CONSERVATION LAWS AND EXACT SOLUTIONS OF KUDRYASHOV-SINELSHCHIKOV EQUATION AND BENNEY-LUKE EQUATION

by

SIVENATHI OSCAR MBUSI (23915242)

Dissertation submitted in fulfilment for the degree of Master of Science in Applied Mathematics in the Department of Mathematical Sciences in the Faculty of Agriculture, Science and Technology at North-West University, Mafikeng Campus

April 2017

Supervisor : Professor B Muatjetjeja

Co-Supervisor : Doctor A R Adem

Contents

Declaration

I SIVENATHI OSCAR MBUSI student number 23915242, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed:..

Mr S. O. MBUSI

Date: 11/10/2017

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF B. MUATJETJEJA

Date: 11/10/2017

Signed:..

DR A. R. ADEM

Date: 11-10-2017

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 2

SO Mbusi, B Muatjetjeja, AR Adem, Conservation laws and exact solutions for a generalized Kudryashov-Sinelshchikov equation. Submitted for publication to Differential Equations and Dynamical System.

Chapter 3

B Muatjetjeja, SO Mbusi, AR Adem, Conservation laws and exact solutions for a generalized Benney-Luke equation. Submitted for publication to Waves in Random and Complex Media.

Dedication

To my loving mother, brother, sister and everyone who showed me support throughout my studies.

Acknowledgements

I would like to thank my supervisor, Professor B. Muatjetjeja and my co-supervisor Doctor A.R. Adem for their guidance and encouragement in compiling this project. My further acknowledgement goes to North-West University postgraduate bursary scheme.

Finally, I would also like to thank DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) and the North-West University, Mafikeng Campus for financial support.

Abstract

In this dissertation we study two nonlinear partial differential equations namely; the Kudryashov-Sinelshchikov equation and the Benney-Luke equation. We employ the multiplier method to find conservation laws and Kudryashov method to obtain exact solutions for the generalized Kudryashov-Sinelshchikov equation. We derive the Noether symmetries of a generalized Benney-Luke equation. Thereafter, we construct the associated conserved vectors. In addition, we search for exact solutions for the generalized Benney-Luke equation via the extended tanh method.

Introduction

In recent years nonlinear partial differential equations (NLPDEs) have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics and geochemistry. Thus, it is important to investigate the exact solutions of NLPDEs. Finding solutions of such equations is a difficult task, only in certain special cases can one write down the solutions explicitly.

There is no doubt that conservation laws play a remarkable role in the study of differential equations. The mathematical idea of conservation laws comes from the formulation of well known physical conserved quantities such as mass, momentum and energy. Finding the conservation laws of differential equations is often the initiating step towards finding the exact solutions. Thus, it is essential to study conservation laws of partial differential equations.

In the last few decades, a variety of effective methods for finding exact solutions, such as homogeneous balance method [1], ansatz method [2, 3], variable separation approach [4], inverse scattering transform method [5], Bäcklund transformation [6], Darboux transformation [7] and Hirota's bilinear method [8] were successfully applied to NLPDEs.

The Kudryashov method was one of the methods for finding exact solutions of nonlinear partial differential equations [9]. Steudel [10] introduced a different approach of constructing conservation laws, that involves writing a conserved vector in a characteristic form, where the characteristics are the multipliers of the differential equation.

In this dissertation we study the generalized Kudryashov-Sinelshchikov equation and the Benney-Luke equation. Firstly, we study the generalized Kudryashov-Sinelshchikov equation that is given by

$$u_t + auu_x + bu_{xxx} + cuu_{xxx} + cu_xu_{xx} + du_xu_{xx} = 0, \tag{1}$$

where $u(t, x)$ is a real valued function and a, b, c and d are arbitrary constants. Equation (1) models the pressure waves in a mixture of a liquid and gas bubbles by taking into account the viscosity of the liquid and the heat transfer. When $b = 1$ and $c = -1$ in equation (1), Kudryashov and Sinelshchikov investigated its peaked solitons and certain other properties in liquid with gas bubbles. Tu et al. [11] studied the generalized Kudryashov-Sinelshchikov equation (1) for its Lie point symmetries.

Lastly, we consider the Benney-Luke equation [12]

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_tu_{xx} + 2u_xu_{xt} = 0, \tag{2}$$

where $u = u(t, x)$ denotes the wave profile and the variables t and x represent time and space respectively. This equation is an approximation of the full water wave equations and formally suitable for describing two-way water wave propagation in presence of surface tension. The positive parameters α and β are related to the inverse bond number $\alpha - \beta = \gamma - 1/3$, which captures the effects of surface tension and gravity forces.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions, theorems and corollaries concerning the Noether theorem and multiplier method are presented.

In Chapter two, the multiplier method is used to construct conservation laws for a generalized Kudryashov-Sinelshchikov equation. Moreover, exact solutions of the generalized Kudryashov-Sinelshchikov equation are obtained with the aid of the

Kudryashov method [13].

In Chapter three, the conservation laws for the Benney-Luke equation are obtained using Noether's theorem [14]. Thereafter, we construct the exact solutions for the Benney-Luke equation using the extended tanh method [15].

In Chapter four, we discuss and conclude what we have done in this dissertation.

A bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter, we present some basic methods on how to obtain conservation laws of differential equations and methods of obtaining exact solutions of differential equations, which will be utilized in this dissertation.

1.1 Fundamental relation of multiplier method

In this section, we present the notation that will be used to construct conservation laws for (1) by the multiplier method [16].

Consider a kth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{1.1}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$ respectively, with the *total derivative operator* with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \tag{1.2}$$

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \qquad \alpha = 1, \dots, m. \tag{1.3}$$

The n-tuple vector $\mathbf{T} = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a *conserved vector* of (1.1) if T^i satisfies

$$D_i T^i|_{(1.1)} = 0. \tag{1.4}$$

The equation (1.4) defines a *local conservation law* of system (1.1).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.5}$$

holds identically. Here we will consider multipliers of the zeroth order,
i.e., $\Lambda_\alpha = \Lambda_\alpha(t, x, u)$. The right hand side of (1.5) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.6}$$

1.2 Fundamental relationship concerning the Noether theorem

In this section we briefly present the notation and pertinent results that will be used in this research. For details the reader is referred to [14,17–22]. Consider the system of qth order partial differential equations

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(q)}) = 0, \quad \alpha = 1, 2, ..., m. \tag{1.7}$$

If there exists a function $L(x, u, u_{(1)}, u_{(2)}, ...u_{(s)}) \in \mathcal{A}$ (space of differential functions), $s < q$ such that system (1.7), is equivalent to

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, ..., m, \tag{1.8}$$

then L is called a Lagrangian of (1.7) and (1.8) are the corresponding Euler-Lagrange differential equations.

In (1.8), $\delta/\delta u^\alpha$ is the Euler-Lagrange operator defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} ... D_{i_s} \frac{\delta}{\delta u^\alpha_{i_1 ... i_s}}, \quad \alpha = 1, ..., m. \tag{1.9}$$

Definition 1.1 (Point symmetry) The vector field

$$\mathbf{X} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.10}$$

is said to be a *point symmetry* of the pth-order partial differential equation (1.7), if

$$\mathbf{X}^{[p]}(E_\alpha) = 0, \tag{1.11}$$

whenever $E_\alpha = 0$. This can also be written as

$$\mathbf{X}^{[p]}\, E_\alpha\big|_{E_\alpha=0} = 0, \tag{1.12}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.2 A Lie-Bäcklund operator $\mathbf{X}$ is a Noether symmetry generator associated with a Lagrangian L of (1.8) if there exists a vector $\mathbf{A} = (A^1, ..., A^n)$, $A^i \in \mathcal{A}$, such that

$$\mathbf{X}(L) + LD_i(\xi^i) = D_i(A^i). \tag{1.13}$$

If in (1.13) $A^i = 0$, $i = 1, ..., n$ then $\mathbf{X}$ is referred to as a strict Noether symmetry generator associated with Lagrangian $L \in \mathcal{A}$.

Theorem 1.1 For each Noether symmetry generator $\mathbf{X}$ associated with a given Lagrangian L, there corresponds a vector $\mathbf{T} = (T^1, T^2, ..., T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = N^i L - A^i, \quad i = 1, ..., n, \tag{1.14}$$

which is a conserved vector of the Euler-Lagrange equations (1.8) and the Noether operator associated with $\mathbf{X}$ is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}...D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{i i_1...i_s}}, \quad i = 1, ..., n, \tag{1.15}$$

in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from equation (1.9) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}...D_{j_s}\frac{\delta}{\delta u_{ij_1...j_s}^\alpha}, \quad i=1,...,n, \quad \alpha=1,...,m.$$

In (1.15), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha, \quad \alpha = 1,...,m.$$

The vector (1.14) is a conserved vector of equation (1.7) if T^i satisfies

$$D_i T^i|_{(1.7)} = 0. \tag{1.16}$$

1.3 Conclusion

In this chapter we briefly discussed the multiplier method. In addition, we presented the fundamental relations concerning Noether symmetries and conservation laws.

Chapter 2

Conservation laws and exact solutions for a generalized Kudryashov-Sinelshchikov equation

Kudryashov and Sinelshchikov proposed a nonlinear evolution model given by

$$u_t + \lambda u u_x + u_{xxx} - (u u_{xx})_x - \chi u_x u_{xx} = 0. \quad (2.1)$$

Here λ and χ are arbitrary constants and it models the pressure waves in a mixture of a liquid and gas bubbles by taking into account the viscosity of the liquid and the heat transfer. Kudryashov and Sinelshchikov investigated its peaked solitons and certain other properties in liquid with gas bubbles. Moreover, Ryabov [23] computed exact solutions of equation (2.1). The generalized Kudryashov-Sinelshchikov equation (1) reduces to the Korteweg-de Vries equation [24]

$$u_t + 6 u u_x + u_{xxx} = 0 \quad (2.2)$$

by taking suitable values of the underlying arbitrary constants and it is commonly studied in the context of shallow water waves in fluid dynamics.

In this chapter, we consider the generalized Kudryashov-Sinelshchikov equation [11] given by

$$u_t + auu_x + bu_{xxx} + c\left(uu_{xx}\right)_x + du_x u_{xx} = 0, \tag{2.3}$$

where a, b, c and d are arbitrary constants. We will employ the multiplier method to derive the conservation laws of equation (2.3). The exact solutions of equation (2.3) will be derived by employing the Kudryashov method.

