

Valuation of credit default swaptions using Finite Difference Method

by
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Dedicated to

My family and late grand mother.

“The Lord is my light and salvation, he is the stronghold of my life” Psalm 27.

Declaration

I declare that apart from the assistance acknowledged, Valuation of credit default swaptions using Finite Difference Method is my unaided own work. It has not been submitted for any degree or examination in any other University, and this research dissertation is submitted in partial fulfilment of the requirements for the degree Master of Science at the Centre for Business Mathematics and Informatics at the North-West University (Potchefstroom campus). The sources I have used and quoted have been indicated and acknowledged by complete references.

Signature.....

Date.....

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Executive Summary

Credit default swaptions (CDS options) are credit derivatives that are widely used by financial institutions such as banks and hedging companies to manage their credit risk. These options are usually priced using Black-Scholes model, but the assumptions underlying this model do not always hold especially when solving complex financial problems. The proposed solution is to use numerical methods such as finite difference method (FDM) to approximate the solution of the Black-Scholes PDE in cases where closed form solutions cannot be obtained.

The pricing of swaptions are important in financial markets, hence we specifically discuss the pricing of interest rate swaptions, CDS options, commodity swaptions and energy swaptions using Black-Scholes model.

Simple parabolic PDE known as *heat equation* given at (Higham, 2004) forms a foundations to understand the application of FDM when solving a PDE. Since, Black-Scholes PDE is also a parabolic equation it is transformed to a form of a heat equation (diffusion equation) by applying change of variables technique.

FDM, specifically Crank-Nicolson method can be applied to the heat equation but in this dissertation it is applied directly to the Black-Scholes PDE to approximate its solution. Therefore, it is preferable to use Crank-Nicolson method because it is known to be second-order accurate, unconditionally stable, very flexible, suitable and can accommodate variations in financial problems, (Duffy, 2008). The stability of this method is investigated using a matrix approach because it accommodates the effect of boundary conditions.

To test the convergence of Crank-Nicolson method, it is compared with the Black-Scholes method used in (Tucker and Wei, 2005) to price CDS options. Conclusively the results obtained by Crank-Nicolson method to price CDS options are similar to those obtained using Black-Scholes method.

Keywords : Finite Difference Methods, Black-Scholes Method, Credit Default swap spreads, Credit Default swaps and swaptions, Interest rate swaps and swaptions, Currency swaps and swaptions.

LIST OF ACRONYMS

SDE	Stochastic Differential Equation.
PDE	Partial Differential Equation.
ODE	Ordinary Differential Equation.
FDM	Finite Difference Method.
CDS	Credit Default Swap.
IRS	Interest Rate Swap.
FTCS	Forward difference in Time Central difference in Space.
BTCS	Backward difference in Time Central difference in Space.
OTC	Over-the-counter

Contents

1	Introduction	1
1.1	Background	1
1.2	Problem statement	3
1.3	Research aims and objectives	4
1.4	Method of investigation	4
1.5	Dissertation overview	5
2	Swaps	7
2.1	Interest rate swaps	7
2.2	Credit default swaps	12
2.2.1	Credit default swap premium	15
2.2.2	Valuation of the fixed leg and floating leg for Credit Default Swap	16
2.3	Currency Swaps	21
2.4	Asset Swaps	24
2.5	Index Swaps	25
2.6	Applications of Swaps	27
2.7	Summary	29
3	Swaptions	31
3.1	Interest Rate Swaption	31
3.2	CDS option	35
3.2.1	CDS spreads	36
3.3	Cross currency swaptions	40
3.4	Commodity Swaptions	41
3.5	Energy Swaptions	42
3.6	Summary	42

4	Black-Scholes method to price swaptions	44
4.1	Valuation of Interest Rate Swaptions	44
4.2	Valuation of the CDS options	48
4.3	Valuation of Commodity Swaptions	51
4.3.1	Pricing Formulae	52
4.4	Valuation of Energy Swaptions	54
4.5	Summary	57
5	Finite Difference methods	58
5.1	Operators of Finite Difference	59
5.1.1	Heat equation	60
5.1.2	Discretization	60
5.1.3	FTCS and BTCS	61
5.1.4	Crank-Nicolson	65
5.2	Finite difference methods for the Black-Scholes PDE	67
5.2.1	FTCS, BTCS and Crank-Nicolson for the Black-Scholes PDE	67
5.3	Characteristics of the Finite Difference Methods	71
5.3.1	Stability	72
5.3.2	Consistency	74
5.4	Summary	74
6	Pricing CDS options using Crank-Nicholson method	76
6.1	Transforming Black-Scholes PDE to the heat equation	76
6.1.1	Transformation of the PDE	77
6.2	Solution for the parabolic PDE by the Crank-Nicolson method	79
6.2.1	LU factorisation	80
6.3	Numerical results	82
6.4	Summary	84
7	Conclusion and recommendations	86
7.1	Conclusion	86
7.1.1	Swaps	86
7.1.2	Swaptions	87
7.1.3	Black-Scholes method to price swaptions	88

7.1.4	Finite Difference method to price CDS options	88
7.2	Recommendations	89
7.2.1	Interest rate swaps	90
7.2.2	CDS and CDS options	90
7.2.3	Crank-Nicolson to price CDS option	90
7.3	Closure	91
A	Mathematical tools	92
A.1	Definitions and theorems	92
B	Black-Scholes model	99
B.1	Derivation of Black-Scholes PDE	99
B.2	Solution of the Black-Scholes PDE to price CDS option	101
B.3	Black-Scholes to price European options and swaptions	103
B.3.1	Pricing put and call options	103
B.3.2	Pricing put and call swaptions	105
C	Finite Difference Methods to price options	109
C.1	Explicit method	109
C.2	Implicit method	111
C.3	Crank-Nicolson method	114
D	Crank-Nicolson method to price CDS options	117

List of Figures

2.1	Initial positions of both companies.	8
2.2	Swap between two companies.	9
2.3	Interest rates increase at time T_n	11
2.4	Interest rates decline at time T_n	12
2.5	Credit Default Swap.	13
2.6	credit default swap without Default.	14
2.7	credit default swap with Default.	15
2.8	Payment contingent on credit event	16
2.9	Default before maturity (time $s \leq T_N$).	17
2.10	Payments time line.	19
2.11	Credit default swap on Daimler Chrysler.	20
2.12	A currency swap.	24
2.13	Asset swap.	26
2.14	Swap Indices.	27
2.15	Interest rate swap.	29
3.1	Decision about the swaption based on interest rate movement	35
3.2	CDS knock-out payer swaption	36
3.3	US and SA CDS spreads	37
3.4	US and SA annual probability of default	38
3.5	Lognormal distribution for the forward CDS spread	40
5.1	Finite difference grid $\{jh, ik\}_{j=0, i=0}^{N_x, N_t}$. Points are spaced at a distance h apart in the x -direction and k apart in the t -direction.	61
5.2	Stencil for FTCS. Solid circles indicate the location of values that must be known to obtain the value located at the open circle.	62

5.3	Stencil for BTCS. Solid circles indicate the location of values that must be known to obtain the value located at the open circle.	64
5.4	Stencil for Crank-Nicolson. Solid circles indicate the location of values that must be known to obtain the value located at the open circle. . .	66
6.1	Crank-Nicolson method when $T=0.5$ and $S = K$	83
6.2	Crank-Nicolson method when $T=0.5$ and $S>K$	84
A.1	A sample path of Brownian motion	92

List of Tables

2.1	Results of a rise in interest rates	10
2.2	Results of a decrease in interest rates	11
2.3	Borrowing index rates provided at the market	26
2.4	Loan rates provided at the market place	27
2.5	Difference between fixed and floating rates	28
4.1	Interest rate data for swaption valuation	47
5.1	Difference operators.	59
6.1	Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 0.5$	83
6.2	Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 0.25$	83
6.3	Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 1$	84
C.1	Comparative table of option prices using numerical Explicit method .	111
C.2	Comparative table of option prices using numerical Implicit method .	114
C.3	Comparative table of option prices using numerical Crank-Nicolson method	116

Chapter 1

Introduction

1.1 Background

An option is a contract that gives its holder the right but not the obligation to trade an asset for a fixed price (known as the strike price) at expiry. Furthermore, there are two types of options, namely a call and a put options: a call option grants its holder the right to buy the asset, while a put gives its holder the right to sell the asset. According to (Hull, 2000): the value of the option must be dependent on the value of the asset.

The first swap contract was negotiated in the early 1980s and thereafter the market has seen phenomenal growth based on different swaps. In addition, swaps are now occupying a position of central importance in over-the-counter derivative's market. A *swap* is a contract between two parties to exchange cash flows in the future and the interest rate should be specified so that it is applicable to each cash payment. Additionally, in this contract the time table for the payments is also defined and the two payment legs are calculated on a different basis, (Eales and Choudhry, 2003) and (Satyajit, 2004).

Besides there are different types of swaps namely; commodity swaps, cross-currency swaps, asset swaps, interest rate swaps, credit default swaps and others. However, there exist options on these swaps known as *swaptions* and this term is defined as an option that grants its owner the right but not obligation to enter into an underlying swap. In addition, swaptions are interest rate options that closely resemble many of the embedded options formed in fixed income securities and insurance liabilities. They are useful instruments for hedging long dated option risk. The existence of swaptions and its efficiency in the market make them increasingly

attractive to asset managers and insurance companies, in order to manage interest rate risk exposure and option risk.

For instance, if a bank expects floating rates to rise at time T , it can buy a payer swaption. Hence, if at time T interest rates rise this swaption can be exercised by paying a fixed rate below market levels and in return receive higher market floating rates. Therefore, the bank is not exposed to the risk of interest rate movements on its floating rate lending but on its fixed-rate lending.

There are two important parties involved in a swaption's contracts namely a payer and a receiver swaptions. A *payer swaption* gives the owner the right to enter into a swap whereby they pay the fixed leg and receive the floating leg; whereas a *receiver swaption* is the reverse of a payer swaption, (Eales and Choudhry, 2003), (Rutkowski and Armstrong, 2009).

The pricing of swaptions in financial markets are important and this is a topic of interest since in early 1973, when Chicago Board of Trade (CBOT) started trading options and other financial derivatives. But, they were faced with a problem of pricing these options because the buyer needed to be informed about the price he's willing to pay in order to buy options. Fortunately, this problem was resolved in a physicist's approach by Black, Scholes and Merton, (Ntwiga, 2005).

The Black-Scholes model was developed in 1973 and until now it has been used as a standard pricing formula for differential kinds of options. This model is based on a theory of geometric Brownian motion and this theory was proposed by Louis Bachelier to speculate prices. The geometric Brownian motion is a mathematical model for price movements and this principle is used by Black, Scholes and Merton when they developed Black-Scholes Merton model, (Ntwiga, 2005) and (Bjork, 2009).

The theory of Brownian motion is based on a random walk in continuous time, but in finance this theory has its flaws because it assumes that the asset's price follows a random variable. Thus, this leads to the disadvantage of the theory since it gives rise to negative asset's value which are unrealistic. However, there are analytical formulae available to obtain the prices of simpler options, for instance binomial method, trinomial method, and others. But sometimes the solution cannot be obtained by these methods, for example when pricing options which are represented by non-linear Partial Differential Equation (PDE) or other complex financial problems.

Even when analytic solutions are available, numerical methods can be used since they

form an important part of the pricing of financial derivatives in cases where there is no analytical solution. Thus it is desirable to have methods that are efficient to provide accurate solutions so that prices for swaptions and other financial derivatives can be obtained. For instance, finite difference methods (FDM) are efficient to approximate the Black-Scholes PDE because of the following reasons by (Haug, 2007) and (Duffy, 2006).

- The greeks can be accurately estimated using finite difference methods.
- They are flexible and can also be “adapted to approximate partial derivatives and difference equations used when underlying prices exhibit jumps”, Haug (2007).
- They are more accurate (for given computational cost) than Monte Carlo method and can be easily programmed on a digital computer.
- In finance the solution of inverse problems can be encountered and finite difference methods is applicable to this types of problems.

The solution of PDE approximated by numerical methods have a broad application in areas of science for many years, and numerical methods to solve PDE's became famous because of its high-speed computational power at low cost. Finite difference methods have been used by pure mathematicians to prove existence and uniqueness to solutions of the boundary value problems.

In the 1950's these methods proved to be of interest in their own right when engineers started to use them to solve engineering and scientific problems. The following is the applications of the finite difference methods: Magnetohydrodynamics, fluid mechanics, chemical reaction theory, numerical weather modelling and financial engineering. (Tavella and Randall, 2000)

1.2 Problem statement

Credit Default Swaptions (CDS Options) are credit derivatives which are widely priced by using Black-Scholes method, binomial method and other methods. However, some problems cannot be solved analytically or their closed-form solutions do not exist specifically when pricing complex financial problems. Hence, these methods have disadvantages and due to them it is not preferable to use these types of methods to obtain prices for swaptions.

In addition, particularly on the Black-Scholes method: the assumption based on this method do not always hold and are not practically realistic. For instance, it assumes

that CDS spread volatility is constant and delta-hedging is continuous, whereas it is known that these actions are unrealistic practise. Hence, this means that the exact solution obtained by this method to price swaptions is not that accurate, (Wilmott, 2006).

Therefore, we resort to FDM but focusing on the Crank-Nicolson method to approximate the solution of Black-Scholes PDE and obtain the prices of the CDS options. Hence, this method replaces continuous partial equations appearing on these PDE by discrete difference equation. Thus, the purpose of this dissertation is to obtain the price of the CDS option using Crank-Nicolson method.

1.3 Research aims and objectives

The following research objectives need to be reached in order to reach the aim of this dissertation:

- Introduce concepts of credit derivatives, definitions and mathematical tools vital in the valuation of these derivatives.
- Investigating various of reasons that drive investors or borrowers to enter into respective swaps and swaptions contracts.
- Apply FDM (i.e. Explicit, Implicit and Crank-Nicolson methods) and Black Scholes methods to obtain the prices of CDS options.
- Implementing Crank-Nicolson method in *Matlab*[®] so that the prices of CDS options can be obtained and then compare them to the those modelled by using Black-Schole's method.

1.4 Method of investigation

Mathematical methods:

- In order to determine the price of swaptions, Black-Scholes method with relevant dynamics following standard Brownian motion will be used. Hence, the main assumptions used by this method is that a CDS spreads are lognormally distributed.
- Finite difference methods (Implicit, Explicit and Crank-Nicholson methods) value a derivative by solving partial differential equation. Additionally, the

continuous differential equation is converted into a set of difference equations so that these equations can be solved iteratively. This approach is illustrated by considering how it is by Higham (2004) to price both a European put and call options.

- Lastly, after pricing European options using Finite Difference Methods, then Crank-Nicholson method will be used to approximate the price of the CDS option. As a result this method is preferable because it is second accurate in space dimension and it is unconditionally stable, (Kolb & Overdahl, 2010).

Computer methods:

Matlab[®] programme is used to implement Black-Scholes and FDM to obtain the prices of European options, CDS options and other types of swaptions.

1.5 Dissertation overview

An overview of the chapters to follow

Chapter 2: Swaps

Different types of swaps with relevant examples are presented in this chapter.

Chapter 3: Swaptions

Interest rate swaptions, Credit Default swaptions, commodity swaptions, energy swaptions and cross-currency swaptions including practical examples are discussed in this chapter.

Chapter 4: Black-Scholes method to price swaptions

The implementation of swaptions (i.e. CDS option, interest rate swaption, commodity swaption and energy swaption) using Black-Scholes method is presented in this chapter.

Chapter 5: Finite Difference Methods

The solution of the heat equation is approximated by FDM and this method is further used to solve the Black-Scholes PDE to obtain the solution of it which is “demon-

strated that any contingent claim must be satisfied when the assumptions they made about stock price dynamics hold”, (Kolb & Overdahl, 2010). And the fundamental concepts of the FDM is introduced in this chapter.

Chapter 6: Pricing Credit Default swaptions using the Crank-Nicholson method

The Black-Scholes PDE is transformed to a form of heat equation because both of these equations are parabolic equations and the solution of these equations can be approximated by Crank-Nicolson method. But this method will be specifically applied to the original Black-Scholes PDE.

Chapter 7: Conclusion

Based on the results of CDS option prices obtained by Crank-Nicolson and Black-Scholes methods, a conclusion is drawn from comparing these methods.

Chapter 2

Swaps

Swaps are widely used by investors for various reasons and in this chapter we introduce different types of swaps including examples based on these swaps. Hence, from the discussion of these swaps different reasons that attract investors to enter into an underlying swaps are investigated.

2.1 Interest rate swaps

An *Interest rate swap* is a private agreement between two parties to exchange one stream of interest payments (fixed-rate) for another stream of interest payments (floating or variable) on a specific notional amount of principal for a specific period of time, (Kim, 2011). For instance, IRS can be used by parties who speculates that interest rates will increase more sharply than expected by the market, and one party could agree to pay fixed (i.e. fixed-rate payer) to the other party (i.e. fixed-rate receiver) and in exchange to receive floating from a swap.

However, if the rates rises as expected then the fixed-rate payments will be exceeded by the floating-rate receipts, and the fixed-rate payer will benefit because interest rate movement went according to his expectations. Hence, interest rate swaps are used by companies that want to hedge their interest rate risks or improve their cost of funding. Furthermore, they can also be used by investing institutions that want to move from a floating-rate of return to a fixed-rate of return or vice-versa. This type of a swap is also known as a *plain vanilla swap*.

Consider the following example whereby two companies want to convert fixed-rate liabilities into floating-rate liabilities and vice-verse so that their interest rate expenses can be reduced, (Stuart, 2001).

Example: 2.1.1

Suppose Company X has R10,000,000 of floating-rate debt outstanding on which it pays $LIBOR + 150bps$ which it is paid quarterly. The company is worried that interest rates (floating-rates) will increase and if they do then the company's interest expenses will increase. The company then decides to convert its debt from floating-rate debt into fixed-rate debt.

Suppose Company Y has R10,000,000 of fixed-rate debt outstanding on which it pays 9% interest which it is paid annually. The company believes that interest rates (floating-rates) will decrease and if they do; the company prefers to have floating-rate debt instead of fixed-rate debt so that its interest expense can decline. Figure 2.1 shows the initial positions of the two companies.

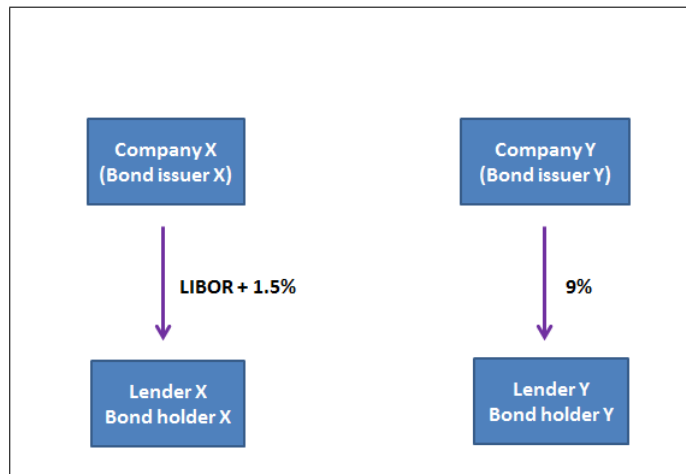


Figure 2.1: Initial positions of both companies.

The two companies can enter into an interest rate swap to convert their existing liabilities into the liabilities they want.

Situation 1:

- Company X might agree to pay Company Y fixed-rate interest payments of 8% computed on a $\frac{1}{12}$ calender and paid annually. Thus, the payment at the end of each year would be:

$$R10,000,000 \times 0.08 \times \frac{1}{12} = R66,666.67$$

- Company Y might agree to pay Company X floating-rate interest payments of *LIBOR* computed on an $\frac{3}{12}$ basis and paid quarterly. Thus, if *LIBOR* is currently at 7.5% and then the payments at the end of the first quarter would be:

$$R10,000,000 \times 0.075 \times \frac{3}{12} = \$187,500.00$$

Therefore, each quarter the amount will change depending on the number of days in that particular calendar quarter and *LIBOR* rate during that particular calendar quarter. The transaction that will be made between these two companies are illustrated in figure 2.2.

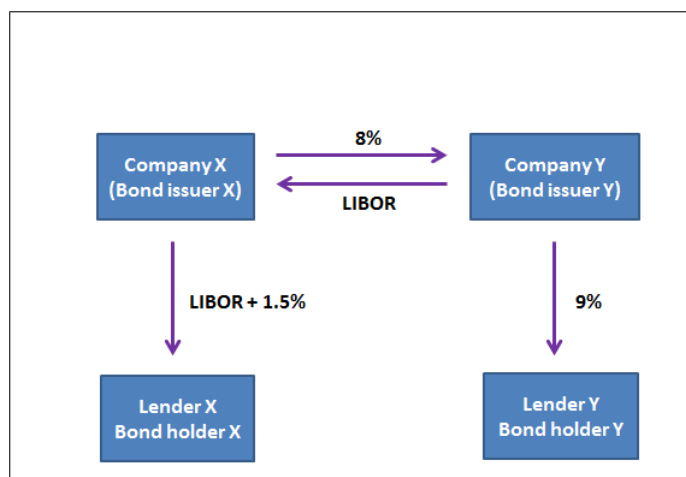


Figure 2.2: Swap between two companies.

Net costs in this swap:

- Company X pays *LIBOR* + 150bps to its original lender and 8% in the swap. The total amount it pays out is *LIBOR* + 950bps or *LIBOR* + 9.50%, and it receives *LIBOR* from the swap such that the two *LIBORs* cancel each other leaving cost of funds of 9.50%, a fixed rate.
- Company Y pays 9% to its original lender and *LIBOR* in the swap. The total amount it pays is *LIBOR* + 9% and it receives 8% in the swap for a cost of *LIBOR* + 1%, a floating rate. And since *LIBOR* is 7.5%, then the cost is (*LIBOR* + 1 = 7.5% + 1% = 8.5%) which is less than it paid originally to its lender (9%), thus this company has reduced its cost by 0.5%.

How will the value of a swap to each counterparty change when interest rates changes?

(Situation 2 market's fixed-rate increase):

1. In a swap:

Since, all swap quotes assume that the floating leg of the swap has no margin attached, thus in this example it is also assumed LIBOR flat (floating-rate). If the market's fixed rate increases from 8% to 8.5%, then Company X (fixed-rate payer) would benefit from this swap since the prevailing market rate is higher than the fixed-rate it pays in the swap. Hence, the company has reduced its interest rate expenses by entering into a swap because it paid less fixed-rate of 8% than the rate offered in the market which is 8.5%.

Company Y (fixed-rate receiver) will suffer a loss because the rise in rates will adversely affect a borrower with floating-rate debt. In other words this company converted its fixed-rate debt to the floating-rate debt anticipating that the interest rate will decline. As a result, this company anticipated the market's fixed-rate movement incorrectly and they will lose because the movement does not favour them. Hence, this company will receive less fixed-rate than the rates offered by the market, thus it will receive 8% instead of 8.5% in exchange for LIBOR.

Table 2.1 shows the results when market's fixed-rates increases.

Table 2.1: Results of a rise in interest rates

Party	Value of swap
Fixed-rate payer	Increases
Fixed-rate receiver	Decreases

2. Not in a swap:

If market fixed-rates increases, Company X would not reduce its interest rate expenses instead they will increase with rising rates because a rise in rates unfavourably affect a borrower with a floating liabilities.

Company Y will not be affected by the rise of interest rates since it has the fixed-rate debt and any party with fixed-rate liabilities can only be unfavourably affected when the rates decline.

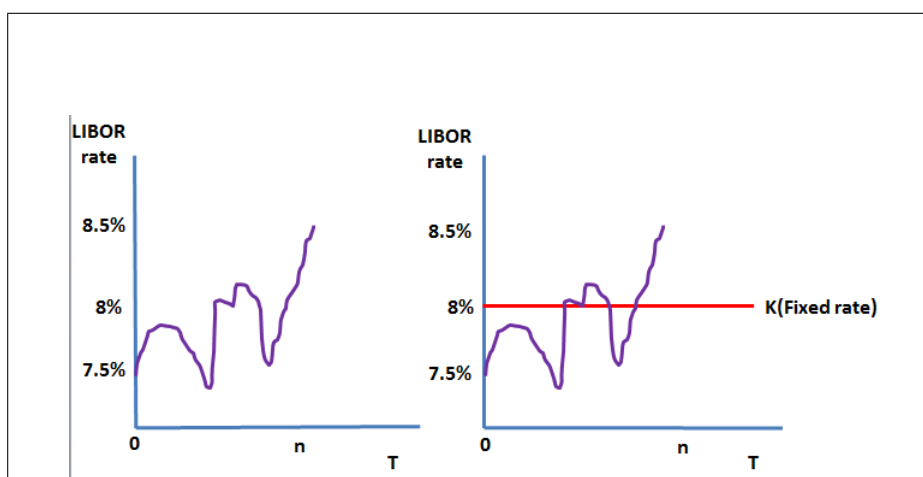
Figure 2.3: Interest rates increase at time T_n .

Figure 2.3 illustrate the interest rates movements in the future at time T_n .

Situation 3 (market's fixed rates decline):

1. In a swap:

If interest rates decline to 7%, then Company X will pay initial swap fixed-rate of 8% instead of 7% to obtain LIBOR. Thus, this company could have payed less rates if it went straight to the market since the prevailing market fixed-rate is less than the rate quoted on the swap agreement. Company X will pay 8% in a swap to obtain LIBOR and it will lose by 1% in this swap.

Company Y will benefit from this swap because it will now receive 8% instead of lower fixed-rates provided on the markets which is 7% in exchange for LIBOR. Therefore interest rate movement went according to their expectations and favours them and they will benefit from this swap by 1%.

The results of a decrease in interest rates are shown in table 2.2.

