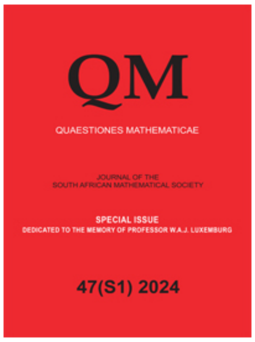

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WEAK AND STRONG TYPE INEQUALITIES FOR ORLICZ SPACES

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Abstract. We revisit the generalisation of Calderon's Transfer Principle as espoused in [7]. This principle is used to generate weak type maximal inequalities for ergodic operators in the setting of σ-compact locally compact Hausdorff groups acting measure-preservingly on σ-finite measure spaces. In particular we develop a much more robust protocol for transferring weak and strong type inequalities from Orlicz spaces in the group setting to Orlicz spaces in the measure space setting. This is an important addition to the protocol developed in [7], which to date has only yielded weak type inequalities. The current approach also places fewer restrictions on the underlying Young functions describing the Orlicz spaces involved.

Mathematics Subject Classification (2020): Primary: 37A15, 37A30, 46E30; Secondary: 22F10, 37A05, 37A46.

Key words: Transfer Principle, strong/weak maximal inequalities, Banach function spaces, pointwise ergodic theorems

1. Introduction. Weak type inequalities feature prominently in establishing pointwise ergodic convergence results (see for example [7, Section 5]). Strong type inequalities on the other hand feature prominently in interpolation theory (see [3, Section 4.1]). Much work has already been done on this topic, but the vast majority of these results focus on L^p-spaces. We revisit the agenda of [7] which is to generalise Calderon's transfer principle [5] both as far as the actions are concerned (here we have in view group actions on measure spaces) and as far as the spaces in view are concerned. A large number of authors have of course already explored various versions of the transfer principle. Readers wishing to have a clear explanation of exactly how the present approach differs from other approaches, may refer to the explanation preceding Definition 1.1 of [7]. In Calderon's transfer principle a given operator T in the group context is used to construct an operator $T^{\#}$ in the measure space context. A further part of this programme then consists of transferring maximal inequalities for T in the group context to inequalities for $T^{\#}$ in the measure space context. If this can be done, then by an application

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of the Banach principle one is able to also transfer convergence properties from the group context, to pointwise ergodic convergence results in the measure space context. In the present setting the transfer of maximal inequalities from the group context to the measure space context, is done by means of a calculus involving fundamental and Young function inequalities. In Section 4 of [7] several such results were established wherein a set of only two fundamental function inequalities were needed. However the approach used there made essential use of delicate changes in the order of integration at various points, which had the unforeseen consequence at times restricting the class of Young functions to which the results applied, and which also failed to yield strong type inequalities. In the present note we revise the strategy for transferring such maximal inequalities. Ultimately we will here need a set of four fundamental function inequalities to effect the transfer. However the advantage of the present approach is that it seems to be more robust holding for a much larger class of Young functions, and also has the advantage of yielding strong type inequalities.

2. Preliminaries. Our primary objective is to establish weak and strong type inequalities for Orlicz spaces. However to appreciate the nature of the spaces involved in these inequalities, a little bit of background regarding Banach function spaces and fundamental functions is needed. For this we shall follow the doctrine of Bennett and Sharpley [3]. Given a measure space (Ω, Σ, μ) we shall write $\mathcal{M}$ for the measurable functions and $\mathcal{M}^0$ for the measurable functions that are finite a.e. For us a simple function will just be a measurable finite valued function. It is not difficult to see that such functions are just linear combinations of characteristic functions χ_E corresponding to measurable sets. We shall for the most part be concerned with what Bennett and Sharpley call resonant measure spaces. These turn out to be σ-finite measure spaces which are either non-atomic, or fully atomic with all atoms having equal measure [3, Theorem 2.2.7]. It is an interesting exercise to show that the product of two σ-finite measure spaces will be non-atomic, if any one of the a priori given spaces is non-atomic. (See the appendix.) Since the product of two atomic measure spaces is clearly atomic, this in turn shows that the product of two resonant measure spaces is again resonant.

A (Banach) function norm $\boldsymbol{\rho}$ on the cone $\mathcal{M}_+(\Omega, \Sigma, \mu)$ of non-negative measurable functions is then defined to be a mapping $\boldsymbol{\rho} : \mathcal{M}_+ \to [0, \infty]$ satisfying

- $\boldsymbol{\rho}(f) = 0$ iff $f = 0$ a.e.
- $\boldsymbol{\rho}(\lambda f) = \lambda \boldsymbol{\rho}(f)$ for all $f \in L^0_+, \lambda > 0$.
- $\boldsymbol{\rho}(f + g) \leq \boldsymbol{\rho}(f) + \boldsymbol{\rho}(g)$ for all f, g.
- $f \leq g$ implies $\boldsymbol{\rho}(f) \leq \boldsymbol{\rho}(g)$ for all $f, g \in L^0_+$, with $\boldsymbol{\rho}(f_n) \nearrow \boldsymbol{\rho}(f)$ if $f_n \nearrow f$ μ-a.e.
- $\mu(E) < \infty$ implies $\boldsymbol{\rho}(\chi_E) < \infty$.
- For any $E \in \Sigma$ with $\mu(E) < \infty$ there exists $0 < C_{E,\boldsymbol{\rho}} < \infty$ such that $\int_E f \, d\mu \leq C_{E,\boldsymbol{\rho}} \boldsymbol{\rho}(f)$.

Such a $\boldsymbol{\rho}$ may be extended to all of $\mathcal{M}$ by setting $\boldsymbol{\rho}(f) = \boldsymbol{\rho}(|f|)$, in which case we may then define $L^{\boldsymbol{\rho}}(\Omega, \Sigma, \mu)$ to be the space $L^{\boldsymbol{\rho}}(\Omega, \Sigma, \mu) = \{f \in \mathcal{M}(\Omega, \Sigma, \mu) : \boldsymbol{\rho}(f) < \infty\}$ equipped with the norm $\|f\|_{\boldsymbol{\rho}} = \boldsymbol{\rho}(|f|)$. We shall refer to this space as the Banach Function Space corresponding to $\boldsymbol{\rho}$. These spaces all satisfy a version of Fatou's lemma, which in turn ensures their completeness [3, Lemma 1.1.5]. Such spaces are said to have an *absolutely continuous norm* if for any $f \in L^{\boldsymbol{\rho}}$ and any sequence (E_n) of measurable sets for which (χ_{E_n}) converges a.e. to 0, we have that $\|f\chi_{E_n}\|_{\boldsymbol{\rho}} \to 0$ as $n \to \infty$. Every Banach function norm $\boldsymbol{\rho}$ as defined above admits an *associate norm* $\boldsymbol{\rho}'$ on $\mathcal{M}_+$ given by $\boldsymbol{\rho}'(g) = \sup\{\int fg\,d\mu\colon f \in \mathcal{M}_+, \boldsymbol{\rho}(f) \leq 1\}$. The associate norm is itself also a Banach function norm, with $L^{\boldsymbol{\rho}'}$ referred to as the *associate space* of $L^{\boldsymbol{\rho}}$. This space is commonly denoted by $(L^{\boldsymbol{\rho}})'$.

Given an a.e. finite measurable function f on (Ω, μ), we may define the distribution function by $s \mapsto m(f, s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\}$. The decreasing rearrangement f^* of f is then defined as

$$f^*(s) = \sup\{t : m(f, t) \leq s\}.$$

Note that if f and g are equimeasurable functions on (Ω, μ) then $f^*(t) = g^*(t)$ for all $t \geq 0$. When the function norm $\boldsymbol{\rho}$ that defines the Banach function space has the property that $\boldsymbol{\rho}(f) = \boldsymbol{\rho}(g)$ for all equi-measurable functions f and g, the Banach space is called *rearrangement invariant* - see [3, Definitions 1.1, 4.1]. Rearrangement invariant Banach function spaces $L^{\boldsymbol{\rho}}$ have the property that for any two finitely measurable sets E_1 and E_2 with the same measure, we have that $\|\chi_{E_1}\|_{\boldsymbol{\rho}} = \|\chi_{E_2}\|_{\boldsymbol{\rho}}$. This fact then enables us to introduce the notion of a *fundamental function* for such spaces. Given such a space $L^{\boldsymbol{\rho}}$, the fundamental function corresponding to $L^{\boldsymbol{\rho}}$ is given by

$$\varphi_{\boldsymbol{\rho}}(t) = \|\chi_E\|_X \text{ for all } E \in \Sigma \text{ with } \mu(E) = t$$

where the domain of $\varphi_{\boldsymbol{\rho}}$ consists of all $t \in [0, \infty)$ for which there is some measurable set E with $\mu(E) = t$.

We briefly summarise the essential facts regarding fundamental functions which we shall need hereafter. The reader will find an excellent survey of fundamental functions in [3, Section 2.5]. We firstly note that the associate space of any rearrangement invariant Banach function space is again rearrangement invariant. It is then a deep fact that the fundamental functions of the pair $(L^{\boldsymbol{\rho}}, (L^{\boldsymbol{\rho}})')$ satisfy the equality $\varphi_{\boldsymbol{\rho}}(t)\varphi_{\boldsymbol{\rho}'}(t) = t$ for all admissible values of $t \geq 0$ [3, Theorem 2.5.2]. It further follows that these functions correspond to the class of so called *quasi-concave* functions which are functions φ that are 0 only at 0, are non-decreasing, and for which $t \to \frac{\varphi(t)}{t}$ is non-increasing. On the basis of what we noted above, the function

$$\varphi'(t) = \begin{cases} 0 & t = 0 \\ \frac{t}{\varphi(t)} & t > 0 \end{cases}$$

will also be a quasi-concave (fundamental) function. This function may be viewed as the "dual" fundamental function of φ.

