function $f \in N^+(\mathbb{D})$ is said to have canonical representation $f = b/a$, a, b are to be understood as functions having the properties as described in Lemma 2.3.9.

2.4 Unbounded operators and Fredholm theory

In this section, we briefly recall some basic definitions and results on unbounded linear operators.

Throughout this section X is a Banach space. We recall that a linear operator $A: \mathcal{D}(A) \subset X \rightarrow X$ is a linear map with $\mathcal{D}(A)$ a subspace of X . We call $\mathcal{D}(A)$ the domain of A . We write $A(X \rightarrow X)$ to mean that A is a linear operator defined on some domain $\mathcal{D}(A) \subset X$. We denote by $\mathcal{R}(A)$ the range of A which is a subspace of X . We also denote by $\ker(A)$ the kernel of A . The operator A is called densely defined if $\mathcal{D}(A)$ is a dense subspace of X .

Definition 2.4.1. Let $A: \mathcal{D}(A) \subset X \rightarrow X$ be a linear operator. Then A is said to be closed if it satisfies the following; If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{D}(A)$ such that $x_n \rightarrow x \in X$ and $Ax_n \rightarrow y$, with convergence in norm, then $x \in \mathcal{D}(A)$ and $Ax = y$.

It is clear that all bounded linear operators are closed.

In the subsequent sections, we will be concerned with the spectral and Fredholm theory of linear operators. In this regard, we will review some basic definitions.

Definition 2.4.2. A linear operator $A(X \rightarrow X)$ is called Fredholm if the following conditions hold;

- (i) $\dim(\ker(A)) < \infty$.

(ii) $\text{codim}(\mathcal{R}(A)) < \infty$.

(iii) $\mathcal{R}(A)$ is closed.

In that case, we call $\text{Index}(A) = \dim(\ker(A)) - \text{codim}(\mathcal{R}(A))$ the *index* of A .

Definition 2.4.3. A linear operator $A(X \rightarrow X)$ is called *invertible* if there exists a bounded linear operator B such that $AB = I_X$, where I_X is the identity operator on X , and $BA = I_{\mathcal{D}(A)}$, where $I_{\mathcal{D}(A)}$ is the identity operator restricted to $\mathcal{D}(A)$.

By the closed graph theorem, if $A(X \rightarrow X)$ is closed and bijective, then A is invertible in the sense of the above definition.

Definition 2.4.4. Let $A(X \rightarrow X)$ be a linear operator. The spectrum of A , denoted by $\sigma(A)$ of A is defined to be the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ not invertible}\}. \quad (2.10)$$

The point spectrum of A , denoted by $\sigma_p(A)$ are the set of eigenvalues of A , which is the set

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not injective}\}.$$

The resolvent of A , denoted by $\rho(A)$, is the defined to be the set

$$\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is invertible with bounded inverse}\}.$$

Definition 2.4.5. Let $A(X \rightarrow X)$ be a linear operator. The essential spectrum of A , denoted by $\sigma_{ess}(A)$, is defined to be the set

$$\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not Fredholm}\}.$$

Chapter 3

Bounded Toeplitz operators

In the preceding chapter, we introduced the Hardy spaces of the unit disk $H^p(\mathbb{D})$. In particular, we showed that the Banach space $H^p(\mathbb{D})$ can be identified with a closed subspace of $L^p(\mathbb{T})$ which we defined as $H^p(\mathbb{T})$, adopting the abbreviated notation H^p due to the identification. In this chapter, we will discuss a special class of operators that act on $H^p(\mathbb{T})$ and we only consider $1 < p < \infty$. These are the classical bounded Toeplitz operators. A bounded Toeplitz operator is the compression of a multiplication operator acting on $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$, where the multiplication is by an essentially bounded function on $\mathbb{T}$, i.e. a function in $L^\infty(\mathbb{T})$. One of the goals of this section is to recall some of the basic results concerning classical Toeplitz operators. In particular, we will discuss their spectral and Fredholm theory. There are numerous sources [Dou72; FM16; NGG20; Nik02] which discuss bounded Toeplitz operators. Many of them only consider Toeplitz operators on the Hardy-Hilbert space H^2 . However, the Böttcher-Silbermann book [BS06] provides a full treatment on the subject for the general case H^p , $1 < p < \infty$.

3.1 Toeplitz operators with L^∞ symbols

In this section, we review the theory of Toeplitz operators with symbols belonging to L^∞ .

3.1.1 Definition and Basic properties

Definition 3.1.1. A matrix $T = (a_{jk})_{j,k \in \mathbb{N}_0}$ is called an infinite *Toeplitz matrix*, if there exists a bilateral sequence $(b_n)_{n \in \mathbb{Z}}$ such that $a_{jk} = b_{k-j}$ for all $j, k \geq 0$. That is, T is of the form

$$T = \begin{bmatrix} b_0 & b_{-1} & b_{-2} & b_{-3} & \dots \\ b_1 & b_0 & b_{-1} & b_{-2} & \ddots \\ b_2 & b_1 & b_0 & b_{-1} & \ddots \\ b_3 & b_2 & b_1 & b_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Definition 3.1.2. A linear map T defined at least on the set of analytic polynomials $\{t^n\}_{n \in \mathbb{N}_0}$ is called a *Toeplitz operator* if its matrix representation with respect to the basis $\{t^n\}_{n \in \mathbb{N}_0}$ is a Toeplitz matrix.

Let $b \in L^\infty$, with Fourier series $b(t) = \sum_{j=-\infty}^{\infty} b_j t^j$. Let T_b be the map defined by

$$T_b : H^p \rightarrow H^p, \quad T_b f = P_+(bf).$$

It is clear that T_b is a linear operator on H^p . In fact, we have that T_b is bounded. Indeed, by Lemma 2.1.12 and Corollary 2.1.10 T_b is bounded with norm

$$\|T_b\| \leq \|P_+\| \|b\|_\infty.$$

Recall that for $f \in H^p$, $g \in H^q$, where $1 < p, q < \infty$ and $1/p + 1/q = 1$, the dual pairing is given by

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t) \overline{g(t)} dm(t).$$

The above pairing shows that

$$\langle T_b t^j, t^k \rangle = b_{k-j} \quad \text{for all } j, k \in \mathbb{N}_0.$$

Hence, we observe that T_b has a Toeplitz matrix, which by Definition 3.1.2 implies that T_b is a Toeplitz operator. The above discussion can be summarized as follows.

Lemma 3.1.3. Let $b \in L^\infty$. The map

$$T_b : H^p \rightarrow H^p, \quad T_b f = P_+(bf)$$

is a bounded Toeplitz operator with norm

$$\|T_b\| \leq \|P_+\| \|b\|_\infty.$$

In the above lemma, the function b is called *the symbol* of the Toeplitz operator T_b . In some instances, we say that T_b is induced by b . The following theorem, due to A. Brown and P. Halmos asserts that every bounded Toeplitz operator on H^p which has a Toeplitz matrix is induced by a function from L^∞ .

Theorem 3.1.4. [BS06] Let T be a bounded linear operator on H^p such that T has a Toeplitz matrix. Then there exists a function $a \in L^\infty$ such that $T_a = T$. Moreover,

$$\|a\|_\infty \leq \|T_a\| \leq \|P_+\| \|a\|_\infty. \quad (3.1)$$

The above theorem is a first sign of an elegant interplay between a Toeplitz operator and the symbol that induces it. In particular, the analysis of Toeplitz operators can be reduced to the behavior of its corresponding symbol and vice versa. The following results are immediate from the definition of Toeplitz operators.

Lemma 3.1.5. Let $a \in L^\infty$. Then T_a is the zero operator on H^p if and only if $a = 0$ a.e. on $\mathbb{T}$.

Proof. Suppose T_a is the zero operator. Then $T_a 1 = 0$. Note that $z^n \in H^p$ for all $n \in \mathbb{N}_0$ and thus we also have $T_a(z^n) = 0$ for all $n \in \mathbb{N}_0$. In particular we have

$$T_a(z^n) = P_+(z^n a) = 0 \quad \text{for all } n \in \mathbb{N}_0.$$

Hence $z^n a \in K^p$ for all $n \in \mathbb{N}_0$. This is possible only if $a = 0$ a.e. on $\mathbb{T}$. The converse is trivial. $\square$

Lemma 3.1.6. Let $a, b \in L^\infty$ and $\lambda, \alpha \in \mathbb{C}$. Then $T_{\lambda a + \alpha b} = \lambda T_a + \alpha T_b$ on H^p .

We recall that the function b that defines the Toeplitz operator T_b is called *the symbol* of T_b . The justification of this terminology is due to the fact that every Toeplitz operator is uniquely defined by a function in L^∞ . Indeed, suppose we have $T_a = T_b$ for some $a, b \in L^\infty$. We then have $T_{a-b} = T_a - T_b = 0$ and by the above lemma, we must have that $a - b = 0$ a.e. or equivalently, $a = b$ a.e. on $\mathbb{T}$.

Let $\tilde{\mathcal{T}}_p$ denote the set of all bounded Toeplitz operators on H^p . By Lemma 3.1.6, it is clear that $\tilde{\mathcal{T}}_p$ is a vector space. In the next lemma, we use the correspondence between Toeplitz operators and their symbols to reveal deeper properties of $\tilde{\mathcal{T}}_p$.

Lemma 3.1.7. The map $V: L^\infty \rightarrow \tilde{\mathcal{T}}_p$ defined by $V(b) = T_b$ is a bijective bounded linear operator. In particular,

$$\|b\|_\infty \leq \|V(b)\| \leq \|P_+\| \|b\|_\infty. \quad (3.2)$$

Furthermore, $\tilde{\mathcal{T}}_p$ is a Banach space.

Proof. By Lemma 3.1.6, we have that V is a linear operator. We will now show that V is bounded. Let $b \in L^\infty$. Then by Theorem 3.1.4 we have

$$\|V(b)\| = \|T_b\| \leq \|P_+\| \|b\|_\infty.$$

Hence $\|V\| \leq \|P_+\| < \infty$, since the Riesz projection is bounded on H^p for $1 < p < \infty$. Lastly, we show that V is a bijective operator.

Let $b \in \ker(V)$. Then $V(b) = T_b = 0$. By Lemma 3.1.5, $b = 0$ a.e. on $\mathbb{T}$ and thus we get that V is injective.

Surjectivity of V follows immediately from Theorem 3.1.4, since every bounded Toeplitz operator $T \in \tilde{\mathcal{T}}_p \subset \mathcal{B}(H^p)$ is induced by a function in L^∞ . The inequalities (3.2) follows from Theorem 3.1.4, since $\|V(b)\| = \|T_b\|$. $\square$

Since L^∞ is an algebra as well, one might expect that the map V in the above lemma is multiplicative, that is $T_{ab} = T_a T_b$ for $a, b \in L^\infty$. However, this is not always the case. Indeed, consider the Toeplitz operators $T_{\bar{z}}, T_z$. The product $T_z T_{\bar{z}}$ is not a Toeplitz operator since it has the matrix representation,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

There is a result due to Brown and Halmos [BH64] which precisely determines when the product of two Toeplitz operators is again a Toeplitz operator.

Theorem 3.1.8. [BH64] Let $a, b \in L^\infty$. Then $T_{ab} = T_a T_b$ if and only if $b \in H^\infty$ or $\bar{a} \in H^\infty$.

In [BH64], Brown and Halmos gave an algebraic characterization of Toeplitz operators acting on the Hardy-Hilbert space H^2 . In particular, they showed that a bounded operator X is a Toeplitz operator if and only if $T_{\bar{z}} X T_z = X$. In fact, this result is true for Toeplitz operators acting on the general Hardy spaces H^p . This result is an immediate consequence of Theorem 3.1.4.

Lemma 3.1.9. A bounded operator X on H^p is a Toeplitz operator if and only if $T_{\bar{z}} X T_z = X$.

Proof. Suppose that X is a bounded Toeplitz operator. Then by definition X has a Toeplitz matrix with respect to the standard basis of H^p . Let $f \in H^p$ with $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and let X have the matrix representation

$$X = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We then have

$$\begin{aligned} T_{\bar{z}}XT_zf &= T_{\bar{z}}X\left(\sum_{n=0}^{\infty}\hat{f}(n)z^{n+1}\right) \\ &= T_{\bar{z}}\sum_{j=0}^{\infty}\left(\sum_{n=0}^{\infty}a_{-n}+j+1\hat{f}(n)\right)z^j \\ &= \sum_{j=0}^{\infty}\left(\sum_{n=0}^{\infty}a_{-n+j}\hat{f}(n)\right)z^j \\ &= Xf. \end{aligned}$$

Since f was arbitrary, we get that $T_{\bar{z}}XT_z = X$.

Conversely, suppose $T_{\bar{z}}XT_z = X$. Writing out X , $T_{\bar{z}}$ and T_z in matrix representation with respect to the basis $\{t^n\}_{n \in \mathbb{N}}$, one easily verifies that the identity $T_{\bar{z}}XT_z = X$ implies that the matrix representation of X is a Toeplitz matrix. Hence X is a Toeplitz operator. $\square$

3.1.2 Kernels of bounded Toeplitz operators

We will now briefly discuss some special kernels of Toeplitz operators. These are *minimal Toeplitz kernels*. Over the past 50 years, the kernels of Toeplitz operators, or commonly called Toeplitz kernels, have garnered significant attention due to their rich structure. The paper [HM16] discusses this in great detail from which one immediately observes that the theory of Toeplitz kernels is quite a broad subject. Unfortunately, despite the rich theory of Toeplitz kernels, their relevance in the subsequent sections is inessential; therefore, we will only include those results which will be necessary.

Lemma 3.1.10. If $a \in L^\infty$ such that $\ker(T_a) \neq \{0\}$, then $\log|a| \in L^1$.

Proof. Suppose $\ker(T_a) \neq \{0\}$. Then there exists $f \in H^p \setminus \{0\}$ such that $T_af = P_+(af) = 0$. This implies that $af \in K^p$. We recall that functions in H^p and K^p are log-integrable,

hence $\log |f| \in L^1$ and $\log |af| \in L^1$. We then have the following;

$$\begin{aligned} \int_{\mathbb{T}} |\log |a(t)|| dm(t) &= \int_{\mathbb{T}} \left| \log \frac{|a(t)f(t)|}{|f(t)|} \right| dm(t) \\ &= \int_{\mathbb{T}} |\log |a(t)f(t)| - \log |f(t)|| dm(t) \\ &\leq \int_{\mathbb{T}} |\log |a(t)f(t)|| dm(t) + \int_{\mathbb{T}} |\log |f(t)|| dm(t) < \infty. \end{aligned}$$

Therefore, $\log |a| \in L^1$. □

Definition 3.1.11. For $f \in H^p \setminus \{0\}$. The *minimal Toeplitz kernel* of f , denoted by $K_{\min}(f)$, is a kernel of a bounded Toeplitz operator T_a which contains f and such that if f is contained in the kernel of a bounded Toeplitz operator T_b , then

$$\ker(T_a) \subset \ker(T_b).$$

Theorem 3.1.12. [HM16] Let $f \in H^p \setminus \{0\}$ with inner-outer factorization $f = f_i f_o$. Then

$$K_{\min}(f) = \ker(T_{\overline{zf/f_o}}).$$

In the above theorem, the fact that $f \in \ker(T_{\overline{zf/f_o}})$ is clear. Indeed, we have

$$\left(\frac{\overline{zf}}{f_o} \right) f = \left(\frac{\overline{zf_i f_o}}{f_o} \right) (f_i f_o) = \overline{zf_i} f_i \overline{f_o} = \overline{zf_o} \in \overline{H_0^p} = K^p.$$

Thus $T_{\overline{zf/f_o}} f = P_+((\overline{zf_i f_o}/f_o)f) = 0$, which shows that $f \in \ker(T_{\overline{zf/f_o}})$.

For $1 < p < \infty$, the dual space of H^p is equal to $(H^p)^* = L^{p'}/K^{p'}$, where $1/p + 1/p' = 1$ but is usually identified with $H^{p'}$ in the following sense; each functional G in $(H^p)^*$ has the form

$$G(f) = \int_{\mathbb{T}} f(t) \overline{g(t)} dm(t)$$

for some g belonging to $H^{p'}$ and this function g is uniquely determined by G . Note that this identification of $(H^p)^*$ and $H^{p'}$ corresponds to the duality pairing in Definition 2.1.16. In the sequel, we won't distinguish between $(H^p)^*$ and $H^{p'}$. For $a \in L^\infty$ the adjoint T_a^* of T_a on H^p can be identified with the Toeplitz operator $T_{\overline{a}}$ on $H^{p'}$.

The following theorem is due to Coburn which states that a nonzero bounded Toeplitz operator on H^p has either a trivial kernel or its adjoint has a trivial kernel.

Theorem 3.1.13. [BS06] Let $a \in L^\infty$ and suppose $a(t) \neq 0$ a.e. on $\mathbb{T}$. Then either the kernel of T_a on H^p is trivial or the kernel of $T_{\overline{a}}$ on $H^{p'}$, where $1/p + 1/p' = 1$, is trivial.

3.1.3 Spectral and Fredholm properties for continuous symbol

In this subsection, we focus on the spectral and Fredholm properties of Toeplitz operators with continuous symbols. We denote $C(\mathbb{T})$ for the space of all continuous functions on $\mathbb{T}$.

Let $\mathbb{T}^\circ$ denote the punctured unit circle $\mathbb{T} \setminus \{-1\}$. We then denote by $C(\mathbb{T}^\circ)$ the space of all continuous functions on $\mathbb{T}^\circ$. Considering continuous symbols for Toeplitz operators allows for better descriptions of the Fredholm and spectral properties compared to Toeplitz operators with general L^∞ functions. This is due to the existence of the winding number for continuous functions without roots on the unit circle.

Suppose $a \in C(\mathbb{T})$ with no roots on $\mathbb{T}$. Then there exists a real-valued function $b \in C(\mathbb{T}^\circ)$ such that $a = |a|e^{2\pi i b}$ [BS06]. The increment of b , which is the result of the curve a that circuits about the origin, is an integer which only depends on a . This integer is referred to as the *winding number* of a relative to 0 and we denote it by $\text{wind}(a, 0)$.

Theorem 3.1.14. Let $a \in C(\mathbb{T})$ and $1 < p < \infty$. Then

- (i) T_a is Fredholm on H^p if and only if a has no roots on $\mathbb{T}$.
- (ii) If T_a is Fredholm on H^p , then $\text{Index}(T_a) = -\text{wind}(a, 0)$.
- (iii) T_a is invertible on H^p with a bounded inverse if and only if a has no roots on $\mathbb{T}$ and $\text{wind}(a, 0) = 0$.

