

INAUGURAL LECTURE

of

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Interpolating through history



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Interpolating through history

Abstract

Several events in the history of interpolation theory have had a crucial impact on the development of mathematics itself. In the present paper some of these events, ranging from the ancient Greek and Babylonian era to the late 20th century, are described in their historical framework, clarifying the interchange with and influence on other fields of mathematics.

1 Introduction

Mathematical interpolation developed from the early Babylonian and Greek mathematical traditions into a very diverse topic which at present can no longer be contained in one specific field of mathematics. Thorvald Thiele characterized interpolation in his book from 1909 as (translation taken from [37]):

“The art of reading between the lines in a numerical table.”

For a modern complex interpolation theorist such a description can almost sounds offensive, but it actually give a reasonable description what interpolation was before the 20th century. In its simplest form, namely linear interpolation, various measurements are connected by straight lines to get an approximation of the underlying system, much like the game connecting-the-dots most people did at some early stage of their live. In more advanced forms of interpolation the interpolants come from a broader class of functions with the possibility of more sophisticated interpolation conditions.

Several aspects keep reoccurring in the history of mathematical interpolation: New developments were often initiated by advancements in astronomy, and the lack of good notation and terminology as well as a limited understanding of the function-concept often restricted progress. On the other hand, interpolation theory has had his influence in other branches of mathematics, most notable in the development of calculus. Taylor series and approximation with Taylor polynomials originate from the interpolation formulas developed in 17th century by Newton, Gregory, Stirling, Gauss and many others.

This paper consists of five sections, not counting the present introduction, and can be divided into two parts. The first part consists of Sections 2, 3 and 4 and deals with the first occurrences of interpolation, predominately linear, during the ancient Greek and Babylonian era (Section 2), the first (documented) higher order interpolation methods (Section 3), which were found in India and China, and the maturation of (polynomial) interpolation during the European scientific revolution (Section 4) from the 16th to the 18th century, and extending somewhat into the 19th century. The remaining sections constitute the second part, which has a more specialized character. The proper development of the theory of functions opened up many new classes of functions suitable as interpolants, and the literature on interpolation from the 20th century is so overwhelming, that it is impossible to even give a remotely complete overview. The focus on the second part will henceforth be limited to a specific topic, namely that of metric constrained interpolation in complex analysis (Section 5) and its interplay with operator theory and system and control theory in the second half of the 20th century (Section 6). To get an idea of the diversity of topics, the account given here is completely disjoint from that in Section IV of [37] on research in interpolation theory in the 20th century related to signal and image processing.

2 Linear interpolation: Greece and Babylon

The first known documented evidence of mathematical interpolation goes back to the Greeks and Babylonians. This was predominantly linear interpolation, although a bit more complicated than our early childhood explorations. There are sources going back to about 200BC, most of which deal with prediction of astronomical phenomena: the positions of the sun, moon, stars and planets, and the various interrelations between these celestial bodies. Those predictions were used for the purpose of agricultural planning, naval navigation, calendar making and time reckoning. Astronomy in these times was in an early developing stage, and went from a “flat earth” model through a model with a cylindrical earth (Anaximander, ca. 610 BC–ca. 546 BC, [33]) to the geocentric sphere model of deferents and epicycles. Heliocentric models existed too (e.g., Aristarchus of Samos, 3th century BC, [27]), but these were not well received.

In the Greek tradition, astronomy was considered part of the mathematical sciences. The earlier work was purely geometrical in nature and aimed at a broad theoretical underpinning of the principles of the movement of celestial bodies, rather than on accurate prediction. The Babylonians on the other hand had a much longer tradition of making precise measurements, although nothing is known about the instruments that were used in making these measurements. The Babylonian approach was based more on arithmetic, working with numerical schemata with the aim of deriving accurate predictions for specific phenomena; there was much less an intension to develop an all encompassing theory to account for the astronomic observations [57, 58]. There are even some indications that the Babylonians used second order interpolation methods, i.e., non-linear interpolation, for instance in their attempts to compute the position of Jupiter [31]. Whether this really is the case, is not easy to prove since the work of the Babylonians is much less documented than that of the Greeks; only few of their clay tablets are preserved, often only partially. An additional complication is that the Babylonians kept their methods and computations on separate clay tablets. For their attempts to describe the position of Jupiter only the computations are preserved, while the tablet describing their methods appears to be lost.

Through Hipparchus of Rhodes (ca. 190 BC–ca. 120 BC), the arithmetic approach of the Babylonians was introduced into the Greek tradition, as argued by Toomer [55] and Jones [32]; there is no evidence that numerical exactness played any role in Greek astronomy before Hipparchus. Combined with the existing geometrical theory, Greek astronomy culminated into the geocentric epicycle model, perfected to its definite form by Claudius Ptolemy (ca. 90 AD–ca. 168 AD) in his book *Almagest* [56] from ca. 140 AD, which is now viewed as the first comprehensive treatise on astronomy. Earlier work of the Greeks developed theories for the moon, or a planet or the sun separately, but never combined into one all encompassing model. The geocentric epicycle model remained the dominant theory in Europe until the arrival of the heliocentric model of Copernicus in the 16th century.

The measurements in the days of Ptolemy were accurate enough to see that circular movements were to simple. As a correction, the celestial bodies revolved on a second circle (epicycle) that in turn revolved on the main circle (deferent). To make the model more accurate several epicycles were occasionally used. A second modification of the model was that the earth was positioned slightly off-center with respect to the deferent (eccentric deferent). Both corrections appear in the lunar model of the *Almagest*, which is given in Figure 1.