2.1 Conservation laws for a generalized Kudryashov-Sinelshchikov equation (2.3)

In this section we derive the conservation laws for equation (2.3). Here we will consider multipliers of the zeroth order $\Lambda(t, x, u)$ defined by

$$\frac{\delta}{\delta u}[\Lambda(t, x, u)(u_t + auu_x + bu_{xxx} + c\left(uu_{xx}\right)_x + du_x u_{xx})] = 0, \tag{2.4}$$

where the Euler-Langrage Operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} \\ &\quad + D_x D_t\frac{\partial}{\partial u_{xt}} + \cdots , \end{aligned} \tag{2.5}$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots ,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots .$$

Expanding equation (2.4) leads to

$$\left[\frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{xt}} - D_x^3\frac{\partial}{\partial u_{xxx}}\right]$$
$$(\Lambda(u_t + auu_x + bu_{xxx} + c\,(uu_{xx})_x + du_xu_{xx})) = 0,$$
$$\Lambda_u(u_t + auu_x + bu_{xxx} + c\,(uu_{xx})_x + du_xu_{xx}) + au_x\Lambda + cu_{xx}\Lambda - D_t\Lambda - D_x(au\Lambda)$$
$$-D_x(cu_{xx}\Lambda) - D_x(du_{xx}\Lambda) + D_x^2(cu_x\Lambda) + D_x^2(du_x\Lambda) - D_x^3(b\Lambda) - D_x^3(cu\Lambda) = 0.$$

Further expansion of the above equation yields

$$\begin{aligned}
&-\Lambda_t - au\Lambda_x + du_x\Lambda_{xx} + 2du_x^2\Lambda_{xu} + du_{xx}\Lambda_x + du_x^3\Lambda_{uu} + 3du_xu_{xx}\Lambda_u\\
&-b\Lambda_{xxx} - bu_x\Lambda_{xxu} - 2bu_x\Lambda_{xxu} - 2bu_x^2\Lambda_{xuu} - 2bu_{xx}\Lambda_{xu} - bu_x^2\Lambda_{xuu} - bu_x^3\Lambda_{uuu}\\
&-2bu_xu_{xx}\Lambda_{uu} - bu_{xx}\Lambda_{xu} - bu_xu_{xx}\Lambda_{uu} - cu\Lambda_{xxx} - cuu_x\Lambda_{xxu} - cuu_x\Lambda_{xxu}\\
&-cuu_x^2\Lambda_{xuu} - cu_x^2\Lambda_{xu} - cuu_{xx}\Lambda_{xu} - cu_x\Lambda_{xx} - cu_x^2\Lambda_{xu} - cu_{xx}\Lambda_x - cuu_x\Lambda_{xxu}\\
&-cuu_x^2\Lambda_{xuu} - cu_x^2\Lambda_{xu} - cuu_{xx}\Lambda_{xu} - cuu_x^2\Lambda_{xuu} - cuu_x^3\Lambda_{uuu} - 2cuu_xu_{xx}\Lambda_{uu}\\
&-cu_x^2 - cu_x^3\Lambda_{uu} - cuu_{xx}\Lambda_{xu} - cuu_xu_{xx}\Lambda_{uu} - cu_x\Lambda_{xx} - cu_x^3\Lambda_{uu} - 2cu_xu_{xx}\Lambda_u\\
&-cu_{xx}\Lambda_x - cu_xu_{xx}\Lambda_u = 0. \qquad (2.6)
\end{aligned}$$

Since Λ depends only on t, x and u, the coefficients of the like derivatives of u can be equated to zero to yield the following system of over determined linear partial differential equations:

$$u_x^3 \;:\; d\Lambda_{uu} - b\Lambda_{uuu} - cu\Lambda_{uuu} - 2c\Lambda_{uu} = 0, \qquad (2.7)$$
$$u_x^2 \;:\; 2d\Lambda_{xu} - 3b\Lambda_{xuu} - 3cu\Lambda_{xuu} - 4c\Lambda_{xu} = 0, \qquad (2.8)$$
$$u_x \;:\; d\Lambda_{xx} - 3b\Lambda_{xuu} - 3cu\Lambda_{xuu} - 2c\Lambda_{xx} = 0, \qquad (2.9)$$
$$u_{xx} \;:\; d\Lambda_x - 3b\Lambda_{xu} - 3cu\Lambda_{xu} - 2c\Lambda_x = 0, \qquad (2.10)$$
$$u_xu_{xx} \;:\; d\Lambda_u - b\Lambda_{uu} - cu\Lambda_{uu} - c\Lambda_u = 0, \qquad (2.11)$$
$$1 \;:\; \Lambda_t + au\Lambda_x + b\Lambda_{xxx} + cu\Lambda_{xxx} = 0. \qquad (2.12)$$

Solving the above system of linear partial differential equations for Λ prompts the following three cases:

Case 1. a, b, c, d arbitrary but not in the form contained in Case 2 and 3.

In this case, we integrate equation (2.7) with respect to u and obtain

$$\Lambda(t,x,u) = \frac{A(t,x)(b+cu)^{\frac{d}{c}}}{d(d-c)} + B(t,x)u + E(t,x), \tag{2.13}$$

where $(d-c) \neq 0$, $A(t,x), B(t,x)$ and $E(t,x)$ are arbitrary functions of t and x. Inserting equation (2.13) into equation (2.8) and solving the resulting equation yields

$$A_x(b+cu)^{\frac{d}{c}-1}(-d-c) + (2d-4c)(d-c)B_x = 0. \tag{2.14}$$

Splitting the above equation on $(b+cu)^{\frac{d}{c}-1}$ yields

$$(b+cu)^{\frac{d}{c}-1} \ : \ (-d-c)A_x = 0, \tag{2.15}$$

$$1 \ : \ (2d-4c)(d-c)B_x = 0. \tag{2.16}$$

Integrating equation (2.15) with respect to x gives

$$A(t,x) = F(t), \tag{2.17}$$

where $d \neq -c$ and $F(t)$ is an arbitrary function of t. Integrating equation (2.16) with respect to x, we obtain

$$B(t,x) = Z(t), \tag{2.18}$$

where $d \neq 2c$ and $Z(t)$ is an arbitrary function of t. We now substitute equation (2.17) and (2.18) into equation (2.13) and we get

$$\Lambda(t,x,u) = \frac{F(t)(b+cu)^{\frac{d}{c}}}{d(d-c)} + Z(t)u + E(t,x). \tag{2.19}$$

By substituting equation (2.19) into equation (2.9), one obtains

$$(d-2c)E_{xx} = 0. \tag{2.20}$$

Integrating the above equation twice with respect to x, we get

$$E(t,x) = h(t)x + p(t), \tag{2.21}$$

where $h(t)$ and $p(t)$ are arbitrary functions of t. Inserting equation (2.21) into equation (2.19) yields

$$\Lambda(t,x,u) = \frac{F(t)(b+cu)^{\frac{d}{c}}}{d(d-c)} + Z(t)u + h(t)x + p(t). \tag{2.22}$$

Now substituting equation (2.22) into equation (2.10), we obtain

$$h(t) = 0. \tag{2.23}$$

Therefore, equation (2.22) reduces to

$$\Lambda(t, x, u) = \frac{F(t)(b + cu)^{\frac{d}{c}}}{d(d - c)} + Z(t)u + p(t). \tag{2.24}$$

Inserting equation (2.24) into (2.12) yields

$$\frac{F'(t)(b + cu)^{\frac{d}{c}}}{d(d - c)} + Z'(t)u + p'(t) = 0. \tag{2.25}$$

Separating the above equation on powers of u, yields

$$(b + cu)^{\frac{d}{c}} \; : \; F'(t) = 0, \tag{2.26}$$

$$u \; : \; Z'(t) = 0, \tag{2.27}$$

$$1 \; : \; p'(t) = 0. \tag{2.28}$$

By integrating equations (2.26), (2.27) and (2.28) with respect to t, we obtain

$$F(t) = R_1, \quad Z(t) = R_2, \quad p(t) = R_3, \tag{2.29}$$

where R_1, R_2 and R_3 are arbitrary constants. Therefore equation (2.24) becomes

$$\Lambda(t, x, u) = \frac{R_1(b + cu)^{\frac{d}{c}}}{d(d - c)} + R_2 u + R_3. \tag{2.30}$$

Substituting equation (2.30) into (2.11) and solving the resulting equation yields

$$(d - c)R_2 = 0. \tag{2.31}$$

Since $(d - c) \neq 0$, we have $R_2 = 0$. Thus

$$\Lambda(t, x, u) = \frac{R_1(b + cu)^{\frac{d}{c}}}{d(d - c)} + R_3. \tag{2.32}$$

Therefore, equation (2.32) yields the following multiplier:

$$\Lambda(t, x, u) = k_1 + k_2(b + cu)^{\frac{d}{c}}, \tag{2.33}$$

where $k_2 = R_1/d(d - c)$ and $k_1 = R_3$.

Integrating equation (2.43) with respect to u_t, we obtain

$$J(t,x,u,u_t,u_x) = I_{u_x}(t,x,u,u_x)u_t + M(t,x,u,u_x), \tag{2.44}$$

where $M(t,x,u,u_x)$ is an arbitrary function of t, x, u and u_x. Therefore equation (2.42) becomes

$$\begin{aligned} T^2(t,x,u,u_t,u_x,u_{xx}) &= k_1(b+cu)u_{xx} + k_2(b+cu)^{\frac{d}{c}+1}u_{xx} + I_{u_x}(t,x,u,u_x)u_t \\ &+ M(t,x,u,u_x). \end{aligned} \tag{2.45}$$

Substituting equations (2.41) and (2.45) into (2.40) yields

$$\begin{aligned} &I_t(t,x,u,u_x) + I_u(t,x,u,u_x)u_t + I_{xu_x}(t,x,u,u_x)u_t + M_x(t,x,u,u_x) \\ &+u_x\left[k_1cu_{xx} + (d+c)k_2(b+cu)^{\frac{d}{c}}u_{xx} + I_{uu_x}(t,x,u,u_x)u_t + M_u(t,x,u,u_x)\right] \\ &+u_{xx}\left[I_{u_xu_x}(t,x,u,u_x)u_t + M_{u_x}(t,x,u,u_x)\right] = [k_1 + k_2(b+cu)^{\frac{d}{c}}] \\ &\times(u_t + auu_x + cu_xu_{xx} + du_xu_{xx}). \end{aligned} \tag{2.46}$$

Separating the above equation on powers of u_{xx}, gives the following:

$$\begin{aligned} u_{xx} \ &: \ k_1cu_x + k_2(d+c)(b+cu)^{\frac{d}{c}}u_x + I_{u_xu_x}(t,x,u,u_x)u_t + M_{u_x}(t,x,u,u_x) \\ &= (k_1 + k_2(b+cu)^{\frac{d}{c}})[(d+c)u_x], \qquad (2.47) \\ 1 \ &: \ I_t(t,x,u,u_x) + I_u(t,x,u,u_x)u_t + I_{xu_x}(t,x,u,u_x)u_t + M_x(t,x,u,u_x) \\ &I_{uu_x}(t,x,u,u_x)u_t + M_u(t,x,u,u_x) = k_1u_t + k_1auu_x + k_2(b+cu)^{\frac{d}{c}}u_t \\ &+k_2(b+cu)^{\frac{d}{c}}auu_x. \qquad (2.48) \end{aligned}$$