3.1.4 Invertibility of Toeplitz operators with unimodular symbols

In parallel to the spectral theory of bounded Toeplitz operators, the invertibility of Toeplitz operators also received a great deal of attention from the 1950s. In the 1960s, Devinatz and Widom studied the invertible bounded Toeplitz operators on the Hardy-Hilbert space H^2 [Dev64; Wid60]. More specifically, they provided the following necessary and sufficient conditions for when a bounded Toeplitz operator with unimodular symbol is invertible.

Theorem 3.1.15. [Nik02] Let $a \in L^\infty$ be a unimodular function. Then the following are equivalent.

- (i) T_a is invertible on H^2 .
- (ii) There exists an outer function $h \in H^2$ such that

$$a = \frac{\bar{h}}{h} \text{ a.e. on } \mathbb{T}.$$

The above result is commonly known as the Devinatz-Widom criterion for invertibility of Toeplitz operators. For further details, we refer to [Dev64; Wid60]. Although this result holds in the Hardy-Hilbert space setting H^2 , it also holds on the general Hardy spaces H^p for $1 < p < \infty$, as was shown by Rochberg [Roc77].

Theorem 3.1.16. [Roc77] Let $b \in L^\infty$ be a unimodular function. Then the following are equivalent.

- (i) T_b is invertible on H^p .
- (ii) There exists an outer function $g \in H^p$ such that

$$b = \frac{\bar{g}}{g} \text{ a.e. on } \mathbb{T}.$$

In the above theorem, since b is a unimodular function we clearly have that $b^{-1} \in L^\infty$. In particular,

$$b^{-1} = \frac{g}{\bar{g}}.$$

From this, it follows that the inverse of T_b can be written as

$$T_b^{-1} = T_g T_{1/\bar{g}}.$$

Since $g, 1/\bar{g} \in L^p$, the Toeplitz operators $T_g, T_{1/\bar{g}}$ are possibly unbounded, but can be densely defined on the set of analytic polynomials. We consider this observation to be a natural motivation to study unbounded Toeplitz operators.

3.2 Toeplitz operators with H^∞ symbols

In this section, we will discuss a special case of Toeplitz operators which was introduced in section 3.1. Here we consider Toeplitz operators with symbols belonging to the closed subspace $H^\infty \subset L^\infty$, which are the bounded analytic functions. In this case, we call the Toeplitz operators with symbols in H^∞ *analytic Toeplitz operators*. The subsections here are written in parallel to the subsections in Section 3.1. In particular, we will compare the various results between the cases L^∞ and H^∞ and demonstrate that the results are much richer for analytic Toeplitz operators. Since analytic Toeplitz operators have symbols in H^∞ , the various complex analysis tools we acquired in Chapter 2 allow us to reveal a deeper structure of analytic Toeplitz operators.

3.2.1 Definitions and Basic properties

Let $a \in H^\infty \subset L^\infty$. From Section 3.1, the *analytic Toeplitz* operator T_a is defined by

$$T_a f = P_+(af) = af, \text{ for } f \in H^p.$$

Since $af \in H^p$, by Lemma 2.1.12, we have $P_+(af) = af$. Hence T_a can also be viewed as the multiplication operator on H^p by a . Due to the absence of the Riesz projection, we have a sharp norm estimate of (3.1).

Theorem 3.2.1. Let $a \in H^\infty$. Then $T_a: H^p \rightarrow H^p$ is a bounded Toeplitz operator. In that case, $\|T_a\| = \|a\|_\infty$.

Proof. Suppose $a \in H^\infty \subset L^\infty$. Then for all $f \in H^p$ and by Lemma 2.1.12 we have

$$\|T_a f\|_p = \|af\|_p \leq \|a\|_\infty \|f\|_p. \quad (3.3)$$

Hence $\|T_a\| \leq \|a\|_\infty$ and thus T_a is bounded. By the first inequality of (3.1) and (3.3), we get $\|T_a\| = \|a\|_\infty$. $\square$

Let Γ_p denote the set of all bounded analytic Toeplitz operators on H^p . Since H^∞ is a vector space, it follows that Γ_p is a subspace of $\mathcal{B}(H^p)$, the set of all bounded linear operators on H^p . Moreover, due to the absence of the Riesz projection and the fact that H^∞ is an algebra, it is clear that for two analytic Toeplitz operators T_a and T_b , we have $T_a T_b = T_{ab}$ and thus Γ_p is an algebra as well. The following lemma provides an elegant correspondence between analytic Toeplitz operators and their defining symbols.

Lemma 3.2.2. The map $J_p: H^\infty \rightarrow \Gamma_p$, defined by $J_p(a) = T_a$ is an isometric algebra isomorphism. In particular, Γ_p is a Banach algebra.

Proof. It is trivial to see that J_p is a linear map. Next we show that J_p is injective. Suppose $a \in \ker(J_p)$. Then $J_p(a) = T_a = 0$. Since $1 \in H^p$, we get $T_a 1 = a = 0$. Hence $\ker(J_p) = \{0\}$.

Surjectivity of J_p follows from the definition of Γ_p . Note that

$$\|J_p(a)\| = \|T_a\| = \|a\|_\infty,$$

which shows that J_p is an isometry. In total, we have that J_p is an isometric isomorphism. Lastly, suppose $a, b \in H^\infty$. Then $ab \in H^\infty$ and then

$$J_p(ab) = T_{ab} = T_a T_b = J_p(a) J_p(b).$$

Since H^∞ is a Banach algebra and J_p is an isometric algebra isomorphism, we have that Γ_p is a Banach algebra as well. $\square$

The above lemma should be compared to Lemma 3.1.7, where we showed the correspondence between L^∞ and $\tilde{\mathcal{T}}_p$, the vector space of all bounded Toeplitz operators on H^p .

The following lemma provides an algebraic characterization of analytic Toeplitz operators, analogous to Lemma 3.1.9. However, for analytic Toeplitz operators, the result is much stronger.

Lemma 3.2.3. [Vuk03, Theorem 4] A bounded operator T on H^p is an analytic Toeplitz operator if and only if T commutes with the forward shift T_z , i.e. $T_z T = T T_z$.

Lemma 3.2.4. For $a \in H^\infty \setminus \{0\}$, the analytic Toeplitz operator T_a on H^p is injective.

Proof. Let $a \in H^\infty \setminus \{0\}$. Suppose $f \in \ker(T_a)$. Then $T_a f = af = 0$, that is $a(t)f(t) = 0$ a.e. on $\mathbb{T}$. By Lemma 2.1.3, a cannot vanish on a set $E \subset \mathbb{T}$ of positive measure. Therefore, we must have that $f = 0$ a.e. on $\mathbb{T}$. Thus $\ker(T_a) = \{0\}$. $\square$

We recall that every function $h \in H^\infty$ admits a RS-factorization $h = SBF$, where S is a singular inner function, B is a Blaschke product and F is an outer function. We will now demonstrate how these factors influence the properties of analytic Toeplitz operators.

Lemma 3.2.5. Let $a \in H^\infty$ be an inner function. Then T_a is an isometry. In particular, T_a has closed range.

Proof. Let a be an inner function. Then $|a(t)| = 1$ a.e. on $\mathbb{T}$. We then have for any $f \in H^p$

$$\begin{aligned} \|T_a f\|_p &= \left(\int_{\mathbb{T}} |a(t)f(t)|^p dm(t) \right)^{1/p} \\ &= \left(\int_{\mathbb{T}} |a(t)|^p |f(t)|^p dm(t) \right)^{1/p} \\ &= \left(\int_{\mathbb{T}} |f(t)|^p dm(t) \right)^{1/p} \\ &= \|f\|_p. \end{aligned}$$

Hence T_a is an isometry and consequently T_a has closed range. $\square$

Lemma 3.2.6. Let $b \in H^\infty$ be an outer function. Then T_b has dense range in H^p .

Proof. Suppose b is an outer function in H^∞ . Since $H^\infty \subset H^p$, b is also an outer function in H^p . By Corollary 2.2.10, the set $\text{span}\{z^n b : n \geq 0\}$ is dense in H^p .

Let $h \in \text{span}\{z^n b : n \geq 0\}$ with $h(z) = \sum_{j=1}^m \alpha_j z^j b(z)$. Since $\rho := \sum_{j=1}^m \alpha_j z^j$ is a polynomial, ρ belongs to H^p . We then have

$$h = \rho b = T_b \rho \in \mathcal{R}(T_b).$$

Therefore, $\text{span}\{z^n b : n \geq 0\} \subset \mathcal{R}(T_b)$. Since $\text{span}\{z^n b : n \geq 0\}$ is dense in H^p , we have that $\mathcal{R}(T_b)$ is dense in H^p . $\square$

3.2.2 Fredholm and spectral properties

In this section, we will briefly discuss the spectral properties of bounded analytic Toeplitz operators.

Lemma 3.2.7. [Dou72, Theorem 7.21] Let $a \in H^\infty$. Then T_a is invertible if and only if a is invertible in H^∞ .

Lemma 3.2.8. [Vuk03] Let $a \in H^\infty$ be non-constant. Then the bounded analytic Toeplitz operator T_a has no eigenvalues.

Proof. Suppose to the contrary that T_a has an eigenvalue $\lambda \in \mathbb{C}$. Let $f \in H^p \setminus \{0\}$ be a corresponding eigenvector. We then have

$$T_a f = a f = \lambda f \text{ a.e. on } \mathbb{T}.$$

Equivalently,

$$(a - \lambda)f = 0 \text{ a.e. on } \mathbb{T}.$$

Since $a - \lambda \in H^\infty \setminus \{0\}$, by the Riesz brothers theorem, $a - \lambda$ cannot vanish on a set of positive measure. Hence we must have that $f(t) = 0$ a.e. on $\mathbb{T}$, but this is a contradiction since $f \neq 0$. Thus T_a has no eigenvalues. $\square$

Lemma 3.2.9. [Vuk03, Theorem 7, pg 107] The spectrum $\sigma(T_a)$ of an analytic Toeplitz operator T_a is equal to $\overline{a(\mathbb{D})}$.

The following theorems will be essential for describing the Fredholm properties for unbounded Toeplitz operators in the subsequent chapter.

Theorem 3.2.10. [Vuk03] Let $a \in H^\infty$. Then T_a has closed range in H^p if and only if there exists a constant $M > 0$ such that the boundary function $|\tilde{a}(t)| > M$, for almost every $t \in \mathbb{T}$.

Theorem 3.2.11. [Dou72, Theorem 7.36] Let $a \in H^\infty$. Then T_a is a Fredholm operator if and only if there exists constants $\epsilon, \gamma > 0$ such that $|\dot{a}(rt)| \geq \epsilon$ for all $t \in \mathbb{T}$ and all $1 - \gamma < r < 1$. In that case, the index of T_a is equal to $-\text{wind}(\dot{a}(rt), 0)$, for any $1 - \gamma < r < 1$.

Remark 3.2.12. In [Dou72], Lemma 3.2.7 and Theorem 3.2.11 is stated for Toeplitz operators on the Hardy-Hilbert space H^2 . Upon inspection of the proof, one can see that the theorem holds for analytic Toeplitz operators on H^p as well.

3.3 Toeplitz operators with polynomial symbols

In this section, we will discuss Toeplitz operators with analytic polynomial symbols. This is essential for the next chapter, where we will study unbounded Toeplitz operators with rational symbols. Moreover, since analytic polynomials are continuous on the unit circle, many results from the preceding subsections allows for concrete descriptions. We will first

introduce some notation which will be used subsequently. We denote by $\mathcal{P}$ the space of all analytic polynomials. For $n \in \mathbb{N}_0$, we denote by $\mathcal{P}_n$ the subspace of $\mathcal{P}$ consisting of all polynomials with degree at most n .

Lemma 3.3.1. Let $s \in \mathcal{P}$ with $s = s_- s_0 s_+$, where s_- , s_0 , s_+ are polynomials having roots only on $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then

$$\mathcal{R}(T_s) = \mathcal{R}(T_{s_- s_0}) \quad \text{and} \quad \overline{\mathcal{R}(T_s)} = \mathcal{R}(T_{s_-})$$

In particular, $\mathcal{R}(T_s)$ is closed if and only if s has no roots on $\mathbb{T}$.

Proof. By definition s_+ has no roots on $\overline{\mathbb{D}}$, hence $1/s_+ \in H^\infty$. This gives us that T_{s_+} is invertible with inverse $T_{s_+}^{-1}$. In particular, $\mathcal{R}(T_{s_+}) = H^p$ which shows that

$$\mathcal{R}(T_s) = s_- s_0 s_+ H^p = s_- s_0 H^p = \mathcal{R}(T_{s_- s_0}).$$

Since s_0 has roots only on $\mathbb{T}$, by Lemma 2.2.17, s_0 is an outer function and thus by Lemma 3.2.6 $\overline{\mathcal{R}(T_{s_0})} = \overline{s_0 H^p} = H^p$. By the boundedness of T_{s_-} , we have

$$\overline{\mathcal{R}(T_s)} = \overline{\mathcal{R}(T_{s_- s_0})} = \overline{\mathcal{R}(T_{s_-})}.$$

Since s_- is continuous and has no roots on $\mathbb{T}$, there exists $M > 0$ such that $|s(t)| > 0$ for all $t \in \mathbb{T}$. By Theorem 3.2.10 we have that T_{s_-} has closed range. Altogether,

$$\overline{\mathcal{R}(T_s)} = \overline{\mathcal{R}(T_{s_-})} = \mathcal{R}(T_{s_-}). \quad \square$$

Lemma 3.3.2. Let $s \in \mathcal{P}$ with $s = s_- s_0 s_+$, where s_- , s_0 , s_+ are polynomials having roots only on $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then $H^p = \mathcal{R}(T_{s_-}) \dot{+} \mathcal{P}_{\deg(s_-)-1}$ forms a direct sum decomposition of H^p . In particular, T_s is Fredholm if and only if s has no roots on $\mathbb{T}$ and in that case $\text{Index}(T_s) = -\deg(s_-)$.

Proof. Since s_- has no roots on $\mathbb{T}$, by Lemma 3.3.1 $\mathcal{R}(T_{s_-})$ is a closed subspace of H^p . On the other hand, since $\mathcal{P}_{\deg(s_-)-1}$ is a finite dimensional subspace of H^p , by [GG81, Theorem 2.5 page 199] the sum $\mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1}$ forms a closed subspace of H^p . Next we show that $\mathcal{P} \subset \mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1}$. Let $\rho \in \mathcal{P}$. By the Euclidean division algorithm, we can write $\rho = r_1 s_- + r_2$, with $\deg(r_2) < \deg(s_-)$. We then observe that

$$\rho = s_- r_1 + r_2 \in s_- H^p + \mathcal{P}_{\deg(s_-)-1} = \mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1},$$

since $r_1 \in H^\infty \subset H^p$. Since $\rho \in \mathcal{P}$ was arbitrary, we have that $\mathcal{P} \subset \mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1}$. Since $\mathcal{P}$ is dense in H^p , we then see that $\mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1}$ is also dense in H^p . In total, we have that

$$\mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1} = \overline{\mathcal{R}(T_{s_-}) + \mathcal{P}_{\deg(s_-)-1}} = H^p.$$

We are left to show that $\mathcal{R}(T_{s_-}) \cap \mathcal{P}_{\deg(s_-)-1} = \{0\}$. Let $r \in \mathcal{R}(T_{s_-}) \cap \mathcal{P}_{\deg(s_-)-1} = s_-H^p \cap \mathcal{P}_{\deg(s_-)-1}$. Then $r = s_-h$ for some $h \in H^p$ and $\deg(r) < \deg(s_-)$. We then have

$$\frac{r}{s_-} = h \in H^p.$$

Since $\deg(r) < \deg(s_-)$ and all the roots of s_- are in $\mathbb{D}$, we have that h has a pole on $\mathbb{D}$. However, this is impossible since $h \in \text{hol}(\mathbb{D})$. Hence, we must have that $r \equiv 0$.

Lastly, we prove the final claim. Suppose that T_s is Fredholm. Then $\mathcal{R}(T_s)$ has closed range and thus s has no roots on $\mathbb{T}$ by Lemma 3.3.1. Conversely, if s has no roots on $\mathbb{T}$, then by Lemma 3.3.1, $\mathcal{R}(T_s)$ is closed. Since T_{s_+} is invertible with bounded inverse, we get that $\mathcal{R}(T_s) = \mathcal{R}(T_{s_-}T_{s_+}) = \mathcal{R}(T_{s_-})$ and hence $\text{codim}(\mathcal{R}(T_s)) = \text{codim}(\mathcal{R}(T_{s_-}))$. Note that by Lemma 3.2.4, $\dim(\ker(T_s)) = 0$ and by the first part of the proof $\text{codim}(\mathcal{R}(T_{s_-})) = \dim(\mathcal{P}_{\deg(s_-)-1}) = \deg(s_-)$. Summing up, we get that T_s is Fredholm with

$$\text{Index}(T_s) = \dim(\ker(T_s)) - \text{codim}(\mathcal{R}(T_s)) = -\deg(s_-).$$

□

Chapter 4

Unbounded Toeplitz operators

In the previous chapter, we considered Toeplitz operators with symbols belonging to L^∞ which precisely yielded bounded Toeplitz operators. Once we leave the realm of bounded symbols, the situation becomes difficult and we get into the theory of unbounded “Toeplitz” operators. We recall that a bounded Toeplitz operator on H^p , $1 < p < \infty$ is defined to be the compression of the multiplication operator, by an essentially bounded function, on H^p . In particular, we showed that the matrix representation of a bounded Toeplitz operator, with respect to the basis of H^p , is a Toeplitz matrix, which justified the name “Toeplitz operator”. In this Chapter, we will study unbounded “Toeplitz” operators whose domains will clearly not be the entire space H^p and may not even contain the polynomials. Consequently, one cannot obtain their matrix representations. However, we continue to use the term “Toeplitz” operator for the unbounded case, as their definitions and actions are akin to that of bounded Toeplitz operators.

We recall Section 3.1.4, where unbounded Toeplitz operators arose as the inverses of bounded Toeplitz operators on H^p . In particular, the symbols of these unbounded Toeplitz operators came from L^p . This raises an interesting problem; How far “beyond” L^∞ can we go and what sense of generality of symbols can be expected which still define in a sense Toeplitz operators on the Hardy spaces H^p ? We noticed that the Hardy space theory brought forth an intimate overlap of measure theory and analytic function theory. In this regard, the idea of a “general symbol” can be understood from two directions. In one direction, we can consider general analytic functions on the unit disc that are conducive to well-defined operators. This approach will be discussed in Section 4.1. In the second direction, we consider general measurable functions on the unit circle and similarly, consider those functions that yield well-defined operators. This approach will be discussed in Section 4.2. There is also a pragmatic approach to unbounded Toeplitz operators. Indeed, one can consider rational functions with poles on the unit circle $\mathbb{T}$ as symbols which naturally induces unbounded Toeplitz operators. This approach will be discussed in Section 4.3. The aim of this section is to find connections between the various classes

of unbounded Toeplitz operators studied throughout the chapter.

