



**SYMMETRY REDUCTIONS, EXACT  
SOLUTIONS AND CONSERVATION  
LAWS OF A VARIABLE  
COEFFICIENT (2+1)-DIMENSIONAL  
ZAKHAROV-KUZNETSOV EQUATION**

by

**LETLHOGONOLO DADDY MOLELEKI (18045510)**

Dissertation submitted for the degree of Master of Science in Applied  
Mathematics in the Department of Mathematical Sciences in the  
Faculty of Agriculture, Science and Technology at North-West

University, Mafikeng Camp

November 2011



North-West University  
Mafikeng Campus Library

**Supervisor: Professor C M Khalique**



# Contents

Declaration . . . . .	4
Dedication . . . . .	5
Acknowledgements . . . . .	6
Abstract . . . . .	7
<b>Introduction</b>	<b>8</b>
<b>1 Lie symmetry methods for differential equations and the conservation theorems</b>	<b>11</b>
1.1 Introduction . . . . .	11
1.2 Local one-parameter Lie group . . . . .	12
1.3 Infinitesimal transformations . . . . .	13
1.4 Group invariants . . . . .	14
1.5 Construction of a symmetry group . . . . .	15
1.5.1 Prolongation of point transformations . . . . .	15
1.5.2 Group admitted by a PDE . . . . .	18
1.6 Lie algebras . . . . .	18
1.7 Fundamental operators and their relationship . . . . .	19
1.8 Variational method for a system and its adjoint . . . . .	21

1.9	Conclusion . . . . .	22
<b>2</b>	<b>Symmetries and Conservation laws of KdV-Burgers equation: Illustrative example</b>	<b>23</b>
2.1	Lie point symmetries of the KdV-Burgers equation . . . . .	24
2.2	Exact group-invariant solutions for KdV-Burgers equation . . . . .	27
2.3	Construction of conservation laws for KdV-Burgers equation . . . . .	29
2.4	Conclusion . . . . .	31
<b>3</b>	<b>Symmetry reductions and exact solutions of a variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation</b>	<b>33</b>
3.1	$f(t) = 1, g(t) = a_0/t$ and $h(t) = b_0/t$ , where $a_0$ and $b_0$ are arbitrary constants . . . . .	34
3.1.1	Lie point symmetries . . . . .	34
3.1.2	Symmetry reductions and exact group-invariant solutions of the equation (3.2) . . . . .	36
3.2	$f(t) = t, g(t) = a/t$ and $h(t) = bt^3$ , where $a$ and $b$ are arbitrary constants	40
3.2.1	Lie point symmetries . . . . .	40
3.2.2	Symmetry reductions and exact group-invariant solutions of the equation (3.34) . . . . .	42
3.3	$f(t) = a, g(t) = b$ and $h(t) = k(t - d)^2$ , where $a, b$ and $k$ are arbitrary constants . . . . .	45
3.3.1	Lie point symmetries . . . . .	45
3.3.2	Symmetry reductions and exact group-invariant solutions of the equation (3.63) . . . . .	47
3.4	Concluding remarks . . . . .	49

<b>4</b>	<b>Conservation laws of a variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation</b>	<b>50</b>
4.1	$f(t) = 1, g(t) = a_0/t$ and $h(t) = b_0/t$ where $a_0$ and $b_0$ are constants. .	51
4.2	$f(t) = t, g(t) = a/t$ and $h(t) = bt^3$ where $a$ and $b$ are constants. . . .	54
4.3	$f(t) = a, g(t) = b$ and $h(t) = k(t - d)^2$ where $a, b$ and $d$ are constants	56
4.4	Concluding Remarks . . . . .	58
<b>5</b>	<b>Concluding remarks</b>	<b>59</b>
	<b>Bibliography</b>	<b>60</b>

## Declaration

I declare that the dissertation for the degree of Master of Science at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

LETLHOGONOLO DADDY MOLELEKI

15 November 2011

## **Dedication**

To my parents and everyone who contributed to my studies.

## **Acknowledgements**

I would like to express my sincere thanks to my supervisor Professor C M Khaliq for his guidance, patience and support throughout this research project.

I would also like to thank Dr A G Johnpillai and Dr B Muatjetjeja for the helpful discussions.

I greatly appreciate the generous financial grant from the North-West University, Mafikeng Campus and the National Research Foundation.

Finally, thanks to my parents for their everlasting love and support.

Above all, I would like to thank the Almighty God, who made this programme successful.

## Abstract

This research studies two nonlinear problems arising in mathematical physics. Firstly the Korteweg-de Vries-Burgers equation is considered. Lie symmetry method is used to obtain the exact solutions of Korteweg-de Vries-Burgers equation. Also conservation laws are obtained for this equation using the new conservation theorem. Secondly, we consider the generalized (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation of time dependent variable coefficients from the Lie group-theoretic point of view. We classify the Lie point symmetry generators to obtain the optimal system of one-dimensional subalgebras of the Lie symmetry algebras. These subalgebras are then used to construct a number of symmetry reductions and exact group-invariant solutions of the ZK equation. We utilize the new conservation theorem to construct the conservation laws of the ZK equation.

# Introduction

Nonlinear evolution equations (NLEEs) have been extensively studied in the past few decades. These equations arise in various branches of applied sciences such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinetics, chemical physics, and so on. In recent years various methods of solving these types of NLEEs have been proposed. Some of the important methods are solitary wave ansatz method [1,2], homogeneous balance method [3], Lie group analysis [4-6], Weierstrass elliptic function expansion method [7],  $F$ -expansion method [8],  $G'/G$  method [9], exponential function method [10], etc.

It is well known that conservation laws play an important role in the solution process of differential equations. Finding the conservation laws of system of differential equations is often the first step towards finding the solution. In fact, the existence of a large number of conservation laws of a system of partial differential equations is a strong indication of its integrability [4].

In this dissertation two such equations will be studied. These are the Korteweg-de Vries-Burgers equation and the generalized (2+1)-dimensional Zakharov-Kuznetsov equation of time dependent variable coefficients.

Firstly, the Korteweg-de Vries-Burgers equation of the form

$$u_t + kuu_x - au_{xx} + bu_{xxx} = 0, \quad (1)$$

where  $k$ ,  $a > 0$  and  $b$  are arbitrary constants is studied. This equation was derived by Su and Gardner [11] and is a model equation for a wide class of nonlinear systems in the weak nonlinearity and long wavelength approximations. It possesses steady-state

solution which has been demonstrated to model weak plasma shocks propagating perpendicular to a magnetic field [12]. This equation has also been used in the study of wave propagation through a liquid-filled elastic tube [13] and for a description of shallow water waves on a viscous fluid [14].

Secondly, the generalized (2+1)-dimensional Zakharov-Kuznetsov equation with time dependent variable coefficients of the form

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_{xyy} = 0, \quad (2)$$

where  $f(t), g(t)$  and  $h(t)$  are arbitrary smooth functions of the variable  $t$  and  $f, g$  and  $h \neq 0$  is discussed. The equation (2) models the nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields [15]. This equation also appears in different forms in many areas of physics, applied mathematics and engineering (see for example [1, 2]).

The outline of this research project is as follows:

In Chapter 1 the basic definitions and theorems concerning the one-parameter groups of transformations are presented. The fundamental operators and their relationship for the conservation laws are given. Also the variational method for a system and its adjoint is presented.

Chapter 2 deals with the construction of exact solutions of the Korteweg-de Vries-Burgers equation. The Lie symmetry method is used to find the exact solution of (1). Furthermore, conservation laws are constructed for (1) by using the new conservation theorem.

In Chapter 3, three special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation (2) are considered and Lie symmetry method is employed to obtain the exact solutions.

Chapter 4 deals with the construction of conservation laws for the three special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation (2) that are studied in Chapter 3. The new conservation theorem is used to construct the

conservation laws.

Chapter 5 summarizes the results of the dissertation and discusses some future possible work.

Bibliography is given at the end.



# Chapter 1

## Lie symmetry methods for differential equations and the conservation theorems

In this chapter we give some basic methods of Lie symmetry analysis of differential equations including the algorithm to determine the Lie point symmetries of partial differential equations (PDEs). Also, we give the fundamental operators and their relationships and the variational approaches to construct conservation laws for a system of PDEs.

### 1.1 Introduction

In the late nineteenth century an outstanding mathematician Sophus Lie (1842-1899) developed a new method, known as Lie group analysis, for solving differential equations and showed that majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Recently, many good books have appeared in the literature in this field. We mention a few here, Bluman and Kumei [4], Ovsiannikov [5], Olver [6], Stephani [16], Ibragimov [17, 18], Cantwell [19]. See also Mahomed [20].

Definitions and results given in this Chapter are taken from the books mentioned above.

## 1.2 Local one-parameter Lie group

Here a transformation will be understood to mean an invertible transformation, i.e. a bijective map. Let  $t$ ,  $x$  and  $y$  be three independent variables and  $u$  be a dependent variable. We consider a change of the variables  $t$ ,  $x$ ,  $y$  and  $u$ :

$$\begin{aligned} T_a : \bar{t} &= f(t, x, y, u, a), \quad \bar{x} = g(t, x, y, u, a), \\ \bar{y} &= g(t, x, y, u, a), \quad \bar{u} = h(t, x, y, u, a) \end{aligned} \quad (1.1)$$

with  $a$  being a real parameter, which continuously ranges in values from a neighborhood  $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$  of  $a = 0$  and  $f$ ,  $g$ ,  $k$  and  $h$  are differentiable functions.

**Definition 1.1** A *continuous one-parameter (local) Lie group of transformations* is a set  $G$  of transformations (1.1) which satisfies the following three conditions:

- (i) For  $T_a, T_b \in G$  where  $a, b \in \mathcal{D}' \subset \mathcal{D}$  then  $T_b, T_a = T_c \in G$ ,  $c = \phi(a, b) \in \mathcal{D}$  (Closure),
- (ii)  $T_0 \in G$  if and only if  $a = 0$  such that  $T_0 T_a = T_a T_0 = T_a$  (Identity),
- (iii) For  $T_a \in G$ ,  $a \in \mathcal{D}' \subset \mathcal{D}$ ,  $T_a^{-1} = T_{a^{-1}} \in G$ ,  $a^{-1} \in \mathcal{D}$  such that  $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$  (Inverse).

From (i) we see that the associativity property is satisfied. Also, if the identity transformation occurs at  $a = a_0 \neq 0$  i.e.  $T_{a_0}$  is the identity, then a shift of the parameter  $a = \bar{a} + a_0$  will give  $T_0$  as above. The property (i) can be written as

$$\begin{aligned} \bar{t} &\equiv f(\bar{t}, \bar{x}, \bar{y}, \bar{u}, b) = f(t, x, y, u, \phi(a, b)), \\ \bar{x} &\equiv g(\bar{t}, \bar{x}, \bar{y}, \bar{u}, b) = g(t, x, y, u, \phi(a, b)), \\ \bar{y} &\equiv g(\bar{t}, \bar{x}, \bar{y}, \bar{u}, b) = k(t, x, y, u, \phi(a, b)), \\ \bar{u} &\equiv h(\bar{t}, \bar{x}, \bar{y}, \bar{u}, b) = h(t, x, y, u, \phi(a, b)). \end{aligned} \quad (1.2)$$

The function  $\phi$  is termed as the *group composition law*. A group parameter  $a$  is called *canonical* if  $\phi(a, b) = a + b$ .