Table 2.2: Results of a decrease in interest rates

Party	Value of swap
Fixed-rate payer	Decreases
Fixed-rate receiver	Increases

2. Not in a swap:

Company X would not be negatively affected by the decrease of fixed-rates because they have a floating-rate debt and a borrower with floating-rate can only be negatively affected by rising rates.

Since company Y has a fixed-rate debt it will be unfavourably affected by the decline of rates, thus this company will suffer a loss and would not reduce its interest rate expenses.

Simple path of interest rates movement in the future at time T_n are shown in Figure 2.4.

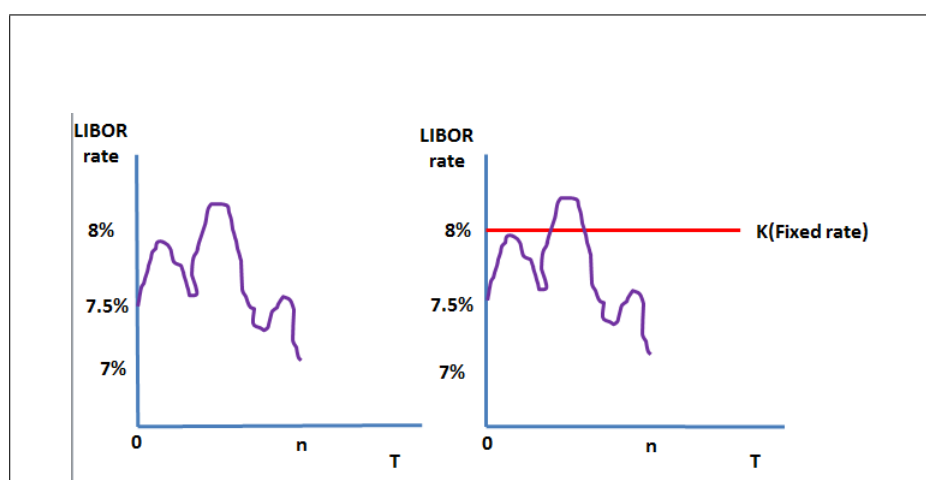


Figure 2.4: Interest rates decline at time T_n .

2.2 Credit default swaps

CDS is a contract in which the protection seller offers the buyer protection against a credit event (i.e. default) of its reference entity for a specific period of time in return for a fixed leg or a CDS spread. Hence, this swap only pays out if the credit event occurs and if it does not occur the buyer will continue making payments until maturity of the contract.

The buyer of protection has two choices to make payments either by paying an upfront amount or making periodic payments to the seller, usually this is a percentage of the notional amount. And this percentage is called the CDS spread, premium or fixed-rate, (Houweling and Vorst, 2005) and (Jankowitsch *et al.* 2008).

Thus, in exchange the buyer receives a payoff from the seller if the reference entity

goes into default or (i.e. bankruptcy or restructuring) for compensations against this credit event. There are two parties involved in this swap:

- The buyer of protection (holder of the bonds or loan), and this party is known as the fixed-rate payer.
- The seller of protection (the writer of CDS), and this party is called the fixed-rate receiver because this party will receive a fixed leg from the buyer in exchange for protection.

The *referenced asset* is defined as the asset that is protected against a credit event, and this asset can be a loan or a bond and the *referenced entity* can be a borrower or issuer of the bond. If during the life of the swap a credit event occurs, then the seller of protection has to take delivery of the referenced asset (i.e. bonds) and pay a set amount of money to the buyer of protection (normally the par value of the asset), (Chisholm, 2004).

Deals are mostly structured such that if a credit event occurs the buyer of protection sells the referenced asset to the seller of protection at a set price. This is often estimated through a series of dealer polls. A holder of a bond buy protection to hedge its risk of default. The default swap can be settled by one of the two following ways, if credit event occurs:

- **Cash Settlement:**

The protection buyer keeps the underlying assets, but is compensated by the protection seller for the loss incurred by the credit event.

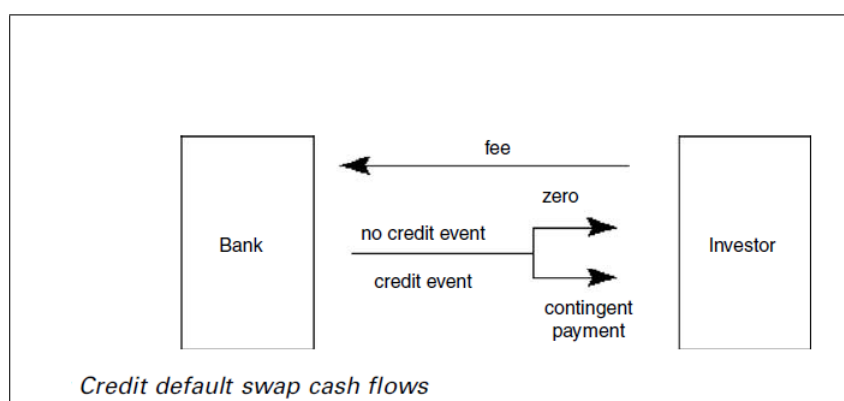


Figure 2.5: Credit Default Swap.

- **Physical Settlement:**

The protection buyer delivers the reference obligations to the seller, (i.e. delivers bonds) and in exchange receives the full notional amount of the delivered bonds.

If in the contract it is agreed on periodic payments and the reference entity defaults on its obligations before maturity, then the protection buyer pays the remaining payment known as accrual payment.

The *objective* for the credit default swaps might be any of the following:

To sell a specific risk, to pick up additional yield assuming the credit risk, to improve portfolio diversification and to gain exposure to credits without buying the assets.

Buyers of protection in CDS include commercial banks who wish to reduce their exposure to credit risk on their loan books; and investing institutions seeking to hedge against the risk of default on a bond or portfolio of bonds. Sellers of protection include banks and insurance companies who earn premium in return for insuring companies against default by their reference entity.

The basic deal (CDS) without default and with default event is illustrated in Figure 2.6 and 2.7, respectively.

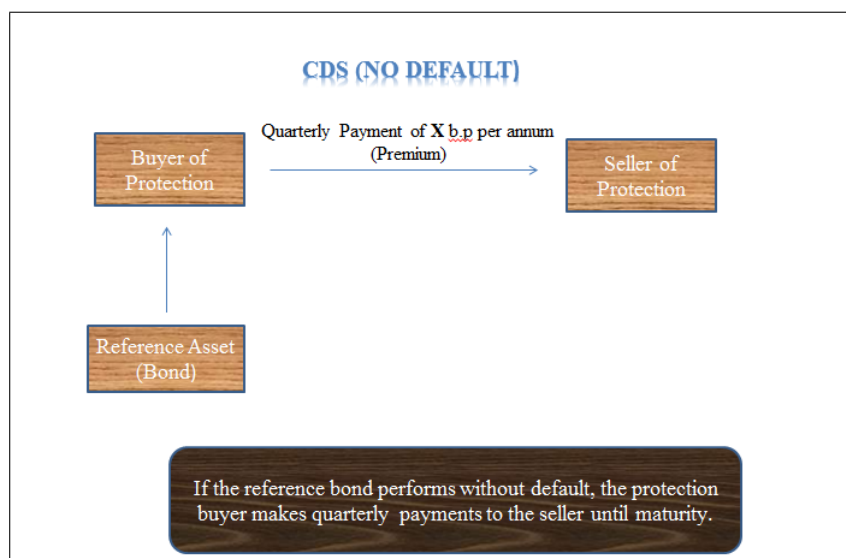


Figure 2.6: credit default swap without Default.

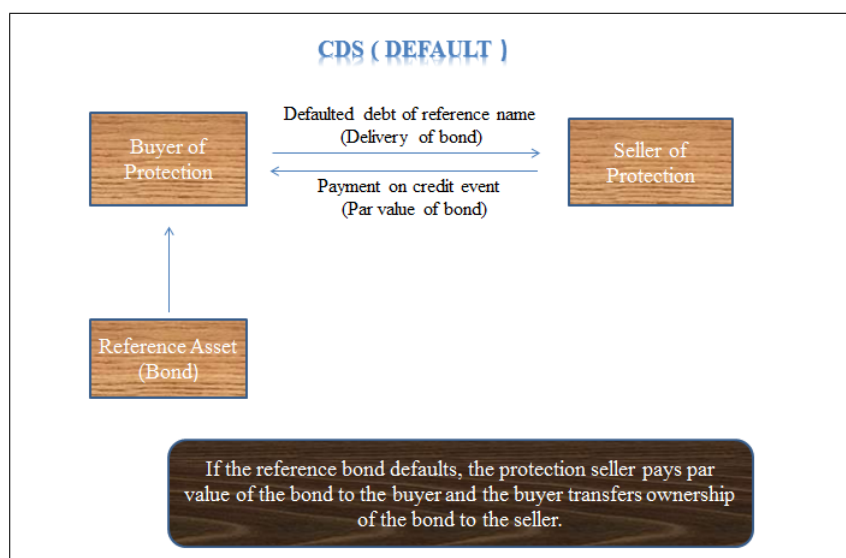


Figure 2.7: credit default swap with Default.

2.2.1 Credit default swap premium

According to (Chisholm, 2004), the premium paid periodically on a credit default swap is related to the credit spread but they are not exactly the same on the reference asset. The credit spread is known as the additional return that the investors can earn on a certain asset and this return is above the return of assets that are free of default risk.

For instance, suppose that a 5-year government bond pays a return of 4% p.a and the return on 5-year Treasuries is 3% p.a. Then it is clear that the credit spread of the bond is 1% p.a. Its size depends on the rating of the bond, because the spread can be used to measure the probability of default. The seller of protection in a CDS speculates that the reference entity of its protection buyer will not default. And this seller needs to be compensated for taking the risk from the protection seller.

Furthermore, an insurance company has invested in risk-free Treasury bonds. The returns are safe but not very exciting. Then it decides to enter into a CDS in which it receives a premium in return for providing default protection against a referenced asset, (Chisholm, 2004). The position of the insurance company is shown in Figure 2.8.

Therefore, the insurance company has moved from a risk-free investment to a situation that involves default risk by entering into a swap. The spread or fixed leg

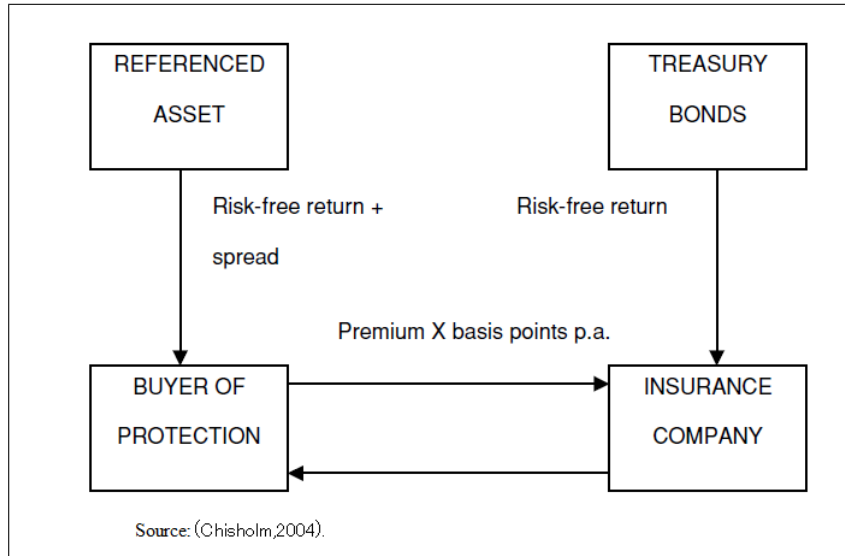


Figure 2.8: Payment contingent on credit event

received by the protection seller from the buyer in this swap should be related to the additional return over the risk-free rate (the credit spread) available on the referenced asset, (Chisholm ,2004).

In this manner parties that are involved in a CDS also acquire a credit exposure to each other which is also known as the counterparty credit risk. The expected recovery rate and probability of default on a certain reference entity is important when CDS premium is determined. The recovery rate is defined as the percentage of the asset par value that is possible to recover in the credit event.

2.2.2 Valuation of the fixed leg and floating leg for Credit Default Swap

In order to obtain the price dynamics of a CDS, the forward floating and the forward fixed legs are required. (Houweling and Vorst, 2005): Valuate the fixed leg and floating leg as follows:

Consider a default swap contract with payment dates $\mathbf{T} = (T_1, \dots, T_N)$, maturing at T_N , premium percentage P and notional 1. The fixed leg is denoted by $V_{fixed}(t, \mathbf{T}, P)$ and the value of the floating leg by $V_{float}(t)$, so that the value of the default swap to the protection buyer equals $V_{float}(t) - V_{fixed}(t, \mathbf{T}, P)$. The premium P is chosen at initiation in order for the value of the default swap to be equal to zero. As a result the premium percentage is to be chosen as $P = \frac{V_{float}(t)}{V_{fixed}(t, \mathbf{T}, 1)}$.

The fixed leg is first determined then at each payment date T_i , the protection buyer has to pay $\alpha(T_{i-1}, T_i)P$ to the protection seller; whereby $\alpha(T_{i-1}, T_i)$ is the year fraction between T_{i-1} and T_i (T_0 is equal to t). If the reference entity does not default during the life of the swap, then the protection buyer makes all payments until maturity.

Consequently, if default occurs before maturity (i.e. at time $s \leq T_N$), and the buyer has made only $I(s)$ payments, where $I(s) = \max(i = 0, \dots, N : T_i < s)$ and the remaining payments $I(s) + 1, \dots, N$ are no longer relevant. Thus, the protection buyer has to make accrual payment of $\alpha(T_{I(s)}, s)P$ at time s and let random time be denoted by τ . Figure 2.9 illustrate the default of the reference entity before maturity of the contract.

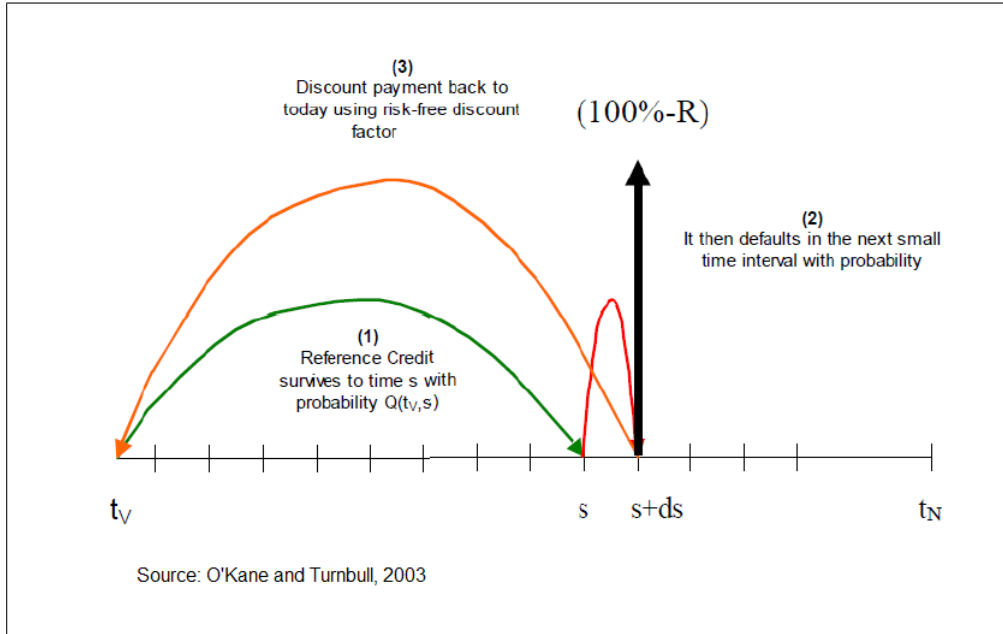


Figure 2.9: Default before maturity (time $s \leq T_N$).

The value of the fixed leg at time t is given by:

$$\begin{aligned}
 V_{fixed}(t, \mathbf{T}, P) &= \sum_{i=1}^N p(t, T_i) \tilde{\mathbb{E}}_t \left[\alpha(T_{i-1}, T_i) P \mathbf{1}_{\tau > T_i} \right] + \tilde{\mathbb{E}}_t \left[p(t, \tau) \alpha(T_{I(\tau)}, \tau) P \mathbf{1}_{\tau \leq T_N} \right] \\
 &= \sum_{i=1}^N p(t, T_i) \alpha(T_{i-1}, T_i) P \tilde{\mathbb{P}}(t, T_i) + \int_t^{T_N} p(t, s) \alpha(T_{I(s)}, s) P f(s) ds.
 \end{aligned} \tag{2.1}$$

The value of the floating leg is calculated as follows:

If the contract specifies cash settlement at default, then the protection buyer keeps the reference obligation and the protection seller pays the difference between the *reference price* and the *final price*. (Houweling and Vorst 2005), assumes that the recovery rate of the reference entity is constant for the sake of simplicity. “The reference price equals to 100% and the final price is the market value of the reference obligation at the default date; under the assumption made about the recovery rate the final price is equal to δ ”, (Houweling and Vorst 2005). Thus, the floating leg value under cash settlement equals :

$$V_{float}(t) = \tilde{\mathbb{E}}_t \left[p(t, \tau) (1 - \delta) P \mathbf{1}_{\tau \leq T_N} \right] = \int_t^{T_N} p(t, s) (1 - \delta) f(s) ds. \quad (2.2)$$

Hence, if the contract specifies *physical settlement*, the protection buyer delivers reference obligations with a total notional 1 to the protection seller and the seller pays 1 in return. The following examples illustrate the credit default swap, whereby a credit event occurs before maturity.

Example: 2.2.1 (Credit default swap on Daimler Chrysler).

At time $t = 0$, Company A and Company B enter into a credit default swap on Daimler Chrysler. Let say Daimler Chrysler issues unsecured USD bonds, and is the reference asset of Company A. Then Company A anticipates or fears that Daimler Chrysler may default to make coupon payments; and it decides to protect itself against the credit default of Daimler Chrysler by entering into a credit default swap with Company B, (Schonbucher, 2003). Thus, Company A is the protection buyer (holder) and Company B is the protection seller (CDS writer). Both companies have agreed on the following:

- The reference credit: Daimler Chrysler AG.
- The term of the credit default swap: 5 years.
- The notional of the credit default swap: \$20m.
- The credit default swap fee: $\bar{s} = 100$ bp.

And the solution of this problem will be discussed using three different situations.

Situation: 1 *The fee payments:*

The credit default swap fee $\bar{s} = 100 \text{ bp}$ is quoted per annum as a fraction of the notional. Company A pays the fee to Company B semi-annually, and for simplicity, let the day count fractions be $\frac{1}{2}$ such that Company A pays Company B the following:

$$100 \text{ bp} \times \frac{20 \times 10^6}{2} = \$100,000 \text{ at } T_1 = 1/2, T_2 = 1, \dots, T_{10} = 5.$$

These payments are stopped and the CDS is unwound as soon as a default of Daimler Chrysler occurs.

The default payments:

First, Company A pays the remaining accrued fee. If the default occurred two months after the last fee payments, (see figure 2.10).

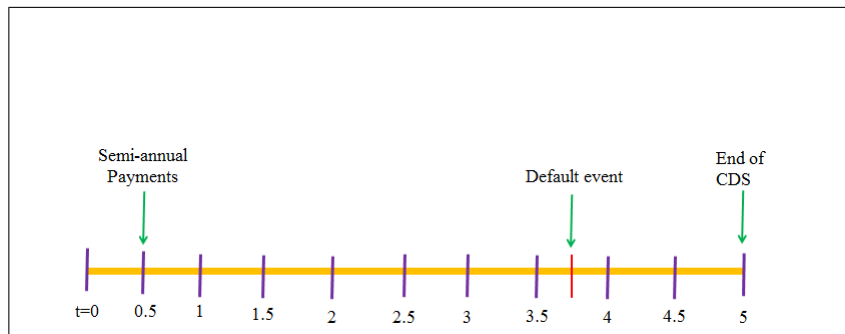


Figure 2.10: Payments time line.

Then, Company A will pay: $\$100,000 \times \frac{2}{6} = \$33,333.33$ to Company B (the protection seller). The default payment that has to be paid by Company B to Company A must be determined since Daimler Chrysler has defaulted on its obligation. Thus, the default payment can be settled by the following two ways:

1. Physical settlement:

Company A will deliver Daimler Chrysler bonds to Company B with a total notional of USD 20m (the notional of the CDS). Naturally, Company A will choose to deliver the bond with the lowest market value, unless it has its underlying position that it needs to unwind. Even then Company A may prefer to sell its position in the market and buy cheaper bonds to deliver them to Company B. Therefore, Company B has to pay the full notional, (i.e. \$20m) to Company A for the delivery of the bonds.

2. Cash settlement:

To determine the market value of the bonds after default, a robust procedure is needed. If there were no liquidity problems, a dealer would be asked to give a price of these bonds or dealers are asked to provide quotes. Thus average from the provided prices is taken but after eliminating the highest and the lowest quotes. This is repeated to eliminate the influence of temporary liquidity holes.

Thus, the price of the defaulted bonds is determined, (i.e. say \$500 for a bond of \$1000 notional). Hence, the protection seller pays the difference between this price and par value for a notional of \$20m to the protection buyer:

$$\frac{20 \times 10^6 \times 1000 - 20 \times 10^6 \times 500}{1000} = \$10\text{million}$$

CDS transaction on Daimler Chrysler is shown in figure 2.11.

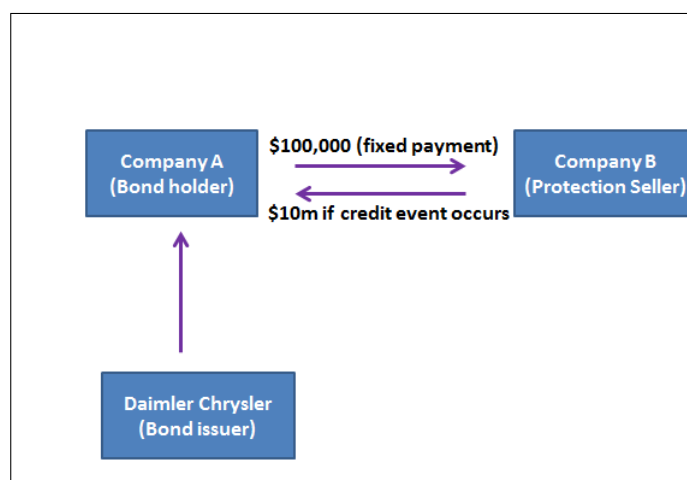


Figure 2.11: Credit default swap on Daimler Chrysler.

Situation: 2 (No default occurrence)

In a swap

If default does not occur then Company A (protection buyer) will continue paying Company B (protection seller) the agreed fixed payments until maturity of the CDS contract. Thus, Company A will suffer a loss because of the bad anticipation made about the default of the bond issuer and this shows that the event is unfavourable to

the protection buyer.

Company B will benefit and receive a profit from the contract since the reference entity did not default and thus no payment for protection will be made to Company A.

Not in a swap

Company A will not suffer any loss but benefit, because Daimler Chrysler was able to meet its obligations by paying its coupons.

Situation: 3(Bond issuer default)

In a swap

If Daimler Chrysler (bond issuer) defaults, then Company B (protection seller) have to compensate for the losses suffered by Company A because the bond issuer has defaulted on its coupon payments. And this payment can either be made by cash settlement or physical settlement. Thus, Company A will benefit from a CDS contract.

Company B will not benefit anything but it will at least receive the accrued payment from Company A because a default has occurred before maturity of the contract. And in return it has to pay Company A the corresponding full amount of the bonds, depending on the value of them at that moment.

Not in a swap

Company A will suffer a huge loss since Daimler Chrysler failed to meet its payments obligation and it does not have any protection against this credit event.

2.3 Currency Swaps

A *currency swap* is an agreement in which one party gives a certain principal in one currency to its counterparty in exchange for the same principal amount but in a different currency. In a currency swap, the complete principal amounts along with the interest payments are exchanged.

At the maturity of the swap the exchange of the principal amount is essential but at the beginning it is optional. The interest rates are also involved in the currency swap and they are expressed on either a fixed or a floating-rate basis in either or both currencies. Different types of currency swaps are formed from combining currency swaps and interest rate swaps namely: (Siddaiah, 2010) and (Rogers, 2004).

- Fixed-to-fixed currency swaps - Company X borrows a certain amount in US dollars at a fixed rate of interest, and Company Y raises a loan in another currency (say in Rands), at a fixed rate of interest. Thus, these companies may agree to exchange the loan amounts and they make periodic interest payments in the same currency. Therefore, at maturity the principal amounts are re-exchanged.
- Floating-to-floating currency swaps - These swaps are known as basis swaps and the transactions involve moving from the floating-rate index of one currency (US dollars) to the floating-rate index of another (Rands). Thus, parties pay a floating rate but with different LIBOR and T-bill rate.
- Fixed-to-floating currency swaps - The exchange is between interest rate payments at a fixed rate in one currency and interest rate payments at a floating rate in another currency.

For instance, a British company may want to swap British pounds for US dollars, and a US company may also want to exchange US dollars for British pounds. Thus, these two companies may enter into a currency swap to exchange their currencies. A standard currency swap requires three transactions, (Rogers, 2004).

1. At the initiation of the swap the two parties will exchange the currencies in which the notional principals are denominated.
2. The parties make periodic interest payments to each other during the life of the swap agreement.
3. At the termination of the swap, the two parties will again exchange the currencies in which the notional principals are denominated.

Currency swap can be related to the interest rate swap but they are different from each other and the following are the differences between them:

- The cash flows exchanged are in two different currencies.
- There are two notional principal amounts which are also exchanged but in currency swaps notional principals are not exchanged.