4.1 Unbounded Toeplitz operators with analytic symbols

In this section we will consider the unbounded Toeplitz operators considered by Sarason [Sar08], which we will designate as Toeplitz operators of *Sarason-type*. This class can be viewed as the most general class of Toeplitz operators on the Hardy spaces H^p with analytic symbols. It turns out that the largest class of symbols one can consider, while having a well-defined operator, is the Nevanlinna class $N(\mathbb{D})$ and on the other hand the largest class of symbols one can consider for densely-defined Toeplitz operators is the Smirnov class $N^+(\mathbb{D})$. Due to the characterization of these classes in terms of H^∞ , a significant portion of the theory of Sarason-type Toeplitz operators can be retained from the well-known bounded case. The aim of this section is to see which results of bounded analytic Toeplitz operators still hold for Sarason-type Toeplitz operators.

4.1.1 Definition and Basic properties

Let $\phi \in \text{hol}(\mathbb{D})$. The Sarason-type Toeplitz operator with symbol ϕ , which we denote by T_ϕ^A , (with the superscript A indicating ‘Analytic’) is defined in the following way

$$\mathcal{D}(T_\phi^A) = \{f \in H^p : \phi f \in H^p\}, \quad T_\phi^A f = \phi f.$$

Lemma 4.1.1. Let $\phi \in \text{hol}(\mathbb{D})$. Then T_ϕ^A is a closed operator.

Proof. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{D}(T_\phi^A)$ with $f_n \rightarrow f \in H^p$ and $T_\phi^A f_n = \phi f_n \rightarrow g$. We show that $f \in \mathcal{D}(T_\phi^A)$ and $T_\phi^A f = g$. Firstly, we recall that the functionals $E_z : H^p \rightarrow \mathbb{C}$ defined by $E_z(f) = f(z)$ are continuous for each $1 \leq p \leq \infty$. Then for each $z \in \mathbb{D}$, since $f_n \rightarrow f \in H^p$, we have

$$f_n(z) = E_z(f_n) \rightarrow E_z(f) = f(z).$$

We also get that $\phi(z)f_n(z) \rightarrow \phi(z)f(z)$. On the other hand, since $\phi f_n \rightarrow g \in H^p$, we have for each $z \in \mathbb{D}$ that

$$\phi(z)f_n(z) = E_z(\phi f_n) \rightarrow E_z(g) = g(z).$$

By uniqueness of limits, we have $\phi(z)f(z) = g(z)$ for each $z \in \mathbb{D}$. Since $g \in H^p$, we have $f \in \mathcal{D}(T_\phi^A)$ and $T_\phi^A f = g$. $\square$

Lemma 4.1.2. Let $\phi \in \text{hol}(\mathbb{D})$. Then $\mathcal{D}(T_\phi^A)$ is nontrivial if and only if ϕ belongs to the Nevanlinna class $N(\mathbb{D})$. In particular, if $\phi = \phi_1/\phi_2 \in N(\mathbb{D})$, then $\phi_2 H^p \subset \mathcal{D}(T_\phi^A)$.

Proof. Suppose $\mathcal{D}(T_\phi^A) \neq \{0\}$. Then there exists a nonzero $f \in H^p$ such that $\phi f \in H^p$. Then $\phi = \phi f / f$ is a quotient of functions in H^p , hence $\phi \in N(\mathbb{D})$.

Conversely, suppose $\phi \in N(\mathbb{D}) \setminus \{0\}$. By Lemma 2.3.2, $\phi = \phi_1 / \phi_2$ with $\phi_1, \phi_2 \in H^\infty \setminus \{0\}$. Then $\phi_2 \in \mathcal{D}(T_\phi^A)$, since

$$\phi \phi_2 = \frac{\phi_1}{\phi_2} \phi_2 = \phi_1 \in H^\infty \subset H^p.$$

Let $f \in H^p$ be arbitrary, then $\phi_2 f \in \phi_2 H^p$. Multiplying by ϕ gives

$$\phi(\phi_2 f) = \frac{\phi_1}{\phi_2}(\phi_2 f) = \phi_1 f \in H^p.$$

Thus $\phi_2 H^p \subset \mathcal{D}(T_\phi^A)$ and $\phi_2 H^p \neq \{0\}$ since $\phi_2 \neq 0$.

For the case when $\phi = 0$, T_ϕ^A is bounded, so that $\mathcal{D}(T_\phi^A) = H^p$. $\square$

Lemma 4.1.3. Let $\phi \in N(\mathbb{D})$. Then $T_\phi^A = 0$ if and only if $\phi = 0$.

Proof. Suppose $T_\phi^A = 0$. Since $\phi \in N(\mathbb{D})$, $\mathcal{D}(T_\phi^A) \neq \{0\}$, so let $f \in \mathcal{D}(T_\phi^A) \setminus \{0\}$. Then $T_\phi^A f = \phi f = 0$. By the Riesz brothers theorem, Theorem 2.1.3 we must have that $\phi = 0$. The converse is trivial. $\square$

Theorem 4.1.4. Let $\phi \in \text{hol}(\mathbb{D})$. Then $\mathcal{D}(T_\phi^A)$ is dense in H^p if and only if ϕ belongs to the Smirnov class $N^+(\mathbb{D})$.

Proof. Suppose $\mathcal{D}(T_\phi^A)$ is dense in H^p . Clearly, $\mathcal{D}(T_\phi^A) \neq \{0\}$. By Lemma 4.1.2, ϕ belongs to the Nevanlinna class and thus we may write $\phi = \psi / \chi$ with $\psi, \chi \in H^\infty$ with the inner factors of ψ, χ being relatively prime. Take $f \in \mathcal{D}(T_\phi^A)$ and set $g = \phi f$. Then $\psi f = \chi g$. Let

$$\psi = \psi_i \psi_o, \quad \chi = \chi_i \chi_o, \quad f = f_i f_o, \quad g = g_i g_o$$

be the inner-outer factorization of the respective functions. We can then write $\psi f = \chi g$ as $\psi_i f_i \psi_o f_o = \chi_i g_i \chi_o g_o$. By uniqueness of the inner-factorization, we have $\psi_i f_i = \chi_i g_i$. Since ψ_i and χ_i are relatively prime, the function χ_i must be a factor of f_i , that is $f_i = \chi_i h$ for some inner function h . Consequently,

$$\psi_o f = \psi_o f_i f_o = \psi_o h \chi_i f_o = \chi_i (\psi_o h f_o) \in \chi_i H^p.$$

Similarly, applying the uniqueness of the inner-outer factorization, we obtain $f \in \chi_i H^p$. Since $f \in \mathcal{D}(T_\phi^A)$ was arbitrary, we have $\mathcal{D}(T_\phi^A) \subset \chi_i H^p$. The hypothesis implies that $\chi_i H^p$ is dense in H^p and by Corollary 2.2.10 we have that χ_i is an outer function. Since χ_i is also an inner function we must have that χ_i is a uni-modular constant λ . So we have $\chi = \lambda \chi_o$ and hence $\phi \in N^+(\mathbb{D})$.

Conversely, assume $\phi \in N^+(\mathbb{D})$. Then we may write $\phi = \psi / \chi$ with $\psi, \chi \in H^\infty$ and χ an outer function. By Lemma 4.1.2 we have $\chi H^p \subset \mathcal{D}(T_\phi^A)$. Since χ is outer, we have by Lemma 3.2.6 that $\mathcal{R}(T_\chi) = \chi H^p$ is dense in H^p and thus $\mathcal{D}(T_\phi^A)$ is dense in H^p . $\square$

We recall Lemma 2.3.9 where it was shown that every function in the Smirnov class admits a canonical factorization. This allows us to provide a concrete description for the domain $\mathcal{D}(T_\phi^A)$ and range $\mathcal{R}(T_\phi^A)$ when ϕ is in the Smirnov class.

Theorem 4.1.5. Let $\phi \in N^+(\mathbb{D}) \setminus \{0\}$ with canonical representation $\phi = b/a$. Then $\mathcal{D}(T_\phi^A) = aH^p$, $\mathcal{R}(T_\phi^A) = bH^p$ and $T_\phi^A : af \mapsto bf$ for each $f \in H^p$.

Proof. The inclusion $aH^p \subset \mathcal{D}(T_\phi^A)$ follows from Lemma 4.1.2. For the reverse inclusion, suppose $f \in \mathcal{D}(T_\phi^A)$. We then have on $\mathbb{T}$

$$|\phi f|^2 = \frac{|b|^2}{|a|^2} |f|^2 = \frac{1 - |a|^2}{|a|^2} |f|^2 = \frac{|f|^2}{|a|^2} - |f|^2.$$

Equivalently,

$$|\phi f|^2 + |f|^2 = \frac{|f|^2}{|a|^2}.$$

By using the inequality, $(a + b)^p \leq 2^p(a^p + b^p)$, where $a \geq 0, b \geq 0$, we have

$$\begin{aligned} \int_{\mathbb{T}} \left(\frac{|f(t)|}{|a(t)|} \right)^p dm(t) &= \int_{\mathbb{T}} \left(\frac{|f(t)|^2}{|a(t)|^2} \right)^{p/2} dm(t) \\ &= \int_{\mathbb{T}} (|\phi(t)f(t)|^2 + |f(t)|^2)^{p/2} dm(t) \\ &\leq 2^{p/2} \int_{\mathbb{T}} (|\phi(t)f(t)|^p + |f(t)|^p) dm(t) \\ &\leq 2^{p/2} \left(\int_{\mathbb{T}} |\phi(t)f(t)|^p dm(t) + \int_{\mathbb{T}} |f(t)|^p dm(t) \right) < \infty, \end{aligned}$$

since $f, \phi f \in L^p$. Thus $f/a \in L^p$. Since a is an outer function in H^∞ and $f \in H^p$, we have $f/a \in N^+(\mathbb{D})$. Therefore, $f/a \in L^p \cap N^+(\mathbb{D}) = H^p$ by Lemma 2.3.8. Equivalently, $f \in aH^p$. Since $f \in \mathcal{D}(T_\phi^A)$ was arbitrary, we have $\mathcal{D}(T_\phi^A) \subset aH^p$. In total, $\mathcal{D}(T_\phi^A) = aH^p$. Lastly, for $af \in aH^p$, $T_\phi^A af = T_{b/a}^A af = bf \in H^p$. Hence $\mathcal{R}(T_\phi^A) = bH^p$. $\square$

4.1.2 Spectral and Fredholm properties

In this section, we will discuss the Fredholm and spectral properties of Toeplitz operators with symbols belonging to the Smirnov class. Since the Smirnov class functions are ratios of bounded analytic functions, the Sarason-type Toeplitz operators are closely related to bounded analytic Toeplitz operators.

Lemma 4.1.6. Let $\phi \in N(\mathbb{D})$ be non-constant. Then T_ϕ^A has no eigenvalues. Thus $\sigma_p(T_\phi^A) = \emptyset$.

Proof. Suppose to the contrary that T_ϕ^A has an eigenvalue $\lambda \in \mathbb{C}$. Let $f \in \mathcal{D}(T_\phi^A) \subset H^p \setminus \{0\}$ be the corresponding eigenvector. We then have

$$T_\phi^A f = \phi f = \lambda f,$$

or equivalently,

$$(\phi - \lambda)f = 0 \text{ a.e. on } \mathbb{T}$$

Since $\phi - \lambda \in N(\mathbb{D})$, by Corollary 2.3.7 $\phi - \lambda$ cannot be zero on a set of positive measure, so we must have that $f = 0$ a.e. on $\mathbb{T}$. This is a contradiction, thus T_ϕ^A has no eigenvalues. $\square$

Corollary 4.1.7. Let $\phi \in N(\mathbb{D}) \setminus \{0\}$. Then T_ϕ^A is injective.

By Theorem 4.1.5, for the canonical factorization $\phi = b/a \in N^+(\mathbb{D})$, we have $\mathcal{D}(T_\phi^A) = aH^p$ and $\mathcal{R}(T_\phi^A) = bH^p$. We note that the range of the bounded analytic Toeplitz operator T_b , with symbol b , is also bH^p . Thus $\mathcal{R}(T_\phi^A) = \mathcal{R}(T_b)$. In effect, due to the injectivity of T_ϕ^A , the Fredholm theory of T_ϕ^A is reduced to the Fredholm theory of the bounded analytic Toeplitz operator T_b .

Theorem 4.1.8. For $\phi = b/a \in N^+(\mathbb{D})$ with canonical factorization, the Toeplitz operator T_ϕ^A is Fredholm if and only if there exists constants $\gamma, \epsilon > 0$ such that $|b(rt)| \geq \epsilon$ for $1 - \gamma < r < 1$. Moreover, the index of T_ϕ^A is given by

$$\text{Index}(T_\phi^A) = -\text{wind}(\dot{b}(rt), 0).$$

Proof. We recall that T_ϕ^A and T_b are both injective. Then T_ϕ^A is Fredholm if and only if $\mathcal{R}(T_\phi^A)$ is closed with a finite dimensional complement and similarly for T_b . By Theorem 4.1.5, $\mathcal{R}(T_\phi^A) = \mathcal{R}(T_b)$ and thus T_ϕ^A is Fredholm if and only if T_b is Fredholm. In that case, $\text{Index}(T_\phi^A) = \text{Index}(T_b)$. By Theorem 3.2.11, T_b is Fredholm if and only if there exists constants $\gamma, \epsilon > 0$ such that $|\dot{b}(rt)| > \epsilon$ for all $1 - \gamma < r < 1$ and $t \in \mathbb{T}$. Moreover, the index is given by $\text{Index}(T_b) = -\text{wind}(\dot{b}(rt), 0)$ with $1 - \gamma < r < 1$ and $t \in \mathbb{T}$. In total, T_ϕ^A is Fredholm if and only if there exists constants $\gamma, \epsilon > 0$ such that $|\dot{b}(rt)| > \epsilon$ for all $1 - \gamma < r < 1$ and $t \in \mathbb{T}$ and with index

$$\text{Index}(T_\phi^A) = \text{Index}(T_b) = -\text{wind}(\dot{b}(rt), 0), \text{ for } 1 - \gamma < r < 1 \text{ and } t \in \mathbb{T}. \quad \square$$

If $\phi = s/q \in \text{Rat}(\mathbb{T})$, with all the roots of s on $\mathbb{D}$, then by the above theorem, T_ϕ^A is Fredholm with $\text{Index}(T_\phi^A) = -\deg(s)$. We now provide necessary and sufficient conditions for T_ϕ^A to be invertible.

Theorem 4.1.9. Let $\phi = b/a \in N^+(\mathbb{D})$ be the canonical representation of ϕ . Then T_ϕ^A is invertible if and only if there exists a $M > 0$ such that $|\phi(z)| \geq M$ on $\mathbb{D}$.

Proof. Suppose that T_ϕ^A is invertible. Then $\mathcal{R}(T_\phi^A) = bH^p = H^p$. This implies that $1/b \in H^\infty$. Equivalently, $\|b\|_\infty \geq M$, where $M := \|1/b\|_\infty$. Consequently,

$$|\phi(z)| = \frac{|b(z)|}{|a(z)|} \geq \frac{M}{\|a\|_\infty}.$$

Conversely, assume that $|\phi(z)| \geq M$ for all $z \in \mathbb{D}$. Then $1/|\phi(z)| \leq 1/M$ for all $z \in \mathbb{D}$. Consequently, $1/\phi \in H^\infty$ hence $T_{1/\phi}$ is a bounded Toeplitz operator. Finally, we show that $T_{1/\phi}$ is the bounded inverse of T_ϕ^A which concludes the proof.

For any $f \in \mathcal{D}(T_\phi^A)$, we have

$$T_{1/\phi} T_\phi^A f = T_{1/\phi} \phi f = \left(\frac{1}{\phi} \right) \phi f = f.$$

This shows that $T_{1/\phi} T_\phi^A = I_{\mathcal{D}(T_\phi^A)}$, the identity operator on $\mathcal{D}(T_\phi^A)$. On the other hand, for $f \in H^p$ we have

$$T_\phi^A T_{1/\phi} f = T_\phi^A (f/\phi) = \phi \left(\frac{f}{\phi} \right) = f.$$

Hence $T_\phi^A T_{1/\phi} = I_{H^p}$ and so altogether, $T_{1/\phi}$ is the bounded inverse of T_ϕ^A . $\square$

4.1.3 Notes

Most of the results in this section come from [Sar08], where in particular, the Sarason-type Toeplitz operators act on the Hardy-Hilbert space H^2 . In this section, we have adjusted the various results to work on the Hardy space H^p for $1 < p < \infty$. We have also added some new results in this section, and some of them may have been omitted in [Sar08] due to their triviality.

The proof of Lemma 4.1.1 is ours, while only the statement is given in [Sar08]. In Lemma 4.1.2, [Sar08] only proves that $\mathcal{D}(T_\phi^A)$ being non-trivial implies that ϕ belongs to the Nevanlinna class. The proof of the converse of this result is ours. Lemma 4.1.3 is our result, however the result draws inspiration from [CMP21, Proposition 3.1]. All the results in Subsection 4.1.2 are ours. We note that Lemma 4.1.6 is just an extension of Lemma 3.2.8. Theorem 4.1.8 and Theorem 4.1.9 are ours.

4.2 Unbounded Toeplitz operators with measurable symbols

In this section we will consider the class of unbounded Toeplitz operators considered by Câmara et al. [CMP21] which we will call Toeplitz operators of CMP-type. This class can be viewed as the most general class of unbounded Toeplitz operators on H^p . The CMP-type Toeplitz operators can also be viewed as the generalization of the classical bounded Toeplitz operators on H^p , where now the symbols are general measurable functions on the unit circle. In their paper [CMP21], Câmara et al. studied the kernels of unbounded Toeplitz operators with measurable symbols. In this section, we will provide some of the basic properties of the CMP-type Toeplitz operators and make a connection with the Sarason-type Toeplitz operators which were studied in Section 4.1. We also consider

the question of the existence of minimal Toeplitz kernels, with unbounded symbols, for functions in the domain of Sarason-type Toeplitz operators.

Throughout this section, we only consider H^p , for $1 < p < \infty$.

