On Modular Quasi-Pseudometric Spaces

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Abstract

In this MSc dissertation, we present modular metric spaces in asymmetric settings (called modular quasi-pseudometric spaces). Our results generalise and extend the concept of a modular metric on an arbitrary set, as presented in the work of Chistyakov. We show that Chistyakov's results also hold in an asymmetric framework. Furthermore, we show that most of Chistyakov's results do not need the symmetry property of a modular metric and prove that the results even hold when the symmetric property is not assumed. Finally, we observe that for any modular quasi-pseudometric which is convex on a set, its conjugate modular quasi-pseudometric is also convex and its symmetrized modular pseudometric preserves convexity.

Preface

The work described in this MSc dissertation was carried out under the supervision of Professor Olivier Olela Otafudu, North-West University, Mafikeng campus, South Africa, and co-supervision of Doctor Zechariah Mushaandja, Botswana International University of Science and Technology, Botswana, from February 2015 to November 2016.

The dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others, it is duly acknowledged in the text.

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Dedication

To my family.

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Notation and Conventions

S : Arbitrary set

S_m : Modular set

$S/\overset{m}{\sim}$: Quotient set

m : Modular metric

w : Modular quasi-pseudometric

d_m : Metric on a modular set

d : Metric on an arbitrary set

$\tilde{d}$: Metric on the quotient set

$\overset{m}{\sim}$: Equivalence relation with respect to a modular metric

$\overset{w}{\approx}$: Equivalence relation with respect to a modular quasi-pseudometric

Contents

Introduction

In 2008, Chistyakov in [8] introduced the concept of a modular metric space induced by F-modulars. For more details about the theory of F-modulars we refer the reader to [15, 19, 26]. Furthermore, Chistyakov developed the theory of a modular on an arbitrary set and studied the notion of a metric space induced by a modular that he called modular metric spaces (see [7, 8]). He defined a modular metric on a set X as a function $m : (0, \infty) \times X \times X \to [0, \infty]$ which for any $x, y, z \in X$ satisfies the following conditions:

1. $m(\delta, x, y) = 0$ if and only if $x = y$ whenever $\delta > 0$,
2. $m(\delta, x, y) = m(\delta, y, x)$ whenever $\delta > 0$,
3. $m(\delta + \gamma, x, y) \leq m(\delta, x, z) + m(\gamma, z, y)$ whenever $\delta, \gamma > 0$.

For any $x_0 \in X$, the set $X_m(x_0) = \{x \in X : \lim\limits_{\delta \to \infty} m(\delta, x, x_0) = 0\}$ is called a modular set. For simplicity, we shall write X_m for $X_m(x_0)$. Furthermore, Chistyakov defined the metric d_m on the modular set X_m by $d_m(x, y) = \inf\{\delta > 0 : m(\delta, x, y) \leq \delta\}$ whenever $x, y \in X_m$ and he called the pair (X_m, d_m) a modular metric space. In [1], it was observed that the physical interpretation of the theory of modular metric is that, while a metric on a set represents nonnegative finite distance between any two points of the set, a modular metric on a set attributes a nonnegative, sometimes infinite, valued field of generalized velocities to each time $\delta > 0$ (the absolute value of) an average velocity $m(\delta, x, y)$ is associated in such a way that in order to cover the distance between two points x and y of X, it takes time δ to move from x to y with velocity $m(\delta, x, y)$. Recently, Abdou in [1] introduced the concept of one-local retract in a more general setting in modular metric spaces and proved the existence of common fixed points for a family of modular non-expansive mappings defined on non-empty m-closed m-bounded subsets in a modular metric space. The main goal of this MSc dissertation is to generalise and extend the concept of modular metric onto the framework of modular quasi-pseudometric. We shall also study the concept of a modular on a real linear space in an asymmetric context. For instance we show that many results of Chistyakov do not use the symmetry axiom of a modular metric. Therefore they still hold in the framework of quasi-metric space.

Outline of the dissertation

Chapter 1. This chapter serves as a preliminary chapter for our dissertation as we recall some of the relevant definitions that are going to be utilized right through the dissertation. In the first section, we present the summary of quasi-pseudometric spaces and give examples. In the second section of this chapter, we recall convergence in asymmetric framework. We revisit a modular on a real linear space in the third section and lastly in the fourth section we look at convergence in modular spaces.

Chapter 2. Within this chapter, we present the results of Chistyakov [7]. The first section of this chapter reviews the definition of a modular metric and considers some examples. In the second section, we discuss the concept of a modular set. We show that a modular set equipped with different metrics turns out to be a metric space. An equivalence relation on a modular metric space and an equivalence class of an element in a set are also reviewed in this section. From the definition of a modular metric (see Definition 2.1.1), the triangle inequality axiom is modified, and this alteration brings to life the concept of convex modular metric which is discussed in the third section. In the fourth section, convergence in a modular metric space is reviewed.

Chapter 3. In this chapter we start with the first part of our own research. We generalise the work of Chistyakov [7] to the setting of quasi-pseudometric spaces. The first section of this chapter is devoted to the definition and discussion of the new concept of a modular quasi-pseudometric spaces. In the second section of this chapter, we discuss the convexity of a non-empty set endowed with a modular quasi-pseudometric. In the last section of this chapter, we learn about convergence in modular quasi-pseudometric spaces.

Chapter 4. We commence with the second part of our own research in this chapter. The first section of this chapter is devoted to the concept of a nonsymmetric modular and its structure. After exploring the nonsymmetric modular, in the next section we compare it with a modular quasi-metric which was investigated in Chapter 3. We show that [7, Theorem 3.11] does not depend on the symmetry property of symmetric modular and modular metric, but still holds in a nonsymmetric setting. We look at convergence of sequences in a modular space in the third section and lastly we investigate a nonsymmetric modular on a normed lattice.

Chapter 5. In this chapter, we summarise our investigations and present some open problems that set the way to future investigations.

1

Preliminaries

In this chapter we give a brief introduction of the concept quasi-pseudometric space and recall the necessary definitions that will be used in this dissertation.

We now define a quasi-pseudometric and other concepts as they form the basis of this dissertation.

1.1. Quasi-pseudometric spaces

Definition 1.1.1. *[17, Definition 2.1] Let S be a non-empty set and let q be a function $q : S \times S \to [0, \infty)$. The function q is called a quasi-pseudometric if it satisfies the following conditions :*

1. *$q(a, a) = 0$ for all $a \in S$,*
2. *$q(a, b) \leq q(a, c) + q(c, b)$ for all $a, b, c \in S$.*

If q is a quasi-pseudometric on a set S, then we shall call the pair (S, q) a quasi-pseudometric space. We say that a quasi-pseudometric q is a T_0-quasi-metric on S if q satisfies the condition $q(a, b) = 0 = q(b, a)$ which implies that $a = b$ for all $a, b \in S$.

Note that in a T_0-quasi-metric space (S, q), we can have $q(a, b) = 0$, which does not imply that $a = b$ whenever $a, b \in S$.
We give an example.

Example 1.1.1. Let $q(a, b)$ be a "distance above sea level" function such that a is a point at sea level and b is some point vertical to a. Now, if $q(a, b) = 10$, it means b is 10 units above a. But if $q(a, b) = 0$, it does not necessarily mean that both a and b are at sea level. It could mean that b is below sea level. So $q(a, b) = 0$ because the function is not defined for below sea level. Hence $q(a, b)$ is a T_0-quasi-metric space (S, q).

Remark 1.1.2. *Let q be a quasi-pseudometric on a non-empty set S. Then the function $q^{-1} : S \times S \to [0, \infty)$ defined as $q^{-1}(a, b) = q(b, a)$ for all $a, b \in S$ is also a quasi-pseudometric, named the conjugate quasi-pseudometric of q.*
Now if $q = q^{-1}$ where q is a quasi-pseudometric, then q is called a pseudometric.
As usual, for any T_0-quasi-pseudometric q, $q^s = \max\{q, q^{-1}\} = q \vee q^{-1}$ is a metric [17].

Example 1.1.2. Let $\mathbb{R}$ be the set of real numbers. The function

$$u(a, b) = \max\{a - b, 0\}$$

for all $a, b \in \mathbb{R}$ is a quasi-pseudometric on $\mathbb{R}$. Moreover

$$u^s = |a - b|$$

for all $a, b \in \mathbb{R}$ is the usual metric on $\mathbb{R}$.

Example 1.1.3. Let $\mathbb{R}$ be the set of real numbers and the function q defined by

$$q(a, b) = \begin{cases} e^a - e^b & \text{if } a \geq b \\ e^b - e^a & \text{if } a < b \end{cases}$$

for all $a, b \in \mathbb{R}$ is a T_0-quasimetric on $\mathbb{R}$.

Remark 1.1.3. *If (S, q) is a quasi-pseudometric space, then*

1. *the set $B_q(a, r) = \{b \in S : q(a, b) < r\}$ for all $a \in S$ and $r > 0$, is called an open ball with centre a and radius r.*
2. *The collection of all open balls forms a base for a topology $\tau(q)$ and is called the topology induced by q on S.*
3. *We also have $C_q(a, r) = \{b \in S : q(a, b) \leq r\}$ for all $a \in S$ and $r > 0$, which is called the closed ball with centre a and radius r.*

Note that $C_q(a, r)$ is $\tau(q^{-1})$-closed but not $\tau(q)$-closed in general.

Remark 1.1.4. *The topology generated by q^s (i.e. $\tau(q^s)$), is finer than other topologies (i.e. $\tau(q)$ and $\tau(q^{-1})$).*

1.2. Convergence in a quasi-pseudometric space

We now recall different concepts of Cauchyness of sequences in a quasi-pseudometric space. For more details we refer the reader to [9, 11].

Definition 1.2.1. *[4] Let (S, q) be a quasi-pseudometric space. A sequence (a_n) in (S, q) is said to be Left-Cauchy if for each $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that*

$$q(a_n, a_m) < \epsilon$$

for all $m \geq n \geq k$.

Definition 1.2.2. *[4] Let (S, q) be a quasi-pseudometric space. A sequence (a_n) in (S, q) is said to be Right-Cauchy if for each $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that*

$$q(a_n, a_m) < \epsilon$$

for all $n \geq m \geq k$.

