

CHAPTER 5

MATHEMATICAL MODELS

In current industrial applications, the column-top heat exchanger is bolted very stiffly to the column with steel bolts located at the top of the heat exchanger, leaving the bottom of the panels free to vibrate. The bottom header pipe is connected to the cooling medium supply line with a bellow (Appendix C). In this case the mode shape of the lowest natural frequency of the heat exchanger was the deflection of the bottom of the plates

The forces caused by both vortex shedding and vortex ejection were directed in the direction normal to the individual plates. The plate pack was also expected to be the most flexible in this direction and, therefore, these forces could easily excite vibration in this direction.

As the damage experienced in the real case coincided with the location of the high stresses due to the deflection of the plates, therefore, the vibration of the plate pack in this direction was selected as a plausible cause of the damage experienced by the column-top condenser at Sasol.

The prevention of a re-occurrence of the failures and the improvement of the reliability of the heat exchangers rested on the fact that this mode shape and the resulting dynamic forces in the internals could be controlled by isolating the heat exchanger from the almost rigid (stiff steel) mounted structure currently used.

To analyse this mode shape further, the assumption was made that the system was only free to vibrate in the one direction in which the chosen mode shape caused displacement. The other directions (bounce and rotation modes) in which the heat exchanger as a whole could vibrate, was assumed to be rigid.

With these assumptions, the case of a dimple plate heat exchanger mounted with soft rubber elements can be graphically illustrated by Figure 32.

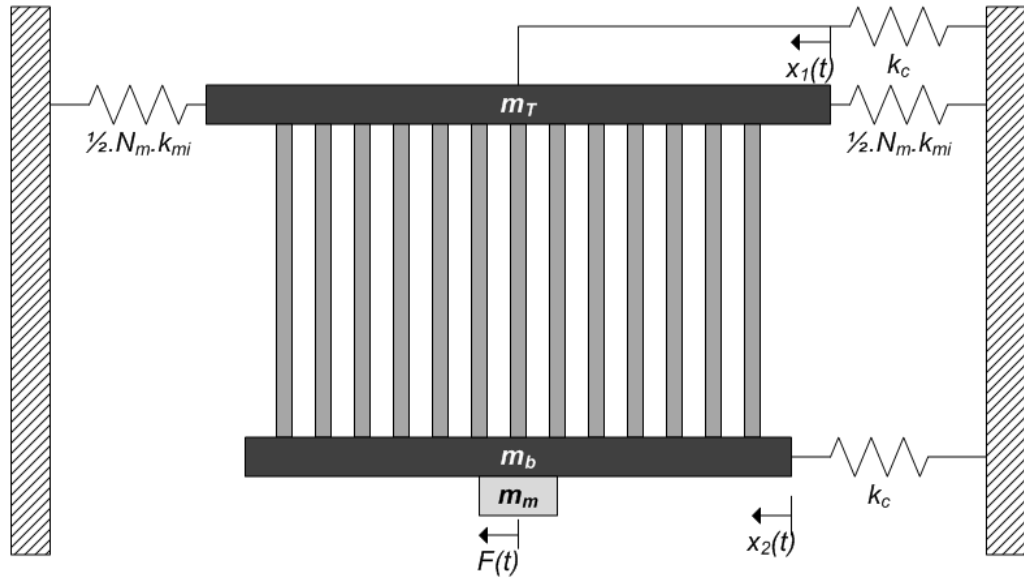


Figure 32: Schematic view of the soft rubber-mounted heat exchanger from the side

As the vibration model that was used did not have flow over the plates that could initiate vibration, an electric vibrating motor, with mass m_m , was connected to the bottom structure, causing an oscillating force ($F(t)$) in the plates in the same way that flow-induced vibrations were excited in the plates. The oscillating force can be described by Equation (5.1).

$$F(t) = F_0 \sin(\omega t) \tag{5.1}$$

The characteristics of the oscillating force $F(t)$ included the amplitude of the oscillating force (F_0) and the frequency of the oscillation (ω).