(Siddaiah, 2010), says that currency swaps are used to lower the cost of funds and can also be used when parties want to:

- Take an advantage on interest rates.
- Issue debt securities in foreign currency at favourable rates.
- Invest in foreign assets without foreign currency exposure.
- Match the assets with liabilities in the same currency.

The following example illustrate a fixed-to-fixed currency swap and it's from (Kim, 2011).

Example: 2.3.1

The current spot rate for British pounds is £0.5 per dollar (\$ 2 per pound) and the British interest rate is 8% and for US is 10%.

At time $t = 0$, British Telecommunications (BT) wants to exchange £ 5 million for dollars, in return for these pounds, Global Markets (GM) would pay \$10 million to BT at the beginning of the swap. The swap has the term of 5 years and these two firms will make their interest payments annually.

Thus, GM will pay 8% on the £5 million it received from BT and the annual payment from GM to BT will be $0.08 \times £5\text{million} = £400,000$. Similarly, BT will pay 10% on the \$10 million it received from GM so BT will pay GM each year the amount of $0.10 \times \$10\text{million} = \1million .

In practise, the two parties will only make net payments. For instance, if the spot rate for pounds changes to £0.45 per dollar at year 1, then £1 will be worth \$2.2. Hence, valuing the interest rate obligation in dollars at the exchange rate of \$2.2, then BT owes \$1 million and GM owes $£400,000 \times \$2.2 = \$888,000$. Therefore, BT will pay the \$112,000 difference.

At other times, say, time $t = 2$ the exchange rate could be different thereby making the net payment reflect the different exchange rate.

And at the maturity of the swap, at time $t = 5$, the two counterparties will again exchange principal. Thus, BT would pay \$10 million and GM would pay £5 million and this final payment terminates the currency swap.

A currency swap between BT and GM firms is illustrated in figure 2.12.

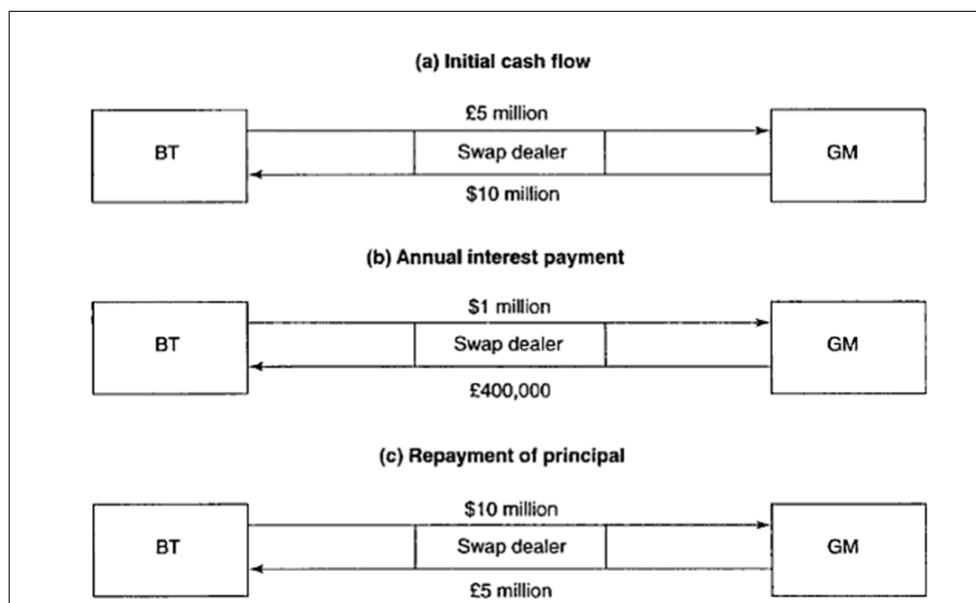


Figure 2.12: A currency swap.

2.4 Asset Swaps

When investors use swaps to increase their returns, they are called asset swaps, and “its package is a combination of a defaultable bond with a swap rate contract that swaps the coupon of the bond into a payoff stream of LIBOR plus a spread” (Schonbucher, 2003). And the bond is a fixed-coupon bond and the type of a swap used is a fixed-for-floating interest rate swap.

Table 2.4 in **example 2.6.1** will be used in this section whereby it provides the rate at which borrowers can borrow. But in this section this rates will be used by investors as a tool to generate profit by lending cash to a certain company. Suppose there are two types of investors:

- *Investor 1* which is adverse to credit risk and believes that the interest rates over the next years will decline.
- *Investor 2* which is willing to accept credit risk and believes that interest rates will rise.

(Stuart,2001) gives or discusses the following example as follows:

Case: 1

Investor 1 could buy the *AA*-rated 4-year 7% fixed-rate note and investor 2 could buy the *BBB+* - rated 4-year floating-rate note yielding $LIBOR + 60bps$. Thus, both investors will be investing on their respective weak side of the market. Investor 1 wants an investment with high credit quality and high credit quality instruments offer a lower yield than a lower credit quality instruments.

The credit quality cost 100 *bps* on the fixed-rate side of the market and on the floating-rate side of the market it cost 50 *bps*. Investor 1 should buy the credit quality on the side of the market where it is cheap, which is on the floating-rate side.

Case: 2

Investor 2 wants a higher return and is willing to take additional credit risk in order to get it and he should be paid for the additional risk. On the floating-rate side of the market the yield premium for accepting credit risk is 50 *bps*, but the premium for accepting the same amount of additional credit risk is 100 *bps* on the fixed-rate side of the market. Thus, investor 2 should invest where the premium for accepting credit risk is high which is on the fixed-rate side of the market.

The investors can invest on their strongest side of the market and then use a swap to convert their investments to the type they want. After investing on the respective strong side of the market and performing or executing the swap, the remaining returns of the investors will be:

- Investor 1 has $LIBOR + 1\% + 7.15\%$ coming in and an outflow of $LIBOR$ for a net income of 7.25%.
- Investor 2 has an inflow of $LIBOR + 8\%$ and outflow of 7.15% for net income of $LIBOR + 0.85\%$.

The *Asset Swap* is further explained by figure 2.14.

2.5 Index Swaps

“An index swap is a combination of a bond and a credit option” (Caouette *et al*, 2005). Moreover in this swap a floating-rate assets/liabilities is exchanged for a fixed-rate assets/liabilities and one floating-rate can be swapped for another. Consider the following example from (Stuart, 2001) and assume that the given rates are the best rates available in the market.

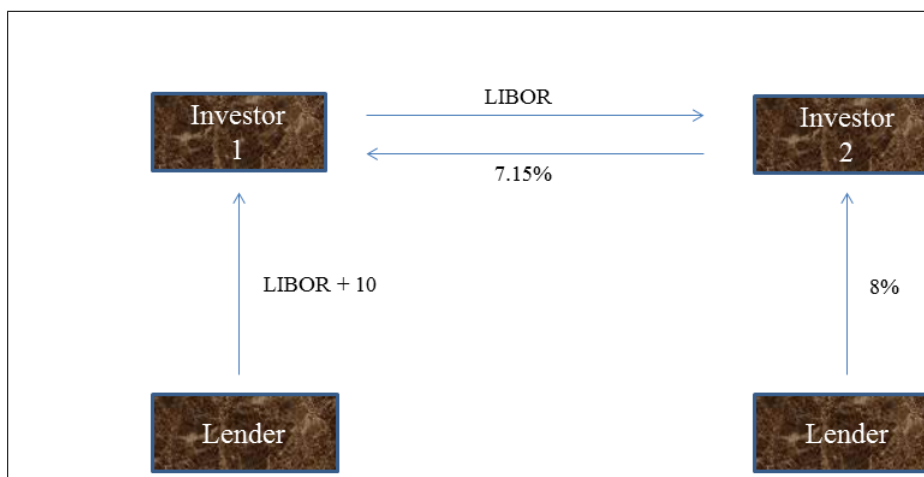


Figure 2.13: Asset swap.

Example: 2.5.1

AA-rated company wants *LIBOR* financing because it has *LIBOR*-based liabilities and wants to match its assets against its liabilities. Similarly *BBB+* - rated company wants to tie its financing to the commercial paper (*CP*) rate because they expect that the *CP* rates will decline relative to other rates. However, if both companies borrow money tied to their desired index rates then they would be borrowing from the weak side of the market. (See table 2.3 which shows provided *floating-rates*).

Table 2.3: Borrowing index rates provided at the market

AA	BBB+	
90 - day US Dollar LIBOR	L	L + 80
90 - day Commercial paper	CP	CP + 100

The *AA*-rated company has a strong credit rating hence it saves 80 *bps* if it borrows at a rate tied to the *LIBOR* rate but if it borrows at a rate tied to the commercial paper rate it would save 100 *bps*. If *BBB+* - rated company borrows on the *LIBOR* side of the market, its penalty would be 80 *bps*. The solution for both companies is to borrow on the respective strong side of the market and then use a swap to obtain a financing tied to the indices they desire. (See Figure 2.14 which shows the *Swap Tied to Indices*).

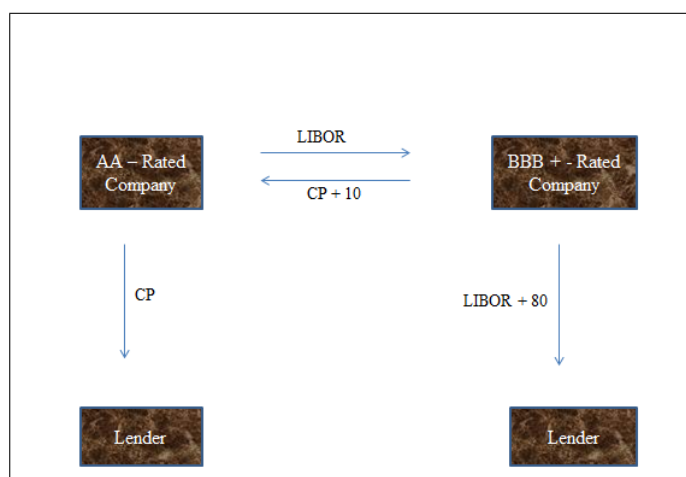


Figure 2.14: Swap Indices.

2.6 Applications of Swaps

Swaps can be used to increase an investor's interest income or decrease a borrower's interest expense, and applications swaps will be illustrated by the following example, and it is given by (Stuart, 2001).

Example: 2.6.1

Table 2.4 shows the existing loan in the marketplace at a particular day and it will be used in this example, (Stuart, 2001).

Table 2.4: Loan rates provided at the market place

Interest rates sides	AA	BBB+
3 - year fixed	7%	8%
3 - year floating	LIBOR + 0.1%	LIBOR + 0.6%

Case: 1

The 'AA rated' company wants to borrow R10,000,000 at a floating rate for 5 years because they believe that the interest rates will decline. The company can borrow at the floating rate ($LIBOR + 10bps$) available for companies with AA credit rating. But that would be a mistake because the AA company has a credit rating and a high credit rating entitles the company to borrow at a lower rate.

The *AA*-rated company can save 50 *bps* in interest expense on the floating-rated side of the market relative to the *BBB+* company. Hence, on the fixed-rated side of the market the *AA*-rated company saves 100 *bps*. The difference between the fixed-rates and floating-rates of the *AA* and *BBB+* credit ratings are given in table 2.4.

Table 2.5: Difference between fixed and floating rates

Interest rates sides	AA	BBB+	Difference
3 - year fixed	7%	8%	100 bp(1%)
3 - year floating	LIBOR + 0.1%	LIBOR + 0.6%	50 bp(0.5%)

Therefore, *AA*-rated company is strong on the fixed-rate side of the market and has an advantage on the fixed-rate side of the market. The *AA*-rated company will save more on the fixed-rate side of the market, thus companies should always borrow on the side of the market where they are the strongest.

Case: 2

The *BBB+* company wants to borrow $R10,000,000$ at a fixed-rate for 5 years because they want to protect the company from the risk of higher rates. They know that they will pay higher rates when they borrow money from the fixed-rate side of the market.

If the *BBB+* company borrows at a fixed-rate it will pay a premium of 100 *bps* relative to the *AA*-rated company, but it will pay 50 basis points (*bps*) more than the *AA*-rated company if it borrows at a floating-rate. Thus, the lower-rated company is stronger and has an advantage on the floating-rate side of the market. Both companies have valid reasons to borrow money for their companies on the weak side of the market.

The companies could borrow on their strongest side of the market, instead on the weak side and then use a swap to convert their respective financing to the type of financing they want. Therefore, by borrowing on the strong side the combined savings in the interest expense can be used to create a “slush fund” that equals the difference of the differences in the yields which is 50 *bps*. And this fund can be used to reduce the total borrowing cost of both parties. Assume that the parties agree to benefit 25 *bps* from this swap.

The *AA*-rated company could lock in a cost of $LIBOR + 10bps$ if borrows from at a floating rate. In order to benefit by $25 bps$, the *AA*-rated company's cost would be $LIBOR - 15bps$; and the *BBB+* - rated company could lock in a cost of 8% if borrows at a fixed rate. Thus, the *BBB+* - rated company's cost of borrowing must be 7.75% in order to benefit by $25 bps$.

In order for these companies to benefit equally, the convention must start with the floating-rate side and set it equal $LIBOR$ flat. Thus, if the fixed rate is set to be 7.15% this will allow both companies to benefit by $25 bps$.

- The *AA*-rated company pays $LIBOR$ and receives a net $15 bps$ from the difference between the two fixed rates for a net cost of $LIBOR - 15bps$.
- The *BBB+* - rated company offset the two $LIBORs$ and is left paying $7.15\% + 0.06\% = 7.75\%$.

Figure 2.13 shows the *interest rate swap*.

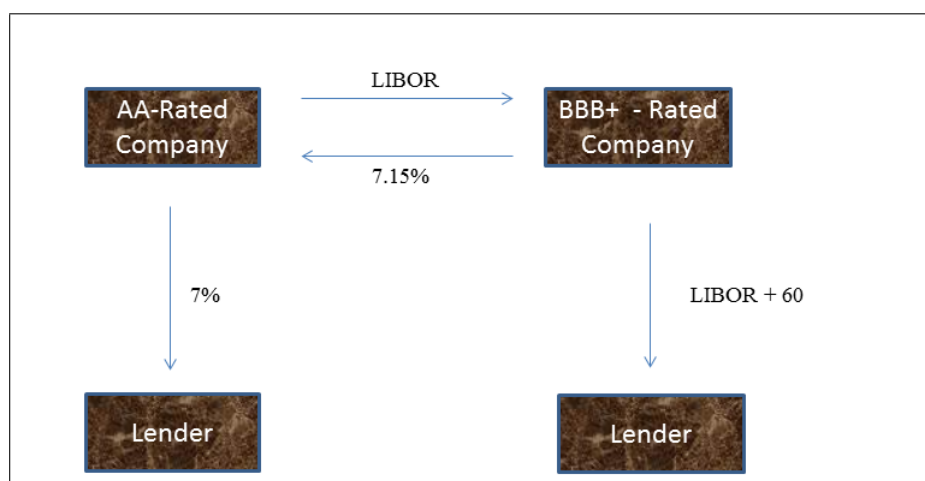


Figure 2.15: Interest rate swap.

2.7 Summary

Swaps are used by parties or investors to convert fixed-rate assets/liabilities into floating-rate assets/liabilities and vice-versa. If two parties prefer to convert their existing assets/liabilities from the floating-rate into the fixed-rate, they should first invest or borrow from their strongest side of the market, (i.e. floating-rate or fixed-rate

side of the market). Then, they can use a swap to convert their respective financing to the type of financing they prefer or speculate.

Therefore, from the discussed examples in each section it is observed that; (i) *In the interest rate swaps*: The fixed-rate payer who speculates that market's fixed-rates will increase, will benefit from a swap, if interest rates increases whereas fixed-rate receiver loses. The fixed-rate payer benefits because market rate's movement went according to their speculations. But if interest rates decline, then fixed-rate payer will suffer a loss whereas the fixed-rate receiver benefits. (ii) *In the credit default swaps*: The protection buyer (fixed-rate payer) will only benefit from a CDS if its reference entity defaults because he will be compensated for that particular loss.

Additionally, if the reference entity does not default, then the protection buyer has to pay the seller until the maturity of the contract whereby the seller will benefit and the buyer will lose from a CDS. To determine whether a party benefit or not in a swap, it depends on the party's speculation regarding the market's interest rates movements in the future. As a result if interest rates move according to the party's expectations then it will benefit from the CDS. But if the movement of the interest rates opposes the party's expectations in the future, it will realise a loss since the movement does not favour them.

Chapter 3

Swaptions

A swaption is an option on an underlying swap whereby it gives its holder the right not an obligation to enter at a future date T into a swap with price G fixed at the outset of the swaption, (Hull ,2000). At time T the value of the swap does not have to be zero since the guaranteed price of swap G was fixed at the time t when the swaption was purchased by the buyer and not at the beginning of the swap period at time T . The buyer of the swaption should only exercise the right granted if the swap market value at time T is positive.

According to (Chance, 2003), swaptions have been around since 1988 and there exist different types of swaptions depending on the respective underlying swap. Therefore, Interest rate swaptions, credit default swaptions, commodity swaptions, cross-currency swaptions and energy swaptions are going to be discussed in this chapter.

3.1 Interest Rate Swaption

An *interest rate swaption* is an option granting its owner the right but not the obligation to enter into an interest rate swap agreement at a future date. And there are two types of swaptions namely a payer and a receiver swaptions:

- A payer swaption gives the holder the right to enter into an interest rate swap as a fixed-rate payer and a floating-rate receiver.
- A receiver swaption gives the holder the right to enter into an interest rate swap as a floating-rate payer and a fixed-rate receiver.

Hence, a *call swaption* grants the holder of the swaption the right to receive fixed-rate (the strike rate), and a receiver's swaption is similar to a call on a bond that pays a

fixed-rate of interest. A receiver swaption is attractive when interest rates are anticipated to decline, because when buying a receiver's swaption, protection is obtained from receiving a lower fixed-rate over the life of the swap.

The owner of the receiver swaption will exercise if the market fixed-rate is less than the strike rate which is the fixed interest payments stipulated in the swaption contract. Thus, the holder of the swaption will enter into a swap to receive fixed interest payments at the strike rate of which is greater than the market fixed-rate in exchange for paying a sequence of floating-rate at the new lower rate of interest.

(Sundaram, 2011) says that swaptions can be used:

- When hedging is necessary.
- To remove an existing swaps when it becomes unattractive.
- To enhance the yield on an underlying position by selling a swaption.
- To obtain access to a swap when borrowers are uncertain of the funding that will be required or when they are willing to anticipate the benefit of an interest rate movement prior to drawing the funds.

The following example (receiver swaption) is from (Kolb, 2003):

Consider a European receiver swaption on a 5-year swap with semi-annual payments and a notional principal of \$10 million. Assume that the fixed-rate of 8% is paid in exchange for LIBOR. This swaption can be exercised by entering into a swap to receive-fixed and pay-floating. The owner should only exercise if the fixed-rate on the swap underlying the swaption exceeds the market fixed-rate.

Assume that at expiration the market fixed-rate increase to 8.5% whereby this circumstance will not favour the owner. Hence, the receiver swaption is worthless and its holder will leave it to expire because this swaption allows the holder to enter a swap to receive 8%. But the market rate of this swap is to receive a fixed-rate of 8.5%.

If the prevailing fixed-rate of this swap decline to 7.5% at the maturity of the swaption, then the holder of the swaption can exercise. Because the swaption allows him to enter into a swap to receive 8% fixed-rate whereas the current market offers only 7.5% fixed-rate. Therefore, this shows that the holder of the receiver swaption will benefit and exercise only if the market rate decline in the future or at maturity of the

swaption contract.

Similarly, a *put swaption* gives the holder the right to pay fixed at strike rate and a payer's swaption is similar to a put on a bond. When the payer swaption is exercised then the owner of this swaption will be in the position of an issuer of a fixed-rate bond. The owner of the payer swaption will be in a position of an issuer because this swaption is exercised by paying a sequence of fixed-rate interest payments in exchange for inflows at a floating-rate.

The owner of a payer swaption will only exercise when interest rates are expected to increase. And protection will be obtained from paying higher fixed rate during the life of the swap a payer's swaption is purchased. This swaption will be exercised if the fixed rate (current market fixed rate) is greater than the strike rate stipulated in the contract. Therefore, the payer swaption owner will enter into a swap by making a sequence of fixed rate interest payments at old lower rate agreed in the swaption contract in exchange of a sequence of floating rate payments.

The following example (payer swaption) is also from (Kolb, 2003):

Assume that the swap underlying this payer swaption has a 5-year tenor with annual payments, a notional principal of \$10 million and assume that the fixed-rate specified in the swap agreement is 8% and the floating-rate is LIBOR. The premium for this payer swaption might be 50 basis points (0.5%) of the principal of \$10 million, thus the premium would be \$50,000.

At expiration date of the swaption, the owner of the payer swaption can either exercise or let the swaption to expire worthless. The owner of the payer swaption will only exercise if the fixed-rate increases to be more than 8%. Hence, if the owner decides to exercise then he will pay a fixed-rate of 8% and receive a floating-rate of LIBOR for the five-year tenor of the swap.

Furthermore, assume that at the expiration, the market fixed-rate swap increases to 8.5% in exchange for LIBOR. Then the holder of the payer swaption will exercise by entering into the swap agreement at terms more favourable than those which are current in the market. And if the market fixed-rate decreases to be 7.5%, then the owner of the payer swaption will not exercise the swaption because it is worthless.

The swaption is considered unattractive because it grants the owner the right to enter a swap to pay fixed-rate of 8% and receive LIBOR, whereas the current market rate allows the owner to pay 7.5% and receive LIBOR. Therefore, the payer swaption will only exercise and benefit if the market fixed-rate increases in the future or at maturity of the swaption contract.

The following example illustrate one of the uses of the interest rate swaption and is from (Sundaram, 2011).

Example: 3.1.1

A borrower is planning a project whose funding is uncertain and there is corporation that is willing to tender this project. If the borrower goes ahead with the project, then the project will be funded at the prevailing interest rate but there is a risk that the interest rates may increase. A typical swap is not appropriate because there would be speculative gains and losses from unwinding the swap if the project does not proceed and when interest rate changes significantly.

The borrower can enter into a swaption instead of a swap. In a swaption the buyer pays an upfront premium and is assured a swap and locks in a certain funding cost if the project proceed and the interest rate increases. Therefore, in this case the cost of borrowing will be equal to (Swap rate + Option premium). Alternatives available to the swaption buyer depends on both the project's success and interest rate movement and they are shown below:

- If the interest rate decreases and the project is successful, the corporation will let the option expire and borrow funds in the market at the market rate. In this case, the cost of funding will be equal to (Market rate added to the Option premium).
- If interest rate decreases and the project is not successful, the corporation will let the option expire and will lose the premium paid on the option.
- If the interest rate increases and the project is successful, the corporation will exercise the option and the cost of funding will be equal to (Swap rate added to the Option premium).
- If the interest rate increases and the project is not successful, the corporation will still exercise the option, because the market rate will be higher than the swap rate and the company will gain the difference between the two rates.

Figure 3.1 shows the decision of the corporation about the swaption based on project success and interest rate movement.

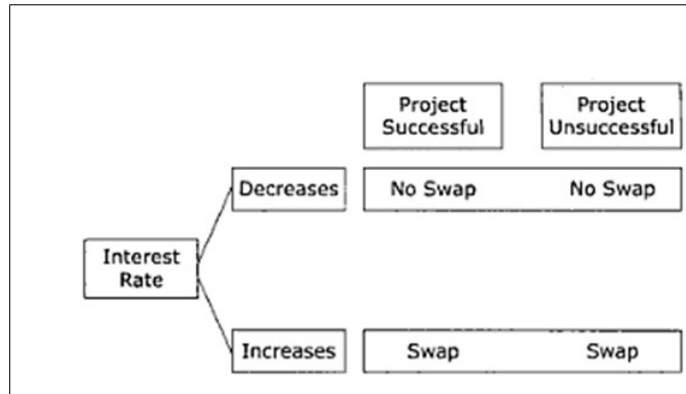


Figure 3.1: Decision about the swaption based on interest rate movement

3.2 CDS option

A CDS option grants its owner the right but the obligation to enter into an underlying CDS by buying (or selling) the protection and the owner pays an upfront fee. And it is assumed that this swaption is European-style, meaning the option can be exercised only on expiry date, (Rutkowski and Armstrong, 2009). Thus, this can be interpreted as a put (or call) option having strike zero written on the market value of the underlying CDS at the maturity of the option.

A *payer swaption* is defined as the option that gives the holder the right but not the obligation to buy CDS protection at the agreed strike rate (K) at expiry date. Thus, the option holder will be in a position of going short on the credit and if an investor buys this option, he will exercise the option at time t_E if the CDS spread $X(t_E, T) > K$. This, implies that this option is valuable to the holder and it will generate him money because the CDS spread widened beyond (K).

Payers are equivalent to *put options* since investors have the right to sell credit risk on bonds at a higher price than market price at maturity (T). And a payer swaption can also be equivalent to a call option if its viewed on spreads.

The *receiver swaption* grants the option holder the right to sell CDS protection at maturity date at the agreed strike rate (K), and the holder will have a right to go

long the credit risk. If at time t_E the CDS spread $X(t_E, T) < K$ then the holder can exercise this option. Therefore, the option is valuable to the investor because the spread is less than the strike rate, and it is equivalent to having the right to buy credit on risky bonds at a lower price than the prevailing market price at maturity. And the receiver is equivalent to bond call option since the holder has the right to buy credit risk, (Banks and Siegel, 2007) and (Barrios *et al*, 2003).

If it happens that the reference entity (i.e. bond issuer) of the underlying CDS defaults before the maturity of the swaption then the contract is knocked out, this implies that the swaption is nullified and it will terminate with a zero value. According to (O’kane, 2008) there exist two difference on the mechanics of a CDS option and they are given as follows:

- A *knock-out swaption* which is known as the swaption that cancels out without payments, if credit event occurs before the maturity date of the option.
- A *non knock-out swaption* does not cancel if credit event occurs before the maturity date of the option.