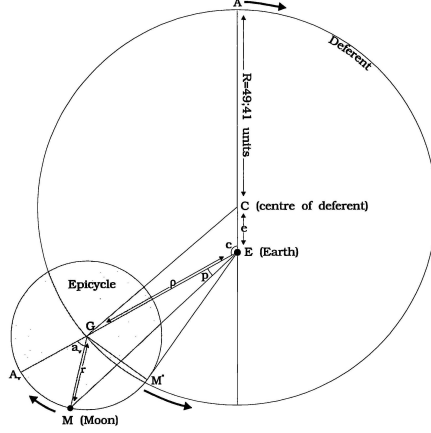


Figure 1: Ptolemy's lunar model (from [13])

The anomaly between the expected position of the moon on the deferent (G) and the actual position of the moon in the model (M) was measured with the angle those two positions make with the position of the earth (E), i.e., $\angle GEM$. Mathematically, this can be expressed as

$$\angle GEM = \sin^{-1} \left(\frac{r \sin a}{\sqrt{r^2 \sin^2 a + (\sqrt{R^2 - e^2 \sin^2 c} + e \cos c + r \cos a)^2}} \right), \quad (1)$$

in terms of the angle $c = \angle AEG$, with A the position on the deferent farthest removed from the Earth, and $a = \angle A_vGM$, with A_v the position on the epicycle farthest removed from the Earth at that specific moment; see [43, 13] for more details. An intriguing formula, but impossible to compute exact by hand, except maybe for a few values. Note also that the ancient Greeks did not have the notation to express it in such a concise form; there was no explicit concept of a function.

In order to make approximations of the anomaly, Ptolemy used an interpolation method in two variables, c and a in this case, as illustrated in Figure 2.

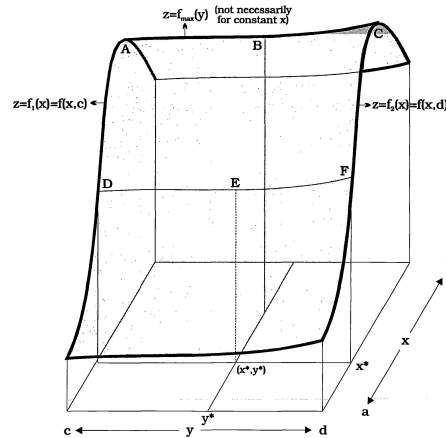


Figure 2: Ptolemy's interpolation technique (from [13])

Only values on the boundary of the domain were computed in this interpolation technique. Then points on the boundary were connected by straight lines, in the direction where the least variation is to be expected, leading to an approximation of the surface defined by the values of the left hand side of (1).

3 Higher order interpolation methods in India and China

Although there is some evidence suggesting that the Babylonians used second order interpolation techniques, the first methods of higher order interpolation of which written evidence is preserved, appeared in India and China. The earliest methods originate from the 7th century BC; at that time there was frequent contact between scientists from the two countries, c.f., Chapter 10 in [36], which might explain why these techniques developed almost simultaneous in both countries. Again, the methods were developed by astronomer-mathematicians, c.f., Chapter 4 in [45], in this case in attempts to approximate values of the trigonometric functions. The following values on the interval $[0, \pi/2]$ are known:

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
$\cos x$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	—

For any other values on $[0, \pi/2]$ one needs to make an approximation. Linear interpolation for $\sin x$ on $[0, \pi/2]$ yields the following approximation.

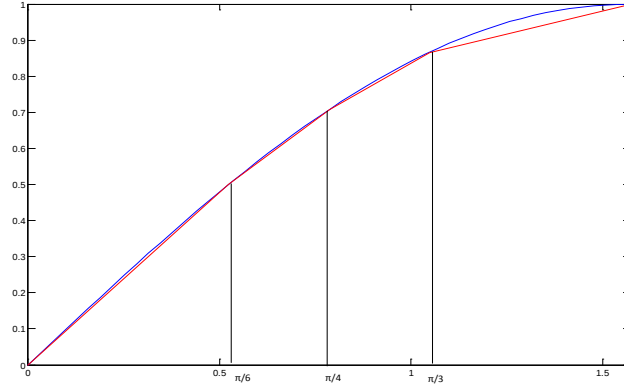


Figure 3: Linear interpolation of $\sin x$

Clearly the approximation is not particularly accurate, especially on the interval $[\pi/3, \pi/2]$.

Brahmagupta (ca. 598-ca. 670) was one of the most prominent 7th century mathematician and astronomer in India. He provided a first second order interpolation method in his book *Dhyāna-Graha-adhikāra*, which is undated, but appeared before his book *Brāhmasphuṭa Siddhānta* from 628 [25]. This method considered only interpolation with equal distance between the interpolation points, a more evolved interpolation method with intervals of unequal length was given by Brahmagupta in his book *Khandra Khādyaka* from 665. The equal length interval interpolation rule in *Dhyāna-Graha-adhikāra*, in Sanskrit, reads as:

गत भोग्य खण्डकान्तर दल विकल वधात् शतैर्नवभिराप्तैः ।
तद्युति दलं युतोर्न भोग्यादूनाधिकं भोग्यम् ॥

Figure 4: Brahmagupta's interpolation rule (from [25])

A translation to English is given in [25], and reads as:

“Multiply half the difference of the tabular differences crossed over and to be crossed over by the residual arc and divide by 900' (=h). By the result (so obtained) increased or decreased by half the sum of the same (two) differences, according as this (semi-sum) is less or greater than the difference to be crossed over. We get the true functional differences to be crossed over.”

In mathematical language this can be rephrased as:

$$f(x_0 + nh) = f(x_0) + \frac{n}{2}\{\Delta f(x_0) + \Delta f(x_0 - h)\} + \frac{n^2}{2}\{\Delta f(x_0) - \Delta f(x_0 - h)\},$$

where the finite difference $\Delta g(x)$ of a function g in the point x is given by

$$\Delta g(x) = g(x + h) - g(x).$$

In this finite difference form one can identify the interpolation rule of Brahmagupta as the equal length interval version of the second order Newton-Stirling formula [51] from 1719. Later Indian work also contained methods that, in modern mathematical language, render second order versions of the Newton-Gauss formulas, by Govindasvāmi (ca. 800–ca. 850), and of Taylor approximation, attributed to Mādhava (ca. 1350–ca. 1410), c.f., [8].