Definition 1.2.3. *[24] Let (S, q) be a quasi-pseudometric space. A sequence (a_n) in (S, q) is said to be a Cauchy sequence if*

$$\lim_{n,m \to \infty} q(a_n, a_m) = 0.$$

Remark 1.2.4. *[24] A sequence (a_n) is a Cauchy sequence in (S, q) if (a_n) is a sequence in the pseudometric space (S, q^s).*

Definition 1.2.5. *[5, Definition 2] A sequence (a_n) in a quasi-pseudometric space (S, q) is said to be left K-Cauchy if for some $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$q(a_m, a_n) < \epsilon \qquad \textit{whenever} \qquad n_\epsilon \leq m \leq n.$$

Definition 1.2.6. *[5, Defnition 2] A sequence (a_n) in a quasi-pseudometric space (S, q) is said to be right K-Cauchy if for some $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$q(a_n, a_m) < \epsilon \qquad \textit{whenever} \qquad n_\epsilon \leq m \leq n.$$

1.3. Asymmetric normed linear space

Definition 1.3.1. *[21] Let* S *be a real linear space and* $||.| : S \to [0, \infty)$ *be a function. Then* $||.|$ *is called an asymmetric norm on* S *if it satisfies the following conditions :*

1. *If* $||a| = || - a| = 0$ *then* $a = 0$ *for all* $a \in S$*,*
2. $||\delta a| = \delta ||a|$ *for all* $a \in S$ *and* $\delta \geq 0$*,*
3. $||a + b| \leq ||a| + ||b|$ *for all* $a, b \in S$*.*

The pair $(S, ||.|)$ *shall be called an asymmetric normed linear space.*

Remark 1.3.2. *Suppose* $||.|$ *is an asymmetric norm on a real linear space* S*. Then* $|.|| : S \to [0, \infty)$ *defined by* $|a|| = || - a|$ *for all* $a \in S$ *is also an asymmetric norm on* S *called the conjugate norm of* $||.|$ *([21]).*
Generally, an asymmetyric norm $||.|$ *on* S *such that* $||.| = |.||$ *is called a norm. Moreover, for any asymmetric norm* $||.|$*, the function*

$$||.|| = \max\{||.|, |.||\}$$

is a norm and the pair $(S, ||.||)$ *is called a normed real linear space. The asymmetric norm induces, in a natural way, a quasi-metric* $q_{||.|}$ *on* S *defined by*

$$q_{||.|}(a, b) = ||a - b|$$

for all $a, b \in S$ *([21]).*

Example 1.3.1. [9, Example 1.1.3] Let the set $\mathbb{R}$ of real numbers be equipped with an asymmetric norm $||a| = a^+ = \max\{a, 0\}$. Then, for any $a \in \mathbb{R}$, $|a|| = a^- = \max\{-a, 0\}$ and $||a|| = |a|$.
The topology $\tau(q_{||.|})$ generated by $q_{||.|}$ is called the upper topology of $\mathbb{R}$ whereas the topology $\tau(q^{-1}_{||.|})$ generated by $q^{-1}_{||.|}$ is called the lower topology of $\mathbb{R}$.

Definition 1.3.3. *[16] A norm* $||.||$ *on a real linear space* S *is said to be equivalent to a norm* $||.||_0$ *on* S *if there are* $x, y \in \mathbb{R}^+$ *such that*

$$x||a||_0 \leq ||a|| \leq y||a||_0$$

for all $a \in S$*.*

Note : Equivalent norms on S define the same topology for S.

Alegre, Ferrer and Gregory [2, 3, 12], introduced an asymmetric norm on a normed lattice and studied the properties of the induced quasi-uniformity and topology in connection with the usual properties of normed lattices.

Definition 1.3.4. *[9] An ordered vector space is a real vector space* S *equipped with a partial order relation such that* $a \leq b \Rightarrow a + c \leq b + c$ *and* $\delta a \leq \delta b$ *for all* $a, b, c \in S$ *and* $\delta \geq 0$*.*

Definition 1.3.5. *[9] An ordered vector space $(S, \leq)$ is called a vector lattice (also known as Riez space) if every pair of elements $a, b \in S$ admits a least upper bound $a \vee b$. Since*

$$a \leq b \Leftrightarrow -b \leq -a,$$

it follows

$$a \wedge b = -((-a) \vee (-b))$$

so that every pair of elements $a, b \in S$ has a greatest lower bound.

Definition 1.3.6. *[9] A norm $||.||$ on an ordered pair $(S, \leq)$ called a lattice norm if it satisfies one of the following equivalent conditions :*

1. *$|a| \leq |b| \Rightarrow ||a|| \leq ||b||$ for all $a, b \in S$,*
2. *(i) $||\,|a|\,|| = ||a||$ for all $a \in S$,*
    (ii) if $0 \leq a \leq b$, then $||a|| \leq ||b||$ for all $a, b \in S$.

An ordered vector space equipped with a lattice norm is called a normed lattice and is denoted by $(S, ||.||, \leq)$.

Definition 1.3.7. *[9] A normed lattice $(S, ||.||, \leq)$ is called an L-space, M-space and E-space respectively, provided that*

1. *$||a + b|| = ||a|| + ||b||$ for all $a, b \in S$, (L)*
2. *$||a \vee b|| = ||a|| \vee ||b||$ for all $a, b \in S$, (M)*
3. *$||a + b||^2 + ||a - b||^2 = 2||a||^2 + 2||b||^2$ for all $a, b \in S$. (E)*

Definition 1.3.8. *[9] Let p be an asymmetric norm on a vector space S. We can associate the following norms, defined by the equalities:*

$$\begin{aligned} p^s_L(a) &= p(a) + p(-a) \qquad for \quad all \quad a \in S, \\ p^s_M(a) &= p^s(a) = p(a) \vee p(-a) \qquad for \quad all \quad a \in S, \\ p^s_E(a) &= \sqrt{p(a)^2 + p(-a)^2} \qquad for \quad all \quad a \in S. \end{aligned} \tag{1.1}$$

The norm p^s_M is the usual norm p^s that we have associated to an asymmetric norm p.

Note : All norms defined in Equation (1.1) are equivalent [9].

1.4. Modular on a linear space

Definition 1.4.1. *[18] Let S be a real linear space. A function $\sigma : S \to [0, \infty]$ is said to be a modular on S if it satisfies*

1. $\sigma(a) = 0$ *if and only if* $a = 0$ *for* $a \in S$,
2. $\sigma(a) = \sigma(-a)$ *for all* $a \in S$,
3. $\sigma(\alpha a + \beta b) \leq \sigma(a) + \sigma(b)$ *for* $a, b \in S$ *and* $\alpha, \beta \geq 0$ *such that* $\alpha + \beta = 1$.

Moreover, Musielak and Orlicz [18] introduced the concept of convexity on a modular and defined it as follows.

Definition 1.4.2. *[18] Let S be a real linear space. A function $\sigma : S \to [0, \infty]$ is said to be a convex modular on S if it satisfies*

1. $\sigma(a) = 0$ *if and only if* $a = 0$ *for* $a \in S$,
2. $\sigma(a) = \sigma(-a)$ *for all* $a \in S$,
3. $\sigma(\alpha a + \beta b) \leq \alpha\sigma(a) + \beta\sigma(b)$ *for* $a, b \in S$ *and* $\alpha, \beta \geq 0$ *such that* $\alpha + \beta = 1$.

Furthermore, Musielak and Orlicz in their paper [18], mentioned that if σ is a modular on S, then the linear subspace S^*_σ (see Definition 1.4.5 on page 9) of S is called a modular space.

Definition 1.4.3. *[18] Let S be a real linear space. If σ is a modular on S, then*

$$S_\sigma = \left\{ a \in S : \lim_{\alpha \to 0} \sigma(\alpha a) = 0 \right\}$$

is called a modular set and is a linear subspace of S.

According to [18], a modular space can be equipped with a norm. We see this in the following definition.

Definition 1.4.4. *[18] Let σ be a modular on a real linear space S and S_σ be a modular set, then*

$$||a||_\sigma = \inf \left\{ \epsilon > 0 : \sigma\left(\frac{a}{\epsilon}\right) \leq \epsilon \quad for \quad a \in S_\sigma \right\}$$

is a norm on S_σ.

Furthermore, if σ is a convex modular on S, then a modular space is defined as follows.

Definition 1.4.5. *[18] Let S be a real linear space and σ be a convex modular on S, then a modular space is defined as*

$$S_\sigma^* = \{a \in S : \exists \alpha > 0,\ \sigma(\alpha a) < \infty\}$$

and the norm on S_σ^* is defined as

$$||a||_\sigma^* = \inf\left\{\epsilon > 0 : \sigma\left(\frac{a}{\epsilon}\right) \leq 1 \quad for \quad a \in S_\sigma^*\right\}.$$

Now, S_σ is equivalent to S_σ^* if $||a||_\sigma \leq 1$ and $||a||_\sigma^* \leq 1$. Moreover, if $||a||_\sigma \leq 1$ or $||a||_\sigma^* \leq 1$, then $||a||_\sigma^* \leq ||a||_\sigma \leq \sqrt{||a||_\sigma^*}$, otherwise $\sqrt{||a||_\sigma^*} \leq ||a||_\sigma \leq ||a||_\sigma^*$.

We now give examples to emphasize the concept of a modular on a linear space.

Example 1.4.1. [25, Example 2.2(a)] Let $\mathbb{R}$ be the real line. Then the function $\sigma : \mathbb{R} \to [0, \infty)$ defined by

$$\sigma(a) = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases} \qquad where \quad a \in \mathbb{R}$$

is a modular.

Example 1.4.2. [20] Let $(S, ||.||)$ be a normed space. Then S_σ is a modular set with modular σ defined as

$$\sigma(a) = ||a|| \quad for \quad a \in S_\sigma.$$

Example 1.4.3. [20] From Example 1.4.2, it is clear that every norm is a modular. Moreover, Example 3 of [20] shows that the converse is not true.

Example 1.4.4. Let $I = [a, b]$ be an interval on the real line $\mathbb{R}$, then

$$\sigma(a) = \int_a^b ||a(s)||ds$$

is a convex modular on I.

We now recall convergence in a modular space.

1.5. Convergence in a modular space

Definition 1.5.1. *[25, Definition 2.4.1] Let S_σ be a modular space. The sequence (a_n) in S_σ is said to converge to $a \in S_\sigma$ and it is denoted by*

$$a_n \to a \qquad whenever \qquad \sigma(a_n - a) \to 0.$$

Definition 1.5.2. *[25, Definition 2.4.2] Let S_σ be a modular space. The sequence (a_n) in S_σ is said to be Cauchy if*

$$\lim_{m,n\to\infty} \sigma(a_m - a_n) = 0.$$

In [7], Chistyakov unveiled the congruity between modular and modular linear spaces. This statement is shown in the next theorem.

Theorem 1.5.3. *[7, Theorem 3.11(a)] Let S be a real linear space. If m is a modular metric on S and σ is a modular on S, then*

$$m(\delta, a, b) = \sigma\left(\frac{a-b}{\delta}\right)$$

for all $a, b \in S$ and $\delta > 0$.

Theorem 1.5.4. *[7, Theorem 3.11(b)] Let S be a real linear space. If m is a convex modular metric on S and ρ is a convex modular on S, then*

$$m(\delta, a, b) = \sigma\left(\frac{a-b}{\delta}\right)$$

for all $a, b \in S$ and $\delta > 0$.

Corollary 1.5.5. *Let S be a real linear space. If Theorem 1.5.3 holds, then*

$$S_\sigma = S_m(0)$$

is a linear subspace of S and

$$||a||_\sigma = d_m(a, 0)$$

is a norm on S for all $a \in S$.

Corollary 1.5.6. *Let S be a real linear space. If Theorem 1.5.4 holds, then*

$$S_\sigma^* = S_m^*(0) = S_\sigma$$

is a linear subspace of S and

$$||a||_\sigma^* = d_m^*(a, 0)$$

is a norm on S_σ^* for all $a \in S_\sigma^*$.

2

Modular metric spaces

In this chapter, we study the concept of modular metric spaces. The notion of modular metric space was introduced by Chistyakov [7]. A modular metric is a function m on a non-empty set S that satisfies certain properties (see Definition 2.1.1).
We recall an equivalence relation on a set S and an equivalence class of an element in a modular set. Moreover, we also look at a convex modular metric and convergence in modular metric spaces.

Modular metric spaces have topological properties like convexity and convergence. This chapter is dedicated to the study of these concepts with examples.

2.1. Modular metric spaces

We now give the definition of a modular metric.

Definition 2.1.1. *[7, Definition 2.1] Let S be a non-empty set. A function $m : (0,\infty)\times S\times S \to [0,\infty]$ is said to be a modular metric on S if it satisfies the following properties:*

1. $m(\delta, a, b) = 0$ *if and only if* $a = b$ *for all* $\delta \in (0,\infty)$ *and* $a, b \in S$, *(non degeneracy),*
2. $m(\delta, a, b) = m(\delta, b, a)$ *for all* $\delta \in (0,\infty)$ *and* $a, b \in S$, *(symmetry),*
3. $m(\delta+\gamma, a, b) \leq m(\delta, a, c)+m(\gamma, c, b)$ *for all* $\delta, \gamma \in (0,\infty)$ *and* $a, b, c \in S$, *(triangle inequality).*

If m is a modular metric on S, then the pair (S, m) will be called a modular metric space.