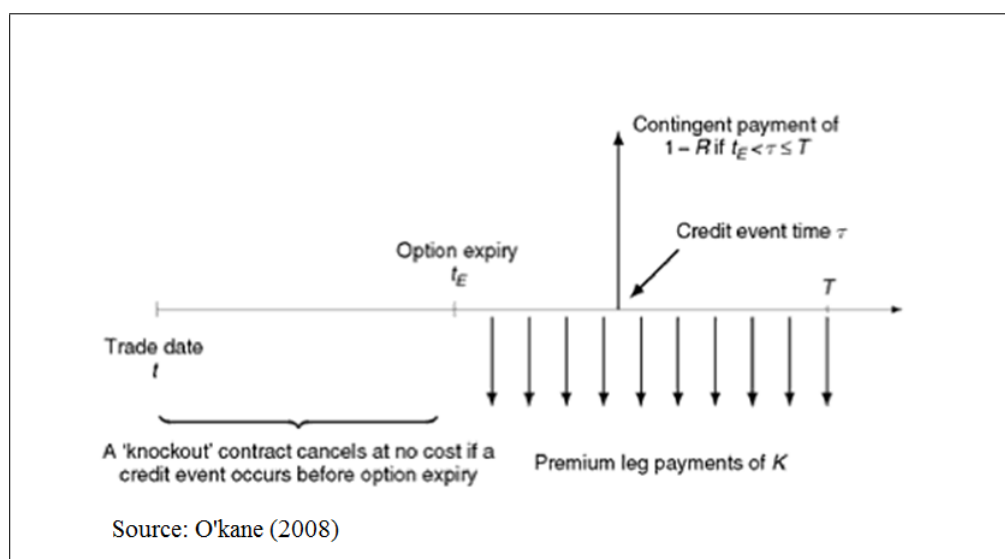


Figure 3.2: CDS knock-out payer swaption

3.2.1 CDS spreads

In a CDS contract, the fixed leg payments that will be made by the protection buyer are known as a *CDS spread*. In practise, the CDS spreads data can be obtained from

the market for various expiry dates based on a certain reference entity. The CDS spread has a link to the recovery rates of the reference entity and probability of default (PD) of the underlying loan or deliverable obligations.

The PD of the entity can be determined from the market CDS spreads and (O’Kane, 2008) gives the equation where PD is calculated and this equation is given in this subsection. CDS spread can be used by investors as a measure of credit risk, the higher the spread the higher the PD of the reference entity. For instance, investor may have prediction on a certain reference entity’s credit quality, that this particular entity CDS spread is increasing; and as spread increases the entity’s credit worthiness decreases.

Any investor can make bids or speculations about a certain entity without having deliverable obligations with it. Therefore, the investor may buy a CDS protection on the entity when speculating that it is about to default; or may sell the CDS protection when speculating that the entity’s credit worthiness is going to improve.

Hence, this investor (known as the protection seller) is known as being long on CDS and the protection buyer is viewed as being short on the CDS. Figures 3.3 and 3.4 illustrate the market CDS spreads and the corresponding annual probabilities of default for USA and SA respectively. The CDS spreads data, assuming the recovery rate of 40% is obtained from Bloomberg.

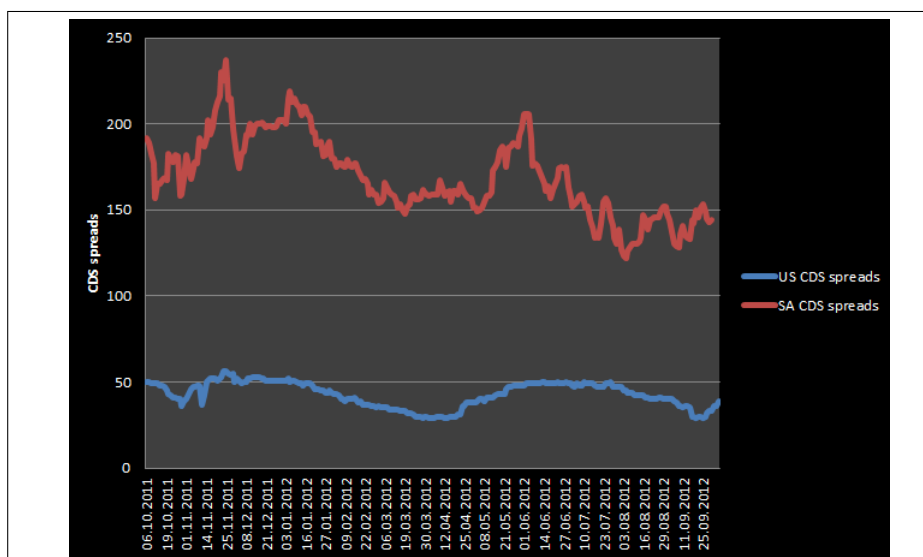


Figure 3.3: US and SA CDS spreads

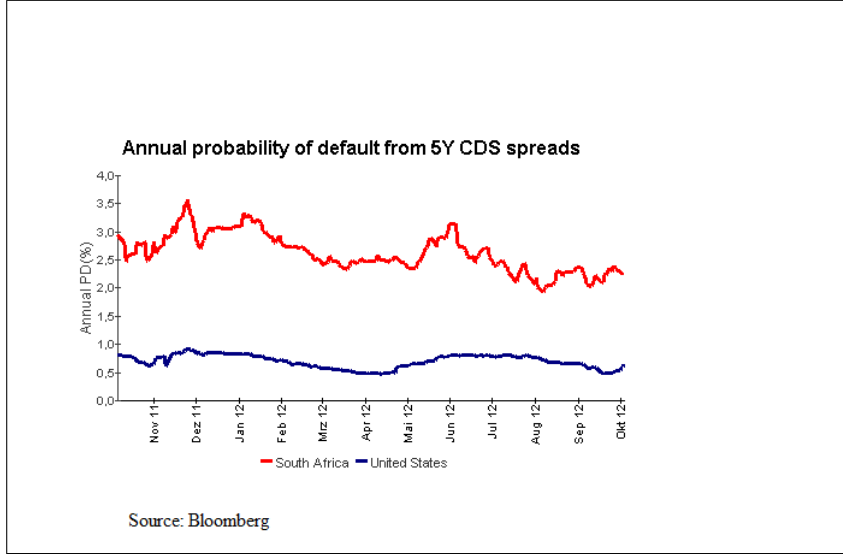


Figure 3.4: US and SA annual probability of default

Determining the forward CDS spread

(O’Kane, 2008) discuss the valuation of the forward CDS spread in details. Let the survival probability from time $t = 0$ to forward time t_f be denoted by $W(t, t_f)$. Suppose that for each payment, there is no probability of a credit event to occur before time t_i which is the occurrence of the payment. The present value (PV) of the fixed leg and the protection leg derived at (O’Kane, 2008) are given as:

$$\boxed{\text{Fixed leg } PV(t) = S(t, t_f, T) \cdot RPV01(t, t_f, T)} \quad (3.1)$$

$$\boxed{\text{Protection leg } PV(t) = (1 - R) \int_{t_f}^T Z(t, s) (-dW(t, s))} \quad (3.2)$$

where $Z(t, s)$ is the LIBOR discount curve which is binded to the CDS effective date. And the forward starting risky $PV01$ together with the accrued payment at default is given by:

$$RPV01(t, t_f, T) = \frac{1}{2} \sum_i^M = 1 \Delta(t_{i-1}, t_i) Z(t, t_i) (W(t, t_{i-1}) + W(t, t_i))$$

By subtracting the fixed leg from the protection leg equation, the market-to-market value for a long protection is obtained:

$$F(t, t_f, T) = (1 - R) \int_{t_f}^T Z(t, s)(-dW(t, s)) - S(t, t_f, T) \cdot RPV01(t, t_f, T) \quad (3.3)$$

Finally, the forward starting CDS spread is obtained by setting $F(t, t_f, T) = 0$:

$$S(t, t_f, T) = \frac{(1 - R) \int_{t_f}^T Z(t, s)(-dW(t, s))}{RPV01(t, t_f, T)} \quad (3.4)$$

When the forward spread is expressed in terms of the CDS spreads and spot starting $RPV01$, the forward spread can be written as:

$$S(t, t_f, T) = \frac{S(t, T)RPV01(t, T) - S(t, t_f)RPV01(t, t_f)}{RPV01(t, T) - RPV01(t, t_f)} \quad (3.5)$$

Estimating CDS spread volatility

(O’Kane, 2008) gives the CDS option model and this model is based on the forward CDS spreads not the spot spreads. And this calibration is performed by taking a time series of a full CDS term structures for a certain reference entity. The forward spread is calculated for each date in the time series and then the volatility is calculated from this time series. The method given by (O’kane, 2008) is described below.

Let series of CDS spreads be denoted by S_i for $i = 1, 2, \dots, N_D$ and N_D is the number of daily observations. And the CDS spreads are assumed to be lognormally distributed thus the volatility will be based on the continuously compounded daily return:

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \approx \frac{S_i - S_{i-1}}{S_{i-1}}. \quad (3.6)$$

And the daily volatility is estimated by:

$$\sigma_{Daily} = \sqrt{\frac{1}{N_D - 1} \sum_{i=1}^{N_D} (r_i - \hat{r})^2}, \quad (3.7)$$

where the average of the returns is given by:

$$\hat{r} = \frac{1}{N_D} \sum_{i=1}^{N_D} r_i. \quad (3.8)$$

Since the volatility was estimated based on daily observations, then it is possible to convert them to an annual volatility and this is done by the following:

$$\sigma_{Annual} = \sqrt{252}\sigma_{Daily} \quad (3.9)$$

where 252 is just the trading days in a year.

Figure 3.5 shows the lognormal distribution for the forward CDS spread and it is assumed that a time to maturity of the option is one year and this figure is for volatility of 10%, 20%, 40% and 80%.

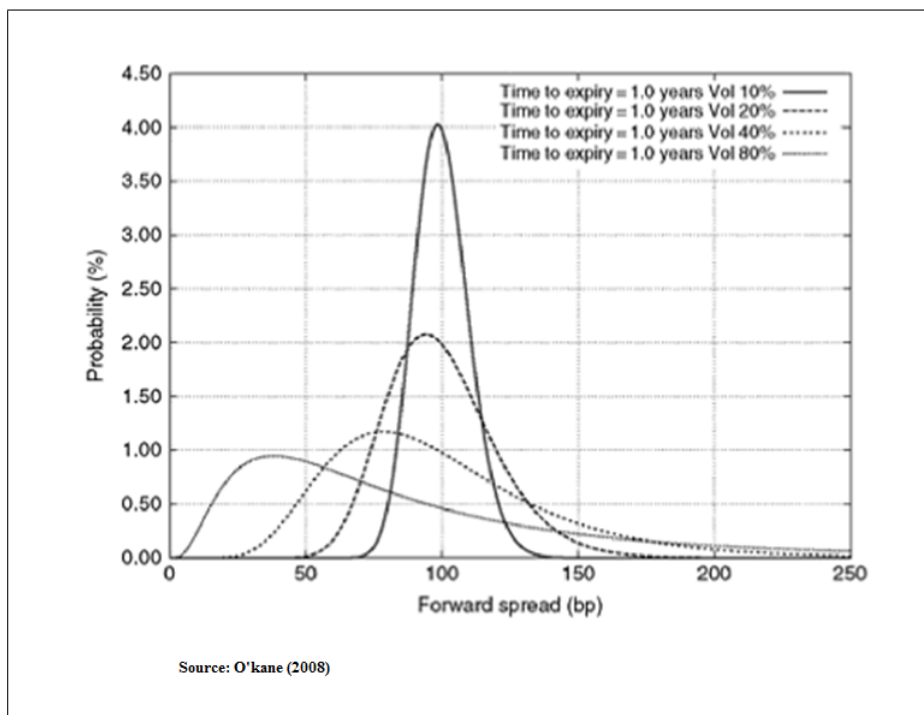


Figure 3.5: Lognormal distribution for the forward CDS spread

3.3 Cross currency swaptions

Cross currency swaption is a type of a swaption that gives the holder the right but not the obligation to enter into a cross currency swap. And here we will introduce some terms and definitions of this swaption from (Beidleman, 1992).

- **Put swaption** which is also known as a payer's swaption, it grants the buyer the right to pay a fixed rate on an agreed notional amount in one currency; and in exchange the holder will receive the floating rate on the same notional. The

holder will only exercise this swaption if the interest rates are higher than the strike rate, since he will be paying a lower fixed rate than the prevailing market fixed rate.

- **Call swaption (or receiver swaption):**

When an investor buys this swaption he will get the right to pay a floating rate and in the same time receives a fixed rate on the specified notional amount. Hence, an investor will exercise a call swaption only if the interest rates falls below the strike rate. This interest rate movement makes this swaption valuable to the holder because he will be receiving a higher fixed-rate.

A swaption can be further be described by the following terms:

- *In-the-money swaption:*

By entering this swaption now, it would result a positive payoff and it has an intrinsic value and this value helps to assume the expiry of the contract immediately rather than in the future. Thus this implies that the swaption allows the holder the right to either receive a fixed-rate that is greater than the forward rate or pay a lower fixed-rate than the prevailing forward rate.

- *Out-of-the-money:*

The holder of this swaption is granted the right to pay a fixed-rate that is greater than the prevailing forward rate or to receive a lower fixed-rate than the forward rate. Hence, this swaption may produce an immediate loss.

- *At-the-money:*

This swaption is formed by setting the forward rate equal to the strike rate (i.e. $S_t = K$).

3.4 Commodity Swaptions

(Jarvinen and Toivonen, 2004) define a commodity swaption as an option that grants the holder a right to pay (payer swaption) or receive (receiver swaption) a fixed price in exchange for the floating price of the underlying commodity. The commodity swaption is similar to the standard interest rate swaption, but the notional amount of the commodity swaption is in tones or barrels or other units whereas the notional

amount of the interest rate swaption is a currency amount. The underlying instrument in the commodity swaption is a spot commodity swap when it matures.

3.5 Energy Swaptions

Energy swaptions also known as European options on energy swaps are options that at expiration grant the holder a delivery of an underlying energy swap at the strike price (it does not have to be a physical delivery of any energy).

Hence, the swap can have either physical or cash settlement, it depends on the contract agreement at time t_0 . For instance, in reality energy swaptions are mostly traded at the Nordic Power Exchange and Nord Pool. If a call swaption at expiration is in-the-money, then the option can be exercised and there will be a delivery of swap but if the swaption is not in-the-money, the option will be allowed to expire worthless.

The pay-off after exercising the option at maturity is not obtained immediately but during the delivery period of the underlying swap. And the delivery period is specified at the beginning of the contract, for example at Nord Pool there exist financial daily settlement in the delivery period of the swap against the daily settlement of the underlying physical market, (Clewlow and Strickland, 1999) and (Haug, 2007).

3.6 Summary

Different types of swaptions with examples was discussed and two parties involved in swaptions were also specified, namely a payer and a receiver swaption. These parties can also be called put and call swaptions, it depends on which swaption an investor chooses to be involved in. For instance, if an investor enter an interest rate swaption and enter it as a payer swaption, then this swaption will be associated with a put swaption. Furthermore, the investor will only exercise this swaption when interest rate are expected to increase as speculated.

Investors have various reasons for using swaptions, for example interest rate swaptions can be used for protection against unfavourable interest rate movements in the market which are anticipated by the investor. Additionally, CDS options can be used to obtain protection against a credit event or take an advantage of a certain reference entity's credit quality by an investor speculating the decline of the entity's credit-worthiness.

Thus, he will accomplish this by buying protection against this movement and the

investor does not have to be the owner of the entity to enter this swaption as a payer swaption. And if the entity defaults then he'll be compensated for the loss by the protection seller.

CDS spreads are known as the fixed leg payments that are made payable by the protection buyer when entering a CDS contract. And this spread is important because it can be used as a measure of credit risk and the default probability of a certain company or reference entity can be determined from it.

Chapter 4

Black-Scholes method to price swaptions

Black-Scholes model can price any derivative and the price of these derivatives are dependent on a non-divided-paying stock and must satisfy the Black-Scholes PDE. In the following sections we will discuss the valuation of European interest rate swaptions, commodity swaptions, CDS option and energy swaptions using Black-Scholes formula. And the derivation of the Black-Scholes-Merton PDE is given on the Appendix B.1.

(Hull, 2000) gives the following assumptions underlying the Black-Scholes model:

- The swap rates follow a geometric Brownian motion with constant drift and volatility.
- Short selling is permitted.
- There are no transactions costs or taxes.
- The underlying swap does not pay dividends.
- There are no risk-less arbitrage opportunities.
- Security trading is continuous.
- Cash can be borrowed and be lend at a known constant risk-free interest rate.

4.1 Valuation of Interest Rate Swaptions

Option pricing models are used to value a European option on a swap, by assuming that the swap rate at maturity of the option is lognormal. The following notations

will be used in the discussion of the method, (Eales *et al*, 2003):

$rs \equiv$ Swap rate at maturity

$rX \equiv$ Swaption strike rate

$T \equiv$ Maturity

$t \equiv$ Start date

F (quarterly, semi-annual or annual) \equiv Pay basis

$M \equiv$ Notional Principal

If the swap rate on expiry of the swaption is denoted by rs , then the payoff for the swaption is as follows:

$$\frac{M}{F} \max(r - r_s, 0), \quad (4.1)$$

The swaption value is essentially given by the difference between the strike rate and the swap rate at the time it is being valued. The payoff at each interest date is given by $(rs - rX) \times M \times F$, if a swaption is exercised. And a call swaption is exercised when the swap rate is higher than the strike rate (i.e. $rs > rX$) and the payoff of the option on any interest payment in the swap is as follows:

$$Swaption_{Interest\ Payment} = \max[0, (rs - rX) \times M \times F]. \quad (4.2)$$

And it can be shown that the value of a call swaption at maturity is given by the following:

$$C_{swaption} = \sum_{n=1}^N Df_{(0,n)}(rs - rX) \times M \times F, \quad (4.3)$$

where $Df_{(0,n)}$ is the spot rate discount factor which starts today and ends at time t . The value of a put swaption is obtained by the same logic and it is given by:

$$P_{swaption} = \sum_{n=1}^N Df_{(0,n)}(rX - rs) \times M \times F \quad (4.4)$$

The value of the swaption can be obtained by taking a sum of each valued call or put option for a single payment in a swap. (Eales *et al*, 2003) gives an assumes that when Black 76 model is used, the LIBOR rate follows a lognormal distribution with constant volatility.

Consider that at time t , a call swaption is priced and it matures at time T , and start by valuing a single payment (assume that the option is exercised) which is made at time T_n , so that we can have $T_n > T > t$. At the time when the option is being valued, the call option maturity is $T - t$ and there is $T_n - t$ until the n th payment. Then the value of this payment is:

$$C_t = MF e^{-r(T_n-t)} [rsN(d_1) - rXN(d_2)], \quad (4.5)$$

where

$C_t \equiv$ Price of the call option on a single payment in the swap.

$r \equiv$ Risk free rate.

$N(.) \equiv$ Cumulative normal distribution.

$\sigma \equiv$ Interest rate volatility.

$$d_1 = \frac{[\ln(\frac{rs}{rX}) + \frac{\sigma^2}{2}(T-t)]}{\sigma\sqrt{T-t}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Lognormal distribution determines the remaining life of the swaption ($T - t$) which governs the probability that it will expire in-the-money. However, the interest payment is discounted (by using $e^{-r(T_n-t)}$) over the time ($T_n - t$) since it is not paid until time T_n .

Thus, we can value the call swaption as a collection of single payment of calls and its value is:

$$C_{swaption} = \sum_{n=1}^N MF e^{-r(T_n-t)} [rsN(d_1) - rXN(d_2)] \quad (4.6)$$

By substituting discrete spot rate discount factor instead of the continuous form given by the above equation, then the equation is rewritten as:

$$C_{swaption} = MF [rsN(d_1) - rXN(d_2)] \sum_{n=1}^N Df_{t,T_n}. \quad (4.7)$$

The corresponding put swaption value is given by the following:

$$P_{swaption} = MF[rXN(-d_2) - rsN(-d_1)] \sum_{n=1}^N Df_{t,T_n}. \quad (4.8)$$

The following is an example from (Eales and Choudhry, 2003) to price interest rate swaption.

Example: 4.1.1

The basic concepts of pricing swaptions will be shown in this example and a forward-starting annual interest swap with a notional of £10 million is also priced, whereby it starts in 2 years and has a maturity of three years. Interest rate data for swaption valuation is shown in table 4.1.

Table 4.1: Interest rate data for swaption valuation

Date	Term (years)	Discount factor	Par yield	Zero-coupon rate	Forward rate
18/02/2001	0	1	5	5	5
18/02/2001	1	0.952381	5.00	5	6.03015
18/02/2001	2	0.898217	5.50	5.51382	7.10333
18/02/2001	3	0.838645	6.00	6.04102	6.66173
18/02/2001	4	0.786132	6.15	6.19602	6.71967
18/02/2001	5	0.736379	6.25	6.30071	8.05230
18/02/2001	6	0.681652	6.50	6.58946	8.70869
18/02/2001	7	0.627192	6.75	6.88862	9.40246
18/02/2001	8	0.573154	7.00	7.20016	10.18050
18/02/2001	9	0.520195	7.25	7.52709	5.80396
18/02/2001	10	0.491660	7.15	7.35361	6.16366

The swap rate is determined by:

$$rs = \frac{\sum_{t=1}^N rf_{(t-1),t} \times Df_{0,t}}{\sum_{t=1}^N Df_{0,t}},$$

where rf is the forward rate. The numerator and denominator is calculated by using the above equation and respectively given as:

$$(0.0666 \times 0.8386) + (0.0672 \times 0.7861) + (0.0805 \times 0.7634) = 0.1701$$

The denominator is:

$$(0.08386 + 0.7861 + 0.7634) = 2.3881$$

Thus, the forward-starting swap rate is $\frac{0.1701}{2.3881} = 0.07123$ and by focusing on the call swaption whereby the buyer has the right to enter into a 3-year swap by paying the fixed-rate swap rate of 7%. Hence, it is determined the values of d_1 and d_2 given the volatility of the forward swap rate which is 20%.

$$d_1 = \frac{\ln(rs/rX) + 0.5 \times \sigma^2 \times (T - t)}{\sigma \sqrt{(T - t)}} = \frac{\ln(0.07123/0.07) + 0.5 \times 0.2^2 \times 2}{0.2 \times \sqrt{(2)}} = 0.20291,$$

$$d_2 = d_1 - \sigma \sqrt{(T - t)} = 0.20291 - 0.2 \times (1.4142) = -0.0799.$$

The cumulative normal values of d_1 and d_2 are respectively given as:

$$N(d_1) = 0.5804, \quad N(d_2) = 0.4681$$

Therefore, by using the above information we can calculate the value of the call swaption which is as follows:

$$\begin{aligned} C_{swaption} &= MF[rsN(d_1) - rXN(d_2)] \sum_{n=1}^N Df_{t,T_n} \\ &= 10,000,000 \times 1 \times [0.07123 \times 0.5804 - 0.07 \times 0.4681] \times 2.3881 = \pounds 219,250 \end{aligned}$$

4.2 Valuation of the CDS options

The valuation of the European CDS call and put options is derived by (Bomfim, 2005). Let the value to a protection buyer of a forward-starting CDS be denoted by $W(t)$. And consider the CDS that will start at a future time $t = T$ and it has payments dates of $T_1, T_2, T_3, \dots, T_n$ and it is assumed that the option is set *at-the-money* (i.e. where the premium is set at the strike K). Thus, we have:

$$W(t) = \sum_{i=1}^n R(t, T_i) P(t, T_i) \delta_i [S^F(t, T, T_n) - K] \quad (4.9)$$

It is also considered a European CDS option which is written at time $t = 0$ whereby a protection is bought on the contract underlying the forward-starting CDS. Then, at time $t = T$ (i.e. at the maturity of the CDS option) the value of the CDS option will be given by:

$$V(T) = \max \left\{ \sum_{i=1}^n R_0^d(t, T_i) \delta_i [S^F(T, T, T_n) - K], 0 \right\} \quad (4.10)$$

whereby $\sum_{i=1}^n R_0^d(t, T_i) \delta_i$ is the annuity factor that gives the value of differences between the CDS fixed leg and the premium specified on the CDS option contract. Thus, the option holder will exercise this option if $S^F(T, T, T_n) > K$, meaning when the current par CDS exceeds the premium written on the CDS option. Let $A(T, T_n)$ denote the annuity factor defined above, then the value of the CDS option is rewritten as:

$$V(T) = \max \left[A(T, T_n) (S^F(T, T, T_n) - K), 0 \right], \quad (4.11)$$

The value of the CDS option at time- t is the present value of the difference between the premium payments of the two CDS:

$$V(T) = A(T, T_n) \max \left[(S^F(T, T, T_n) - K), 0 \right]. \quad (4.12)$$

By introducing the expected value on the above equation, it is obtained the risk-adjusted expected present value of the CDS cash flow and is as follows:

$$V(t) = A(t, T_n) E_t \left[\max (S^F(T, T, T_n) - K, 0) \right]. \quad (4.13)$$

In order to derive a pricing formula for this CDS option, it is assumed that the CDS spread $S^F(t, T, T_n)$ is lognormally distributed and from this assumption the Black-Scholes formula can be used.