Although a fundamental function may a priori not be defined on all of $[0, \infty)$, it is for resonant measure spaces always possible to extend a fundamental function

φ corresponding to some space $\mathcal{X}$ to a quasi-concave function defined on all of $[0,\infty)$. If say $\mu(\Omega)$ is finite with φ defined on $[0,\mu(\Omega))$, then for any $\alpha \in [0, \frac{\|\chi_\Omega\|}{\mu(\Omega)}]$, one may extend φ by defining its action on $[\mu(\Omega),\infty)$ by means of the formula $\varphi(t) = \alpha(t-\mu(\Omega)) + \|\chi_\Omega\|_{\mathcal{X}}$. If on the other hand (Ω,μ) is discrete with atoms having a measure of c, and with φ say defined on $c\mathbb{N}$, then one simply assigns φ the value $(1-\lambda)\varphi(cn)+\lambda\varphi(c(n+1))$ at any point $(1-\lambda)cn+\lambda c(n+1)$ $(0<\lambda<1)$ in the interval $[cn, c(n+1)]$. It is an exercise to see that the resultant extension will in either of these cases still be quasi-concave. Hence we shall hereafter assume all fundamental functions to be defined on all of $[0,\infty)$. We finally note that although fundamental functions may not actually be concave, any fundamental function is always equivalent to a concave fundamental function [3, Proposition 2.5.10]. Any Banach function space on a resonant measure space may correspondingly be equivalently renormed in such a way that the fundamental function is in fact concave [3, Proposition 2.5.10].

There may be several Banach function spaces corresponding to any specific quasi-concave function φ, but in the family of rearrangement invariant Banach function spaces corresponding to a such a function, there is in the case where (Ω,Σ,μ) is resonant always a largest and smallest space - the so-called maximal and minimal Lorentz spaces corresponding to φ (see [3, Corollary 2.5.14] and the comment following it). Assuming φ to actually be concave these spaces are respectively defined by

$$M_\varphi(\Omega,\Sigma,\mu) = \{f \in \mathcal{M}^0\colon \sup_{0<t<\infty} \frac{\varphi(t)}{t}\int_0^t f^*(s)\,ds < \infty\}$$

and

$$\Lambda_\varphi(\Omega,\Sigma,\mu) = \{f \in \mathcal{M}^0\colon \int_0^\infty f^*(s)\,d\varphi(s) < \infty\}$$

with the norms given by

$$\|f\|_{M_\varphi} = \sup_{0<t<\infty} \frac{\varphi(t)}{t}\int_0^t f^*(s)\,ds \quad\text{and}\quad \|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(s)\,d\varphi(s).$$

Given a fundamental function φ there is in fact a larger space we may construct from this function, namely

$$M^*_\varphi(\Omega,\Sigma,\mu) = \{f \in \mathcal{M}^0\colon \sup_{0<t<\infty} \varphi(t)f^*(t) < \infty\},$$

but this space is in general at best a quasi-normed space (with the quasi-norm given by $q_\varphi(f) = \sup_{0<t<\infty}\varphi(t)f^*(t)$).

In the family of rearrangement invariant spaces corresponding to a specific fundamental function, the so-called *Orlicz spaces* occupy a special place. We proceed with introducing this class and justifying this claim. By the term *Young function* we understand a convex function $A : [0,\infty) \to [0,\infty]$ satisfying $A(0) = 0$ and $\lim_{u\to\infty} A(u) = \infty$, which is neither identically zero nor infinite valued on all of $(0,\infty)$, and which is left continuous at $b_A = \sup\{u > 0 : A(u) < \infty\}$. It is worth pointing out that any Young function must also be increasing, and continuous on $[0, b_A]$. In the family of Young functions, the so-called N-functions form a particularly elegant subclass. A Young function is said to be an N-function if it is continuous on all of $[0, \infty)$ and also satisfies the additional conditions that $\lim_{t\to\infty} \frac{A(t)}{t} = \infty$ and $\lim_{t\searrow 0} \frac{A(t)}{t} = 0$.

There are various growth conditions by means of which Young functions are classified in the literature. One very common growth condition which will be relevant for us, is the so-called Δ_2 condition. A Young function A is said to *satisfy* Δ_2 *globally* if there exists a constant $c > 0$ such that $A(2s) \le cA(s)$ for all $s \ge 0$.

The so-called right-continuous inverse $A^{-1} : [0, \infty) \to [0, \infty]$ of a Young function is given by the formula

$$A^{-1}(t) = \sup\{s : A(s) \le t\}.$$

We further note that each Young function A induces a complementary Young function A^* which is defined by $A^*(u) = \sup_{v>0}(uv - A(v))$. The pair A and A^* satisfy the following Hausdorff-Young inequality:

$$st \le A(s) + A^*(t) \qquad s, t \ge 0.$$

The Orlicz space $L^A(\Omega, \Sigma, \mu)$ associated with A is defined to be the set

$$L^A = \{f \in \mathcal{M}^0(\Omega, \Sigma, \mu) : A(\lambda|f|) \in L^1 \quad \text{for some} \quad \lambda = \lambda(f) > 0\}.$$

This space turns out to be a linear subspace of $\mathcal{M}^0(\Omega, \Sigma, \mu)$ which becomes a Banach space when equipped with the so-called *Luxemburg-Nakano norm*

$$\|f\|_A = \inf\{\lambda > 0 : \|A(|f|/\lambda)\|_1 \le 1\}.$$

An equivalent norm (the *Orlicz norm*) is given by the formula

$$\|f\|_A^O = \sup\{|\textstyle\int fg\, d\mu| : g \in L^{A^*}, \|g\|_{A^*} \le 1\}.$$

As Banach function spaces the space $(L^A, \|\cdot\|_{A^*}^O)$ is precisely the associate Banach function space of $(L^A, \|\cdot\|_A)$ [3, Corollary 4.8.15]. From our discussion of Banach function spaces it therefore follows that the fundamental function of the space $(L^A, \|\cdot\|_{A^*}^O)$, is nothing but φ'_A.

The Lebesgue L^p spaces (with $1 \le p < \infty$) are of course just Orlicz spaces corresponding to the Young function $\psi(t) = t^p$. The astute reader will have noted that the notational conventions for Banach function spaces, Orlicz spaces and Lebesgue spaces are all different. Rather than overturning well established conventions for the sake of harmonising the notation, we shall leave the notation as presented above in the hope that this warning will suffice to ward of confusion that might have arisen.

Although the Luxemburg and Orlicz norms are equivalent, we shall hereafter prefer the Luxemburg norm for the simple fact that it encodes a strong link between the given Young function A, and the fundamental function of the Orlicz space $L^A(\Omega, \Sigma, \mu)$ in that

$$\varphi_A(t) = \frac{1}{A^{-1}(1/t)} \tag{2.1}$$

(see [3, Lemma 4.8.17].) Thus the fundamental function can be constructed from the given Young function. However more is true. Assuming φ to be a concave fundamental function, it is an exercise to see that $A(t) = \frac{1}{\varphi^{-1}(1/t)}$ then defines a Young function for which $\frac{1}{A^{-1}(1/t)} = \varphi(t)$. So there is an almost bijective correspondence between fundamental functions and Young functions (and hence also Orlicz spaces). In our analysis we may therefore use Young functions rather than fundamental functions as our starting point. Given a fundamental function φ for which $\varphi = \varphi_A$ for some Young function, we shall therefore write Λ_A, M_A and M_A^* for the spaces Λ_φ, M_φ and M_φ^*.

If we wish to give meaning to the notion of strong and weak type in this context, then given a pair of Young functions (A, B) and a sublinear operator T defined on the locally integrable functions $L^{loc}(\Omega, \Sigma, \mu)$, it is natural to define *strong type* (A, B) to be the fact that $T(f) \in L^B$ for every $f \in L^A$ with $\|T(f)\|_B \leq c\|f\|_A$ for some $c > 0$ (independent of f), and *weak type* (A, B) to be the fact that $T(f) \in M_B$ for every $f \in L^A$ with $\|T(f)\|_{M_B} \leq c\|f\|_A$ for some $c > 0$ (independent of f). In the literature weak type (A, B) is sometimes also defined as the claim that T restricts to a bounded operator from L^A to M_B^*. We shall in this note however prefer the slightly stronger definition given above. We will hereafter consistently write A, B, C, and X, Y, Z for Young functions.

In our analysis we shall have occasion to make use of Orlicz-Bochner spaces. For the sake of the reader we briefly survey the essentials. Given a measure space (Ω, Σ, μ) and a Banach space $\mathcal{X}$, a vector valued function $F : \Omega \to \mathcal{X}$ is said to be *Borel measurable* if it is measurable with respect to Σ and the Borel σ-algebra of $\mathcal{X}$. The function F is said to be strongly measurable if it is both Borel measurable and separably valued. It is not difficult to see that if F is strongly measurable, then $t \to \|F_t\|_{\mathcal{X}}$ is a μ-measurable real-valued function. (See the introduction to Appendix E of [6].) For a given Young function A, the space $L^A((\Omega, \Sigma, \mu), \mathcal{X})$ is defined to be the space of all (equivalence classes of) strongly measurable functions F from Ω to $\mathcal{X}$ for which the function $t \to \|F_t\|_{\mathcal{X}}$ belongs to $L^A(\Omega, \Sigma, \mu)$, with the norm given by $\|F\|_{A,\mathcal{X}} = \| \, \|F(\cdot)\|_{\mathcal{X}} \, \|_A$. Where there is no possibility of confusion, we will simply write $\|F\|_A$ for the norm. Readers will find references to these spaces throughout the literature.