Equation (2.47) simplifies to

$$I_{u_xu_x}(t,x,u,u_x)u_t + M_{u_x}(t,x,u,u_x) = k_1du_x. \tag{2.49}$$

Splitting the above equation on powers of u_t, we obtain

$$\begin{aligned} u_t \ &: \ I_{u_xu_x}(t,x,u,u_x) = 0, \qquad (2.50) \\ 1 \ &: \ M_{u_x}(t,x,u,u_x) = k_1du_x. \qquad (2.51) \end{aligned}$$

Integrating equation (2.50) twice with respect to u_x gives

$$I(t,x,u,u_x) = N(t,x,u)u_x + Q(t,x,u), \tag{2.52}$$

We now apply equation (1.5) to construct the conservation laws of equation (2.33)

$$\Lambda_\alpha E_\alpha = D_i T^i. \tag{2.34}$$

From equation (2.34) we have

$$\begin{aligned}
&[k_1 + k_2(b+cu)^{\frac{d}{c}}](u_t + auu_x + bu_{xxx} + c\,(uu_{xx})_x + du_x u_{xx}) \\
&= \left[\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{txx}\frac{\partial}{\partial u_{xx}}\right] T^1(t,x,u,u_t,u_x,u_{xx}) \\
&+ \left[\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}}\right] T^2(t,x,u,u_t,u_x,u_{xx}),
\end{aligned}$$

which gives

$$\begin{aligned}
&[k_1 + k_2(b+cu)^{\frac{d}{c}}](u_t + auu_x + bu_{xxx} + c\,(uu_{xx})_x + du_x u_{xx}) = T^1_t + u_t T^1_u + u_{tx}T^1_{u_x} \\
&+ u_{tt}T^1_{u_t} + u_{txx}T^1_{u_{xx}} + T^2_x + u_x T^2_u + u_{tx}T^2_{u_t} + u_{xx}T^2_{u_x} + u_{xxx}T^2_{u_{xx}}.
\end{aligned} \tag{2.35}$$

Splitting equation (2.35) on u_{tt},u_{xxx},u_{tx} and u_{txx} yields

$$u_{tt} \;:\; T^1_{u_t} = 0, \tag{2.36}$$

$$u_{xxx} \;:\; T^2_{u_{xx}} = [k_1 + k_2(b+cu)^{\frac{d}{c}}](b+cu), \tag{2.37}$$

$$u_{tx} \;:\; T^1_{u_x} + T^2_{u_t} = 0, \tag{2.38}$$

$$u_{txx} \;:\; T^1_{u_{xx}} = 0, \tag{2.39}$$

$$\begin{aligned}
1 \;:\; & T^1_t + u_t T^1_u + T^2_x + u_x T^2_u + u_{xx}T^2_{u_x} = [k_1 + k_2(b+cu)^{\frac{d}{c}}] \\
& \times (u_t + auu_x + cu_x u_{xx} + du_x u_{xx}).
\end{aligned} \tag{2.40}$$

We can now solve the above equations for T^1 and T^2. From equations (2.36) and (2.39), we obtain

$$T^1(t,x,u,u_x) = I(t,x,u,u_x), \tag{2.41}$$

where $I(t,x,u,u_x)$ is an arbitrary function of t,x,u and u_x. Integrating equation (2.37) with respect to u_{xx}, we obtain

$$T^2(t,x,u,u_t,u_x,u_{xx}) = k_1(b+cu)u_{xx} + k_2(b+cu)^{\frac{d}{c}+1}u_{xx} + J(t,x,u,u_t,u_x), \tag{2.42}$$

where $J(t,x,u,u_t,u_x)$ is an arbitrary function of t,x,u,u_t and u_x. Substituting the values of T^1 and T^2 into equation (2.38) gives

$$I_{u_x}(t,x,u,u_x) + J_{u_t}(t,x,u,u_t,u_x) = 0. \tag{2.43}$$

where $N(t,x,u)$ and $Q(t,x,u)$ are arbitrary functions of t, x and u. Integrating equation (2.51) with respect to u_x gives

$$M(t,x,u,u_x) = \frac{1}{2}k_1 d u_x^2 + S(t,x,u), \tag{2.53}$$

where $S(t,x,u)$ is an arbitrary function of t, x and u. Thus we have

$$T^1(t,x,u,u_x) = N(t,x,u)u_x + Q(t,x,u), \tag{2.54}$$

$$\begin{aligned} T^2(t,x,u,u_t,u_x,u_{xx}) &= k_1(b+cu)u_{xx} + k_2(b+cu)^{\frac{d}{c}+1}u_{xx} + N(t,x,u)u_t \\ &+ \frac{1}{2}k_1 d u_x^2 + S(t,x,u). \end{aligned} \tag{2.55}$$

Inserting equations (2.52) and (2.53) into (2.48), we obtain

$$\begin{aligned} &N_t(t,x,u)u_x + Q_t(t,x,u) + N_u(t,x,u)u_x u_t + u_t Q_u(t,x,u) + N_x(t,x,u)u_t \\ &S_x(t,x,u) + N_u(t,x,u)u_x u_t + S_u(t,x,u)u_x = k_1 u_t + k_1 a u u_x + k_2(b+cu)^{\frac{d}{c}}u_t \\ &+k_2(b+cu)^{\frac{d}{c}} a u u_x. \end{aligned} \tag{2.56}$$

Separating the above equation on u_x and u_t yields

$$\begin{aligned} u_t u_x &: \ N_u(t,x,u) = 0, & (2.57)\\ u_t &: \ N_x(t,x,u) + Q_u(t,x,u) = k_1 + k_2(b+cu)^{\frac{d}{c}}, & (2.58)\\ u_x &: \ N_t(t,x,u) + S_u(t,x,u) = k_1 a u + k_2(b+cu)^{\frac{d}{c}} a u, & (2.59)\\ 1 &: \ Q_t(t,x,u) + S_x(t,x,u) = 0. & (2.60) \end{aligned}$$

Integrating equation (2.57) with respect to u gives

$$N(t,x,u) = V(t,x), \tag{2.61}$$

where $V(t,x)$ is an arbitrary function of t and x. Inserting equation (2.61) into (2.58) and integrating with respect to u yields

$$Q(t,x,u) = k_1 u + \frac{k_2(b+cu)^{\frac{d}{c}+1}}{(d+c)} + V_x(t,x)u + Z(t,x), \tag{2.62}$$

where $Z(t,x)$ is an arbitrary function of t and x. Substituting equation (2.61) into (2.59) and integrating with respect to u gives

$$\begin{aligned} S(t,x,u) &= \frac{1}{2}k_1 a u^2 + \frac{k_2 a u(b+cu)^{\frac{d}{c}+1}(d+2c) - a k_2(b+cu)^{\frac{d}{c}+2}}{(d+c)(d+2c)} \\ &+ V_t(t,x)u + W(t,x), \end{aligned} \tag{2.63}$$

where $W(t,x)$ is an arbitrary function of t and x. Substituting the values of Q and S into equation (2.60) yields

$$Z_t(t,x) + W_x(t,x) + 2V_{tx}(t,x)u = 0. \tag{2.64}$$

Splitting the above equation on powers of u, we obtain

$$u \ : \ V_{tx}(t,x) = 0, \tag{2.65}$$

$$1 \ : \ Z_t(t,x) + W_x(t,x) = 0. \tag{2.66}$$

Equation (2.65) simplifies to

$$V(t,x) = \int Y(x)dx + P(t), \tag{2.67}$$

where $P(t)$ and $Y(x)$ are arbitrary functions of t and x respectively. Therefore we have

$$\begin{aligned} T^1(t,x,u,u_x) &= \left[\int Y(x)dx + P(t)\right] u_x + k_1 u + \frac{k_2(b+cu)^{\frac{d}{c}+1}}{(d+c)} \\ &\quad + Y(x)u + Z(t,x), \end{aligned} \tag{2.68}$$

$$\begin{aligned} T^2(t,x,u,u_t,u_x,u_{xx}) &= k_1(b+cu)u_{xx} + k_2(b+cu)^{\frac{d}{c}+1}u_{xx} + \left[\int Y(x)dx + P(t)\right] u_t \\ &\quad + \frac{1}{2}k_1 d u_x^2 + \frac{1}{2}k_1 a u^2 + \frac{k_2 a u(b+cu)^{\frac{d}{c}+1}(d+2c) - ak_2(b+cu)^{\frac{d}{c}+2}}{(d+c)(d+2c)} \\ &\quad P'(t)u + W(t,x). \end{aligned} \tag{2.69}$$

Substituting equation (2.68) and (2.69) into equation (2.38), we obtain

$$\int Y(x)dx + P(t) = 0. \tag{2.70}$$

Differentiating the above equation with respect to t yields

$$P'(t) = 0. \tag{2.71}$$

Integrating the above equation with respect to t, we obtain

$$P(t) = k_3, \tag{2.72}$$

where k_3 is an arbitrary constant of integration. Inserting equation (2.72) into (2.70) yields

$$\int Y(x)dx = -k_3. \tag{2.73}$$

Thus, from equations (2.68) and (2.69) we obtain

$$T^1(t,x,u,u_x) = k_1 u + \frac{k_2(b+cu)^{\frac{d}{c}+1}}{(d+c)} + Z(t,x), \tag{2.74}$$

$$T^2(t,x,u,u_t,u_x,u_{xx}) = k_1(b+cu)u_{xx} + k_2(b+cu)^{\frac{d}{c}+1}u_{xx} + \frac{1}{2}k_1 d u_x^2 + \frac{1}{2}k_1 a u^2 + \frac{k_2 a u(b+cu)^{\frac{d}{c}+1}(d+2c) - a k_2 (b+cu)^{\frac{d}{c}+2}}{(d+c)(d+2c)} + W(t,x). \tag{2.75}$$

Therefore, the components of the conserved vectors are

$$T_1^1 = u, \tag{2.76}$$

$$T_1^2 = (b+cu)u_{xx} + \frac{1}{2}du_x^2 + \frac{1}{2}au^2; \tag{2.77}$$

$$T_2^1 = \frac{(b+cu)^{\frac{d}{c}+1}}{(d+c)}, \tag{2.78}$$

$$T_2^2 = (b+cu)^{\frac{d}{c}+1}u_{xx} + \frac{au(b+cu)^{\frac{d}{c}+1}(d+2c) - a(b+cu)^{\frac{d}{c}+2}}{(d+c)(d+2c)}, \tag{2.79}$$

associated with the multiplier (2.33).