**Theorem 1.1** For any  $\phi(a, b)$ , there exists the canonical parameter  $\tilde{a}$  defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \quad \text{where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

We now give the definition of a symmetry group for the third-order PDE

$$u_t = F(t, x, y, u, u_t, u_x, u_{xx}, u_{xy}, u_{xxx}, u_{xyy}, \dots), \quad \frac{\partial F}{\partial u_{xxx}} \neq 0. \quad (1.3)$$

**Definition 1.2 (Symmetry group)** A one-parameter group  $G$  of transformations (1.1) is called a *symmetry group* of (1.3) if it is form-invariant (has the same form) in the new variables  $\bar{t}, \bar{x}, \bar{y}$  and  $\bar{u}$ , i.e. ,

$$\bar{u}_t = F(\bar{t}, \bar{x}, \bar{y}, \bar{u}, \bar{u}_t, \bar{u}_x, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{y}}, \bar{u}_{\bar{x}\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{y}\bar{y}}, \dots), \quad (1.4)$$

where the function  $F$  is the same as in (1.3).

### 1.3 Infinitesimal transformations

Lie's theory tells us that the construction of the symmetry group  $G$  is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\begin{aligned} \bar{t} &\approx t + a \tau(t, x, y, u), & \bar{x} &\approx x + a \xi(t, x, y, u), \\ \bar{y} &\approx y + a \psi(t, x, y, u), & \bar{u} &\approx u + a \eta(t, x, y, u) \end{aligned} \quad (1.5)$$

obtained from (1.1) by expanding the functions  $f, g, k$  and  $h$  into Taylor series in  $a$  about  $a = 0$  and also taking into account the initial conditions

$$f|_{a=0} = t, \quad g|_{a=0} = x, \quad k|_{a=0} = y, \quad h|_{a=0} = u.$$

Thus, we have

$$\begin{aligned} \tau(t, x, y, u) &= \left. \frac{\partial f}{\partial a} \right|_{a=0}, & \xi(t, x, y, u) &= \left. \frac{\partial g}{\partial a} \right|_{a=0}, \\ \psi(t, x, y, u) &= \left. \frac{\partial k}{\partial a} \right|_{a=0}, & \eta(t, x, y, u) &= \left. \frac{\partial h}{\partial a} \right|_{a=0}. \end{aligned} \quad (1.6)$$

Now one can write (1.5) as

$$\bar{t} \approx (1 + aX)t, \quad \bar{x} \approx (1 + aX)x, \quad \bar{y} \approx (1 + aX)y, \quad \bar{u} \approx (1 + aX)u,$$

where

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}. \quad (1.7)$$

This differential operator  $X$  is known as the *infinitesimal operator (generator)* of the group  $G$ . If the group  $G$  is admitted by (1.3), we say that  $X$  is an *admitted operator* of (1.3) or  $X$  is an *infinitesimal symmetry* of (1.3).

## 1.4 Group invariants

**Definition 1.3** A function  $F(t, x, y, u)$  is called an *invariant of the group of transformation* (1.1) if

$$\begin{aligned} F(\bar{t}, \bar{x}, \bar{y}, \bar{u}) &\equiv F(f(t, x, y, u, a), g(t, x, y, u, a), k(t, x, y, u, a), h(t, x, y, u, a)) \\ &= F(t, x, y, u), \end{aligned} \quad (1.8)$$

identically in  $t, x, y, u$  and  $a$ .

**Theorem 1.2 (Infinitesimal criterion of invariance)** A necessary and sufficient condition for a function  $F(t, x, u)$  to be an invariant is that

$$\begin{aligned} XF &\equiv \tau(t, x, y, u) \frac{\partial F}{\partial t} + \xi(t, x, y, u) \frac{\partial F}{\partial x} \\ &+ \psi(t, x, y, u) \frac{\partial F}{\partial y} + \eta(t, x, y, u) \frac{\partial F}{\partial u} = 0. \end{aligned} \quad (1.9)$$

From the above theorem it follows that every one-parameter group of point transformations (1.1) has three functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(t, x, y, u) = c_1, \quad J_2(t, x, y, u) = c_2, \quad J_3(t, x, y, u) = c_3,$$

of the characteristic equations

$$\frac{dt}{\tau(t, x, y, u)} = \frac{dx}{\xi(t, x, y, u)} = \frac{dy}{\psi(t, x, y, u)} = \frac{du}{\eta(t, x, y, u)}.$$

**Theorem 1.3** Given the infinitesimal transformation (1.5) or its symbol  $X$ , the corresponding one-parameter group  $G$  is obtained by solving the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{da} &= \tau(\bar{t}, \bar{x}, \bar{y}, \bar{u}), & \frac{d\bar{x}}{da} &= \xi(\bar{t}, \bar{x}, \bar{y}, \bar{u}), \\ \frac{d\bar{y}}{da} &= \psi(\bar{t}, \bar{x}, \bar{y}, \bar{u}), & \frac{d\bar{u}}{da} &= \eta(\bar{t}, \bar{x}, \bar{y}, \bar{u}) \end{aligned} \quad (1.10)$$

subject to the initial conditions

$$\bar{t}|_{a=0} = t, \quad \bar{x}|_{a=0} = x, \quad \bar{y}|_{a=0} = y, \quad \bar{u}|_{a=0} = u.$$

## 1.5 Construction of a symmetry group

Here we describe the algorithm to determine a symmetry group for a given PDE but first we give some definitions.

### 1.5.1 Prolongation of point transformations

Consider a third-order PDE

$$E(t, x, y, u, u_t, u_x, u_{xx}, u_{xy}, u_{xxx}, u_{xyy}, \dots) = 0, \quad (1.11)$$

where  $t$ ,  $x$  and  $y$  are three independent variables and  $u$  is a dependent variable. Let

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}, \quad (1.12)$$

be the infinitesimal generator of the one-parameter group  $G$  of transformation (1.1).

The *third prolongation* of the operator  $X$  is denoted by  $X^{[3]}$  and is given by

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{222} \frac{\partial}{\partial u_{xxx}} + \zeta_{233} \frac{\partial}{\partial u_{xyy}} + \dots,$$

where

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\psi), \\
\zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{yx} D_x(\psi), \\
\zeta_{23} &= D_y(\zeta_2) - u_{tx} D_y(\tau) - u_{xx} D_y(\xi) - u_{yx} D_y(\psi), \\
\zeta_{222} &= D_x(\zeta_{22}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi) - u_{yxx} D_x(\psi), \\
\zeta_{233} &= D_y(\zeta_{23}) - u_{txy} D_y(\tau) - u_{xxy} D_y(\xi) - u_{xyy} D_y(\psi).
\end{aligned}$$

Here, the total derivatives  $D_t$ ,  $D_x$  and  $D_y$  are given by, respectively

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tt} \frac{\partial}{\partial u_t} + \dots, \quad (1.13)$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + \dots, \quad (1.14)$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \dots. \quad (1.15)$$

Applying the definitions of  $D_t$ ,  $D_x$  and  $D_y$  given above, we obtain

$$\zeta_1 = \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u - u_y \psi_t - u_y u_t \psi_u, \quad (1.16)$$

$$\zeta_2 = \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u - u_y \psi_x - u_y u_x \psi_u, \quad (1.17)$$

$$\begin{aligned}
\zeta_{22} &= \eta_{xx} + 2u_x \eta_{xu} + (u_x)^2 \eta_{uu} + u_{xx} \eta_u - u_t \tau_{xx} - 2u_t u_x \tau_{xu} - 2u_{tx} \tau_x \\
&\quad - u_t (u_x)^2 \tau_{uu} - u_t u_{xx} \tau_u - 2u_{tx} u_x \tau_u - u_x \xi_{xx} - 2(u_x)^2 \xi_{xu} \\
&\quad - 2u_{xx} \xi_x - (u_x)^2 \xi_{uu} - 3u_x u_{xx} \xi_u - u_y \psi_{xx} - 2u_y u_x \psi_{xu} \\
&\quad - 2u_{xy} \psi_x - u_y (u_x)^2 \psi_{uu} - 2u_{xy} u_x \psi_u - u_{xx} u_y \psi_u,
\end{aligned} \quad (1.18)$$

$$\begin{aligned}
\zeta_{23} &= \eta_{xy} + u_y \eta_{xu} + u_x \eta_{uy} + u_x u_y \eta_{uu} + u_{xy} \eta_u - u_t \tau_{xy} - u_t u_y \tau_{xu} \\
&\quad - u_{ty} \tau_x - u_t u_x \tau_{uy} - u_t u_x u_y \tau_{uu} - u_x u_{ty} \tau_u - u_t u_{xy} \tau_u - u_x \xi_{xy} \\
&\quad - u_x u_y \xi_{xu} - u_{xy} \xi_x - (u_x)^2 \xi_{uy} - (u_x)^2 u_y \xi_{uu} - 2u_x u_{xy} \xi_u - u_y \psi_{xy} \\
&\quad - (u_y)^2 \psi_{xu} - u_{yy} \psi_x - u_y u_x \psi_{uy} - u_x (u_y)^2 \psi_{uu} - u_x u_{yy} \psi_u \\
&\quad - 2u_y u_{xy} \psi_u - u_{tx} \tau_y - u_y u_{tx} \tau_u - u_{xx} \xi_y - u_{xx} u_y \xi_u - u_{xy} \psi_y,
\end{aligned} \quad (1.19)$$

$$\begin{aligned}
\zeta_{222} = & \eta_{xxx} + 3u_x\eta_{xxu} + 3(u_x)^2\eta_{xuu} + 3u_{xx}\eta_{xu} + (u_x)^3\eta_{uuu} + 3u_xu_{xx}\eta_{uu} \\
& + u_{xxx}\eta_u - u_t\tau_{xxx} - 3u_tu_x\tau_{xxu} - 3u_{tx}\tau_{xx} - 3u_t(u_x)^2\tau_{xuu} \\
& - 6u_{tx}u_x\tau_{xu} - 3u_{xx}u_t\tau_{xu} - 3u_{xxt}\tau_x - u_t(u_x)^3\tau_{uuu} - 3u_{tx}(u_x)^2\tau_{uu} \\
& - 3u_tu_xu_{xx}\tau_{uu} - 3u_{tx}u_{xx}\tau_u - u_tu_{xxx}\tau_u - u_xu_{xxt}\tau_u - u_x\xi_{xxx} \\
& - 3(u_x)^2\xi_{xxu} - 3u_{xx}\xi_{xx} - 2(u_x)^3\xi_{xuu} - 9u_xu_{xx}\xi_{xu} - 3u_{xxx}\xi_x \\
& - (u_x)^2\xi_{xuu} - (u_x)^3\xi_{uuu} - 2u_xu_{xx}\xi_{uu} - 3(u_x)^2u_{xx}\xi_{uu} - 3(u_{xx})^2\xi_u \\
& - 4u_xu_{xxx}\xi_u - u_y\psi_{xxx} - 3u_xu_y\psi_{xxu} - 3u_y(u_x)^2\psi_{xuu} - 6u_xu_{xy}\psi_{xu} \\
& - 3u_{xy}\psi_{xx} - 3u_yu_{xx}\psi_{xu} - 3u_{xxy}\psi_x - u_y(u_x)^3\psi_{uuu} - 3u_{xy}(u_x)^2\psi_{uu} \\
& - 3u_{xxy}u_x\psi_u - 3u_{xy}u_{xx}\psi_u - u_{xx}u_yu_x\psi_{uu} - u_{xxx}u_y\psi_u, \tag{1.20}
\end{aligned}$$