Definition 2.1.2. *[7] Let S be a non-empty set. A function $m : (0,\infty) \times S \times S \to [0,\infty]$ is said to be a modular pseudometric on S if it satisfies the following properties :*

1. $m(\delta, a, a) = 0$ *for all* $\delta > 0$ *and* $a \in S$,
2. $m(\delta, a, b) = m(\delta, b, a)$ *for all* $\delta > 0$ *and* $a, b \in S$,
3. $m(\delta + \gamma, a, b) \leq m(\delta, a, c) + m(\gamma, c, b)$ *for all* $\delta, \gamma > 0$ *and* $a, b, c \in S$.

Remark 2.1.3. *In the axiom (3) of Definition 2.1.1, if we set* $a = b$ *and* $\gamma = \delta > 0$*, then we find*

$$0 = m(2\delta, a, a) \leq 2m(\delta, a, c)$$

for all $a, c \in S$ *and* $\delta > 0$*.*

Remark 2.1.4. *Let* m *be a modular metric on a set* S*, then :*

1. *If* $m(\delta, a, b)$ *does not depend on* $a, b \in S$*, then* $m \equiv 0$*,*
2. *If* $m(\delta, a, b)$ *does not depend on* $\delta > 0$ *(i.e.*$m(\delta, a, b) = m(a, b)$*) and furthermore assume* $m(\delta, a, b) \in [0, \infty)$*, then* m *in Definition 2.1.1 is a metric on* S *and* m *in Definition 2.1.2 is a pseudometric on* S*.*

The following lemma can be compared to Lemma 3.1.4 (see page 42).

Lemma 2.1.5. *[7] Let* m *be a modular metric on a non-empty set* S*, then the function* $\lambda : (0, \infty) \to [0, \infty]$ *defined by* $\lambda(\delta) = m(\delta, a, b)$ *whenever* $a, b \in S$ *and* $\delta \in (0, \infty)$ *is non-increasing.*

Proof. Let $a, b \in S$ and $\delta, \eta \in (0, \infty)$ with $\delta > \eta$. Since $\delta - \eta > 0$, we have

$$\begin{aligned}\lambda(\delta) &= m(\delta, a, b)\\ &= m(\delta - \eta + \eta, a, b)\\ &\leq m(\delta - \eta, a, a) + m(\eta, a, b).\end{aligned}$$

Since $m(\delta - \eta, a, a) = 0$, by property (1) of Definition 2.1.1 as m is a modular metric on S. It follows that

$$\lambda(\delta) = m(\delta, a, b) \leq m(\eta, a, b) = \lambda(\eta)$$

which means $\lambda(\delta) \leq \lambda(\eta)$ for $0 < \eta < \delta$. Therefore $\lambda(\delta)$ is a non-increasing function. ☐

Here are some examples of a modular metric on a non-empty set S.

Example 2.1.1. [7, Example 2.4. (a)] Let S be a non-empty set. Define $m : (0, \infty) \times S \times S \to [0, \infty]$ by

$$m(\delta, a, b) = \begin{cases} \infty & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

for all $a, b \in S$ and $\delta > 0$, then m is a modular metric on S.

Proof. We show that m satisfies the properties of a modular metric.

1. If $a = b$, then we have $m(\delta, a, b) = 0$.
   Moreover if $m(\delta, a, b) = 0$, by definition of m then we have $a = b$.

2. We show that the symmetry condition holds. We have two cases (i.e. $a = b$ and $a \neq b$).

   (a) If $a = b$, then we have
   $$m(\delta, a, b) = 0 = m(\delta, b, a).$$

   (b) If $a \neq b$, then we have
   $$m(\delta, a, b) = \infty = m(\delta, b, a).$$

3. We show the triangle inequality property. Let $a, b, c \in S$ and $\delta, \gamma > 0$. We consider different cases. If we fix $a = b$, then we have two cases (i.e. ($a = c$ and $b = c$) and ($a \neq c$ and $b \neq c$)).

   (A) Suppose $a = b$.

   (i) If $b = c$ and $a = c$, then we have
   $$\begin{aligned} m(\delta, a, b) &= 0 \\ &= 0 + 0 \\ &= m(\delta, a, c) + m(\delta, b, c). \end{aligned}$$

   (ii) If $a \neq c$ and $b \neq c$, then we have
   $$\begin{aligned} m(\delta + \gamma, a, b) &= 0 \\ &< \infty + \infty \\ &= m(\delta, a, c) + m(\gamma, c, b). \end{aligned}$$

   Furthermore, if $a \neq b$, then we have three cases (i.e. ($a = c$ and $b \neq c$), ($a \neq c$ and $b = c$) and ($a \neq c$ and $b \neq c$)).
   Note that we cannot have $a = c$ and $b = c$ but $a \neq b$.

   (B) Suppose $a \neq b$.

   (i) If $a = c$ and $b \neq c$, then we have
   $$\begin{aligned} m(\delta + \gamma, a, b) &= \infty \\ &= 0 + \infty \\ &= m(\delta, a, c) + m(\gamma, c, b). \end{aligned}$$

(ii) If $a \neq c$ and $b = c$, then we have

$$\begin{aligned} m(\delta+\gamma, a, b) &= \infty \\ &= \infty + 0 \\ &= m(\delta, a, c) + m(\gamma, c, b). \end{aligned}$$

(iii) If $a \neq c$ and $b \neq c$, then we have

$$\begin{aligned} m(\delta+\gamma, a, b) &= \infty \\ &= \infty + \infty \\ &= m(\delta, a, c) + m(\gamma, c, b). \end{aligned}$$

After considering all possible cases, we have noted that

$$m(\delta+\gamma, a, b) \leq m(\delta, a, c) + m(\gamma, c, b).$$

Therefore $m(\delta, a, b)$ is a modular metric on S. □

Example 2.1.2. [7, Example 2.4. (b)] Let S be a non-empty set. Suppose the pair (S, d) is a metric space. Then $m : (0, \infty) \times S \times S \to [0, \infty]$ defined by

$$m(\delta, a, b) = \frac{d(a, b)}{f(\delta)}$$

for any $\delta > 0$ and $a, b, c \in S$ where $f : (0, \infty) \to (0, \infty)$ is a non-decreasing function, is a modular metric on S.

Proof. We show that the axioms of Definition 2.1.1 are satisfied.

1. We show for the symmetry condition. Since d is a metric on S, then

$$\begin{aligned} m(\delta, a, b) &= \frac{d(a, b)}{f(\delta)} \\ &= \frac{d(b, a)}{f(\delta)} \\ &= m(\delta, b, a) \end{aligned}$$

   for all $a, b \in S$ and $\delta > 0$. Hence $m(\delta, a, b)$ is symmetric.

2. Since d is a metric on S, then if $a = b$, then we have $d(a, b) = 0$. Therefore $m(\delta, a, b) = 0$ if $a = b$. Also if $m(\delta, a, b) = 0 = \dfrac{d(a, b)}{f(\delta)}$, then $d(a, b) = 0$ implying $a = b$.

3. We show that the triangle inequality holds. Let $\delta > 0$ and $\eta > 0$, then we have $\eta + \delta > \eta$ and $\eta + \delta > \delta$. Since f is a non-decreasing function on $(0, \infty)$, then we have $f(\delta + \eta) \geq f(\delta)$ and $f(\delta + \eta) \geq f(\eta)$. Therefore we have

$$\frac{1}{f(\delta+\eta)} \leq \frac{1}{f(\delta)} \tag{2.1}$$

and furthermore we have

$$\frac{1}{f(\delta+\eta)} \leq \frac{1}{f(\eta)}. \tag{2.2}$$

If we multiply (2.1) by $d(a, c)$ and multiply (2.2) by $d(c, b)$, then we have

$$\frac{d(a,c)}{f(\delta+\eta)} \leq \frac{d(a,c)}{f(\delta)} \qquad and \qquad \frac{d(c,b)}{f(\delta+\eta)} \leq \frac{d(c,b)}{f(\eta)}.$$

Moreover since d is a metric on S, which means we have $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in S$, so

$$\begin{aligned} m(\delta+\eta, a, b) &= \frac{d(a,b)}{f(\delta+\eta)} \\ &\leq \frac{d(a,c)+d(c,b)}{f(\delta+\eta)} \\ &= \frac{d(a,c)}{f(\delta+\eta)} + \frac{d(c,b)}{f(\delta+\eta)} \\ &\leq \frac{d(a,c)}{f(\delta)} + \frac{d(c,b)}{f(\eta)}. \end{aligned}$$

Finally we have

$$\begin{aligned} m(\delta+\eta, a, b) &= \frac{d(a,b)}{f(\delta+\eta)} \\ &\leq \frac{d(a,c)}{f(\delta)} + \frac{d(c,b)}{f(\eta)} \\ &= m(\delta, a, c) + m(\eta, c, b). \end{aligned}$$

Hence $m(\delta + \eta, a, b) \leq m(\delta, a, c) + m(\eta, c, b)$ for all $a, b, c \in S$ and $\delta, \eta > 0$.
Thus $m(\delta, a, b)$ is a modular metric on S.

□

Example 2.1.3. [7, Example 2.4. (c)] Consider a non-empty set S. Suppose (S, d) is a metric space and $m : (0, \infty) \times S \times S \to [0, \infty]$ is defined by

$$m(\delta, a, b) = \begin{cases} \infty & \text{if } \delta \leq d(a,b) \\ 0 & \text{if } \delta > d(a,b) \end{cases}$$

for all $a, b \in S$ and $\delta > 0$. Then $m(\delta, a, b)$ is a modular metric on S.

Proof. We show that m satisfies the properties in Definition 2.1.1.

1. If $m(\delta, a, b) = 0$, then we have $\delta > d(a, b)$. The condition where δ can be strictly greater than $d(a, b)$ is when $d(a, b) = 0$ since $\delta > 0$. Therefore $a = b$ since d is a metric on S.
   If $a = b$, then $d(a, b) = 0$ and since $\delta > d(a, b) = 0$, then by definition of m we have $m(\delta, a, b) = 0$.

2. To show the symmetry condition, we consider two cases (i.e. $\delta \leq d(a, b)$ and $\delta > d(a, b)$).
   (a) If $\delta \leq d(a, b)$, then we have $m(\delta, a, b) = \infty$. Now since $\delta \leq d(a, b) = d(b, a)$ because d is a metric on S, then
   $$m(\delta, a, b) = \infty = m(\delta, b, a).$$
   (b) If $\delta > d(a, b)$, then we have $m(\delta, a, b) = 0$ and since $\delta > d(a, b) = d(b, a)$ because d is a metric on S, then
   $$m(\delta, a, b) = 0 = m(\delta, b, a).$$

3. We now show that the triangle inequality holds. Let $\delta, \mu > 0$ and $a, b, c \in S$. We consider three cases (i.e. ($\delta > d(a, c)$ and $\mu > d(c, b)$), ($\delta > d(a, c)$ and $\mu \leq d(c, b)$) and ($\delta \leq d(a, c)$ and $\mu \leq d(c, b)$)).
   (a) If $\delta > d(a, c), \mu > d(c, b)$ and $\delta + \mu > d(a, b)$, then we have
   $$\begin{aligned} m(\delta + \mu, a, b) &= 0 \\ &= 0 + 0 \\ &= m(\delta, a, c) + m(\mu, c, b). \end{aligned}$$
   (b) If $\delta > d(a, c)$, $\mu \leq d(c, b)$ and $\delta + \mu > d(a, b)$, then we have
   $$\begin{aligned} m(\delta + \mu, a, b) &= 0 \\ &< 0 + \infty \\ &= m(\delta, a, c) + m(\mu, c, b). \end{aligned}$$
   (c) If $\delta \leq d(a, c)$, $\mu \leq d(c, b)$ and $\delta + \mu \leq d(a, b)$, then we have
   $$\begin{aligned} m(\delta + \mu, a, b) &= \infty \\ &\leq \infty + \infty \\ &= m(\delta, a, c) + m(\mu, c, b). \end{aligned}$$

Thus $m(\delta + \mu, a, b) \leq m(\delta, a, c) + m(\mu, c, b)$. Therefore m is a modular metric on S. □

2.2. Modular set

In this section we study the concept of a modular set equipped with a modular metric m. We recall the definitions of an equivalence relation on a modular metric space and an equivalence class of an element. Furthermore, it turns out that a modular set equipped with a metric defined on it becomes a metric space (see Theorem 2.2.7 and Theorem 2.2.8). We also illustrate the theory of a modular set by some basic examples.