In the cases where soft rubber mounts were used to mount the heat exchanger to the frame, the stiffness of the mounting system (k_m) was determined by the combined stiffness of all the individual soft rubber mounts (k_{mi}) and the number of mounts (N_m) in the mounting system, as can be seen from Equation (5.2).

$$k_m = N_m \cdot k_{mi} \tag{5.2}$$

The rest of the stiffness evident in Figure 32 was the stiffness of the pipe compensator used to isolate the top header when the mounting system was used (k_{ci}).

The model indicated in Figure 32 can be simplified by modelling all the dimpled plates as one cantilevered beam, with a stiffness of k_p combined with the equivalent masses on the two edges of the beam. The equivalent mass of the plates connected to the bottom structure, while vibrating, was modelled by m_e and was moved to the bottom structure. The balance of the mass (m_r) was moved onto the top structure, with m_r defined by the mass of the plates (m_p) and the effective mass of the vibration (m_e) in Equation (5.3).

$$m_r = m_p - m_e \quad (5.3)$$

With these assumptions the model was simplified to the model illustrated by Figure 33.

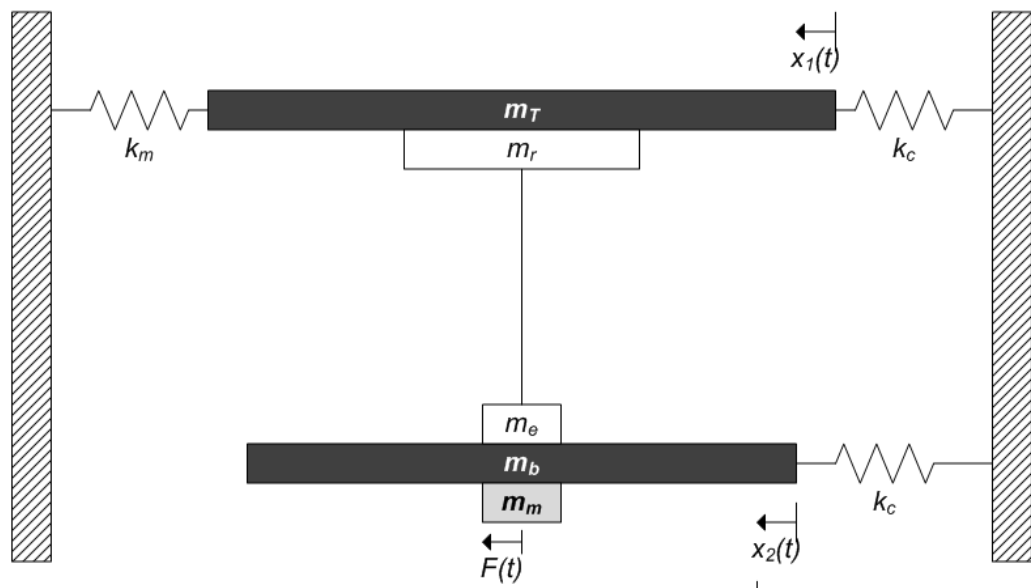


Figure 33: Replacement of plates with cantilever beam

The cantilevered beam with stiffness of k_p could further be substituted with a spring in the horizontal direction and the same stiffness.

As the plates and the bottom structure of the heat exchanger model could not be detached from each other without affecting the stiffness of the plate pack (k_p). The equivalent mass of the plate pack vibrated in conjunction with the bottom structure (m_e) and the mass of the bottom

structure itself, was combined to form mass m_B . The formula used for the calculation of m_B can be seen in Equation (5.4).

$$m_B = m_b + m_e \quad (5.4)$$

With these assumptions, the model was simplified to the model illustrated in Figure 34.

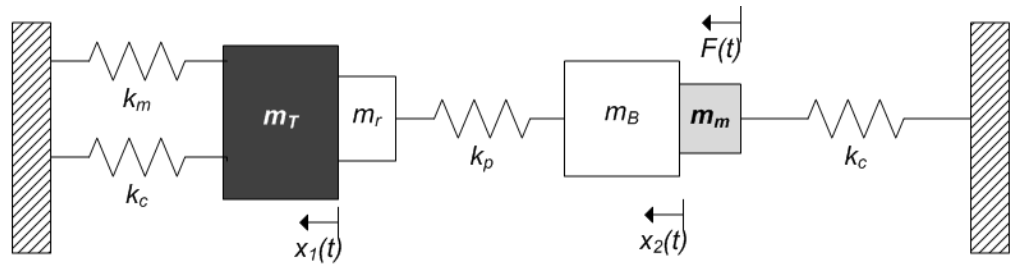


Figure 34: Replacement of cantilever beam with spring element

The mass of the top and bottom frame could be incorporated in the other mass elements connected to them, to provide only two masses for the analysis. The mass m_1 was the sum of all the mass elements moving together with the top of the heat exchanger, as defined by Equation (5.5).