$$\begin{aligned}
\zeta_{223} = & \eta_{xyy} + 2u_y\eta_{xyu} + (u_y)^2\eta_{xuu} + u_{yy}\eta_{xu} + u_x\eta_{yyu} + 2u_xu_y\eta_{yuu} + 2u_{xy}\eta_{yu} \\
& + u_x(u_y)^2\eta_{uuu} + u_xu_{yy}\eta_{uu} + 2u_yu_{xy}\eta_{uu} + u_{xyy}\eta_u - u_t\tau_{xyy} - 2u_tu_y\tau_{xyu} \\
& - 2u_{ty}\tau_{xy} - u_t(u_y)^2\tau_{xuu} - u_{yy}u_t\tau_{xu} - 2u_{ty}u_y\tau_{xu} - u_{tyy}\tau_x - u_xu_t\tau_{yyu} \\
& - 2u_xu_tu_y\tau_{yuu} - 2u_{ty}u_t\tau_{yu} - 2u_{xy}u_t\tau_{yu} - u_tu_x(u_y)^2\tau_{uuu} - u_{yy}u_xu_t\tau_{uu} \\
& - 2u_{ty}u_xu_y\tau_{uu} - 2u_{ty}u_{xy}\tau_u - 2u_{xy}u_tu_y\tau_{uu} - u_xu_{tyy}\tau_u - u_tu_{xyy}\tau_u \\
& u_x\xi_{xyy} - 2u_xu_y\xi_{xyu} - 2u_{xy}\xi_{xy} - u_x(u_y)^2\xi_{xuu} - u_xu_{yy}\xi_{xu} - 2u_yu_{xy}\xi_{xu} \\
& - u_{xyy}\xi_x - (u_x)^2\xi_{yyu} - 2(u_x)^2u_y\xi_{yuu} - 4u_xu_{xy}\xi_{yu} - (u_x)^2(u_y)^2\xi_{uuu} \\
& - (u_x)^2u_{yy}\xi_{uu} - 4u_xu_yu_{xy}\xi_{uu} - 2(u_{xy})^2\xi_u - 2u_xu_{xyy}\xi_u - u_y\psi_{xyy} - 2u_y\psi_{xyu} \\
& - 2u_{yy}\psi_{xy} - (u_y)^3\psi_{xuu} - 3u_yu_{yy}\psi_{xu} - u_{yyy}\psi_x - u_xu_y\psi_{yyu} - 2u_x(u_y)^2\psi_{yuu} \\
& - 2u_xu_{yy}\psi_{yu} - 4u_yu_{xy}\psi_{yu} - u_x(u_y)^3\psi_{uuu} - 3u_xu_yu_{yy}\psi_{uu} - 3(u_y)^2u_{xy}\psi_{uu} \\
& - 3u_{xy}u_{yy}\psi_u - u_{yyy}u_x\psi_u - 3u_yu_{xyy}\psi_u - u_{tx}\tau_{yy} - 2u_yu_{tx}\tau_{yu} - 2u_{txy}\tau_y \\
& - (u_y)^2u_{tx}\tau_{uu} - u_{yy}u_{tx}\tau_u - 2u_yu_{txy}\tau_u - u_{xx}\xi_{yy} - 2u_yu_{xx}\xi_{yu} - 2u_{xxy}\xi_y \\
& - (u_y)^2u_{xx}\xi_{uu} - u_{yy}u_{xx}\xi_u - 2u_yu_{xxy}\xi_u - u_{xy}\psi_{yy} - 2u_{xyy}\psi_y. \tag{1.21}
\end{aligned}$$

## 1.5.2 Group admitted by a PDE

The operator

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u} \quad (1.22)$$

is a *point symmetry* of the third-order PDE (1.11) if

$$X^{[3]}(E) = 0 \quad (1.23)$$

whenever  $E = 0$ . This can also be written as (symmetry condition)

$$X^{[3]}(E)|_{E=0} = 0, \quad (1.24)$$

where the symbol  $|_{E=0}$  means evaluated on the equation  $E = 0$ .

**Definition 1.4** Equation (1.24) is called the *determining equation* of (1.11), because it determines all the infinitesimal symmetries of (1.11).

The theorem below enables us to construct some solutions of (1.11) from known one.

**Theorem 1.4** A symmetry of (1.11) transforms any solution of (1.11) into another solution of the same equation.

## 1.6 Lie algebras

Let us consider two operators  $X_1$  and  $X_2$  defined by

$$X_1 = \tau_1(t, x, y, u) \frac{\partial}{\partial t} + \xi_1(t, x, y, u) \frac{\partial}{\partial x} + \psi_1(t, x, y, u) \frac{\partial}{\partial y} + \eta_1(t, x, y, u) \frac{\partial}{\partial u},$$

and

$$X_2 = \tau_2(t, x, y, u) \frac{\partial}{\partial t} + \xi_2(t, x, y, u) \frac{\partial}{\partial x} + \psi_2(t, x, y, u) \frac{\partial}{\partial y} + \eta_2(t, x, y, u) \frac{\partial}{\partial u}.$$

**Definition 1.5 (Commutator)** The *commutator* of  $X_1$  and  $X_2$ , written as  $[X_1, X_2]$ , is defined by  $[X_1, X_2] = X_1(X_2) - X_2(X_1)$ .

**Definition 1.6 (Lie algebra)** A Lie algebra is a vector space  $L$  of operators with the following property : For all  $X_1, X_2 \in L$ , the commutator  $[X_1, X_2] \in L$ .

The dimension of a Lie algebra is the dimension of the vector space  $L$ .

**Theorem 1.5** The set of all solutions of any determining equation forms a Lie algebra.

## 1.7 Fundamental operators and their relationship

In this section we briefly present the notation and pertinent results, which we utilize later in this dissertation.

Consider a  $k$ th-order system of PDEs of  $n$  independent variables  $x = (x^1, x^2, \dots, x^n)$  and  $m$  dependent variables  $u = (u^1, u^2, \dots, u^m)$ , viz.,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (1.25)$$

where  $u_{(1)}, u_{(2)}, \dots, u_{(k)}$  denote the collections of all first, second,  $\dots$ ,  $k$ th-order partial derivatives, that is,  $u_i^\alpha = D_i(u^\alpha)$ ,  $u_{ij}^\alpha = D_j D_i(u^\alpha)$ ,  $\dots$ , respectively, with the *total derivative operator* with respect to  $x^i$  given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (1.26)$$

and the summation convention is used whenever appropriate.

The following are known (see for e.g., [21] and the references therein).

The *Euler-Lagrange operator*, for each  $\alpha$ , is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (1.27)$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \quad (1.28)$$

where  $\mathcal{A}$  is the space of *differential functions*.

The operator (1.28) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad (1.29)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \quad (1.30)$$

in which  $W^\alpha$  is the *Lie characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha. \quad (1.31)$$

One can write the Lie-Bäcklund operator (1.29) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}. \quad (1.32)$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator  $X$  are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (1.33)$$

where the Euler-Lagrange operators with respect to derivatives of  $u^\alpha$  are obtained from (1.27) by replacing  $u^\alpha$  by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u_{i j_1 j_2 \dots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \quad (1.34)$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \quad (1.35)$$

The  $n$ -tuple vector  $T = (T^1, T^2, \dots, T^n)$ ,  $T^j \in \mathcal{A}$ ,  $j = 1, \dots, n$ , is a *conserved vector* of (1.25) if  $T^i$  satisfies

$$D_i T^i|_{(E_\alpha(x, u, u_{(1)}, \dots, u_{(k)})=0)} = 0. \quad (1.36)$$

The equation (1.36) defines a *local conservation law* of system (1.25).

## 1.8 Variational method for a system and its adjoint

The system of *adjoint equations* to the system of  $k$ th-order differential equations (1.25) is defined by [22]

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (1.37)$$

where

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha, \beta = 1, \dots, m, \quad v = v(x) \quad (1.38)$$

and  $v = (v^1, v^2, \dots, v^m)$  are new dependent variables.

We recall here the following results as given in Ibragimov [23].

A system of equations (1.25) is said to be *self-adjoint* if the substitution of  $v = u$  into the system of adjoint equations (1.37) yields the same system (1.25).

Assume the system of equations (1.25) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \quad (1.39)$$

Then the system of adjoint equations (1.37) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)], \quad (1.40)$$

where the operator (1.40) is an extension of (1.39) to the variable  $v^\alpha$  and the  $\lambda_\beta^\alpha$  are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \quad (1.41)$$

*Theorem 3.1.* We now state the new conservation theorem due to Ibragimov [23]. Every Lie point, Lie-Bäcklund and non local symmetry (1.39) admitted by the system of equations (1.25) gives rise to a conservation law for the system consisting of the equation (1.25) and the adjoint equation (1.37), where the components  $T^i$  of the conserved vector  $T = (T^1, \dots, T^n)$  are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta L}{\delta u_{i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (1.42)$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \quad (1.43)$$

*Remark:* If we consider the differentiation of  $L$  up to third-order derivative only then the equation (1.42) can be written as

$$\begin{aligned} T^i = & \xi^i L + W^\alpha \left\{ \frac{\partial L}{\partial u_i^\alpha} - D_j \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right\} \\ & + D_j (W^\alpha) \left\{ \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) - D_k \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right\} + D_j D_k (W^\alpha) \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right). \end{aligned} \quad (1.44)$$

## 1.9 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis of PDEs and gave some results which will be used throughout this project. We also gave the algorithm to determine the Lie point symmetries of PDEs. The fundamental operators and their relationship for the conservation laws are given, also the variational method for a system and its adjoint.

## Chapter 2

# Symmetries and Conservation laws of KdV-Burgers equation: Illustrative example

In this chapter we consider the Korteweg-de Vries-Burgers equation

$$u_t + kuu_x - au_{xx} + bu_{xxx} = 0, \quad (2.1)$$

where  $k$ ,  $a$  and  $b$  are arbitrary constant. Equation (2.1) was constructed when electron inertia effects in the description of weak nonlinear plasma waves were included. In one hand, when the parameter  $a = 0$  equation (2.1) will be the KdV equation. The KdV equation has been focus considerable recent studies for finding exact solution in [24–26] as well as numerical solution in [27–29].

Recently, exact solutions of equation (2.1) were obtained in [30] by using Exp-function method.

In this chapter we use Lie symmetry analysis to obtain the exact solutions of equation (2.1). Also, we apply the new conservation theorem [23] to calculate the conservation laws of equation (2.1).

## 2.1 Lie point symmetries of the KdV-Burgers equation

In this section we first determine the Lie point symmetries of (2.1) and then use them to construct some exact solutions.

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Determining all the symmetries of a differential equation is a formidable task. However, the Norwegian mathematician Sophus Lie (1842-1899) realized that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. The symmetry group of KdV-Burgers equation (2.1) will be generated by the vector field of the form

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}. \quad (2.2)$$

Applying the third prolongation to (2.1) yields the following over determined system of linear PDEs:

$$\tau_u = 0, \quad (2.3)$$

$$\tau_x = 0, \quad (2.4)$$

$$\xi_u = 0, \quad (2.5)$$

$$\eta_{uu} = 0, \quad (2.6)$$

$$3\xi_x - \tau_t = 0, \quad (2.7)$$

$$\eta_t + ku\eta_x - a\eta_{xx} + b\eta_{xxx} = 0, \quad (2.8)$$

$$-a\tau_t + 2a\xi_x + 3b\eta_{xu} - 3b\xi_{xx} = 0, \quad (2.9)$$

$$ku\tau_t - \xi_t - ku\xi_x - 2a\eta_{xu} + a\xi_{xx} + k\eta + 3b\eta_{xxu} - b\xi_{xxx} = 0. \quad (2.10)$$

Equations (2.3) and (2.4) imply that

$$\tau = A(t). \quad (2.11)$$

Equation (2.5) implies that

$$\xi = B(t, x). \quad (2.12)$$

Equation (2.6) gives

$$\eta = C(t, x)u + D(t, x). \quad (2.13)$$

Substituting the above values of  $\xi$  and  $\eta$  in (2.7), we obtain

$$\xi = \frac{1}{3}A'(t)x + E(t). \quad (2.14)$$

Now substituting the results of  $\xi$ ,  $\eta$  and  $\tau$  in (2.9), we get

$$C(t, x) = \frac{a}{9b}A'(t)x + F(t) \quad (2.15)$$

which give us

$$\eta = \frac{a}{9b}A'(t)xu + uF(t) + D(t, x) \quad (2.16)$$

and inserting the above value of  $\eta$  in (2.8), we obtain

$$\frac{a}{9b}A''(t)xu + uF'(t) + D_t + \frac{u^2ak}{9b}A'(t) + ukD_x - aD_{xx} + bD_{xxx} = 0. \quad (2.17)$$