Definition 2.2.1. *[7] Let S be a non-empty set and m be a modular metric on S. Define a binary relation $\overset{m}{\sim}$ on S by*

$$a \overset{m}{\sim} b \quad if\ and\ only\ if \quad \lim_{\delta\to\infty} m(\delta, a, b) = 0$$

for all $a, b \in S$ and $\delta > 0$.

Lemma 2.2.2. *[7] Let S be a non-empty set. Then the binary relation $a \overset{m}{\sim} b$ where $a, b \in S$ in Definition 2.2.1 is an equivalence relation on S.*

Proof. To show that $\overset{m}{\sim}$ is an equivalence relation, we need to show that the properties of an equivalence relation are satisfied (i.e. symmetry, transitivity and reflexivity).

(i) Let $a \in S$ and $\delta > 0$. Then $m(\delta, a, a) = 0$ since m is a modular metric. Therefore

$$\lim_{\delta\to\infty} m(\delta, a, a) = 0$$

hence $a \overset{m}{\sim} a$.

(ii) Let $a, b \in S$ and $\delta > 0$. If $a \overset{m}{\sim} b$, then

$$\lim_{\delta\to\infty} m(\delta, a, b) = 0 = \lim_{\delta\to\infty} m(\delta, b, a)$$

since m is a modular metric (i.e. $m(\delta, a, b) = m(\delta, b, a)$). Hence $b \overset{m}{\sim} a$.

(iii) Now suppose that $a \overset{m}{\sim} c$ and $c \overset{m}{\sim} b$ for all $a, b, c \in S$. We want to show that $a \overset{m}{\sim} b$. Since m is a modular metric, then we have

$$\begin{aligned} m(\delta, a, b) &= m\left(\frac{\delta}{2} + \frac{\delta}{2}, a, b\right) \\ &\leq m\left(\frac{\delta}{2}, a, c\right) + m\left(\frac{\delta}{2}, c, b\right). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{\delta\to\infty} m(\delta, a, b) &= \lim_{\delta\to\infty} m\left(\frac{\delta}{2} + \frac{\delta}{2}, a, b\right) \\ &\leq \lim_{\delta\to\infty} m\left(\frac{\delta}{2}, a, c\right) + \lim_{\delta\to\infty} m\left(\frac{\delta}{2}, c, b\right) \\ &= 0 + 0. \end{aligned}$$

Now

$$\lim_{\delta\to\infty} m(\delta, a, b) = 0$$

hence $a \overset{m}{\sim} b$. Thus $\overset{m}{\sim}$ is an equivalence relation on S.

□

Definition 2.2.3. *[7] Let m be a modular metric on a non-empty set S with the equivalence relation $\overset{m}{\sim}$ defined on S above. We define an equivalence class of an element $a \in S$ as*

$$\begin{aligned} S_m(a) &= \{b \in S : b \overset{m}{\sim} a\} \\ &= \left\{ b \in S : \lim_{\delta\to\infty} m(\delta, a, b) = 0 \right\} \end{aligned}$$

and the quotient set of S with respect to $\overset{m}{\sim}$ denoted by $S/\overset{m}{\sim}$ is

$$S/\overset{m}{\sim} = \{S_m(a) : a \in S\}.$$

Definition 2.2.4. *Let m be a modular metric on a non-empty set S and $\overset{m}{\sim}$ be an equivalence relation. If we fix $a \in S$, set $S_m(a) = S_m$, then we call the set S_m a* modular set.

Now we give examples of a modular set S_m equipped with a modular metric m on a non-empty set S.

Example 2.2.1. Let S be a non-empty set. Given m as a modular metric on S defined as

$$m(\delta, a, b) = \begin{cases} \infty & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

for all $a, b \in S$ and $\delta > 0$. Then its modular set is given by $S_m(a) = \{a\}$.

Proof. From Definition 2.2.4, the modular set is defined as

$$S_m(a) = \left\{ b \in S : \lim_{\delta\to\infty} m(\delta, a, b) = 0 \right\}.$$

Since m is a modular metric on S then

$$\lim_{\delta\to\infty} m(\delta, a, b) = 0 \quad if \quad m(\delta, a, b) = 0$$

which implies that $a = b$. Therefore

$$\begin{aligned} S_m(a) &= \left\{ b \in S : \lim_{\delta\to\infty} m(\delta, a, b) = 0 \right\} \\ &= \{b \in S : a \overset{m}{\sim} b\} \\ &= \{a \in S : b \overset{m}{\sim} a\} \\ &= \{a\}. \end{aligned}$$

Thus the modular set $S_m(a)$ is given by $S_m(a) = \{a\}$. □

Example 2.2.2. [7] Consider the modular metric m in Example 2.1.2 on a metric space (S, d). Let $a \in S$, then a modular set of a is given by

$$S_m(a) = \left\{ b \in S : \lim_{\delta \to \infty} \frac{d(a,b)}{f(\delta)} = 0 = \lim_{\delta \to \infty} \frac{d(b,a)}{f(\delta)} \right\}$$

for any $\delta > 0$. Now there are two possibilities for $\lim\limits_{\delta \to \infty} \dfrac{d(a,b)}{f(\delta)} = 0$, either $f(\delta) \to \infty$ as $\delta \to \infty$ (i.e. $f(\delta)$ is not bounded from above) and $d(a,b) > 0$ which would imply $S_m(a) = S$ or $f(\delta)$ is bounded from above and $d(a,b) = 0$, which would imply $S_m(a) = \{a\}$ if d is a T_0-quasi-metric on S.

Proof. **(a)** If $f(\delta) \to \infty$ as $\delta \to \infty$, then

$$\begin{aligned} S_m(a) &= \left\{ b \in S : \lim_{\delta \to \infty} m(\delta, a, b) = 0 \right\} \\ &= \left\{ b \in S : \lim_{\delta \to \infty} \left[\frac{d(a,b)}{f(\delta)} \right] = 0 \right\} \\ &= \left\{ b \in S : \lim_{\delta \to \infty} \left[\frac{d(b,a)}{f(\delta)} \right] = 0 \right\}. \end{aligned}$$

Since $f(\delta) \to \infty$ as $\delta \to \infty$ and $\lim\limits_{\delta \to \infty} m(\delta, a, b) = 0$, then $d(a,b) > 0$. Because $d(a,b) \neq 0$, then definitely $a \neq b$. Hence $S_m(a)$ cannot be a singleton but the whole set S. Thus $S_m(a) = S$.

(b) If $f(\delta) \nrightarrow \infty$ as $\delta \to \infty$, meaning $f(\delta) \in (0, \infty)$ is bounded from above, then

$$\lim_{\delta \to \infty} f(\delta) < \infty \quad \textit{whenever} \quad \delta \to \infty$$

hence

$$\lim_{\delta \to \infty} \frac{1}{f(\delta)} < \infty \quad \textit{whenever} \quad \delta \to \infty$$

since $f(\delta) \in (0, \infty)$. We have

$$\lim_{\delta \to \infty} \frac{d(a,b)}{f(\delta)} = 0$$

which implies that $d(a,b) = 0$ and $d(b,a) = 0$ for all $a, b \in S$. Now if d is a T_0-quasi-metric on S, then $d(a,b) = 0$ and $d(b,a) = 0$ implies $a = b$ hence

$$S_m(a) = \{a\}.$$

But if d is not a T_0-quasi-metric on S, then

$$S_m(a) \neq \{a\}.$$

□

Remark 2.2.5. *From Example 2.2.1 and Example 2.2.2, we can conclude that the modular set* $S_m \subseteq S$ *where* S *is a non-empty set.*

Lemma 2.2.6. *[7] Let S be a non-empty set and m be a modular metric on S. The function $\tilde{d} = (S/\overset{m}{\sim}) \times (S/\overset{m}{\sim}) \to [0,\infty]$ defined by*

$$\tilde{d}\,(S_m(a), S_m(b)) = \lim_{\delta\to\infty} m(\delta, a, b)$$

whenever $S_m(a), S_m(b) \in S/\overset{m}{\sim}$ is a metric on the quotient set $S/\overset{m}{\sim}$.

Proof. We refer the reader to [7]. ☐

The following theorem is one of the main theorems in Chistyakov's paper [7]. It shows that a modular set equipped with a metric on it is a metric space.

Theorem 2.2.7. *[7, Theorem 2.6.] Let S be a non-empty set and m be a modular metric on S. The function $d_m : S_m \times S_n \to [0,\infty]$ defined by*

$$d_m(a,b) = \inf\{\delta > 0 : m(\delta, a, b) \leq \delta\}$$

for all $a, b \in S_m$, then d_m is a metric on S_m.

Proof. For us to show that d_m is a metric on S_m, we have to show that the function d_m satisfies all the properties of a metric. But we start by checking if d_m is well defined.
Suppose $a, b \in S_m$. Then $a \overset{m}{\sim} b$ (i.e $\lim\limits_{\delta\to\infty} m(\delta,a,b) = 0$). Hence the smallest value of d_m is $d_m(a,b) = 0$.
Furthermore for any $\delta > 0$ we have some $\delta_0 > 0$ such that when $\delta > \delta_0$, we have $m(\delta,a,b) \leq 1$.
If we let $\delta_1 = \max\{1, \delta_0\}$, then $m(\delta,a,b) \leq 1 \leq \delta_1$. So we have

$$d_m(a,b) = \inf\{\delta > 0 : m(\delta,a,b) \leq \delta\} \leq \delta_1 < \infty.$$

This implies that $d_m(a,b)$ is finite. Therefore $d_m(a,b) \in [0,\infty)$ for all $a, b \in S_m$. Thus the function d_m is well defined. We now prove that d_m is a metric.

1. If $a = b$ where $a, b \in S_m$, then $m(\delta,a,b) = 0$ for all $\delta > 0$ since m is a modular metric on S. Then we have

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : m(\delta,a,b) \leq \delta\} \\ &= 0. \end{aligned}$$

   Therefore $d_m(a,b) = 0$. If $d_m(a,b) = 0$, then

$$d_m(a,b) = \inf\{\delta > 0 : m(\delta,a,b) \leq \delta\}$$

$$0 = \inf\{\delta > 0 : m(\delta,a,b) \leq \delta\}.$$

   This implies that $m(\delta,a,b) = 0$ since $\delta \neq 0$. This means that $a = b$ because m is a modular metric on S.

2. We prove the symmetry condition. Since m is a modular metric on S, then m possesses the symmetry property. Thus we have

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : m(\delta, a, b) \leq \delta\} \\ &= \inf\{\delta > 0 : m(\delta, b, a) \leq \delta\} \\ &= d_m(b,a) \end{aligned}$$

   Therefore $d_m(a,b) = d_m(b,a)$.