$$m_1 = m_T + m_r \quad (5.5)$$

In the same way the mass m_2 was defined as the sum of the masses moving with the bottom structure of the heat exchanger, as indicated by Equation (5.6). This mass was comprised of the mass of the bottom frame (m_B) defined in Equation (5.4) and the mass of the electric vibrating motor (m_m).

$$m_2 = m_B + m_m \quad (5.6)$$

The value for the effective mass of the plate pack (m_e) had to be determined from the natural frequency of the first mode shape of the assembled system, together with the measured response of the system under a forcing frequency.

To ensure that only the stiffness of the plate pack and the resulting mode shape had an effect, the model was bolted tightly to the frame and the compensators removed.

The natural frequency measured from a bump test could then be defined as in Equation (5.7).

$$\omega_n = \sqrt{\frac{k_p}{m_1}} \quad (5.7)$$

This equation can be transformed to make k_p the object of the equation as illustrated in Equation (5.8).

$$k_p = \omega_n^2 \cdot m_1 \quad (5.8)$$

The forced vibration of the system can be described as defined in Equation (5.9).

$$m_1 \ddot{X} + k_p X = F_0 \quad (5.9)$$

The displacement X can be transformed into \ddot{X} by using the frequency of the exciting vibration (ω) and Equation (5.10).

$$X = \frac{\ddot{X}}{-\omega^2} \quad (5.10)$$

The exciting force, F_0 , is a function of the unbalanced mass (m_u), the radius of rotation (r) and the forcing frequency (ω), as can be seen in Equation (5.11).

$$F_0 = m_u r \cdot \omega^2 \quad (5.11)$$

By substituting Equations (5.8), (5.10) and (5.11) into Equation (5.9), the mass, m_1 , can be calculated as illustrated in Equation (5.12).

$$m_1 = \frac{m_u r \cdot \omega^2}{\ddot{X} \left(1 - \frac{\omega_n^2}{\omega^2} \right)} \quad (5.12)$$

This value (m_1) forms the basis of the characterisation of the other masses and respective stiffness. The characterisation of the different parameters will be discussed further in Section 7.

As with the masses, the individual stiffness were also grouped together according to which points they were connected.

The stiffness k_1 was defined as the stiffness between the heat exchanger frame and the column. This stiffness is controlled by the way in which the heat exchanger is mounted to the frame. To differentiate between two different mounting cases, the stiffness, k_1 , becomes k_{1M} for the soft rubber-mounted system ($k_1 = k_{1M}$) or k_{1R} when the heat exchanger is mounted with stiff steel bolts directly to the frame ($k_1 = k_{1R}$).

In the case where a soft rubber mounting system had been used, this stiffness became k_{1M} , as defined by Equation (5.13) This stiffness was controlled by characteristics of the mounts (k_m), as defined in Equation (5.2) and the additional spring elements between the top frame and the reference structure – in this case the top compensator (k_{ci}).

$$k_{1M} = k_m + k_{ci} \quad (5.13)$$

This mounting system (mounts and compensators) was only used in the configuration where the top of the model was connected with soft rubber mounts to the structure.

In the current design (used in the industry), the heat exchangers were bolted tightly to the column (assumed to be a rigid frame and reference point); therefore, the stiffness was relatively very large.

An assumption was made that the mounting was sufficiently stiffer than the plate pack, to seem almost rigid in comparison with the mounted case as described in Equation (5.14).

$$k_{1R} = 1000 \cdot k_2 \quad (5.14)$$

The other stiffness (k_2 and k_3) were exactly the same for both the current stiff steel mounted case and the proposed soft rubber-mounted case.