4.2.1 Definition and basic properties

The unbounded Toeplitz operator considered by [CMP21] is a natural extension of the bounded Toeplitz operators with L^∞ symbols, that is, a compression of a multiplication operator defined on a “sensible” domain in H^p . More explicitly, for a measurable function g on $\mathbb{T}$, we define the CMP-type unbounded Toeplitz operator, T_g^M with symbol g (here M stands for measurable) to be the operator with domain

$$\mathcal{D}(T_g^M) = \{f \in H^p : gf \in L^p\} \quad (4.1)$$

and

$$T_g^M f = P_+(gf), \quad f \in \mathcal{D}(T_g^M).$$

We attempt to characterize the domain of CMP-type Toeplitz operators in the same manner as was done for Sarason-type Toeplitz operators in Section 4.1. However, we will need to define the Nevanlinna and Smirnov class “Analogues” for CMP-type Toeplitz operators.

Definition 4.2.1. For $1 \leq p \leq \infty$, we define the following set of measurable functions;

$$M_p(\mathbb{T}) := \left\{ \frac{f}{a} : f \in L^p, a \in H^p \setminus \{0\} \right\}. \quad (4.2)$$

Lemma 4.2.2. Let g be a measurable function on $\mathbb{T}$. Then $\mathcal{D}(T_g^M) \neq \{0\}$ if and only if $g \in M_p(\mathbb{T})$.

Proof. Suppose $\mathcal{D}(T_g^M) \neq \{0\}$. Then there exists $f \in H^p \setminus \{0\}$ such that $gf \in L^p$. Then $g = gf/f \in M_p(\mathbb{T})$.

Conversely, suppose that $g \in M_p(\mathbb{T}) \setminus \{0\}$, say $g = f/a$ with $f \in L^p \setminus \{0\}$ and $a \in H^p \setminus \{0\}$. Then $ga = (f/a)a = f \in L^p$ which implies that $0 \neq a \in \mathcal{D}(T_g^M)$.

For the case when $g \equiv 0$, T_g^M is a bounded operator and thus $\mathcal{D}(T_g^M) = H^p$. $\square$

Lemma 4.2.3. Let $g \in M_p(\mathbb{T})$. Then $T_g^M = 0$ if and only if $g = 0$.

Proof. Suppose $T_g^M = 0$. Since $g \in M_p(\mathbb{T})$ by Lemma 4.2.2, there exists $f \in H^p \setminus \{0\}$ such that $gf \in L^p$. Clearly $zf \in \mathcal{D}(T_g^M)$ and so $z^n f \in \mathcal{D}(T_g^M)$ for all $n \in \mathbb{N}$. Since $T_g^M f = 0$, this implies that $gf \in K^p$ and consequently, $z^n gf \in K^p$ for all $n \in \mathbb{N}$. Let $(gf)(z) = \sum_{n=1}^{\infty} \hat{g}f(-n)z^{-n}$ be the Taylor expansion of gf . Then we have that $\hat{g}f(-n) = 0$ for all $n \geq 1$. Thus $gf \equiv 0$. Since $f \neq 0$, by the Riesz brothers theorem, Theorem 2.1.3 we must have that $g \equiv 0$. The converse is trivial. $\square$

For the subsequent text, we shall only consider nonzero symbols, i.e. $g \neq 0$.

Definition 4.2.4. For $1 \leq p \leq \infty$, we define the following set of measurable functions;

$$M_p^+(\mathbb{T}) := \left\{ \frac{f}{a} : f \in L^p, a \in H^p \text{ and outer} \right\}.$$

It is clear that $M_\infty(\mathbb{T})$ contains the Nevanlinna class and $M_\infty^+(\mathbb{T})$ contains the Smirnov class.

Lemma 4.2.5. Let $g \in M_\infty^+(\mathbb{T})$. Then $\mathcal{D}(T_g^M)$ is dense in H^p .

Proof. Suppose $g \in M_\infty^+(\mathbb{T})$, say $g = f/a$ where $f \in L^\infty$ and $a \in H^\infty$ and outer. We will now show that the subspace aH^p is contained in $\mathcal{D}(T_g^M)$. Let $ah \in aH^p$. Then

$$g(ah) = \frac{f}{a}(ah) = fh \in L^p.$$

Thus $ah \in \mathcal{D}(T_g^M)$. Since $ah \in aH^p$ was arbitrary, we have that $aH^p \subset \mathcal{D}(T_g^M)$. Since $a \in H^p$ is outer, by Lemma 3.2.6 $\mathcal{R}(T_a) = aH^p$ is dense in H^p . Therefore, by the inclusion $aH^p \subset \mathcal{D}(T_g^M)$, we have that $\mathcal{D}(T_g^M)$ is dense in H^p . $\square$

The above result should be compared with Theorem 4.1.4. The following lemma gives an extension of Lemma 3.1.10. Moreover, this result shows how some functions in $M_p^+(\mathbb{T})$ naturally arise from non-trivial kernels of CMP-type Toeplitz operators.

Theorem 4.2.6. Let g be a measurable function on $\mathbb{T}$ such that $\ker(T_g^M) \neq \{0\}$. Then g is log-integrable. Moreover, $g \in M_\infty^+(\mathbb{T})$.

Proof. Suppose $\ker T_g^M \neq \{0\}$. Then there exists $f \in H^p \setminus \{0\}$ such that $gf = K^p$. Since K^p functions are log-integrable, we have that $\log |gf| \in L^1$ and similarly since H^p functions are log-integrable by Theorem 2.1.3, we also have that $\log |f| \in L^1$. Hence, $\log |g| = \log |gf/f| = \log |gf| - \log |f|$. Thus

$$\int_{\mathbb{T}} |\log |g(t)|| dt \leq \int_{\mathbb{T}} |\log |g(t)f(t)|| dt + \int_{\mathbb{T}} |\log |f(t)|| dt < \infty.$$

Hence $\log |g| \in L^1$. For the final claim, the condition $\log |g| \in L^1$ allows us to define the outer spectral factor F_g which by Lemma 2.3.6 belongs to the Smirnov class. We can then write $F_g = O_1/O_2$ where $O_1, O_2 \in H^\infty$ and outer. We then have

$$g = \frac{g}{F_g} F_g = \left(\frac{g}{F_g} \right) \left(\frac{O_1}{O_2} \right). \quad (4.3)$$

Since $|F_g(t)| = |g(t)|$ a.e. on $\mathbb{T}$, we have that $|g(t)/F_g(t)| = 1$ a.e. and by Corollary 2.3.7, F_g cannot vanish on a set of positive measure. We then have $Q := g/F_g \in L^\infty$. Hence (4.3) gives

$$g = \left(\frac{g}{F_g} \right) \left(\frac{O_1}{O_2} \right) = \frac{QO_1}{O_2} \in M_\infty^+(\mathbb{T}). \quad \square$$

For convenience, we define Ω to be the following set

$$\Omega := \{g : \mathbb{T} \rightarrow \mathbb{C} : g \text{ is measurable, } \log |g| \in L^1\}.$$

By Theorem 4.2.6 it is clear that $\Omega \subset M_\infty^+(\mathbb{T})$. We also note that every function $g \in \Omega$ has an outer spectral factor F_g which belongs to the Smirnov class by Lemma 2.3.6. Moreover, since $M_\infty^+(\mathbb{T})$ contains the Smirnov class, CMP-type Toeplitz operators with non-trivial kernels can be described in terms of Sarason-type Toeplitz operators.

Theorem 4.2.7. Let $g \in \Omega$ with outer spectral factor F_g . Then

$$\mathcal{D}(T_g^M) = \{f \in H^p : F_g f \in H^p\} = \mathcal{D}(T_{F_g}^A).$$

Moreover, $g = \phi F_g$ with $\phi \in L^\infty$ and

$$T_g^M = T_\phi T_{F_g}^A.$$

Proof. Let $f \in \mathcal{D}(T_{F_g}^A)$ then $F_g f \in H^p \subset L^p$. Since $|F_g(t)| = |g(t)|$ a.e on $\mathbb{T}$, we have

$$\int_{\mathbb{T}} |g(t)f(t)|^p dm(t) = \int_{\mathbb{T}} |F_g(t)f(t)|^p dm(t) < \infty.$$

Hence $gf \in L^p$ so that $f \in \mathcal{D}(T_g^M)$. Thus we have the inclusion $\mathcal{D}(T_{F_g}^A) \subset \mathcal{D}(T_g^M)$.

To show the reverse inclusion, let $f \in \mathcal{D}(T_g^M)$, then $gf \in L^p$. Since $g \in \Omega$, the outer spectral factor F_g exists and belongs to Smirnov class $N^+(\mathbb{D})$ by Lemma 2.3.6. Since $|g| = |F_g|$ a.e on $\mathbb{T}$, we also have that $F_g f \in L^p$. Since $N^+(\mathbb{D})$ is an algebra and $f \in H^p \subset N^+(\mathbb{D})$, we have $F_g f \in N^+(\mathbb{D})$. In total, we have $F_g f \in L^p \cap N^+(\mathbb{D}) = H^p$, by Lemma 2.3.8. Thus $f \in \mathcal{D}(T_{F_g}^A)$.

Since $g \in \Omega$, g is log-integrable. By the proof of Theorem 4.2.6, we can write

$$g = \left(\frac{g}{F_g} \right) F_g = \phi F_g,$$

where $\phi := g/F_g \in L^\infty$.

We finally show that $T_g^M = T_\phi T_{F_g}^A$.

Let $f \in \mathcal{D}(T_g^M) = \mathcal{D}(T_{F_g}^A)$. Then

$$T_g^M f = P_+(gf) = P_+ \left(\frac{g}{F_g} F_g f \right) = P_+(\phi T_{F_g}^A f) = T_\phi T_{F_g}^A f.$$

Hence, $T_g^M = T_\phi T_{F_g}^A$. □

The above lemma shows that certain questions regarding CMP-type Toeplitz operators can be reduced to questions in terms of bounded and Sarason-type Toeplitz operators.

4.2.2 Minimal kernels

We recall from Subsection 3.1.2 that for any $f \in H^p \setminus \{0\}$ for $1 < p < \infty$, there exists a unimodular function $u \in L^\infty$ such that $f \in \ker(T_u)$ and the kernel $\ker(T_u)$ is the smallest bounded Toeplitz kernel containing f . We also observe that for a minimal Toeplitz kernel $K_{\min}(f) = \ker(T_u)$, it is necessary that $u \notin H^\infty$, for otherwise $\ker(T_u) = \{0\}$. Intuitively, to obtain a smaller Toeplitz kernel $\ker(T)$ containing a given function f and $\ker(T) \subset K_{\min}(f)$, we have to consider a larger class of symbols, especially containing L^∞ . The latter is necessary, as we have observed that the Sarason-type Toeplitz operator is always injective.

Let Q be an outer function in $N^+(\mathbb{D})$. Consider the class of symbols;

$$\sigma_Q = \{h \text{ is measurable on } \mathbb{T} : |h(t)| \leq c|Q(t)| \text{ a.e. on } \mathbb{T}, \text{ for some } c > 0\}.$$

For $f \in H^p \setminus \{0\}$, we denote by $K_{\min}^Q(f)$, the minimal Toeplitz kernel containing f with symbol belonging to σ_Q , if it exists. In other words, if there exists a $g \in \sigma_Q$ such that $f \in \ker(T_g^M)$, then

$$K_{\min}^Q(f) \subset \ker(T_g^M).$$

Theorem 4.2.8. Let Q be an outer function in $N^+(\mathbb{D})$. If $f \in \mathcal{D}(T_Q^A) \setminus \{0\}$, then $K_{\min}^Q(f)$ exists. In particular, if $f = f_i f_o$ is the inner-outer factorization of f with f_i inner and f_o outer, then

$$K_{\min}^Q(f) = \ker(T_{\frac{f_i f_o}{zQ}}^M).$$

Proof. Let $f \in \mathcal{D}(T_Q^A) \setminus \{0\} \subset H^p$ with inner-outer factorization $f = f_i f_o$, where f_i is an inner function and f_o is outer. Then $Qf \in H^p \setminus \{0\}$ and likewise let $Qf = f_i(Qf_o)$ be the inner-outer factorization of Qf . Then

$$\overline{zQ} \frac{\overline{f_i f_o}}{f_o} f = \overline{zQ} \frac{\overline{f_i f_o}}{f_o} f_i f_o = \overline{zQ f_o}.$$

Since $|\overline{zQ} \frac{\overline{f_i f_o}}{f_o} f| = |Qf|$ a.e. on $\mathbb{T}$, we have that $\overline{zQ f_o} \in L^p$. On the other hand note that $Qf_o \in N^+(\mathbb{D})$ and thus $\overline{zQ f_o} \in \overline{N_0^+}$. Summing up, we get, $\overline{zQ f_o} \in \overline{N_0^+} \cap L^p = \overline{H_0^p} = K^p$. Hence

$$T_{\frac{f_i f_o}{zQ}}^M f = P_+ \left(\overline{zQ} \frac{\overline{f_i f_o}}{f_o} f \right) = P_+(\overline{zQ f_o}) = 0.$$

Set $h := \overline{zQ} \frac{\overline{f_i f_o}}{f_o}$. Then $f \in \ker(T_h^M)$. We are left to show that $\ker(T_h^M)$ is the minimal Toeplitz kernel with symbol belonging to σ_Q . We first note that

$$|h| = \left| \frac{\overline{zQ f_i f_o}}{f_o} \right| = |Q| \text{ a.e. on } \mathbb{T},$$

hence $h \in \sigma_Q$.

Let $g \in \sigma_Q$ such that $f \in \ker(T_g^M)$. Then by definition $|g(t)| \leq c|Q(t)|$ a.e. on $\mathbb{T}$. Equivalently,

$$|Q^{-1}(t)g(t)| \leq c \text{ a.e. on } \mathbb{T}.$$

Hence $Q^{-1}g \in L^\infty$. We also have that $Qf \in \ker(T_{Q^{-1}g})$. By Theorem 3.1.12,

$$\ker(T_{Q^{-1}g}) \supset K_{\min}(Qf) = \ker(T_{\frac{zQf_i f_o}{Qf_o}}) = \ker(T_{Q^{-1}h}). \quad (4.4)$$

We also note that

$$y \in \ker(T_h^M) \iff y \in H^p \text{ and } hy = v \in K^p \quad (4.5)$$

$$\iff y \in H^p \text{ and } hQ^{-1}(Qy) = v \in K^p. \quad (4.6)$$

If $y \in \ker(T_h^M) \subset H^p$, we have on the one hand

$$Qy \in N^+(\mathbb{D}),$$

since $Q \in N^+(\mathbb{D})$ by Lemma 2.3.6 and $N^+(\mathbb{D})$ is an algebra, while on the other hand, $Qy = Qv/h = v/Q^{-1}h \in L^p$, since $|Q^{-1}h| = 1$ a.e. on $\mathbb{T}$ and $v \in H^p$. Hence, $Qy \in L^p \cap N^+(\mathbb{D}) = H^p$. We then have that $Qy \in \mathcal{D}(T_{Q^{-1}h})$. By (4.4) and (4.6), we get

$$Qy \in \ker(T_{Q^{-1}h}^M) \subset \ker(T_{Q^{-1}g}^M).$$

In total, we have

$$y \in \ker(T_h^M) \iff Qy \in \ker(T_{Q^{-1}h}) \implies Qy \in \ker(T_{Q^{-1}g}).$$

This implies that

$$yg = Q^{-1}gQy \in K^p \implies y \in \ker(T_g^M).$$

Since $y \in \ker(T_h^M)$ was arbitrary, we have $\ker(T_h^M) \subset \ker(T_g^M)$. Since $g \in \sigma_Q$ was arbitrary, we conclude that

$$K_{\min}^Q(f) = \ker(T_h^M). \quad \square$$

If Q is an outer function in $N^+(\mathbb{D})$ which is bounded below, then σ_Q contains L^∞ . Indeed, suppose $b \in L^\infty$. If Q is bounded below, then there exists $M > 0$ such that $|Q(t)| \geq M$ a.e. on $\mathbb{T}$. We then have

$$|Q^{-1}(t)b(t)| \leq M\|b\|_\infty := C \text{ a.e. on } \mathbb{T}.$$

Hence, $|b(t)| \leq C|Q(t)|$ a.e. on $\mathbb{T}$ and thus $b \in \sigma_Q$. Moreover, the assumption that Q is bounded below allows us to discuss the relationship between the minimal kernels $K_{\min}(f)$ and $K_{\min}^Q(f)$ for a given function f .

Lemma 4.2.9. Let Q be an outer function in $N^+(\mathbb{D})$ which is bounded below. If $f \in \mathcal{D}(T_Q^A) \setminus \{0\}$, then

$$K_{\min}^Q(f) = K_{\min}(f) \cap \mathcal{D}(T_Q^A).$$

Proof. Since Q is bounded below, by the above discussion we have that $L^\infty \subset \sigma_Q$. Let $f \in \mathcal{D}(T_Q^A)$, by Theorem 4.2.8 we have

$$K_{\min}^Q(f) \subset K_{\min}(f).$$

In particular, $K_{\min}^Q(f) = \ker(T_{\frac{zQf_i f_o}{f_o}})$. Note that

$$\ker(T_{\frac{zQf_i f_o}{f_o}}^M) \subset \mathcal{D}(T_{\frac{zQf_i f_o}{f_o}}^M) \subset \mathcal{D}(T_Q^M) = \mathcal{D}(T_Q^A),$$

where the last equality follows from Theorem 4.2.7. Hence we get

$$K_{\min}^Q(f) \subset K_{\min}(f) \cap \mathcal{D}(T_Q^A)$$

We are left to show the reverse inclusion. Suppose $v \in K_{\min}(f) \cap \mathcal{D}(T_Q^A)$. Let $f = f_i f_o$ be the RS-factorization of f , then since $v \in K_{\min}(f) = \ker(T_{\frac{zQf_i f_o}{f_o}})$, there exists a $k \in K^p$ such that

$$\frac{z f_i f_o}{f_o} v = k.$$

Therefore

$$\frac{Q z f_i f_o}{f_o} v = \bar{Q} k = \frac{k}{Q^{-1}}.$$

Since $v \in \mathcal{D}(T_Q^A)$ and $|(\overline{zQf_i f_o}/f_o)v| = |Qv|$ a.e. on $\mathbb{T}$ shows that $(\overline{zQf_i f_o}/f_o)v \in L^p$. On the other hand, $k/\bar{Q}^{-1} \in N_0^+(\mathbb{D})$, since $k \in K^p$ and Q^{-1} is an outer function in H^∞ . In total, we have

$$\frac{zQf_i f_o}{f_o} v \in L^p \cap \overline{N_0^+(\mathbb{D})} = K^p.$$

Hence $v \in \ker(T_{\frac{zQf_i f_o}{f_o}}) = K_{\min}^Q(f)$ and we get $K_{\min}(f) \cap \mathcal{D}(T_Q^A) \subset K_{\min}^Q(f)$. This concludes the proof. $\square$

4.2.3 Notes

The primary source used in this section is [CMP21], where the CMP-type Toeplitz operator acts on the Hardy space of the upper half plane $H^p(\mathbb{C}_+)$. In this section, we have translated the results in [CMP21] from $H^p(\mathbb{C}_+)$ to $H^p(\mathbb{D})$. Definition 4.2.1 and Definition 4.2.4 are natural extensions of the Nevanlinna and the Smirnov class respectively and does not explicitly appear in [CMP21]. Lemma 4.2.2 and Lemma 4.2.5 are our own results. In Theorem 4.2.7, only the equivalence of the domain is given in [CMP21]. In particular, the connection between CMP-type Toeplitz operators and Sarason-type Toeplitz operators, given in this result, is our own. Theorem 4.2.8 comes from [CMP21], where it was assumed that the given outer function Q is not necessarily bounded below. We adjusted this theorem to the case when the outer function Q is not bounded below. Moreover, Theorem 4.2.8 and Lemma 4.2.9 provides the existence of minimal Toeplitz kernels for functions in the domain of Sarason-type Toeplitz operators. This has not been explicitly mentioned in [CMP21].