A closely related concept, is that of a *mixed norm space.* For any two rearrangement invariant Banach function spaces $\mathcal{X}$ and $\mathcal{Y}$ on resonant measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ respectively, the mixed norm space $\mathcal{X}[\mathcal{Y}]$ (in the sense of Benedek and Panzone [2]) is defined to be the space of all $\mu_1 \times \mu_2$-measurable functions $(x, y) \to f(x, y)$ on $\Omega_1 \times \Omega_2$ for which $x \to \|f_x\|_{\mathcal{Y}}$ belongs to $\mathcal{X}$. (Here f_x is for any fixed x of course the function $y \to f(x, y)$. The canonical norm on such a mixed norm space $\mathcal{X}[\mathcal{Y}]$ is just $\|f\|_{\mathcal{X}[\mathcal{Y}]} = \| \, \|f_x\|_{\mathcal{Y}} \, \|_{\mathcal{X}}$. The reader will find a very useful discussion of such spaces in [4]. In the case where $\mathcal{X} = L^A$ for some Young function, the Bochner space $L^A(\Omega_1, \mathcal{Y})$ may also be regarded as a space of $\mu_1 \times \mu_2$-measurable functions $(x, y) \to f(x, y)$ where each vector-valued function $F \in L^A(\Omega_1, \mathcal{Y})$ corresponds to the function $f_F : \Omega_1 \times \Omega_2$ given by $f_F \colon (x, y) \to [F(x)](y)$. With $L^A(\Omega_1, \mathcal{Y})$ represented in this form, the μ_1-measurability of $x \to \|F(x)\|_{\mathcal{Y}}$ then ensures that $L^A(\Omega_1, \mathcal{Y})$ is isometrically a subspace of $L^A[\mathcal{Y}]$.

The question now arises as to when the spaces $L^A(\Omega_1, \mathcal{Y})$ and $L^A[\mathcal{Y}]$ are equal. It is an easy exercise to see that characteristic functions of measurable rectangles (christened rectangular simple functions by Blozinski [4]) all actually belong to the "smaller" space $L^A(\Omega_1, \mathcal{Y})$. So whenever the rectangular simple functions are dense in $L^A[\mathcal{Y}]$, we must have that $L^A(\Omega_1, \mathcal{Y}) = L^A[\mathcal{Y}]$. For such density to actually hold true, one must at least have absolute continuity of the norms involved. See for example Proposition 3.4.

In our analysis we shall be concerned with operators $T : \mathcal{E} \to \mathcal{F}$ between locally convex spaces of measurable which are ordered spaces with respect to the order structure inherited from the measurable functions. We allow T to be either linear or *sublinear*, where sublinearity means that $|T(f+g)| \leq |T(f)| + |T(g)|$ and $|T(\lambda f)| = |\lambda|\,|T(f)|$ for all f and g in $\mathcal{E}$ and all scalars λ. Such an operator is said to be *quasi-bounded* if for some set of seminorms $\{p_\alpha\}$ determining the topology of $\mathcal{E}$, we can find a set of seminorms $\{q_\beta\}$ determining the topology of $\mathcal{F}$ and a positive scalar c, such that for any α we can find an α' such that $p_\alpha(T(x) - T(y)) \leq c q_{\alpha'}(x - y)$ for all $x, y \in \mathcal{E}$. Clearly, quasi-bounded operators are continuous.

Denoting the space of *locally integrable functions* by L^{loc} (that is the space of measurable functions which are integrable on measurable sets of finite measure), the crucial information we shall need regarding such maps is contained in the result below. Apart from the claim regarding L^A, this result corresponds to [7, Proposition 2.8]. The proof of the claim regarding L^A runs along similar lines to that of the first claim, and hence we forgo the proof.

PROPOSITION 2.1. *Let T, $\mathcal{E}$ and $\mathcal{F}$ be as before with T quasi-bounded. Further let (Ω, μ) be a σ-finite measure and A a Young function. Writing $\mathcal{S}(\Omega, \mathcal{E})$ for the $\mathcal{E}$-valued simple functions, the map $\widetilde{T} : \mathcal{S}(\Omega, \mathcal{E}) \to \mathcal{S}(\Omega, \mathcal{F})$ canonically defined by $\widetilde{T}(f) := T \circ f$, extends to a continuous mapping from $L^{loc}(\Omega, \mathcal{E})$ to $L^{loc}(\Omega, \mathcal{F})$, and also to a continuous mapping from $L^A(\Omega, \mathcal{E})$ to $L^A(\Omega, \mathcal{F})$. (In the case of a linear map, the first extension corresponds to the linear map $I \otimes T : L^{\mathrm{loc}}(\Omega)\widehat{\otimes}_\pi \mathcal{E} \to L^{\mathrm{loc}}(\Omega)\widehat{\otimes}_\pi \mathcal{F}$.)*

For us a *dynamical system* consists of a multiplicative locally compact group G acting on a measure space (Ω, μ) via an integrable action $g \to \alpha_g$ of G on Ω by α. The action is assumed to be measure-preserving with $\tilde{\alpha} : \Omega \times G \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ in addition being measurable. Our interest will be in identifying ways to transfer dynamical phenomena in the group context, to the measure space context where in the group context such phenomena are encoded by what we call *transferable* operators T from $L^{loc}(G)$ to $C(G)$. Unless otherwise stated we shall generally assume G to also be σ-compact. The "transfer" hinted at above consists of constructing an operator $T^\# : (L^1 + L^\infty)(\Omega) \to L^{loc}(\Omega)$ from T. With the term *transferable operator* we have in mind the following concept:

DEFINITION 2.2. An operator $T : L^{loc}(G) \to C(G)$ is called *transferable* if

(1) T is a linear or positive valued sublinear operator which is also quasi-bounded.

(2) T is *semilocal* in that there exists an open neighbourhood U of $1 \in G$ with compact closure such that if $\mathrm{supp}(f)$ is contained in a set V, then $\mathrm{supp}(T(f))$ is contained in VU;

(3) T is *right translation invariant* in that for all $t \in G$ and $f \in L^{loc}(G)$,

$$\tau_t \circ T(f) = T \circ \tau_t(f),$$

where $\tau_t(f)$ is the operator defined by $s \mapsto f(st)$ for all measurable f and $s \in G$.

We pass to describing how a transferable operator T induces a related operator $T^{\#}$ in the measure space context. For this we shall need to expand our menagerie of spaces (see [7] for details). We specifically introduce the space $L^{r-loc}(\Omega_1 \times \Omega_2)$ which is the space of all measurable functions which are locally integrable on finitely measurable rectangles. It is a locally convex space whose topology is generated by seminorms of the form $p_{E\times F}(f) = \int_{E\times F} |f|\, d(\mu_1 \times \mu_2)$ where $E \times F$ is a finitely measurable rectangle. The space $L^{r-loc}(\Omega \times G)$ may be identified with $L^{loc}(\Omega)\widehat{\otimes}_\pi L^{loc}(G)$, which in turn may be identified with $L^{loc}(\Omega, L^{loc}(G))$ [7, Propositions 2.7 & 2.10].

With the foundational concepts in place we pass to the construction of $T^{\#}$. For any $f \in (L^1 + L^\infty)(\Omega)$ and any measurable subset $K \subset G$, define $f \otimes_\alpha \chi_K$ by $(\omega, t) \to f(\alpha_t(\omega))\chi_K(t)$ for all $\omega \in \Omega$ and $t \in G$.

- For a fixed measurable subset K of G the prescription $f \to f \otimes_\alpha \chi_K$ yields a map $\iota_{\alpha,K}$ from $(L^1 + L^\infty)(\Omega)$ to $L^{r-loc}(\Omega \times G) \equiv L^{loc}(\Omega, L^{loc}(G))$.
- As described in Proposition 2.1 above, we may extend T to the map $\widetilde{T} : L^{loc}(\Omega, L^{loc}(G)) \to L^{loc}(\Omega, C(G))$ (equivalently to a map from $L^{loc}(\Omega)\widehat{\otimes}_\pi L^{loc}(G)$ to $L^{loc}(\Omega)\widehat{\otimes}_\pi C(G)$).
- With δ_e denoting point evaluation at the group unit e, $\mathbb{1} \otimes \delta_e$ will now map $L^{loc}(\Omega)\widehat{\otimes}_\pi C(G) \equiv L^{loc}(\Omega, C(G))$ to $L^{loc}(\Omega)\widehat{\otimes}_\pi \mathbb{C} \equiv L^{loc}(\Omega)$.
- The composition $(\mathbb{1} \otimes \delta_e) \circ \widetilde{T} \circ \iota_{\alpha,K}$ then yields a sublinear map $T^{\#}_K$ from $(L^1 + L^\infty)(\Omega)$ into $L^{loc}(\Omega)$. In the case where $K = G$ we write $T^{\#}$ for $T^{\#}_K$.

As is pointed out in Remark 2.4, there is a slight challenge to working with $T^{\#}$, which is the fact that elements of the form $f \otimes_\alpha \mathbb{1}$ lie in a much smaller class of spaces than those of the form $f \otimes_\alpha \chi_K$ where K is compact. The good news is that we may obtain $T^{\#}$ as a limit of maps of the form $T^{\#}_K$ with K a compact set.