Separating (2.17) with respect to  $u$ , we obtain

$$u^2 : A'(t) = 0, \quad (2.18)$$

$$u : F'(t) + kD_x = 0, \quad (2.19)$$

$$u^0 : D_t - aD_{xx} + bD_{xxx} = 0. \quad (2.20)$$

This implies that

$$\tau = c_1, \quad (2.21)$$

$$\eta = uF(t) + D(t, x), \quad (2.22)$$

$$\xi = E(t). \quad (2.23)$$

Now substituting the new results of  $\xi$ ,  $\eta$  and  $\tau$  in (2.10), we get

$$-E'(t) + kuF(t) + kD(t, x) = 0 \quad (2.24)$$

and splitting (2.24) with respect to  $u$ , we have

$$u : F(t) = 0, \quad (2.25)$$

$$u^0 : -E'(t) + kD(t, x) = 0. \quad (2.26)$$

Substituting (2.25) into (2.19), we obtain  $D_x(t, x) = 0$ . This gives

$$D(t, x) = G(t) \quad (2.27)$$

and substituting (2.27) into (2.20), we get  $G'(t) = 0$ . This implies that  $G(t) = c_2$ , where  $c_2$  is an arbitrary constant and so  $D(t, x) = c_2$ . Hence

$$\eta = c_2. \quad (2.28)$$

Then equation (2.26) gives  $E'(t) = c_2k$ , which implies that  $E(t) = c_2kt + c_3$ , where  $c_3$  is an arbitrary constant. Thus we obtain the value of  $\xi$  as

$$\xi = c_2kt + c_3. \quad (2.29)$$

Thus

$$\tau = c_1, \quad (2.30)$$

$$\xi = c_2kt + c_3, \quad (2.31)$$

$$\eta = c_2 \quad (2.32)$$

and so the infinitesimal symmetries of the KdV-Burgers equation are

$$X_1 = \partial_t,$$

$$X_2 = kt \partial_x + \partial_u,$$

$$X_3 = \partial_x.$$

We note that  $X_1$  is the time translation symmetry,  $X_2$  is a galilean boost and  $X_3$  is the space translation symmetry.

## 2.2 Exact group-invariant solutions for KdV-Burgers equation

We now construct group-invariant solutions under the symmetry operators of the KdV-Burgers equation. We start with the operator  $X_1$ .

### Case 1.

Let us calculate the invariant solution under the operator  $X_1$ , namely

$$X_1 = \partial_t. \quad (2.33)$$

Let

$$X_1 J = \frac{\partial J}{\partial t} + 0 \frac{\partial J}{\partial x} + 0 \frac{\partial J}{\partial u} = 0.$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which gives us two invariants  $J_1 = x$  and  $J_2 = u$ . Thus the invariant solution is given by  $J_2 = f(J_1)$ , i.e,

$$u = f(x). \quad (2.34)$$

Differentiating (2.34), and substituting the results in (2.2) we get

$$bf''' - af'' + kff' = 0. \quad (2.35)$$

Integration of (2.35) gives

$$bf'' - af' + \frac{k}{2}f^2 + C_1 = 0, \quad (2.36)$$

where  $C_1$  is an arbitrary constant on integration.

### Case 2.

Let us now calculate the invariant solution under the operator  $X_2$ , namely

$$X_2 = kt\partial_x + \partial_u. \quad (2.37)$$

Let

$$X_2 J = 0 \frac{\partial J}{\partial t} + kt \frac{\partial J}{\partial x} + \frac{\partial J}{\partial u} = 0.$$

The characteristic equations are

$$\frac{dt}{0} = \frac{dx}{kt} = \frac{du}{1}.$$

We obtain  $J_1 = t$  as one invariant. Solving

$$\frac{dx}{kt} = \frac{du}{1}$$

gives us the second invariant as  $J_2 = u - \frac{x}{kt}$ . Thus the invariant solution is given by

$J_2 = f(J_1)$ , i.e,

$$u = \frac{x}{kt} + f(t). \quad (2.38)$$

Differentiating (2.38), and substituting the results in (2.2) we get

$$\frac{df}{dt} + \frac{f}{t} = 0. \quad (2.39)$$

Solving for  $f$ , we obtain

$$f(t) = \frac{C}{t}, \quad (2.40)$$

where  $C$  is an arbitrary constant and hence the invariant solution of (2.2) is

$$u(t, x) = \frac{x + kC}{kt}. \quad (2.41)$$

### Case 3.

The symmetry  $X_3$  leads to the group-invariant solution  $J_2 = f(J_1)$ , where  $J_1 = t$  and  $J_2 = u$ . Substitution of this solution into the equation (2.2) gives the solution

$$u(t, x) = C, \quad (2.42)$$

where  $C$  is an arbitrary constant.

### Case 4.

Finally, we construct the invariant solutions under a linear combination of the symmetry operators  $X_1$  and  $X_3$ , namely,

$$X = 0\partial_t + (\delta + kt)\partial_x + \partial_u, \quad (2.43)$$

where  $\delta$  is an arbitrary constant. Let

$$XJ = 0\frac{\partial J}{\partial t} + (\delta + kt)\frac{\partial J}{\partial x} + \frac{\partial J}{\partial u} = 0.$$

The characteristic equations are

$$\frac{dt}{0} = \frac{dx}{(\delta + kt)} = \frac{du}{1}.$$

We obtain  $J_1 = t$  as one invariant. Solving

$$\frac{dx}{\delta + kt} = \frac{du}{1}$$

gives us the second invariant as  $J_2 = x - (\delta + kt)u$ . Thus the invariant solution is given by  $J_2 = f(J_1)$ , i.e,

$$u = \frac{x}{\delta + kt} - \frac{1}{\delta + kt}f(t). \quad (2.44)$$

Differentiating (2.44), and substituting the results in (2.2) we get

$$\frac{df}{dt} = 0. \quad (2.45)$$

Solving for  $f$ , we obtain

$$f(t) = C_4, \quad (2.46)$$

where  $C_4$  is an arbitrary constant and hence the invariant solution of (2.2) is

$$u(t, x) = \frac{x - C_4}{\delta + kt}. \quad (2.47)$$

## 2.3 Construction of conservation laws for KdV-Burgers equation

In this section we construct the conservation laws of the Korteweg-de Vries-Burgers equation

$$u_t + kuv_x - au_{xx} + bu_{xxx} = 0 \quad (2.48)$$

using the new conservation theorem due to Ibragimov [23].

The equation (2.48) admits the following Lie point symmetry generators

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= kt\partial_x + \partial_u, \\ X_3 &= \partial_x. \end{aligned}$$

The adjoint equation of (2.48), by invoking (1.38), is

$$E^*(t, x, u, v, \dots, u_{xxx}, v_{xxx}) = \frac{\delta}{\delta u} [v(u_t + kuu_x - au_{xx} + bu_{xxx})] = 0, \quad (2.49)$$

where  $v = v(t, x)$  is a new dependent variable and (2.49) gives

$$v_t + kuv_x + av_{xx} + bv_{xxx} = 0. \quad (2.50)$$

It is obvious from the adjoint equation (2.50) that equation (2.48) is not self-adjoint. By recalling (1.43), we get the following Lagrangian for the system of equations (2.48) and (2.50):

$$L = v(u_t + kuu_x - au_{xx} + bu_{xxx}). \quad (2.51)$$

(i) We first consider the Lie point symmetry generator  $X_1 = \partial_t$ . It can be verified from (1.40) that the operator  $Y_1$  is the same as  $X_1$  and hence the Lie characteristic function is  $W = -u_t$ . Thus by using (1.44), the components  $T^i$ ,  $i = 1, 2$ , of the conserved vector  $T = (T^1, T^2)$  are given by

$$\begin{aligned} T^1 &= (kuu_x - au_{xx} + bu_{xxx})v, \\ T^2 &= (au_{tx} - kuu_t - bu_{txx})v + (bu_{tx} - au_t)v_x - bu_tv_{xx}. \end{aligned}$$

*Remark:* The conserved vector  $T$  contains the arbitrary solution  $v$  of the adjoint equation (2.50) and hence gives an infinite number of conservation laws.

The same remark applies to all the following cases where we use the new conservation theorem.

(ii) Now for the symmetry generator  $X_2 = kt\partial_x + \partial_u$ , we have  $W = 1 - ktu_x$ . Hence, by invoking (1.44), the symmetry generator  $X_2$  gives rise to the following

components of the conserved vector

$$\begin{aligned} T^1 &= (1 - kt u_x)v, \\ T^2 &= (kt u_t + ku)v + (a - kat u_x + kbt u_{xx})v_x - kbt u_x v_{xx}. \end{aligned}$$

(iii) Finally, we consider the Lie symmetry generator  $X_3 = \partial_x$  has the Lie characteristic function  $W = -u_x$ . Hence using (1.44), one can obtain the conserved vector  $T$  whose components are given by

$$\begin{aligned} T^1 &= -u_x v, \\ T^2 &= u_t v + (bu_{xx} - au_x)v_x - bu_x v_{xx}. \end{aligned}$$

Verification for the first conserved vectors that we obtained from symmetry  $X_1$ .

$$D_i T^i = D_t T^1 + D_x T^2$$

$$\begin{aligned} D_i T^i &= \left( \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \dots \right) \left( kuvvu_x - au_{xx} + bu_{xxx} \right) \\ &+ \left( \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots \right) \left( avu_{tx} - kuvu_t - bu_{txx}v \right. \\ &\quad \left. + bu_{tx}v_x - au_tv_x - bu_tv_{xx} \right) \\ &= kuvu_x v_t + ku_tv_x v + kuvu_{tx} - av_tv_{xx} - avu_{xxt} + bu_{xxx}v_t + bu_tv_{xxx} \\ &\quad + avu_x u_{tx} + avu_{txx} - kuv_x u_t - kuvu_{tx} - bv_x u_{txx} - bu_{txx}v_x - au_tv_{xx} \\ &\quad - av_x u_{tx} + bu_{xx}u_{tx} + bu_x u_{txx} - bu_{tx}v_{xx} - bu_tv_{xxx} = 0 \end{aligned}$$

on equations (2.48) and (2.50). Likewise, it can be verified that the other two conserved vectors, which we derived above, satisfy the equation  $D_i T^i = 0$  whenever equations (2.48) and (2.50) are satisfied.

## 2.4 Conclusion

In this chapter we studied the KdV-Burgers equation using the Lie symmetry group method. Firstly, we derived the Lie point symmetries of the KdV-Burgers equation

and then used them to obtain the group-invariant solutions. Secondly, we employed the new conservation theorem to construct the conservation laws of the KdV-Burgers equation.

## Chapter 3

# Symmetry reductions and exact solutions of a variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation

In this chapter we consider three special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation of time dependent variable coefficients

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_{xyy} = 0. \quad (3.1)$$

We classify the Lie point symmetry generators to obtain the optimal system of one-dimensional subalgebras of the Lie symmetry algebras. These subalgebras are then used to construct a number of symmetry reductions and exact group-invariant solutions.

Part of this work has been accepted for publication in [31].