Case 2. $d = -c$.

In this case we follow the same procedure as in Case 1 above and obtain the following multiplier:

$$\Lambda(t,x,u) = \frac{c}{(b+cu)}\left[k_2 \cos\left(\sqrt{\frac{a}{c}}\right) x + k_1 \sin\left(\sqrt{\frac{a}{c}}\right) x\right] \sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right) t + \frac{c}{(b+cu)}\left[k_2 \sin\left(\sqrt{\frac{a}{c}}\right) x - k_1 \cos\left(\sqrt{\frac{a}{c}}\right) x\right] \cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right) t + \frac{(k_3 tu - k_5)c}{(b+cu)} + k_4 - \frac{k_3 x c}{a(b+cu)}, \tag{2.80}$$

for the Kudryashov-Sinelshchikov equation (2.3). The corresponding conservation laws for the above multiplier are

$$
\begin{aligned}
T_1^1 &= \ln(b+cu)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad +L(t,x), \qquad (2.81) \\
T_1^2 &= \left[\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]cu_{xx} \\
&\quad -\sqrt{ac}\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)+\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]u_x \\
&\quad +\left(\frac{ba}{c}\right)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad -\left(\frac{ba}{c}\right)\ln(b+cu)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad +M(t,x) \qquad (2.82)
\end{aligned}
$$

with $L_t+M_x=-\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right)\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]$;

$$
\begin{aligned}
T_2^1 &= \ln(b+cu)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)+\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad +L(t,x), \qquad (2.83) \\
T_2^2 &= \left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)+\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]cu_{xx} \\
&\quad -\sqrt{ac}\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)-\sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]u_x \\
&\quad +\left(\frac{ba}{c}\right)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)+\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad -\left(\frac{ba}{c}\right)\ln(b+cu)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)+\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right] \\
&\quad +M(t,x), \qquad (2.84)
\end{aligned}
$$

with $L_t + M_x = -\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right)\left[\sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]$.

$$
\begin{aligned}
T_3^1 &= \frac{bt}{c} + tu - \frac{bt}{c}\ln(b+cu) - \frac{x}{a}\ln(b+cu), && (2.85)\\
T_3^2 &= ctuu_{xx} - \frac{cx}{a}u_{xx} + \frac{c}{a}u_x - \frac{ct}{2}u_x^2 + \frac{at}{2}u^2 - \frac{3ab^2t}{2c^2} - \frac{bat}{c}u + \frac{ab^2t}{c^2}\ln(b+cu)\\
&\quad -\frac{b}{c}x - xu + \frac{bx}{c}\ln(b+cu); && (2.86)\\
T_4^1 &= u, && (2.87)\\
T_4^2 &= bu_{xx} + cuu_{xx} - \frac{1}{2}cu_x^2 + \frac{1}{2}au^2; && (2.88)\\
T_5^1 &= -\ln(b+cu), && (2.89)\\
T_5^2 &= -cu_{xx} - \frac{ba}{c} - au + \frac{ba}{c}\ln(b+cu). && (2.90)
\end{aligned}
$$

Case 3. $d = 2c$.

This case provides us with the multiplier of the form

$$
\begin{aligned}
\Lambda(t,x,u) &= k_1\left(u+\frac{b}{c}\right)^2 + k_2 + \left[k_3\cos\left(\sqrt{\frac{a}{c}}\right)x + k_4\sin\left(\sqrt{\frac{a}{c}}\right)x\right]\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right)t\\
&\quad + \left[k_3\sin\left(\sqrt{\frac{a}{c}}\right)x - k_4\cos\left(\sqrt{\frac{a}{c}}\right)x\right]\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}\right)t, && (2.91)
\end{aligned}
$$

and the asssociated conservation laws of the generalized Kudryashov-Sinelshchikov equation (2.3) are

$$
\begin{aligned}
T_1^1 &= \frac{1}{6}u^3 + \frac{1}{2c}bu^2 + \frac{b^2}{2c^2}u, && (2.92)\\
T_1^2 &= \frac{1}{2}bu^2u_{xx} + \frac{1}{2}cu^3u_{xx} + \frac{b^2}{c}uu_{xx} + bu^2u_{xx} + \frac{b^3}{2c^2}u_{xx} + \frac{b^2}{2c}uu_{xx} + \frac{b^2}{4c^2}du_x^2\\
&\quad + \frac{b^2}{4c}u_x^2 - \frac{1}{2}cu^2u_x^2 - buu_x^2 - \frac{3b^2}{4c}u_x^2 + \frac{1}{4}du^2u_x^2 + \frac{1}{2c}bduu_x^2 + \frac{1}{2}au^4 + \frac{b}{c}au^3\\
&\quad + \frac{b^2}{8c^2}au^2; && (2.93)
\end{aligned}
$$

$$
\begin{aligned}
T_2^1 &= u, && (2.94)\\
T_2^2 &= (b+cu)u_{xx} + \frac{1}{2}du_x^2 + \frac{1}{2}au^2; && (2.95)\\
T_3^1 &= \left[-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]u, && (2.96)\\
T_3^2 &= \left[-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right](b+cu)u_{xx}\\
&\quad +\frac{1}{2}du_x^2\left[-\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]\\
&\quad -\sqrt{\frac{a}{c}}(b+cu)u_x\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]\\
&\quad +\frac{ba}{c}u\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) - \sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]; && (2.97)\\
T_4^1 &= \left[\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]u, && (2.98)\\
T_4^2 &= \left[\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right](b+cu)u_{xx}\\
&\quad +\frac{1}{2}du_x^2\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]\\
&\quad -\sqrt{\frac{a}{c}}(b+cu)u_x\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) - \sin\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]\\
&\quad -\frac{ba}{c}u\left[\cos\left(\sqrt{\frac{a}{c}}x\right)\sin\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right) + \sin\left(\sqrt{\frac{a}{c}}x\right)\cos\left(\frac{ba^{\frac{3}{2}}}{c^{\frac{3}{2}}}t\right)\right]. && (2.99)
\end{aligned}
$$

2.2 Exact solutions using Kudryashov method

The purpose of this segment is to present the algorithm of the Kudryashov technique for finding exact solutions of the nonlinear evolution equations. The Kudryashov method was one of the initial methods for finding exact solutions of nonlinear partial differential equations. [9, 25, 26].

Let us recall the basic idea of the Kudryashov method. Consider the nonlinear partial differential equation in the form

$$E_1[u_t, u_x, \cdots] = 0. \tag{2.100}$$

We use the following ansatz

$$u(x,t) = F(z) \quad z = k_1 x + k_2 t + k_3. \tag{2.101}$$

From equation (2.100), we obtain the ordinary nonlinear differential equation

$$E_2[k_1 F'(z), k_2 F'(z), k_1^2 F''(z), k_2^2 F''(z), \cdots] = 0, \tag{2.102}$$

which has a solution of the form

$$F(z) = \sum_{i=0}^{M} A_i (H(z))^i, \tag{2.103}$$

where

$$H(z) = \frac{1}{1 + \cosh(z) + \sinh(z)}$$

satisfies the equation

$$H'(z) = H(z)^2 - H(z), \tag{2.104}$$

and M is a positive integer while $A_0, \cdots, A_M$ are parameters to be determined.

2.2.1 Application of the Kudryashov method

Making use of anstaz (2.101), we obtain the following nonlinear ordinary differential equation

$$ak_1 F(z)F' + bk_1^3 F''' + c\left(k_1^3 F(z)F''' + k_1^3 F'F''\right) + dk_1^3 F'F'' + k_2 F' = 0. \tag{2.105}$$

By letting $M = 1$, the solutions of equation (2.105) are of the form

$$F(z) = A_0 + A_1 H. \tag{2.106}$$

Substituting equation (2.106) into equation (2.105) and making use of equation (2.104) and then equating all coefficients of the functions H^i to zero, we obtain the following overdetermined system of algebraic equations in terms of A_0, A_1:

$$
\begin{aligned}
&8\,cA_1{}^2k_1{}^3 + 2\,dA_1{}^2k_1{}^3 = 0,\\
&6\,cA_1k_1{}^3A_0 - 17\,cA_1{}^2k_1{}^3 - 5\,dA_1{}^2k_1{}^3 + 6\,bA_1k_1{}^3 = 0,\\
&-cA_1k_1{}^3A_0 - bA_1k_1{}^3 - aA_1k_1A_0 - A_1k_2 = 0,\\
&-12\,cA_1k_1{}^3A_0 + 11\,cA_1{}^2k_1{}^3 + 4\,dA_1{}^2k_1{}^3 - 12\,bA_1k_1{}^3 + aA_1{}^2k_1 = 0,\\
&7\,cA_1k_1{}^3A_0 - 2\,cA_1{}^2k_1{}^3 - dA_1{}^2k_1{}^3 + 7\,bA_1k_1{}^3 + aA_1k_1A_0\\
&-aA_1{}^2k_1 + A_1k_2 = 0.
\end{aligned}
$$

On solving the resultant system of algebraic equations, we obtain

$$
\begin{aligned}
a &= -ck_1{}^2,\\
A_0 &= \frac{k_2}{2ck_1^3} - \frac{3b}{2c},\\
d &= -4\,c,\\
A_1 &= \frac{k_3}{ck_1^3} + \frac{b}{c}.
\end{aligned}
$$

Consequently a solution of equation (2.3) is

$$
u(x,t) \quad = \quad A_0 + A_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\} \tag{2.107}
$$

where $z = k_1x + k_2t + k_3$.

