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FINITE 2-GROUPS IN WHICH DISTINCT NONLINEAR IRREDUCIBLE CHARACTERS HAVE DISTINCT KERNELS

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Dedicated to Professor Jamshid Moori on the occasion of his seventieth birthday

Abstract. In this paper, we study finite 2-groups in which distinct nonlinear irreducible characters have distinct kernels. We prove several results concerning these groups and completely classify 2-groups with at most five nonlinear irreducible characters satisfying this property.

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Key words: Irreducible character kernels, CM-groups, CM_1-groups.

1. Introduction and preliminaries. Finite groups in which distinct ordinary representations have distinct kernels, were introduced by Zhmud in [12]. He has called these groups CM-groups and proved several results on these groups. For example, it is proved that any non-trivial CM-group is an extension of a 3-group by a nontrivial CM-2-group. Also CM-groups with a trivial center are completely classified (for proofs and more results on CM-group, see [12, 13] and [2, Chapter 9.3]). A natural generalization of this notion may be considered by concentrating only on nonlinear irreducible characters. In other words we have:

Definition 1.1. Let G be a finite group. Then we say that G is a CM_1-group if distinct *nonlinear* irreducible ordinary characters of G have distinct kernels.

It is convenient to assume that an abelian group is a CM_1-group. By [12, Theorem 2.4], abelian CM-groups are just elementary abelian 2-groups. So the family of CM_1-groups properly contains the family of CM-groups. Other examples of CM_1-groups are the family of groups with only one nonlinear irreducible character. An old paper of Seitz [10] asserts that a group with only one nonlinear irreducible character is either an extraspecial 2-group or a doubly transitive Frobenius group with abelian kernel and complement. Note that the epimorphic image of a CM_1-group is also a CM_1-group. As we mentioned above, if G is a CM-group, then

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$\pi(G) \subseteq \{2,3\}$, where $\pi(G)$ is the set of prime divisors of $|G|$. Seitz's groups show that this is not true for CM_1-groups in general. However, we will show in this section that CM_1-groups are solvable. Recall that G is a Q-group (respectively, Q_1-group) if all irreducible (respectively, nonlinear irreducible) characters of G are rational valued.

LEMMA 1.2. *Each* CM-*group* (CM_1-*group*) *is a* Q-*group* (Q_1-*group*).

Proof. Let χ be an arbitrary (nonlinear) irreducible character of G and assume that $\sigma \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$, where ϵ is a primitive root of unity. Hence $\ker\chi^\sigma = \ker\chi$. As G is a CM-group (CM_1-group), we conclude that $\chi^\sigma = \chi$. So χ is a rational valued character. □

LEMMA 1.3. *Let G be a finite group. Then the following are equivalent.*

1. *G is a* CM-*group.*
2. *G is both a* CM_1-*group and a Q-group.*
3. *G is a* CM_1-*group and G/G' is an elementary abelian* 2-*group.*

Proof. It is clear that (1) implies (2). Also if (2) holds, then G/G' is an abelian Q-group. But, abelian Q-groups are elementary abelian 2-groups. This proves (3). Finally, we show that (3) implies (1). Assume that (3) holds. Let χ_1 and χ_2 be arbitrary irreducible characters of G with $\ker\chi_1 = \ker\chi_2$. As G is a CM_1-group, we may assume that χ_1 and χ_2 are linear. So they may be viewed as irreducible characters of G/G'. Now G/G' is a CM-group and consequently we have $\chi_1 = \chi_2$. This completes the proof. □

COROLLARY 1.4. *Let G be a* CM_1-*group. Then G is solvable.*

Proof. Assume that G is a non-solvable CM_1-group. Then by Lemma 1.2, G is a Q_1-group. Now [4, Theorem 3.10], implies that G is a Q-group. Therefore by Lemma 1.3 we conclude that G is a CM-group. Since CM-groups are solvable, we produced a contradiction. □

If G is a CM_1-group, then according to the results of [4], $Z(G)$ is an elementary abelian 2-group. Also, nilpotent CM_1-groups are forced to be 2-groups. Throughout the rest of this paper, we only consider the nilpotent case and obtain many results concerning CM_1-2-groups. Specifically, we completely classify 2-groups with at most five nonlinear irreducible characters (surprisingly, we show that a CM_1-group can not have exactly five nonlinear irreducible characters). Throughout the rest of this paper, all groups are assumed to be finite 2-groups. The set of the irreducible character degrees and nonlinear irreducible character kernels of G are denoted by $\mathrm{cd}(G)$ and $\mathrm{Kern}(G)$, respectively. Also $c(G)$ is the nilpotency class of

G. Denote the set of the irreducible characters of G by $\mathrm{Irr}(G)$ and the set of the nonlinear irreducible characters of G by $\mathrm{Irr}_1(G)$. Note that if G is a CM_1-group, then $|\mathrm{Kern}(G)| = |\mathrm{Irr}_1(G)|$.