3. Now we prove the triangle inequality (i.e. $d_m(a,b) \leq d_m(a,c) + d_m(c,b)$ for all $a,b,c \in S_m$).

   By the definition of d_m, for any $\delta, \gamma > 0$ we have

$$d_m(a,b) < \delta \qquad \mathit{and} \qquad d_m(c,b) < \gamma$$

   and furthermore we have

$$m(\delta, a, b) \leq \delta \qquad \mathit{and} \qquad m(\gamma, c, b) \leq \gamma.$$

   Since m is a modular metric on S, then

$$\begin{aligned} m(\delta+\gamma, a, b) &\leq m(\delta, a, c) + m(\gamma, c, b) \\ &\leq \delta + \gamma. \end{aligned}$$

   Moreover by the definition of $d_m(a,b)$ it follows that

$$d_m(a,b) \leq \delta + \gamma.$$

   Furthermore

$$\begin{aligned} d_m(a,b) &= \inf\{\delta, \gamma > 0 : m(\delta+\gamma, a, b) \leq \delta + \gamma\} \\ &\leq \inf\{\delta > 0 : m(\delta, a, c) \leq \delta\} + \inf\{\gamma > 0 : m(\gamma, c, b) \leq \gamma\} \\ &= d_m(a,c) + d_m(c,b). \end{aligned}$$

   Therefore we have

$$d_m(a,b) \leq d_m(a,c) + d_m(c,b) \quad \mathit{for\,all} \quad a,b,c \in S_m \quad \mathit{and} \quad \delta, \gamma > 0.$$

Thus d_m is a metric on the set on S_m. □

We now give examples for Theorem 2.2.7.

Example 2.2.3. [7, Example 2.7. (a)] Let S be a non-empty set and m be a modular metric on S. Consider the metric d_m on a modular set S_m in Theorem 2.2.7. If we let $S_m = \{a\}$, then $d_m(a,b) = 0$ for all $a,b \in S_m$.

Proof. Let $a, b \in S_m$ and $\delta > 0$. If $S_m = \{a\}$, then $a = b$ since S_m is a singleton. Now, by definition of d_m, we have

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : m(\delta, a, b) \leq \delta\} \\ &= \inf\{\delta > 0 : m(\delta, a, a) \leq \delta\} \\ &= 0 \end{aligned}$$

since $m(\delta, a, a) = 0$ as m is a modular metric on S.
Thus $d_m(a, b) = 0$ for all $a, b \in S_m$. □

Example 2.2.4. [7, Example 2.7. (b)] Consider the modular metric m in Example 2.1.2 on a metric space (S, d). If $f(\delta)$ is bounded from above, then $S_m = S_m(a) = \{a\}$. Hence $d_m(a, b) = 0$ for all $a, b \in S_m$.

Proof. The proof is the same as that of Example 2.2.2. □

Example 2.2.5. [7, Example 2.7. (c)] Given the modular metric m in Example 2.1.2 on a metric space (S, d). Let $a, b \in S_m$. If $f(\delta)$ is not bounded from above (i.e. $f(\delta) \to \infty$ as $\delta \to \infty$), then

$$S_m = S \qquad \text{and} \qquad d_m(a,b) = g^{-1}[d(a,b)]$$

where g is a strictly increasing function defined on $g : (0, \infty) \to (0, \infty)$ by

$$g(\delta) = \delta f(\delta)$$

for all $\delta > 0$ and g^{-1} is the inverse function of g.

Proof. We leave the proof of $S_m = S$ to the reader. We show that $d_m(a, b) = g^{-1}[d(a, b)]$.
Let $a, b \in S_m$. From Example 2.1.2, $m(\delta, a, b) = \dfrac{d(a, b)}{f(\delta)}$ and by definition of d_m, we have

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : m(\delta, a, b) \leq \delta\} \\ &= \inf\left\{\delta > 0 : \frac{d(a,b)}{f(\delta)} \leq \delta\right\} \\ &= \inf\{\delta > 0 : d(a,b) \leq \delta f(\delta)\}. \end{aligned}$$

Since $g(\delta) = \delta f(\delta)$, then we have

$$d_m(a,b) = \inf\{\delta > 0 : d(a,b) \leq g(\delta)\}.$$

Since g is strictly an increasing function on $(0, \infty)$, then $d_m(a, b) \leq g(\delta)$ implies that $g^{-1}[d(a, b)] \leq \delta$. Therefore

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : g^{-1}[d(a,b)] \leq \delta\} \\ &= g^{-1}[d(a,b)]. \end{aligned}$$

Thus we have established that $d_m(a, b) = g^{-1}[d(a, b)]$. □

Example 2.2.6. [7, Example 2.7. (d)] Consider the modular metric m in Example 2.1.2 on a metric space (S, d), where $S = \mathbb{R}$ and $d(a, b)$ is the usual metric on $\mathbb{R}$. Then

$$m(\delta, a, b) = \frac{|a-b|}{f(\delta)}$$

is a modular metric on $\mathbb{R}$ for all $a, b \in \mathbb{R}$ and $\delta > 0$.
Let $f(\delta) = \delta^u$ where $u > 0$ is an arbitrary constant such that $f(\delta) \to \infty$ as $\delta \to \infty$, then

$$d_m(a, b) = \Big[d(a, b)\Big]^{\dfrac{1}{u+1}}.$$

Proof. We refer the reader to Example 2.1.2 for the proof that

$$m(\delta, a, b) = \frac{|a-b|}{f(\delta)}$$

is a modular metric on $\mathbb{R}$.

Now we want to show that $d_m(a, b) = |a-b|^{\frac{1}{u+1}}$.

From definition of d_m, we have

$$\begin{aligned} d_m(a, b) &= \inf\{\delta > 0 : m(\delta, a, b) \le \delta\} \\ &= \inf\left\{\delta > 0 : \frac{d(a, b)}{f(\delta)} \le \delta\right\} \\ &= \inf\left\{\delta > 0 : \frac{|a-b|}{\delta^u} \le \delta\right\} \\ &= \inf\{\delta > 0 : |a-b| \le \delta^{u+1}\} \\ &= \inf\{\delta > 0 : |a-b|^{\frac{1}{u+1}} \le \delta\}. \end{aligned}$$

Thus

$$\begin{aligned} d_m(a, b) &= |a-b|^{\frac{1}{u+1}} \\ &= \Big[d(a, b)\Big]^{\frac{1}{u+1}}. \end{aligned}$$

□

Example 2.2.7. [7, Example 2.7. (d)] Let $\mathbb{R}$ be a set of real numbers and $q(a, b)$ be a usual metric in $\mathbb{R}$ for all $a, b \in \mathbb{R}$. Let u be a positive constant and let $f(\delta) = \delta^u$, then

$$m(\delta, a, b) = \frac{q(a, b)}{f(\delta)} = \frac{q(a, b)}{\delta^u}$$

is a modular metric on $\mathbb{R}$. Since $a \in \mathbb{R}$, then $\mathbb{R}_m(a) = \mathbb{R}$ and

$$d_m(a, b) = \big[q(a, b)\big]^{\frac{1}{u+1}}$$

The next theorem is one of the main theorems in Chistyakov's paper [7]. It shows that a modular set equipped with a metric defined on it is a metric space. Later we will present a corollary (see Corollary 2.2.9 on page 26) where we show that the metric d_m in Theorem 2.2.7 is equivalent to metric D_m in the Theorem 2.2.8. Moreover, we give examples to illustrate these theorems.

Theorem 2.2.8. *[7, Theorem 2.8.] Let S be a non-empty set and m be a modular metric defined on S. The function $D_m : S_m \times S_m \to [0, \infty)$ defined by*

$$D_m(a, b) = \inf\{\delta > 0 : \delta + m(\delta, a, b)\}$$

for all $a, b \in S_m$ and $\delta > 0$ is a metric on S_m.

Proof. We want to show that (S_m, D_m) is a metric space. Firstly we need to show that D_m is well defined then proceed to show that the function D_m satisfies the properties of a metric.
Let us show that D_m is well defined. Suppose $a, b \in S_m$. Then by Definition 2.2.3, we get that $m(\delta, a, b)$ in S_m is finite. Thus the set $\{\delta > 0 : \delta + m(\delta, a, b)\}$ is a non-empty subset of $\mathbb{R}^+$ (i.e. the set of positive real numbers) and is bounded below. Hence $D_m(a, b) \in [0, \infty)$. Therefore D_m is well defined.

We now show that D_m is a metric on S_m.

(i) If $a = b$, then

$$D_m(a, b) = \inf\{\delta > 0 : \delta + m(\delta, a, b) = \delta\}$$

because m is a modular metric and $m(\delta, a, b) = 0$ if and only if $a = b$ for all $a, b \in S_m$. Hence $D_m(a, b) = 0$.

We want to show that if $D_m(a, b) = 0$ then $a = b$.
Let $a, b \in S_m$ and $D_m(a, b) = 0$. For us to show that $a = b$, it is sufficient to show that $m(\delta, a, b) = 0$ for all $\delta > 0$. We show this by contradiction. Suppose there exists some $\delta_0 > 0$ such that for $\delta \geq \delta_0$, we have

$$m(\delta_0, a, b) > 0.$$

Therefore $\delta + m(\delta, a, b) \geq \delta_0$. But if $\delta < \delta_0$, then from Lemma 2.1.5 we have

$$m(\delta_0, a, b) \leq m(\delta, a, b) \leq \delta + m(\delta, a, b).$$

Let $\delta_1 = \min\{\delta_0, m(\delta_0, a, b)\}$ for all $\delta > 0$. Then we have $\delta + m(\delta, a, b) \geq \delta_1$. But by definition of D_m, we have

$$D_m(a, b) = \inf\{\delta > 0 : \delta + m(\delta, a, b) \geq \delta_1\}.$$

This is a contradiction to the assumption that $m(\delta_0, a, b) > 0$. Thus $m(\delta, a, b) = 0$ for all $\delta > 0$. Hence $a = b$ since m is a modular metric on S_m.

(ii) We show the symmetry property for D_m for $a, b \in S_m$.

$$\begin{aligned} D_m(a,b) &= \inf\{\delta > 0 : \delta + m(\delta, a, b)\} \\ &= \inf\{\delta > 0 : \delta + m(\delta, b, a)\} \\ &= D_m(b,a) \end{aligned}$$

since m is a modular metric and possesses the property $m(\delta, a, b) = m(\delta, b, a)$. Therefore $D_m(a,b) = D_m(b,a)$.

(iii) We show the triangle inequality. According to the definition of D_m, for any $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 0$ such that $\delta + m(\delta, a, c) \leq D_m(a,c) + \epsilon$ and $\gamma + m(\gamma, a, c) \leq D_m(a,c) + \epsilon$. Thus

$$\begin{aligned} D_m(a,b) &\leq (\delta + \gamma) + m(\delta + \gamma, a, b) \\ &\leq \delta + \gamma + m(\delta, a, c) + m(\gamma, c, b) \\ &\leq D_m(a,c) + \epsilon + D_m(c,b) + \epsilon \\ &= D_m(a,c) + D_m(c,b) + 2\epsilon \\ &= D_m(a,c) + D_m(c,b) \qquad as \qquad \epsilon \to 0. \end{aligned}$$

Therefore $D_m(a,b) \leq D_m(a,c) + D_m(c,b)$ for all $a, b, c \in S_m$.