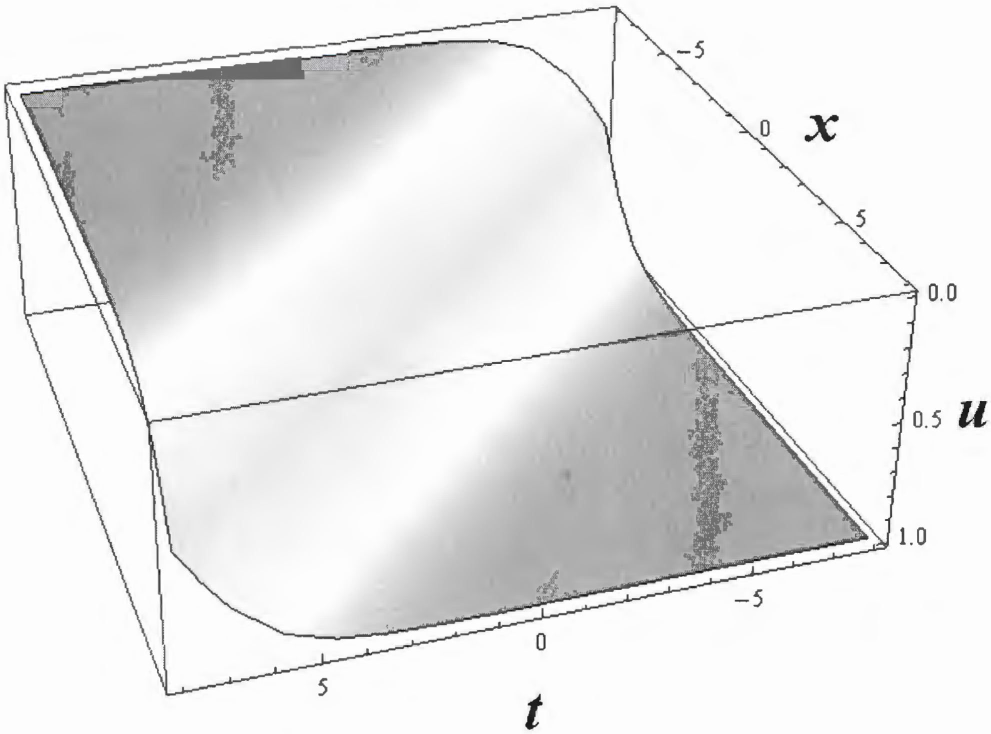

Figure 2.1: Evolution of travelling wave solution (2.107).

Similarly by letting $M = 2$, we obtain the following overdetermined system of algebraic equations:

$$
\begin{aligned}
&36\, c{k_1}^3{A_2}^2 + 12\, d{k_1}^3{A_2}^2 = 0,\\
&40\, c{k_1}^3A_1A_2 - 86\, c{k_1}^3{A_2}^2 + 10\, d{k_1}^3A_1A_2 - 32\, d{k_1}^3{A_2}^2 = 0,\\
&-c{k_1}^3A_0A_1 - b{k_1}^3A_1 - ak_1A_0A_1 - k_2A_1 = 0,\\
&24\, c{k_1}^3A_0A_2 + 8\, c{k_1}^3{A_1}^2 - 92\, c{k_1}^3A_1A_2 + 66\, c{k_1}^3{A_2}^2 + 2\, d{k_1}^3{A_1}^2\\
&-26\, d{k_1}^3A_1A_2 + 28\, d{k_1}^3{A_2}^2 + 24\, b{k_1}^3A_2 + 2\, ak_1{A_2}^2 = 0,
\end{aligned}
$$

$$7\,ck_1{}^3A_0A_1 - 8\,ck_1{}^3A_0A_2 - 2\,ck_1{}^3A_1{}^2 - dk_1{}^3A_1{}^2 + 7\,bk_1{}^3A_1$$
$$-8\,bk_1{}^3A_2 + ak_1A_0A_1 - 2\,ak_1A_0A_2 - ak_1A_1{}^2 + k_2A_1 - 2\,k_2A_2 = 0,$$
$$6\,ck_1{}^3A_0A_1 - 54\,ck_1{}^3A_0A_2 - 17\,ck_1{}^3A_1{}^2 + 67\,ck_1{}^3A_1A_2 - 16\,ck_1{}^3A_2{}^2$$
$$-5\,dk_1{}^3A_1{}^2 + 22\,dk_1{}^3A_1A_2 - 8\,dk_1{}^3A_2{}^2 + 6\,bk_1{}^3A_1 - 54\,bk_1{}^3A_2$$
$$+3\,ak_1A_1A_2 - 2\,ak_1A_2{}^2 = 0,$$
$$-12\,ck_1{}^3A_0A_1 + 38\,ck_1{}^3A_0A_2 + 11\,ck_1{}^3A_1{}^2 - 15\,ck_1{}^3A_1A_2 + 4\,dk_1{}^3A_1{}^2$$
$$-6\,dk_1{}^3A_1A_2 - 12\,bk_1{}^3A_1 + 38\,bk_1{}^3A_2 + 2\,ak_1A_0A_2 + ak_1A_1{}^2 - 3\,ak_1A_1A_2$$
$$+2\,k_2A_2 = 0.$$

By solving the above resultant algebraic equations, we obtain

$$d = -3\,c,$$
$$A_2 = -A_1,$$
$$k_1 = \kappa,$$
$$k_2 = -\frac{a\kappa\left(12\,cA_0{}^2 + 2\,cA_0A_1 + 12\,bA_0 + bA_1\right)}{12\,cA_0 + cA_1 + 12\,b},$$

where κ, is any root of $(12\,cA_0 + cA_1 + 12\,b)\,\kappa^2 - aA_1 = 0$ and subsequently the desired solution takes the form

$$\begin{aligned} u(x,t) = \quad & A_0 + A_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\} \\ + \quad & A_2\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^2, \end{aligned} \tag{2.108}$$

where $z = k_1x + k_2t + k_3$.

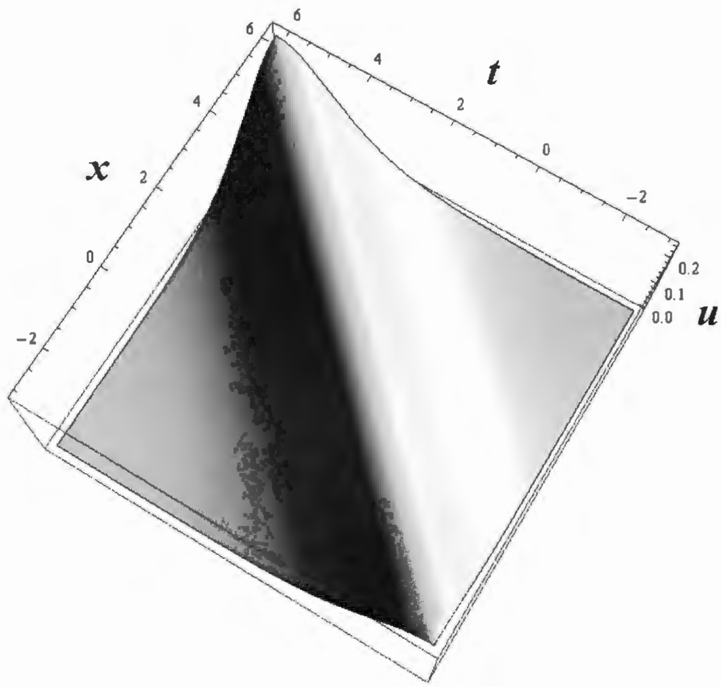

Figure 2.2: Evolution of travelling wave solution (2.108).

Following the same procedure as before and taking $M = 4$, we get the following overdetermined system of algebraic equations:

$$200\,ck_1{}^3A_4{}^2 + 80\,dk_1{}^3A_4{}^2 = 0,$$

$$288\,ck_1{}^3A_3A_4 - 524\,ck_1{}^3A_4{}^2 + 108\,dk_1{}^3A_3A_4 - 224\,dk_1{}^3A_4{}^2 = 0,$$

$$-ck_1{}^3A_0A_1 - bk_1{}^3A_1 - ak_1A_0A_1 - k_2A_1 = 0,$$

$$208\,ck_1{}^3A_2A_4+96\,ck_1{}^3A_3{}^2-744\,ck_1{}^3A_3A_4+452\,ck_1{}^3A_4{}^2+64\,dk_1{}^3A_2A_4$$
$$+36\,dk_1{}^3A_3{}^2-300\,dk_1{}^3A_3A_4+208\,dk_1{}^3A_4{}^2+4\,ak_1A_4{}^2=0,$$
$$7\,ck_1{}^3A_0A_1-8\,ck_1{}^3A_0A_2-2\,ck_1{}^3A_1{}^2-dk_1{}^3A_1{}^2+7\,bk_1{}^3A_1-8\,bk_1{}^3A_2$$
$$+ak_1A_0A_1-2\,ak_1A_0A_2-ak_1A_1{}^2+k_2A_1-2\,k_2A_2=0,$$
$$154\,ck_1{}^3A_1A_4+126\,ck_1{}^3A_2A_3-530\,ck_1{}^3A_2A_4-243\,ck_1{}^3A_3{}^2+631\,ck_1{}^3A_3A_4$$
$$-128\,ck_1{}^3A_4{}^2+28\,dk_1{}^3A_1A_4+42\,dk_1{}^3A_2A_3-176\,dk_1{}^3A_2A_4-99\,dk_1{}^3A_3{}^2$$
$$+276\,dk_1{}^3A_3A_4-64\,dk_1{}^3A_4{}^2+7\,ak_1A_3A_4-4\,ak_1A_4{}^2=0,$$
$$-12\,ck_1{}^3A_0A_1+38\,ck_1{}^3A_0A_2-27\,ck_1{}^3A_0A_3+11\,ck_1{}^3A_1{}^2-15\,ck_1{}^3A_1A_2$$
$$+4\,dk_1{}^3A_1{}^2-6\,dk_1{}^3A_1A_2-12\,bk_1{}^3A_1+38\,bk_1{}^3A_2-27\,bk_1{}^3A_3+2\,ak_1A_0A_2$$
$$-3\,ak_1A_0A_3+ak_1A_1{}^2-3\,ak_1A_1A_2+2\,k_2A_2-3\,k_2A_3=0,$$
$$120\,ck_1{}^3A_0A_4+84\,ck_1{}^3A_1A_3-388\,ck_1{}^3A_1A_4+36\,ck_1{}^3A_2{}^2-312\,ck_1{}^3A_2A_3$$
$$+442\,ck_1{}^3A_2A_4+201\,ck_1{}^3A_3{}^2-175\,ck_1{}^3A_3A_4+18\,dk_1{}^3A_1A_3-76\,dk_1{}^3A_1A_4$$
$$+12\,dk_1{}^3A_2{}^2-114\,dk_1{}^3A_2A_3+160\,dk_1{}^3A_2A_4+90\,dk_1{}^3A_3{}^2-84\,dk_1{}^3A_3A_4$$
$$+120\,bk_1{}^3A_4+6\,ak_1A_2A_4+3\,ak_1A_3{}^2-7\,ak_1A_3A_4=0,$$
$$60\,ck_1{}^3A_0A_3-300\,ck_1{}^3A_0A_4+40\,ck_1{}^3A_1A_2-204\,ck_1{}^3A_1A_3+319\,ck_1{}^3A_1A_4$$
$$-86\,ck_1{}^3A_2{}^2+251\,ck_1{}^3A_2A_3-120\,ck_1{}^3A_2A_4-54\,ck_1{}^3A_3{}^2+10\,dk_1{}^3A_1A_2$$
$$-48\,dk_1{}^3A_1A_3+68\,dk_1{}^3A_1A_4-32\,dk_1{}^3A_2{}^2+102\,dk_1{}^3A_2A_3-48\,dk_1{}^3A_2A_4$$
$$-27\,dk_1{}^3A_3{}^2+60\,bk_1{}^3A_3-300\,bk_1{}^3A_4+5\,ak_1A_1A_4+5\,ak_1A_2A_3-6\,ak_1A_2A_4$$
$$-3\,ak_1A_3{}^2=0,$$
$$6\,ck_1{}^3A_0A_1-54\,ck_1{}^3A_0A_2+111\,ck_1{}^3A_0A_3-64\,ck_1{}^3A_0A_4-17\,ck_1{}^3A_1{}^2$$
$$+67\,ck_1{}^3A_1A_2-40\,ck_1{}^3A_1A_3-16\,ck_1{}^3A_2{}^2-5\,dk_1{}^3A_1{}^2+22\,dk_1{}^3A_1A_2$$
$$-12\,dk_1{}^3A_1A_3-8\,dk_1{}^3A_2{}^2+6\,bk_1{}^3A_1-54\,bk_1{}^3A_2+111\,bk_1{}^3A_3-64\,bk_1{}^3A_4$$
$$+3\,ak_1A_0A_3-4\,ak_1A_0A_4+3\,ak_1A_1A_2-4\,ak_1A_1A_3-2\,ak_1A_2{}^2+3\,k_2A_3$$
$$-4\,k_2A_4=0,$$