Interpolation methods in China developed simultaneously to those in India. However, they were always presented in the context of specific applications, often in calendar making, which makes it hard to point out the mathematical aspect of the method [36, Chapter 17]. Second order methods appeared in the work of Liu Zhuo (544–610), with constant step size, and Yi Xing (683–727) with non-constant step size. Later work by Guo Shoujing (1231–1316) hints at the use of cubic interpolation using finite differences. The Chinese textbook Siyuan Yujian (1303) by Zhu Shijie includes an exercise whose solution involves taking finite differences up to order 4:

“Soldiers are recruited in cubes. On the first day the size of the cube is three. On the following days, it is one more per day. At present, it is fifteen. Each soldier receives 250 guan per day. What is the number of soldiers recruited after n days and what is the total amount paid out?”

The solution given by Zhu Shijie, in modern mathematical notation, for the number of soldier $F(n)$ and salary paid $S(n)$ on day n can be expressed as:

$$\begin{aligned} F(n) &= \binom{n}{1}\Delta F(0) + \binom{n}{2}\Delta^2 F(0) + \binom{n}{3}\Delta^3 F(0) + \binom{n}{4}\Delta^4 F(0), \\ S(n) &= 250 \left(\binom{n+1}{2}\Delta F(0) + \binom{n+1}{3}\Delta^2 F(0) + \binom{n+1}{4}\Delta^3 F(0) + \binom{n+1}{5}\Delta^4 F(0) \right). \end{aligned}$$

Here $\binom{m}{k}$ denotes the binomial coefficient of m over k and the higher order finite differences $\Delta^n g(x)$ of a function g in the point x are computed recursively via

$$\Delta g(x) = g(x + 1) - g(x), \quad \Delta^{n+1} g(x) = \Delta(\Delta^n g(x)).$$

Phrased in this way, it looks very similar to the modern interpretation of Brahmagupta’s rule, but then with finite differences up to order four.

4 The scientific revolution in Europe

After Ptolemy, the way science was carried out on the European continent followed in the footprints set out in the Greek era, until the 16th century, in what is now called the European Scientific Revolution [26]. Although some dispute the term Scientific Revolution, there certainly was a rapid accumulation of developments in the natural sciences, with many scientific discoveries following one another, which shaped much of the way science is conducted nowadays. Some prominent events among these developments were the heliocentric theory of cosmology proposed by Nicolaus Copernicus (1473–1543) in his book *De Revolutionibus Orbium Coelestium* (1543), supported by the work of Kepler (1571–1630) and Galilei (1564–1642), the discovery of blood circulation by Andreas Vesalius (1514–1564), and the theories of magnetism and electricity, whose foundations were published by William Gilbert (1544–1603) in 1600, and of light as a wave (Christian Huygens (1629–1695) in 1678) or as a particle (Isaac Newton (1642–1727) in 1666). The inventions of the first mechanical calculator in 1642 by Blaise Pascal (1623–1662) and the single-lens microscope by Antonie

van Leeuwenhoek (1632–1723) were late discoveries during the scientific revolution, but with huge impacts on science in the centuries to come. The ‘scientific method’ initiated by the work of Francis Bacon (1561–1626) in *Novum Organum* (1620) and established in René Descartes’ *Discours de la Méthode* (1637) formalized the new way of science, which until the 16th century was based on the work of Aristotle.

Many of the developments in Mathematics were again initiated by discoveries in astronomy, which necessitated more and more accurate approximations of the values of the trigonometric functions and the logarithmic function. Interpolation turned out to be one of the main tools in making these approximations. See [24] for a detailed account of the development of various methods. Initial work in this direction was done by Sir Thomas Harriot (1560–1621) and Henry Briggs (1561–1630). Harriot did groundbreaking work on the calculus of finite differences, but left much of his work unpublished and as a result his contributions are often neglected. Briggs used several sophisticated interpolation methods to compute the values of the logarithm function in his *Arithmetica Logarithmica* (1624); this book contains the logarithms of the numbers from 1 to 20000 and from 90001 to 100000 approximated up to fourteen decimals. See [46] for a reconstruction of the tables in Briggs’ *Arithmetica Logarithmica*. Both Harriot and Briggs used higher order interpolation methods but they did not give proofs for their methods, nor did they explain the origins of their interpolation rules. Hence, it is not clear whether they would have been able to use methods of arbitrary order. This extensive tabulation of function values was common practice in these days, others like John Napier (1550–1617) and Johannes Kepler (1571–1630) also published lists of approximations of values of the trigonometric functions and the logarithm function.

In mathematics, part of the progress can probably be explained through the refinement of notation and terminology, which made it possible to obtain a much deeper understanding of many mathematical concepts. Franciscus Vieta (1540–1603) played an important role in this. He used the term ‘coefficient’ in the way we currently understand it and he introduced the concept of a parameter or variable. The notation improved along the way. The polynomial $1 + 3x^2 - 5x^3$ was written as $1 + 3(2) - 5(3)$ in Vieta’s time, Briggs’ would write $1 + 3\textcircled{2} - 5\textcircled{3}$, in Newton’s work we would find the notation $1 + 3xx - 5x^3$, using xx instead of x^2 , and with Leonard Euler (1707–1783) the mathematical notation had set to its current form, not just for polynomials.