Thus D_m is a metric on S_m and the pair (S_m, D_m) is a metric space of modular m. □

Example 2.2.8. [7, Example 2.9. (a)] Let S be a non-empty set. Given the modular metric m defined in Example 2.1.2 as

$$m(\delta, a, b) = \frac{d(a,b)}{f(\delta)}.$$

If $f(\delta) = \delta^u$ where $u > 0$ is some constant, then

$$D_m(a,b) = (u+1)u^{\frac{-u}{u+1}} \bigg[d(a,b) \bigg]^{\frac{1}{u+1}}$$

for all $a, b \in S_m$ and $\delta, \gamma > 0$.

Proof. According to Theorem 2.2.8, D_m is defined as

$$D_m(a,b) = \inf\{\delta > 0 : \delta + m(\delta, a, b)\}.$$

Substituting m and $f(\delta)$ we get

$$\begin{aligned} D_m(a,b) &= \inf\left\{\delta > 0 : \delta + \frac{d(a,b)}{f(\delta)}\right\} \\ &= \inf\left\{\delta > 0 : \delta + \frac{d(a,b)}{\delta^u}\right\}. \end{aligned}$$

Let $\phi(\delta) = \delta + \dfrac{d(a,b)}{\delta^u}$. So we have

$$D_m(a,b) = \inf\{\delta > 0 : \phi(\delta)\}.$$

The first derivative of ϕ with respect to δ yields

$$\phi'(\delta) = 1 - u\frac{d(a,b)}{\delta^{u+1}}.$$

The derivative $\phi'(\delta) = 0$ only at the point $\delta_0 = \big[ud(a,b)\big]^{\frac{1}{u+1}}$. Now, $\phi'(\delta) < 0$ for all $\delta < \delta_0$ and $\phi'(\delta) > 0$ for all $\delta > \delta_0$. This implies that the slope of the $\phi(\delta)$ changes from negative to positive at the point $(\delta_0, \phi(\delta_0))$. This implies that δ_0 is a global minimum on the interval $(0, \infty)$. Therefore at the global minimum, we have

$$\begin{aligned} D_m(a,b) &= \inf\left\{\delta > 0 : \delta_0 + \frac{d(a,b)}{\delta_0^u}\right\} \\ &= \inf\left\{\delta > 0 : \big[ud(a,b)\big]^{\frac{1}{u+1}} + \frac{d(a,b)}{\big[ud(a,b)\big]^{\frac{u}{u+1}}}\right\} \\ &= \inf\left\{\delta > 0 : u^{\frac{1}{u+1}}\big[d(a,b)\big]^{\frac{1}{u+1}} + \big[d(a,b)\big]^{\frac{1}{u+1}}\big[u\big]^{\frac{-u}{u+1}}\right\} \\ &= \inf\left\{\delta > 0 : (u+1)u^{\frac{-u}{u+1}}\big[d(a,b)\big]^{\frac{1}{u+1}}\right\}. \end{aligned}$$

Thus at the global minimum,

$$D_m(a,b) = \inf\left\{\delta > 0 : (u+1)u^{\frac{-u}{u+1}}\big[d(a,b)\big]^{\frac{1}{u+1}}\right\}.$$

□

Corollary 2.2.9. *[7, Theorem 2.8. (b)] If d_m is the metric in Theorem 2.2.7 and D_m is the metric in Theorem 2.2.8, then we have*

$$d_m(a,b) \le D_m(a,b) \le 2d_m(a,b)$$

for all $a, b \in S_m$.

Proof. We want to show that $d_m(a,b) \le D_m(a,b) \le 2d_m(a,b)$. Firstly we show that $d_m(a,b) \le D_m(a,b)$ and then show that $D_m(a,b) \le 2d_m(a,b)$.

(i) We show that $d_m(a,b) \le D_m(a,b)$. Let $\delta > 0$ and if $m(\delta,a,b) \le \delta$, then by definition of d_m in Theorem 2.2.7, we have $d_m(a,b) \le \delta$. If $m(\delta,a,b) > 0$, then

$$d_m(a,b) \le m(\delta,a,b).$$

Let $\gamma = m(\delta,a,b)$. Therefore we have $0 < \delta < \gamma$ and $d_m(a,b) \le \gamma$. From Lemma 2.1.5, we have $m(\gamma,a,b) \le m(\delta,a,b) = \gamma$ which implies that $d_m(a,b) \le \gamma = m(\gamma,a,b)$.

Thus for any $\delta > 0$, we have

$$\begin{aligned} d_m(a,b) &\leq \max\{\delta, m(\delta,a,b)\} \\ &= \delta + m(\delta,a,b). \end{aligned}$$

Taking the infimum over all $\delta > 0$, we get

$$d_m(a,b) \leq \inf\{\delta > 0 : \delta + m(\delta,a,b)\}$$

for all $a, b \in S_m$. Hence $d_m(a,b) \leq D_m(a,b)$.

(ii) Now we show that $D_m(a,b) \leq 2d_m(a,b)$.
From Theorem 2.2.7, we have $d_m(a,b) < \delta$ and $m(\delta,a,b) \leq \delta$ for $\delta > 0$. From Theorem 2.2.8, we have

$$D_m(a,b) \leq \delta + m(\delta,a,b) \leq 2\delta$$

since $m(\delta,a,b) \leq \delta$. Taking the limit as $\delta \to d_m(a,b)$, we get

$$D_m(a,b) \leq 2d_m(a,b).$$

Thus $d_m(a,b) \leq D_m(a,b) \leq 2d_m(a,b)$ for all $a, b \in S_m$.

□

In the next example, we want to illustrate Corollary 2.2.9 using the metric space (S_m, D_m) in Example 2.2.8 and the metric space (S_m, d_m) in Example 2.2.6.

Example 2.2.9. If D_m is a metric in Example 2.2.8 and d_m is a metric in Example 2.2.6, then we have

$$d_m(a,b) \leq D_m(a,b) \leq 2d_m(a,b)$$

for all $a, b \in S_m$.

Proof. We want to show that $d_m(a,b) \leq D_m(a,b) \leq 2d_m(a,b)$. We first show that $d_m(a,b) \leq D_m(a,b)$ and then show that $D_m(a,b) \leq 2d_m(a,b)$.

(i) From Example 2.2.6, d_m is given by

$$d_m(a,b) = \big[d(a,b)\big]^{\frac{1}{u+1}}$$

and from Example 2.2.8, D_m is given by

$$D_m(a,b) = (u+1)u^{\frac{-u}{u+1}}\big[d(a,b)\big]^{\frac{1}{u+1}}.$$

Comparing $d_m(a,b)$ and $D_m(a,b)$, they both have the part $\big[d(a,b)\big]^{\frac{1}{u+1}}$. Now looking at the factor $(u+1)u^{\frac{-u}{u+1}}$ in $D_m(a,b)$, where $u > 0$, then we have

$$1 \leq (u+1)u^{\frac{-u}{u+1}} \leq 2 \qquad for \quad all \quad u > 0.$$

This implies that $D_m(a,b)$ is obtained by multiplying $d_m(a,b)$ by $(u+1)u^{\frac{-u}{u+1}}$ which is greater than one. Hence $d_m(a,b) \leq D_m(a,b)$.

(ii) We want to show that $D_m(a,b) \leq 2d_m(a,b)$. If $u = 1$, then by definition of $d_m(a,b)$ in Example 2.2.6 we have

$$d_m(a,b) = \big[d(a,b)\big]^{\frac{1}{2}} = \sqrt{d(a,b)}$$

and by definition of $D_m(a,b)$ in Example 2.2.8 we have

$$D_m(a,b) = (1+1)(1)^{\frac{-1}{2}}[d(a,b)]^{\frac{1}{2}} = 2\sqrt{d(a,b)}$$

for all $a, b \in S_m$. Hence $D_m(a,b) \leq 2d_m(a,b)$.

Thus $d_m(a,b) \leq D_m(a,b) \leq 2d_m(a,b)$ for all $a, b \in S_m$.

□

Theorem 2.2.10. *[7, Theorem 2.10. (a)] Let m be a modular metric on a non-empty set S. Let d_m be a metric on a modular set S_m. If $d_m(a,b) < \delta$, then*

$$m(\delta,a,b) \leq d_m(a,b) < \delta$$

for all $a, b \in S_m$ and $\delta > 0$.

Proof. Consider $\gamma > 0$ such that $d_m(a,b) < \gamma < \delta$ for all $a, b \in S_m$ and $\delta > 0$. By definition of d_m in Theorem 2.2.7, then

$$m(\gamma,a,b) \leq \gamma$$

and by Lemma 2.1.5, we have

$$m(\delta,a,b) \leq m(\gamma,a,b)$$

hence

$$m(\delta,a,b) \leq \gamma.$$

Now, as $\gamma \to d_m(a,b)$, we have

$$m(\delta,a,b) \leq d_m(a,b) < \delta$$

□

Corollary 2.2.11. *[7, Theorem 2.10. (b)] Let S be a non-empty set and m be a modular metric defined on S. Consider d_m to be a metric on a modular set S_m. If $m(\delta,a,b) = \delta$, then*

$$d_m(a,b) = \delta$$

for all $a, b \in S_m$ and $\delta > 0$.

Proof. By the definition of d_m in Theorem 2.2.7,

$$d_m(a,b) \leq \delta.$$

According to the previous theorem, Theorem 2.2.10, if we let $m(\delta,a,b)=\delta$, then

$$d_m(a,b)=\delta.$$

□

2.3. Convex modular metric space

In this section, we look at the convexity of a non-empty set S endowed with a modular metric.

Definition 2.3.1. *Let S be a non-empty set. A function $m:(0,\infty)\times S\times S\to[0,\infty]$ is said to be a convex modular metric on S if it satisfies*

1. $m(\delta,a,b)=m(\delta,b,a)$ *for* $\delta>0$ *and* $a,b\in S$,
2. $m(\delta,a,b)=0$ *if and only if* $a=b$ *for all* $a,b\in S$ *and* $\delta>0$,
3. $m(\delta+\mu,a,b)\leq \dfrac{\delta}{\delta+\mu}m(\delta,a,c)+\dfrac{\mu}{\delta+\mu}m(\mu,c,b)$ *for all* $\delta,\mu>0$ *and* $a,b,c\in S$.

We note that axiom (3) of Definition 2.3.1 implies property (3) of Definition 2.1.1 hence convex modular metric is a modular metric which is convex.

Lemma 2.3.2. *Let m be a modular metric on a non-empty set S. Then m is convex if and only if the function $\hat{m}(\delta,a,b)=\delta m(\delta,a,b)$ is a modular metric on S for all $a,b\in S$ and $\delta>0$.*

Proof. Suppose m is convex. We prove that $\hat{m}$ is a modular metric on S.

(i) If $a,b\in S$ and $\delta>0$, then

$$\begin{aligned}\hat{m}(\delta,a,b)&=\delta m(\delta,a,b)\\&=\delta m(\delta,b,a)\\&=\hat{m}(\delta,b,a).\end{aligned}$$

(ii) If $\hat{m}(\delta,a,b)=0=\delta m(\delta,a,b)$ then $a=b$ and it is obvious that if $a=b$ then $\hat{m}(\delta,a,b)=0$.

(iii) We now prove the triangle inequality.

$$\begin{aligned}\hat{m}(\delta+\mu,a,b)&=(\delta+\mu)m(\delta+\mu,a,b)\\&\leq(\delta+\mu)\left(\frac{\delta}{\delta+\mu}m(\delta,a,c)+\frac{\mu}{\delta+\mu}m(\mu,c,b)\right).\end{aligned}$$

Since m is convex, then

$$\begin{aligned}\hat{m}(\delta+\mu,a,b) &\leq \delta m(\delta,a,c)+\mu m(\mu,c,b)\\ \hat{m}(\delta+\mu,a,b) &\leq \hat{m}(\delta,a,c)+\hat{m}(\mu,c,b)\end{aligned}$$

Now we suppose that $\hat{m}$ is a modular metric on S and we prove that m is convex on S. Since $\hat{m}$ is a modular metric on S, then we have

$$\hat{m}(\delta+\mu,a,b) \leq \hat{m}(\delta,a,c)+\hat{m}(\mu,c,b)$$

which implies that

$$(\delta+\mu)m(\delta+\mu,a,b) \leq \delta m(\delta,a,c)+\mu m(\mu,c,b).$$

Dividing throughout with $(\delta+\mu)$, then we have

$$m(\delta+\mu,a,b) \leq \frac{\delta}{\delta+\mu}m(\delta,a,c)+\frac{\mu}{\delta+\mu}m(\mu,c,b).$$

Therefore m is convex on S. □

We now give examples of a convex modular metric on S.