The stiffness between the top and bottom structure (k_2) was only determined by the characteristics of the plate pack when deflected normal to the plate surface (k_p) as defined by Equation (5.15).

$$k_2 = k_p \quad (5.15)$$

The stiffness between the heat exchanger and the column (reference point) was determined by the characteristics of the compensator (k_c) that connected the bottom header to the frame in order to isolate the bottom header, as defined by Equation (5.16).

$$k_3 = k_c \quad (5.16)$$

These assumptions simplified the model illustrated in Figure 34 to the final two DOF model illustrated in Figure 35. As can be seen from the figure, no damping elements were taken into account.

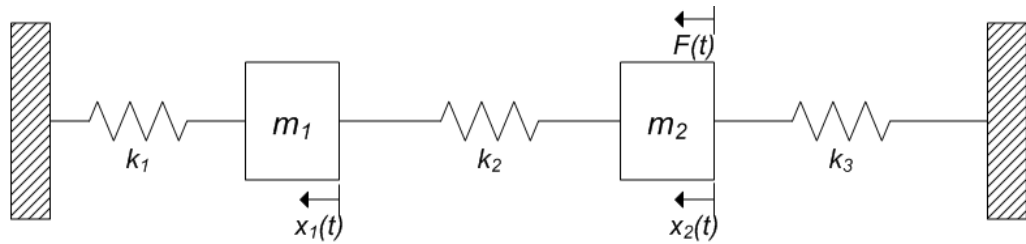


Figure 35: Final 2 DOF model for heat exchanger

This heat exchanger model could be modelled mathematically by using matrices, creating a two Degrees of Freedom (DOF) mathematical model that could be analysed by using matrix algebra.

The mass matrix ($[M]$) combined all the mass elements of the system, defined by Equation (5.17).

$$[M] = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (5.17)$$

In the same way, the stiffness matrix ($[K]$) of the system is defined by Equation (5.18).

$$[K] = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \quad (5.18)$$

In the case where a soft rubber mounting system was used, the value of k_1 became k_{1M} , as defined by Equation (5.13). The resulting stiffness matrix ($[K]_M$) could therefore also be defined by Equation (5.19).

$$[K]_M = \begin{bmatrix} k_{1M} + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (5.19)$$

In the case where the top structure of the model was mounted stiffly with steel bolts to the frame, the stiffness value k_1 became k_{1R} , as defined by Equation (5.14). The resulting stiffness matrix ($[K]_R$) is defined by Equation (5.20).

$$[K]_R = \begin{bmatrix} k_{1R} + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (5.20)$$

The mass matrix and specified stiffness matrices were enough to specify the system for a two DOF model when damping is ignored for both the mounting cases.

5.1. Two DOF model without damping

Negating the damping up to this point made it possible to analyse the system mathematically in the frequency domain, without the need for time-steps in the simulation. This assumption greatly reduced the computing time needed for the model and made it possible to solve the equation for the whole range of frequencies in one run.

According to Rao, the equation of motion could be written in matrix form by Equation (5.21) (Rao, 2004).

$$-[M] \cdot \omega^2 \cdot \mathbf{x} + [K] \cdot \mathbf{x} = \mathbf{F} \quad (5.21)$$

This equation was in a linear form that could be solved with matrix algebra to determine the displacement vector (\mathbf{X}), consisting of the displacement of the top and bottom structure (x_1 and x_2).

After the displacement had been determined, the force (F_k) in a spring element with an arbitrary stiffness (k) and relative displacement (Δx) is defined by Equation (5.22).

$$F_k = k \cdot \Delta x \quad (5.22)$$

In the case of the spring element k_1 the relative displacement over the element was x_1 , therefore, the force could be determined by Equation (5.23)

$$F_{k1} = k_1 \cdot x_1 \quad (5.23)$$

This equation could be implemented for both mounting cases by substituting the general stiffness k_1 with either k_{1M} from Equation (5.13) or k_{1R} from Equation (5.14).

In the case of the second spring, with a stiffness of k_2 , the relative displacement (Δx) was the resultant displacement ($x_2 - x_1$) between the two masses; therefore, the force could be calculated, using Equation (5.24).

$$F_{k2} = k_2 \cdot (x_2 - x_1) \quad (5.24)$$

The force in the final spring element, with a stiffness of k_3 , could be determined in the same way as the other two spring elements. The resultant displacement for the element was equal to the displacement of the bottom mass (x_2) relative to the reference frame. According to this argument, the force could be determined by using Equation (5.25).

$$F_{k3} = k_3 \cdot x_2 \quad (5.25)$$

As the forces, whether positive or negative, will generate the exact same stresses in the plates, the decision was taken to ignore the direction of the force and only compare the numerical size of the forces and ignoring the direction.