While Harriot and Briggs used interpolation methods up to some specific order without providing a theoretical foundation, a few decades later the first general formulas started appearing, in the form that they are still used in today. The first of such formulas is attributed to James Gregory (1738–1675) and were presented in a letter he wrote in 1670. See [54] for a historical perspective on the interpolation formulas appearing in Gregory’s work. Newton first made reference to the interpolation methods he had developed, but at that stage not yet published, in a letter dated on October 24, 1676, which includes the following passage (translation taken from [23]):

“But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem. To describe a geometrical curve which shall pass through any given points.”

A little further he proceeds on the same topic:

“... but the above problem is of another kind; and although it may seem to be intractable at first sight, it is nevertheless quite the contrary. Perhaps indeed it is one of the prettiest problems that I can ever hope to solve.”

The first interpolation methods published by Newton appear in his book *Philosophiae Naturalis Principia Mathematica* (1687), without proof. *Principia* contains two interpolation formulas, one with equal interval length and one with variable interval length. The formula for the equal-length intervals coincides with the formula of Gregory, and reads

$$\begin{aligned} f(x_0 + \rho T) = & f(x_0) + \rho \Delta f(x_0) + \frac{1}{2} \rho(\rho - 1) \Delta^2 f(x_0) + \frac{1}{3} \rho(\rho - 1)(\rho - 2) \Delta^3 f(x_0) + \\ & + \frac{1}{4} \rho(\rho - 1)(\rho - 2)(\rho - 3) \Delta^4 f(x_0) + \dots \end{aligned}$$

with finite difference notation as before. It is now known as the Gregory-Newton formula. There is no evidence suggesting that Newton was aware of Gregory's work. In these centuries it was common for mathematicians to communicate their work in letters and only publish it years later, and sometimes decades later, occasionally making it hard to attribute results to a specific person. Several controversies resulted from this practice, among which claim by both Newton and Leibnitz of being the originator of calculus. Proofs and alternative formulae for the interpolation methods were provided in Newton's *Methodis Differentialis* (1711) and further expended on by James Stirling (1692–1770) in 1719 [51] and 1730 [52]. The Newton-Stirling formulas, of which Brahmagupta's rule is a special case, appear in these works.

Again decades later interpolation results of a different kind appeared, what is now refereed to as Lagrange interpolation, after Joseph-Louis Lagrange (1736–1813) although it was later found out that the formulas appeared sixteen years earlier in work by Waring. Lagrange had a high opinion on interpolation [35]:

“The method of interpolation is, after logarithms, the most useful discovery in calculus.”

The formula provided by Waring and Lagrange determines a polynomial, of minimal degree, that takes prescribed values at prescribed points, and does not use finite differences in any form. If the given (distinct) points are $x_1, \dots, x_N$ and the associated values $y_1, \dots, y_N$ then the polynomial

$$p(x) = \sum_{j=1}^N L_j(x) \quad \text{with} \quad L_j(z) = \prod_{i \neq j} \frac{z - x_i}{x_j - x_i} y_j$$

satisfies $p(x_i) = y_i$ for $i = 1, \dots, N$, indeed, one easily sees that $L_j(x_i) = 0$ whenever $i \neq j$ and $L_j(x_j) = y_j$ from which the claim follows immediately. It is interesting to note that for a long time the formula was considered a beautiful theoretical result but of little practical use, since it could be highly instable with the wrong choice of interpolation points. As F.S Acton states it in his book *Numerical Methods That Work* [1] from 1990:

“Lagrangian interpolation is praised for analytic utility and beauty but deplored for numerical practice.”

This was the common opinion, until Berrut and Trefethen [11] in 2004 introduced a specific way of choosing the interpolation points, referred to as Barycentric Lagrange Interpolation, which remedies the instability.

In current day calculus books, interpolation hardly plays a role, but, in fact, it had a huge impact on the development of calculus. The well known Taylor series formula is a logical extension of many of the results discussed above. Indeed, Taylor's formula appears from many of the interpolation formulas discussed above if one lets the degree of the approximating polynomial go to infinity and the interval length go to zero. In this way there is a direct line to the study of power series, i.e., functions that have an representation as an infinite polynomial $f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n$. In the early days of calculus the concept of a function still was not well understood. The trigonometric functions, exponential and logarithm functions and polynomials were known, but in solving general problems many mathematicians worked with functions in the form of power series, manipulating them algebraically, like polynomials, and not worrying about convergence of the series representation. As calculus developed, and along with it the function concept, interpolation shifted from polynomial interpolation to interpolation for other classes of functions. Already in 1821 Augustin-Louis Cauchy (1789–1857) worked on interpolation problems in the spirit of Lagrange but then for fractions of polynomials, i.e., rational functions. These functions also admit power series representations in all point except for the roots of the polynomial in the denominator. Interpolation for functions given by power series appeared in the context of complex analysis, which will be discussed in the next section.

5 Complex interpolation in the early 20th century

Although complex numbers first appeared in the 16th century formulas for the roots of polynomials of degree larger than three, for instance in the work of Rafael Bombelli (1526-1572) from 1572, it was still a long path

to the study of complex functions. The first occurrences of complex numbers, initially often referred to as “impossible numbers”, were only implicitly and for over a century their usage was only tolerated in a limited algebraic sense. Before the theory of complex functions was properly developed, complex numbers were used as variables, but this was often just with the aim of deriving concise identities. Several of these identities appeared in the correspondence between Gottfried Leibniz (1646–1716) and Johann Bernoulli (1667–1748) and later between Leonhard Euler (1707–1783) and Jean-Baptiste le Rond d’Alembert (1717–1783), in their discussions on the complex logarithm [14]. This started with Bernoulli’s 1702 identity

$$\frac{dx}{1+x^2} = \frac{dx}{2(1+x\sqrt{-1})} + \frac{dx}{2(1-x\sqrt{-1})},$$

which would now be written more concisely as $\tan^{-1} x = \frac{1}{2i} \log \frac{i-x}{i+x}$. In 1714 Roger Cotes (1682–1716) discovered the relation between the complex logarithm and the trigonometric functions given by $\log(\cos x + i \sin x) = ix$. This can be viewed as an early incarnation of Euler’s famous formula

$$e^{ix} = \cos x + i \sin x,$$

which he derived in a 1748 publication by comparing the power series expansions of e^{ix} , $\cos x$ and $\sin x$. With this identity the formula $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ obtained by Abraham de Moivre (1667–1754) in 1730 is a simple consequence of the exponential addition-multiplication formula $e^{v+w} = e^v e^w$, extended to complex numbers $v, w \in \mathbb{C}$.