4.3 Unbounded Toeplitz operators with rational symbols

In this section, we will discuss unbounded Toeplitz operators whose symbols are rational functions that have poles on the unit circle, which were studied in [Gro+18; Gro+19a; Gro+19b]. We will call these the GHJR-type Toeplitz operators. In particular, these series of papers provided a full treatment to the spectral and Fredholm theory, and the adjoint of GHJR-type Toeplitz operators with rational symbols. Our aim in this section is to consider symbols which are rational functions with poles only on the unit circle. We note that rational functions with poles only on the unit circle belong to the Smirnov class $N^+(\mathbb{D})$. The key distinction of GHJR-type Toeplitz operators is the domain of definition and its action in comparison to CMP-type and Sarason-type Toeplitz operators. Due to this, GHJR-type Toeplitz operators do not immediately display any correlation to CMP-type or Sarason-type Toeplitz operators. We do, however, have the result from [Gro+19b] that Sarason-type Toeplitz operators with proper rational symbols that have poles only on $\mathbb{T}$ are the adjoints of GHJR-type Toeplitz operators which we present at the end of this section.

Before we begin, we introduce some notation which will be used throughout this section. We denote by Rat , the vector space of all rational functions and Rat_0 the subspace of strictly proper rational functions. Moreover, $\text{Rat}(\mathbb{T})$ will denote the space of all rational functions with poles only on $\mathbb{T}$ and $\text{Rat}_0(\mathbb{T})$ the subspace of all strictly proper rational functions with poles only on $\mathbb{T}$. Throughout this section, we only consider H^p for $1 < p < \infty$.

4.3.1 Definition and basic properties

For $w \in \text{Rat}$ with poles on $\mathbb{T}$, we define the unbounded Toeplitz operator, denoted by T_w^R , on the domain

$$\mathcal{D}(T_w^R) = \{g \in H^p : wg = f + \rho, \text{ where } f \in L^p, \rho \in \text{Rat}_0(\mathbb{T})\} \text{ by } T_w^R g = P_+ f.$$

The superscript R refers to 'rational' and is added to distinguish the operator from the unbounded Toeplitz operators which were considered in the preceding sections. It is clear that $\mathcal{D}(T_w^R)$ is a linear subspace of H^p and that T_w^R is a linear map. It is not immediate that T_w^R is a well-defined map on $\mathcal{D}(T_w^R)$. To show this, we will require the following lemma.

Lemma 4.3.1. If $\phi \in \text{Rat}$ has a pole $\alpha \in \mathbb{T}$, then $\phi \notin L^p$. In particular, $\text{Rat}_0(\mathbb{T}) \cap L^p = \{0\}$.

Proof. Suppose $\alpha \in \mathbb{T}$ is a pole of ϕ with order n . Then notice that $|\phi(z)| \sim |z - \alpha|^{-n}$ as

$z \longrightarrow \alpha$. Consequently, the integral

$$\int_{\mathbb{T}} |\phi(z)|^p dz$$

diverges. $\square$

Lemma 4.3.2. For $w \in \text{Rat}(\mathbb{T})$, T_w^R is a well-defined map on $\mathcal{D}(T_w^R)$.

Proof. We have to prove that every function in $\mathcal{D}(T_w^R)$ admits a unique decomposition. Suppose $g \in \mathcal{D}(T_w^R)$ with $wg = f_1 + \rho_1$ and $wg = f_2 + \rho_2$ with $f_1, f_2 \in L^p$ and $\rho_1, \rho_2 \in \text{Rat}_0(\mathbb{T})$. Then $f_1 + \rho_1 = wg = f_2 + \rho_2$ which implies that

$$f_1 - f_2 = \rho_2 - \rho_1 \in L^p \cap \text{Rat}_0(\mathbb{T}) = \{0\}, \quad (4.7)$$

by Lemma 4.3.1. Hence $f_1 = f_2$, $\rho_1 = \rho_2$ and thus $g \in \mathcal{D}(T_w^R)$ is uniquely defined. $\square$

In the subsequent subsections, we will only focus on symbols belonging to $\text{Rat}(\mathbb{T})$. In this regard, it is convenient to have an alternative formula for the domain $\mathcal{D}(T_w^R)$ which is given in the following Lemma.

Lemma 4.3.3. Let $w \in \text{Rat}(\mathbb{T})$ with $w = s/q$ where $s, q \in \mathcal{P}$ co-prime. Let $\deg(s) = n$ and $\deg(q) = m$. Then

$$\mathcal{D}(T_w^R) = \{g \in H^p : wg = h + r/q, h \in H^p, r \in \mathcal{P}_{m-1}\} =: D$$

and

$$T_w^R g = h.$$

Moreover, $\mathcal{D}(T_w^R)$ is invariant under the forward shift T_z and

$$T_{z^{-1}} T_w^R T_z f = T_w^R f \quad \text{for } f \in \mathcal{D}(T_w^R).$$

Proof. Suppose $g \in D$. Then $g \in H^p$ and $wg = h + r/q$ with $h \in H^p$, $r \in \mathcal{P}_{m-1}$. Since $H^p \subset L^p$, we have $g \in L^p$ and $r/q \in \text{Rat}_0(\mathbb{T})$, where the latter follows from the fact that $w \in \text{Rat}(\mathbb{T})$. Thus $g \in \mathcal{D}(T_w^R)$. Conversely, suppose that $g \in \mathcal{D}(T_w^R)$. By definition this means that $wg = f + \rho$ where $f \in L^p$ and $\rho \in \text{Rat}_0(\mathbb{T})$. Assume that $\rho = u/v$, where $u, v \in \mathcal{P}$ with $\deg(u) < \deg(v)$. By using the Euclidean division algorithm we have

$$qu = vr_0 + \rho_0, \quad \deg(\rho_0) < \deg(v).$$

Consequently,

$$q\rho = \frac{qu}{v} = \frac{vr_0 + \rho_0}{v} = r_0 + \frac{\rho_0}{v}.$$

Note that $\deg(qu) = \deg(vr_0)$. Since $\deg(u) < \deg(v)$, we have

$$\deg(v) + \deg(r_0) = \deg(q) + \deg(u) \quad \text{and thus} \quad \deg(r_0) = \deg(q) + (\deg(u) - \deg(v)) < \deg(q) = m.$$

We then have

$$sg = qwg = q(f + \rho) = qf + q\rho = qf + r_0 + \frac{\rho_0}{v}.$$

Equivalently,

$$\frac{\rho_0}{v} = sg - qf - r_0 \in L^p \cap \text{Rat}_0(\mathbb{T}) = \{0\},$$

by Lemma 4.3.1. Hence $\rho_0/v \equiv 0$. Thus $sg = qf + r_0$. Since $f \in L^p = H^p + K^p$, we may write $f = h + k$, with $h \in H^p$ and $k \in K^p$. The latter may be written in its Fourier expansion, $k(t) = \sum_{n=-1}^{-\infty} \hat{k}_n t^n$. Multiplying by the polynomial $q(t) = q_0 + q_1 t + \dots + q_m t^m$, gives

$$\begin{aligned} q(t)k(t) &= (q_0 + q_1 t + \dots + q_m t^m) \left(\frac{\hat{k}_{-1}}{t} + \frac{\hat{k}_{-2}}{t^2} + \dots \right) \\ &= c_0 + c_1 t + \dots + c_{m-1} t^{m-1} + \left(\frac{c_m}{t} + \frac{c_{m+1}}{t^2} + \dots \right) \end{aligned}$$

for some sequence $(c_k)_{k \in \mathbb{N}} \subset \mathbb{C}$. So we may write $qk = r_1 + k_1$, where $r_1 \in \mathcal{P}_{m-1}$ and $k_1 \in K^p$. Thus all together,

$$sg = qf + r_0 = qh + qk + r_0 = qh + k_1 + r_1 + r_0.$$

Equivalently,

$$k_1 = sg - qh - r_1 - r_0 \in H^p \cap K^p = \{0\}.$$

Hence $k_1 \equiv 0$. In total, we see that $sg = qh + r$, with $r := r_1 + r_0 \in \mathcal{P}_{m-1}$. Dividing by q gives us

$$wg = h + \frac{r}{q}, \text{ with } h \in H^p, r \in \mathcal{P}_{m-1}.$$

So $g \in \mathcal{D}$ and we have $\mathcal{D}(T_w^R) = D$. We are left to show that $\mathcal{D}(T_w^R)$ is invariant under T_z and the identity $T_{z^{-1}} T_w^R T_z = T_w^R$. Let $f \in \mathcal{D}(T_w^R)$. By definition, $wf = h + r/q$ with $h \in H^p$ and $r \in \mathcal{P}_{m-1}$ or equivalently, $sf = qh + r$. Multiplying by z gives $zsf = zqh + zr$. By Euclidean division we can write $zr = cq + q_0$, where $c \in \mathbb{C}$, $q_0 \in \mathcal{P}_{m-1}$. Consequently,

$$zsf = qzh + qc + q_0 = q(zh + c) + q_0 \in qH^p + \mathcal{P}_{m-1}.$$

Equivalently, dividing by q gives $wzf = (zh + c) + q_0/q$ and thus $zf \in \mathcal{D}(T_w^R)$ and $T_w^R T_z f = T_w^R z f = zh + c$. We also have $T_{z^{-1}} T_w^R T_z f = T_{z^{-1}}(zh + c) = P_+(h + cz^{-1}) = h = T_w^R f$. This proves the final claim. $\square$

Lemma 4.3.4. Let $w \in \text{Rat}$. Then $w = w_0 + w_1$ with $w_0 \in \text{Rat}_0(\mathbb{T})$ and $w_1 \in \text{Rat}$ with no poles on $\mathbb{T}$. Moreover, the pair w_0, w_1 is uniquely determined by w and the poles of w_0 and w_1 correspond to the poles of w on and off $\mathbb{T}$ respectively. Furthermore, $T_w^R = T_{w_0}^R + T_{w_1}$.

Proof. Suppose $w = s/q$ where $s, q \in \mathcal{P}$ and co-prime. We can then write

$$q(z) = (z - \alpha_1)^{n_1} (z - \alpha_2)^{n_2} \dots (z - \alpha_j)^{n_j} (z - \beta_1)^{m_1} (z - \beta_2)^{m_2} \dots (z - \beta_k)^{m_k},$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ denotes the roots of q on and off $\mathbb{T}$ respectively. By partial fraction decomposition we have

$$\frac{s(z)}{q(z)} = \rho(z) + \sum_{i=1}^k \frac{r_i(z)}{(z - \beta_i)^{m_i}} + \sum_{i=1}^j \frac{s_i(z)}{(z - \alpha_i)^{n_i}}, \quad r_i, s_i \in \mathcal{P} \quad \text{and} \quad \deg(s_i) < n_i, \deg(r_i) < m_i.$$

By letting $w_0(z) = \sum_{i=1}^j \frac{s_i(z)}{(z - \alpha_i)^{n_i}}$ and $w_1(z) = \rho(z) + \sum_{i=1}^k \frac{r_i(z)}{(z - \beta_i)^{m_i}}$ we obtain the desired decomposition of w . We are left to show that such a decomposition is unique. Firstly, by the Euclidean division algorithm we can write $w_1 = v_1 + \rho_1$ where $\rho_1 \in \mathcal{P}$ and $v_1 \in \text{Rat}_0$. We then have $w = w_0 + v_1 + \rho_1$. Suppose that $w = w'_0 + v'_1 + \rho'_1$ where $w'_0 \in \text{Rat}_0(\mathbb{T})$, $v'_1 \in \text{Rat}_0 \setminus \text{Rat}_0(\mathbb{T})$ and $\rho'_1 \in \mathcal{P}$. We then have

$$(w_0 - w'_0) + (v_1 - v'_1) = \rho'_1 - \rho_1 \in \text{Rat}_0 \cap \mathcal{P} = \{0\}.$$

Hence $\rho_1 = \rho'_1$. We then get $w_0 - w'_0 = v'_1 - v_1 \in \text{Rat}_0$. Moreover $v'_1 - v_1$ has no poles on $\mathbb{T}$, while $w_0 - w'_0$ has all its poles on $\mathbb{T}$. Hence we must have that $w_0 - w'_0 = v'_1 - v_1 \equiv 0$, or equivalently, $w_0 = w'_0$ and $v'_1 = v_1$. All together, we see that w_0 and w_1 are uniquely determined. We now prove the final claim.

We first show that $\mathcal{D}(T_w^R) = \mathcal{D}(T_{w_0}^R + T_{w_1})$. Since $w_1 \in L^\infty$, $\mathcal{D}(T_{w_1}) = H^p$ and thus it suffices to show that $\mathcal{D}(T_w^R) = \mathcal{D}(T_{w_0}^R)$. Let $g \in \mathcal{D}(T_w^R)$. By definition, $wg = f + \rho$ where $f \in L^p$ and $\rho \in \text{Rat}_0(\mathbb{T})$. Then $w = w_0 + w_1$ gives $w_0g + w_1g = wg = f + \rho$ or equivalently, $w_0g = (f - w_1g) + \rho$ and $f - w_1g \in L^p$. Thus $g \in \mathcal{D}(T_{w_0}^R)$.

Conversely, let $g \in \mathcal{D}(T_{w_0}^R)$, then by definition $w_0g = f + \rho$ with $f \in L^p$ and $\rho \in \text{Rat}_0(\mathbb{T})$. We then have $wg = w_0g + w_1g = (f + w_1g) + \rho$ with $f + w_1g \in L^p$ and thus $g \in \mathcal{D}(T_w^R)$. We then have $T_w^R g = P_+(f + w_1g) = P_+f + P_+(w_1g) = T_{w_0}^R g + T_{w_1}g$. $\square$

Theorem 4.3.5. For $w \in \text{Rat}$, T_w^R is a densely-defined closed operator on $\mathcal{D}(T_w^R)$.

Proof. Let $w \in \text{Rat}$. By Lemma 4.3.4 we can write $w = w_0 + w_1$ with $w_0 \in \text{Rat}_0(\mathbb{T})$ and $w_1 \in \text{Rat}$ with no poles on $\mathbb{T}$. Note that $w_1 \in L^\infty$. We now show that T_w^R is a closed operator. From Lemma 4.3.4, $T_w^R = T_{w_0}^R + T_{w_1}$ with $w_0 \in \text{Rat}_0(\mathbb{T})$ and $w_1 \in L^\infty$. In particular, T_{w_1} is bounded. We then have T_w^R is closed if and only if $T_{w_0}^R$ is closed. So we may assume without the loss of generality that $w \in \text{Rat}_0(\mathbb{T})$. Suppose that $w = s/q$ with $s, q \in \mathcal{P}$ with s, q co-prime, q has root only on $\mathbb{T}$ and $\deg(s) < \deg(q)$. Let $(g_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{D}(T_w^R)$ such that

$$g_j \longrightarrow g \in H^p \quad \text{and} \quad T_w^R g_j \longrightarrow h \in H^p, \quad (4.8)$$

with convergence in the H^p norm. We need to show that $g \in \mathcal{D}(T_w^R)$ and $T_w^R g = h$. Since $g_j \in \mathcal{D}(T_w^R)$ for each j , we have by Lemma 4.3.3, $wg_j = h_j + r_j/q$ with $h_j \in H^p$ and $r_j \in \mathcal{P}_{\deg(q)-1}$ for each j . Furthermore, $h_j = T_w^R g_j \longrightarrow h$. On the other hand (4.8) implies

$$r_j = qwg_j - qh_j = sg_j - qh_j \longrightarrow sg - qh,$$

since $s, q \in H^\infty$. Set $r := sg - qh$. Since $r_j \rightarrow r$ and $r_j \in \mathcal{P}_{\deg(q)-1}$, which is a closed subspace of H^p , we have $r \in \mathcal{P}_{\deg(q)-1}$. Thus $r/q \in \text{Rat}_0(\mathbb{T})$. We also get $sg = qh + r$ or equivalently, $wg = h + r/q$. Therefore $g \in \mathcal{D}(T_w^R)$ and $T_w^R g = h$. Hence T_w^R is closed.

Lastly, we show that T_w^R is densely defined for $w \in \text{Rat}$. By Lemma 4.3.4, we may write $w = w_0 + w_1$ with $w_0 \in \text{Rat}_0(\mathbb{T})$ and $w_1 \in \text{Rat}$ with no poles on $\mathbb{T}$. We will show that $\mathcal{P} \subset \mathcal{D}(T_w^R)$. Let $r \in \mathcal{P}$, then $wr = w_0 r + w_1 r$. Note that $w_1 r \in L^p$, since $r \in L^\infty$. Suppose $w_0 = u/v$ with $\deg(u) < \deg(v)$. By using the Euclidean division algorithm

$$ur = vp + l, \quad \text{for some } \rho \in \mathcal{P}, \deg(l) < \deg(v).$$

Consequently,

$$w_0 r = \frac{ur}{v} = \frac{v\rho + l}{v} = \rho + \frac{l}{v}.$$

Thus $r \in \mathcal{D}(T_{w_0}^R) = \mathcal{D}(T_w^R)$. Since r was arbitrary, $\mathcal{P} \subset \mathcal{D}(T_w^R)$. Since $\mathcal{P}$ is dense in H^p , it follows that $\mathcal{D}(T_w^R)$ is dense in H^p . $\square$

4.3.2 Concrete description for T_w^R for w in $\text{Rat}(\mathbb{T})$

In this subsection, we will provide a concrete description for the domain and range of T_w^R when $w \in \text{Rat}(\mathbb{T})$. Furthermore, we will provide a complement of the range and an explicit formula for the kernel of T_w^R . This enables us to describe the Fredholm properties of T_w^R which we present in the subsequent section. Before we prove this result, we first introduce some further notation.