Proposition 2.3. *Let $T : L^{loc}(G) \to C(G)$ be a transferable operator, E be a compact neighbourhood of $e \in G$ and U the neighbourhood guaranteed by semilocality. For any compact set K containing $U^{-1}E$, we have that $T^{\#}_K = T^{\#}$.*

Proof. For the sake of clarity we will here write δ_e for point evaluation at e on $C(G)$, and $\widetilde{\delta}_e$ for the same process on $L^{loc}(\Omega, C(G))$.

We start by analysing the effect of semilocality on the action of T. Let E be a compact neighbourhood of $e \in G$. For any compact set $K \supset U^{-1}E$, the support of $T(\chi_{K^c})$ (where $K^c = G - K$) will be contained in K^cU. On arguing as in [7, Lemma 2.14], it follows that $K^cU \cap E$ is empty, and hence that we will for any $f \in L^{loc}(G)$ have that $\delta_e(T(\chi_{K^c}f)) = 0$. For any simple tensor $f \otimes a$ (with $f \in L^{loc}(G)$ and $a \in L^{loc}(\Omega)$) direct computation therefore shows that $\widetilde{\delta}_e\widetilde{T}(f\chi_K \otimes a) = \widetilde{\delta}_e\widetilde{T}(f \otimes a)$. Next select a net $\{P_n\}$ of linear combinations of simple tensors converging to $\chi_K \otimes_\alpha a$ in $L^{loc}(\Omega)\widehat{\otimes}_\pi L^{loc}(G)$. If say $P_n = \sum_{k=1}^m \gamma_k(f_k \otimes a_k)$ we may make sense of $(\chi_K \otimes \mathbb{1})P_n$ by identifying it with $\sum_{k=1}^m \gamma_k(\chi_K f_k \otimes a_k)$. Observe that $\{(\chi_K \otimes \mathbb{1})P_n\}$ is then still a net of simple tensors. For the sake of simplicity we will in the following simply write $P_n(t)$ for $\sum_{k=1}^m \gamma_k(f_k(t)a_k)$. Similarly $[(\chi_K \otimes \mathbb{1})P_n](t)$ means $\sum_{k=1}^m \gamma_k(\chi_K(t)f_k(t)a_k)$. The canonical seminorms determining the topology on $L^{loc}(\Omega)\widehat{\otimes}_\pi L^{loc}(G) \equiv L^{loc}(\Omega, L^{loc}(G))$ are known to be of the form $\widetilde{p}_{A,B} = \int_A \int_B (\cdot)\, dh\, d\mu$ where A and B respectively range over finitely measurable subsets of Ω and G (see [7]). For any $F \in L^{loc}(\Omega, L^{loc}(G))$ we clearly have that

$$\begin{aligned} \widetilde{p}_{A,B}((\chi_K \otimes \mathbb{1})F) &= \int_A \int_B \chi_K |F|\, dh\, d\mu \\ &= \int_B \int_{K\cap A} |F|\, dh\, d\mu \\ &\leq \int_B \int_A |F|\, dh\, d\mu \\ &= \widetilde{p}_{A,B}(F). \end{aligned}$$

This proves that the net $\{(\chi_K \otimes \mathbb{1})P_n\}$ converges to $\chi_K \otimes_\alpha a$. From what we showed earlier for simple tensors, it is clear that $\widetilde{\delta}_e\widetilde{T}(P_n) = \widetilde{\delta}_e\widetilde{T}((\chi_K \otimes \mathbb{1})P_n)$ for each n. Hence for any $a \in L^{1+\infty}(\Omega)$ we have by continuity that $T_K^\#(a) = \widetilde{\delta}_e\widetilde{T}(\chi_K \otimes_\alpha a) = \widetilde{\delta}_e\widetilde{T}(\mathbb{1} \otimes_\alpha a) = T^\#(a)$. □

The usefulness of the above proposition is highlighted by the ensuing remark.

Remark 2.4. One can show that for any compact set K and $f \in L^{loc}(\Omega)$, $f \otimes \chi_K$ and $f \otimes_\alpha \chi_k$ are equimeasurable. So if for example one of them is in $L^p(\Omega \times G)$, so is the other one. The reason why in a case like this we work with $f \otimes_\alpha \chi_K$ and not $f \otimes_\alpha 1$, is that if $f \in L^p(\Omega)$, it will for non-compact groups then be $f \otimes \chi_K$ that belongs to $L^p(\Omega \times G)$, rather than $f \otimes 1$. For simplicity of notation we will often write F for $f \otimes_\alpha 1$ and $F_{KU^{-1}}$ for $f \otimes_\alpha \chi_{KU^{-1}}$.

We close the discussion regarding $\widetilde{T}$ as defined above by investigating its action on spaces of measurable functions. For this task we introduce two new spaces (fully described in [7]). We firstly write $L^{k-loc}(G)$ for the space of measurable functions on G which are integrable on compact subsets of G. This space is also a locally convex space with the topology generated by seminorms of the form $p_K(f) = \int_k |f|\, dh$ (where K ranges over the compact sets). It is clear that $C(K)$ continuously embeds into $L^{k-loc}(G)$. This fact then ensures that we may also regard $\widetilde{T}$ as a continuous map from $L^{loc}(\Omega, L^{loc}(G))$ to $L^{loc}(\Omega, L^{k-loc}(G))$. However the space $L^{loc}(\Omega, L^{k-loc}(G))$ is known to be equivalent to the space $LK^{r-loc}(\Omega \times G)$ which is the space of measurable functions on $\Omega \times G$ which are integrable on measurable rectangles of the form $E \times K$ where E has finite measure and K is compact. The relevance of this is that for any triple A, B, C of Young functions, the spaces $L^A[L^B](\Omega \times G)$ and $L^C(\Omega \times G)$ are contained in both $L^{loc}(\Omega \times G)$ and $LK^{r-loc}(\Omega \times G)$. So if for example we know that T maps $L^B(G)$ into $L^C(G)$ and then use Proposition 2.1 to from T construct a map from $L^A(\Omega, L^B(G))$ to $L^A(\Omega, L^C(G))$, that map will of necessity be a restriction of the map $\widetilde{T}$ from $L^{loc}(\Omega, L^{loc}(G))$ to $L^{loc}(\Omega, L^{k-loc}(G))$. When this does happen we shall retain the same notation for the restriction. We may then of course also interpret this map as either a vector valued map on L^A or as the restriction of a map on $L^{loc}(\Omega \times G)$, as the situation demands.

The actual process of using the above structure to obtain ergodic convergence results runs as described below:

REMARK 2.5. At ground level the process for obtaining ergodic convergence results follows a three step programme. We summarise the essentials from [7, Section 5].

(1) One starts with a sequence (T_n) of transferable linear operators in the group context. From this one constructs the sublinear operator $T : f \to \sup_n |T_n(f)|$ checking if it is still transferable.

(2) Given the weak (alt. strong) type of the operator T, one then uses that information to obtain the weak (alt. strong) type of $T^\#$. It is this step that is the focus of this paper.

(3) The next step is to identify a dense subset D of the domain of $T^\#$ computed in step (2), for which the pointwise convergence of $(T_n^\# f)$ can be verified for all $f \in D$. This step is achieved by results like [7, Proposition 5.3].

(4) The final step is to then use an appropriate version of Banach's Principle (like for example [7, Proposition 5.1]) to extend the a.e. convergence noted in step (3) to the whole domain of $T^\#$.

We illustrate the process by means of a simple example.

EXAMPLE 2.6. Let G be the additive group $\mathbb{R}$ and T_n the operator $T_n : v \to \frac{1}{n}\chi_{[-n,0]} * v$ where $v \in L^{loc}(\mathbb{R})$. We know from [7, Lemma 2.16] that each T_n is a transferable operator. For each $f \in (L^1 + L^\infty)(\Omega)$ and each compactum $K \subset \mathbb{R}$, the operator $\widetilde{T}_n$ will map the simple tensor $F(\omega, t) = (f \otimes_\alpha \chi_K)(\omega, t) \equiv \chi_K(t) f(\alpha_t(\omega))$ to the function $\frac{1}{n}\widetilde{T}(F)$ given by $\widetilde{T}(F)(\omega, t) = T(F_\omega)(t) = \int_{\mathbb{R}} \chi_{[-n,0]}(t-s)\chi_K(t-s) f(\alpha_{t-s}(\omega))\, ds$. Point evaluation at the group unit $t = 0$ yields $(T_n)_K^\#(f)(\omega) = \int_{\mathbb{R}} \chi_{[-n,0]}(s)\chi_K(-s) f(\alpha_{-s}(\omega))\, ds$. Letting K increase we then obtain $T_n^\#(f)(\omega) = \int_{\mathbb{R}} \chi_{[-n,0]}(s) f(\alpha_{-s}(\omega))\, ds = \int_0^n f(\alpha_s(\omega))\, ds$. The process described above may then be used to show that under acceptable restrictions, convergence of the T_n's at the group level, ensures a.e. convergence of the averages $T_n^\#(f)(\omega) = \frac{1}{n}\int_0^n f(\alpha_s(\omega))\, ds$.

In refining the above theory we shall as in [7] develop a theory where ergodic convergence results may be obtained by means of inequalities involving either fundamental functions or Young functions. For this the following two facts are useful.