5.2. Two DOF model with damping

The proposed concept for isolating the model from the frame was the use of soft rubber mounts and compensators. As rubber mounts normally have a significant amount of damping, the damping of the mounts and compensators could not be neglected.

To incorporate damping into the previous model, the schematic drawing illustrated in Figure 34 had to be adapted to include the damping inherent to any elastic element, modifying the model to the one illustrated in Figure 36.

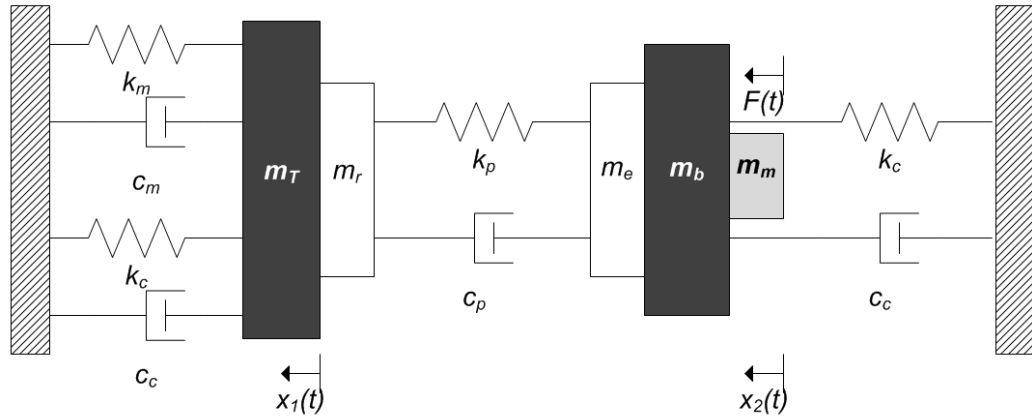


Figure 36: Schematic model of heat exchanger with damping

The value c_m in Figure 36 was the combined damping coefficient of all the mounts. This value was determined by the number of mounts used (N_m) and the damping coefficient of the individual mounts (c_{mi}), as defined by Equation (5.26).

$$c_m = N_m \cdot c_{mi} \tag{5.26}$$

For the case of the two DOF model without damping, the model was simplified by grouping together all the stiffness and mass elements that interacted between the same two points. This same process was repeated on the stiffness and mass elements in the two DOF model with damping, illustrated in Figure 36.

The damping coefficient (c_1) was defined as the damping of the elastic elements constituting the mounting system between the top structure and the column, if installed.

In the current industrial configuration that was used for the column top condensers, the top structure was stiffly connected with steel bolts to the column. Because the relative motion between the frame and the top

structure was assumed to be negligible, the relevant damping coefficient (c_{1R}) was assumed to be zero, as described in Equation (5.27).

$$c_{1R} = 0 \quad (5.27)$$

For the alternative case, where the heat exchanger was mounted on a soft rubber mounting system, the damping coefficient had to be calculated. The combined damping value (c_{1M}) for the mounted system is influenced by the characteristics of the compensator (c_{ci}) and the total damping coefficients of all the mounts (c_m) together, as described by Equation (5.28).

$$c_{1M} = c_m + c_{ci} \quad (5.28)$$

The elements between the top structure of the heat exchanger and the bottom structure were modelled together to become a combined damping element (c_2), the value of c_2 was only affected by the characteristics of plate pack, namely c_p as defined by Equation (5.29).

$$c_2 = c_p \quad (5.29)$$

The damping element located between the bottom structure of the heat exchanger and the column is defined as c_3 . This damping coefficient was only affected by the damping inherent to the compensator (c_{ci}) connecting the bottom of the heat exchanger to the column as described by Equation (5.30).

$$c_3 = c_{ci} \quad (5.30)$$

All these assumptions, together with the assumptions stated for the two DOF model without damping, simplify the model illustrated in Figure 36 to the final two DOF model that takes damping into account, illustrated in Figure 37.