Throughout this section, $w = s/q \in \text{Rat}(\mathbb{T})$ and we let $\deg(s) = n$ and $\deg(q) = m$. We define by $\tilde{\mathcal{P}}$ the subspace of $\mathcal{P}$ given by

$$\tilde{\mathcal{P}} = \{r \in \mathcal{P} : rq = r_1 s + r_2, r_1, r_2 \in \mathcal{P}_{m-1}\} \subset \mathcal{P}_{n-1}.$$

For $w = s/q \in \text{Rat}(\mathbb{T})$. We denote by $\tilde{\mathcal{Q}}_s$, the subspace of $\mathcal{P}$ defined by

$$\tilde{\mathcal{Q}}_s = \begin{cases} \mathcal{P}_{n-m-1} & \text{if } n > m \\ \{0\} & \text{if } n \leq m \end{cases}.$$

We begin with providing a concrete description for the domain and range of T_w^R .

Theorem 4.3.6. Let $w \in \text{Rat}(\mathbb{T})$ with $w = s/q$, $s, q \in \mathcal{P}$ co-prime. Then

$$\mathcal{D}(T_w^R) = qH^p + \mathcal{P}_{m-1} \quad \text{and} \quad \mathcal{R}(T_w^R) = sH^p + \tilde{\mathcal{P}}. \quad (4.9)$$

Proof. The proof of this result is divided into two parts.

Part 1

In the first part, we show the following inclusions;

$$qH^p + \mathcal{P}_{m-1} \subset \mathcal{D}(T_w^R) \quad (4.10)$$

$$T_w^R(qH^p + \mathcal{P}_{m-1}) = sH^p + \tilde{\mathcal{P}} \subset \mathcal{R}(T_w^R), \quad (4.11)$$

We begin with the inclusion (4.10). Suppose $g \in qH^p + \mathcal{P}_{m-1}$. Then $g = qh + r_1$ where $h \in H^p$ and $\deg(r_1) < m$. By the Euclidean division algorithm we may write $sr_1 = rq + r_2$ with $\deg(r_2) < \deg(q) = m$ and $r \in \mathcal{P}$. Consequently,

$$\begin{aligned} \deg(r) + \deg(q) &= \deg(rq) \\ &= \deg(rq + r_2) \\ &= \deg(sr_1) \\ &= \deg(s) + \deg(r_1) \\ &< \deg(s) + \deg(q). \end{aligned}$$

Thus $\deg(r) < \deg(s)$ and so $r \in \mathcal{P}_{n-1}$. We then have

$$\begin{aligned} wg &= \frac{s(qh + r_1)}{q} = sh + \frac{sr_1}{q} = sh + \frac{rq + r_2}{q} \\ &= (sh + r) + \frac{r_2}{q} \in H^p + \text{Rat}_0(\mathbb{T}). \end{aligned}$$

By Lemma 4.3.3, $g \in \mathcal{D}(T_w^R)$ and we have the desired inclusion.

We also get $T_w^R g = sh + r \in sH^p + \mathcal{P}_{n-1}$. Furthermore, notice that $sr_1 = rq + r_2$ implies $rq = sr_1 - r_2$ and $\deg(r_1) < \deg(q) = m$ and $\deg(r_2) < \deg(q) = m$. Hence $r \in \tilde{\mathcal{P}}$. We then have $T_w^R g = sh + r \in sH^p + \tilde{\mathcal{P}}$ and we also have the inclusion $T_w^R(qH^p + \mathcal{P}_{m-1}) \subset sH^p + \tilde{\mathcal{P}}$. We are left to show the reverse inclusion.

Suppose $f \in sH^p + \tilde{\mathcal{P}}$. Then $f = sh + r$ with $h \in H^p$ and $r \in \tilde{\mathcal{P}}$. The latter implies that $qr = r_1s + r_2$ for some $r_1, r_2 \in \mathcal{P}_{m-1}$. Consider the function $g := qh + r_1 \in qH^p + \mathcal{P}_{m-1}$ and the polynomial $\tilde{r} := -r_2$. We then have

$$wg = \frac{s}{q}(qh + r_1) + sh + \frac{sr_1}{q} = (sh + r) + \frac{\tilde{r}}{q}.$$

By the first part of the proof $g \in \mathcal{D}(T_w^R)$ and by the above formula we have $T_w^R g = sh + r = f$. This gives the inclusion $sH^p + \tilde{\mathcal{P}} \subset T_w^R(qH^p + \mathcal{P}_{m-1})$. So in total, $sH^p + \tilde{\mathcal{P}} = T_w^R(qH^p + \mathcal{P}_{m-1})$.

Part 2

In the second part, we achieve the identities given by (4.9). Note that it suffices to show that $\mathcal{D}(T_w^R) \subset qH^p + \mathcal{P}_{m-1}$, since by Part 1 we would have $\mathcal{R}(T_w^R) = T_w^R(qH^p + \mathcal{P}_{m-1}) = sH^p + \tilde{\mathcal{P}}$.

Suppose $g \in \mathcal{D}(T_w^R)$. By definition there exists $h \in H^p$ and $r \in \mathcal{P}_{m-1}$ such that $wg = h + r/q$ or equivalently, $sh = qh + r$. Since $s, q \in \mathcal{P}$ are co-prime, by Bezout's identity there exists $a, b \in \mathcal{P}$ such that $sa + qb \equiv 1$. By the Euclidean division algorithm we can

write $ar = qr_1 + r_2$ where $r_2 \in \mathcal{P}_{m-1}$. We then have

$$\begin{aligned} sg &= qh + r = qh + (sa + qb)r \\ &= qh + qbr + sar \\ &= qh + qbr + s(qr_1 + r_2) \\ &= q(h + br + sr_1) + sr_2. \end{aligned}$$

Dividing throughout by s gives

$$g = q \frac{h + br + sr_1}{s} + r_2.$$

The aim is to show that $h' := (h + br + sr_1)/s$ belongs to H^p . We first address the simpler case which is when s has no roots on $\mathbb{T}$. Factor $s = s_- s_+$, where $s_-, s_+ \in \mathcal{P}$ with roots only on $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then $s_+^{-1} \in H^\infty$ and we have

$$g = q \frac{h + br + sr_1}{s_- s_+} = q \frac{s_+^{-1}(h + br + sr_1)}{s_-}.$$

Let $s_+^{-1}(h + br + sr_1) = BSF$ be the RS-factorization with B the Blaschke product, S the singular inner function and F the outer function. Since $g \in H^p \subset \text{hol}(\mathbb{D})$, s_- must be a factor of the Blaschke product B , say $B = B_1 s_-$, with $B_1 \in H^\infty$. Consequently,

$$h' = (s_+^{-1}(h + br + sr_1))/s_- = BSF/s_- = B_1 s_- SF/s_- = B_1 SF \in H^p.$$

We will now address the general case. If $g \in \mathcal{D}(T_w^R)$, then $g = qh' + r_2$ with h' defined as before. Then

$$h' = \frac{g - r_2}{q} \in N^+(\mathbb{D}),$$

since q is a polynomial with roots only on $\mathbb{T}$ and thus an outer function by Lemma 2.2.17. For convenience let $\hat{h} := h + br + sr_1$ and so $h' = \hat{h}/s$. We now show that h' belongs to L^p . Suppose to the contrary that $\int_{\mathbb{T}} |h'(t)|^p dm(t) = \infty$. Since $s, q \in \mathcal{P}$ are co-prime and have finitely many roots, we can decompose $\mathbb{T}$ as $\mathbb{T} = \mathbb{T}_s \cup \mathbb{T}_q$ with $\mathbb{T}_q \cap \mathbb{T}_s = \emptyset$, where the set $\mathbb{T}_q$ is a union of closed arcs on $\mathbb{T}$ containing the roots of q as interior points and similarly, $\mathbb{T}_s$ is the union of closed arc on $\mathbb{T}$ containing the roots of s as interior points. Then there exists constants N_1, N_2 such that $|q(t)| > N_1$ for all $t \in \mathbb{T}_s$ and $|s(t)| > N_2$ for all $t \in \mathbb{T}_q$. We then have

$$\int_{\mathbb{T}_q} |h'(t)|^p dm(t) = \int_{\mathbb{T}_q} \frac{|\hat{h}(t)|^p}{|s(t)|^p} dm(t) \leq N_2^{-p} \int_{\mathbb{T}_q} |\hat{h}(t)|^p dm(t) = N_2^{-p} \|\hat{h}\|_p^p < \infty.$$

Since we assumed that $\int_{\mathbb{T}} |h'(t)|^p dm(t) = \infty$ and $\int_{\mathbb{T}_q} |h'(t)|^p dm(t) < \infty$ shown above, we must have that $\int_{\mathbb{T}_s} |h'(t)|^p dm(t) = \infty$. However, on the other hand $|q(t)| > N_1$ implies

that

$$\begin{aligned}
\|g - r_2\|_p^p &= \int_{\mathbb{T}} |g(t) - r_2(t)|^p dm(t) \\
&= \int_{\mathbb{T}} |q(t)h'(t)|^p dm(t) \\
&\geq \int_{\mathbb{T}_s} |q(t)h'(t)|^p dm(t) \\
&\geq \int_{\mathbb{T}_s} N_1^{-p} |h'(t)|^p dm(t) = \infty.
\end{aligned}$$

This is contradiction since $g - r_2 \in H^p \subset L^p$. Hence, we conclude that $h' \in L^p$ and $h' \in N^+(\mathbb{D})$ from the preceding result. Summing up, we get $h' \in L^p \cap N^+(\mathbb{D}) = H^p$. Since $g = qh' + r_2 \in \mathcal{D}(T_w^R)$ was arbitrary, we have that $\mathcal{D}(T_w^R) \subset qH^p + \mathcal{P}_{m-1}$. $\square$

The following lemma will be used to provide a complement to the range $\mathcal{R}(T_w^R)$, in which case we can describe the Fredholm and spectral properties of T_w^R .

Lemma 4.3.7. Let $w \in \text{Rat}(\mathbb{T})$ with $w = s/q$, $s, q \in \mathcal{P}$ co-prime. Let $\deg(s) = n$ and $\deg(q) = m$. Then $\mathcal{P}_{n-1} = \tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s$ is a direct sum decomposition of $\mathcal{P}_{n-1}$. In particular, $\mathcal{P}_{n-1} = \tilde{\mathcal{P}}$ if and only if $n \leq m$.

Proof. We first consider the case when $n \leq m$. Then by definition $\tilde{\mathcal{Q}}_s = \{0\}$, from which $\tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s$ would clearly be a direct sum. We are left to show that $\mathcal{P}_{n-1} = \tilde{\mathcal{P}}$. Suppose $r \in \mathcal{P}_{n-1}$. By the Euclidean division algorithm we have

$$qr = sv + u \text{ with } \deg(u) < \deg(s) = n.$$

we also have $\deg(v) < \deg(q) = m$. Consequently, $qr \in s\mathcal{P}_{m-1} + \mathcal{P}_{n-1} \subset s\mathcal{P}_{m-1} + \mathcal{P}_{m-1}$. This shows that $r \in \tilde{\mathcal{P}}$ and we have the inclusion $\mathcal{P}_{n-1} \subset \tilde{\mathcal{P}}$. In fact, we have $\mathcal{P}_{n-1} = \tilde{\mathcal{P}}$ as the reverse inclusion holds from the definition of $\tilde{\mathcal{P}}$. In total, we get that $\mathcal{P}_{n-1} = \tilde{\mathcal{P}} + \{0\}$ is a direct sum decomposition.

Finally, we address the case when $n > m$. We first show that $\tilde{\mathcal{P}} \cap \tilde{\mathcal{Q}}_s = \{0\}$. Suppose that $r \in \tilde{\mathcal{Q}}_s$. By definition, $\deg(r) < \deg(s) - \deg(q) = n - m$. Equivalently, $\deg(rq) < \deg(s) = n$. Assume we can write rq as $rq = r_1s + r_2$ with $\deg(r_2) < m$, then we must have that $r_1 \equiv 0$ which gives $rq = r_2$ and $\deg(r_2) = \deg(rq) \geq \deg(q) = m$. Thus r cannot belong to $\tilde{\mathcal{P}}$ unless $r \equiv 0$. Hence we get $\tilde{\mathcal{P}} \cap \tilde{\mathcal{Q}}_s = \{0\}$.

We are left to show that $\mathcal{P}_{n-1} = \tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s$. The inclusion $\tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s \subset \mathcal{P}_{n-1}$ is clear since $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_s \subset \mathcal{P}_{n-1}$. Suppose $r \in \mathcal{P}_{n-1}$. By the Euclidean division algorithm we may write $rq = r_3s + r_4$ where $r_3, r_4 \in \mathcal{P}$ with $\deg(r_3) < \deg(q) = m$ and $\deg(r_4) < \deg(s) = n$. By using the Euclidean division algorithm once more, we can write $r_4 = \tilde{r}_3q + \tilde{r}_4$ with $\deg(\tilde{r}_4) < \deg(q) = m$. We then have

$$\deg(\tilde{r}_3) + \deg(q) = \deg(r_4) < \deg(s) \implies \deg(\tilde{r}_3) < \deg(s) - \deg(q) = n - m.$$

This gives $\tilde{r}_3 \in \tilde{\mathcal{Q}}_s$. Moreover, we have

$$rq = r_3s + r_4 = r_3s + \tilde{r}_4q + \tilde{r}_4 = (r_3s + \tilde{r}_4) + \tilde{r}_3q.$$

Equivalently, $(r - \tilde{r}_3)q = sr_3 + \tilde{r}_4 \in s\mathcal{P}_{m-1} + \mathcal{P}_{m-1}$. This shows that $r - \tilde{r}_3 \in \tilde{\mathcal{P}}$. In total, we may write

$$r = (r - \tilde{r}_3) + \tilde{r}_3 \in \tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s.$$

Since $r \in \mathcal{P}_{n-1}$ was arbitrary, we have the inclusion $\mathcal{P}_{n-1} \subset \tilde{\mathcal{P}} + \tilde{\mathcal{Q}}_s$. This concludes the proof. $\square$

Theorem 4.3.8. Let $w = s/q \in \text{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Factor $s = s_-s_0s_+$, where $s_-, s_0, s_+ \in \mathcal{P}$ with roots only on $\mathbb{D}, \mathbb{T}$ and $\mathbb{C} \setminus \{\overline{\mathbb{D}}\}$. Let $\deg(s) = n$ and $\deg(q) = m$. Then $H^p = \overline{\mathcal{R}(T_w^R)} \dot{+} \tilde{\mathcal{Q}}_{s_-}$ is a direct sum decomposition of H^p .

Proof. By Theorem 4.3.6 we have that $\mathcal{R}(T_w^R) = sH^p + \tilde{\mathcal{P}}$. Note that since $\tilde{\mathcal{P}}$ is a finite dimensional subspace by [GG81, Theorem 2.5 page 199] we get that

$$\overline{\mathcal{R}(T_w^R)} = \overline{sH^p + \tilde{\mathcal{P}}} = \overline{sH^p} + \tilde{\mathcal{P}} = s_-H^p + \tilde{\mathcal{P}}, \quad (4.12)$$

where the last identity follows from Lemma 3.3.1. Next we show the following identity;

$$\overline{\mathcal{R}(T_w^R)} = s_-H^p + \tilde{\mathcal{P}} = s_-H^p + \tilde{\mathcal{P}}_-, \quad (4.13)$$

where $\tilde{\mathcal{P}}_-$ is the subspace defined by

$$\tilde{\mathcal{P}}_- := \{r \in \mathcal{P} : rq = s_-r_1 + r_2, r_1, r_2 \in \mathcal{P}_{m-1}\}.$$

We first show the inclusion $s_-H^p + \tilde{\mathcal{P}} \subset s_-H^p + \tilde{\mathcal{P}}_-$.

Let $g = s_-h + r \in s_-H^p + \tilde{\mathcal{P}}$ with $h \in H^p$ and $r \in \tilde{\mathcal{P}}$, say $rq = r_1s + r_2$, with $r_1, r_2 \in \mathcal{P}_{\deg(q)-1}$. By the Euclidean division algorithm we can write

$$r_1s_0s_+ = r_3q + r_4 \quad \text{with} \quad \deg(r_4) < \deg(q).$$

We then have

$$\begin{aligned} qr &= r_1s + r_2 \\ &= (r_1s_0s_+)s_- + r_2 \\ &= (r_3q + r_4)s_- + r_2 \\ &= qr_3s_- + r_4s_- + r_2, \end{aligned}$$

or equivalently, $q(r - r_3s_-) = r_4s_- + r_2$. This implies that $r - r_3s_- \in \tilde{\mathcal{P}}_-$ since $r_4, r_2 \in \mathcal{P}_{m-1}$. We then have

$$g = s_-h + r = s_-h + s_-r_3 - s_-r_3 + r = s_-(h + r_3) + (r - r_3s_-) \in s_-H^p + \tilde{\mathcal{P}}_-.$$

So we have $\overline{\mathcal{R}(T_w^R)} \subset s_-H^p + \tilde{\mathcal{P}}_-$.