In the next half century the fundamental theorem of algebra was proved. It was D’Alembert who came with a first attempt in 1746 and the by many considered first satisfactory proof was given by Carl Friedrich Gauss (1777–1855) in 1799, although both proofs would be considered incomplete in today’s rigorous mathematical standards. Two aspect were still missing, the geometric interpretation of the complex numbers via the identification of $x + iy$ with the point (x, y) in $\mathbb{R}^2$ and the notion of continuity. The geometric approach to $\mathbb{C}$ was first published in an anonymous pamphlet by Jean-Robert Argand (1768–1822) [12, p. 139] in 1806. The notion of continuity came with the rigorous approach to (complex) analysis initiated by Cauchy in his book *Cours d’Analyse* (1821) and later perfected in the geometric approach of Bernhard Riemann (1826–1866) and the arithmetic approach of Karl Weierstrass (1815–1897), see [12]. In *Cours d’Analyse* Cauchy sets out the first $\varepsilon - \delta$ proofs, and although his work was more focussed on complex integration and the residue-theorem, his name is also connected to complex differentiation, in the form of the Cauchy-Riemann equations. One of the main results in complex analysis states that a complex function f is complex differentiable if and only if when viewed as a function of two real variables, via the identification of $\mathbb{C}$ and $\mathbb{R}^2$, the real part f_1 and imaginary part f_2 of f , which are then real functions of two real variables, satisfy the partial differential equations

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x},$$

which are now known as the Cauchy-Riemann equations. This in turn is also equivalent to the existence of a power series representation of f and the fact that f can be differentiated infinitely many times. The Cauchy-Riemann equations were in fact already present in the work of D’Alembert, who observed that in several applications in hydrodynamics there were pairs of real functions P and Q of two variables that together formed a complex function $f(x + iy) = P(x, y) + iQ(x, y)$, i.e., only dependent on the quantity $x + iy$, and noted that they satisfies the Cauchy-Riemann equations. Complex functions that (locally) satisfy the Cauchy-Riemann equations and hence one of the other three equivalent conditions, are called analytic functions. With Weierstrass the focus in complex analysis was permanently shifted towards analytic functions, and with his results on uniform convergence of complex power series the time was ripe for complex interpolation, i.e., interpolation of analytic functions.

The theory of complex interpolation was initiated by a new generation of complex analysts, which included Georg Pick (1859–1942), Constantin Carathéodory (1873–1950), Issai Schur (1875–1941), Leopold Fejér (1880–1959) and Rolf Nevanlinna (1895–1980). They investigated interpolation problems for analytic self-maps of

the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, or of the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, as well as for analytic functions mapping $\mathbb{D}$ into $\mathbb{C}_+$, or the other way around. Since the domains $\mathbb{D}$ and $\mathbb{C}_+$ can be bijectively transferred into one another via a linear fractional transformation, it suffices to consider analytic self-maps of $\mathbb{D}$, and we will do so in the sequel, translating results to this context where necessary. Analytic self-maps of $\mathbb{D}$ are nowadays referred to as Schur functions, after the seminal work of Schur in [48, 49]; the Schur algorithm, and more generally Schur analysis, is currently still an active topic of extensive research [34].

Complex interpolation starts with the papers of Carathéodory [15] and Carathéodory and Fejér [16] on Schur functions of which the first values of the first few derivative at 0 are prescribed, in other words, the first part of the power series around 0 is given:

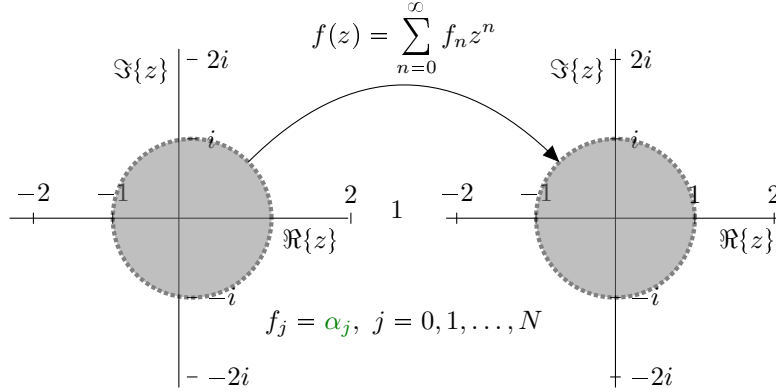


Figure 4: Carathéodory-Fejér interpolation.

The solution criterion obtained in [16] states that such a Schur function exists if and only if the $N \times N$ lower triangular matrix

$$\begin{bmatrix} \alpha_0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \alpha_N & \alpha_{N-1} & \cdots & \alpha_0 \end{bmatrix}$$

is contractive, that is, as a linear transformation it maps vectors of length 1 to vectors of length at most one.