PROPOSITION 2.7. ([3]) *For an arbitrary Young function A we have that $A(A^{-1}(t)) \leq t \leq A^{-1}(A(t))$ for any $t \geq 0$.*

COROLLARY 2.8. *Let A, B and C be Orlicz spaces. Then the validity of $B(u)C(v) \leq A(\theta uv)$ for some $\theta > 0$ and all $u, v \geq 0$ implies that $\varphi_C(s)\varphi_B(t) \leq \theta\varphi_A(st)$ for all $s, t \geq 0$. Analogously the validity of $\varphi_C(s)\varphi_B(t) \geq \theta\varphi_A(st)$ for some $\theta > 0$ all $s, t \geq 0$ implies that $B(u)C(u) \geq A(\theta uv)$ for all $u, v \geq 0$.*

3. Continuity of $\widetilde{T}$.

Given Young functions A and Z, our primary goal in this section is to provide criteria under which $\widetilde{T}$ canonically induces a bounded linear map from $L^Z(\Omega \times G)$ into $L^A(\Omega \times G)$ (respectively into $M_A(\Omega \times G)$). The strategy we shall use to achieve this objective is to combine Proposition 2.1 with results comparing mixed norm spaces of the form $L^A[L^B](\Omega_1 \times \Omega_2)$ and $L^A[M_B](\Omega_1 \times \Omega_2)$ with spaces of the form $L^C(\Omega_1 \times \Omega_2)$ and $M_C(\Omega_1 \times \Omega_2)$. Apart from the works of Maligranda [11], Milman [13, 12] and O'Neil [14] and some earlier work by Ando, there is - almost surprisingly - very little in the literature regarding the type of inclusion results we need. There does not seem to be any operator theoretic results of the type we need in print. It is therefore incumbent on us to clearly show how what we need may be extracted from what is available. The following two theorems are the crucial ingredients in achieving this objective.

THEOREM 3.1. (Theorem 3; [11]) *Let X, Y and Z be Young functions. Then the following holds:*

(1) *The space $L^Z(\Omega_1 \times \Omega_2)$ continuously embeds into $L^X[L^Y](\Omega_1 \times \Omega_2)$ for any pair of σ-finite measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ if and only if $X(u)Y(v) \leq Z(\theta uv)$ for all $u, v \geq 0$.*

(2) *The space $L^X[L^Y](\Omega_1 \times \Omega_2)$ continuously embeds into $L^Z(\Omega_1 \times \Omega_2)$ for any pair of σ-finite measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ if and only if $Z(\theta uv) \leq X(u)Y(v)$ for all $u, v \geq 0$.*

The claim below corresponds to the implication (i)$\Rightarrow$(iii) in [13, Theorem 1]. Milman's result relies on an inequality involving inverses of Young functions. It is however clear from Proposition 2.7 that the Young function inequality given in the result below implies the inequality used by Milman. Theorem 1 of [13] is actually formulated for the Lebesgue space $([0, \infty), \mathscr{B}[0, \infty), \lambda)$ only. We claim that the implication just mentioned holds whenever the measure spaces involved are resonant σ-finite measure spaces. To see this note that in [13], Milman uses [13, Lemma 2.2] to prove this implication. This lemma in turn corresponds to Theorem 3.1 of [12]. Although also formulated for $([0, \infty), \mathscr{B}[0, \infty), \lambda)$ only, a careful perusal of the proof of [12, Theorem 3.1] shows that all we really need for the proof to go through, is Fubini's theorem for σ-finite measure spaces, a refined formula for the "second decreasing rearrangement" as given in [3, Proposition 2.3.3(a)], and a Hölder inequality up to the task (see [3, Corollaries 2.4.3 & 2.4.5]). As just noted all of these are also available for resonant σ-finite spaces. We therefore obtain the following

PROPOSITION 3.2. *Let A, B and C be Young functions and let $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ be resonant σ-finite measure spaces. If $\theta\varphi_C(uv) \leq \varphi_A(u)\varphi_B(u)$ for some $\theta > 0$ and all $u, v \geq 0$, then $L^A[M_B](\Omega_1 \times \Omega_2)$ continuously embeds into $M_C(\Omega_1 \times \Omega_2)$.*

For our purposes we actually need versions of the above results that hold for Orlicz-Bochner spaces rather than mixed norm spaces. Hence a bit of refinement is needed. The first step in this refinement is given by the following result.

PROPOSITION 3.3. *Let X, Y and Z be Young functions satisfying $X(u)Y(v) \leq Z(\theta uv)$ for some $\theta > 0$ and all $u, v \geq 0$. Suppose that $L^Z(\Omega_1 \times \Omega_2)$ is an Orlicz space in which the span of characteristic functions corresponding to measurable rectangles of finite measure is dense. Then $L^Z(\Omega_1 \times \Omega_2)$ continuously embeds into $L^X(\Omega_1, L^Y(\Omega_2))$.*

Proof. We know from Theorem 3.1 that $L^Z(\Omega_1 \times \Omega_2)$ continuously embeds into $L^X[L^Y]$. However each simple function corresponding to characteristic functions of measurable rectangles of finite measure is easily shown to belong to the "smaller" subspace $L^X(\Omega_1, L^Y(\Omega_2))$. When combined with the density assumption, the continuity of the embedding into $L^X[L^Y]$ now ensures that $L^Z(\Omega_1 \times \Omega_2)$ actually maps into $L^X(\Omega_1, L^Y(\Omega_2))$. $\square$

Having noted the above result, the question of how large the class of Young functions Z are that actually satisfy the criteria stipulated in the above result now arises. We pass to answering this question. The first step in answering this question is provided by the following result:

PROPOSITION 3.4. *Let $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ be resonant σ-finite measure spaces and let $\mathcal{X} = L^\rho(\Omega_1 \times \Omega_2)$ be a rearrangement-invariant Banach function space with absolutely continuous norm. If $\varphi_{\mathcal{X}}(t) \to 0$ as $t \to 0$, the span of the characteristic functions corresponding to measurable rectangles with finite measure, will then be norm dense in $L^\rho(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$.*

Although arguably well-known, we provide details for the sake of the reader.

Proof. Suppose that $\varphi_{\mathcal{X}}(t) \to 0$ as $t \to 0$. We know from [3, Theorems II.5.4 & II.5.5] that the integrable simple functions will then be norm dense in X. (To see the veracity of this claim, note that it is clear from for example the first paragraph of the proof of [3, Theorem I.1.4], that Bennett and Sharpley reserve the term "simple function" for what we call integrable simple functions.) The theorem will therefore follow if we can show that for any measurable subset $E \subset \Omega_1 \times \Omega_2$ with finite measure, the function χ_E is the norm limit of integrable simple functions composed of characteristic functions of measurable rectangles with finite measure.

Given such an E we know from the outer measure approach to constructing $\mu_1 \times \mu_2$ [8, 251] that $(\mu_1 \times \mu_2)(E) = \inf\{\sum_{n\geq 1} \mu_1(A_n)\mu_2(B_n)\colon E \subset \cup_{n\geq 1}(A_n \times B_n)\}$. So given $\epsilon > 0$ we may select $\{A_n\} \subset \Sigma_1$ and $\{B_n\} \subset \Sigma_2$ so that $E \subset \cup_{n\geq 1}(A_n \times B_n)$ and $(\mu_1 \times \mu_2)(\cup_{n\geq 1}(A_n \times B_n) \leq \sum_{n\geq 1} \mu_1(A_n)\mu_2(B_n) \leq (\mu_1 \times \mu_2)(E) + \epsilon$. This then in particular implies that $(\mu_1 \times \mu_2)(E - \cup_{n\geq 1}(A_n \times B_n)) \leq \epsilon$ and $(\mu_1 \times \mu_2)(\cup_{n\geq 1}(A_n \times B_n) - \cup_{N\geq n\geq 1}(A_n \times B_n)) \leq (\mu_1 \times \mu_2)(\cup_{n\geq N+1}(A_n \times B_n) \leq \sum_{n\geq N+1} \mu_1(A_n)\mu_2(B_n) \to 0$ as $N \to \infty$. We may then apply [3, Lemma I.3.4] to firstly see that $(\chi_{\cup_{N\geq n\geq 1}(A_n\times B_n)})$ converges to $\chi_{\cup_{n\geq 1}(A_n\times B_n)}$ in norm and that on letting $\epsilon \searrow 0$, one may select a sequence composed of terms of the form $\chi_{\cup_{n\geq 1}(A_n\times B_n)}$ which converges in norm to χ_E. The claim then follows from noting that it is a straightforward exercise to, if necessary, write $\cup_{N\geq n\geq 1}(A_n \times B_n)$ as a finite union of disjoint rectangles, in which case we will then have that $\chi_{\cup_{N\geq n\geq 1}(A_n\times B_n)} = \sum_{N\geq n\geq 1} \chi_{(A_n\times B_n)}$. □

Examples of Orlicz spaces L^Z satisfying the criteria of Proposition 3.3 are therefore those with absolutely continuous norm, for which we also have that $\varphi_Z(t) \to 0$ as $t \searrow 0$. As is shown by the following two results, all Orlicz spaces constructed from N-functions satisfying the Δ_2-property satisfy these criteria.