Example 2.3.1. Consider the modular metric in Example 2.1.2.
If $f(\delta)=\delta$ (and not $f(\delta)\equiv 1$ for all $\delta>0$), then

$$m(\delta,a,b)=\frac{d(a,b)}{\delta}$$

is a convex modular metric on S for all $a,b\in S$ and $\delta>0$.

Proof. To show that m is a convex modular metric on S, we need to show that m satisfies the axioms in Definition 2.3.1.

1. If $a=b$, then

   $$m(\delta,a,b)=\frac{d(a,b)}{\delta}=0$$

   because $d(a,b)$ is a metric on S and $d(a,b)=0$ if and only if $a=b$.
   Then $m(\delta,a,b)=0$.

   If $m(\delta,a,b)=0$, then

   $$0=\frac{d(a,b)}{\delta}$$

   which implies that $d(a,b)=0$. Now since $d(a,b)$ is a metric on S, then $a=b$.

2. For the symmetry condition, we have

$$\begin{aligned} m(\delta, a, b) &= \frac{d(a,b)}{\delta} \\ &= \frac{d(b,a)}{\delta} \\ &= m(\delta, b, a) \end{aligned}$$

since $d(a,b)$ is a metric on S and $d(a,b) = d(b,a)$. Hence $m(\delta, a, b) = m(\delta, b, a)$.

3. Since $d(a,b)$ is a metric on S, then $d(a,b) \leq d(a,c) + d(c,b)$. Then we have

$$\begin{aligned} m(\delta+\mu, a, b) &= \frac{d(a,b)}{\delta+\mu} \\ &\leq \frac{d(a,c)+d(c,b)}{\delta+\mu} \\ &= \frac{d(a,c)}{\delta+\mu} + \frac{d(c,b)}{\delta+\mu} \\ &= \frac{\delta}{\delta+\mu}\frac{d(a,c)}{\delta} + \frac{\mu}{\delta+\mu}\frac{d(c,b)}{\mu} \\ &= \frac{\delta}{\delta+\mu} m(\delta, a, c) + \frac{\mu}{\delta+\mu} m(\mu, c, b). \end{aligned}$$

Thus $m(\delta+\mu, a, b) \leq \dfrac{\delta}{\delta+\mu} m(\delta, a, c) + \dfrac{\mu}{\delta+\mu} m(\mu, c, b)$.

Hence $m(\delta, a, b)$ is a convex modular metric on S. ☐

Example 2.3.2. Given the modular metric in Example 2.1.2, if $f(\delta) = 1$, then $m(\delta, a, b)$ is not a convex modular metric on S for all $a, b \in S$ and $\delta > 0$.

Proof. If $f(\delta) = 1$, for any $\delta > 0$, then we have

$$m(\delta, a, b) = d(a,b).$$

Since $d(a,b)$ is only a metric on S and $d(a,b)$ does not posses the convexity property, thus $m(\delta, a, b)$ is not a convex modular metric. ☐

Example 2.3.3. Let (S,d) be a metric space and the function $g : [0,\infty] \to [0,\infty]$ be a convex function. Then

$$m(\delta, a, b) = g\left(\frac{d(a,b)}{\delta}\right)$$

is a convex modular metric on S for all $a, b \in S$ and $\delta > 0$.

Proof. To show that $m(\delta, a, b)$ is a convex modular metric, we need to prove that $m(\delta, a, b)$ satisfies the properties of Definition 2.3.1.

1. If $a = b$, then
$$\begin{aligned} m(\delta, a, b) &= g\left(\frac{d(a, b)}{\delta}\right) \\ &= g(0) \\ &= 0 \end{aligned}$$
since $d(a, b)$ is a metric on S and $d(a, b) = 0$ if $a = b$. Thus $m(\delta, a, b) = 0$.
If $m(\delta, a, b) = 0$, then
$$0 = g\left(\frac{d(a, b)}{\delta}\right).$$
Now $g\left(\frac{d(a, b)}{\delta}\right) = 0$ if and only if $\frac{d(a, b)}{\delta} = 0$, which implies that $d(a, b) = 0$. Moreover, if d is a metric on S, then $d(a, b) = 0$ implies that $a = b$.
Thus $a = b$.

2. For the symmetry condition
$$\begin{aligned} m(\delta, a, b) &= g\left(\frac{d(a, b)}{\delta}\right) \\ &= g\left(\frac{d(b, a)}{\delta}\right) \\ &= m(\delta, b, a) \end{aligned}$$
since d is a metric on S and $d(a, b) = d(b, a)$.

3. Lastly, we want to show that $m(\delta + \mu, a, b) \leq \frac{\delta}{\delta + \mu} m(\delta, a, c) + \frac{\mu}{\delta + \mu} m(\mu, c, b)$. We have
$$\begin{aligned} m(\delta + \mu, a, b) &= g\left(\frac{d(a, b)}{\delta + \mu}\right) \\ &\leq g\left(\frac{d(a, c) + d(c, b)}{\delta + \mu}\right) \\ &= g\left(\frac{d(a, c)}{\delta + \mu} + \frac{d(c, b)}{\delta + \mu}\right) \\ &= g\left(\frac{\delta}{\delta + \mu}\frac{d(a, c)}{\delta} + \frac{\mu}{\delta + \mu}\frac{d(c, b)}{\mu}\right) \end{aligned}$$
since d is a metric on S and $d(a, b) \leq d(a, c) + d(c, b)$ and g is convex. Moreover, we have
$$\begin{aligned} m(\delta + \mu, a, b) &\leq \frac{\delta}{\delta + \mu} g\left(\frac{d(a, c)}{\delta}\right) + \frac{\mu}{\delta + \mu} g\left(\frac{d(c, b)}{\mu}\right) \\ &= \frac{\delta}{\delta + \mu} m(\delta, a, c) + \frac{\mu}{\delta + \mu} m(\mu, c, b). \end{aligned}$$
Thus we have $m(\delta + \mu, a, b) \leq \frac{\delta}{\delta + \mu} m(\delta, a, c) + \frac{\mu}{\delta + \mu} m(\mu, c, b)$.
Hence $m(\delta, a, b)$ is a convex modular metric on S for all $a, b \in S$ and $\delta > 0$.

□

The following lemma can be compared to Lemma 3.2.5 (see page 47).

Lemma 2.3.3. *Let S be a non-empty set and m be a convex modular metric on S. The function $\lambda : (0, \infty) \to [0, \infty]$ defined by*

$$\lambda(\delta) = \delta m(\delta, a, b)$$

for all $\delta > 0$ and $a, b \in S$ is non-increasing.

Proof. Let $a, b \in S$ and $\delta, \mu \in (0, \infty)$ with $\delta > \mu$. Since $\delta - \mu > 0$ and m is a convex modular metric on S, we have

$$\begin{aligned} m(\delta, a, b) &= m(\delta - \mu + \mu, a, b) \\ &\leq \frac{\delta - \mu}{\delta} m(\delta - \mu, a, a) + \frac{\mu}{\delta} m(\mu, a, b). \end{aligned}$$

Therefore we have

$$m(\delta, a, b) \leq \frac{\mu}{\delta} m(\mu, a, b).$$

Hence

$$\lambda(\delta) = \delta m(\delta, a, b) \leq \mu m(\mu, a, b).$$

□

In the next section, we look at modular sets defined in a convex setting which Chistyakov called modular sets of a convex modular metric on set S and denoted them as S^*_m. Subsequently we look at the relationship between a modular set of a modular metric and a modular set of a convex modular metric.

2.4. Modular set of a convex modular metric space

Definition 2.4.1. *Let m be a modular metric on an empty set S together with the modular set S_m. Let $a \in S$, we define the set $S^*_m(a)$ by*

$$S^*_m(a) = \{b \in S : \exists \delta > 0, m(\delta, a, b) < \infty\}.$$

For all $a \in S$, we set $S^*_m = S^*_m(a)$.

Remark 2.4.2. *It is clear that from Section 2.2, the modular set $S_m \subset S^*_m$, that is, in general the inclusion is strict.*

Example 2.4.1. Given the modular metric m in Example 2.1.2 on a metric space (S, d), if $S = \{a\}$, then we have

$$S_m = \{a\} \qquad \text{and} \qquad S^*_m(a) = S^*_m = S$$

Proposition 2.4.3. *If m is a convex modular metric on a non-empty set S, then*

$$S_m(a) = S^*_m(a)$$

for all $a \in S$.

Proof. From Remark 2.4.2, we have $S_m(a) \subset S^*_m(a)$, then we have only to show that $S_m(a) \supseteq S^*_m(a)$ to show that $S_m(a) = S^*_m(a)$. Let $b \in S^*_m(a)$. Then we have $m(\gamma, a, b) < \infty$ for some $\gamma > 0$. For any $\delta > \gamma$, we have $m(\delta, a, b) \leq \frac{\gamma}{\delta} m(\gamma, a, b)$. Since $\lambda(\delta)$ is a non-increasing function from Lemma 2.3.3, then

$$\lim_{\delta \to \infty} m(\delta, a, b) \leq \lim_{\delta \to \infty} \frac{\gamma}{\delta} m(\gamma, a, b) = 0.$$

Therefore $b \in S_m(a)$. This shows that $S_m(a) \supseteq S^*_m(a)$, hence $S_m(a) = S^*_m(a)$. □

Remark 2.4.4. *If m is a convex modular metric on a non-empty set S and $a \in S$, then*

$$\begin{aligned} S^*_{\hat{m}}(a) &= \{b \in S : \exists \delta > 0, \hat{m}(\delta, a, b) < \infty\} \\ &= \{b \in S : \exists \delta > 0, \delta m(\delta, a, b) < \infty\} \\ &= \{b \in S : \exists \delta > 0, m(\delta, a, b) < \infty\} \\ &= S^*_m. \end{aligned}$$

Hence $S^*_{\hat{m}}(a) = S^*_m(a)$.

Remark 2.4.5. *If m is a convex modular metric on a non-empty set S and $a, b \in S$, then*

$$\begin{aligned} d_{\hat{m}}(a, b) &= \inf\{\delta > 0 : \hat{m}(\delta, a, b) \leq \delta\} \\ &= \inf\{\delta > 0 : \delta m(\delta, a, b) \leq \delta\} \\ &= \inf\{\delta > 0 : m(\delta, a, b) \leq 1\}. \end{aligned}$$

Theorem 2.4.6. *Let m be a convex modular metric on a non-empty set S. Then the function d^*_m defined by*

$$d^*_m = \inf\{\delta > 0 : m(\delta, a, b) \leq 1\}$$

for all $a, b \in S^*_m$ is a metric on S^*_m.