2. Some results on $\mathbf{CM_1}$-groups. In this section we prove some results on CM_1-2-groups. We start with the following lemma.

Lemma 2.1. *Let G be a CM_1-2-group. Then the following statements are equivalent.*

1. $|Z(G)| = 2$.
2. *$Z(G)$ is cyclic.*
3. $1 \in \mathrm{Kern}(G)$.
4. *G contains a faithful character of degree $\sqrt{|G|/2}$.*

Proof. By [7, Lemma 2.32] and the fact that the center of G is elementary abelian, we deduce that $(1), (2)$ and (3) are equivalent. Also (3) is an immediate consequence of (4). It remains to prove that (3) implies (4). Let t be the sum of the squares of the nonlinear non-faithful irreducible character degrees of G. We can write:

$$|G| = |G : G'| + t + \chi(1)^2,$$

where χ is the nonlinear faithful irreducible character of G. Also writing the same equality for $G/Z(G)$ we get

$$|G : Z(G)| = |G : G'| + t.$$

Combining the equalities, one gets $|G| = |G : Z(G)| + \chi(1)^2$. Since (3) implies (1), then $|Z(G)| = 2$ and we have $\chi(1)^2 = |G|/2$. □

Lemma 2.2. *Assume that $Z_k(G)$ is the kth term of the upper central series of the 2-group G. If for some $k \geq 2$ we have $|Z_k(G) : Z_{k-2}(G)| = 4$, then G is a not a CM_1-group.*

Proof. Replacing $G/Z_{k-2}(G)$ by G, we may assume that $|Z_2(G)| = 4$. Then $|Z(G)| = |Z(G/Z(G))| = 2$. If G is a CM_1-group, then by Lemma 2.1(4), both G and $G/Z(G)$ have non-square orders. This is clearly a contradiction. □

The following result is an immediate consequent of Lemma 2.2:

Corollary 2.3. *Let G be a 2-group of maximal class. Then G is a not a CM_1-group unless $|G| = 8$. In particular, Q_8 and D_8 are the only CM_1-groups among the families of Dihedral, Quaternion and semi-Dihedral 2-groups.*

Two subgroups H and K of G are said to be incident if either $H < K$ or $K \leq H$. If each pair of the elements of Kern(G) are non-incident, then following [3], we call G a J-group. By [3, page 252], a p-group G is a J-group if and only if G is of class 2 and G' is elementary abelian. So for CM_1-2-groups, we have the following lemma.

LEMMA 2.4. *The* CM_1-2-*group* G *is a* J-*group if and only if* G *is of nilpotency class* 2.

PROPOSITION 2.5. *The* 2-*group* G *is a* CM_1-*group if and only if for each nonlinear irreducible character* χ *of* G, *we have* $|G : \ker\chi| = 2\chi(1)^2$.

Proof. Let G be a CM_1-group and $\chi \in \mathrm{Irr}(G)$. Then, $G/\ker\chi$ is a CM_1-group with a cyclic center. Hence by Lemma 2.1, $\chi(1) = 1/2|G/\ker\chi|$. Conversely, suppose that χ_1 and χ_2 are distinct nonlinear irreducible characters of G with $\ker\chi_1 = \ker\chi_2$. Consider the group G/K where $K = \ker\chi_1$. We put $t = |G/K : (G/K)'|$ and we deduce

$$|G : K| \geq t + \chi_1(1)^2 + \chi_2(1)^2 = t + 2\chi_1(1)^2 = t + |G : K|,$$

which is a contradiction. □

As we mentioned in introduction, the center of a CM_1-group is an elementary abelian 2-group. Next lemma shows that the converse of this assertion holds for the family of groups with two extreme character degrees, that is, groups in which $\mathrm{cd(G)} = \{1, |G : Z(G)|^{1/2}\}$. These groups have been completely characterized in [6]. In particular it proved that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ if and only if every normal subgroup N of G not containing G' is central and $Z(G/N) = Z(G)/N$. We use this fact in proof of next lemma.