Lemma 3.5. *Let $L^A(\Omega, \Sigma, \mu)$ be an Orlicz space for which A satisfies the Δ_2 condition. Then $\varphi_A(t) \to 0$ as $t \to 0$.*

Proof. Suppose that A satisfies the Δ_2 condition. That is suppose that there exists a constant $c \geq 0$ such that $A(2s) \leq cA(s)$ for every $s \geq 0$. We may clearly assume that $c > 1$. Recalling that $A^{-1}(t)$ is defined as $A^{-1}(t) = \sup\{s\colon A(s) \leq t\}$, the Δ_2 condition clearly ensures that $\{s\colon A(s) \leq \frac{t}{c}\} \subset \{s\colon A(2s) \leq t\}$ which in turn ensures that $A^{-1}(t/c) \leq \sup\{s\colon A(2s) \leq t\} = \frac{1}{2}A^{-1}(t)$. Given that the fundamental function of L^A when equipped with the Luxemburg norm is exactly $\varphi_A(t) = \frac{1}{A^{-1}(1/t)}$, it now easily follows that $\varphi_A(t/c) \leq \frac{1}{2}\varphi(t)$ for every $t \geq 0$. This clearly shows that as claimed $\varphi_A(t) \to 0$ as $t \to 0$. □

The following theorem was noted by Krasnoselskii and Rutitskii in the discussion following [9, Theorem II.10.3]

Theorem 3.6. *Let A be an N-function and (Ω, Σ, μ) a σ-finite measure space. Then A satisfies the Δ_2 property if and only if $L^A(\Omega, \Sigma, \mu)$ has absolutely continuous norm.*

We need to take note of one more result before we are able to state the main theorem of this section.

Proposition 3.7. *Let (Ω, ν) be a σ-finite measure space admitting an integrable group action of some locally compact group G and T a transferable operator from $L^{loc}(G)$ to $C(G)$. Let A, B and C be Young functions, and suppose that T maps $L^B(G)$ into $L^C(G)$ (respectively $L^B(G)$ into $M_C(G)$). Then $\widetilde{T}$ maps $L^A(\Omega, L^B(G))$ into $L^A(\Omega, L^C(G))$ (respectively into $L^A(\Omega, M_C(G))$).*

Proof. The argument of Corollary 2 on p 134 of [16] goes through as is. □

With all the preparation done, the main theorem of this section is now an easy consequence. We remind the reader that a σ-locally compact group is a topological group which is both locally compact and σ-compact. It is well-known that the σ-compactness will then force the associated Haar measure to be σ-finite.

THEOREM 3.8. *Let (Ω, ν) be a σ-finite measure space admitting an integrable group action of some σ-locally compact group G and T a transferable operator from $L^{loc}(G)$ to $C(G)$. Let A, B, X, Y and Z be Young functions satisfying $X(u)Y(v) \leq Z(\theta_1 uv)$ and $\varphi_X(s)\varphi_B(t) \geq \theta_2\varphi_A(st)$ for all $u, v, s, t \geq 0$, and suppose that $L^Z(\Omega\times G)$ is an Orlicz space in which the span of characteristic functions corresponding to measurable rectangles of finite measure is dense. If T maps $L^Y(G)$ into $L^B(G)$ (respectively $L^Y(G)$ into $M_B(G)$), then $\widetilde{T}$ maps $L^Z(\Omega\times G)$ into $L^A(\Omega\times G)$ (respectively into $M_A(\Omega\times G)$).*

Proof. The validity of the first claim follows from the continuity of each of the following maps:

$$L^Z(\Omega\times G) \hookrightarrow L^X(\Omega, L^Y(G))$$
$$\widetilde{T} : L^X(\Omega, L^Y(G)) \to L^X(\Omega, L^B(G))$$
$$L^X(\Omega, L^B(G)) \hookrightarrow L^A(\Omega\times G).$$

The second claim follows similarly. □

4. Maximal inequalities for $T^{\#}$.

THEOREM 4.1. *Let (Ω, ν) be a measure space admitting an integrable group action of some locally compact group G and T a transferable operator from $L^{loc}(G)$ to $C(G)$. Suppose that $B(u)C(u) \leq A(\theta uv)$ for all $u, v \geq 0$. Then the following holds:*

(1) $\widetilde{T}(F_{KU^{-1}}) \in L^A(\Omega\times G) \;\Rightarrow\; \varphi_C(|K|)\|T^{\#}(f)\|_B \leq \theta\|\widetilde{T}(F_{KU^{-1}})\|_A;$

(2) $\widetilde{T}(F_{KU^{-1}}) \in M_A(\Omega\times G) \;\Rightarrow\; \varphi_C(|K|)\|T^{\#}(f)\|_{M_B} \leq \theta\|\widetilde{T}(F_{KU^{-1}})\|_{M_A};$

(3) $\widetilde{T}(F_{KU^{-1}}) \in M_A^*(\Omega\times G) \;\Rightarrow\; \varphi_C(|K|)\|T^{\#}(f)\|_{M_B^*} \leq \theta\|\widetilde{T}(F_{KU^{-1}})\|_{M_A^*}.$

Here the spaces M_A^* and M_B^* are the "weaker" Lorentz type spaces defined in the introduction. Since our definition of weak type does not make use of these spaces, the final claim above is therefore not really relevant for us, but rather mentioned for the sake of completeness.

Proof. *Claims 2 and 3:* We know from [7, Lemma 4.6] that $|K|m(T^{\#}(f), s) \leq m(\widetilde{T}(F_{KU^{-1}}), s)$ for all $s > 0$ which in turn implies that $(T^{\#}(f))^*(t/|K|) \leq (\widetilde{T}(F_{KU^{-1}}))^*(t)$ for all $t > 0$. But this last inequality further ensures that $\frac{|K|}{t}\int_0^t (T^{\#}(f))^*(s)\,ds \leq \frac{1}{t}\int_0^t (\widetilde{T}(F_{KU^{-1}}))^*(s)\,ds$ for all $t > 0$.

Since by Corollary 2.8 we have that $\varphi_C(s)\varphi_B(t) \leq \theta\varphi_A(st)$ for all $t, s > 0$, it therefore follows that

$$\theta\|\widetilde{T}(F_{KU^{-1}})\|_{M_A^*} = \theta \sup_{s>0} \varphi_A(s)\mu_{\widetilde{T}(F_{KU^{-1}})}(s) \geq \theta \sup_{s>0} \varphi_A(s)\mu_{T^\#(f)}(s/|K|)$$

$$\geq \varphi_C(|K|) \sup_{s>0} \varphi_B(s/|K|)\mu_{T^\#(f)}(s/|K|) = \varphi_C(|K|).\|T^\#(f)\|_{M_B^*}.$$

On using the second inequality noted above, a similar proof shows that $\theta\|\widetilde{T}(F_{KU^{-1}})\|_{M_A} \geq \varphi_C(|K|).\|T^\#(f)\|_{M_B}$

Claim 1: For any $\epsilon > 0$ we will by the definition of the Luxemburg norm have that

$$\int_0^\infty B(\mu_{T^\#(f)}(t)/(\|T^\#(f)\|_B + \epsilon))\,dt > 1.$$

From the above we conclude that

$$\begin{aligned}
&\int_0^\infty A(\mu_{\theta\widetilde{T}(F_{KU^{-1}})}(t)/(\varphi_C(|K|)\|T^\#(f)\|_B + \epsilon))\,dt \\
&\quad\geq \int_0^\infty A(\mu_{\theta T^\#(f)}(t/|K|)/(\varphi_C(|K|)\|T^\#(f)\|_B + \epsilon))\,dt \\
&\quad\geq \int_0^\infty A(A^{-1}(C(1/\varphi_C(|K|))B(\mu_{T^\#(f)}(t/|K|)/\|T^\#(f)\|_B + \epsilon))\,dt \\
&\quad= \int_0^\infty C(C^{-1}(1/|K|))B(\mu_{T^\#(f)}(t/|K|)/\|T^\#(f)\|_B + \epsilon))\,dt \\
&\quad= \int_0^\infty \tfrac{1}{|K|}B(\mu_{T^\#(f)}(t/|K|)/\|T^\#(f)\|_B + \epsilon))\,dt \\
&\quad= \int_0^\infty B(\mu_{T^\#(f)}(s)/\|T^\#(f)\|_B + \epsilon))\,ds \\
&\quad> 1.
\end{aligned}$$

It now clearly follows from the definition of the Luxemburg norm that $\theta\|\widetilde{T}(F_{KU^{-1}})\|_A < \varphi_C(|K|)(\|T^\#(f)\|_B + \epsilon)$. The fact that $\epsilon > 0$ was arbitrary now ensures that $\theta\|\widetilde{T}(F_{KU^{-1}})\|_A \leq \varphi_C(|K|)\|T^\#(f)\|_B$. □

LEMMA 4.2. *Let (Ω, ν) be a measure space admitting an integrable group action of some locally compact group G. Suppose that $\theta\varphi_X(s)\varphi_Y(t) \geq \varphi_Z(st)$ for all $t, s > 0$. For any $f \in L^Y(\Omega)$ and any compact $K \subset G$, we have that $F_{KU^{-1}} = f \otimes_\alpha \chi_{KU^{-1}} \in L^Z(\Omega \times G)$ with $\|F_{KU^{-1}}\|_Z \leq \theta\varphi_X(|KU^{-1}|).\|f\|_Y$*

Proof. By [7, Lemma 2.4] $f \otimes_\alpha \chi_{KU^{-1}}$ and $f \otimes \chi_{KU^{-1}}$ are equimeasurable and hence have the same $\|\cdot\|_Z$-norm. The lemma now follows from [14, Theorem 8.15]. See also [1, Theorem 6]. □

The following theorem now clearly follows.

THEOREM 4.3. *Let* (Ω, ν) *be a measure space admitting an integrable group action of some amenable locally compact group* G *and* T *a transferable operator from* $L^{loc}(G)$ *to* $C(G)$. *Suppose that* $B(u)C(v) \leq A(\theta uv)$ *for all* $u, v \geq 0$, *and that* $\theta_2\varphi_X(s)\varphi_Y(t) \geq \varphi_Z(st)$ *for all* $t, s > 0$. *If* $\varphi_X \leq \theta_3\varphi_C$, *then the following holds:*

(1) *If* $\widetilde{T}$ *boundedly maps* $L^Z(\Omega \times G)$ *into* $L^A(\Omega \times G)$, *then* $T^\#$ *boundedly maps* $L^Y(\Omega)$ *into* $L^B(\Omega)$ *with* $\|T^\#\| \leq \theta_1\theta_2\theta_3\|\widetilde{T}\|$.