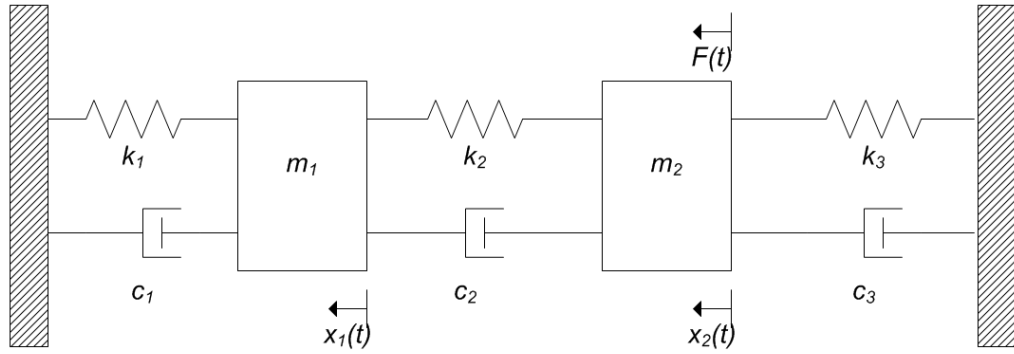


Figure 37: 2 DOF model with damping for heat exchanger

With the addition of damping to the two DOF model, the equations of motion changed to incorporate the damping matrix (Rao, 2004). The resulting equation describing the system can be seen in Equation (5.31).

$$-[M] \cdot \ddot{\mathbf{x}}(t) + [C] \cdot \dot{\mathbf{x}}(t) + [K] \cdot \mathbf{x}(t) = \mathbf{F}(t) \quad (5.31)$$

With $[M]$ and $[K]$ defined in Equations (5.17) and (5.18), while the damping matrix is defined by Equation (5.32).

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad (5.32)$$

In the same way as the stiffness matrix, the damping ratio of the element between the frame and the top of the heat exchanger (c_1) varied between the case where the model was connected with stiff steel bolts to the frame and when the heat exchanger was mounted with soft rubber mounts.

In the case where a soft rubber mounting system was used, the value of c_1 became c_{1M} , as defined by Equation (5.28). The resulting stiffness matrix ($[C]_M$) could therefore also be defined by Equation (5.33).

$$[C]_M = \begin{bmatrix} c_{1M} + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad (5.33)$$

In the case where the top structure of the model was mounted with stiff steel bolts, the stiffness value c_1 became c_{1R} , as defined by

Equation (5.27). The resulting stiffness matrix ($[K]_R$) is defined by Equation (5.34).

$$[C]_R = \begin{bmatrix} c_{1R} + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad (5.34)$$

The mass matrix and specified stiffness matrices were enough to specify the system for a two DOF model when damping is ignored for both the mounting cases.

Once damping had been taken into account in the modelling of the system, calculation was done in the time domain, using a numerical integration technique such as Runga-Kutta over small time steps (Δt).

To determine a single, effective value for the time-dependent parameters such as $\mathbf{x}(t)$ (consisting of x_1 and x_2) and $\mathbf{F}(t)$, the route-mean-square (RMS) average value of the parameter was calculated over a time interval after the transient behaviour had dissipated.

From signal processing theory, the effective displacement value of a time-dependant wave ($x(t)$), that would transfer the same energy as the original wave, could be determined by the Route Mean Square (RMS) approach (Smith & Dorf, 1992). This approach was implemented in Equation (5.35) on an arbitrary variable x .

$$x_{RMS} = \sqrt{\frac{\int_0^T x^2 dt}{T}} \quad (5.35)$$

If this equation were rewritten in discrete terms, this equation becomes Equation (5.36) (Smith & Dorf, 1992).

$$x_{RMS} = \sqrt{\frac{\sum x^2 \cdot \Delta t}{T}} \quad (5.36)$$

The discrete equation incorporates the time step (Δt) in the simulation and the total time (T) over which the effective value is calculated.