### Lie point symmetries and symmetry reductions

### 3.1 $f(t) = 1$ , $g(t) = a_0/t$ and $h(t) = b_0/t$ , where $a_0$ and $b_0$ are arbitrary constants

#### 3.1.1 Lie point symmetries

Therefore, the equation that is going to be studied in this section takes the form

$$u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} = 0. \quad (3.2)$$

The symmetry group of ZK equation (3.2) will be generated by vector field of the form

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}. \quad (3.3)$$

Applying the third prolongation to (3.2) yields the following over determined system of linear PDEs:

$$\tau_u = 0, \quad (3.4)$$

$$\tau_x = 0, \quad (3.5)$$

$$\xi_u = 0, \quad (3.6)$$

$$\psi_u = 0, \quad (3.7)$$

$$\tau_y = 0, \quad (3.8)$$

$$\psi_x = 0, \quad (3.9)$$

$$\xi_y = 0, \quad (3.10)$$

$$\psi_t = 0, \quad (3.11)$$

$$\xi_{xx} = 0, \quad (3.12)$$

$$\eta_{xu} = 0, \quad (3.13)$$

$$\eta_{uu} = 0, \quad (3.14)$$

$$2\eta_{yu} - \psi_{yy} = 0, \quad (3.15)$$

$$-1/t^2 \tau + 1/t \tau_t - 3/t \xi_x = 0, \quad (3.16)$$

$$\eta + \tau_t u - \xi_t - \xi_x u + b_0/t \eta_{yyu} = 0, \quad (3.17)$$

$$\eta_t + \eta_x u + a_0/t \eta_{xxx} + b_0/t \eta_{xyy} = 0, \quad (3.18)$$

$$-1/t^2 \tau + 1/t \tau_t - 1/t \xi_x - 2/t \psi_y = 0. \quad (3.19)$$

Solving the determining equations (3.4)-(3.19) for  $\tau, \xi, \psi$  and  $\eta$ , we obtain the following symmetry group generators given by

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_y, \\ X_3 &= t \partial_x + \partial_u, \\ X_4 &= t \partial_t - u \partial_u. \end{aligned} \quad (3.20)$$

### 3.1.2 Symmetry reductions and exact group-invariant solutions of the equation (3.2)

Here we first construct the optimal system of one-dimensional subalgebras of the Lie algebra admitted by the equation (3.2). The classification of the one-dimensional subalgebras are then used to reduce the equation (3.2) into a PDE having two independent variables. Then we also study the symmetry properties of the reduced PDE to derive further symmetry reductions and exact group-invariant solutions for the underlying equation.

The results on the classification of the Lie point symmetries of the equation (3.2) are summarized by the Tables 1, 2 and 3. The commutator table of the Lie point symmetries of the equation (3.2) and the adjoint representations of the symmetry group of (3.2) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (3.2) which is given in Table 3 (for more details of the approach see [6] and the references therein).

**Table 1.** Commutator table of the Lie algebra of equation (3.2)

	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	0	0	0	0
$X_2$	0	0	0	0
$X_3$	0	0	0	$-X_3$
$X_4$	0	0	$X_3$	0

**Table 2.** Adjoint table of the Lie algebra of equation (3.2)

Ad	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	$X_1$	$X_2$	$X_3$	$X_4$
$X_2$	$X_1$	$X_2$	$X_3$	$X_4$
$X_3$	$X_1$	$X_2$	$X_3$	$X_4 + \epsilon X_3$
$X_4$	$X_1$	$X_2$	$e^{-\epsilon} X_3$	$X_4$

**Table 3.** Subalgebra, group-invariant solutions of (3.2)

N	X	$\alpha$	$\beta$	Group – invariant solution
1	$X_4 + \lambda X_1 + \mu X_2$	$x - \lambda \ln t$	$y - \mu \ln t$	$u = \frac{1}{t} h(\alpha, \beta)$
2	$X_2 + \nu X_1$	$t$	$x - \nu y$	$u = h(\alpha, \beta)$
3	$X_3 + \epsilon X_1$	$t$	$y$	$u = \frac{x}{(t+\epsilon)} + h(\alpha, \beta)$
4	$X_3 + \delta X_2 + \epsilon X_1$	$t$	$\delta x - (t + \epsilon)y$	$u = \frac{x}{(t+\epsilon)} + h(\alpha, \beta)$
5	$X_1$	$t$	$y$	$u = h(\alpha, \beta)$

Here  $\epsilon = 0, \pm 1$ ,  $\delta = \pm 1$  and  $\lambda, \mu$  and  $\nu$  are arbitrary constants.

**Case 1.1** In this case, the group-invariant solution corresponding to the symmetry generator  $X_4 + \lambda X_1 + \mu X_2$  reduces the equation (3.2) to the PDE

$$\lambda h_\alpha + \mu h_\beta - h h_\alpha + h - a_0 h_{\alpha\alpha} - b_0 h_{\alpha\beta\beta} = 0. \quad (3.21)$$

Now the equation (3.21) admits the following symmetry generators given by

$$X_1 = \partial_\alpha, \quad X_2 = \partial_\beta. \quad (3.22)$$

- (a) The group-invariant solution corresponding to  $X_1$  is  $h = H(\beta)$ , the substitution of this solution into the equation (3.21) and solving we obtain a solution  $u(t, x, y) = C e^{-y/\mu}$  for (3.2), here  $C$  is a constant.
- (b) The generator  $X_1 + \rho X_2$ , where  $\rho$  is a constant, leads to the group-invariant solution  $h = H(\beta - \rho\alpha)$ . Substitution of this solution into the equation (3.21) gives rise to the ordinary differential equation (ODE)

$$(\rho^3 a_0 + \rho b_0) H''' + \rho H H' + (\mu - \lambda\rho) H' + H = 0. \quad (3.23)$$

**Case 1.2** The group-invariant solution arising from  $X_2 + \nu X_1$  where  $\nu$  is a constant, reduces the equation (3.2) to the PDE

$$h_\alpha + h h_\beta + \frac{(a_0 + b_0 \nu^2)}{\alpha} h_{\beta\beta\beta} = 0. \quad (3.24)$$

The equation (3.24) admits the following three Lie point symmetry generators

$$X_1 = \partial_\beta, \quad X_2 = \alpha \partial_\beta + \partial_h, \quad X_3 = -\alpha \partial_\alpha + h \partial_h. \quad (3.25)$$

The optimal system of one-dimensional subalgebras are  $X_3 + cX_1, X_2 + dX_1, X_1$ , where  $c$  is an arbitrary real constant and  $d = 0, \pm 1$ .

- (a) The group-invariant solution corresponding to  $X_3 + cX_1$  where  $c$  is a constant, is  $h = \frac{1}{\alpha}H(\beta + c \ln \alpha)$ , the substitution of this solution into the equation (3.24) results in the following ODE

$$(a_0 + b_0 \nu^2)H''' + HH' + a_0H' - H = 0. \quad (3.26)$$

- (b) The generator  $X_2 + dX_1$  where  $d$  is a constant, leads to the group-invariant solution  $h = \frac{\beta}{\alpha+d} + H(\alpha)$ . Substitution of this solution into the equation (3.24) gives the solution

$$u(t, x, y) = \frac{x - \nu y + C}{(t + d)}, \quad (3.27)$$

where  $C$  is a constant.

- (c) The symmetry generator  $X_1$  gives the trivial solution  $u(t, x, y) = C$ , where  $C$  is a constant.

**Case 1.3** The group-invariant solution that corresponds to  $X_3 + \epsilon X_1$  where  $\epsilon$  is a constant, reduces the equation (3.2) to the PDE

$$h_\alpha + \frac{h}{\alpha + \epsilon} = 0. \quad (3.28)$$

Hence the solution of the equation (3.2) is given by

$$u(t, x, y) = \frac{x + H(y)}{(t + \epsilon)}, \quad (3.29)$$

where  $H(y)$  is an arbitrary function of its argument.

**Case 1.4** The invariant solution that corresponds to  $X_3 + \delta X_2 + \epsilon X_1$  where  $\delta$  and  $\epsilon$  are constants, reduces the equation (3.2) to the PDE

$$h_\alpha + \frac{\beta}{(\alpha + \epsilon)}h_\beta + \delta h h_\beta + \frac{h}{(\alpha + \epsilon)} + \left[ \frac{a_0 \delta^3}{\alpha} + \frac{b_0 \delta (\alpha + \epsilon)^2}{\alpha} \right] h_{\beta\beta\beta} = 0. \quad (3.30)$$

- (a) For  $\epsilon = 0$ .

In this case, the PDE (3.30) becomes

$$h_\alpha + \frac{\beta}{\alpha} h_\beta + \delta h h_\beta + \frac{h}{\alpha} + \left( \frac{a_0 \delta^3}{\alpha} + b_0 \delta \alpha \right) h_{\beta\beta\beta} = 0. \quad (3.31)$$

The equation (3.31) admits the Lie algebra spanned by the following symmetry generators

$$X_1 = \alpha \partial_\beta, \quad X_2 = \delta \partial_\beta - 1/\alpha \partial_h. \quad (3.32)$$

(i) The group-invariant solution corresponding to  $X_1$  is  $h = H(\alpha)$ , the substitution of this solution into the equation (3.31) and solving we obtain the solution  $u(t, x, y) = \frac{x+C}{t}$ , where  $C$  is a constant.

(ii) The group-invariant solution corresponding to  $X_1 + \omega X_2$  where  $\omega$  is a constant, is  $h = -\beta/\alpha(\omega \alpha + \delta) + H(\alpha)$ , the substitution of this solution into the equation (3.31) and solving we obtain the solution

$$u(t, x, y) = \frac{\omega x + y + C}{\omega t + \delta},$$

where  $C$  is a constant.

(b) For  $\epsilon \neq 0$ , the PDE (3.30) admits the following symmetry generators

$$X_1 = \delta \partial_\beta - 1/(\alpha + \epsilon) \partial_h, \quad X_2 = \delta \alpha \partial_\beta + \epsilon/(\alpha + \epsilon) \partial_h. \quad (3.33)$$

(i) The group-invariant solution corresponding to  $X_1$  is  $h = -\beta/\delta(\alpha + \epsilon) + H(\alpha)$ , the substitution of this solution into the equation (3.30) and solving we obtain the solution  $u(t, x, y) = \frac{y}{\delta} + C$ , where  $C$  is a constant.

(ii) The  $X_2 + \omega X_1$  where  $\omega$  is a constant, give us the invariant solution as  $h = (\epsilon - \omega)\beta/\delta(\alpha + \omega)(\alpha + \epsilon) + H(\alpha)$ , the substitution of this solution into the equation (3.30) and solving we obtain the solution

$$u(t, x, y) = \frac{\delta x - (\epsilon - \omega)y + C}{\delta(t + \omega)},$$

where  $C$  is a constant.

**Case 1.5** The  $X_1$ -invariant solution reduces the equation (3.2) to  $h_\alpha = 0$ . Hence the solution of the equation (3.2) is given by  $u(t, x, y) = H(y)$ , where  $H(y)$  is an arbitrary function of its argument.

## 3.2 $f(t) = t$ , $g(t) = a/t$ and $h(t) = bt^3$ , where $a$ and $b$ are arbitrary constants

### 3.2.1 Lie point symmetries

Then, equation that is going to be studied in this section takes the form

$$u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy} = 0. \quad (3.34)$$

The symmetry group of ZK equation (3.34) will then be generated by the vector field of the form

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}. \quad (3.35)$$

Applying the third prolongation to (3.34) yields the following over determined system of linear PDEs:

$$\tau_u = 0, \quad (3.36)$$

$$\tau_x = 0, \quad (3.37)$$

$$\xi_u = 0, \quad (3.38)$$

$$\psi_u = 0, \quad (3.39)$$

$$\tau_y = 0, \quad (3.40)$$

$$\psi_x = 0, \quad (3.41)$$

$$\xi_y = 0, \quad (3.42)$$

$$\psi_t = 0, \quad (3.43)$$

$$\eta_{uu} = 0, \quad (3.44)$$

$$\xi_{xx} = 0, \quad (3.45)$$

$$\eta_{xu} = 0, \quad (3.46)$$

$$2\eta_{yu} - \psi_{yy} = 0, \quad (3.47)$$

$$\frac{a}{t}\tau_t - \frac{3a}{t}\xi_x + \frac{a}{t^2}\tau = 0, \quad (3.48)$$

$$\eta_t + \frac{a}{t}\eta_{xxx} + bt^3\eta_{xyy} + tu\eta_x = 0, \quad (3.49)$$

$$bt^3\tau_t - bt^3\xi_x - 2bt^3\psi_y + 3bt^2\tau = 0, \quad (3.50)$$

$$tu\tau_t - \xi_t + t\eta - tu\xi_x + u\tau + bt^3\eta_{yyu} = 0. \quad (3.51)$$

Solving the determining equations (3.36)-(3.51) for  $\tau, \xi, \psi$  and  $\eta$ , we obtain the following symmetry group generators given by

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_y, \\ X_3 &= t^2 \partial_x + 2\partial_u, \\ X_4 &= -t\partial_t - 2y\partial_y + 2u\partial_u. \end{aligned} \quad (3.52)$$

### 3.2.2 Symmetry reductions and exact group-invariant solutions of the equation (3.34)

Here we first construct the optimal system of one-dimensional subalgebras of the Lie algebra admitted by the equation (3.34). The classification of the one-dimensional subalgebras are then used to reduce the equation (3.34) into a PDE having two independent variables. Then we also study the symmetry properties of the reduced PDE to derive further symmetry reductions and exact group-invariant solutions for the underlying equation.