PROPOSITION 2.6. *Let* G *be a* 2-*group with* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. *Then* G *is a* CM_1-*group if and only if* $Z(G)$ *is elementary abelian.*

Proof. Assume that $Z(G)$ is elementary abelian. We use induction on $|G|$. If $|Z(G)| = 2$, then G must be an extraspecial 2-group. So we may assume that $|Z(G)| > 2$. Let χ_1, χ_2 be nonlinear irreducible characters of G with $K = \ker\chi_1 = \ker\chi_2$. Now G/K satisfies the hypothesis of the induction. Indeed $Z(G/K) = Z(G)/K$ is elementary abelian. Also note that $K \neq 1$ because $Z(G)$ is not cyclic. Therefore, G/K is a CM_1-group by induction, which implies that $\chi_1 = \chi_2$. This completes the proof. □

EXAMPLE 2.7. Let G be a p-group with $|G'| = p$. Then it is easy to see that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ and G has exactly $\frac{p-1}{p}\,|Z\,(G)|$ nonlinear irreducible characters. So 2-groups with $|G'| = 2$ and elementary abelian center are CM_1-groups with $|Z(G)|/2$ irreducible characters.

EXAMPLE 2.8. Another family of CM_1-groups are semi-extraspecial 2-groups (see [1]). These are groups in which for every maximal subgroup M of $Z(G)$, $Z(G/M)$ is extraspecial. According to [8, Lemma 5.4], G is a semi-extraspecial p-group if and only if $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ and $G' = Z(G)$. To see that these groups are CM_1-group, it suffices to show that G' is elementary abelian, which is the case by [6, Theorem D]. It is also easy to see that if G a semi-extraspecial 2-group, then it contains exactly $|Z(G)| - 1$ irreducible characters.

3. $\mathbf{CM_1}$-groups with few nonlinear irreducible characters.

In this section we completely classify CM_1-groups with at most five nonlinear irreducible characters. Throughout the rest of this paper, we use the following notation. A nonlinear irreducible character kernel of G is said to be a $\mathcal{K}$-maximal if it is a maximal element of $\mathrm{Kern}(G)$, with respect to inclusion. Our main result is the following:

THEOREM 3.1. *Let G be a CM_1-2-group and let $t \leq 5$ be the number of the nonlinear irreducible characters of G. Then $t \leq 4$ and one of the following cases occur:*

- $t = 1$ *if and only if G is an extraspecial* 2*-group.*
- $t = 2$ *if and only if $Z(G)$ is is elementary abelian of order* 4 *and* $|G'| = 2$.
- $t = 3$ *if and only if G is either a* 2*-group with $Z(G) = G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and all normal subgroups of G not containing G' are central or G is a group of order* 32 *with* $|Z(G)| = 2$ *and* $|G'| = 4$.
- $t = 4$ *if and only if $Z(G)$ is elementary abelian of order* 8 *and* $|G'| = 2$.

In [5], Doostie and the author classified finite p-groups with at most three nonlinear irreducible character kernels. Using the results of [5], we can easily complete our classification for $t \leq 3$. To see this, observe that if $t = 1$, then by [5] we have $|G'| = 2$ and $Z(G)$ is cyclic. Since the center of a CM_1-group is elementary abelian, we conclude that $|Z(G)| = 2$. That is, G is an extraspecial 2-group. A similar argument works for $t = 2$ with the additional observation that by Corollary 2.3, G can not be of maximal class. Next assume that $t = 3$. According to [5], six types of groups $(a) - (f)$ satisfy this property. Since we only work with 2-groups with elementary abelian center which are not of maximal class, groups of type $(a), (c)$ and (e) may be ignored. Now it suffices to see that according to Lemma 3.2 below, groups of type (b) and (f) coincide.

LEMMA 3.2. ([6, Lemma 2.5]) *For a p-group G, if $|(G/Z(G))'| = p$, then $|G : Z_2(G)| = p^2$.*

LEMMA 3.3. ([5, Lemma 2.1]) *Let H be a non-abelian finite group. Then,* $\bigcap_{K \in \mathrm{Kern}(H)} K = 1$.

LEMMA 3.4. *Let G be a non-abelian p-group. Assume that the nonlinear irreducible character kernels of G constitute a chain with respect to inclusion. Then G is not a CM_1-group, unless* $\mathrm{Kern}(G) = \{1\}$.

Proof. By the main theorem of [9], G is of maximal class or G' is a unique minimal normal subgroup of G. In both cases, we must have $\mathrm{Kern}(G) = \{1\}$. □

The following lemma may be obtained using the proof of the main theorem of [5]

LEMMA 3.5. *Let G be a 2-group with 2 or 3 nonlinear irreducible character kernels. If G is of class 2 then all irreducible character kernels are of order 2. Also if G is of class 3 then $|G| = 32$, $|Z(G)| = 2$ and G contains two irreducible character kernels of order 4.*

LEMMA 3.6. *Let G be a CM_1-2-group with $|\mathrm{cd}(G)| = 2$. If K is a nonlinear irreducible character kernel of G, then*

$$|\mathrm{Kern}(G)| = 2\frac{|K|}{|G'|}(|G'| - 1).$$

Proof. Let $t = |\mathrm{Kern}(G)|$. By Proposition 2.5 we have:

$$|G| = |G : G'| + t\frac{|G|}{2|K|}.$$

Therefore, $t = 2\frac{|K|}{|G'|}(|G'| - 1)$. □

REMARK 3.7. Assume that G is a CM_1-2-group with t nonlinear irreducible characters. Then, we may verify the followings by GAP [11]:

- If $|G| = 32$, then $t \in \{1, 3, 4, 6\}$. Moreover if $t = 4$, then $|G'| = 2$.
- If $|G| = 64$, then $t \in \{2, 3, 6, 8, 9, 12\}$.