$$24\,ck_1{}^3A_0A_2 - 144\,ck_1{}^3A_0A_3 + 244\,ck_1{}^3A_0A_4 + 8\,ck_1{}^3{A_1}^2 - 92\,ck_1{}^3A_1A_2$$
$$+160\,ck_1{}^3A_1A_3 - 85\,ck_1{}^3A_1A_4 + 66\,ck_1{}^3{A_2}^2 - 65\,ck_1{}^3A_2A_3 + 2\,dk_1{}^3{A_1}^2$$
$$-26\,dk_1{}^3A_1A_2 + 42\,dk_1{}^3A_1A_3 - 20\,dk_1{}^3A_1A_4 + 28\,dk_1{}^3{A_2}^2 - 30\,dk_1{}^3A_2A_3$$
$$+24\,bk_1{}^3A_2 - 144\,bk_1{}^3A_3 + 244\,bk_1{}^3A_4 + 4\,ak_1A_0A_4 + 4\,ak_1A_1A_3$$
$$-5\,ak_1A_1A_4 + 2\,ak_1{A_2}^2 - 5\,ak_1A_2A_3 + 4\,k_2A_4 = 0.$$

Solving the above system of algebraic equations, we obtain

$$\begin{aligned}
a &= -2\,ck_1{}^2,\\
d &= -\frac{5}{2}\,c,\\
A_0 &= -\frac{cA_3 + 72\,b}{72\,c},\\
A_1 &= \frac{1}{6}\,A_3,\\
A_2 &= -\frac{2}{3}\,A_3,\\
A_4 &= -\frac{1}{2}\,A_3,\\
k_2 &= -\frac{cA_3k_1{}^3}{72} - 2\,bk_1{}^3.
\end{aligned}$$

As a result, the solution of equation (2.3) is

$$\begin{aligned}
u(x,t) &= A_0 + A_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\} + A_2\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^2\\
&+ A_3\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^3 + A_4\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^4, \quad (2.109)
\end{aligned}$$

where $z = k_1x + k_2t + k_3$.

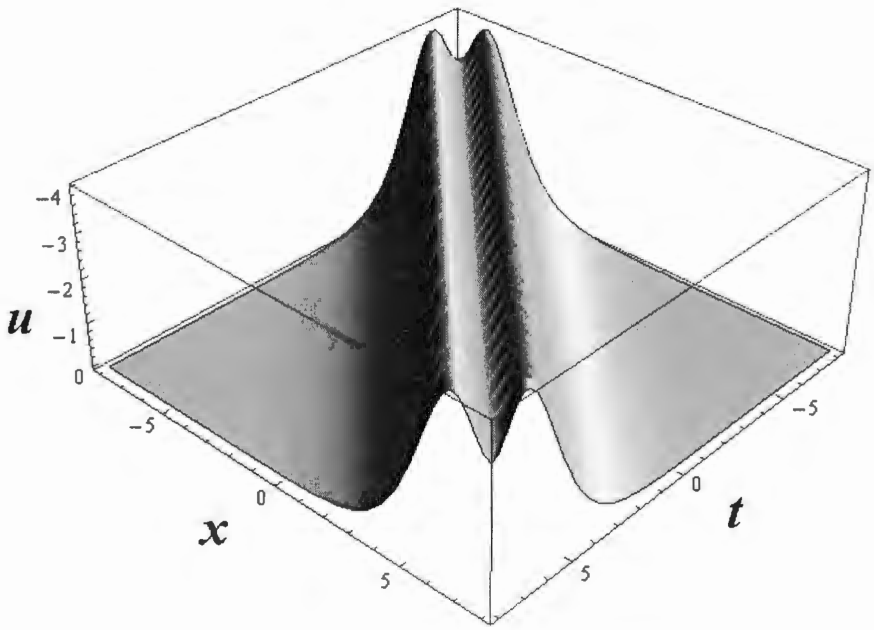

Figure 2.3: Evolution of travelling wave solution (2.109).

2.3 Concluding remarks

New exact solutions and conservation laws of a generalized Kudryashov-Sinelshchikov equation were computed. Kudryashov method was employed to compute solitary wave solutions while conservation laws were computed via the multiplier approach.

Chapter 3

Lagrangian formulation, Conservation laws, Travelling wave solutions of a generalized Benney-Luke equation

In this chapter, we study the generalized Benney-Luke equation in the form

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{3.1}$$

In 1964, D.J. Benney and J.C. Luke derived the above equation [27], where α, β are positive constants. Benney-Luke equation (3.1) models waves propagating on the surface of a fluid in a shallow channel of constant depth taking into consideration the surface tension effect. The Benney-Luke equation and its generalizations have been extensively investigated [24, 28–31]. The approaches used in the investigation include stability analysis, Cauchy problem, existence and analyticity of solutions, etc. We refer the interested reader to references [24, 28–31] and references therein. However, in this present work, our goal is to compute conservation laws and exact solutions of equation (3.1).

We use the Noether theorem [14] to construct conservation laws for equation (3.1).

Furthermore, we will obtain exact solutions of the Benney-Luke equation via the extended tanh method.

3.1 Construction of conservation laws for Benney-Luke equation (3.1)

Consider the Benney-Luke equation (3.1), viz.,

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0.$$

It can be verified that the second-order Langragian given by

$$L = \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 - \frac{1}{2}u_t u_x^2, \tag{3.2}$$

satisfies the Euler-Lagrange equation (1.26). Thus

$$\frac{\delta L}{\delta u} = 0, \tag{3.3}$$

where the Euler-Langrage Operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} \\ &\quad + D_x D_t \frac{\partial}{\partial u_{xt}} + \cdots , \end{aligned} \tag{3.4}$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots ,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots .$$

We now verify that equation (3.2) satisfies equation (3.3)

$$\begin{aligned} \frac{\delta L}{\delta u} &= D_t(u_t) - D_x(u_x) - D_t\left(-\frac{1}{2}u_x^2\right) + D_x^2(\alpha u_{xx}) + D_x D_t(\beta u_{tx}) + D_x(u_t u_{xx}) \\ &= u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} \\ &= 0. \end{aligned} \tag{3.5}$$

As a result the Langragian (3.2) is the Langragian of (3.1).

Consider the vector field

$$\mathbf{X} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.6}$$

which has the second-order prolongation given by

$$\mathbf{X}^{[2]} = \mathbf{X} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}, \tag{3.7}$$

where

$$\zeta_t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \tag{3.8}$$

$$\zeta_x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \tag{3.9}$$

$$\zeta_{tt} = D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \tag{3.10}$$

$$\zeta_{tx} = D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \tag{3.11}$$

$$\zeta_{xx} = D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi). \tag{3.12}$$

The vector field $\mathbf{X}$, defined in equation (3.7), is a called Noether symmetry corresponding to the Lagrangian L if it satisfies

$$\mathbf{X}^{[2]}(L) + \{D_t(\tau) + D_x(\xi)\}L = D_t(B^1) + D_x(B^2), \tag{3.13}$$

where $B^1(t,x,u)$ and $B^2(t,x,u)$ are the gauge terms. Using the definition of $\mathbf{X}^{[2]}$ from equation (3.7) and inserting L from equation (3.2) into equation (3.13) yields

$$\begin{aligned}&\left[\zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}\right]\left(\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 - \frac{1}{2}u_t u_x^2\right)\\ &+\left[\left(\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u}\right)(\tau) + \left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u}\right)(\xi)\right]\left(\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 - \frac{1}{2}u_t u_x^2\right)\\ &= \left(\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u}\right)B^1 + \left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u}\right)B^2,\end{aligned} \tag{3.14}$$

which gives

$$\begin{aligned}&-u_t\zeta_t - \frac{1}{2}u_x^2\zeta_t + u_x\zeta_x - u_t u_x\zeta_x - \beta u_{tx}\zeta_{tx} + \alpha u_{xx}\zeta_{xx} + (\tau_t + u_t\tau_u + \xi_x + u_x\xi_u)\\ &\times\left(\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 - \frac{1}{2}u_t u_x^2\right) = B_t^1 + u_t B_u^1 + B_x^2 + u_x B_u^2.\end{aligned} \tag{3.15}$$