We are left to prove the reverse inclusion of (4.12). Suppose $g = s_-h + r \in s_-H^p + \tilde{\mathcal{P}}_-$, with

$h \in H^p$ and $r \in \tilde{\mathcal{P}}_-$, where we can write the latter as $qr = r_1s_- + r_2$ with $r_1, r_2 \in \mathcal{P}_{m-1}$. Since s and q are co-prime, certainly s_0s_+ and q are co-prime and $\deg(r_1) < \deg(q)$, by Bezout's identity there exists $r'_1, r'_2 \in \mathcal{P}$ with $\deg(r'_1) < \deg(q)$ and $\deg(r'_2) < \deg(s_0s_+)$ which satisfy the equation $r'_1s_0s_+ + r'_2q = r_1$. We then have

$$\begin{aligned} r'_1s + r_2 &= r'_1s_-s_0s_+ + r_2 \\ &= (r_1 - r'_2q)s_- + r_2 \\ &= r_1s_- + r_2 - qr'_2s_- \\ &= qr - qr'_2s_- \\ &= q(r - r'_2s_-). \end{aligned}$$

Since $r'_1, r_2 \in \mathcal{P}_{m-1}$, we have that, $r - r'_2s_- \in \tilde{\mathcal{P}}$. In conclusion,

$$g = s_-h + r = s_-h + s_-r'_2 - s_-r'_2 + r = s_-(h + r'_2) + (r - s_-r'_2) \in s_-H^p + \tilde{\mathcal{P}}.$$

Since $g \in s_-H^p + \tilde{\mathcal{P}}_-$ was arbitrary, we have $s_-H^p + \tilde{\mathcal{P}}_- \subset s_-H^p + \tilde{\mathcal{P}} = \overline{\mathcal{R}(T_w^R)}$ and thus we have the identity (4.13). To conclude the proof, we note that $\mathcal{P}_{\deg(s_-)-1} = \tilde{\mathcal{P}}_- \dot{+} \tilde{\mathcal{Q}}_{s_-}$ is a direct sum by Lemma 4.3.7 applied to $s = s_-$. We then have

$$\overline{\mathcal{R}(T_w^R)} + \tilde{\mathcal{Q}}_{s_-} = s_-H^p + \tilde{\mathcal{P}}_- + \tilde{\mathcal{Q}}_{s_-} = s_-H^p + \tilde{\mathcal{P}}_- \dot{+} \tilde{\mathcal{Q}}_{s_-} = s_-H^p \dot{+} \mathcal{P}_{\deg(s_-)-1} = H^p.$$

The fact that $\overline{\mathcal{R}(T_w^R)} \dot{+} \tilde{\mathcal{Q}}_{s_-}$ is a direct sum now follows. $\square$

Lemma 4.3.9. Let $w \in \text{Rat}(\mathbb{T})$ and $w = s/q$ where $s, q \in \mathcal{P} \setminus \{0\}$ are co-prime. If $s = s_-s_0s_+$, where s_-, s_0, s_+ are polynomials with roots only $\mathbb{D}, \mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then

$$\ker(T_w^R) = \left\{ \frac{r}{s_+} : r \in \mathcal{P}, \deg(r) < \deg(q) - (\deg(s_-) + \deg(s_0)) \right\} := \mathcal{K}.$$

Proof. We first show the inclusion $\mathcal{K} \subset \ker(T_w^R)$. Suppose $g \in \mathcal{K}$. Then $g = r/s_+$ where $\deg(r) < \deg(q) - (\deg(s_-) + \deg(s_0))$. We then have $sg = s_-s_0s_+r/s_+ = s_-s_0r \in \mathcal{P}$. Furthermore,

$$\deg(s_-s_0r) = \deg(s_-s_0) + \deg(r) < \deg(s_-s_0) + \deg(q) - (\deg(s_-s_0)) = \deg(q).$$

Thus $wg = sg/q = s_-s_0r/q \in \text{Rat}_0(\mathbb{T})$, which implies that $g \in \mathcal{D}(T_w^R)$ by Lemma 4.3.3. We also have $T_w^R g = 0$ which shows that $g \in \ker(T_w^R)$. Since $g \in \mathcal{K}$ was arbitrary, we have $\mathcal{K} \subset \ker(T_w^R)$. We are left to show that $\ker(T_w^R) \subset \mathcal{K}$.

Suppose $g \in \ker(T_w^R)$. Then $T_w^R g = 0$ and so by Lemma 4.3.3 this means that $wg = r/q$, or equivalently $sg = r$, for some $r \in \mathcal{P}_{\deg(q)-1}$. We then have $s_-s_0(s_+g) = sg = r$ and we set $\tilde{r} = s_+g \in H^p$. Since $s_-s_0s_+g = r$, we have

$$\frac{r}{s_-s_0s_+} = g = \frac{\tilde{r}}{s_+} \implies \tilde{r} = \frac{r}{s_-s_0}.$$

Since $\tilde{r} \in H^p$ and s_-s_0 have roots only on $\overline{\mathbb{D}}$, we must have that s_-s_0 divides r , in other words $r = s_-s_0r_1$ for some $r_1 \in \mathcal{P}$. Consequently, $\tilde{r} = r/s_-s_0 = s_-s_0r_1/s_-s_0 = r_1$ and we also have

$$\deg(\tilde{r}) = \deg(s_+g) = \deg(r) - \deg(s_-s_0) < \deg(q) - \deg(s_-) + \deg(s_0).$$

In total, we have $g = \tilde{r}/s_+ \in \mathcal{K}$. Hence we have the inclusion $\mathcal{K} \subset \ker(T_w^R)$. $\square$

4.3.3 Fredholm properties of T_w^R for $w \in \text{Rat}(\mathbb{T})$

Lemma 4.3.10. Let $w \in \text{Rat}(\mathbb{T})$, with $w = s/q$ and $s, q \in \mathcal{P}$ co-prime. Let $\deg(s) = n$ and $\deg(q) = m$. Then T_w^R has closed range if and only if s has no roots on $\mathbb{T}$.

Proof. By Theorem 4.3.6 we have $\mathcal{R}(T_w^R) = sH^p + \tilde{\mathcal{P}}$. So we have to show that s has no roots on $\mathbb{T}$ if and only if $sH^p + \tilde{\mathcal{P}}$ is closed in H^p .

Suppose that $sH^p + \tilde{\mathcal{P}}$ is closed. We then have

$$sH^p + \tilde{\mathcal{P}} = \overline{sH^p + \tilde{\mathcal{P}}} = s_-H^p \dot{+} \tilde{\mathcal{P}}_- = s_-H^p \dot{+} \tilde{\mathcal{P}}_- \dot{+} \tilde{Q}_{s_-}.$$

So we have $H^p = sH^p + \tilde{\mathcal{P}} + \tilde{Q}_{s_-}$. We can take a finite dimensional subspace of $\tilde{\mathcal{P}} + \tilde{Q}_{s_-}$ that serves as a complement for sH^p . Then by the fact that every finite dimensional subspace of a Banach space is complemented by a closed subspace, we have that sH^p is a closed subspace. By Lemma 3.3.1 we have that s has no roots on $\mathbb{T}$.

Conversely, assume s has no roots on $\mathbb{T}$. Since s is a polynomial, this is equivalent to $|s(t)| > m$ on $\mathbb{T}$ for some $m > 0$. Hence the bounded analytic Toeplitz operator T_s has closed range by Theorem 3.2.10. We have that $\mathcal{R}(T_s) = sH^p$ is a closed subspace of H^p . Since $\tilde{\mathcal{P}}$ is a finite dimensional subspace of H^p , the sum $sH^p + \tilde{\mathcal{P}}$ is a closed subspace of H^p by [GG81, Theorem 2.5 page 199]. $\square$

The following Corollaries are immediate consequences of the preceding lemma

Corollary 4.3.11. Let $w = s/q \in \text{Rat}(\mathbb{T})$, where $s, q \in \mathcal{P}$ are co-prime. If $s = s_-s_0s_+$, where s_- , s_0 , s_+ are polynomials with roots only $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then

$$\dim(\ker(T_w^R)) = \max\{0, \deg(q) - \deg(s_-s_0)\}.$$

In particular, T_w^R is injective if and only if $\deg(s_-s_0) \geq \deg(q)$ that is, when the number of zeros, counting multiplicities, of w on $\overline{\mathbb{D}}$ is greater than or equal to the number of poles, counting multiplicities, of w on $\mathbb{T}$.

Proof. Let $w = s/q$ where $s, q \in \mathcal{P} \setminus \{0\}$ co-prime. There are two cases that need to be addressed. Consider the case $\deg(q) < \deg(s_-s_0)$. Then by Lemma 4.3.9, $\ker(T_w^R) = \{0\}$, hence $\dim(\ker(T_w^R)) = 0$.

Finally, consider the case $\deg(s_-s_0) < \deg(q)$. Then by Lemma 4.3.9, $\ker(T_w^R) \neq \{0\}$. So let $r/s_+ \in \ker(T_w \setminus \{0\})$ with $\deg(r) = p$. We then have

$$\frac{r}{s_+} = \frac{r_0}{s_+} + \frac{r_1 z}{s_+} + \cdots + \frac{r_p z^l}{s_+} \in \text{span}\left\{\frac{z^j}{s_+} : j = 0, \dots, i\right\},$$

where $i = \deg(q) - \deg(s_-s_0) - 1 \geq l$. Thus $\ker(T_w^R) \subset \text{span}\{z^j/s_+ : j = 0, \dots, n\}$. In fact, by Lemma 4.3.9, the reverse inclusion holds as well and we get $\ker(T_w^R) = \text{span}\{z^j/s_+ : j = 0, \dots, n\}$. Thus $\dim(\ker(T_w^R)) = n + 1 = \deg(q) - \deg(s_-s_0)$. Collectively, combining the two cases gives

$$\dim(\ker(T_w^R)) = \max\{0, \deg(q) - \deg(s_-s_0)\}.$$

The rest of the results follow from the above formula. $\square$

In the following Corollary we give an explicit formula for the kernel of T_w^R when $w \in \text{Rat}(\mathbb{T})$.

Corollary 4.3.12. Let $w = s/q \in \text{Rat}(\mathbb{T})$, where $s, q \in \mathcal{P}$ co-prime and s has all its roots in $\overline{\mathbb{D}}$. Let $\deg(s) = n$ and $\deg(q) = m$. Then

$$\ker(T_w^R) = \{r : \deg(r) < m - n\} \subset \mathcal{P}.$$

Proof. If s has all its roots in $\overline{\mathbb{D}}$, then $s_+ = 1$. Then the result follows by Lemma 4.3.9. $\square$

Corollary 4.3.13. Let $w = s/q \in \text{Rat}_0(\mathbb{T})$, where $s, q \in \mathcal{P} \setminus \{0\}$ co-prime. Let $\deg(s) = n$ and $\deg(q) = m$. Then $\mathcal{P}_{m-n-1} \subset \ker(T_w^R)$. In particular, T_w^R is not injective.

Proof. Suppose $r \in \mathcal{P}_{n-m-1}$. We then have

$$\deg(sr) = \deg(s) + \deg(r) < \deg(s) + \deg(q) - \deg(s) = \deg(q).$$

Consequently, $wr = sr/q \in \text{Rat}_0(\mathbb{T})$ and thus by Lemma 4.3.3 $r \in \mathcal{D}(T_w^R)$ and we have $T_w^R r = 0$. $\square$

Corollary 4.3.14. Let $w = s/q \in \text{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime. Factor $s = s_-s_0s_+$, with $s_-, s_0, s_+ \in \mathcal{P}$ having roots only on $\mathbb{D}$ and $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then

$$\text{codim}(\overline{\mathcal{R}(T_w^R)}) = \max\{0, \deg(s_-) - \deg(q)\}.$$

In particular, T_w^R is surjective if and only if the number of roots of w on $\mathbb{D}$, counting multiplicities, is less than or equal to the number of poles of w on $\mathbb{T}$ counting multiplicities.

Proof. By Theorem 4.3.8 we have $H^p = \overline{\mathcal{R}(T_w^R)} \dot{+} \tilde{\mathcal{Q}}_{s_-}$ is a direct sum decomposition of H^p . Hence $\text{codim}(\mathcal{R}(T_w^R)) = \dim(\tilde{\mathcal{Q}}_{s_-}) = \max\{0, \deg(s_-) - \deg(q)\}$. $\square$

The following corollary is immediate from Lemma 4.3.14

Corollary 4.3.15. Let $w = s/q \in \text{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Let $s = s_- s_0 s_+$, where $s_-, s_0, s_+ \in \mathcal{P}$ with roots only on $\mathbb{D}, \mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then T_w^R is Fredholm if and only if s has no roots on $\mathbb{T}$, that is $s_0 \equiv 1$. In that case, the Fredholm index is given by

$$\text{Index}(T_w^R) = \deg(q) - \deg(s_-).$$

Proof. If T_w^R is Fredholm, then by definition $\mathcal{R}(T_w^R)$ is closed. By Lemma 4.3.10, we have that s has no roots on $\mathbb{T}$, hence $s_0 \equiv 1$.

Conversely, suppose $s_0 \equiv 1$. By Corollary 4.3.14 and Corollary 4.3.11 we have that T_w^R is Fredholm.

Suppose $\deg(q) > \deg(s_-)$. Then by Corollary 4.3.14 and Corollary 4.3.11, we have

$$\text{Index}(T_w^R) = \dim(\ker(T_w^R)) - \text{codim}(\mathcal{R}(T_w^R)) = \deg(q) - \deg(s_-).$$

Similarly, suppose $\deg(q) < \deg(s_-)$. Then by the same corollaries as used above we have

$$\text{Index}(T_w^R) = \dim(\ker(T_w^R)) - \text{codim}(\mathcal{R}(T_w^R)) = 0 - \deg(s_-) - \deg(q) = \deg(q) - \deg(s_-).$$

Combining the two cases, we achieve the desired formula. $\square$

4.3.4 The Adjoint T_w^* for $w \in \text{Rat}(\mathbb{T})$

In this section, we will discuss the adjoint of the Toeplitz operator T_w^R , where $w \in \text{Rat}(\mathbb{T})$. The results in this section come from [Gro+19b] where full treatment was given to the theory of the adjoint T_w^{R*} , for $w \in \text{Rat}(\mathbb{T})$ and $w \in \text{Rat}$. The main aim of this section is to establish a connection between Sarason-type Toeplitz operators and GHJR-type Toeplitz operators. For this purpose, we will only discuss the adjoint T_w^{R*} for the case where $w \in \text{Rat}(\mathbb{T})$ is proper.

In Chapter 3, we discussed that the adjoint of a bounded Toeplitz operator T_a can be identified with the Toeplitz operator $T_{\bar{a}}$ which acts on $H^{p'}$, where $1 < p' < \infty$ satisfies $1/p + 1/p' = 1$.

We first introduce some new notation which will be used in this section. For $r \in \mathcal{P}$ with $\deg(r) = k$, say $r(z) = r_0 + r_1 z + \dots + r_k z^k$, we define $r^\sharp \in \mathcal{P}$ by

$$r^\sharp(z) = z^k \overline{r(1/\bar{z})} = \bar{r}_0 z^k + \bar{r}_1 z^{k-1} + \dots + \bar{r}_k.$$

We note that $\deg(r) = \deg(r^\sharp)$ only in the case when $r_0 \neq 0$. We also have the following

consequence. For $z \in \mathbb{T}$,

$$\begin{aligned} \overline{r(z)} &= \overline{r_0 + r_1 z + \cdots + r_k z^k} \\ &= \overline{r_0} + \overline{r_1 z} + \cdots + \overline{r_k z^k} \\ &= \overline{r_0} + \overline{r_1} \frac{1}{z} + \cdots + \overline{r_k} \frac{1}{z^k} \\ &= \frac{1}{z^k} [\overline{r_0} z^k + \overline{r_1} z^{k-1} + \cdots + \overline{r_k}] \\ &= \frac{1}{z^k} r^\sharp(z). \end{aligned}$$

Let $w = s/q \in \text{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime and $\deg s = n$, $\deg q = m$. We define the rational function $w^*(z)$ on $\mathbb{T}$ by

$$w^*(z) = \overline{w(z)} = \frac{\overline{s(z)}}{\overline{q(z)}}.$$

By the preceding formulas, we then have on $\mathbb{T}$

$$w^*(z) = \frac{z^{-n} s^\sharp(z)}{z^{-m} q^\sharp(z)} = \frac{z^{m-n} s^\sharp(z)}{q^\sharp(z)}. \quad (4.14)$$

It can be shown that $\alpha \in \mathbb{C} \setminus \{0\}$ is a root of q , if and only if $1/\overline{\alpha} \in \mathbb{C} \setminus \{0\}$ is a root of $q^\sharp$. In particular, if $\alpha \in \mathbb{T}$, then $1/\overline{\alpha} \in \mathbb{T}$ so the roots on $\mathbb{T}$ don't change. Consequently, if q is a polynomial with all roots on $\mathbb{T}$, then $q^\sharp$ also has roots only on $\mathbb{T}$. In view of (4.14), we see that if $w \in \text{Rat}(\mathbb{T})$ is proper, i.e. when $m \geq n$, then w^* is a ratio of two polynomials with $q^\sharp$ having roots only on $\mathbb{T}$ and thus $w^* \in \text{Rat}(\mathbb{T})$.

In the case when $w \in \text{Rat}(\mathbb{T})$ is not proper, i.e. $m \leq n$, then we have

$$w^*(z) = \frac{s^\sharp(z)}{z^{n-m} q^\sharp(z)}$$

and thus w^* has a pole at $z = 0$ so that $w^* \notin \text{Rat}(\mathbb{T})$.

So far we have described the basic properties of w^* for $w \in \text{Rat}(\mathbb{T})$. In the subsequent text, we will only focus on proper rational functions $w \in \text{Rat}(\mathbb{T})$ with poles on $\mathbb{T}$.

Theorem 4.3.16. Let $w = s/q \in \text{Rat}(\mathbb{T})$ be proper, with $s, q \in \mathcal{P}$ co-prime, $\deg(s) = n$, $\deg(q) = m$. If $s = s_- s_0 s_+$, where $s_-, s_0, s_+ \in \mathcal{P}$ with roots only on $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. Then for $T_{w^*}^R$ on $H^{p'}$ we have

$$\mathcal{D}(T_{w^*}^R) = q^\sharp H^{p'} + \mathcal{P}_{m-1} \quad \text{and} \quad \mathcal{R}(T_{w^*}^R) = z^{m-n} s^\sharp H^{p'} + \tilde{\mathcal{P}}_\sharp$$

where

$$\tilde{\mathcal{P}}_\sharp = \{r \in \mathcal{P} : r q^\sharp = z^{m-n} r_1 s^\sharp + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1}\}.$$

Furthermore, $\mathcal{R}(T_{w^*}^R)$ is dense in $H^{p'}$.

Proof. Since $m \geq n$, we can write

$$w^*(z) = \frac{z^{m-n}s^\sharp(z)}{q^\sharp(z)} =: \frac{\tilde{s}(z)}{\tilde{q}(z)},$$

where $\tilde{s}(z) := z^{m-n}s^\sharp$ and $\tilde{q}(z) := q^\sharp(z)$. Note that all roots of $\tilde{q}$ belong to $\mathbb{T}$, hence $w^* \in \text{Rat}(\mathbb{T})$. On the other hand, we can factor $\tilde{s}$ as follows

$$\tilde{s}(z) = (z^{m-n}(s_+)^\sharp)(s_0)^\sharp(s_-)^\sharp,$$

where $(z^{m-n}(s_+)^\sharp)$, $(s_0)^\sharp$ and $(s_-)^\sharp$ have roots only on $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively. We note that $\deg(q) = \deg(q^\sharp)$ and $\deg((s_+)^\sharp) = \deg(s_+)$. We now obtain the formula for the domain and range using Theorem 4.3.6.

$$\begin{aligned} \mathcal{D}(T_{w^*}^R) &= q^\sharp H^{p'} + \mathcal{P}_{m-1} \\ \mathcal{R}(T_{w^*}^R) &= z^{m-n}s^\sharp H^{p'} + \tilde{\mathcal{P}}_\sharp. \end{aligned}$$

Lastly, we show that $\mathcal{R}(T_{w^*}^R)$ is dense in $H^{p'}$. We recall from Theorem 4.3.8 that a complement of the closure of $\mathcal{R}(T_{w^*}^R)$ in $H^{p'}$ is given by $\tilde{Q} = \mathcal{P}_{k-1}$, where

$$k = \max\{0, \deg(z^{m-n}(s_+)^\sharp) - m\}.$$

We have

$$\deg(z^{m-n}(s_+)^\sharp) - m = m - n + \deg((s_+)^\sharp) - m = \deg((s_+)^\sharp) - n \leq 0.$$

Hence $\mathcal{P}_{k-1} = \{0\}$, which shows that $\mathcal{R}(T_{w^*}^R)$ is dense in $H^{p'}$. $\square$

We are now ready to discuss the adjoint of T_w^R , which we denote by T_w^{R*} acting on $H^{p'}$.