Nevanlinna [41, 42] and Pick [44] worked independently on interpolation for Schur functions with a Lagrangian type of interpolation conditions, i.e, prescribed values $w_1, \dots, w_N$ at given distinct points $z_1, \dots, z_N$ in $\mathbb{D}$:

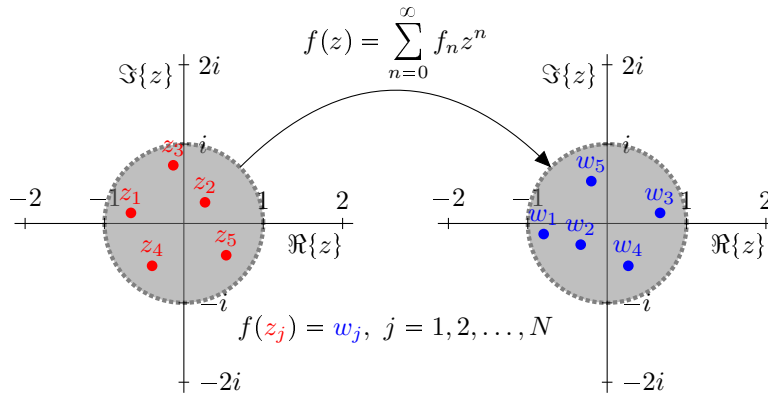


Figure 5: Nevanlinna-Pick interpolation.

Pick was the first to publish on this problem in [44] where he obtained the necessary and sufficient solution criterion that the Hermitian $N \times N$ matrix

$$P := \left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right]_{i,j=1}^N$$

be positive semidefinite, that is, all eigenvalues be nonnegative. The paper also contains the observation that there exists a unique solution in case the matrix P , nowadays referred to as the Pick-matrix [6], is singular and a description of this unique solution. Unaware of Pick's work, Nevanlinna worked on the problem in [41] which later resulted in a description of all solutions for the case that the matrix P is nonsingular in [42].

For both problems there exist explicit descriptions of the interpolants, but unlike for Lagrangian interpolation, as described above, it is much less transparent why the functions in the descriptions are actually interpolants. The reason for this is the constraint that the modulus of the values of the solutions f must be bounded by one. This additional metric constraint makes these types of interpolation problems hard to solve in general; without it there would always be a solution, even in the form of a complex polynomial.

6 Influences on complex interpolation in the second half of the 20th century

After the formative period (ca. 1910–ca. 1930) of complex interpolation, many variations on and combinations of Carathéodory-Fejér and Nevanlinna-Pick interpolation were studied. A new impetus came from the at first sight seemingly unrelated work of Zeev Nehari (1915–1978) on bilinear forms of Hankel type [40] from 1957. In this paper, given an infinite sequence $\alpha_0, \alpha_1, \alpha_2, \dots \in \mathbb{C}$ the question is when the bilinear form

$$A(a, b) = \sum_{\nu, \mu=0}^{\infty} \alpha_{\nu+\mu} a_{\nu} b_{\mu}, \quad a = (a_{\nu})_{\nu=0}^{\infty}, \quad b = (b_{\mu})_{\mu=0}^{\infty},$$

is bounded, that is, when does there exist an $M > 0$ such that $|A(a, b)| < M$ for $\sum_{\nu=0}^{\infty} |a_{\nu}|^2 = \sum_{\mu=0}^{\infty} |b_{\mu}|^2 = 1$. The answer presented by Nehari is that this happens if and only if the numbers $\alpha_0, \alpha_1, \alpha_2, \dots$ appear as the negatively indexed Fourier coefficients of a function in $L^{\infty}(\mathbb{T})$, the Banach space of essentially bounded measurable function on the unit circle $\mathbb{T}$ in $\mathbb{C}$. In this case one can take M to be the infimum of the essential supremum norms $\|f\|_{\infty}$ of all functions $f \in L^{\infty}(\mathbb{T})$ with this property; the result of Nehari also includes the observation that this infimum is in fact a minimum. Alternatively, the problem considered by Nehari can be rephrased as finding the minimal norm of a function in $L^{\infty}(\mathbb{T})$ with the prescribed negatively indexed Fourier coefficients, or, with some more work, as the problem of finding the best $\|\cdot\|_{\infty}$ -norm approximant of a function $f \in L^{\infty}(\mathbb{T})$ by one who has all negatively indexed Fourier coefficients equal to zero. The latter subset of $L^{\infty}(\mathbb{T})$ can be identified with the Banach space $H^{\infty}(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|\cdot\|_{\infty}$ over $\mathbb{D}$, through the almost everywhere existing nontangential boundary limits on $\mathbb{T}$.

The connection with the interpolation problems in the previous section is that the set of Schur functions is precisely the closed unit ball of the Banach space $H^{\infty}(\mathbb{D})$. However, with the infinite number of data in the Nehari problem, this problem could not be dealt with by matrix analysis tools, and operator theory methods entered into the picture. The solution criterion of what is now called the Nehari problem, but which is actually the reverse of the problem considered in [40], can be rephrased as saying that the ‘infinite structured matrix’

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

defines a bounded operator on ℓ_+^2 , the Hilbert space of square summable unilateral complex sequences, i.e., sequences $(a_\nu)_{\nu=0}^\infty$ with $a_\nu \in \mathbb{C}$ and $\sum_{\nu=0}^\infty |a_\nu|^2 < \infty$. Operators generated by such structured infinite matrices are called Hankel operators.

The many similarities, both in results and the applied techniques, to various metric constrained interpolation problems, initiated the work of Vadim Adam'yan (1938–), Damir Arov (1934–) and Mark Kreĭn (1907–1989) [2, 3, 4, 5] on a uniform approach, leading to general solution criteria and formulas for the solution. This approach made extensive use of Hankel operators and is still used in many branches of applied mathematics today, under the moniker of AAK-theory, cf., [7]. Around the same time Donald Sarason (1933–) observed that many of these interpolation problems can be interpreted as norm bounded extension problems of structured Hilbert space operators. This observation relies heavily on the fact that $H^\infty(\mathbb{D})$ is the multiplier algebra of the Hilbert space $H^2(\mathbb{D})$, the Hardy space of square integrable complex functions on $\mathbb{T}$ with an analytic extension to $\mathbb{D}$. With the structure encoded in a commutative relation on the operators, the solution obtained by Sarason in [47] became the first version on the commutant lifting theorem. Combined with unitary and isometric dilation theory, the commutant lifting theorem was perfected by Béla Szőkefalvi-Nagy (1913-1998) and Ciprian Foiaş (1933–) into its definitive form in the paper [38] and their monograph [39]. This result is one of the landmarks in operator theory, and the combination of the commutant lifting theorem with the techniques of Adam'yan, Arov and Kreĭn culminated in what is now known as commutant lifting theory [20, 17, 21].