The Black-Scholes formula for a CDS call option is the same as the model derived by (Tucker and Wei, 2005) and is given by:

$$V_{call}(t) = A(t, T_n) \left[S^F(t, T, T_n) N(d_1) - K N(d_2) \right] \quad (4.14)$$

and the corresponding value of a CDS put option is obtained by using a put-call parity:

$$V_{put}(t) = A(t, T_n) \left[K N(-d_2) - S^F(t, T, T_n) N(-d_1) \right], \quad (4.15)$$

where

$$N(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^x e^{-\left(\frac{x^2}{2}\right)} dx;$$

$$d_1 = \frac{\left[\ln\left(\frac{S^F(t,T,T_n)}{K}\right) + \frac{1}{2}\sigma(T,T_n)^2(T-t)\right]}{\sigma\sqrt{T-t}};$$

$$d_2 = d_1 - \sigma(T,T_n)\sqrt{T-t}.$$

Example: 4.2.1

Suppose the forward CDS swap rate is 150 bp compounded semi-annually, so the call swaption is struck at the money. Assume that the forward volatility is 12%. And lastly, assume that the interest rate swap curve is flat at 3% per year compounded continuously.

Solution

We are given the following: $L = \$100$ million, $m = 2$, $n = 3$, $R_0 = 1.5\%$ (s.a.), $R_K = 1.5\%$ (s.a.), $\sigma = 0.12$, and $T = 0.5$;

$$A = \left(\frac{1}{2}\right) \left[e^{-(0.03 \times 0.5)} + e^{-(0.03 \times 1)} + e^{-(0.03 \times 1.5)} + e^{-(0.03 \times 2)} + e^{-(0.03 \times 2.5)} + e^{-(0.03 \times 3)} \right] = 2.8475$$

$$d_1 = \frac{\left[\ln\left(\frac{0.015}{0.015}\right) + (0.12)^2 \times \frac{0.5}{2}\right]}{(0.12)\sqrt{0.5}} = 0.04246$$

$$d_2 = 0.04246 - (0.12)\sqrt{0.5} = -0.04239$$

$$N(0.04246) = 0.51696 \text{ and } N(-0.04239) = 0.48307$$

Thus, substituting these values to the formula of the European call and put swaption we obtain the following values for call and put respectively.

$$C^E = \$144,510 \text{ and } P^E = \$144,510.$$

The values for the call and put swaptions are the same because both are struck at the money.

4.3 Valuation of Commodity Swaptions

In order to value a commodity swaption, the forward fixed and forward floating legs are required and these legs are determined from (Jarvinen and Toivonen, 2004) as follows :

Let a collection of future dates be fixed $T_0 = T_1 < \dots < T_N$ and the forward swap is a financial contract entered at time $t < T_0$ with settlement dates $T_1 < T_N$. The *spot price* of the commodity is often used as the settlement price.

One party pays the difference $S(T_i) - X$ at each settlement date of the swap, where $S(T_i)$ is the floating price for the settlement T_i and X is the agreed fixed price.

The present value of the floating leg is:

$$V_{float} = \sum_{i=1}^N F(t, T_i)P(t, T_i), \quad (4.16)$$

and the present value of the fixed leg can be obtained from the following:

$$V_{fixed} = \sum_{i=1}^N P(t, T_i). \quad (4.17)$$

Fixed leg value is the sum of the discount factors for the settlement dates of the swap contract multiplied by the fixed price of the swap. The actual currency dominated values are obtained by multiplying leg values by notional per fixing. And the notional quantity is assumed to be equal to 1, in equation 4.16 and 4.17 for simplification.

The forward swap which makes the present value of the contract zero is obtained by setting the floating leg and fixed leg equal. The par forward swap price (X) is solved as follows:

$$X = \frac{\sum_{i=1}^N F(t, T_i)P(t, T_i)}{\sum_{i=1}^N P(t, T_i)}. \quad (4.18)$$

The payoff of the European commodity swaption is defined as follows:

Note: The payoff on a European call option is:

$$F(S_T) = \max \{S_T - K, 0\} \text{ at time } T \quad (4.19)$$

and this option can be exercised only at expiry date and if $S_T > K$. Thus, the following equation is the payoff for the *payer swaption*:

$$C(T_0) = \max[V_{float}(T_0) - V_{fixed}(T_0), 0] \quad (4.20)$$

The payer swaption can be exercised if $V_{float}(T_0) > V_{fixed}(T_0)$, therefore it will be exercised if the floating price leg for the settlement T_0 is greater than the agreed fixed price leg.

Note: The payoff on a European put option is:

$$F(S_T) = \max\{K - S_T, 0\} \text{ at time } T \quad (4.21)$$

and this option can be exercised only at expiry date and if $S_T < K$. Thus, the following equation is the payoff for the *receiver swaption*:

$$P(T_0) = \max[V_{fixed}(T_0) - V_{float}(T_0), 0] \quad (4.22)$$

The receiver swaption can be exercised if $V_{float}(T_0) < V_{fixed}(T_0)$. Therefore, it will be exercised if the floating leg for the settlement T_0 is less than the agreed fixed price leg. Hence, *payer swaption* is a call option on the floating price leg and *receiver swaption* is a put option on the floating price leg.

The above payoffs can be written as follows:

$$C(T_0) = \sum_{i=1}^N P(T_0, T_i) \times \max[S(T_0) - X, 0] \quad (4.23)$$

and

$$P(T_0) = \sum_{i=1}^N P(T_0, T_i) \times \max[X - S(T_0), 0] \quad (4.24)$$

4.3.1 Pricing Formulae

(Jarvinen and Toivonen, 2004) further determine the price dynamics of European commodity swaption following Black-Scholes model. In order to value a commodity swaption, the forward floating and the forward fixed legs are required. The dynamics are assumed to be under the real-world probability \mathbf{P} and are given by:

$$dV_{float}(t) = \mu_1(t)V_{float}(t)dt + \sigma_1V_{float}(t)dW_t^1 \quad (4.25)$$

$$dV_{fixed}(t) = \mu_2(t)V_{fixed}(t)dt + \sigma_2V_{fixed}(t)dW_t^2 \quad (4.26)$$

where dW_t is the standard Brownian motion under the probability \mathbf{P} and $\mu_{1,2}(t)$, $\sigma_{1,2}(t)$ are the drift and volatility (diffusion) coefficients, respectively.

Under the risk neutral measure, the drift coefficients of the above equations become $r(t)$ which is the continuously compounded risk-free rate of return. The simple formulae are derived from the model for the forward floating leg and forward fixed leg values, thus all the associated elements $F(t, T_i)$ and $P(t, T_i)$ are summed respectively. And it is assumed that the volatility of the forward swap price is deterministic and it follows the lognormal law.

The expected value of the option payoff is calculated using the forward fixed price leg divided by the strike price as numeraire (also known as annuity). The following expression for the expectation is obtained by:

$$\frac{V_{float}(0)}{A_{T_1, T_N}(0)} = \mathbf{E}^A \left[\frac{V_{float}(t)}{A_{T_1, T_N}(t)} \right] \quad (4.27)$$

where \mathbf{E}^A is the expectation under the annuity measure and $A_{T_1, T_N} = \sum_{i=1}^N P(t, T_i)$. The forward swap price is a martingale and the expectation spot swap price equals the forward swap price, under this change of numeraire.

The contingent claims valuation function is:

$$F(0) = A_{T_1, T_N}(0) \times \mathbf{E}^A \left[\frac{F(T_0)}{A_{T_1, T_N}(T_0)} \right], \quad (4.28)$$

where F is any attainable simple claim having T_0 as a maturity date. Then substitute equation 4.20 into 4.28 to obtain the following:

$$C(0) = A_{T_1, T_N}(0) \times \mathbf{E}^A \left[\frac{A_{T_1, T_N}(T_0) \times \max[S(T_0) - X, 0]}{A_{T_1, T_N}} \right] = A_{T_1, T_N}(0) \times \mathbf{E}^A [\max[S(T_0) - X, 0]], \quad (4.29)$$

Equation 4.29 is the payoff of the standard European payer swaption under the change of numeraire technique. The assumption of deterministic volatility of the forward swap price gives the familiar results:

$$C(t) = \sum_{i=1}^N P(t, T_i) [S(t, T_0, T_N)N(d_1) - XN(d_2)] \quad (4.30)$$

and

$$P(t) = \sum_{i=1}^N P(t, T_i) [XN(-d_2) - S(t, T_0, T_N)N(-d_1)] \quad (4.31)$$

where

$$d_1 = \frac{\ln\left(\frac{S(t, T_0, T_N)}{X}\right) + \frac{1}{2} \int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du}{\sqrt{\int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du}},$$

$$d_2 = d_1 - \left(\sqrt{\int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du} \right)$$

where $\|\cdot\|$ is the Euclidean norm. Equation 4.30 and 4.31 are solutions to the payer (call) and receiver(put) prices for commodity swaption under the assumptions used.

4.4 Valuation of Energy Swaptions

(Clewlow and Strickland, 1999) defines a European energy swaption as an option to a swap with a stream of floating price payments indexed to the market spot price of the energy commodity for a stream of fixed price payments. The following notation will be used for the energy swaption formula and this formula is discussed on (Haug, 2007):

W = Forward swap price observed in the market.

j = Number of compounding per year (number of settlements in a 1-year forward contract).

n = Number of settlements in the delivery period (i.e. this will be the number of trading days in the forward period).

r_j = Swap rate starting at the beginning of the delivery period and ending at the end of the delivery period with j compounding per year, equal to the number of fixings in the delivery period.

r_b = A risk-free continuous compounding zero coupon rate with T_b years to maturity.

r_e = A risk-free continuous compounding zero coupon rate with time to maturity starting now to the end of the delivery period.

r_p = A risk-free continuous compounding zero coupon rate with forward starting at the option maturity T and ending at the beginning of the delivery period T_b .

T_b = Beginning time of the forward delivery period.

T_m = Time in years from now to the middle of the delivery period.

The European energy swaptions are priced using modified Black-76 formula and the energy call swaption formula is given by:

$$C = \frac{\left(1 - \frac{1}{(1+r_j)^n}\right)}{r_j} \frac{j}{n} e^{-r_p(T_b-rT)} e^{-rT} [WN(d_1) - XN(d_2)] = \frac{\left(1 - \frac{1}{(1+r_j)^n}\right)}{r_j} \frac{j}{n} e^{-r_p T_b} [WN(d_1) - XN(d_2)], \quad (4.32)$$

Similarly, energy put swaption formula is given by:

$$P = \frac{\left(1 - \frac{1}{(1+r_j)^n}\right)}{r_j} \frac{j}{n} e^{-r_p T_b} [XN(-d_2) - WN(-d_1)], \quad (4.33)$$

where

$$d_1 = \frac{\ln(S/X) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}. \quad (4.34)$$

The above formulae for the energy swaptions (both call and put) can be approximated by the following:

$$C \approx e^{-r_e T_m} [WN(d_1) - XN(d_2)], \quad (4.35)$$

$$P \approx e^{-r_e T_m} [XN(-d_2) - WN(-d_1)]. \quad (4.36)$$

(Haug, 2007) gives the example of a call energy swaption using the equation 4.35.

Example: 4.4.1

Consider a call on a quarterly electricity swap, with 6 months to maturity, and the start of the delivery period is 17 days after the option maturity and the delivery period is 92 days.

The swap trades at 33 EUR/MwH, with the strike price of 35 EUR/MwH. The risk-free rate from now until the beginning of the delivery period is 5% and the daily compounding swap rate starting at the beginning of the delivery period and ending at the end of the delivery period is 5%. The volatility of the swap is 18%. Calculate the value of the call swaption.

Solution

Given: $T = 0.5$, $T_b = 0.5 + \frac{17}{365} = 0.5466$, $r_b = 0.05$, $r_j = 0.05$, $j = 365$, $n = 92$ and $\sigma = 0.18$.

The approximations for calculating the call swaption is given by:

$$C \approx e^{-r_e T_m} [W N(d_1) - X N(d_2)], \quad (4.37)$$

The time from now to the middle of the delivery period is required in order to use the above formula to calculate the call swaption value and is given by: $T_m = 0.5 + \frac{17}{365} + 92/2/365 = 0.6726$ and assume that the rate from now to the end of the delivery period is $r_e \approx 0.05$.

$$C \approx e^{-0.05 \times 0.6260} [33N(d_1) - 35N(d_2)], \quad (4.38)$$

$$d_1 = \frac{\ln(33/35) + (0.18)^2 0.5/2}{0.18 \times \sqrt{0.5}} = -0.39866, \quad N(d_1) = 0.345072;$$

$$d_2 = -0.39866 - 0.18 \times \sqrt{0.5} = -0.525934, \quad N(d_2) = 0.299467.$$

Therefore, the value for the call swaption is:

$$C = 0.8761$$

4.5 Summary

Black-Scholes model is a widely used method in the pricing of financial derivatives because of its unique assumptions. Hence, the prices of interest rate swaptions, CDS options, commodity swaptions and energy swaptions were determined using this model. The examples which illustrate the use of this method is implemented in *Matlab*[®] for CDS options and energy swaptions.

The prices of these swaptions are determined from solving the Black-Scholes PDE and assuming appropriate boundary conditions. The function that is obtained after solving this PDE must satisfy the Black-Scholes PDE and is used as a pricing function for both payer and receiver swaptions. Hence, this function can be determined by either applying a risk neutral valuation or martingale approach.

Chapter 5

Finite Difference methods

Numerical methods can be used to solve challenging and complex financial problems (i.e. non-linear PDE, or solving stochastic variables) that cannot be solved analytically to obtain the solution. Hence, numerical schemes are mostly applied to approximate the solution of that particular PDE.

(Duffy, 2006), describes the points considered when the PDE is approximated by discrete methods in general: The partial derivatives in the PDE (in space and time), the payoff function and boundary conditions of the financial problems are approximated by the discrete schemes. Some numerical methods can be better than the others but their efficiency in solving problems are uniquely defined by various properties.

There exist different discrete models but in this work, finite difference methods (FDM), (i.e. Crank-Nicolson scheme) is chosen to approximate the Black-Scholes PDE in particular to price CDS options and this will be discussed further in chapter 6. However, FDM is chosen since it is very flexible and thus can be applied to approximate a range of problems. It is also suitable to approximate and accommodate variations in the financial problems, (Duffy, 2006) and (Haug, 2007).

Implicit, Explicit and Crank-Nicholson methods including their properties are going to be introduced in this chapter to solve the basic heat equation and Black-Scholes PDE. These methods used to obtain the prices of options are developed by (Higham, 2004) and (Tavella and Randall, 2000).

5.1 Operators of Finite Difference

Suppose we are given a smooth function $y: \mathbb{R} \rightarrow \mathbb{R}$, and from the definition of a derivative for small h is given by: (See Higham, 2004) and (Tavella and Randall, 2000).

$$\frac{y(x+h) - y(x)}{h} \approx \frac{dy}{dx}(x). \quad (5.1)$$

Equation (5.1) can be written as follows where y_m denote the value $y(mh)$:

$$y_{m+1} - y_m \approx h \frac{dy}{dx}(mh). \quad (5.2)$$

It is assumed that functions are evaluated at $x = mh$, and by using Taylor's series expansion, Equation (5.2) can be extended to

$$y_{m+1} - y_m = hy' + \frac{1}{2}h^2y'' + \dots \quad (5.3)$$

The forward difference is given by $y_{m+1} - y_m$ and the forward operator (Δ) is defined by:

$$\Delta y_m = y_{m+1} - y_m. \quad (5.4)$$

A number of finite difference operators is defined in table 5.1, (Higham, 2004).

<i>Difference operators</i>			
Operator	Symbol	Definition	Taylor series
Forward difference	Δ	$y_{m+1} - y_m$	$hy' + \frac{1}{2}h^2y'' + \dots$
Backward difference	∇	$y_m - y_{m-1}$	$hy' - \frac{1}{2}h^2y'' + \dots$
Half central difference	δ	$y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}}$	$hy' - \frac{1}{24}h^2y'' + \dots$
Full central difference	Δ_0	$\frac{1}{2}(y_{m+1} - y_{m-1})$	$hy' + \frac{1}{6}h^3y''' + \dots$
Second order central difference	δ^2	$y_{m+1} - 2y_m + y_{m-1}$	$h^2y'' - \frac{1}{12}h^4y'''' + \dots$
Shift	E	y_{m+1}	$y + hy' + \dots$
Average	μ	$\frac{1}{2}(y_{m+\frac{1}{2}} + y_{m-\frac{1}{2}})$	$y + \frac{1}{8}h^2y'' + \dots$

Table 5.1: Difference operators.

The main building blocks of finite difference methods are formed by these operators of which act on grid values $y_m = y(mh)$.

5.1.1 Heat equation

On the following simple mathematical problem, a function of two variables $u(x, t)$ must be determined so that it satisfies the PDE which is known as the *heat equation*, (Tavella and Randall, 2000).

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 \leq x \leq L \quad \text{and} \quad 0 \leq t \leq T, \quad (5.5)$$

which is subjected to the initial condition:

$$u(x, 0) = g(x), \quad (5.6)$$

and the boundary conditions:

$$u(0, t) = a(t) \quad \text{and} \quad u(L, t) = b(t). \quad (5.7)$$

The function $u(x, t)$ represents the temperature at time t of the point x on a thin metal bar with initial temperature values given in equation 5.6 and with endpoints from equation (5.7). We use this PDE because, (i) We can develop ideas of finite difference methods in a simpler way, (ii) Basic Black-Scholes PDE can be translated in the form of this methods, and this transformation is discussed in chapter 6.

In order to perform the convention, we will use x as space and t as time. Consider the case where $L = \pi$ with

$$g(x) = \sin(x), \quad a(t) = b(t) = 0, \quad (5.8)$$

and it can be verified that

$$u(x, t) = e^{-t} \sin(x) \quad (5.9)$$

will solve Equation (5.5),(5.6) and (5.7).

5.1.2 Discretization

According to (Higham, 2004): The first step to approximate the solution to the PDE from Equation (5.5), (5.6) and (5.7) is to discretize. The aim is to determine the approximations to the PDE solution only at a finite set of points. The space axis is divided into $N_x + 1$ equally spaced points $\{jh\}_{j=0}^{N_x}$, whereby $h = \frac{L}{N_x}$ and divide time axis into $N_t + 1$ equally spaced points $\{ik\}_{i=0}^{N_t}$ where $k = \frac{L}{T_t}$.

And the spaced points (jh, ik) are known as the grid or mesh. We want to find values U_j^i that will be used to approximate the solution on the grid,

$$U_j^i \approx u(jh, ik), \quad 0 \leq j \leq N_x \quad \text{and} \quad 0 \leq i \leq N_t. \quad (5.10)$$

The grid is shown in Figure 5.1 (Higham, 2004), whereby the open circles indicate grid points where the solution yet unknown and the filled circles are the points used to determine the solution of Equation (5.6) and (5.7). Hence the aim is to obtain numbers to fill the points marked \circ , and this will be filled by using finite difference operators to form equations that the grid values U_j^i must satisfy.

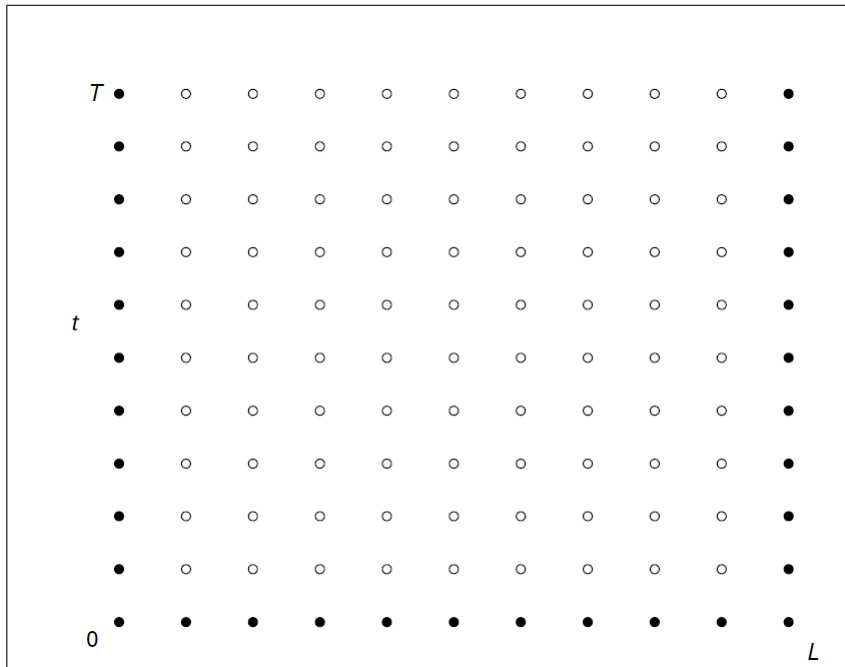


Figure 5.1: Finite difference grid $\{jh, ik\}_{j=0, i=0}^{N_x, N_t}$. Points are spaced at a distance h apart in the x -direction and k apart in the t -direction.

5.1.3 FTCS and BTCS

FTCS is the *forward difference in time*, *central difference in space* and BTCS is *backward difference in time*, *central difference in space*. The two independent variables, $0 \leq x \leq L$ and $0 \leq t \leq T$, are involved in the problem domain. The difference operators in the x and t directions are distinguished with the following subscript: (Higham, 2004).

$$\delta_t U_j^i = U_j^{i+1} - U_j^i \quad \text{and} \quad \delta_x U_j^i = U_{j+1}^i - U_j^i. \quad (5.11)$$

A method that approximates $\frac{\partial}{\partial t}$ by the scaled forward difference in time, is $k^{-1}\Delta_t$ and $\frac{\partial^2}{\partial x^2}$ by scaled second order central difference in space which is $h^{-2}\delta_x^2$. And approximates the solution of the heat equation:

$$k^{-1}\delta_t U_j^i - h^{-2}\delta_x^2 U_j^i = 0, \quad (5.12)$$

and can be expanded as

$$\frac{U_j^{i+1} - U_j^i}{k} - \frac{U_{j+1}^i - 2U_j^i + U_{j-1}^i}{h^2} = 0. \quad (5.13)$$

Equation (5.11) can be re-written as

$$U_j^{i+1} = \nu U_{j+1}^i + (1 - 2\nu)U_j^i + \nu U_{j-1}^i, \quad (5.14)$$

whereby a mesh ratio is denoted by $\nu = \frac{k}{h^2}$. Suppose $\{U_j^i\}_{j=0}^{N_x}$ is known and it is the approximate solution values at time level i . The boundary conditions in Equation (5.7) gives $U_0^{i+1} = a((i+1)k)$ and $U_{N_x}^{i+1} = b((i+1)k)$, thus the formula for computing other approximate values at time $i+1$ which is $\{U_j^{i+1}\}_{j=1}^{N_x-1}$ is given by Equation (5.12).

We are given the time-zero values, $U_j^0 = g(jh)$ by Equation (5.6), implying that the complete set of approximations $\{U_j^i\}_{j=0, i=0}^{N_x, N_t}$ can be computed by stepping forward time. Figure 5.2 shows the stencil for FTCS, the solid circles is the location of values U_{j-1}^i , U_j^i and U_{j+1}^i that must be known to determine the value U_j^{i+1} which is located at the open circle.

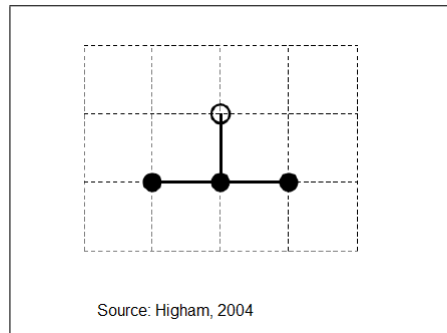


Figure 5.2: Stencil for FTCS. Solid circles indicate the location of values that must be known to obtain the value located at the open circle.

$$\mathbf{U}^i = \begin{bmatrix} U_1^i \\ U_2^i \\ \vdots \\ \vdots \\ U_{N_x-1}^i \end{bmatrix} \in \mathbb{R}^{N_x-1},$$

And FTCS may be written as

$$\mathbf{U}^{i+1} = f\mathbf{U}^i + \mathbf{p}^i, \quad \text{for } 0 \leq i \leq N_t - 1, \quad (5.15)$$

with

$$\mathbf{U}^0 = \begin{bmatrix} g(h) \\ g(2h) \\ \vdots \\ \vdots \\ g((N_x - 1)h) \end{bmatrix} \in \mathbb{R}^{N_x-1},$$

where the form of matrix F is given by

$$F = \begin{bmatrix} 1 - 2\nu & \nu & 0 & \cdots & \cdots & 0 \\ \nu & (1 - 2\nu) & \nu & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 - 2\nu & \nu \\ 0 & \cdots & \cdots & 0 & \nu & 1 - 2\nu \end{bmatrix} \in \mathbb{R}^{(N_x-1) \times (N_x-1)},$$

and vector \mathbf{p}^i has the following form

$$\mathbf{p}^i = \begin{bmatrix} \nu a(ik) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \nu b(ik) \end{bmatrix} \in \mathbb{R}^{N_x-1},$$

The backward difference is given by:

$$\frac{U_j^i - U_j^{i+1}}{k} - \frac{U_{j+1}^i - 2U_j^i + U_{j-1}^i}{h^2} = 0. \quad (5.16)$$

The BTCS can be obtained by re-writing the above equation (i.e. start from time level i to $i + 1$ and increase the time index by 1):

$$U_j^{i+1} = \nu U_{j+1}^i + (1 - 2\nu)U_j^i + \nu U_{j-1}^i. \quad (5.17)$$

BTCS has no explicit way to determine $\{U_j^{i+1}\}_{j=1}^{N_x-1}$ from $\{U_j^i\}_{j=1}^{N_x-1}$. Thus, Figure 5.3 shows the stencil for BTCS.

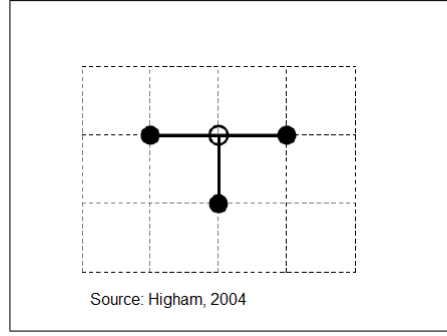


Figure 5.3: Stencil for BTCS. Solid circles indicate the location of values that must be known to obtain the value located at the open circle.