The results on the classification of the Lie point symmetries of the equation (3.34) are summarized by the Tables 1, 2 and 3. The commutator table of the Lie point symmetries of the equation (3.34) and the adjoint representations of the symmetry group of (3.34) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (3.34) which is given in Table 3

**Table 1.** Commutator table of the Lie algebra of equation (3.34)

	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	0	0	0	0
$X_2$	0	0	0	0
$X_3$	0	0	0	$2X_3$
$X_4$	0	0	$-2X_3$	0

**Table 2.** Adjoint table of the Lie algebra of equation (3.34)

Ad	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	$X_1$	$X_2$	$X_3$	$X_4$
$X_2$	$X_1$	$X_2$	$X_3$	$X_4$
$X_3$	$X_1$	$X_2$	$X_3$	$X_4 - 2\epsilon X_3$
$X_4$	$X_1$	$X_2$	$e^{2\epsilon} X_3$	$X_4$

**Table 3.** Subalgebra, group-invariant solutions of (3.34)

N	X	$\alpha$	$\beta$	Group – invariant solution
1	$X_1$	$t$	$y$	$u = h(\alpha, \beta)$
2	$X_2 + \mu X_1$	$t$	$x - \mu y$	$u = h(\alpha, \beta)$
3	$X_3$	$t$	$y$	$u = \frac{2x}{t^2} - \frac{1}{t^2} h(\alpha, \beta)$
4	$X_4$	$x$	$\frac{t^2}{y}$	$u = \frac{1}{t^2} h(\alpha, \beta)$
5	$\rho X_1 + \mu X_2 + X_4$	$x + \mu \ln t$	$\frac{\mu - 2y}{t^2}$	$u = \frac{1}{t^2} h(\alpha, \beta)$

**Case 2.1** The  $X_1$ -invariant solution reduces the equation (3.34) to  $h_\alpha = 0$ . Hence the solution of the equation (3.34) is given by  $u(t, x, y) = H(y)$ , where  $H(y)$  is an arbitrary function of its argument.

**Case 2.2** In this case, the group-invariant solution corresponding to the symmetry generator  $X_2 + \mu X_1$  where  $\mu$  is just a constant reduces the equation (3.34) to the PDE

$$h_\alpha + \alpha h h_\beta + \left[ \frac{a}{\alpha} + b\alpha^3 \mu^2 \right] h_{\beta\beta\beta} = 0. \quad (3.53)$$

Now the equation (3.53) admits the following symmetry generators given by

$$X_1 = \partial_\beta, \quad X_2 = \alpha^2 \partial_\beta + 2\partial_h. \quad (3.54)$$

- (a) The symmetry generator  $X_1$  gives the trivial solution  $u(t, x, y) = C$ , where  $C$  is a constant.
- (b) The generator  $X_2$  leads to the group-invariant solution  $h = \frac{2\beta}{\alpha^2} - \frac{1}{\alpha^2} H(\alpha)$ . Substitution of this solution into the equation (3.53) gives the solution

$$u(t, x, y) = \frac{2x - 2y\mu - C}{t^2}, \quad (3.55)$$

where  $C$  is a constant.

- (c) The group-invariant solution corresponding to  $X_2 + \rho X_1$  where  $\rho$  is a constant, is  $h = \frac{2\beta}{\rho + \alpha^2} + H(\alpha)$ , the substitution of this solution into the equation (3.53) and solving we obtain the solution

$$u(t, x, y) = \frac{2x - 2\rho y - C}{\rho + t^2},$$

where  $C$  is a constant.

**Case 2.3** The  $X_3$ -invariant solution reduces the equation (3.34) to  $h_\alpha = 0$ . Hence the solution of the equation (3.34) is given by

$$u(t, x, y) = \frac{2x - H(y)}{t^2}, \quad (3.56)$$

where  $H(y)$  is an arbitrary function of its argument.

**Case 2.4** The  $X_4$ -invariant solution reduces the equation (3.34) to the PDE

$$-2h + 2\beta h_\beta + hh_\alpha + ah_{\alpha\alpha\alpha} + b\beta^3 h_{\alpha\beta} + b\beta^4 h_{\alpha\beta\beta} = 0. \quad (3.57)$$

The equation (3.57) admits the Lie algebra spanned by the following symmetry generator

$$X_1 = \partial_\alpha. \quad (3.58)$$

The generator  $X_1$  leads to the group-invariant solution  $h = f(\beta)$ . Substitution of this solution into the equation (3.57) gives the solution

$$u(t, x, y) = \frac{c}{y}. \quad (3.59)$$

where  $c$  is a constant.

**Case 2.5** In this case, the group-invariant solution corresponding to the symmetry generator  $\rho X_1 + \mu X_2 + X_4$  where  $\rho$  and  $\mu$  are just a constants reduces the equation (3.34) to the PDE

$$-2h + \rho h_\alpha - 2\beta h_\beta + hh_\alpha + ah_{\alpha\alpha\alpha} + 4bh_{\alpha\beta\beta} = 0. \quad (3.60)$$

The equation (3.60) admits the Lie algebra spanned by the following symmetry generator

$$X_1 = \partial_\alpha. \quad (3.61)$$

The symmetry generator  $X_1$  leads to the group-invariant solution  $h = f(\beta)$ . Substitution of this solution into the equation (3.60) gives the solution

$$u(t, x, y) = \frac{c}{\mu - 2y}, \quad (3.62)$$

where  $c$  is a constant.

### 3.3 $f(t) = a$ , $g(t) = b$ and $h(t) = k(t - d)^2$ , where $a$ , $b$ and $k$ are arbitrary constants

#### 3.3.1 Lie point symmetries

Therefore, we study the equation

$$u_t + auu_x + bu_{xxx} + k(t - d)^2 u_{xyy} = 0. \quad (3.63)$$

The symmetry group of ZK equation (3.63) will then be generated by the vector field of the form

$$X = \tau(t, x, y, u) \frac{\partial}{\partial t} + \xi(t, x, y, u) \frac{\partial}{\partial x} + \psi(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}. \quad (3.64)$$

Applying the third prolongation to (3.63) yields the following over determined system of linear PDEs:

$$\tau_u = 0, \quad (3.65)$$

$$\tau_x = 0, \quad (3.66)$$

$$\xi_u = 0, \quad (3.67)$$

$$\psi_u = 0, \quad (3.68)$$

$$\tau_y = 0, \quad (3.69)$$

$$\psi_x = 0, \quad (3.70)$$

$$\xi_y = 0, \quad (3.71)$$

$$\psi_t = 0, \quad (3.72)$$

$$\eta_{xu} = 0, \quad (3.73)$$

$$\xi_{xx} = 0, \quad (3.74)$$

$$\eta_{uu} = 0, \quad (3.75)$$

$$b\tau_t - 3b\xi_x = 0, \quad (3.76)$$

$$2\eta_{yu} - \psi_{yy} = 0, \quad (3.77)$$

$$\eta_t + b\eta_{xxx} + k(t-d)^2\eta_{xyy} + au\eta_x = 0, \quad (3.78)$$

$$au\tau_t - \xi_t + a\eta - au\xi_x + k(t-d)^2\eta_{yyu} = 0, \quad (3.79)$$

$$k(t-d)^2\tau_t - k(t-d)^2\xi_x - 2k(t-d)^2\psi_y + 2k(t-d)\tau = 0. \quad (3.80)$$

Solving the determining equations (3.65)-(3.80) for  $\tau, \xi, \psi$  and  $\eta$ , we obtain the following symmetry group generators given by

$$\begin{aligned} X_1 &= \partial_y, \\ X_2 &= \partial_x, \\ X_3 &= at\partial_x + \partial_u. \end{aligned} \quad (3.81)$$

### 3.3.2 Symmetry reductions and exact group-invariant solutions of the equation (3.63)

We now construct group-invariant solutions under the symmetry operators of the ZK equation (3.63). The group-invariant solutions are illustrated by the following table.

**Table 1.** Group-invariant solutions of (3.63)

N	X	$\alpha$	$\beta$	Group – invariant solution
1	$X_2$	$t$	$y$	$u = h(\alpha, \beta)$
2	$X_1 + \rho X_2$	$t$	$x - \rho y$	$u = h(\alpha, \beta)$
3	$X_3$	$t$	$y$	$u = \frac{x}{a\alpha} + h(\alpha, \beta)$
4	$\mu X_1 + \delta X_2 + X_3$	$t$	$\mu x - (\delta + at)y$	$u = \frac{y}{\mu} + \frac{1}{\mu}h(\alpha, \beta)$

**Case 3.1.** The  $X_2$ -invariant solution reduces the equation (3.63) to  $h_\alpha = 0$ . Hence the solution of the equation (3.63) is given by  $u(t, x, y) = H(y)$ , where  $H(y)$  is an arbitrary function of its argument.

**Case 3.2.** In this case, the group-invariant solution corresponding to the symmetry generator  $X_1 + \rho X_2$  where  $\rho$  is just a constant reduces the equation (3.63) to the PDE

$$h_\alpha + ah h_\beta + [b + k\rho^2(\alpha - d)^2]h_{\beta\beta\beta} = 0. \quad (3.82)$$

Now the equation (3.82) admits the following symmetry generators given by

$$X_1 = \partial_\beta, \quad X_2 = a\alpha\partial_\beta + \partial_h. \quad (3.83)$$

(a) The symmetry generator  $X_1$  gives the trivial solution  $u(t, x, y) = C$ , where  $C$  is a constant.

(b) The generator  $X_2$  leads to the group-invariant solution  $h = \frac{\beta}{a\alpha} - \frac{1}{a\alpha}H(\alpha)$ . Substitution of this solution into the equation (3.82) gives the solution

$$u(t, x, y) = \frac{x - \rho y - C}{at}, \quad (3.84)$$

where  $C$  is a constant.

- (c) The group-invariant solution corresponding to  $X_1 + rX_2$  where  $r$  is a constant, is  $h = \frac{\beta}{r+a\alpha} + H(\alpha)$ , the substitution of this solution into the equation (3.82) and solving we obtain the solution

$$u(t, x, y) = \frac{x - \rho y + C}{r + at},$$

where  $C$  is a constant.

**Case 3.3.** The group-invariant solution that corresponds to  $X_3$  reduces the equation (3.63) to the PDE

$$h_\alpha + \frac{h}{\alpha} = 0. \quad (3.85)$$

Hence the solution of the equation (3.63) is given by

$$u(t, x, y) = \frac{x + aC}{at}, \quad (3.86)$$

where  $C$  is a constant.