PROPOSITION 3.8. *Let G be a CM_1-2-group with 4 nonlinear irreducible characters. Then $|G'| = 2$.*

Proof. Let $\mathrm{Kern}(G) = \{K_1, K_2, K_3, K_4\}$. We consider the following cases.

Case 1. $1 \in \mathrm{Kern}(G)$.

By Lemma 2.1, $|Z(G)| = 2$. Also $\overline{G} = G/Z(G)$ has three nonlinear irreducible characters. So by Lemma 3.5, we conclude that all non-trivial nonlinear irreducible character kernelss of $\overline{G}$ are of order 2. Thus by Lemma 3.6 we get $\left|\overline{G'}\right| = 1$. That is, $|G'| = 2$.

Case 2. $1 \notin \mathrm{Kern}(G)$ and $c(G) > 2$.

Since $c(G) > 2$, at least one element of $\mathrm{Kern}(G)$, say K_1 is not a $\mathcal{K}$-maximal element. By Lemma 3.3 and Lemma 3.4, K_1 must be contained in exactly two elements of $\mathrm{Kern}(G)$, say K_2, K_3. Now G/K_1 is of class 3 with three nonlinear irreducible character kernels. We get $|G/K_3| = 32$. Hence $|G| = 64$, which is a contradiction by Remark 3.7. So this case may not happen.

Case 3. $1 \notin \mathrm{Kern}(G)$ and $c(G) = 2$.

By Lemma 2.4, all elements of $\mathrm{Kern}(G)$ are $\mathcal{K}$-maximal. Let N be a minimal normal subgroup of G. By Lemma 3.3, G/N has at most three nonlinear irreducible character kernels. So by Lemma 3.5, G has 2 or 3 character kernels of order 4. By Lemma 2.5, G can not simultaneously contain character kernels of order 2 and 4. Hence all character kernels of G are of order 4. Now Lemma 2.5 implies that $|\mathrm{cd}(G)| = 2$. Therefore by Lemma 3.6 we have $|G'| = 2$. □

To complete the proof of Theorem 3.1, we only need to prove the following proposition.

PROPOSITION 3.9. *Let G be a* CM_1*-2-group. Then* $|\mathrm{Kern}(G)| \neq 5$.

Proof. Assume by contradiction that $|\mathrm{Kern}(G)| = 5$. First suppose that $c(G) > 2$. Hence, we may choose a non-$\mathcal{K}$-maximal element $K \in \mathrm{Kern}(G)$. Then K is properly contained in exactly s elements of $\mathrm{Kern}(G)$, for a positive integer s. By Lemma 3.4, $s \neq 1$. Also if $s = 2$, then a similar argument to the proof of Proposition 3.8 shows that $|G| = 32$, which is impossible by Remark 3.7. If $s = 3$ then $c(G/K) > 2$ and $|\mathrm{Kern}(G/K)| = 4$, contradicting Proposition 3.8. Finally By Lemma 3.3, $s \neq 4$. Therefore, we may assume that $c(G) = 2$. Let N be a minimal normal subgroup of G, properly contained in exactly s elements of $\mathrm{Kern}(G)$, $s \geq 1$. Note that if such N does not exists, then all elements of $\mathrm{Kern}(G)$ must be of order 2. Hence by Lemma 3.6 we have $|G'| = -4$, which is impossible. If $s = 2$ or 3, then by Lemma 3.5 we have $|G : N| = 32$, which is a contradiction by Remark 3.7. Also $s \neq 1$ and $s \neq 5$ by Lemma 3.4 and Lemma 3.3, respectively. Finally, assume that $s = 4$. Then by Proposition 3.8, all nonlinear irreducible character kernels of G containing N are of order 8. Let L be a minimal normal subgroup of G, $L \neq N$ and $L \notin \mathrm{Kern}(G)$. According to the above argument, L is also contained in 4 elements of $\mathrm{Kern}(G)$. Therefore, all nonlinear irreducible character kernels of G are of order 8. Hence, Lemma 3.6 yield that $|G'| = 11/16$ which is our final contradiction. □

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