BTCS can be represented in matrix form and is given as:

$$BU^{i+1} = U^i + \mathbf{q}^i, \quad \text{for } 0 \leq i \leq N_t - 1, \quad (5.18)$$

where matrix B form is given by:

$$B = \begin{bmatrix} 1 + 2\nu & -\nu & 0 & \cdots & \cdots & 0 \\ -\nu & 1 + 2\nu & -\nu & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\nu & 1 + 2\nu & -\nu \\ 0 & \cdots & \cdots & 0 & -\nu & 1 + 2\nu \end{bmatrix} \in \mathbb{R}^{(N_x-1) \times (N_x-1)},$$

and vector \mathbf{q}^i has the following form

$$\mathbf{q}^i = \begin{bmatrix} \nu a((i+1)k) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \nu b((i+1)k) \end{bmatrix} \in \mathbb{R}^{N_x-1}.$$

The formulation of BTCS in a matrix form shows that if we are given U^i , then we can determine U^{i+1} by solving a system of linear equations which is known as the standard problem in numerical analysis.

5.1.4 Crank-Nicolson

Crank-Nicolson method is derived by using a temporarily idea of an intermediate time level at $(i + \frac{1}{2})k$, and the heat Equation (5.5) can be approximated by

$$k^{-1}\delta_t U_j^{i+\frac{1}{2}} - h^{-2}\delta_x^2 U_j^{i+\frac{1}{2}} = 0. \quad (5.19)$$

Time averaging operator μ_t can be applied to overcome the difficulty of introducing points that are not on the grid.

$$k^{-1}\delta_t U_j^{i+\frac{1}{2}} - h^{-2}\delta_x^2 \mu_t U_j^{i+\frac{1}{2}} = 0. \quad (5.20)$$

or

$$k^{-1}(U_j^{i+1} - U_j^i) - h^{-2}\delta_x^2 \frac{1}{2}(U_j^{i+1} + U_j^i) = 0. \quad (5.21)$$

And Crank-Nicolson method is given by:

$$2(1 + \nu)U_j^{i+1} = \nu U_{j+1}^{i+1} + \nu U_{j-1}^{i+1} + \nu U_{j+1}^i + 2(1 + \nu)U_j^i + \nu U_{j-1}^i. \quad (5.22)$$

The method has local accuracy $O(k^2) + O(h^2)$ because it has an appealing symmetry and Figure (5.4) shows the stencil for Crank-Nicolson method.

A system of linear equations must be solved in order to determine U^{i+1} from U^i , and these equations may be written as:

$$\hat{B}U^{i+1} = \hat{F}U^i + \mathbf{r}^i, \quad \text{for } 0 \leq i \leq N_t - 1, \quad (5.23)$$

where matrices \hat{B} and \hat{F} have the following form:

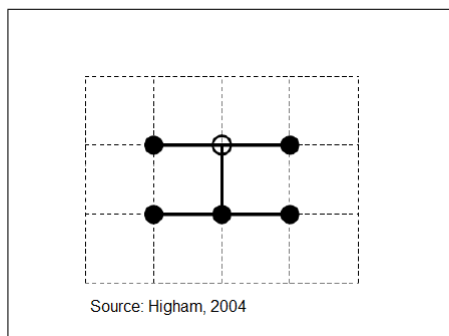


Figure 5.4: Stencil for Crank-Nicolson. Solid circles indicate the location of values that must be known to obtain the value located at the open circle.

$$\hat{B} = \begin{bmatrix} 1 + \nu & -\frac{1}{2}\nu & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2}\nu & 1 + \nu & -\frac{1}{2}\nu & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\frac{1}{2}\nu & 1 + \nu & -\frac{1}{2}\nu \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2}\nu & 1 + \nu \end{bmatrix} \in \mathbb{R}^{(N_x-1) \times (N_x-1)},$$

$$\hat{F} = \begin{bmatrix} 1 - \nu & \frac{1}{2}\nu & 0 & \cdots & \cdots & 0 \\ \frac{1}{2}\nu & 1 - \nu & -\frac{1}{2}\nu & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1}{2}\nu & 1 - \nu & \frac{1}{2}\nu \\ 0 & \cdots & \cdots & 0 & \frac{1}{2}\nu & 1 - \nu \end{bmatrix} \in \mathbb{R}^{(N_x-1) \times (N_x-1)},$$

and vector \mathbf{r}^i has the following form

$$\mathbf{r}^i = \begin{bmatrix} \frac{1}{2}\nu(a(ik) + a((i+1)k)) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{2}\nu(b(ik) + b((i+1)k)) \end{bmatrix} \in \mathbb{R}^{N_x-1}.$$

5.2 Finite difference methods for the Black-Scholes PDE

The finite difference methods are introduced in the previous section to solve heat equation and in this section we will discuss the methods to approximate the Black-Scholes PDE. (See Higham, 2004)

Finite difference methods also known as grid methods are numerical techniques used to value a derivative by solving partial differential equations (PDE) that the derivative satisfies. The differential equation is converted into a set of difference equations and the differential equations are solved iteratively.

This approach is illustrated by considering how it might be used to value an European put and call options. Here we will discuss the common finite difference methods in option pricing, namely:

- Explicit finite difference
- Implicit finite difference
- Crank-Nicolson finite difference

5.2.1 FTCS, BTCS and Crank-Nicholson for the Black-Scholes PDE

The same principal used in section 5.1 to build finite difference methods will also be followed in this section. According to (Haug, 2007) a grid should be first build with time along the x-axis and price along the y-axis, and discretise the time and price movements just as in a tree model.

The relevant PDE on this grid is approximated using finite difference methods, and the finite difference methods can be used to approximate the solution of a large class of options. Hence, it is assumed that the asset follows a geometric Brownian motion so that we can find Black-Scholes-Merton PDE (see Appendix B.1 for the derivation of this PDE).

The Black-Scholes PDE is augmented with a final time condition, and we will make the change of variables $\tau = T - t$ because convention dictates that problems should be specified in initial time condition form. Hence, τ represents the time to expiry and it is from T to 0 where t is from 0 to T . Thus, the Black-Scholes under this transformation is written as:

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV = 0. \quad (5.24)$$

The main focus is on the European call and put options, and let $C(S, t)$ denotes the European call option value and $P(S, t)$ be the put option value, whereby E represent the expected value, $t = T$ at expiry and the payoffs are respectively given as:

$$C(S, T) = \max(S(T) - E, 0). \quad (5.25)$$

$$P(S, T) = \max(E - S(T), 0). \quad (5.26)$$

Equations (5.25) and (5.26) are known as *final condition* because it applies at the final time $t = T$. If $\tau = 0$ then Equation (5.25) becomes

$$C(S, 0) = \max(S(0) - E, 0). \quad (5.27)$$

The corresponding put option in Equation (5.26) changes to

$$P(S, 0) = \max(E - S(0), 0). \quad (5.28)$$

When the boundary conditions are involved, then the European call and put use the PDE on the domain $S \in [0, \text{inf}]$. However, this range is difficult to use thus a finite set of options should be used as a range. In order to fix this, it will be truncated to the domain $S \in [0, L]$ where L presents a large value. By using the following equations

$$C(0, t) = 0, \quad \text{for all } 0 < t < T; \quad (5.29)$$

$$C(S, t) \approx S, \quad \text{for all large } S; \quad (5.30)$$

The call and put option's boundary conditions (payoffs) are respectively given as:

$$C(0, \tau) = 0, \quad \text{and} \quad C(L, \tau) = L. \quad (5.31)$$

$$P(0, t) = Ee^{-r(T-t)}, \quad \text{for all } 0 < t < T. \quad (5.32)$$

$$P(S, t) \approx 0, \quad \text{for all large } S. \quad (5.33)$$

$$P(0, \tau) = Ee^{-r\tau}, \quad \text{and} \quad P(L, \tau) = 0. \quad (5.34)$$

The finite difference grid $\{jh, ik\}_{j=0, i=0}^{N_x, N_t}$ can be used. Let the numerical solution at time level i be denoted by

$$\mathbf{V}^i = \begin{bmatrix} V_1^i \\ V_2^i \\ \vdots \\ \vdots \\ 0 \\ V_{N_x-1}^i \end{bmatrix} \in \mathbb{R}^{N_x-1}$$

whereby \mathbf{V}^0 is specified by the initial data in Equation (5.26) or (5.27) and the boundary values V_0^i and $V_{N_x}^i \forall 1 \leq i \leq N_t$ which are specified by the boundary conditions in Equation (5.30) or (5.33). The full central difference operator from the difference operator table for $\frac{\partial V}{\partial S}$ term is used to obtain a generalised FTCS for the PDE in Equation in (5.24). And then evaluate V term at (jh, ik) to get the following difference equation

$$\frac{V^{i+1} - V^i}{k} - \frac{1}{2}\sigma^2(jh)^2 \frac{(V_{j+1}^i - 2V_j^i + V_{j-1}^i)}{h^2} - rjh \left(\frac{V_{j+1}^i - V_{j-1}^i}{2h} \right) + rV_j^i = 0. \quad (5.35)$$

The corresponding generalised BTCS is:

$$\frac{V^{i+1} - V^i}{k} - \frac{1}{2}\sigma^2(jh)^2 \frac{(V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1})}{h^2} - rjh \left(\frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2h} \right) + rV_j^{i+1} = 0. \quad (5.36)$$

If we define the following, the matrix-vector representation of FTCS remains valid:

$$F = (1 - rk)I + \frac{1}{2}k\sigma^2 D_2 T_2 + \frac{1}{2}kr D_1 T_1 \quad (5.37)$$

and

$$\mathbf{p}^i = \begin{bmatrix} \frac{1}{2}k(\sigma^2 - r)V_0^i \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x - 1)(\sigma^2(N_x - 1) + r)V_{N_x}^i \end{bmatrix}$$

where

$$D_1 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 0 & \ddots & \vdots \\ \vdots & 0 & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & N_x - 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1^2 & 0 & \cdots & \cdots & 0 \\ 0 & 2^2 & 0 & \ddots & \vdots \\ \vdots & 0 & 3^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & (N_x - 1)^2 \end{bmatrix}$$

and

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ \vdots & \ddots & \ddots & 0 & 1 & -2 \end{bmatrix}.$$

Similarly, BTCS takes the following form:

$$B = (1 + rk)I - \frac{1}{2}k\sigma^2 D_2 T_2 - \frac{1}{2}kr D_1 T_1 \quad (5.38)$$

and

$$\mathbf{q}^i = \begin{bmatrix} \frac{1}{2}k(\sigma^2 - r)V_0^{i+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x - 1)(\sigma^2(N_x - 1) + r)V_{N_x}^{i+1} \end{bmatrix}.$$

The generalised Crank-Nicolson scheme is obtained from taking the average of the FTCS and BTCS formulae and is given by

$$(I + B)\mathbf{V}^{i+1} = (I + F)\mathbf{V}^i + (\mathbf{p}^i + \mathbf{q}^i). \quad (5.39)$$

5.3 Characteristics of the Finite Difference Methods

The fundamental concepts of the FDM are further discussed in (Duffy, 2006). We will first introduce the following notations:

Let the operator L denote PDE from some space of functions and consider a pricing equation given by:

$$\frac{\partial u}{\partial t} = Lu, \quad (5.40)$$

where

$$\boxed{Lu = \mu(x, t) \frac{\partial u}{\partial x} + \sigma(x, t) \frac{\partial^2 u}{\partial x^2} + a(x, t)u} \quad (5.41)$$

Additionally, the following examples are deduced from the above PDE:

Convection equation:

$$Lu \equiv \mu(x, t) \frac{\partial u}{\partial x} + a(x, t)u \quad (5.42)$$

Diffusion equation:

$$Lu \equiv \sigma(x, t) \frac{\partial^2 u}{\partial x^2} \quad (5.43)$$

Reaction-diffusion equation:

$$Lu \equiv \sigma(x, t) \frac{\partial^2 u}{\partial x^2} + a(x, t)u \quad (5.44)$$

Convection-diffusion equation:

$$Lu \equiv \mu(x, t) \frac{\partial u}{\partial x} + \sigma(x, t) \frac{\partial^2 u}{\partial x^2} + a(x, t)u \quad (5.45)$$

5.3.1 Stability

The stability of the finite difference algorithms is deduced from the effects of both space and time discretization schemes used when constructing the algorithms.

Given a general one-step initial value problem:

$$\boxed{\boldsymbol{\nu}^{n+1} = E\boldsymbol{\nu}^n, \quad n \geq 0} \quad (5.46)$$

where E is an operator.

By using a definition of stability (Appendix A.1), Equation (5.46) can be generalised to include an inhomogeneous term:

$$\boldsymbol{\nu}^{n+1} = E\boldsymbol{\nu}^n + k\mathbf{G}^n$$

The above difference scheme is said to be consistent with the PDE in Equation (5.41) if the solution of this PDE satisfies:

$$\mathbf{v}^{n+1} = E\mathbf{v}^n + k\mathbf{G}^n + k\boldsymbol{\tau}^n$$

where \mathbf{v} is the vector whose j th component is $u(x_j, t_n)$ and $\boldsymbol{\tau}^n$ is known as the truncation error.

The inhomogeneous equation is accurate of order (r, s) to the given PDE in Equation (5.41) if:

$$\|\boldsymbol{\tau}^n\| = O(h^r) + O(k^s) \quad (5.47)$$

where h and k are positive constants.

(Tavella and Randall, 2000) discuss the stability analysis using both matrix and fourier approaches. In the following example it is illustrated how the framework development (focusing on stability analysis: matrix approach) is used to analyse the

Crank-Nicolson algorithm.

The analysis of this algorithm starts with the time discretization part and then deduce from the λ_j complex plane to determine whether the scheme is stable or unstable. Whereby, λ_j is the eigenvalues determined by the space discretization scheme.

Example: 5.3.1

The Crank-Nicolson method is popular by its unconditional stability and second order accuracy. This method is constructed by combining the explicit and implicit schemes. Consider the ordinary differential equation in eigenvector space deduced from space discretization scheme:

$$\frac{\partial \nu}{\partial t} = \lambda \nu. \quad (5.48)$$

Equation (5.48) is approximated by the components of the explicit and implicit schemes combined:

$$\frac{1}{2} \left(\frac{d\nu}{dt} \Big|_n + \frac{d\nu}{dt} \Big|_{n+1} \right) = \frac{\nu^{n+1} - \nu^n}{\Delta t} \quad (5.49)$$

The above equation can be represented by:

$$\nu^{n+1} = \nu^n + \frac{1}{2} \lambda \Delta t (\nu^{n+1} + \nu^n) \quad (5.50)$$

which is equivalent to the following shift polynomial:

$$P(E)\nu^n = 0 \quad (5.51)$$

whereby the shift polynomial is given by:

$$P(E) = \left(1 - \frac{1}{2} \lambda \Delta t\right) - \left(1 + \frac{1}{2} \lambda \Delta t\right) \quad (5.52)$$

Thus, this method has one (r) root which can be expanded using Taylor expansion in terms of $\lambda \Delta t$:

$$\nu = \frac{\left(1 + \frac{1}{2} \lambda \Delta t\right)}{\left(1 - \frac{1}{2} \lambda \Delta t\right)} \quad (5.53)$$

$$\nu = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \frac{1}{4}(\lambda\Delta t)^3 + O(\lambda\Delta t)^4 \quad (5.54)$$

The requirement for the stability which is given by $|r| \leq 1$ is satisfied \forall values of $\lambda\Delta t$ from Equation (5.54) and this shows that this method is unconditional stable.

5.3.2 Consistency

Consider the following general initial value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + Lu &= F, & -\infty < x < \infty, & \quad t > 0 \\ u(x, 0) &= f(x), & -\infty < x < \infty. & \end{aligned}$$

Given the following finite difference scheme:

$$\begin{aligned} L_h^k u_j^n &= G_j^n \\ u_j^0 &= f(x_j) \end{aligned}$$

where $F(x_j, t_n)$ is approximated by G_j^n and L_h^k is a discrete approximation to L . Then the finite difference scheme is said to be point-wise consistent if the following relationship holds for any function $\nu = \nu(x, t)$:

$$\left(\frac{\partial \nu}{\partial t} + L\nu - F \right)_j^n - \left[L_n^k \nu(x_j, t_n) - G_j^n \right] \rightarrow 0 \quad \text{as } h, k \rightarrow 0 \quad \text{and } (x_j, t_{n+1}) \rightarrow (x, t) \quad (5.55)$$

And the above can be written as:

$$\boxed{\left(\frac{\partial}{\partial t} + L - L_k^n \right) \nu(x_j, t_n) + G_j^n - F_j^n = 0} \quad (5.56)$$

5.4 Summary

FDM (i.e. Explicit, Implicit and Crank-Nicolson methods) are widely used in solving complex financial problems and in problems whereby analytical solution cannot be obtained. Hence, this method is chosen to be used because of its accuracy and stability and thus the properties of this method was introduced and also the analysis of

the method's stability was determined by using a matrix approach.

In this work, this method is used to approximate the numerical solution of two parabolic equations namely, heat equation and Black-Scholes PDE. Therefore, the continuous partial derivatives in the Black-Scholes are replaced by discrete finite difference equations and it is accomplished by first discretizing the PDE in both time and space.

Additionally, this method can be used to approximate the payoff function and boundary conditions of the financial problems. These boundary conditions are discussed and there exist different boundary conditions when applying numerical methods, thus appropriate boundary conditions should be chosen carefully in order for the method to give an accurate solutions.

Chapter 6

Pricing CDS options using Crank-Nicholson method

CDS options are commonly priced using Black-Scholes formula and this method has its flaws, refer to (Wilmott, 2006) for the detailed discussion of the defects in the Black-Scholes Model. Due to its defects Crank-Nicolson scheme is used in approximating this PDE. As stated in chapter 1, this scheme is chosen based on its properties of being second-order accurate and it is also unconditional stable.

Since the Black-Scholes PDE is a parabolic PDE like the heat equation, it is simplified by transforming it into a form of heat equation and approximate its solution by Crank-Nicolson scheme. In order to transform the PDE, change of variables technique need to be applied. However, Crank-Nicolson can also be used to approximate the Black-Scholes PDE directly or approximate this PDE after transforming it into the heat equation form.

6.1 Transforming Black-Scholes PDE to the heat equation

Heat equation is introduced in Chapter 5 whereby this equation was solved using Finite Difference Methods and this equation and the Black-Scholes PDE are both prototype of a problem known as the *parabolic partial differential equations*. Thus, in this section this PDE is transformed to the heat equation and then solve by Crank-Nicolson scheme; whereby a model to price the CDS option is obtained in a discrete form instead in a continuous form. According to (Wilmott, 1998) the Black-Scholes PDE can be reduced to the heat equation using a suitable change of variables.

6.1.1 Transformation of the PDE

Lets recall the Black-Scholes PDE whereby its solution is approximated by Crank-Nicolson method after transforming it to the heat equation.

$$\frac{\partial F}{\partial t} + rX \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} = rF. \quad (6.1)$$

Since change of variables is applied to convert it to the initial time condition form, where $\tau = T - t$.

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} + rX \frac{\partial F}{\partial X} - rF. \quad (6.2)$$

Consider the following respective transformation of independent and dependent variables.

$$X = Ke^y \quad \text{and} \quad t = T - \frac{\tau}{\sigma^2/2}; \quad (6.3)$$

$$\nu(y, \tau) = \frac{1}{K}F(X, t) = \frac{1}{K}F(Ke^y, T; -\frac{\tau}{\sigma^2/2}) \quad (6.4)$$

where

$$F(X, t) = K\nu(y, \tau). \quad (6.5)$$

Differentiation of the PDE is performed by applying chain rule so that the changes of variables can give the following:

$$\begin{aligned} \frac{\partial F}{\partial t} &= K \frac{\partial \nu}{\partial \tau} \frac{\partial \tau}{\partial t} = -K \frac{\partial \nu}{\partial \tau} \frac{\sigma^2}{2} \\ \frac{\partial F}{\partial X} &= K \frac{\partial \nu}{\partial y} \frac{\partial y}{\partial X} = \frac{K}{X} \frac{\partial \nu}{\partial y} \\ \frac{\partial^2 F}{\partial X^2} &= \frac{\partial}{\partial X} \left(\frac{\partial F}{\partial X} \right) = \frac{\partial}{\partial X} \left(\frac{K}{X} \frac{\partial \nu}{\partial y} \right) = \frac{K}{X^2} \left(\frac{\partial^2 \nu}{\partial y^2} - \frac{\partial \nu}{\partial y} \right) \end{aligned}$$

We get the following after substituting the above equations into the Black-Scholes PDE:

$$\begin{aligned}
- K \frac{\sigma^2}{2} \nu_\tau &= \frac{1}{2} \sigma^2 X^2 \left(\frac{K}{X^2} \nu_{yy} - \frac{K}{X^2} \nu_y \right) + rX \left(\frac{K}{X} \right) \nu_y - r\nu \\
- K \frac{\sigma^2}{2} \nu_\tau &= \frac{1}{2} \sigma^2 K \nu_{yy} - \frac{1}{2} \sigma^2 K \nu_y + rK \nu_y - r\nu
\end{aligned}$$

After simplifying the above equation further, we get the following results:

$$\nu_\tau = \nu_{yy} + \left(\frac{r}{\sigma^2/2} - 1 \right) \nu_y - \frac{r}{\sigma^2/2} \nu \quad (6.6)$$

Let $Q = \frac{r}{\sigma^2/2}$, then the above equation becomes:

$$\nu_\tau = \nu_{yy} + (Q - 1) \nu_y - Q \nu \quad (6.7)$$

For the final transformation of the dependent variable ν , the following is defined:

$$\alpha = \frac{1}{2}(Q - 1) \quad \text{and} \quad \beta = \frac{1}{2}(Q + 1) \quad (6.8)$$

such that $\beta^2 = \alpha^2 + Q$.

The transformation in terms of new constants is defined as follows:

$$\nu(y, \tau) = Cu(y, \tau), \quad \forall(y, \tau) \quad (6.9)$$

where $C = e^{-\alpha y - \beta^2 \tau}$.

Hence, partial differential derivatives with respect to τ and y can be determined as:

$$\begin{aligned}
\nu_\tau &= \{-\beta^2 u + u_\tau\}C \\
\nu_y &= \{-\alpha u + u_y\}C \\
\nu_{yy} &= \{\alpha^2 - 2\alpha u_y + u_{yy}\}C
\end{aligned}$$

By substituting the above equations into Equation (6.7):

$$(-\beta^2 u + u_\tau)C = (\alpha^2 - 2\alpha u_y + u_{yy})C + (Q - 1)(-\alpha u + u_y)C \quad (6.10)$$

After simplifying Equation (6.10) we obtain:

$$u_\tau = u_{yy} + (-2\alpha + Q - 1)u_y + (2\alpha^2 - Q\alpha + \alpha)u \quad (6.11)$$

Then substitute $\alpha = \frac{1}{2}(Q - 1)$ to Equation (6.11).

$$\boxed{u_\tau = u_{yy} \Rightarrow \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}}. \quad (6.12)$$

therefore, the coefficients of terms u and u_y in Equation (6.10) vanishes when α is chosen to equal to $\frac{1}{2}(Q - 1)$. Equation (6.12) is the same as the dimensionless heat equation with the transformed dependent variable $u = u(y, \tau)$.

6.2 Solution for the parabolic PDE by the Crank-Nicolson method

In Chapter 5, the solution of heat equation and Black-Scholes parabolic PDE was discussed. Hence, in this section the continuous partial derivatives appearing in the heat equation and Black-Scholes PDE, are replaced by discrete difference equations using Crank-Nicolson method.

Heat equation and the Black-Scholes PDE can be approximated by Crank-Nicolson and this method is applied directly to Black-Scholes PDE to approximate its solution in *Matlab*[®]. Hence, the price of both the call and put swaptions are obtained from this implementation. These numerical solutions will be compared to that of the analytical solution obtained by Black-Scholes model and example from the article by (Tucker and Wei, 2005) will be used to carry out the comparisons.

The heat equation and Black-Scholes PDE in a form of difference equations and approximated by Crank-Nicolson scheme are respectively given as follows: (Higham, 2004), (Duffy, 2006) and (Haug, 2007).

$$2(1 + \nu)U_j^{i+1} = \nu U_{j+1}^{i+1} + \nu U_{j-1}^{i+1} + \nu U_{j+1}^i + 2(1 + \nu)U_j^i + \nu U_{j-1}^i. \quad (6.13)$$

System of linear equations can be solved (i.e. LU and QR factorization, Cholesky decomposition and other decomposition methods) in order to determine U^{i+1} from U^i , and this equations may be written as:

$$\hat{B}U^{i+1} = \hat{F}U^i + \mathbf{r}^i, \quad \text{for } 0 \leq i \leq N_t - 1, \quad (6.14)$$

where matrices \hat{B} and \hat{F} and vector \mathbf{r}^i are given in Chapter 5.

Continuous partial derivatives appearing in Black-Scholes PDE represented in difference equations are given by:

$$(I + B)\mathbf{V}^{i+1} = (I + F)\mathbf{V}^i + (\mathbf{p}^i + \mathbf{q}^i). \quad (6.15)$$

Equation (6.15) is the generalised Crank-Nicolson scheme and is obtained from taking the average of the FTCS and BTCS formulae, where B and F in the above equation denotes the equations for FTCS and BTCS respectively. These equations are included in Chapter 5.

In order to determine U^{i+1} appearing on the above equations, methods to solve a system of linear equation of the form $Ax = b$ are required. Since, matrices appearing from the above equations are tridiagonal, then less computational effort is used than when solving a full matrix \mathbf{A} of the same size.