**Case 3.4.** The  $\mu X_1 + \delta X_2 + X_3$ -invariant solution reduces the equation (3.63) to the PDE

$$h_\alpha + ah h_\beta + \left[ b\mu^3 + k(\alpha - d)^2(\delta + a\alpha)^2 \right] h_{\beta\beta\beta} = 0. \quad (3.87)$$

The equation (3.87) admits the Lie algebra spanned by the following symmetry generators

$$X_1 = \partial_\beta, \quad X_2 = a\alpha\partial_\beta + \partial_h. \quad (3.88)$$

- (a) The  $X_1$ -invariant solution reduces the equation (3.87) to  $h_\alpha = 0$ . Hence the solution of the equation (3.63) is given by

$$u(t, x, y) = \frac{1}{\mu}[y + C],$$

where  $C$  is a constant.

- (b) The symmetry generator  $X_2$  leads to the group-invariant solution  $h = \frac{\beta - C}{a\alpha}$  is the group invariant. Substitution of this solution into the equation (3.82) gives the solution

$$u(t, x, y) = \frac{\mu x - \delta y - C}{\mu at}, \quad (3.89)$$

where  $C$  is a constant.

### 3.4 Concluding remarks

In this chapter we studied the generalized (2+1)-dimensional ZK equation with time dependent variable coefficients using the Lie symmetry group method. We derived the Lie point symmetry generators of three special form of the underlying class of equations. The Lie symmetry classification with respect to the special form of the time dependent variable coefficients equation was presented. We used this classification of optimal system of one-dimensional subalgebras of the Lie symmetry algebras to construct symmetry reductions and exact group-invariant solutions for the special forms of the equations.

## Chapter 4

# Conservation laws of a variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation

In this chapter, we study the conservation laws for some special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_{xyy} = 0, \quad (4.1)$$

of time dependent variable coefficients. Here  $f(t)$ ,  $g(t)$  and  $h(t)$  are arbitrary smooth functions of the variable  $t$  and  $f$ ,  $g$  and  $h \neq 0$ . We consider the three cases for the functions of  $f$ ,  $g$  and  $h$ , namely

**Case 1.**  $f(t) = 1$ ,  $g(t) = a_0/t$  and  $h(t) = b_0/t$  where  $a_0$  and  $b_0$  are constants,

**Case 2.**  $f(t) = t$ ,  $g(t) = a/t$  and  $h(t) = bt^3$  where  $a$  and  $b$  are constants,

**Case 3.**  $f(t) = a$ ,  $g(t) = b$  and  $h(t) = k(t - d)^2$  where  $a$ ,  $b$  and  $d$  are constants.

**4.1  $f(t) = 1$ ,  $g(t) = a_0/t$  and  $h(t) = b_0/t$  where  $a_0$  and  $b_0$  are constants.**

In this section equation (4.1) takes the form

$$u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} = 0. \quad (4.2)$$

As seen in Chapter 3, the equation (4.2) admits the following Lie point symmetry generators

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_y, \\ X_3 &= t\partial_x + \partial_u, \\ X_4 &= t\partial_t - u\partial_u. \end{aligned} \quad (4.3)$$

The adjoint equation of (4.2), by invoking (1.38), is

$$\begin{aligned} E^*(t, x, y, u, v, \dots, u_{xxx}, u_{xyy}, v_{xxx}, v_{xyy}) = \\ \frac{\delta}{\delta u} \left[ v(u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy}) \right] = 0, \end{aligned} \quad (4.4)$$

where  $v = v(t, x, y)$  is a new dependent variable and (4.4) gives

$$v_t + uv_x + \frac{a_0}{t}v_{xxx} + \frac{b_0}{t}v_{xyy} = 0. \quad (4.5)$$

It is obvious from the adjoint equation (4.5) that equation (4.2) is a self-adjoint. By recalling (1.43), we get the following third order Lagrangian for the system of equations (4.2) and (4.5):

$$\bar{L} = v(u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy}). \quad (4.6)$$

By reducing this third-order Lagrangian to second-order Lagrangian, we obtain

$$L = vu_t + vuu_x - \frac{a_0}{t}v_xu_{xx} - \frac{b_0}{t}v_xu_{yy}. \quad (4.7)$$

(i) We first consider the Lie point symmetry generator

$$X_1 = \partial_x.$$

It can be verified from (1.40) that the operator  $Y_1$  is the same as  $X_1$  and hence this will give us two Lie characteristic functions  $W^1 = -u_x$  and  $W^2 = -v_x$ . Thus by using (1.44) and the second order Lagrangian, the components  $T^i$ ,  $i = 1, 2, 3$ , of the conserved vector  $T = (T^1, T^2, T^3)$  are given by

$$\begin{aligned} T^1 &= -vu_x, \\ T^2 &= vu_t - \frac{a_0}{t}u_xv_{xx} + \frac{a_0}{t}v_xu_{xx}, \\ T^3 &= \frac{b_0}{t}v_xu_{xy} - \frac{b_0}{t}u_xv_{xy}. \end{aligned}$$

*Remark:* The conserved vector  $T$  contains the arbitrary solution  $v$  of the adjoint equation (4.5) and hence gives an infinite number of conservation laws.

The same remark applies to all the following cases where we use the conservation theorem.

(ii) Now for the symmetry generator

$$X_2 = \partial_y,$$

we have  $W^1 = -u_y$  and  $W^2 = -v_y$ . Hence, by invoking (1.44) and the second order Lagrangian, the symmetry generator  $X_2$  gives rise to the following components of the conserved vector

$$\begin{aligned} T^1 &= -vu_y, \\ T^2 &= -vuu_y - \frac{a_0}{t}u_yv_{xx} + \frac{a_0}{t}v_yu_{xx} + \frac{b_0}{t}v_yu_{yy} + \frac{a_0}{t}v_xu_{xy}, \\ T^3 &= vu_t + vuu_x - \frac{a_0}{t}v_xu_{xx} - \frac{b_0}{t}u_yv_{xy}. \end{aligned}$$

(iii) The symmetry generator

$$X_3 = t\partial_x + \partial_u$$

has the Lie characteristic functions  $W^1 = 1 - tu_x$  and  $W^2 = v - tvu_x$ . Hence using (1.44) and the second order Lagrangian, one can obtain the conserved vector  $T$  whose

components are given by

$$\begin{aligned} T^1 &= v - tvu_x, \\ T^2 &= tvu_t + vu + \frac{a_0}{t}v_{xx} - a_0u_xv_{xx} + a_0v_xu_{xx}, \\ T^3 &= \frac{b_0}{t}v_{xy} - b_0u_xv_{xy} + b_0v_xu_{xy}. \end{aligned}$$

(iv) Finally, we consider the Lie point symmetry generator

$$X_4 = t \partial_t - u \partial_u.$$

From (1.41), it can easily be shown that

$$X_4 \left( u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} \right) = -2 \left( u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} \right),$$

hence we get  $\lambda = -2$ . Since  $D_i(\xi^i) = 1$ , from (3.31) we obtain  $\eta_* = v$ . Therefore the extending operator (3.31) of  $X_4$  is,

$$Y_4 = t \partial_t - u \partial_u + v \partial_v. \quad (4.8)$$

It can be verified that the system of equations (4.2) and (4.5) admit the symmetry generator  $Y_4$ . Now the symmetry generator  $X_4$  has the Lie characteristic functions  $W^1 = -u - tu_t$  and  $W^2 = v - tv_t$ . Hence using (1.44) and the second-order Lagrangian, we obtain the following components of the conserved vector  $T$ :

$$\begin{aligned} T^1 &= tvuu_x - a_0v_xu_{xx} - b_0v_xu_{yy} - vu, \\ T^2 &= -vu^2 - \frac{a_0}{t}uv_{xx} - tvuu_t - a_0u_tv_{xx} - \frac{a_0}{t}vu_{xx} \\ &\quad - \frac{b_0}{t}vu_{yy} + a_0v_tv_{xx} + b_0v_tv_{yy} + \frac{a_0}{t}v_xu_x + a_0v_xu_{tx}, \\ T^3 &= -\frac{b_0}{t}uv_{xy} - b_0u_tv_{xy} + \frac{b_0}{t}u_yv_x + b_0v_xu_{ty}. \end{aligned}$$

One can easily verify, like we did in section 2.3, that the conserved vectors which we derived above satisfy the equation  $D_iT^i = 0$  whenever equations (4.2) and (4.5) are satisfied.

**4.2**  $f(t) = t$ ,  $g(t) = a/t$  and  $h(t) = bt^3$  where  $a$  and  $b$  are constants.

In this section equation (4.1) becomes

$$u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy} = 0. \quad (4.9)$$

The equation (4.9) admits the four Lie point symmetry generators (see Chapter 3)

$$\begin{aligned} X_1 &= \partial_y, \\ X_2 &= \partial_x, \\ X_3 &= t^2 \partial_x + 2\partial_u, \\ X_4 &= -t\partial_t - 2y\partial_y + 2u\partial_u. \end{aligned}$$

The adjoint equation of (1.25), by invoking (1.38), is

$$E^*(t, x, y, u, v, \dots, u_{xxx}, u_{xyy}, v_{xxx}, v_{xyy}) = \frac{\delta}{\delta u} \left[ v(u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy}) \right] = 0, \quad (4.10)$$

where  $v = v(t, x, y)$  is a new dependent variable and (4.10) gives

$$v_t + tvv_x + \frac{a}{t}v_{xxx} + bt^3v_{xyy} = 0. \quad (4.11)$$

From equations (4.9) and (4.11), we see that equation (4.9) is a self-adjoint.

By using (1.43), the third-order Lagrangian for the system of equations (4.9) and (4.11) is given by

$$\bar{L} = v(u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy}), \quad (4.12)$$

which can be reduced to a second-order Lagrangian,

$$L = vu_t + tvuu_x - \frac{a}{t}v_xu_{xx} - bt^3v_xu_{yy}. \quad (4.13)$$

(i) Consider the Lie point symmetry generator

$$X_1 = \partial_y.$$



From (1.40) we note that the operator  $Y_1$  is the same as  $X_1$  and hence this will give us two Lie characteristic functions  $W^1 = -u_y$  and  $W^2 = -v_y$ . Thus by using (1.44) and the second order Lagrangian, the components  $T^i$ ,  $i = 1, 2, 3$ , of the conserved vector  $T = (T^1, T^2, T^3)$  are given by

$$\begin{aligned} T^1 &= -vu_y, \\ T^2 &= -tuvu_y - \frac{a}{t}u_yv_{xx} + \frac{a}{t}v_yu_{xx} + bt^3u_{yy}v_y + \frac{a}{t}v_xu_{xy}, \\ T^3 &= vu_t + tuv u_x - \frac{a}{t}v_xu_{xx} - bt^3u_yv_{xy}. \end{aligned}$$

(ii) For the symmetry generator

$$X_2 = \partial_x,$$

we have  $W^1 = -u_x$  and  $W^2 = -v_x$ . Hence, using (1.44) and the second-order Lagrangian, the symmetry generator  $X_2$  gives rise to the conserved vector  $T^i = (T^1, T^2, T^3)$  where  $T^1$ ,  $T^2$  and  $T^3$  are given by

$$\begin{aligned} T^1 &= -vu_x, \\ T^2 &= vu_t - \frac{a}{t}u_xv_{xx} + \frac{a}{t}v_xu_{xx}, \\ T^3 &= bt^3v_xu_{xy} - bt^3u_xv_{xy}. \end{aligned}$$

(iii) The symmetry generator

$$X_3 = t^2\partial_x + 2\partial_u,$$

gives  $W^1 = 2 - t^2u_x$  and  $W^2 = -t^2v_x$ . By using (1.44) and the second-order Lagrangian, the symmetry generator  $X_3$  gives rise to the following components of the conserved vector

$$\begin{aligned} T^1 &= 2v - t^2vu_x, \\ T^2 &= t^2vu_t + 2tuv + \frac{2a}{t}v_{xx} - at u_xv_{xx} + av_xu_{xx}, \\ T^3 &= 2bt^3v_{xy} - bt^5u_xv_{xy} + bt^5v_xu_{xy}. \end{aligned}$$