Substituting the values of $\zeta_t, \zeta_x, \zeta_{tx}$ and ζ_{xx} into equation (3.15), we obtain

$$
\begin{aligned}
&-u_t\eta_t - u_t^2\eta_u + u_t^2\tau_t + u_t^3\tau_u + u_tu_x\xi_t + u_t^2u_x\xi_u - \frac{1}{2}u_x^2\eta_t - \frac{1}{2}u_tu_x^2\eta_u + \frac{1}{2}u_tu_x^2\tau_t \\
&+\frac{1}{2}u_xu_t^2\tau_u + \frac{1}{2}u_x^3\xi_t + \frac{1}{2}u_tu_x^3\xi_u + u_x\eta_x + u_x^2\eta_u - u_tu_x\tau_x - u_tu_x^2\tau_u - u_x^2\xi_x - u_x^3\xi_u \\
&-u_tu_x\eta_x - u_tu_x^2\eta_u + u_xu_t^2\tau_x + u_x^2u_t^2\tau_u + u_tu_x^2\xi_x + u_tu_x^3\xi_u - \beta u_{tx}\eta_{tx} - \beta u_xu_{tx}\eta_{tu} \\
&-\beta u_tu_{tx}\eta_{xu} - \beta u_{tx}^2\eta_u - \beta u_xu_tu_{tx}\eta_{uu} + \beta u_{tx}^2\tau_t + \beta u_xu_{tx}\eta_{tu} + \beta u_{tx}^2\xi_x + \beta u_tu_{tx}\tau_{tx} \\
&+\beta u_{tt}u_{tx}\tau_x + \beta u_tu_xu_{tx}\tau_{tu} + \beta u_xu_tu_{tx}\xi_{xu} + \beta u_t^2\tau_{xu} + 2\beta u_tu_{tx}^2\tau_u + \beta u_xu_{tt}u_{tx}\tau_u \\
&+\beta u_xu_t^2u_{tx}\tau_{uu} + \beta u_xu_{tx}\xi_{tx} + \beta u_{xx}u_{tx}\xi_t + \beta u_x^2u_{tx}\xi_{tu} + 2\beta u_xu_{tx}^2\xi_u + \beta u_tu_{xx}u_{tx}\xi_u \\
&+\beta u_tu_{xx}u_{tx}\xi_u + \beta u_tu_x^2u_{tx}\xi_{uu} + \alpha u_{xx}\eta_{xx} + 2\alpha u_xu_{xx}\eta_{xu} + \alpha u_{xx}^2\eta_u + \alpha u_x^2u_{xx}\eta_{uu} \\
&-2\alpha u_{xx}^2\xi_x - \alpha u_uu_{xx}\xi_{xx} - 2\alpha u_x^2u_{xx}\xi_{xu} - 3\alpha u_xu_{xx}^2\xi_u - \alpha u_x^3u_{xx}\xi_{uu} - 2\alpha u_{tx}u_{xx}\tau_x \\
&-\alpha u_tu_{xx}\tau_{xx} - 2\alpha u_tu_xu_{xx}\tau_{xu} - \alpha u_tu_{xx}\tau_u - 2\alpha u_xu_{tx}u_{xx}\tau_{uu} - \alpha u_tu_x^2u_{xx}\tau_{uu} + \frac{1}{2}u_x^2\tau_t \\
&-\frac{1}{2}u_t^2\tau_t + \frac{1}{2}\alpha u_{xx}^2\tau_t - \frac{1}{2}\beta u_{tx}^2\tau_t - \frac{1}{2}u_tu_x^2\tau_t + \frac{1}{2}u_tu_x^2\tau_u - \frac{1}{2}u_x^3\tau_u + \frac{1}{2}\alpha u_tu_{xx}^2\tau_u \\
&-\frac{1}{2}u_tu_{tx}^2\tau_u - \frac{1}{2}u_t^2u_x^2\tau_u + \frac{1}{2}u_x^2\xi_x - \frac{1}{2}u_t^2\xi_x + \frac{1}{2}\alpha u_{xx}^2\xi_x - \frac{1}{2}\beta u_{tx}^2\xi_x - \frac{1}{2}u_tu_x^2\xi_x + \frac{1}{2}u_x^3\xi_u \\
&-\frac{1}{2}u_xu_t^2\xi_u + \frac{1}{2}\alpha u_xu_{xx}^2\xi_u - \frac{1}{2}\beta u_xu_{tx}^2\xi_u - \frac{1}{2}u_tu_x^3\xi_u = B_t^1 + u_tB_u^1 + B_x^2 + u_xB_u^2.
\end{aligned}
\tag{3.16}
$$

Splitting the above equation with respect to the derivatives of u, yields the following overdetermined system of linear PDEs:

$$\tau_u = 0, \tag{3.17}$$

$$\tau_x = 0, \tag{3.18}$$

$$\xi_t = 0, \tag{3.19}$$

$$\xi_u = 0, \tag{3.20}$$

$$\eta_x = 0, \tag{3.21}$$

$$\eta_{uu} = 0, \tag{3.22}$$

$$\eta_{tu} = 0, \tag{3.23}$$

$$B_u^2 = 0, \tag{3.24}$$

$$\xi_x - 3\eta_u = 0, \tag{3.25}$$

$$B_u^1 + \eta_t = 0, \tag{3.26}$$

$$2\eta_u - 3\xi_x + \tau_t = 0, \tag{3.27}$$

$$2\eta_u - \xi_x - \tau_t = 0, \tag{3.28}$$

$$2\eta_u - \xi_x - \eta_t + \tau_t = 0, \tag{3.29}$$

$$B_t^1 + B_x^2 = 0. \tag{3.30}$$

We now solve the above system of linear partial differential equations for τ, ξ, η, B^1 and B^2. Equations (3.17) and (3.18) imply that

$$\tau(t, x, u) = a(t), \tag{3.31}$$

where $a(t)$ is an arbitrary function of t. From equations (3.19) and (3.20), we obtain

$$\xi(t, x, u) = b(x), \tag{3.32}$$

where $b(x)$ is an arbitrary function of x. Integrating equation (3.21) with respect to x gives

$$\eta(t, x, u) = c(t, u), \tag{3.33}$$

where $c(t, u)$ is an arbitrary function of t and u. Substituting the value of η from equation (3.33) into equation (3.22) and integrating twice with respect to u yields

$$c(t, u) = d(t)u + e(t), \tag{3.34}$$

where $d(t)$ and $e(t)$ are arbitrary functions of t. Thus

$$\eta(t, x, u) = d(t)u + e(t). \tag{3.35}$$

Inserting equation (3.35) into (3.23) and solving the resulting equation gives

$$\eta(t, x, u) = k_1 u + e(t), \tag{3.36}$$

where k_1 is an arbitrary constant of integration. Integrating equation (3.24) with respect to u yields

$$B^2(t,x,u) = F(t,x), \tag{3.37}$$

where $f(t,x)$ is an arbitrary function of t and x. Substituting the values of ξ and η into equation (3.25) and integrating with respect to x, we obtain

$$b(x) = 3k_1x + k_2, \tag{3.38}$$

where k_1 and k_2 are arbitrary constants of integration. Thus

$$\xi(t,x,u) = 3k_1x + k_2. \tag{3.39}$$

Inserting equations (3.31), (3.36) and (3.39) into (3.27) and solving gives

$$a(t) = 7k_1t + k_3, \tag{3.40}$$

where k_3 is an arbitrary constant of integration and so we have

$$\tau(t,x,u) = 7k_1t + k_3. \tag{3.41}$$

Substituting equations (3.36), (3.39) and (3.41) into (3.28) and solving the resulting equation gives

$$k_1 = 0. \tag{3.42}$$

As a result equations (3.36), (3.39) and (3.41) reduces to the following:

$$\tau(t,x,u) = k_3, \tag{3.43}$$

$$\xi(t,x,u) = k_2, \tag{3.44}$$

$$\eta(t,x,u) = e(t). \tag{3.45}$$

By substituting equations (3.43), (3.44) and (3.45) into equation (3.29), we obtain

$$e(t) = k_4, \tag{3.46}$$

where k_4 is an arbitrary constant of integration. Thus

$$\eta(t,x,u) = k_4. \tag{3.47}$$

From equation (3.26), we have

$$B^1(t,x,u) = G(t,x). \tag{3.48}$$

Thus equation (3.30) gives

$$G_t(t,x) + F_x(t,x) = 0. \tag{3.49}$$

Consequently we have the following:

$$\tau(t,x,u) = k_3, \qquad \xi(t,x,u) = k_2, \qquad \eta(t,x,u) = k_4, \qquad B^1(t,x,u) = G(t,x),$$
$$B^2(t,x,u) = F(t,x), \qquad G_t(t,x) + F_x(t,x) = 0.$$

We choose $G(t,x) = F(t,x) = 0$ as they only contribute to the trivial part of the conserved vectors. Hence the Noether symmetries and the associated gauge functions are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\ X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0, \\ X_3 &= \frac{\partial}{\partial u}, \quad B^1 = 0, \quad B^2 = 0. \end{aligned}$$

We use the above results to find the components of conserved vectors. Applying Noether's theorem leads to the following nontrivial conserved vectors associated with three Noether point symmetries:

$$\begin{aligned} T_1^1 &= \frac{1}{2}u_x^2 + \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 + \frac{1}{2}\beta u_{tx}^2, \\ T_1^2 &= -u_t u_x + u_t^2 u_x - \beta u_t u_{ttx} + \alpha u_t u_{xxx} - \alpha u_{tx} u_{xx}; \end{aligned} \tag{3.50}$$

$$\begin{aligned} T_2^1 &= u_x u_t + \frac{1}{2}u_x^3 + \beta u_{tx} u_{xx}, \\ T_2^2 &= -\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 + \frac{1}{2}u_t u_x^2 - \beta u_x u_{ttx} + \alpha u_x u_{xxx}; \end{aligned} \tag{3.51}$$

$$\begin{aligned} T_3^1 &= -u_t - \frac{1}{2}u_x^2, \\ T_3^2 &= u_x - u_t u_x + \beta u_{ttx} - \alpha u_{xxx}. \end{aligned} \tag{3.52}$$

3.2 Exact solutions using the extended tanh method

In this section we use the extended tanh function method which was introduced by Wazwaz [32]. We use the following ansatz

$$u(x,t) = F(z), \quad z = x - \omega t. \tag{3.53}$$

Making use of (3.53), equation (3.1) is reduced to the following nonlinear ordinary differential equation:

$$\alpha F''''(z) - \beta\omega^2 F''''(z) + \omega^2 F''(z) - F''(z) - 3\omega F'(z)F''(z) = 0. \tag{3.54}$$

The basic idea in this method is to assume that the solution of (3.54) can be written in the form

$$F(z) = \sum_{i=-M}^{M} A_i H(z)^i, \tag{3.55}$$

where $H(z)$ satisfies an auxiliary equation, say for example the Riccati equation

$$H'(z) = 1 - H^2(z), \tag{3.56}$$

whose solution is given by

$$H(z) = \tanh(z). \tag{3.57}$$

The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (3.54). A_i are parameters to be determined. In our case, the balancing procedure gives $M = 1$ and so the solutions of (3.54) are of the form

$$F(z) = A_{-1}H^{-1} + A_0 + A_1 H. \tag{3.58}$$

Substituting equation (3.58) into equation (3.54) and making use of the Riccati equation (3.56) and then equating the coefficients of the functions H^i to zero, we obtain the following algebraic system of equations:

$$8\,cA_1{}^2k_1{}^3 + 2\,dA_1{}^2k_1{}^3 = 0,$$
$$6\,cA_1k_1{}^3A_0 - 17\,cA_1{}^2k_1{}^3 - 5\,dA_1{}^2k_1{}^3 + 6\,bA_1k_1{}^3 = 0,$$
$$-cA_1k_1{}^3A_0 - bA_1k_1{}^3 - aA_1k_1A_0 - A_1k_2 = 0,$$
$$-12\,cA_1k_1{}^3A_0 + 11\,cA_1{}^2k_1{}^3 + 4\,dA_1{}^2k_1{}^3 - 12\,bA_1k_1{}^3 + aA_1{}^2k_1 = 0,$$
$$7\,cA_1k_1{}^3A_0 - 2\,cA_1{}^2k_1{}^3 - dA_1{}^2k_1{}^3 + 7\,bA_1k_1{}^3 + aA_1k_1A_0 - aA_1{}^2k_1 + A_1k_2 = 0.$$

Solving the resultant system of algebraic equations leads to the following three cases:

Case 1

$$\omega = k,$$
$$A_{-1} = 0,$$
$$A_1 = -\frac{-4\,k\alpha + 4\,k\beta}{4\,\alpha - 1};$$

Case 2

$$\omega = k,$$
$$A_{-1} = -\frac{-4\,k\alpha + 4\,k\beta}{4\,\alpha - 1},$$
$$A_1 = 0;$$

Case 3

$$\omega = p,$$
$$A_{-1} = -\frac{-4\,p\alpha + 4\,p\beta}{16\,\alpha - 1},$$
$$A_1 = -\frac{-4\,p\alpha + 4\,p\beta}{16\,\alpha - 1},$$

where k and p are any roots of $(4\,\beta - 1)\,k^2 - 4\,\alpha + 1 = 0$ and $(-1 + 16\,\beta)\,p^2 - 16\,\alpha + 1 = 0$ respectively. As a result, a solution of (3.1) is

$$u(x,t) \;=\; A_{-1}\mathrm{coth}(z) + A_0 + A_1\tanh(z), \qquad (3.59)$$

where $z = x - \omega t$.