LU factorisation is chosen to carry out the task because this factorisation method is used to solve linear system of equations efficiently, especially when it is applied to tridiagonal matrices. This method is introduced and discussed in the following section.

6.2.1 LU factorisation

According to (Fausset, 2007) and (Yang *et al.* 2005) LU factorisation is implemented by following the steps summarised below. In LU decomposition, a nonsingular matrix \mathbf{A} is expressed as the multiplication of a lower triangular matrix \mathbf{L} and upper triangular matrix \mathbf{U} such that $\mathbf{A} = \mathbf{LU}$.

By using LU factorisation of a tridiagonal matrix \mathbf{T} , less computational effort is used. Gaussian elimination or direct computation methods can be applied to find LU factorisation. Hence, in Gaussian elimination an upper and lower triangular matrices with 1's on the diagonal are constructed. Direct decomposition is more general, whereby LU factorisation is allowed to be unique.

Given a tridiagonal matrix \mathbf{T} and by following the basic Gaussian elimination process, at each step of the elimination there exist three vectors:

$$\mathbf{a} = [a_1 \ a_2 \ \dots \ a_{n-1} \ a_n] \quad (\text{the main diagonal}),$$

$$\mathbf{b} = [b_1 \ b_2 \ \dots \ b_{n-1} \ 0] \quad (\text{above diagonal})$$

$$\mathbf{c} = [0 \ c_2 \ \dots \ c_{n-1} \ c_n] \quad (\text{below diagonal})$$

For the first stage the multiplier used is: $m_{2,1} = -\frac{c_2}{d_1}$.

Then the new diagonal element is obtained: $D_2 = d_2 + b_1 m_{2,1}$.

Lastly, the observed transformation is expressed as $\mathbf{MT} = \mathbf{U}$:

where matrices \mathbf{M} , \mathbf{T} and \mathbf{U} are respectively given as:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} d_1 & b_1 & 0 \\ c_2 & d_2 & b_2 \\ 0 & c_3 & d_3 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} d_1 & b_1 & 0 \\ 0 & D_2 & b_2 \\ 0 & c_3 & d_3 \end{bmatrix}$$

Algorithm: Crank-Nicolson

1. Specify a region where approximation will be carried out: $H = (x, t) : s < x < p, t \geq 0$, given

$$h = \frac{p - s}{n + 1}, \quad x_0 = s, \quad x_i = x_0 + ih, \quad \text{where } i = 1, 2, \dots, n, \quad x_{n+1} = p$$

and

$$t_j = jk, \quad \text{where } j = 0, 1, \dots$$

2. Let $u_{i,j}$ be the values required to approximate the values of the solution at the points (x_i, t_j) , i.e. $u_{i,j} \approx u(x_i, t_j)$, where $i = 1, \dots, n$ and $j = 1, 2, \dots$

3. By using the initial and boundary values it can be stated that

$$u_{i,0} = g(x_i), \quad \text{with } i = 1, 2, \dots, n$$

$$u_{0,j} = f_1(t_j), \quad u_{n+1} = f_2(t_j), \quad j = 0, 1, 2, \dots$$

4. For u_{xx} approximation is made at the point midway between time levels, at the two time steps j and $j + 1$.
5. Discretize the PDE (heat or diffusion equation) in both time and space to obtain the following:

$$-\frac{q}{2}u_{i-1,j+1} + (1 + q)u_{i,j+1} - \frac{q}{2}u_{i+1,j+1} = \frac{q}{2}u_{i-1,j} + (1 - q)u_{i,j} + \frac{q}{2}u_{i+1,j}$$

where $q = \frac{k}{h^2}$ and $i = 1, 2, \dots, n$.

6.3 Numerical results

The example that illustrate the implementation of the Black-Scholes method to obtain the price of the CDS option is provided in Chapter 4. Therefore, the same example is used to approximate the solution of the Black-Scholes PDE using Crank-Nicolson method and price CDS option.

Table 6.1 shows the results obtained when Black-Scholes and Crank-Nicolson methods are used to find the price of CDS option for both call and put swaptions. Hence, in this example it is assumed that the maturity (T) of the CDS option is semi-annually with the 3-years CDS expiry, and the same process is followed for quarterly and annually maturities. The results for these expiry dates are illustrated in Tables 6.2 and 6.3 respectively.

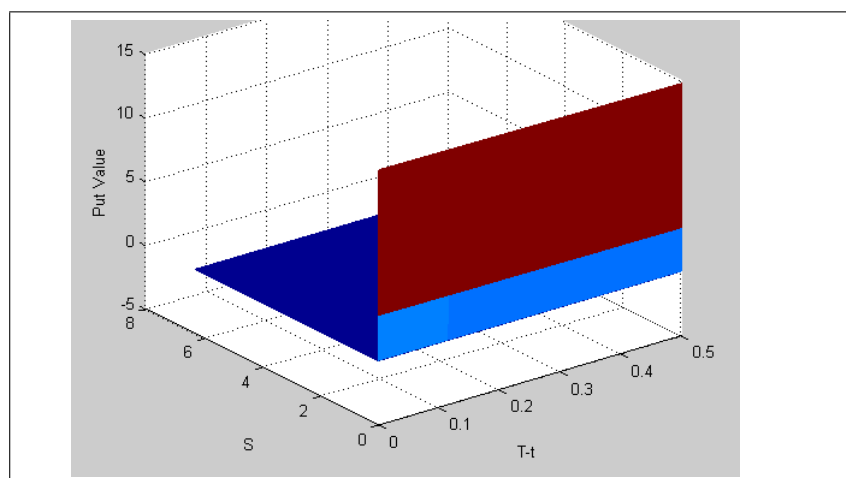
Thus, Figure 6.1, 6.2 illustrate the Crank-Nicolson method whereby they represent results in Table 6.1.

Table 6.1: Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 0.5$

Black-Scholes			Crank-Nicolson		$N_x = N_t(x_{max} = 600)$
	Call	Put	Call	Put	
$S = K (S = K = 0.015)$	0.0014	0.0012	0.0014	0.0012	550
$S > K (S = 0.021, K = 0.015)$	0.0169	0.0107	0.0165	0.0103	150
$S < K (0.015, K = 0.020)$	-2.2795e-004	0.0045	3.2593e-007	0.0047	100

Table 6.2: Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 0.25$

Black-Scholes			Crank-Nicolson		$N_x = N_t(x_{max} = 600)$
	Call	Put	Call	Put	
$S = K (S = K = 0.015)$	0.0010	9.099e-004	0.0010	8.914e-004	750
$S > K (S = 0.021, K = 0.015)$	0.0161	0.0100	0.0160	0.0099	200
$S < K (0.015, K = 0.020)$	-0.0010	0.0038	7.393e-005	0.0049	500

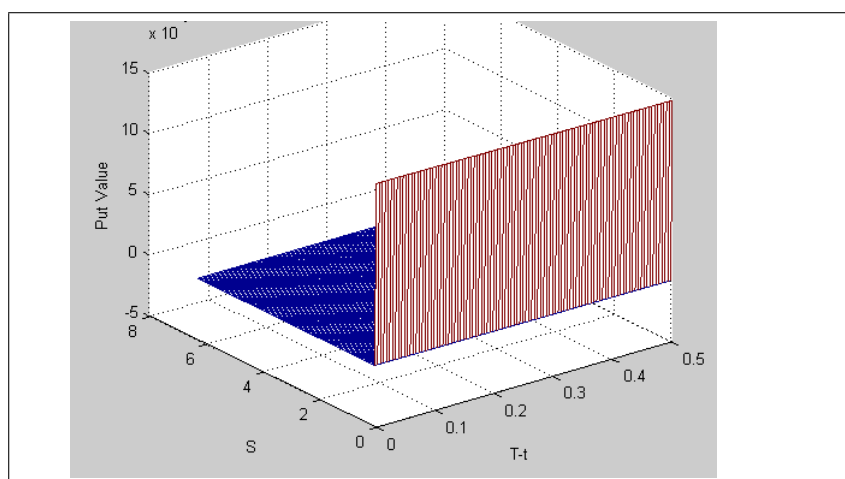
Figure 6.1: Crank-Nicolson method when $T=0.5$ and $S = K$

The prices of the CDS option are determined for the option when is assumed to be at-the-money ($S = K$), for CDS spread greater than the strike and for the CDS spread less than the strike. It is observed from the results in Table 6.1, that the solution obtained by Crank-Nicolson method converges to the solution obtained by Black-scholes method.

Additionally, it is also observed that when the CDS price exceeds the strike, the call

Table 6.3: Comparative table of CDS option prices using Black-Scholes and Crank-Nicolson methods, for $T = 1$

Black-Scholes			Crank-Nicolson		
	Call	Put	Call	Put	$N_x = N_t(x_{max} = 600)$
$S = K (S = K = 0.015)$	0.0020	0.0016	0.0021	0.0016	700
$S > K (S = 0.021, K = 0.015)$	0.0171	0.0106	0.0173	0.0109	150
$S < K (0.015, K = 0.020)$	1.6206e-005	0.0044	1.1622e-005	0.0044	300

Figure 6.2: Crank-Nicolson method when $T=0.5$ and $S > K$

swaption price is more than that of the put swaption.

But when the CDS spread is less than the strike, the price of the call swaption becomes less than the put swaption price. *Matlab*[®] codes for both Black-Scholes and Crank-Nicolson methods are provided in Appendix B.3.2 and Appendix D respectively.

6.4 Summary

Black-Scholes method is commonly used to price CDS option but since this method has its advantages based on its assumptions, it is preferred to replace continuous partial equations appearing on the Black-Scholes PDE by discrete difference equations, thus by applying numerical method (i.e FDM).

FDM became important in finance because in some cases, the option price problems cannot be solved in closed-form but can be solved by FDM especially when these problems are extended beyond standard assumptions.

Hence, Crank-Nicolson method is chosen to approximate the solution of the Black-

Scholes PDE because this method is known to be faster and more reliable than other FDM. It gives accurate results even when larger steps in time are chosen and it is also used because it is second-order accurate and unconditionally stable.

Heat equation is also known as diffusion equation in finance, is similar to the Black-Scholes method because both of them are parabolic PDE's. Thus, this similarity is illustrated by transforming the Black-Scholes PDE into a form of heat equation using a suitable change of variable technique.

Crank-Nicolson can be applied to the transformed Black-Scholes PDE or directly to the original Black-Scholes PDE to approximate the solution of this PDE.

This method is implemented in *Matlab*[®] to determine the price of the CDS option. In order to compare the results of this method to that obtained using Black-Scholes method, it is used an example from (Tucker and Wei, 2005) because they modelled the CDS option by Black-Scholes method.

Therefore, it is observed that for a CDS option maturing at different dates, the prices approximated by Crank-Nicolson method converges to CDS option's prices obtained using Black-Scholes method.

Chapter 7

Conclusion and recommendations

7.1 Conclusion

7.1.1 Swaps

Swaps play an important role in over-the-counter (OTC) derivatives and are defined as the contract between parties to exchange cash flows in the future. Furthermore, there exist different types of swaps in the OTC derivatives namely, interest rate swaps, currency swaps, CDS and others. Swaps have various applications, for example they can be used by parties or investors to convert the fixed-rate assets/liabilities into floating-rate assets/liabilities; and vice-versa so that investors can increase their income or decrease borrower's interest rates expenses.

As a result if two parties want to convert their existing assets/liabilities from the floating-rate into the fixed rate, they should first invest or borrow from their respective strongest side of the market, (i.e. floating-rate or fixed-rate side of the market). Then, they can use a swap to convert their respective financing to the type of financing they prefer.

If a borrower has a floating-rate debt and prefers to have a fixed-rate, because he speculates that interest rates will increase then he can enter into IRS as a fixed-rate payer. Hence, he will benefit from the swap if indeed interest rates rise, if not he will suffer a loss because interest rates movements went opposite to his speculations.

Moreover, in a swap contract there is a payer and a receiver whereby these parties play different roles in swaps: a payer swap is known as a fixed-rate payer and a float-rate receiver whereas for a receiver swap receives the fixed-rate and pays the

floating-rate. Credit derivatives are also important OTC derivatives but CDS is one of these derivatives that are widely used by banks or insurance companies for various reasons.

For instance, insurance company can use a CDS to generate profit through selling protection to the buyer that will protect him against a credit event. Hence, this can only be beneficial to the seller if the reference entity of the buyer does not default; because payments will be made by the buyer until the maturity of the CDS.

Similarly, if the reference entity default on its obligation before the maturity of the contract, then the insurance company will only get accrued payments from the buyer. However, the insurance company has to compensate the protection buyer for the losses suffered when the credit event occurred.

In addition, the insurance company sells protection speculating that the reference entity of the protection buyer will not default on its obligation; whereas the protection buyer has speculations opposite to those of the protection seller. The objective for the CDS is to sell a specific risk to the protection seller or to improve portfolio diversification and to also gain exposure to credits without actually buying assets.

7.1.2 Swaptions

Swaptions give the holder the right but not the obligation to enter into an underlying swap. Different types of swaptions are discussed in Chapter 3, whereby any buyer of these swaptions should only exercise the right granted if the underlying swap market value at maturity is positive. There is payer and receiver swaptions involved in all existing types of swaptions. Thus, in a CDS option a payer swaption is defined as the option that gives the holder the right but not the obligation to buy CDS at the agreed strike rate at maturity.

The option holder will be in a position of going short on the credit if he purchases the payer swaption and he will exercise this option if the CDS spread is greater than the strike rate. A payer swaption is similar to put option since investors have the right to sell credit risk on bonds at a higher price than market price at maturity. Furthermore, this swaption is similar to a call option if its viewed on spreads.

Similarly, in interest rate swaption, there exist payer and receiver swaptions; a payer swaption gives the holder the right to enter into an interest rate swap as a fixed-rate payer and a floating-rate receiver. A receiver swaption is the opposite of the payer swaption. Hence, a receiver swaption is similar to a call on a bond that pays a fixed

rate of interest.

A receiver swaption is attractive when market fixed-rates are anticipated to decline, because when buying a receiver swaption protection is obtained from receiving a lower fixed rate over the life of the swap. Therefore, the holder of the receiver swaption will exercise if the market fixed rate is less than the strike rate which is the fixed interest payments stipulated on the swaption contract.

Additionally, a payer swaption is equivalent to a put on a bond and the holder will only exercise this swaption when market fixed rates are expected to increase. Thus, protection will be obtained from paying higher fixed rate during the life of the swap when a payers swaption is purchased. This swaption will be exercised if the fixed rate (current market fixed rate) is greater than the strike rate stipulated on the contract.

7.1.3 Black-Scholes method to price swaptions

Black-Scholes model has a huge influence on the way traders price and hedge options and it is widely used to model prices of swaptions. In Chapter 4 this model is implemented to obtain the prices of interest rate swaptions, CDS options, commodity swaptions and energy swaptions. This model assumes that the percentages in the asset price are lognormally distributed, (Hull, 2000).

In order to determine the price of swaptions, appropriate boundary conditions are defined so that a function used to price swaptions can be determined from the Black-Scholes PDE. Hence, this function can be determined by applying a risk neutral valuation or martingale approach, (Bjork, 2009). This model was implemented in *Matlab*[®] to obtain the prices for the above mentioned swaptions.

Assumption underlying this model makes it easier for traders to apply it and its a successful model in financial engineering. Counterparty risk on the pricing of CDS options were not considered in this work, this implies that the risk of the protection seller to default is defined to be zero.

7.1.4 Finite Difference method to price CDS options

Numerical methods are mostly applied to challenging and complex financial problems that cannot be solved analytically to obtain the solution. In addition, these methods become an important component in obtaining prices of financial instruments numerically. Furthermore, they play a role in techniques available in practice for modern

quantitative finances.

FDM are used to approximate the prices of financial derivatives with formulations involving parabolic PDE. For instance, heat equation and Black-Scholes PDE are both parabolic equations and the solution of these equations are obtained by applying FDM. These numerical techniques are used because of their high-speed computational power at low cost and they are known to be accurate and stable.

Crank-Nicolson method is used to approximate the solution of the Black-Scholes PDE as stated in Chapter 6, whereby this method is chosen based on its properties of being second-order accurate and it is also unconditionally stable. Since, the Black-Scholes PDE and heat equation are the prototypes of parabolic equation, this PDE is simplified by transforming it into a form of heat equation using change of variables technique.

However, Crank-Nicolson can also be used to approximate the Black-Scholes PDE directly or approximate this PDE after transforming it into the heat equation form. This method approximates the solution of the Black-Scholes PDE by replacing the continuous derivatives appearing in this PDE with discrete difference equations. Hence, this PDE is discretized in both time and space; and it gives accurate results even when larger steps in time are chosen.

Crank-Nicolson method is applied directly to the original Black-Scholes PDE to approximate the solution of this PDE and it is implemented in *Matlab*[®] to determine the price of the CDS option. The example from (Tucker and Wei, 2005) is used to compare the results obtained using this method with those obtained using Black-Scholes method. Furthermore, this example is extended to determine the prices of CDS options with different maturities. It is observed that prices approximated by Crank-Nicolson method indeed converge to CDS option's prices obtained using Black-Scholes method.

7.2 Recommendations

The recommendations for future work related to numerical methods are stated in this chapter namely FDM whereby this method should be investigated further to clarify some uncertainties we came across during the course of this dissertation.

7.2.1 Interest rate swaps

Suppose there are two companies one with floating-liability and the other with fixed-liability and both of these companies want to convert their liabilities to the ones they prefer. In addition, these companies have different speculations about market interest rates in the future. For instance, the company with a floating-liability may speculate that interest rate may increase and prefers to have fixed-liability.

Therefore, these companies should first check from the market the side on which the company is the strongest and if this company is strong at the floating side, then it should invest or borrow from this side and later use a swap to convert their respective financing to the type of financing they prefer. Similarly, this applies to the company with fixed-liability but prefers a floating-liability.

7.2.2 CDS and CDS options

In this work, the impact of counterparty risk was not considered and practically CDS can be exposed not only to the credit risk but also to the counterparty risk that may arise from the protection seller. Thus, this risk is defined as the probability of default by the protection seller and on the valuation of CDS it is recommended that a default correlation framework should be considered in order to capture counterparty risk, (Pleus and Wasterfelt, 2012).

7.2.3 Crank-Nicolson to price CDS option

In financial literature it is observed that Crank-Nicolson is mostly used to price European options and not swaptions. Thus, in this work this method is chosen to approximate the solution of the Black-Scholes PDE so that the price of the CDS option can be obtained. This method is chosen because it is flexible, second order accurate and unconditional stable but remains uncertainties for practical application of this method to price CDS option.

Hence, it is recommended that Greeks for CDS options should be computed by FDM since Black-Scholes model assume that delta hedging is continuous whereas by (Wilmott, 2006) this is impossible practically and thus this makes Black-Scholes model inaccurate.

7.3 Closure

The aim of this dissertation was to approximate the solution of the Black-Scholes PDE by replacing continuous derivatives appearing in this PDE by discrete difference equation using Crank-Nicolson method. Thus, this method was implemented in *Matlab*[®] to approximate the price of the CDS options. Both prices of the CDS options by Black-Scholes and Crank-Nicolson were compared using an example from (Tucker and Wei, 2005), because prices of the CDS options on this paper was modelled by Black-Scholes model.

Based on this example, the solution of the PDE obtained by Crank-Nicolson to determine price of the CDS options converged to solution modelled using Black-Scholes model.

Appendix A

Mathematical tools

A.1 Definitions and theorems

Definition: 1 (Brownian motion)

Brownian motion is defined as a continuous-time stochastic process with a continuous state space and standard Brownian (also known as Wiener process) motion is the simplest form of Brownian motion whereby is obtained when $\mu = 0$ and $\sigma = 1$. (Hull, 2000)

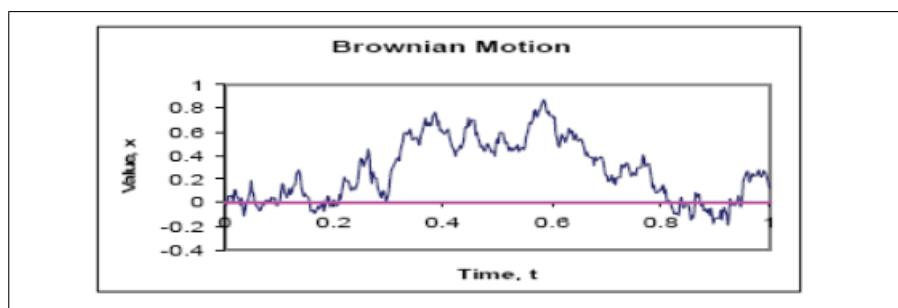


Figure A.1: A sample path of Brownian motion

Definition: 2 (Geometric Brownian motion)

Consider the following model:

$$S_t = e^{W_t}$$

,
whereby W is known as the Brownian process given by $W_t = W_0 + \sigma B_t + \mu t$. Furthermore, S_t is a geometric Brownian motion which is distributed lognormally with

mean $W_0 + \mu t$ and variance $\sigma^2 t$, (Hull, 2000)

Definition: 3 (Stochastic Differential Equation(SDE))

Let X_t be defined by the following equation:

$$X_t = X_{t-1} + \sigma Z_t$$

where Z_t is called *white noise* from the above equation and is a standard normal random variable. The above equation is a stochastic differential equation because it contains the difference $X_t - X_{t-1}$ and stochastic white noise terms, and its solution is as follows: (Bjork, 2009).

$$X_t = X_0 + \sigma \sum_{s=1}^t Z_s$$

Theorem: 4 (Itô's formula)

Let f be a $C^{1,2}$ -function and assume that the process X has a stochastic differential given by:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

where μ and σ are adapted processes and let the process Z be defined by $Z(t) = f(t, X(t))$. Thus, Z has a stochastic differential given as: (Bjork, 2009)

$$df(t, X(t)) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} + \sigma \frac{\partial f}{\partial x} dW(t).$$

Proposition: 5 (Feynman-Kac)

Let us assume that F is a solution to the following boundary value problem: (Bjork, 2009)

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \mu \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) &= 0, \\ F(T, x) &= \Phi(x). \end{aligned}$$

And also assume that the process $e^{-rs} \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)$ is in L^2 , where X satisfies the SDE:

$$\begin{aligned} dX_s &= \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_s &= x. \end{aligned}$$

Thus, F is represented as follows:

$$F(t, x) = e^{-r(T-t)} E_{t,x}[\Phi(X_T)].$$

Definition: 6 (Payer and receiver swap)

Consider a $T_N \times (T_N - T_n)$ swap and its payments are as follows: (Bjork, 2009).

- Payments will be payed and received at time $T_{n+1}, T_{n+2}, \dots, T_N$.
- The LIBOR rate $L_{i+1}(T_i)$ is set for every period $[T_i, T_{i+1}]$, where $i = n, \dots, N - 1$ and the following floating leg is received by the receiver swap at $T_{i=1}$:

$$\alpha_{i+1} \cdot L_{i+1}(T_i)$$

- And the following fixed leg is also payed at $T_{i=1}$:

$$\alpha_{i+1} \cdot K$$

- The arbitrage free value at $t \leq T_n$ of the total floating side and fixed side are respectively given as:

$$\sum_{i=n}^{N-1} [p(t, T_i) - p(t, T_{i+1})] = p_n(t) - p_N(t)$$

$$\sum_{i=n}^{N-1} p(t, T_{i+1}) \alpha_{i+1} K = K \sum_{i=n+1}^N \alpha_i p_i(t)$$

And the net value of the $T_N \times (T_N - T_n)$ payer swap at $t < T_n$ is:

$$PSN_n^N(t; K) = p_n(t) - p_N(t) - K \sum_{i=n+1}^N \alpha_i p_i(t)$$

Definition: 7 (Forward swap rate)

The forward swap rate ($R_n^N(t)$) of the $T_N \times (T_N - T_n)$ swap is the value of the strike rate K for which $\mathbf{PSN}_n^N(t; K) = 0$:

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{K \sum_{i=n+1}^N \alpha_i p_i(t)}.$$

Definition: 8 (Payer swaption)

A $T_N \times (T_N - T_n)$ payer swaption given the swaption strike K is defined as a contract that allows the holder the right but not the obligation to enter into a $T_N \times (T_N - T_n)$ swap by paying the fixed swap rate K at expiry (T_N), (Bjork, 2009).

Thus, we have:

$$X_n^N = \max \left[R_n^N(T_n) - K; 0 \right] S_n^N(T_n)$$

where $S_n^N(T_n)$ is known as the present value of a basis point.

Definition: 9

Lets say that the short rate is deterministic, then the forward and the future price processes coincide and is given by:

$$F(t, T, \chi) = E_{t,s}^Q[\chi]$$

,

where χ denotes the contingent claim.

Definition: 10 (Put-call parity)

We have the following two portfolios, whereby:

Portfolio A: one European call option plus a cash amount of (Ke^{-rT})

Portfolio B: one European call option plus one share

Thus, both of the above portfolios at the maturity of the options are worth: $\max(S_T, K)$. Since, the options are European the portfolios must have same values today at time

$t = 0$. Therefore, the following gives the relationship known as *put-call parity*: (Hull, 2000).

$$\boxed{C + Ke^{-rT} = P + S_0} \quad (\text{A.1})$$

Definition: 11 (Black's formula for Swaptions)

The price, at time t for a $T_N \times (T_N - T_n)$ payer swaption with exercise date T_N and strike K is defined as: (Bjork, 2009).

$$PSN_n^N(t) = S_n^N(t) \left\{ R_n^N(t)N(d_1) - KN(d_2) \right\},$$

where

$$d_1 = \frac{\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2(T_n - t)}{\sigma_{n,N}\sqrt{T_n - t}}$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t}.$$

Definition: 12 (Consistency of a numerical method)

A numerical method is consistent if the finite difference scheme converges to the solution of the PDE as the space and time steps goes to zero.