(iv) Finally, we consider the Lie point symmetry generator

$$X_4 = -t\partial_t - 2y\partial_y + 2u\partial_u.$$

From (1.41), it can easily be shown that

$$X_3 \left( u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy} \right) = 3 \left( u_t + tuu_x + \frac{a}{t}u_{xxx} + bt^3u_{xyy} \right),$$

hence we get  $\lambda = 3$ . Since  $D_i(\xi^i) = -3$ , from (3.31) we obtain  $\eta_* = 0$ . Therefore the extending operator (3.31) of  $X_4$  is,

$$Y_4 = -t\partial_t - 2y\partial_y + 2u\partial_u + 0\partial_v.$$

It can be verified that the system of equations (4.9) and (4.10) admit the symmetry generator  $Y_4$ . Now the symmetry generator  $X_4$  has the Lie characteristic functions  $W^1 = 2u + tu_t + 2yu_y$  and  $W^2 = tv_t + 2yVy$ . Hence using (1.44) and the second-order Lagrangian, we obtain the following components of the conserved vector  $T$ :

$$\begin{aligned} T^1 &= -t^2vuu_x + av_xu_{xx} + bt^4v_xu_{yy} + 2vu + 2yvu_y, \\ T^2 &= 2tu^2v + \frac{2a}{t}uv_{xx} + t^2uvu_t + au_tv_{xx} + 2ytuvu_y + \frac{2ay}{t}u_yv_{xx} \\ &\quad - av_tv_{xx} - bt^4v_tu_{yy} - \frac{2ay}{t}v_yu_{xx} - 2byt^3v_yu_{yy} - \frac{2a}{t}v_xu_x \\ &\quad - av_xu_{tx} - \frac{2ay}{t}v_xu_{xy}, \\ T^3 &= -2yvv_t - 2ytuvu_x + \frac{2ay}{t}v_xu_{xx} + 2bt^3uv_{xy} + bt^4u_tv_{xy} \\ &\quad + 2ybt^3u_yv_{xy} - 4bt^3u_yv_x - bt^4v_xu_{xy}. \end{aligned}$$

Again it can be easily verified, like we did in section 2.3, that the conserved vectors which we derived above satisfy the equation  $D_iT^i = 0$  whenever equations (4.9) and (4.11) are satisfied.

### 4.3 $f(t) = a$ , $g(t) = b$ and $h(t) = k(t - d)^2$ where $a$ , $b$ and $d$ are constants

Then for this section equation (4.1) becomes

$$u_t + auu_x + bu_{xxx} + k(t - d)^2u_{xyy} = 0. \quad (4.14)$$

The equation (4.14) admits the three Lie point symmetry generators

$$\begin{aligned} X_1 &= \partial_y, \\ X_2 &= \partial_x, \\ X_3 &= at\partial_x + \partial_u. \end{aligned}$$

The adjoint equation of (1.25), by invoking (1.38), is

$$\frac{\delta}{\delta u} \left[ E^*(t, x, y, u, v, \dots, u_{xxx}, u_{xyy}, v_{xxx}, v_{xyy}) \right] = 0, \quad (4.15)$$

where  $v = v(t, x, y)$  is a new dependent variable and (4.15) gives

$$v_t + auv_x + bv_{xxx} + k(t-d)^2 v_{xyy} = 0. \quad (4.16)$$

It is obvious from the adjoint equation (4.16) that equation (4.14) is a self-adjoint. By recalling (1.43), we get the following third order Lagrangian for the system of equations (4.14) and (4.16):

$$\bar{L} = v(u_t + auu_x + bu_{xxx} + k(t-d)^2 u_{xyy}). \quad (4.17)$$

By reducing this third-order Lagrangian to second-order Lagrangian, this give us

$$L = vu_t + avuu_x - bv_x u_{xx} - k(t-d)^2 v_x u_{yy}. \quad (4.18)$$

(i) Consider the Lie point symmetry generator

$$X_1 = \partial_y.$$

Here the operator  $Y_1$  is the same as  $X_1$  and hence this will give us two Lie characteristic functions  $W^1 = -u_y$  and  $W^2 = -v_y$ . Thus by using (1.44) and the second-order Lagrangian, the components  $T^i$ ,  $i = 1, 2, 3$ , of the conserved vector  $T = (T^1, T^2, T^3)$  are given by

$$\begin{aligned} T^1 &= -vu_y, \\ T^2 &= -auvu_y - bu_y v_{xx} + bv_y u_{xx} + k(t-d)^2 v_y u_{yy} + bv_x u_{xy}, \\ T^3 &= vu_t + avvu_x - bv_x u_{xx} - k(t-d)^2 u_y v_{xy}. \end{aligned}$$

(ii) For the symmetry generator

$$X_2 = \partial_x,$$

we have  $W^1 = -u_x$  and  $W^2 = -v_x$ . Hence, by invoking (1.44) and the second-order Lagrangian, the symmetry generator  $X_2$  gives rise to the following components of the conserved vector

$$\begin{aligned} T^1 &= -vu_x, \\ T^2 &= vu_t - bu_xv_{xx} + bv_xu_{xx}, \\ T^3 &= -k(t-d)^2u_xv_{xy} + k(t-d)^2v_xu_{xy}. \end{aligned}$$

(iii) Finally, we consider the Lie point symmetry generator

$$X_3 = at\partial_x + \partial_u,$$

we have  $W^1 = 1 - atu_x$  and  $W^2 = -atv_x$ . Hence, by invoking (1.44) and the second-order Lagrangian, the symmetry generator  $X_3$  gives rise to the following components of the conserved vector

$$\begin{aligned} T^1 &= v - atvu_x, \\ T^2 &= atvu_t + auv + bv_{xx} - abtu_xv_{xx} + abtv_xu_{xx}, \\ T^3 &= k(t-d)^2v_{xy} - atk(t-d)^2u_xv_{xy} + atk(t-d)^2v_xu_{xy}. \end{aligned}$$

It can easily be verified, like we did in section 2.3, that the conserved vectors which we derived above satisfy the equation  $D_iT^i = 0$  whenever equations (4.14) and (4.16) are satisfied.

## 4.4 Concluding Remarks

In this chapter we constructed conservation laws for some special cases of the variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation. We consider three special cases and used the new conservation theorem due to Ibragimov to construct a number of conservation laws.

# Chapter 5

## Concluding remarks

In this research project we first reviewed some important definitions and results from Lie group theory, which were later used in the dissertation.

In Chapter 2 we obtained the exact solutions of the Korteweg-de Vries-Burgers equation using the Lie symmetry method. Also the conservation laws of the Korteweg-de Vries-Burgers equation were obtained by using the new conservation theorem.

In Chapter 3 we used the Lie symmetry group method and obtained exact solutions of three special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation of time dependent variable coefficients.

Finally, in Chapter 4 we used the new conservation theorem to construct the conservation laws for three special cases of the generalized (2+1)-dimensional Zakharov-Kuznetsov equation of time dependent variable coefficients.

In future we will construct conservation laws for the generalized (2+1)-dimensional Zakharov-Kuznetsov equation of time dependent variable coefficients using other approaches.

# Bibliography

- [1] A. Biswas and E. Zerrad, Solitary wave solution of the Zakharov-Kuznetsov equation in plasmas with power law nonlinearity, *Nonlinear Analysis: Real World Applications*. 11 (2010) 3272-3274.
- [2] A. Biswas, 1-Soliton solution of the generalized Zakharov-Kuznetsov equation with nonlinear dispersion and time-dependent coefficients, *Phys. Lett. A*. 373 (2009) 2931-2934.
- [3] M. Wang, Y. Zhou and Z. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, *Phys. Lett. A*. 216 (1996) 67-65.
- [4] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.
- [5] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
- [6] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Springer, New York, 1993.
- [7] Y. Chen and Z. Yan, The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations, *Chaos Soliton Fract.* 29 (2006) 948-964.
- [8] M. Wang and X. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, *Chaos Soliton Fract.* 24 (2005) 1257-1268.

- [9] M. Wang, L.X. Xiangzheng and Z.J. Jinliang, The  $(G'/G)$ -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A.* 372 (2008) 417-423.
- [10] S. Zhang, Application of Exp-function method to high-dimensional nonlinear evolution equation, *Chaos Soliton Fract.* 38 (2008) 270-276.
- [11] C.H. Su, C.S. Gardner, Derivation of the Korteweg-de Vries and Burgers equation, *J. Math. Phys.* 10 (1969) 536-539.
- [12] H. Grad, P.N. Hu, Unified shock profile in a plasma, *Phys. Fluids.* 10 (1967) 2596-2601.
- [13] R.S. Jonson, A nonlinear equation incorporating damping and dispersion, *J. Phys. Mech.* 42 (1970) 49-60.
- [14] R.S. Jonson, Shallow water wave in a viscous fluid-the undular bore, *Phys. Fluids.* 15 (1972) 1693-1699.
- [15] V.E. Zakharov and E.A. Kuznetsov, On three-dimensional solitons, *Sov. Phys.* 39 (1974) 285-288.
- [16] H. Stephani, *Differential Equations: their solutions using symmetries*, Cambridge University Press, Cambridge, 1989.
- [17] N.H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons Chichester, 1999.
- [18] N.H. Ibragimov (Ed.), *CRC Handbook of Lie Group Analysis of Differential Equations*, Vols.1-3, CRC Press, Boca Raton, FL, 1994-1996.
- [19] B.J. Cantwell, *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge, 2002.
- [20] F.M. Mahomed, Recent trends in symmetry analysis of differential equations, *Notices of SAMS.* 33 (2002) 11-40.

- [21] N.H. Ibragimov, A.H. Kara, F.M. Mahomed, Lie-Bäcklund and Noether symmetries with applications, *Nonlinear Dynam.* 15 (1998) 115-136.
- [22] R.W. Atherton, G.M. Homsy, On the existence and formulation of variational principles for nonlinear differential equations, *Studies Appl. Math.* 54 (1975) 31-60.
- [23] N.H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.* 333 (2007) 311-328.
- [24] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Kortewegde Vries equation and generalizations. IV. Method for exact solution, *Commun. Pure Appl. Math.* XXVII (1974) 97-133.
- [25] A.H. Khater, M.A. Helal, O.H. El-Kalaawy, Bäcklund transformations: Exact solutions for the KdV and the Calogero-Degasperis-Fokas mKdV equations, *Math. Meth. Appl. Sci.* 21 (1998) 713 -719.
- [26] D. Kaya, M. Aassila, Decomposition method for the solution of the nonlinear Korteweg-de Vries equation, *Phys. Lett. A.* 299 (2002) 201-206.
- [27] D. Kaya, On the solution of a Korteweg-de Vries like equation by the decomposition method, *Int. J. Comput. Math.* 72 (1999) 531-539.
- [28] M. Rasulo, E. Coskun, An efficient numerical method for solving KdV equation by in a class of discontinuous functions, *Appl. Math. Comput.* 102 (1999) 139-154.
- [29] S.I. Zaki, A quintic B-spline finite elements scheme for the KdVB equation, *Comput. Meth. Appl. Mech. Eng.* 188 (2000) 531-539.
- [30] A.A. Soliman, Exact solutions of KdV-Burgers equation by Exp-function method, *Chaos Soliton Fract.* 41 (2009) 1034-1039.
- [31] L.D. Moleleki, A.G. Johnpillai and C.M. Khalique, Symmetry reductions and exact solutions of a variable coefficient (2+1)-dimensional Zakharov-Kuznetsov equation, *Int. J. Phy Sci.* 35 (2011) 7921-7925.