In the late 1970s and early 1980s there was a renewed interest in interpolation theory, when it was observed that many control and electrical engineering problems, like optimal control, robust stabilization, sensitivity minimization, model matching and wideband disturbance attenuation, could be modeled as metric constrained interpolation problems; see the pioneering papers by George Zames (1934–1997) [59, 60] and Bill Helton (1944–) [28], and the lecture notes [22] of Bruce Francis (1947–) for further reference. This new impulse created a demand for non-scalar versions of the classical interpolation problems discussed in the previous section, as well as more explicit and better applicable formulas for their solutions [9]. A matrix-valued version of the Nehari problem, which was already solved as part of the Adam'yan-Arov-Kreĭn legacy, proved to be particularly useful in Francis' solution to the model-matching problem and the H^∞ disk method of Helton, which in turn could be used for certain control problems [30] and equalization in circuits [29]. New methods, better geared toward these ends, like the band method [19] of Harry Dym (1939–) and Israel Gohberg (1928–2009), were developed. As a form of cross-fertilization, state space formulas, known from engineering and the theory of characteristic operator functions, gained a prominent role in operator theory, and not only for solving interpolation problems, cf., [10].

Although the focus has shifted in various new directions since the 1980s, the connections with operator theory on the one side and system and control theory on the other side made in the second part of the 20th century are still very much present in today's research on metric constrained interpolation.

References

- [1] F.S. Acton, *Numerical methods that work*, Mathematical Association of America, Washington, DC, 1990.
- [2] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz. (Russian), *Funkcional. Anal. i Prilozhen* **2** (1968), no. 1, 1–19.
- [3] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, Infinite Hankel matrices and generalized Carathéodory-Fejér and I. Schur problems. (Russian), *Funkcional. Anal. i Prilozhen* **2** (1968), no. 4, 1–17.
- [4] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem (Russian), *Mat. Sb. (N.S.)* **86(128)** (1971), 34–75.
- [5] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, Infinite Hankel block matrices and related problems of extension. (Russian) *Izv. Akad. Nauk ArmjanSSR Ser. Mat.* **6** (1971), no. 2-3, 87-112.

- [6] J. Agler and J.E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics **44**. American Mathematical Society, Providence, RI, 2002.
- [7] F. Andersson, M. Carlsson, and M.V. de Hoop, Sparse approximation of functions using sums of exponentials and AAK theory, *Journal of Approximation Theory* **163**, no. 2 (2011), 213–248.
- [8] A.K. Bag, Madhava's sine and cosine series, *Indian Journal of History of Science* **11**, (1975), 54–57.
- [9] J.A. Ball, I. Gohberg, and L. Rodman, Interpolation of rational matrix functions, *Oper. Theory Adv. Appl.* **45**, Birkhäuser Verlag, Basel, 1990.
- [10] H. Bart, I. Gohberg, and M.A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, *Oper. Theory Adv. Appl.* **01**, Birkhäuser Verlag, Basel, 1979.
- [11] J.-P. Berrut and L.N. Trefethen, Barycentric Lagrange interpolation, *SIAM review* **46** (2004), 501–517.
- [12] U. Bottazzini, *The higher calculus: a history of real and complex analysis from Euler to Weierstrass*, Translated from the Italian by Warren Van Egmond, Springer-Verlag, New York, 1986.
- [13] G. van Brummelen, Lunar and planetary interpolation tables in Ptolemy's Almagest, *J. Hist. Astron.* **25**, no. 4, (1994), 297–311.
- [14] F. Cajori, History of the Exponential and Logarithmic Concepts, *Amer. Math. Monthly* **20**, No. 3 (1913), 75–84.
- [15] C. Carathodory, Über den variabilitätsbereich der koeffizienten von potenzreihen, die gegebene werte nicht annehmen, *Mathematische Annalen* **64**, no. 1 (1907), 95–115.
- [16] C. Caratheodory and L. Fejér, Über den zusammenhang der extremen von harmonischen funktionen mit ihren koeffizienten und über den Picard-Landauschen Satz, *Rendiconti del Circolo Matematico di Palermo* **32**, no. 1 (1911), 218–239.
- [17] T. Constantinescu, *Schur parameters, factorization and dilation problems*, *Oper. Theory Adv. Appl.* **82**, Birkhäuser Verlag, Basel, 1996.
- [18] C. Cullen, An eighth century Chinese table of tangents, *Chinese Science* **5** (1982), 1–33.
- [19] H. Dym and I. Gohberg, Extensions of kernels of Fredholm operators, *J. d'Analyse Math.* **42** (1982/83), 51–97.
- [20] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, *Oper. Theory Adv. Appl.* **44**, Birkhäuser Verlag, Basel, 1990.
- [21] C. Foias, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems* *Oper. Theory Adv. Appl.* **100**, Birkhäuser Verlag, Basel, 1998.
- [22] B.A. Francis, *A Course in H_∞ Control Theory*, Lecture Notes in Control and Information Sciences **88**, Springer-Verlag, Berlin, 1987.
- [23] D.C. Fraser, Newtons Interpolation Formulas, *Journal of thew Institute of Actuaries (1886–1994)* **51**, no. 2, (1918), 77–106.
- [24] H.H. Goldstine, *A history of numerical analysis from the 16th through the 19th century*, Studies in the History of Mathematics and Physical Sciences **2**, Springer-Verlag, New York-Heidelberg, 1977.
- [25] R.C. Gupta, Second order interpolation in Indian mathematics up to the fifteenth century, *Ind. J. Hist. Sci.* **4**, (1969), no. 1–2, 86–98.