Definition: 13 (Stability of a numerical method)

If the difference between the numerical solution obtained using finite difference scheme and the exact solution from the analytical method remains bounded as the number of time steps is increased to infinity, then this numerical scheme is said to be stable.

Definition: 14 (Convergence of a numerical method)

Given the numerical solution and the exact solution at a fixed point, then a scheme converges as the difference between these solutions goes to zero uniformly as the space and time discretizations both go to zero.

Theorem: 15 (Lax Equivalence theorem)

Given a linear value problem and a consistent finite difference scheme, the convergence of these method to the exact solution will occur if its stable, therefore the stability is required in order for a numerical method to converge.

Appendix B

Black-Scholes model

B.1 Derivation of Black-Scholes PDE

Let the CDS spread follow the SDE given by: (Bjork, 2009) and (Ntwiga, 2005).

$$dX = \alpha X dt + \sigma X dW$$

where α is the drift term, σ is the diffusion term of the CDS spread and W is a Wiener process.

Let assume that f is a price of a CDS and it is a function of X and t . Hence, our problem is to determine f using Ito's formula:

$$df = \left[\frac{\partial f}{\partial t} + \alpha X \frac{\partial f}{\partial X} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} \right] dt + \frac{\partial f}{\partial X} \sigma X dW$$

.

At each time step, the Wiener process underlying f and X will be eliminated by choosing the following portfolio:

$$V(X(t), t) = -f(X(t), t) + \Delta(t) \times X(t)$$

.

whereby $V(X(t), t)$ is the value of the portfolio whereby an investor has short one CDS and the underlying credit and long an amount of $\frac{\partial f}{\partial X}$ of bonds.

The change (dV) in the portfolio value over the period $(t, t + dt)$ is given by:

$$\begin{aligned}
dV &= -df + \Delta dX \\
&= -\left(\frac{\partial f}{\partial t} + \alpha X \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2}\right)dt - \frac{\partial f}{\partial X}\sigma X dW + \Delta(\alpha X dt + \sigma X dW) \\
&= -\left(\frac{\partial f}{\partial t} + \alpha X \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} - \Delta\alpha X\right)dt + \left(-\frac{\partial f}{\partial X} + \Delta\right)\sigma X dW
\end{aligned}$$

The stochastic part is removed by choosing Δ such that:

$$\begin{aligned}
-\frac{\partial f}{\partial X} + \Delta &= 0 \\
\Delta &= \frac{\partial f}{\partial X}.
\end{aligned}$$

Then after substitution we get the following formula for (dV):

$$dV = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2}\right)dt$$

In order to avoid arbitrage opportunities and since the portfolio is risk-less due to the removal of the stochastic part, then the portfolio must earn a risk-free rate (r).

Therefore, we have:

$$\begin{aligned}
dV &= rV dt, \quad \text{where } r \text{ is the risk-free interest rate} \\
\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2}\right)dt &= r\left(-f + \frac{\partial f}{\partial X}\right)dt
\end{aligned}$$

Finally, we have the Black-Scholes PDE:

$$\boxed{\frac{\partial f}{\partial t} + rX \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} = rf}$$

Therefore, in order to solve the above PDE to determine the value of the CDS option, we need to specify appropriate boundary conditions.

The boundary conditions are:

$$\boxed{f(T_n, X(T_n)) = \max[X(T_n) - K, 0]S_n^N(T_n)}$$

$$\boxed{f(T_n, X(T_n)) = \max[K - X(T_n)]S_n^N(T_n)}$$

B.2 Solution of the Black-Scholes PDE to price CDS option

The Black-Scholes PDE will be solved by using the above given boundary conditions since these conditions are satisfied by the PDE. Hence, Black-Scholes PDE contains final time conditions and in order to solve this PDE, the convention should be made so that the problem is specified in initial time condition form. Let the change of variables $\tau = T - t$ be made whereby τ is the time to expiry, (Higham, 2004).

Then Black-Scholes PDE can be written as:

$$\frac{\partial f}{\partial \tau} - \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} - rX \frac{\partial f}{\partial X} + rf = 0,$$

$$\boxed{\frac{\partial f}{\partial \tau} = \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} + rX \frac{\partial f}{\partial X} - rf}$$

Assume that the CDS spread is lognormally distributed and let:

$$y = \ln X$$

By applying chain rule to differentiate the above equation with respect to X and y we get:

$$\frac{\partial f}{\partial X} = \frac{1}{X} \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial X^2} = -\frac{1}{X} \frac{\partial f}{\partial y} + \frac{1}{X^2} \frac{\partial^2 f}{\partial y^2}. \tag{B.1}$$

For simplicity let $\psi(y, \tau) = e^{r\tau} f(y, \tau)$ and by using equation B.1 Black-Scholes PDE then becomes:

$$\boxed{\frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} + \left[r - \frac{1}{2}\sigma^2 \right] \frac{\partial \psi}{\partial y}}. \tag{B.2}$$

Equation (B.2) has the solution given as follows:

$$\psi(y, \tau) = \int_{-\infty}^{\infty} \psi(\xi, 0) \kappa(y - \xi, \tau) d\xi. \tag{B.3}$$

Equation (B.3) is solved further in (Ntwiga, 2005) and the boundary conditions (pay-offs) are used in order to get the following equation:

$$\begin{aligned}\psi(y, \tau) &= \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^{\xi} \exp\left[-\frac{-(\xi + Q)^2}{2\sigma^2\tau}\right] d\xi \\ &\quad - \frac{K}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} \exp\left[-\frac{-(\xi + Q)^2}{2\sigma^2\tau}\right] d\xi\end{aligned}$$

whereby Q is given by:

$$\begin{aligned}Q &= y + \left(r - \frac{1}{2}\sigma^2\right)\tau \\ &= \ln X + \left(r - \frac{1}{2}\sigma^2\right)\tau.\end{aligned}$$

The final solution of (ψ) is: (see Ntwiga, 20005)

$$\psi(y, \tau) = e^{r\tau} X N\left[\frac{\ln\left(\frac{X}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right] - KN\left[\frac{\ln\left(\frac{X}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right], \quad (\text{B.4})$$

whereby gives the following price of the European call option with strike K and maturity T at time $t \in [0, T]$

$$\boxed{C^E = SN(d_1) - Ke^{-r(T-t)}N(d_2)}. \quad (\text{B.5})$$

And the following European put option is obtained by using put-call parity:

$$\boxed{P^E = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)}, \quad (\text{B.6})$$

whereby

$$\begin{aligned}d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}.\end{aligned}$$

We will determine the price of the CDS option by using the following proposition: 5 given at Appendix A.1 and from it we can write the contingent claim as follows: (Björk, 2009)

$$\chi = e^{r(T_1-T)} \max[S - e^{-r(T_1-T)}K, 0]$$

Thus, by denoting the futures CDS call swaption by C at time T and following the Black-formula we get:

$$C = \left[S e^{r(T_1-T)} N(d_1) - K e^{-r(T-t)} N(d_2) \right]$$

Finally, by substituting $S e^{r(T_1-T)} = F$ into the above equation, we get the same model derived by (Tucker and Wei, 2005) to value a European CDS call swaption which is given in chapter 4:

$$C = L e^{-r(T-t)} \left[F N(d_1) - K N(d_2) \right] \quad (\text{B.7})$$

whereby from (Tucker and Wei, 2005) we have the following value of a European CDS call swaption:

$$C = LA \left[R_0 N(d_1) - R_k N(d_2) \right] \quad (\text{B.8})$$

where $A = \left(\frac{1}{m}\right) \sum P(0, T_i)$ and L denotes the notional principal of the underlying CDS and (d_1) and (d_2) is respectively given by:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

And the corresponding model to value a European CDS put swaption is obtained from put-call parity:

$$P = L e^{-r(T-t)} \left[F N(d_1) - K N(d_2) \right] \quad (\text{B.9})$$

B.3 Black-Scholes to price European options and swaptions

B.3.1 Pricing put and call options

The following example computes the values of the put and call options and we are provided by the following information:

Stock price (S_T) is 4, with the strike price (K) of 4, the volatility of 30%, interest rate of 3% and the maturity of the option is 1 year. And the value of the European options computed using analytical Black-Scholes will be compared to value of the options that is computed by FDM (Explicit, Implicit and Crank-Nicolson) in the following sections using the same information.

```

function [C, P] = Black(S,E,r,sigma,tau)

% Input arguments: S = stock price at time t
% E = Exercise price
% r = interest rate
% sigma = volatility
% tau = time to expiry (T-t)
%
% Output arguments: C = call value, P = Put value
%\\
%
% Source:
%
% Higham, D. J. An Introduction to Financial Option Valuation: Mathematics, Stochastic
% and Computation. Cambridge University Press, 2004.
%\\

if tau > 0
d1 = (log(S/E) + (r + 0.5*sigma^2)*(tau))/(sigma*sqrt(tau));

d2 = d1 - sigma*sqrt(tau);

N1 = 0.5*(1+erf(d1/sqrt(2)));

N2 = 0.5*(1+erf(d2/sqrt(2)));

C = S*N1-E*exp(-r*(tau))*N2;

P = C + E*exp(-r*tau) - S;

else
C = max(S-E,0);

P = max(E-S,0);

```

end

An example of the function in use is

```
>> E = 4; S = 4; sigma = 0.3; r = 0.03; tau = 1;
>> [C, P] = Black(S,E,r,sigma,tau)
```

which outputs the following call and put prices respectively

C =

0.5313

P =

0.4131

B.3.2 Pricing put and call swaptions

Energy swaptions

The following programme determines the values of both call and put energy swaptions using Black-Scholes model and the example used here is discussed in chapter 4.

```
function [C, P] = bl(S,E,r,sigma,tau,T)

% Input arguments: S = swap price at time t
% E = Strike price
% r = risk-free interest rate
% sigma = volatility
% tau = time to expiry
% T = time from from now to the middle of delivery period
%
% Output arguments: C = call value, P = Put value
%
% S = 33; E = 35; r = 0.05; sigma = 0.18; tau = 0.5; T = 0.6260; \\Example
```

```

if tau > 0
d1 = (log(S/E) + (0.5*sigma^2)*(tau))/(sigma*sqrt(tau));

d2 = d1 - sigma*sqrt(tau);

N1 = 0.5*(1+erf(d1/sqrt(2)));

N2 = 0.5*(1+erf(d2/sqrt(2)));

C = exp(-r*T)*(S*N1 - E*N2);      %Computes energy call swaption

P = exp(-r*T)*(S*-N2 - E*-N1);    %Computes energy put swaption

else
C = max(S-E,0);

P = max(E-S,0);

end
end

```

CDS option

The following is the example of CDS options priced using Black-Scholes model and its discussed in details in chapter 4.

```

function [call_price, put_price] =blaCDS_european_call(R_0, R_k, r, sigma, T,L)

%-----
%
% DESCRIPTION:
%

```

```

% European put and call CDS options using Black-Scholes' formula
%
%
% Reference:
%
% Tucker, A.L. & Wei, J. "Credit default swaptions". The Journal Of Fixed Income
% (June 2005), 8891.
%
%-----
%
% INPUTS:
%
% R_0:    swap rate
% R_k:    strike rate
% r:      interest rate
% sigma:  volatility
% T:      time to maturity
%
%-----
%
% OUTPUT:
%
% call_price: price of a call CDS option
% put_price:  price of a put CDS option
%
%-----
%
%-----

d1 = (log(R_0/R_k) + (0.5*sigma^2)*(T))/(sigma*sqrt(T));

d2 = d1 - sigma*sqrt(T);

A = 0.5*(exp(-r*0.5) + exp(-r*1) + exp(-r*1.5) + exp(-r*2) + exp(-r*2.5) + exp(-r*3))

call_price = L*A*(R_0*normcdf(d1)-R_k*normcdf(d2));    %Computes the call CDS option
put_price  = L*A*(R_k*normcdf(-d2)- R_0*normcdf(-d1)); %Computes the put CDS option

```


Appendix C

Finite Difference Methods to price options

C.1 Explicit method

```
function price = BlScholesv2(r, K, T, S, sigma, Nx, Nt)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%  
%Explicit method to find the value of a european put option  
%using the Black-Scholes formula and the call is calculated  
%by using put-call parity.
```

```
%We are using the following parameters
```

```
%r = interest rate  
%K = Strike price  
%T = Time to maturity  
%S = Asset price  
%sigma = volatility  
%Nx = Number of steps in space  
%Nt = Number of steps in time
```

```
%Source: W. Henao. Put Option pricing. http://www.mathworks.com/matlabcentral/fileexchange/matlabcentral/fileexchange/contributed/matlabcentral/contributed/put-option-pricing-plain-vanilla/content  
%/BlScholescnv2.m, 2012. Date of Access September 27, 2012.
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%  
xmin = 0*S; xmax=2.5*S; %% spatial interval <-----It's a good idea to play with
```

```

L=xmax-xmin;
%% number of spatial discretization intervals
ds=L/Nx;
x=xmin:ds:xmax;
dt = T/Nt;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
v = zeros(Nx+1, Nt+1);
v(:,1)=[max(K-x',0)];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
alpha = sig^2*(xmin+ds:ds:xmax-ds)'.^2*(dt/(ds^2*2));
beta = (r-q)*(xmin+ds:ds:xmax-ds)'.*dt/(ds*2);
l = alpha - beta;
d = (1 - r*dt)*ones(Nx-1,1) - 2*alpha;
u = alpha + beta;
%This is what I change since the last program
A = spdiags([u d l], -1:1, Nx-1, Nx-1)';
A(1,1) = d(1) + 2 * l(1);
A(1,2) = u(1) - l(1);
A(Nx-1,Nx-2) = l(Nx - 1) - u(Nx - 1);
A(Nx-1,Nx-1) = d(Nx - 1) + 2 * u(Nx - 1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for i = 2:Nt+1
    v(2:Nx,i) = A*v(2:Nx,i-1);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
price = findprice(S, xmin, ds, v(:,Nt+1));
[parity] = put_call_parity(price,S,K,r,T)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function price = findprice(S0, S_min, dS, V)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function interpolates the price of the option to yield
%an approximated value and to correct any error in the
%discretization that may appear.

```

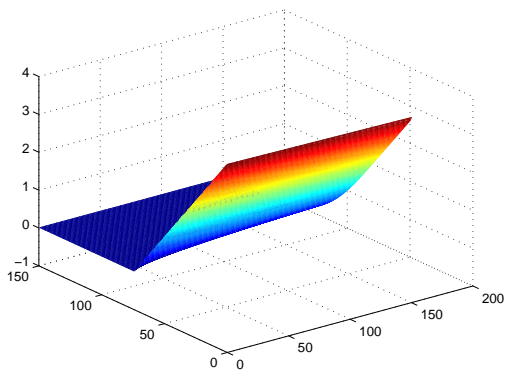
```

%%Source: W. Henao. Put Option pricing. http://www.mathworks.com/matlabcentral/fileex
%ut-option-pricing-plain-vanilla/content/BlScholescnv2.m, 2012. Date of
%Access September 27, 2012.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a = V(floor((S0-S_min)/dS)+1);
b = V(ceil((S0-S_min)/dS)+1);
price = a + ((S0-S_min)/dS - floor((S0-S_min)/dS))*(b-a);
    
```

Table C.1: Comparative table of option prices using numerical Explicit method

Method	European Call	European Put
Black-Scholes(Analytical)	0.5313	0.4131
Explici(10 intervals)	0.4923	0.3741
Explicit(15 intervals)	0.5172	0.3990
Explicit(30 intervals)	0.5290	0.4107

The following figure shows the put prices approximated by Explicit method.



C.2 Implicit method

```
function [price] = BlScholesimply2(r,K, T, S, sigma, Nx, Nt)
```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Implicit method to find the value of a european put option using the%
%Black-Scholes formula and the call is calculated by using put-call parity.
    
```

```

%We are using the following parameters

%r = interest rate
%K = Strike price
%T = Time to maturity
%S = Asset price
%sig = volatility
%Nx = Number of steps in space axis... the grid
%Nt = Number of steps in time axis

%Source: W. Henao. Put Option pricing. http://www.mathworks.com/matlabcentral/fileexchange/put-option-pricing-plain-vanilla/content
%/BlScholescnv2.m, 2012. Date of Access September 27, 2012.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
xmin = 0.1*S; xmax=2.5*S; %% spatial interval
L=xmax-xmin;

%% number of spatial discretization intervals
ds=L/Nx;
x=xmin:ds:xmax;
dt = T/Nt;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%Initial conditions%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
v = zeros(Nx+1, Nt+1);
v(:,1)=[max(K-x',0)];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Setting up the matrix A %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
alpha = sig^2*(xmin+ds:ds:xmax-ds)'.^2*(dt/(ds^2*2));
beta = (r-q)*(xmin+ds:ds:xmax-ds)'.*dt/(ds*2);
l = beta - alpha;
d = 2*ones(Nx-1,1)-(1 - r*dt)*ones(Nx-1,1) + 2*alpha;
u = -(alpha + beta);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%Changing the matrix A %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A = spdiags([u d l], -1:1, Nx-1, Nx-1)';
A(1,1) = d(1) + 2 * l(1);
A(1,2) = u(1) - l(1);

```

```

A(Nx-1,Nx-2) = l(Nx - 1) - u(Nx - 1);
A(Nx-1,Nx-1) = d(Nx - 1) + 2 * u(Nx - 1);

%%%%%%%%%%Calculating the price %%%%%%%%%%
for i = 2:Nt+1
    v(2,i-1) = v(2,i-1) - l(1)*v(1,i-1);
    v(Nx, i-1) = v(Nx, i-1) - u(Nx-1)*v(Nx+1, i-1);
    v(2:Nx,i) = inv(A)*v(2:Nx,i-1);
end

%%%%%%%%%% Final price %%%%%%%%%%

price = findprice(S, xmin, ds, v(:,Nt+1))
[parity] = put_call_parity(price,S,K,r,T)

%%%%%%%%%% findprice Function%%%%%%%%%%

function price = findprice(S0, S_min, dS, V)

%%%%%%%%%%
% This function interpolates the price of the option to yield
% a very approximated value and to correct any error in the
% discretization that may appear.

%%Source: W. Henao. Put Option pricing. http://www.mathworks.com/matlabcentral/fileexchange/ut-option-pricing-plain-vanilla/content
%/BlScholescnv2.m, 2012. Date of Access September 27, 2012.
%%%%%%%%%%

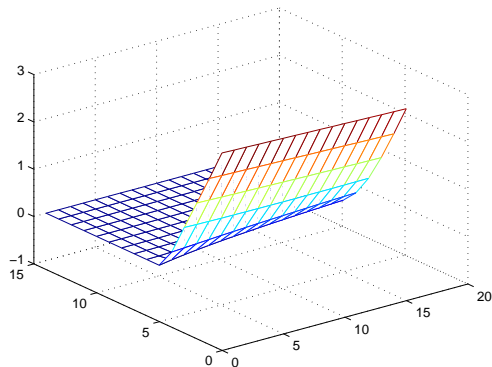
a = V(floor((S0-S_min)/dS)+1);
b = V(ceil((S0-S_min)/dS)+1);
price = a + ((S0-S_min)/dS - floor((S0-S_min)/dS))*(b-a);

```

The figure below illustrate the put prices approximated by Implicit method.

Table C.2: Comparative table of option prices using numerical Implicit method

Method	European Call	European Put
Black-Scholes (Analytical)	0.5313	0.4131
Implicit (10 intervals)	0.5500	0.4317
Implicit (15 intervals)	0.5433	0.4251
Implicit (30 intervals)	0.5312	0.4130



C.3 Crank-Nicolson method

```
%Crank-Nicolson to price a put option
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%Source:
```

```
% Higham, D. J. An Introduction to Financial Option Valuation: Mathematics, Stochastic  
% and Computation. Cambridge University Press, 2004.
```

```
%%%%%%%%% Problem and method parameters %%%%%%%%%
```

```
%Crank-Nicolson method to find the value of a European call and  
%put swaptions using the raw Black-Scholes formula, with parameters:
```

```
%r = interest rate
```

```
%E = Strike
```

```
%T = Time to maturity
```

```
%S = CDS spread
```

```
%sigma = volatility of the CDS spread
```

```
%Nx = Number of steps in space
```

```
%Nt = Number of steps in time
```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
E = 4; sigma = 0.3; r = 0.03; T = 1;
L = 10; Nx = 50; Nt = 50; k = T/Nt; h = L/Nx;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

T1 = diag(ones(Nx-2,1),1) - diag(ones(Nx-2,1),-1);
T2 = -2*eye(Nx-1,Nx-1) + diag(ones(Nx-2,1),1) + diag(ones(Nx-2,1),-1);
mvec = [1:Nx-1];

% Set up coefficients

D1 = diag(mvec);
D2 = diag(mvec.^2);

% Put: Payoff that is initial condition

F = (1-r*k)*eye(Nx-1,Nx-1) + 0.5*k*sigma^2*D2*T2 + 0.5*k*r*D1*T1;
B = (1+r*k)*eye(Nx-1,Nx-1) - 0.5*k*sigma^2*D2*T2 - 0.5*k*r*D1*T1;
A1 = 0.5*(eye(Nx-1,Nx-1) + F);
A2 = 0.5*(eye(Nx-1,Nx-1) + B);
U = zeros(Nx-1,Nt+1);
U(:,1) = max(E-[h:h:L-h]',0);

for i = 1:Nt
tau = (i-1)*k;
p1 = k*(0.5*sigma^2 - 0.5*r)*E*exp(-r*(tau));
q1 = k*(0.5*sigma^2 - 0.5*r)*E*exp(-r*(tau+k));
rhs = A1*U(:,i) + [0.5*(p1+q1); zeros(Nx-2,1)];
X = A2\rhs;
U(:,i+1) = X;

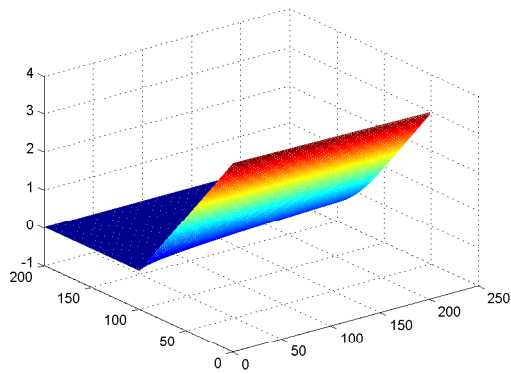
end
bca = E*exp(-r*[0:k:T]);
bcb = zeros(1,Nt+1);
U = [bca;U;bcb];
mesh([0:k:T],[0:h:L],U)
xlabel('T-t'), ylabel('S'), zlabel('Put Value')

```

Table C.3: Comparative table of option prices using numerical Crank-Nicolson method

Method	European Call	European Put
Black-Scholes (Analytical)	0.5313	0.4131
Crank-Nicolson (10 intervals)	0.5031	0.3849
Crank-Nicolson (100 intervals)	0.5311	0.4129
Crank-Nicolson (150 intervals)	0.5312	0.4130
Crank-Nicolson (200 intervals)	0.5313	0.4131

The following figure shows the put prices approximated by Crank-Nicolson method.



Appendix D

Crank-Nicolson method to price CDS options

```
function price = EuPutcrank(S,E,r,T,sigma,Nx,Nt,Notional)

%%%%%%%%% Problem and method parameters %%%%%%%%%%
%Crank-Nicolson method to find the value of a European call and
%put swaptions using the raw Black-Scholes formula, with parameters:
%r = interest rate
%E = Strike
%T = Time to maturity
%S = CDS spread
%sigma = volatility of the CDS spread
%Nx = Number of steps in space
%Nt = Number of steps in time
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xmin = 0*S; xmax=2*S; %% spatial interval
L=xmax-xmin;

h = L/Nx;
k = T/Nt;

T1 = diag(ones(Nx-2,1),1) - diag(ones(Nx-2,1),-1);

T2 = -2*eye(Nx-1,Nx-1) + diag(ones(Nx-2,1),1) + diag(ones(Nx-2,1),-1);
```

```

mvec = [1:Nx-1];
D1 = diag(mvec);

D2 = diag(mvec.^2);

F = (1-r*k)*eye(Nx-1,Nx-1) + 0.5*k*sigma^2*D2*T2 + 0.5*k*r*D1*T1;

B = (1+r*k)*eye(Nx-1,Nx-1) - 0.5*k*sigma^2*D2*T2 - 0.5*k*r*D1*T1;
A1 = 0.5*(eye(Nx-1,Nx-1) + F);

A2 = 0.5*(eye(Nx-1,Nx-1) + B);

U = zeros(Nx-1,Nt+1);

Svec = [h:h:L-h]';

U(:,1) = max(E-Svec,0);

for i = 1:Nt

    tau = (i-1)*k;
    p1 = k*(0.5*sigma^2 - 0.5*r)*E*exp(-r*(tau));
    q1 = k*(0.5*sigma^2 - 0.5*r)*E*exp(-r*(tau+k));
    rhs = A1*U(:,i) + [0.5*(p1+q1); zeros(Nx-2,1)];
    X = A2\rhs;
    U(:,i+1) = X;

end

bca = E*exp(-r*[0:k:T]);
bcb = zeros(1,Nt+1);
U = [bca;U;bcb];

price = findprice(S, xmin, h, U(:,i+1),Notional);
[parity] = put_call_parity(price,S,E,r,T)

```

The following is the function that interpolates the price of the CDS option to to give an approximated value and it also correct any error that may arose in the discretization.

```
function price = findprice(S0, S_min, dS, V,Notional)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author: Wilmer Henao - Columbia University
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Notional = 100*10^6;
a = V(floor((S0-S_min)/dS)+1);
b = V(ceil((S0-S_min)/dS)+1);
price = (a + ((S0-S_min)/dS - floor((S0-S_min)/dS))*(b-a))*Notional;
```

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