According to Equation (5.22), the effective force of a time-varying displacement due to the stiffness of the element can be determined by using Equation (5.37).

$$F_{kRMS} = \sqrt{\frac{\sum (k \cdot \Delta x)^2 \cdot \Delta t}{T}} \quad (5.37)$$

Due to the effect of damping, there was an additional force due to the velocity of the masses relative to each other ($\Delta \dot{x}$). This force is defined in terms of the relative velocity by Equation (5.38).

$$F_c = c \cdot \Delta \dot{x} \quad (5.38)$$

In the same way as in Equation (5.37), the effective force due to the damping for a time-varying velocity could be estimated by Equation (5.39)

$$F_{cRMS} = \sqrt{\frac{\sum (c \cdot \Delta \dot{x})^2 \cdot \Delta t}{T}} \quad (5.39)$$

As the damping and the stiffness were included in the one elastic element, the forces due to the damping (F_{cRMS}) and the forces due to the stiffness (F_{kRMS}) reacted on the same element.

Due to difference in the phase angle between the force induced by the stiffness (F_{kRMS}) and the force induced by the damping (F_{cRMS}), the two values could not be simply added to each other. The resultant force (F_{TRMS}), which was the total effective force in an elastic element with both stiffness and damping values, is defined by Equation (5.40) (Rao, 2004).

$$F_{TRMS} = \sqrt{F_{kRMS}^2 + F_{cRMS}^2} \quad (5.40)$$

This approach had to be the same for all the individual elastic elements; therefore, for all the elastic elements, the forces can be determined by Equations (5.41) to (5.43).

$$F_{k1RMS} = \sqrt{\frac{\sum (k_1 \cdot x_1)^2 \cdot \Delta t}{T}} \quad (5.41)$$

$$F_{k2RMS} = \sqrt{\frac{\sum (k_2 \cdot (x_2 - x_1))^2 \cdot \Delta t}{T}} \quad (5.42)$$

$$F_{k3RMS} = \sqrt{\frac{\sum (k_3 \cdot x_2)^2 \cdot \Delta t}{T}} \quad (5.43)$$

With k_1 , k_2 and k_3 being the stiffness of the stiffness elements, and x_1 and x_2 being the displacement of m_1 or m_2 at a certain time step.

In the case of the damping elements, the equivalent forces can be determined with Equations (5.44) to (5.46).

$$F_{c1RMS} = \sqrt{\frac{\sum (c_1 \cdot \dot{x}_1)^2 \cdot \Delta t}{T}} \quad (5.44)$$

$$F_{c1RMS} = \sqrt{\frac{\sum (c_2 \cdot (\dot{x}_2 - \dot{x}_1))^2 \cdot \Delta t}{T}} \quad (5.45)$$

$$F_{c1RMS} = \sqrt{\frac{\sum (c_3 \cdot \dot{x}_2)^2 \cdot \Delta t}{T}} \quad (5.46)$$

The total effective force can, therefore, be determined for each of the elements by Equations (5.47) to (5.49).

$$F_{1RMS} = \sqrt{F_{k1RMS}^2 + F_{c1RMS}^2} \quad (5.47)$$

$$F_{2RMS} = \sqrt{F_{k2RMS}^2 + F_{c2RMS}^2} \quad (5.48)$$

$$F_{3RMS} = \sqrt{F_{k3RMS}^2 + F_{c3RMS}^2} \quad (5.49)$$

The respective RMS values for the resultant forces once determined could be compared for different combinations of stiffness, damping and forcing frequency to theoretically predict the effect of the change on the forces in the different elements.

5.3. Conclusion

Two mathematical models were constructed to theoretically predict the effect of the changing of the mounting stiffness on the dynamic forces in the elements of the dimple plate heat exchanger.

Both models were simplified by combining all the elements that experienced the same acceleration (masses) or displacements (stiffness elements) and velocities (damping elements) together.

The first model disregarded the effect of damping in the system in order to analyse the model in the frequency domain over an entire frequency range. This would be helpful in the initial selection of a mounting system, as less computing time is required and the predicted effects could be seen for the whole range of frequencies.

The second model included the damping of the elements in the model, thereby theoretically becoming more accurate. The model, however, required more computing resources and computing time. This model, if proven accurate) can be used in the detail design of the mounting system, especially to ensure that the new natural frequencies do not cause large dynamic forces in the internals.