(2) *If* $\widetilde{T}$ *boundedly maps* $L^Z(\Omega \times G)$ *into* $M_A(\Omega \times G)$, *then* $T^\#$ *boundedly maps* $L^Y(\Omega)$ *into* $M_B(\Omega)$ *with* $\|T^\#\| \leq \theta_1\theta_2\theta_3\|\widetilde{T}\|$.

(3) *If* $\widetilde{T}$ *boundedly maps* $L^Z(\Omega \times G)$ *into* $M_A^*(\Omega \times G)$, *then* $T^\#$ *boundedly maps* $L^Y(\Omega)$ *into* $M_B^*(\Omega)$ *with* $\|T^\#\| \leq \theta_1\theta_2\theta_3\|\widetilde{T}\|$.

Proof. We prove only the first claim. The other cases are entirely analogous. The preceding lemmata ensure that for any compact subset K of G and any $f \in L^Y(\Omega)$ we will have that

$$\begin{aligned}\varphi_C(|K|)\|T^\#(f)\|_B \leq \theta_1\|\widetilde{T}(F_{KU^{-1}})\|_A &\leq \theta_1\|\widetilde{T}\|.\|F_{KU^{-1}}\|_Z \\ &\leq \theta_1\theta_2\varphi_X(|KU^{-1}|)\|\widetilde{T}\|.\|f\|_Y.\end{aligned}$$

Hence

$$\frac{\varphi_X(|K|)}{\varphi_X(|KU^{-1}|)}\|T^\#(f)\|_B \leq \theta_3\frac{\varphi_C(|K|)}{\varphi_X(|KU^{-1}|)}\|T^\#(f)\|_B \leq \theta_1\theta_2\theta_3\|\widetilde{T}\|.\|f\|_Y.$$

If we are able to show that $\sup_K \frac{\varphi_X(|K|)}{\varphi_X(|KU^{-1}|)} = 1$, the claim will follow. Since φ_X is non-decreasing, we clearly have that $\frac{\varphi_X(|KU^{-1}|)}{\varphi_X(|K|)} \geq 1$ for any compact set K, and hence that $\inf_K \frac{\varphi_X(|KU^{-1}|)}{\varphi_X(|K|)} \geq 1$. However we also have that $t \to \frac{\varphi_X(t)}{t}$ is non-increasing on $(0, \infty)$ [3, Corollary 2.5.3]. This fact then ensures that we will for any compact set K have that $\frac{\varphi_X(|KU^{-1}|)}{\varphi_X(|K|)} = \frac{\varphi_X(|KU^{-1}|)}{|KU^{-1}|}\frac{|KU^{-1}|}{\varphi_X(|K|)} \leq \frac{\varphi_X(|K|)}{|K|}\frac{|KU^{-1}|}{\varphi_X(|K|)} = \frac{|KU^{-1}|}{|K|}$. Now recall that it is shown in the proof of [7, Theorem 4.8] that $\sup_K \frac{|K|}{|KU^{-1}|} = 1$. We therefore also have that $\inf_K \frac{\varphi_X(|KU^{-1}|)}{\varphi_X(|K|)} \leq 1$, which then ensures that $\inf_K \frac{\varphi_X(|KU^{-1}|)}{\varphi_X(|K|)} = 1$ as required. □

5. An illustrative application. Throughout this section T will be a transferable operator in the group context and $T^\#$ the resultant operator in the measure space context, where we assume the ambient measure space (Ω, Σ, μ) to be σ-finite. Assuming T to be of weak type (A, B), Theorems 3.8 and 4.3 provide us with a mechanism for computing the allowed weak types of $T^\#$ from that of T. The challenge in doing such a computation is to write down a set of Young/fundamental function inequalities that give us the required access to these theorems. So the effectiveness with which we can compute the weak type of $T^\#$ is in some sense dependent on our skill in writing down such inequalities. The "tighter" the inequalities the better the transfer. For Young functions like $t \to t^p$ (where $1 \le p < \infty$) with very regular polynomial growth these required inequalities become equalities in the sense that $(ts)^p = t^p s^p$. Hence here we can exercise very tight control with operators of weak type (p,p) transferring to operators of the same weak type. However for Orlicz spaces like L^{exp} or $L\log(L)$ built using Young functions with extremely rapid or extremely slow growth, controlling this process tightly becomes all the more difficult.

To start off with we note that for any Young function A satisfying $0 = \inf\{t\colon A(t) > 0\}$ and $\infty = \sup\{t\colon A(t) < \infty\}$, there is a set of 2-variable inequalities of "last resort" we always have access to. These are formulated in terms of functions associated with $L^1 + L^\infty$ and $L^1 \cap L^\infty$. These spaces are known to be Orlicz spaces. On employing the short-hand $Y_{1+\infty}$ and $Y_{1\cap\infty}$ for the associated Young functions, we remind the reader that

$$Y_{1+\infty}(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ t-1 & \text{otherwise} \end{cases} \text{ and } Y_{1\cap\infty}(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ \infty & \text{otherwise} \end{cases}.$$

When both spaces are equipped with their Luxemburg norms, we may use equation 2.1 to see that for any allowed valued $t > 0$ we have that $\varphi_{1+\infty}(t) = \frac{t}{t+1}$ and $\varphi_{1\cap\infty}(t) = \max(1,t)$. We shall henceforth assume that these functions are defined on all of $[0,\infty)$ (which will be the case if our measure space is $([0,\infty)\mathscr{B}([0,\infty)))$ equipped with Lebesgue measure).

LEMMA 5.1. *For any Young function A with $0 = \inf\{t\colon A(t) > 0\}$ and $\infty = \sup\{t\colon A(t) < \infty\}$ we have that*

$$A(s)Y_{1+\infty}(t) \le A(2st), \qquad A(\tfrac{1}{2}st) \le A(s)Y_{1\cap\infty}(t)$$

and

$$\varphi_A(s)\varphi_{1+\infty}(t) \le 2\varphi_A(st), \qquad \varphi_A(st) \le 2\varphi_A(s)\varphi_{1\cap\infty}(t)$$

for all $s,t \ge 0$.

Proof. Each quasi-concave (fundamental) function φ admits a so-called concave majorant $\widetilde{\varphi}$ for which we have that $\frac{1}{2}\widetilde{\varphi} \le \varphi \le \widetilde{\varphi}$. (See for example the exposition on fundamental functions in [3].) The function $\widetilde{\varphi}$ is the smallest concave quasi-concave function majorising φ. Using these facts the concavity of $\widetilde{\varphi}_A$ when combined with the fact that $\varphi_{1+\infty}(t) \le \min(1,t)$ for all $t \ge 0$, ensures that $\varphi_{1+\infty}(t)\varphi_A(s) \le \varphi_{1+\infty}(t)\widetilde{\varphi}_A(s) \le \widetilde{\varphi}_A(\varphi_{1+\infty}(t)s) \le \widetilde{\varphi}_A(ts) \le 2\varphi_A(ts)$ for all $t,s \ge 0$. Similarly $\varphi_A(ts) \le 2\varphi_{1\cap\infty}(t)\varphi_A(s)$ for all $t,s \ge 0$. The inequality $A(\frac{1}{2}st) \le A(s).Y_{1\cap\infty}(t)$ now follows from an application of Corollary 2.8 to $\varphi_A(st) \le 2\varphi_A(s).\varphi_{1\cap\infty}(t)$. For the remaining inequality notice that by equation 2.1 and the facts noted in the preceding discussion, we may in the case where $t > 0$ rewrite $\varphi_A(s)\varphi_{1+\infty}(t) \le 2\varphi_A(st)$ as $A^{-1}(\frac{1}{st}) \le 2\left(\frac{t+1}{t}\right)A^{-1}(\frac{1}{s})$. The assumptions on A ensure that in this case $A^{-1}(A(v)) = v = A(A^{-1}(v))$. Thus on respectively replacing $\frac{1}{t}$ and $\frac{1}{s}$ with u and $A(v)$, the resultant inequality can then be seen to lead to the fact that $uA(v) \le A(2(1+u)v)$ for all $u,v > 0$. On setting $w = u+1$, this yields the fact that $Y_{1+\infty}(w).A(v) = (w-1)A(v) \le A(2vw)$ for all $v > 0$ and

$w > 1$, which then verifies the remaining claim. □

To simplify computations we now take $A = Z$ in both Theorems 3.8 and 4.3 assuming also that this Young function satisfies the criteria mentioned in the hypothesis of Theorem 3.8. If we are able to find Young functions B, X and Y such that

$$\theta_1 A(st) \geq X(s)Y(t) \qquad \text{and} \qquad \theta_2 \varphi_A(uv) \leq \varphi_X(u)\varphi_B(u)$$

for all allowable s, t, u and v, then Theorems 3.8 and 4.3 ensure that any transferable operator T of weak(strong) type (Y, B), ultimately yields an operator $T^\#$ of weak(strong) type (B, Y). As mentioned earlier if we hope to prove convergence of ergodic averages on a space corresponding to a Young function with either very rapid or very slow growth, finding such a pair may be very difficult. In the case where $A = Z$, obtaining such a pair can be a significant challenge. So in such settings one may either need additional restrictions for the result to go through, or else be willing to slightly modify the Young function in view in order to obtain a useful result.