- [26] A.R. Hall and G. Dunstan, *The scientific revolution, 1500-1800: the formation of the modern scientific attitude, Vol. 1*, Longmans, 1962.
- [27] T.L. Heath, *Aristarchus of Samos: The Greek Copernicus*, Oxford: Clarendon Press, 1913.
- [28] J.W. Helton, Operator theory, and broad band matching, *Proc. Allerton Conf. Circuits and Systems Theory* (1976), 91–98.
- [29] J.W. Helton, Broadbanding: gain equalization directly from data, *IEEE Trans. Circuits and Systems* **28** (1981), 1125–1137.
- [30] J.W. Helton, Worst case analysis in the frequency domain: the H^∞ approach to control, *IEEE Trans. Automat. Control* **30** (1985), 1154–1170.
- [31] J.P. Hogendijk, Babylonische astronomie: een vergeten hoofdstuk uit de geschiedenis van de wiskunde, In: *Kaleidoscoop van de wiskunde 1*, pp. 161–180, Epsilon Press, Utrecht, 1990.
- [32] A. Jones, The adaptation of Babylonian methods in Greek numerical astronomy, *Isis* **82.3**, (1991), 441–453.
- [33] C.H. Kahn, *Anaximander and the origins of Greek cosmology*, Hackett Publishing, 1994.
- [34] V.E. Katsnelson and B. Kirstein, 25 Years of Schur Analysis in Leipzig, *Complex Anal. Oper. Theory* **5**, (2011), 325–330.
- [35] J.-L. Lagrange, Memoire sur la Méthode d’Interpolation, *Nouveaux Mémoires de l’Académie Royale des Sciences et Belle-Lettres de Berlin*, annes 1792 & 1793.
- [36] J.-C. Martzloff, *A history of Chinese Mathematics*, Springer–Verlag, New York, 2006.
- [37] E. Meijering, A chronology of interpolation: From aincient astronomy to modern signal and image processing, *Proc. IEEE* **90**, No. 3, (2002), 319–342.
- [38] B. Sz.-Nagy and C. Foiaş, Dilation des commutants d’opérateurs, *C. R. Acad. Sci. Paris, Serie A*, **266** (1968), 493–495.
- [39] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland Publishing Co., Amsterdam-Budapest, 1970.
- [40] Z. Nehari, On bounded bilinear forms, *Ann. of Math. (2)* **65** (1957), 153–162.
- [41] R.H. Nevanlinna, Über beschränkte funktionen die in gegebenen punkten vorgeschriebene werte annehmen, *Ann. Acad. Schi. Fenn. Ser. A* **32**, 1919, no. 7.
- [42] R.H. Nevanlinna, Über beschränkte analytische funktionen, *Ann. Acad. Schi. Fenn. Ser. A* **32**, 1929, no. 1.
- [43] V.M. Petersen, The three lunar models of Ptolemy, *Centaurus* **14**, (1969), 142–171.
- [44] G. Pick, Über die beschränkungen analytischer funktionen, welche durch vorgegebene funktionswerte bewirkt werden, *Mathematische Annalen* **77**, no. 1 (1915), 7–23.
- [45] Kim Plofker, *Mathematics in India*, Princeton University Press, 2009.
- [46] D. Roegel, *A reconstruction of the tables of Briggs’ Arithmetica logorithmica*, LOCOMAT project, <http://locomat.loria.fr/>.
- [47] D. Sarason, Generalized interpolation in H^∞ , *Trans. Amer. Math. Soc.* **127** (1967), 179–203.

- [48] I. Schur, Über potenzreihen, die im innern des einheitskreises beschränkt sind, *Journal für die reine und angewandte Mathematik* **147** (1917), 205–232.
- [49] I. Schur, Über potenzreihen, die im innern des einheitskreises beschränkt sind, *Journal für die reine und angewandte Mathematik* **148** (1918), 122–145.
- [50] J. Stillwell, *Mathematics and its history. Second edition*, Undergraduate Texts in Mathematics, Springer–Verlag, New York, 2002.
- [51] J. Stirling, Methodus differentialis Newtoniana illustrata, *Philos. Trans.* **30**, no. 362, (1719), 1050–1070.
- [52] J. Stirling, *Methodus Differentialis sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*, London, U.K., 1730.
- [53] T.N. Thiele, *Interpolationsrechnung*, Leipzig, Germany: B.G. Teubner, 1909.
- [54] H.W. Turnbull, James Gregory: A study in the early history of interpolation, *Proc. Edinburgh Math. Soc.* **3** (1932), 151–172.
- [55] G.J. Toomer, Hipparchus and Babylonian astronomy, In: *A Scientific Humanist: Studies in Memory of Abraham Sachs*, Philadelphia: American Philosophical Society (1988), 353–62.
- [56] G.J. Toomer, *Ptolemy's Almagest*, Princeton University Press, 1998.
- [57] B.L. van der Waerden, Babylonian Astronomy II. The Thirty-Six Stars, *Journal of Near Eastern Studies* **8**, No. 1, (1949), 6–26.
- [58] B.L. van der Waerden, Babylonian Astronomy III. The Earliest Astronomical Computations, *Journal of Near Eastern Studies* **10**, No. 1, (1951), 20–34.
- [59] G. Zames, Optimal sensitivity and feedback: weighted seminorms, approximate inverses, and plant invariant schemes, *Proc. Allerton Conf. Circuits and Systems Theory* (1979).
- [60] G. Zames, Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, *IEEE Trans. Automat. Control* **26** (1981), 301–320.