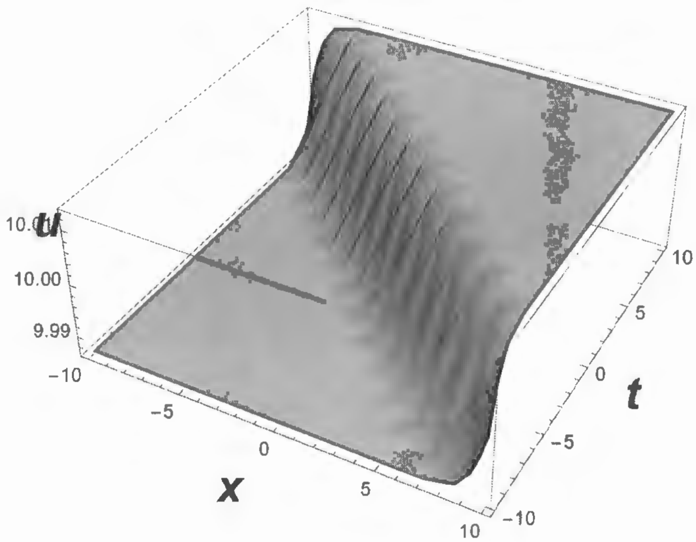

Figure 3.1: Evolution of the solution of (3.1) for Case 1.

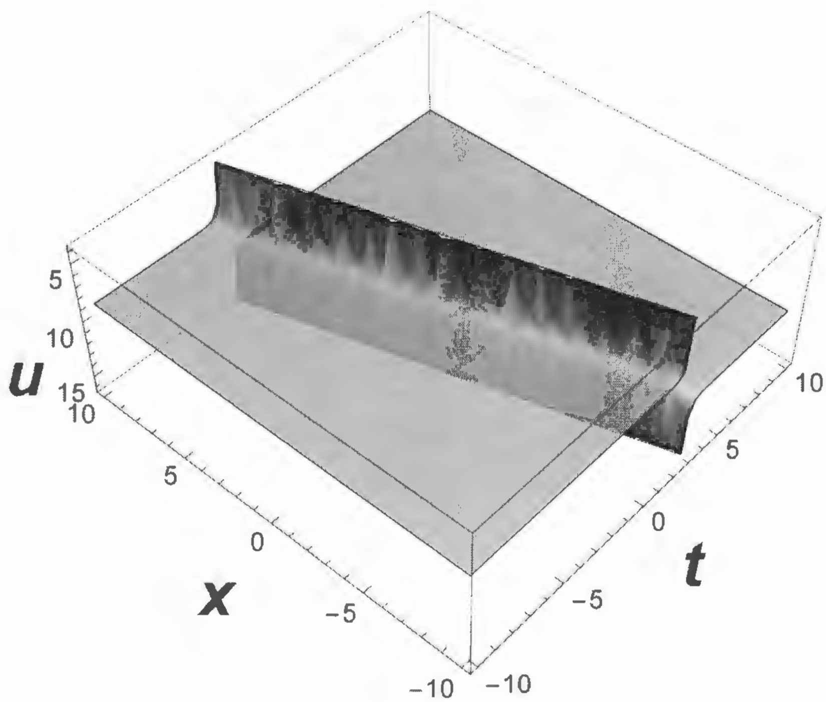

Figure 3.2: Evolution of the solution of (3.1) for Case 3.

3.3 Concluding remarks

In this chapter the Noether symmetries of a generalized Benney-Luke equation were computed. Thereafter, we constructed the associated conservation laws. Moreover, we derived exact solutions for the generalized Benney-Luke equation via the extended tanh method.

Chapter 4

Conclusions and Discussions

In this dissertation we first briefly introduced the basic concepts which were used through out the dissertation. In Chapter two we constructed the conservation laws for the generalized Kudryashov-Sinelshchikov equation (2.3) by applying the multiplier method. Thereafter, Kudryashov method was employed to compute exact solutions for the generalized Kudryashov-Sinelshchikov equation (2.3).

In Chapter three the Noether theorem was used to derive the conservation laws for the Benney-Luke equation (3.1). We then employed the extended tanh method to find the exact solutions for the Benney-Luke equation (3.1). Finally, in Chapter four we summarized the work done in the dissertation.

Bibliography

[1] M. Wang, Y. Zhou, Z. Li, Application of homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A. 216 (1996) 67–75.

[2] J. L. Hu, Explicit solutions to three nonlinear physical models, Phys. Lett. A. 287 (2001) 81–89.

[3] J. Hu, H. Zhang, A new method for finding exact traveling wave solutions to nonlinear partial differential equations, Phys. Lett. A. 286 (2001) 175–179.

[4] S. Lou, J. Lu, Special solutions from variable separation approach: Davey–Stewartson equation, J. Phys. A: Math. Gen. 29 (1996) 4209–4215.

[5] M. J. Ablowitz, P. A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge University Press, Cambridge, 1991.

[6] C. H. Gu, Soliton Theory and its Application, Zhejiang Science and Technology Press, Zhejiang, 1990.

[7] V. B. Matveev, M. A. Salle, Darboux transformations and solitons, Springer, Berlin, 1991.

[8] R. Hirota, The Direct Method in Soliton theory, Cambridge University Press, Cambridge, 2004.

[9] N. A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2248–2253.

[10] H. Steudel, Uber die zuordnung zwischen invarianzeigenschaften und erhaltungssatzen, Z. Naturforsch. 17A (1962) 129–132.

[11] J. M. Tu, S. F. Tian, M. J. Xu, T. T. Zhang, On Lie symmetries, optimal systems and explicit solutions to the Kudryashov-Sinelshchikov equation, Appl. Math. Comput. 275 (2016) 345–352.

[12] J. Akter, M. A. Akbar, Exact solutions to the Benney-Luke equation and the Phi-4 equations by using modified simple equation method, Results Phys. 5 (2015) 125–130.

[13] A. R. Adem, B. Muatjetjeja, Conservation laws and exact solutions for a 2D ZakharovKuznetsov equation, Appl. Math. Lett. 48 (2015) 109–117.

[14] E. Noether, Invariante variationsprobleme, Nachr. König. Gesell. Wiss. Göttingen Math. Phys. Kl. Heft. 2 (1918) 235–257.

[15] A. M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, Appl. Math. Comput. 184 (2007) 1002–1014.

[16] S. C. Anco, G. W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications, Eur. J. Appl. Math. 13 (2002) 545–566.

[17] N. H. Ibragimov, A. H. Kara, F. M. Mahomed, Lie-Bäcklund and Noether symmetries with applications, Nonlinear Dynam. 15 (1998) 115–136.

[18] A. H. Kara, F. M. Mahomed, Relationship between symmetries and conservation laws, Int. J. Theor. Phys. 39 (2000) 23–40.

[19] A. Sjöberg, Double reduction of partial differential equations from the association of symmetries with conservation laws with application, Appl. Math. Comput. 184 (2007) 608–616.

[20] A. H. Bokhari, A. Y. Al-Dweik, F. D. Zaman, A. H. Kara, F. M. Mahomed, Generalization of the double reduction theory, Nonlinear. Anal. Real. 11 (2010) 3763–3769.

[21] R. Naz, Z. Ali, I. Naeem, Reductions and new exact solutions of ZK, Garden KP and Modified KP equations via generalized doubled reduction theorem, Abstr. Appl. Anal. ID: 340564 (2013) 11 pages.

[22] A. H. Kara, F. M. Mahomed, Relationship between symmetries and conservation laws, Inter. J. Theor. Phys. 9 (2002) 60–72.

[23] H. Stephani, Differential Equations: Their Solutions Using Symmetries, Cambridge University Press, Cambridge, 1989.

[24] L. Ji-bin, Exact traveling wave solutions to 2D-generalized Benney-Luke equation, Appl. Math. Mech. 29 (2008) 1391–1398.

[25] N. A. Kudryashov, Exact soliton solutions of the generalized evolution equation of wave dynamics, J. Appl. Math. Mech. 52 (1988) 361-365.

[26] N. A. Kudryashov, On types of nonlinear nonintegrable differential equations with exact solutions, Phys. Lett. A. 155 (1991) 269-275.

[27] D. J. Benney, J. C. Luke, Interactions of permanent waves of finite amplitude, J. Math. Phys. 43 (1964) 309–313.

[28] J. Carlos, M. Grajales, Instability and long-time evolution of cnoidal wave solutions for a Benney–Luke equation, Int. J. Nonlinear Mech. 44 (2009) 999–1010.

[29] S. Wang, G. Xu, G. Chen Cauchy problem for the generalized Benney–Luke equation, J. Math. Phys. 48 (2007) 073521.

[30] J. R. Quintero, Nonlinear stability of solitary waves for a 2-D Benney-Luke equation, Discret. Cont. Dyn. S. 13 (2005) 203–218.

[31] J. R. Quintero, J. C. M. Grajales, Instability of solitary waves for a generalized Benney-Luke equation, Nonlinear. Anal. Theor. Meth. Appl. 68 (2008) 3009–3033.

[32] A. M. Wazwaz, New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations, Appl. Math. Comput. 182 (2006) 1642–1650.