Lemma 4.3.17. Let $w \in \text{Rat}(\mathbb{T})$ be proper with $s, q \in \mathcal{P}$ co-prime, $\deg(q) = m$, $\deg(s) = n$. Then

$$\begin{aligned} q^\sharp H^{p'} &\subset \mathcal{D}(T_{w^*}^R) \cap \mathcal{D}(T_w^{R*}) \\ T_w^{R*}|_{q^\sharp H^{p'}} &= T_{w^*}^R|_{q^\sharp H^{p'}}. \end{aligned}$$

Moreover, $T_w^{R*}(q^\sharp u) = z^{m-n}s^\sharp u$ for each $u \in H^{p'}$.

Proof. By Theorem 4.3.16, we have

$$\mathcal{D}(T_{w^*}^R) = q^\sharp H^{p'} + \mathcal{P}_{m-1} \supset q^\sharp H^{p'}.$$

Next we show that $q^\sharp H^{p'} \subset \mathcal{D}(T_w^{R*})$ and $T_w^{R*}|_{q^\sharp H^{p'}} = T_{w^*}^R|_{q^\sharp H^{p'}}$. Let $g \in q^\sharp H^{p'}$ be arbitrary and we write $g = q^\sharp u$ for $u \in H^{p'}$. To show that $g \in \mathcal{D}(T_w^{R*})$ and that $T_w^{R*}g = T_{w^*}^Rg$, we need to show that for any $f \in \mathcal{D}(T_w^R)$, we have

$$\langle T_w^R f, g \rangle = \langle f, T_w^{R*} g \rangle.$$

Note that $T_w^R g = T_w^R(q^\sharp u) = z^{m-n}s^\sharp u$, because $m \geq n$.

Let $f \in \mathcal{D}(T_w^R)$ be arbitrary and let $h := T_w^R f \in H^p$. By Theorem 4.3.3, we can write $sf = qh + r$ where $r \in \mathcal{P}_{m-1}$. Consequently,

$$\begin{aligned} \langle T_w^R f, g \rangle &= \langle h, q^\sharp u \rangle \\ &= \langle h, z^m \bar{q} u \rangle \\ &= \langle qh, z^m u \rangle \\ &= \langle sf - r, z^m u \rangle \\ &= \langle sf, z^m u \rangle - \langle r, z^m u \rangle \\ &= \langle sf, z^m u \rangle. \end{aligned}$$

The last equality follows since $r \in \mathcal{P}_{m-1}$ and so $z^{-m}r \in K^p$ and $u \in H^{p'}$.

$$\begin{aligned} \langle T_w^R f, g \rangle &= \langle sf, z^m u \rangle \\ &= \langle f, \bar{s} z^m u \rangle \\ &= \langle f, z^{m-n} s^\sharp u \rangle \\ &= \langle f, T_w^R g \rangle. \end{aligned}$$

This completes the proof. $\square$

Lemma 4.3.18. Let $w = s/q \in \text{Rat}(\mathbb{T})$ be proper with $s, q \in \mathcal{P}$ co-prime, $\deg q = m$ and $\deg s = n$. Let $g \in \mathcal{D}(T_w^{R*})$ and $k := T_w^{R*} g \in H^{p'}$. Then for any $r \in \mathcal{P}_{n-1}$ and $r_1 \in \mathcal{P}_{m-1}$ for which

$$sr_1 = qr + r_2 \text{ for some } r_2 \in \mathcal{P}_{m-1}. \quad (4.15)$$

we have

$$\langle r_1, k \rangle = \langle r, g \rangle.$$

Moreover, we have

$$z^{m-n}s^\sharp g - q^\sharp k \in \mathcal{P}_{m-1}. \quad (4.16)$$

In particular, $\mathcal{D}(T_w^{R*}) \subset \mathcal{D}(T_w^R)$ and $T_w^{R*} = T_w^R|_{\mathcal{D}(T_w^{R*})}$.

Proof. Let $g \in \mathcal{D}(T_w^{R*})$ and set $k := T_w^{R*} g$. We then have $\langle T_w^R f, g \rangle = \langle f, T_w^{R*} g \rangle = \langle f, k \rangle$, for each $f \in \mathcal{D}(T_w^R)$. By Theorem 4.3.6 we have $\mathcal{D}(T_w^R) = qH^p + \mathcal{P}_{m-1}$ and $\mathcal{R}(T_w^R) = sH^p + \tilde{\mathcal{P}}$. Let $f \in \mathcal{D}(T_w^R)$. Then we may write, $f = qh + r_1$, where $h \in H^p$ and $r_1 \in \mathcal{P}_{m-1}$ and $T_w^R f = sh + r$ for some $r \in \tilde{\mathcal{P}} \subset \mathcal{P}_{n-1}$. Note in particular that r satisfies (4.15). We then have

$$\begin{aligned} \langle sh, g \rangle + \langle r, g \rangle &= \langle sh + r, g \rangle \\ &= \langle T_w^R f, g \rangle \\ &= \langle f, T_w^{R*} g \rangle \\ &= \langle f, k \rangle \\ &= \langle qh + r_1, k \rangle \\ &= \langle qh, k \rangle + \langle r_1, k \rangle. \end{aligned}$$

Equivalently,

$$\langle sh, g \rangle - \langle qh, k \rangle = \langle r_1, k \rangle - \langle r, g \rangle.$$

Since $f \in \mathcal{D}(T_w^R)$ is arbitrary, the above identity must hold when $h \equiv 0$ or $r_1 \equiv 0$. Note that the latter immediately implies that $r \equiv 0$. We then obtain the following identities;

$$\langle sh, g \rangle = \langle qh, k \rangle \text{ for } h \in H^p, \quad (4.17)$$

$$\langle r_1, k \rangle = \langle r, g \rangle, \quad (4.18)$$

where the latter identity holds for any $r \in \mathcal{P}_{n-1}$ and $r_1 \in \mathcal{P}_{m-1}$ which satisfy (4.15), since $f \in \mathcal{D}(T_w^R)$ is arbitrary. This proves the first claim of the lemma.

Note that (4.17) yields the following;

$$0 = \langle h, \bar{s}g - \bar{q}k \rangle = \langle h, z^{-n}s^\sharp g - z^{-m}q^\sharp k \rangle \text{ for } h \in H^p.$$

This implies that $z^{-n}s^\sharp g - z^{-m}q^\sharp k \in K^p$. Since $m \geq n$, it is clear that $z^{m-n}s^\sharp g - q^\sharp k \in H^p$. We then have

$$T_{z^{-m}}(z^{m-n}s^\sharp g - q^\sharp k) = P_+(z^{-n}s^\sharp g - z^{-m}q^\sharp k) = 0.$$

Hence $z^{m-n}s^\sharp g - q^\sharp k \in \ker(T_{z^{-m}}) = \mathcal{P}_{m-1}$. This proves the second claim.

We finally show the last claim of the lemma.

Let $g \in \mathcal{D}(T_w^{R*})$ be as before with $T_w^{R*}g = k$. We show that $g \in \mathcal{D}(T_w^R)$ and $T_w^Rg = k$. Since $w^* = z^{m-n}s^\sharp/q^\sharp \in \text{Rat}(\mathbb{T})$, by (4.16), we have $z^{m-n}s^\sharp g - q^\sharp k =: r_3 \in \mathcal{P}_{m-1}$. Then $w^*g = k + r_3/q^\sharp$. This shows that $g \in \mathcal{D}(T_w^R)$ and $T_w^{R*}g = T_w^Rg$. This concludes the proof. $\square$

We are now ready to prove the main result of the subsection. Analogous to the Toeplitz operator T_w^R with $w \in \text{Rat}(\mathbb{T})$, we give explicit formulas for the domain and range of the adjoint T_w^{R*} for $w \in \text{Rat}(\mathbb{T})$.

Theorem 4.3.19. Let $w = s/q \in \text{Rat}(\mathbb{T})$ be proper with $s, q \in \mathcal{P}$ co-prime and $1 < p < \infty$, $\deg(q) = m$, $\deg(s) = n$ and $1 < p' < \infty$ with $1/p + 1/p' = 1$. Then

$$\mathcal{D}(T_w^{R*}) = q^\sharp H^{p'} \subset \mathcal{D}(T_w^R)$$

and

$$T_w^{R*} = T_{w^*|_{q^\sharp H^{p'}}}^R.$$

In that case,

$$\mathcal{R}(T_w^{R*}) = z^{m-n}s^\sharp H^{p'} \text{ and } T_w^{R*}(q^\sharp u) = z^{m-n}s^\sharp u, \text{ for } u \in H^{p'}.$$

Proof. By virtue of Lemma 4.3.17, it suffices to show that $\mathcal{D}(T_w^{R*}) \subset q^\sharp H^{p'}$. Throughout the proof, we denote by $P_+(k)$ the projection of H^p or $H^{p'}$ onto $\mathcal{P}_{k-1}$ and we also denote

$Q(k) = I - P_+(k)$. The $k \times k$ truncation of a Toeplitz operator T_a on H^p , or $H^{p'}$ is denoted by $T_{a,k}$, in other words $T_{a,k} = P_+(k)T_a|_{\mathcal{P}_{k-1}}$.

Let $r_1 \in \mathcal{P}_{m-1}$ be given. By the Euclidean division algorithm we can write

$$sr_1 = qr + r_2 \quad \text{with } \deg(r_2) < m \quad (4.19)$$

and r, r_2 are uniquely determined. It follows that for $r_1 \in \mathcal{P}_{m-1}$ and $r_2 \in \mathcal{P}_{n-1}$, (4.19) holds for some r_2 if and only if

$$T_s r_1 - T_q r = sr_1 - qr \in \mathcal{P}_{m-1}.$$

Equivalently,

$$T_{z^{-m}}(T_s r_1 - T_q r) = 0,$$

which is the same as

$$T_{z^{-m}} T_s r_1 = T_{z^{-m}} T_q r. \quad (4.20)$$

By Theorem 3.1.8, we have that $T_{z^{-m}} T_s r_1 = T_{z^{-m}s} r_1$ and $T_{z^{-m}} T_q r = T_{z^{-m}q} r$. We recall that $q^\sharp(z) = z^m \overline{q(z)}$ and $s^\sharp = z^n \overline{s(z)}$. We then have

$$T_{z^{-m}s} r_1 = T_{z^{-m+n}(z^{-n}s)} r_1 = T_{z^{-m+n}} T_{s^\sharp} r_1 = T_{z^{m-n}}^* T_{s^\sharp}^* r_1 \quad \text{and} \quad T_{z^{-m}q} r = T_{q^\sharp}^- r.$$

This gives us that (4.20) is equivalent to

$$T_{s^\sharp}^* T_{z^{m-n}}^* r_1 = T_{q^\sharp}^- r.$$

Since $r_1, r \in \mathcal{P}_{m-1}$, we can take the $m \times m$ truncation of the above identity, leading to

$$T_{s^\sharp, m}^* T_{z^{m-n}, m}^* r_1 = T_{q^\sharp, m}^- r.$$

Notice that $T_{q^\sharp, m}^*$ is an upper triangular Toeplitz matrix with nonzero diagonal. Thus the Toeplitz matrix $T_{q^\sharp, m}^*$ is invertible, and its inverse is an upper triangular Toeplitz matrix as well. We then have

$$r = (T_{q^\sharp, m}^*)^{-1} T_{s^\sharp, m}^* T_{z^{m-n}, m}^* r_1 = T_{s^\sharp, m}^* T_{z^{m-n}, m}^* (T_{q^\sharp, m}^*)^{-1} r_1, \quad (4.21)$$

where the second identity follows since all the above Toeplitz operators are upper triangular and thus commute.

Suppose $g \in \mathcal{D}(T_w^{R*})$ and let $k := T_w^{R*} g$. By Lemma 4.3.18, for any $r_1 \in \mathcal{P}_{m-1}$, with r as in (4.21), we have

$$\begin{aligned} \langle r_1, P_+(m)k \rangle_{\mathcal{P}_{m-1}} &= \langle r_1, k \rangle \\ &= \langle r, g \rangle \\ &= \langle r, P_+(m)g \rangle_{\mathcal{P}_{m-1}} \\ &= \langle T_{s^\sharp, m}^* T_{z^{m-n}, m}^* (T_{q^\sharp, m}^*)^{-1} r_1, P_+(m)g \rangle_{\mathcal{P}_{m-1}} \\ &= \langle r_1, (T_{q^\sharp, m}^*)^{-1} T_{z^{m-n}, m}^* T_{s^\sharp, m}^* P_+(m)g \rangle_{\mathcal{P}_{m-1}} \\ &= \langle r_1, (T_{q^\sharp, m}^*)^{-1} P_+(m) T_{z^{m-n}} T_{s^\sharp} g \rangle \\ &= \langle r_1, (T_{q^\sharp, m}^*)^{-1} P_+(m) z^{m-n} s^\sharp g \rangle. \end{aligned}$$

Since $r_1 \in \mathcal{P}_{m-1}$ was arbitrary, we have that $P_+(m)k = (T_{q^\sharp, m})^{-1}P_+(m)z^{m-n}s^\sharp g$. Consequently,

$$P_+(m)z^{m-n}s^\sharp g = T_{q^\sharp, m}P_+(m)k = P_+(m)T_{q^\sharp}k = P_+(m)q^\sharp k.$$

By (4.16) in Lemma 4.3.18 we have $z^{m-n}s^\sharp g - q^\sharp k \in \mathcal{P}_{m-1}$. Hence, we now proved

$$q^\sharp k = z^{m-n}s^\sharp g.$$

The above identity can also be written as

$$\frac{g}{q^\sharp} = \frac{k}{z^{m-n}s^\sharp} \in N^+(\mathbb{D}),$$

since $q^\sharp$ is a polynomial with roots only on $\mathbb{T}$ and thus is an outer function by Lemma 2.2.17. Using the same technique used in the proof of Theorem 4.3.6 (where we showed that $h' \in L^p$), it follows that $k/z^{m-n}s^\sharp \in L^{p'}$. In total, we obtain $g/q^\sharp \in N^+(\mathbb{D}) \cap L^{p'} = H^{p'}$ and thus $g \in q^\sharp H^{p'}$. We conclude that $\mathcal{D}(T_w^{R*}) \subset q^\sharp H^{p'}$. $\square$

4.3.5 Relation to the Sarason-type Toeplitz operator

In this subsection, we provide a connection between Sarason-type Toeplitz operators and GHJR-type Toeplitz operators.

Theorem 4.3.20. Let $w = s/q \in \text{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime. Then $\mathcal{D}(T_w^A) = qH^p$ and $T_w^A = T_w^R|_{qH^p}$. In particular, if $w \in \text{Rat}(\mathbb{T})$ is proper, then $T_w^{R*} = T_w^{A*}$.

Proof. We first show that $\mathcal{D}(T_w^A) = qH^p$. By hypothesis, $w \in \text{Rat}(\mathbb{T})$, which means that q has roots only on $\mathbb{T}$. By Lemma 2.2.17, q is an outer function of H^∞ . Consequently, $w \in N^+(\mathbb{D})$. Let $w = b/a$ be the canonical form for w . Recall that the property $|a(t)|^2 + |b(t)|^2 = 1$ a.e on $\mathbb{T}$ yields

$$|b(t)|^2 = \frac{|q(t)|^2}{|s(t)|^2 + |q(t)|^2} \text{ a.e on } \mathbb{T}.$$

In particular, the function $|s(t)|^2 + |q(t)|^2$ is a non-negative trigonometric polynomial on $\mathbb{T}$. By Fejer-Riesz Theorem, there exists a polynomial r with no roots on $\mathbb{D}$ and with $r(0) > 0$ such that

$$|r(t)|^2 = |s(t)|^2 + |q(t)|^2 \text{ a.e on } \mathbb{T}.$$

Moreover, we see that r has no roots on $\mathbb{T}$ as well. Indeed, suppose $t_0 \in \mathbb{T}$ is a root of r . Then the above identity shows

$$0 = |r(t_0)|^2 = |s(t_0)|^2 + |q(t_0)|^2.$$

This is only possible if $s(t_0) = 0$ and $q(t_0) = 0$. This contradicts the fact that s, q are co-prime. Thus $r \in \mathcal{P}$ has no roots on $\overline{\mathbb{D}}$ which gives us $r^{-1} \in H^\infty$. Consequently, $s/r, q/r \in$

H^∞ . By uniqueness of the canonical form, $b = s/r$ and $a = q/r$. Since $r^{-1}H^p = H^p$, we have $\mathcal{D}(T_w^A) = aH^p = qH^p$ by Theorem 4.1.5. Next, we show the penultimate claim. Let $f \in \mathcal{D}(T_w^A)$. Then $f = qh$ for some $h \in H^p$ and so $T_w^A f = wf = sh$. On the other hand, since $wf = sh \in H^p$, by definition of T_w^R we have $T_w^R f = P_+(sh) = sh$. Summing up the preceding equations, we get that $T_w^A = T_w^R|_{qH^p}$. Lastly, we prove the final claim. Suppose $w \in \text{Rat}(\mathbb{T})$ is proper, say $w = s/q$. Then we can write $w^*(z) = \overline{w(z)} = z^{m-n}s^\sharp/q^\sharp \in \text{Rat}(\mathbb{T})$. By Theorem 4.3.19 $\mathcal{D}(T_w^{R*}) = q^\sharp H^{p'} = \mathcal{D}(T_{w^*}^A)$ and $T_{w^*}^A = T_{w^*}^R|_{q^\sharp H^{p'}} = T_w^{R*}$ $\square$

4.3.6 Notes

The results in this section all come from [Gro+18; Gro+19a; Gro+19b]. The proofs of some of the results are in close resemblance to what is done in the aforementioned papers. However, in some cases we have added further detail to the proofs and in some instances, we made use of results in Chapter 3. As the interest of this section was mostly for symbols in $\text{Rat}(\mathbb{T})$, only essential results were obtained from the articles which were adapted to obtain the various conclusions.

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