We shall illustrate these principles for the Young function $A_{\log}(t) = t\log(1+t)$ which generates the space $L\log(L+1)$. The significance of this Young function for quantum physics has already been pointed out in [10]. In [7, Section 6] it was shown how under mild additional assumptions one may obtain ergodic convergence results for the space $L\log(L+1)$. We now show how convergence results may be obtained by modifying the function $t \to t\log(t+1)$. A Young function is said to satisfy the Δ' condition globally if there exists some $\theta > 0$ such that $A(st) \leq \theta A(s)A(t)$ for all $s, t \geq 0$. Recall that two Young functions A_0 and A_1 are said to be equivalent if there exists a constant $\theta > 0$ so that $\theta^{-1}A_1 \leq A_0 \leq \theta A_1$. The function $t \to t\log(t+1)$ can be shown to be equivalent to the Young function $t \to (t+1)\log(t+1) - t$ which in turn is known to fail the Δ' condition globally (see the example on page 33 of [15]). [1] We specifically modify the behaviour of $A_{\log}$ near 0 to try and ensure more regular behaviour. Consider the Young function $A_{\log+\epsilon}$ given by

$$A_{\log+\epsilon}(t) = t\log(1+\epsilon+t) \text{ for all } t \geq 0.$$

We clearly have that $A_{\log+\epsilon} \geq A_{\log}$. Though more regular than $A_{\log}$, this class of Young functions nevertheless do approximate $A_{\log}$ very closely. We record further facts in the following lemma.

LEMMA 5.2. *The Young function $A_{\log+\epsilon}$ satisfies the Δ_2 condition globally. In addition*

$$A_{\log+\epsilon}((\tfrac{\log(1+\epsilon)}{2})st) \leq A_{\log+\epsilon}(t).A_{\log+\epsilon}(s) \text{ for all } s, t \geq 0$$

[1] The claimed equivalence follows from the fact that $\lim_{t\searrow 0} \frac{(t+1)\log(t+1)-t}{t\log(t+1)} = \frac{1}{2}$ and $\lim_{t\nearrow\infty} \frac{(t+1)\log(t+1)-t}{t\log(t+1)} = 1$. These limits together with the fact that the function $t \to \frac{(t+1)\log(t+1)-t}{t\log(t+1)}$ must have a positive minimim and maximum on each interval of the form $[a, b]$ where $0 < a < b$ ensures the existence of a constant $\theta > 0$ such that $\theta^{-1}(t+1)\log(t+1) - t \leq t\log(t+1) \leq \theta(t+1)\log(t+1) - t$ for all t.

and

$$\left(\tfrac{\log(1+\epsilon)}{2}\right)\varphi_{A_{\log+\epsilon}}(st) \leq \varphi_{A_{\log+\epsilon}}(t).\varphi_{A_{\log+\epsilon}}(s) \text{ for all } s,t \geq 0.$$

Proof. To see that $A_{\log+\epsilon}$ satisfies the Δ_2 condition globally, notice that when combined with continuity, the fact that for any $k > 0$ we have that $\lim_{t\searrow 0} \frac{A_{\log+\epsilon}(kt)}{A_{\log+\epsilon}(t)}$ $= k$ and $\lim_{t\nearrow\infty} \frac{A_{\log+\epsilon}(kt)}{A_{\log+\epsilon}(t)} = k\lim_{t\nearrow\infty} \frac{\log(1+\epsilon+kt)}{\log(1+\epsilon+x)} = k^2$, guarantees the existence of a constant θ_k such that $A_{\log+\epsilon}(kt) \leq \theta_k A_{\log+\epsilon}(t)$ for all $t \geq 0$. Since here we have that $A^{-1}_{\log+\epsilon}(A_{\log+\epsilon}(t)) = t = A_{\log+\epsilon}(A^{-1}_{\log+\epsilon}(t))$, the second claimed inequality will follow from the first. The first inequality follows on noting that we will for all $s,t \geq 0$ have that

$$\begin{aligned}\log(1+\epsilon+st) &\leq \log((1+\epsilon+s)(1+\epsilon+t)) = \log(1+\epsilon+s) + \log(1+\epsilon+t)\\ &\leq \left(\tfrac{2}{\log(1+\epsilon)}\right)\log(1+\epsilon+s).\log(1+\epsilon+t). \qquad \square\end{aligned}$$

Our efforts to date have now yielded the pair of equations

$$A_{\log+\epsilon}(st) \geq \frac{2}{\log(1+\epsilon)} A_{\log+\epsilon}(s) Y_{1+\infty}(t)$$

and

$$\left(\tfrac{\log(1+\epsilon)}{2}\right)\varphi_{\log+\epsilon}(uv) \leq \varphi_{\log+\epsilon}(u).\varphi_{\log+\epsilon}(v)$$

valid for all allowable s, t, u and v. So in this case Theorems 3.8 and 4.3 ensure that any transferable operator T of weak(strong) type $(Y_{1+\infty}, A_{\log+\epsilon})$, yields an operator $T^{\#}$ of weak(strong) type $(A_{\log+\epsilon}, Y_{1+\infty})$.

In conclusion then, the results obtained in this paper are not a silver bullet which magically make all problems disappear. What they do offer is on the one hand an accessible calculus which will enable anyone who is willing to get his/her hands dirty, to get a very good idea of the kinds of convergence results that may reasonably be expected to hold for a particular Orlicz space, and on the other to present an algorithm for then proving such convergence.

Appendix.

LEMMA. *The product of two σ-finite measure spaces is nonatomic if one of the factors is nonatomic*

Proof. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite with say $(\Omega_1, \Sigma_1, \mu_1)$ nonatomic. To prove the claim we need to show that for any measurable subset $E \subset (\Omega_1 \times \Omega_2)$ with positive measure, we can find a measurable subset E_0 with $0 < (\mu_1 \times \mu_2)(E_0) < (\mu_1 \times \mu_2)(E_0)$. Since both $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite we can find countable families $\{A_n\} \subset \Omega_1$ and $\{B_n\} \subset \Omega_2$ of disjoint sets of finite measure such that $\Omega_1 \times \Omega_2 = \cup_{m,n\geq 1}(A_n \times B_m)$. By countable additivity at least one of the sets $E \cap (A_n \times B_m)$ must then have positive (finite) measure. If a priori we had that $\mu(E) = \infty$ we are done as we may then take E_0 to be such a $E \cap (A_n \times B_m)$. We may therefore assume that $\mu(E) < \infty$. By the outer measure approach to constructing $\mu_1 \times \mu_2$ [8, 251] we have that $(\mu_1 \times \mu_2)(E) = \inf\{\sum_{n\geq 1} \mu_1(C_n)\mu_2(D_n)\colon E \subset \cup_{n\geq 1}(C_n \times D_n)\}$. On selecting some $\epsilon > 0$ which is at least less that $\mu(E)/2$, we may therefore select a family $\{C_n \times D_n\}$ of rectangles with $E \subset \cup_{n\geq 1}(C_n \times D_n)$ and $\sum_{n\geq 1} \mu_1(C_n)\mu_2(D_n) < (\mu_1 \times \mu_2)(E) + \epsilon/2$. We therefore further have that $(\mu_1 \times \mu_2)(\cup_{n\geq N}(C_n \times D_n)) \leq \sum_{n\geq N} \mu_1(C_n)\mu_2(D_n) \to 0$ as $N \to \infty$. Hence we may select $N \in \mathbb{N}$ so that $(\mu_1 \times \mu_2)(\cup_{n\geq 1}(C_n \times D_n)) - \epsilon/2 \leq (\mu_1 \times \mu_2)(\cup_{N\geq n\geq 1}(C_n \times D_n))$. Together these two facts ensure that $(\mu_1 \times \mu_2)(\cup_{n\geq 1}(C_n \times D_n) - [E \cap (\cup_{N\geq n\geq 1}(C_n \times D_n))]) < \epsilon$, guaranteeing that $(\mu_1 \times \mu_2)(E \cap (\cup_{N\geq n\geq 1}(C_n \times D_n)) > 0$. It is not difficult to see that if necessary we may rewrite $\cup_{N\geq n\geq 1}(C_n \times D_n)$ as a union of disjoint rectangles. We may now use the non-atomicity of $(\Omega_1, \Sigma_1, \mu_1)$ to partition each C_n into two disjoint measurable subsets X_n and Y_n, such that $\mu_1(X_n) = \mu_1(Y_n) = \frac{\mu(A_n)}{2}$ for each n. This partitions $\cup_{N\geq n\geq 1}(C_n \times D_n)$ into two disjoint sets $\cup_{N\geq n\geq 1}(X_n \times D_n)$ and $\cup_{N\geq n\geq 1}(Y_n \times D_n)$ each with half the measure of the original set $\cup_{N\geq n\geq 1}(C_n \times D_n)$. The intersection of E with at least one of the disjoint sets $\cup_{N\geq n\geq 1}(X_n \times D_n)$ and $\cup_{N\geq n\geq 1}(Y_n \times D_n)$ must then have positive measure. If say $E \cap (\cup_{N\geq n\geq 1}(X_n \times D_n))$ has positive measure, the halving we have just noted ensures that $(\mu_1 \times \mu_2)(E \cap (\cup_{N\geq n\geq 1}(X_n \times D_n))) \leq \frac{1}{2}\sum_{N\geq n\geq 1} \mu_1(C_n)\mu_2(D_n) < \frac{1}{2}[(\mu_1 \times \mu_2)(E) + \epsilon/2] < \mu(E)$ as required. □

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