Proof. We firstly show that d^*_m is well defined and then show that d^*_m satisfies the properties of a metric. Let $a, b \in S^*_m$. Then there exists $\delta, \gamma > 0$ such that

$$m(\delta, a, b) < \infty \qquad \textit{and} \qquad m(\gamma, a, b) < \infty.$$

Let $\mu > 0$ such that $\mu \geq \delta + \gamma$. Since m is convex, we have

$$\mu m(\mu, a, b) \geq (\delta + \gamma) + m(\delta + \gamma, a, b)$$

and

$$m(\delta+\gamma, a, b) \leq \frac{\delta}{\delta+\gamma} m(\delta, a, c) + \frac{\gamma}{\delta+\gamma} m(\gamma, c, b)$$

then

$$m(\mu, a, b) \leq \frac{\delta}{\mu} m(\delta, a, c) + \frac{\gamma}{\mu} m(\gamma, c, b).$$

Moreover

$$\lim_{\mu\to\infty} m(\mu, a, b) \leq \lim_{\mu\to\infty} \frac{\delta}{\mu} m(\delta, a, c) + \lim_{\mu\to\infty} \frac{\gamma}{\mu} m(\gamma, c, b).$$

Hence $\lim\limits_{\mu\to\infty} m(\mu, a, b) = 0$. By Lemma 2.3.3, $\lambda(\delta)$ is non-increasing, then there exists $\mu > \mu_0$ such that $m(\mu_0, a, b) \leq 1$. So $d_m^*(a, b) \leq \mu_0 < \infty$. Therefore d_m^* is well defined.

Now we show that d_m^* satisfies the properties of a metric.

(i) The symmetry condition is obvious i.e. $d_m^*(a, b) = d_m^*(b, a)$.

(ii) If $a = b$, then $m(\delta, a, b) = 0$, hence $d_m^*(a, b) = 0 < 1$. If $d_m^*(a, b) = 0$, then $m(\delta, a, b) \leq 1$ for some $\delta > 0$. Let $\mu > 0$ such that $\delta > \mu$. It follows that

$$m(\delta, a, b) \leq \frac{\mu}{\delta} m(\mu, a, b) \leq \frac{\mu}{\delta}.$$

Since $\lambda(\delta)$ is non-increasing. Now we have

$$\lim_{\mu\to 0} m(\delta, a, b) \leq \lim_{\mu\to 0} \frac{\mu}{\delta} = 0.$$

Thus $m(\delta, a, b) = 0$ for all $\delta > 0$. Hence $a = b$.

(iii) Now we show that the triangle inequality holds. Let $\delta > d_m^*(a, c)$ and $\gamma > d_m^*(c, b)$. By definition of d_m^*, we have

$$m(\delta, a, c) \leq 1 \qquad \textit{and} \qquad m(\gamma, c, b) \leq 1.$$

Since m is a convex modular metric, we have

$$m(\delta+\gamma, a, b) \leq \frac{\delta}{\delta+\gamma} m(\delta, a, c) + \frac{\gamma}{\delta+\gamma} m(\gamma, c, b).$$

Therefore we have

$$m(\delta+\gamma, a, b) \leq \frac{\delta}{\delta+\gamma} + \frac{\gamma}{\delta+\gamma} = 1.$$

This implies $d_m^*(a, b) \leq \delta + \gamma$. Hence $d_m^*(a, b) \leq d_m^*(a, c) + d_m^*(c, b)$.

□

Theorem 2.4.7. *Let m be a convex modular metric on a non-empty set S. Then the function $D_m^* : S_m^* \times S_m^* \to [0, \infty)$ defined by*

$$D_m^*(a, b) = \inf\{\delta > 0 : \delta + \delta m(\delta, a, b)\}$$

for all $a, b \in S_m^*$ is a metric on S_m^*. Hence the pair (S_m^*, D_m^*) is a metric space.

Proof. Before we prove that D_m^* satisfies the axioms of a metric we start by showing that D_m^* is well defined.

The set $\{\delta > 0 : \delta + \delta m(\delta, a, b)\}$ is non-empty and bounded from above (i.e $0 \leq D_m^*(a,b) < \infty$) for all $a, b \in S_m^*$. Hence the function D_m^* is well defined.

1. If $a = b$, then by definition of D_m^*, we have

$$\begin{aligned} D_m^*(a,b) &= \inf\{\delta > 0 : \delta + \delta m(\delta, a, b)\} \\ &= \inf\{\delta > 0 : \delta + \delta(0)\} \\ &= \inf\{\delta > 0 : \delta\} \\ &= 0 \end{aligned}$$

since $m(\delta, a, b)$ is a convex modular metric $m(\delta, a, b) = 0$ if $a = b$.

If $D_m^*(a,b) = 0$, then we have

$$0 = \inf\{\delta > 0 : \delta + \delta m(\delta, a, b)\}.$$

Since $\delta \neq 0$, then $m(\delta, a, b) = 0$ which implies $a = b$ for $\delta > 0$ and S_m^*. Suppose on the contrary, we have $m(\delta_0, a, b) > 0$ for some δ_0. Now if $\delta \geq \delta_0$, then $\delta + \delta m(\delta, a, b) > 0$. So we have

$$\{\delta > 0 : \delta + \delta m(\delta, a, b)\} \geq \delta_0.$$

If $0 < \delta < \delta_0$, then by Lemma 2.3.3, we have

$$\delta_0 m(\delta_0, a, b) \leq \delta m(\delta, a, b) \leq \delta + \delta m(\delta, a, b).$$

Now, for all $\delta > 0$, we have

$$\delta + \delta m(\delta, a, b) \geq \min\{\delta_0, \delta_0 m(\delta_0, a, b)\}.$$

Let $\min\{\delta_0, \delta_0 m(\delta_0, a, b)\} = \delta_1$, then we have

$$D_m^*(a,b) \geq \delta_1$$

which is a contradiction since $D_m^*(a,b) = 0$. Thus $m(\delta, a, b) = 0$ which suggests that $a = b$ for all $\delta > 0$. Thus $a = b$.

2. We have $m(\delta, a, b) = m(\delta, b, a)$ since m is a convex modular metric. By definition of D_m^*, we have

$$\begin{aligned} D_m^*(a,b) &= \inf\{\delta > 0 : \delta + \delta m(\delta, a, b)\} \\ &= \inf\{\delta > 0 : \delta + \delta m(\delta, b, a)\} \\ &= D_m^*(b,a). \end{aligned}$$

3. We want to show that $D_m^*(a,b) \le D_m^*(a,c) + D_m^*(c,b)$. Let $a,b \in S_m^*$ and $\delta > 0$, by definition of D_m^*, we can find that $\delta = \delta(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 0$ for some $\epsilon > 0$ such that

$$\delta + \delta m(\delta,a,c) \le D_m^*(a,b) + \epsilon$$

and

$$\gamma + \gamma m(\gamma,c,b) \ge D_m^*(c,b) + \epsilon.$$

Since m is a convex modular metric, then we have

$$\begin{aligned}
D_m^*(a,b) &= \inf\{\delta > 0 : \delta + \delta m(\delta,a,b)\} \\
&\le \{(\delta+\gamma) + (\delta+\gamma)m(\delta+\gamma,a,b)\} \\
&\le \left\{(\delta+\gamma) + (\delta+\gamma)\left[\frac{\delta}{\delta+\gamma}m(\delta,a,c) + \frac{\gamma}{\delta+\gamma}m(\gamma,c,b)\right]\right\} \\
&= \delta + \delta m(\delta,a,c) + \gamma + \gamma m(\gamma,c,b) \\
&D_m^*(a,c) + D_m^*(c,b) + 2\epsilon \\
&= D_m^*(a,c) + D_m^*(c,b) \quad as \quad \epsilon \to 0.
\end{aligned}$$

Hence $D_m^*(a,b) \le D_m^*(a,c) + D_m^*(c,b)$ for all $a,b \in S_m^*$.
Thus D_m^* is a metric on S_m^* and the pair (S_m^*, D_m^*) is a metric space.

□

Corollary 2.4.8. *Let m be a convex modular metric on S. Suppose d_m^* is a metric defined in Theorem 2.4.6 and D_m^* is a metric defined in Theorem 2.4.7, then*

$$d_m^*(a,b) \le D_m^*(a,b) \le 2d_m^*(a,b)$$

for all $a,b \in S_m^*$.

Proof. We will firstly show that $d_m^*(a,b) \le D_m^*(a,b)$ then later show that $D_m^*(a,b) \le 2d_m^*(a,b)$.

1. We want to illustrate that $d_m^*(a,b) \le D_m^*(a,b)$. Suppose $\delta > 0$ and $m(\delta,a,b) \le 1$, then by definition of $d_m^*(a,b)$ we have $d_m^*(a,b) \le \delta$. But if $m(\delta,a,b) > 1$ for some $\delta > 0$, then $d_m^*(a,b) \le \delta m(\delta,a,b)$.
Let $\mu = \delta m(\delta,a,b)$ such that $0 < \delta < \mu$, then by Lemma 2.3.3,

$$\mu m(\mu,c,b) \le \delta m(\delta,a,b) = \mu$$

since $\mu = \delta m(\delta,a,b)$. Dividing by μ both sides we get that

$$m(\mu,a,b) \le 1.$$

Thus $d_m^*(a,b) \le \mu = \delta m(\delta,a,b)$, which implies that

$$\begin{aligned}
d_m^*(a,b) &\le \max\{\delta, \delta m(\delta,a,b)\} \\
&\le \delta + \delta m(\delta,a,b) \quad for \quad all \quad \delta > 0.
\end{aligned}$$

But since by definition of D_m^*, $D_m^*(a,b) = \inf\{\delta > 0 : \delta + \delta m(\delta, a, b)\}$, then we have $d_m^* \leq D_m^*$ for all $a, b \in S_m^*$.

2. Now we want to show that $D_m^*(a,b) \leq 2d_m^*(a,b)$.

   From definition of $d_m^*(a,b)$, we note that $m(\delta, a, b) \leq 1$ for all $\delta > d_m^*(a,b)$. When we use the same observation to the definition of $D_m^*(a,b)$, we get that

$$\begin{aligned} D_m^*(a,b) &\leq \delta + \delta m(\delta, a, b) \\ &\leq \delta + \delta(1) \\ &\leq 2\delta. \end{aligned}$$

   Now we let $\delta \to d_m^*(a,b)$, then $D_m^*(a,b) \leq 2\delta = 2d_m^*(a,b)$. Hence

$$D_m^*(a,b) \leq 2d_m^*(a,b).$$

   Finally we have

$$d_m^*(a,b) \leq D_m^*(a,b) \leq 2d_m^*(a,b)$$

   for all $a, b \in S_m^*$.

□

Theorem 2.4.9. *[7, Theorem 3.9] Suppose m is a convex modular metric on a non-empty set S and let d_m^* and d_m be metrics defined in Theorem 2.4.6 and Theorem 2.2.7 respectively. If $d_m^* < 1$ or $d_m < 1$, then*

$$d_m^*(a,b) \leq d_m(a,b) \leq \sqrt{d_m^*(a,b)}$$

otherwise

$$\sqrt{d_m^*(a,b)} \leq d_m(a,b) \leq d_m^*(a,b)$$

for all $a, b \in S_m^*$

Proof. We refer the reader to [7]. □

In the next example, we want to illustrate that in Theorem 2.4.9 if m is not convex, then the results do not hold.

Example 2.4.2. Let (S, d) be a metric space. Let

$$m(\delta, a, b) = \frac{d(a,b)}{\delta + d(a,b)}$$

be a non-convex modular metric on S. Then the inequalities in Theorem 2.4.9 do not hold.

Proof. We start by substituting a non-convex modular metric in the definition of d_m^* and d_m then compare.

By definition of $d_m(a,b)$, we have

$$\begin{aligned} d_m(a,b) &= \inf\{\delta > 0 : m(\delta,a,b) \leq \delta\} \\ &= \inf\left\{\delta > 0 : \frac{d(a,b)}{\delta + d(a,b)} \leq \delta\right\} \\ &= \inf\{\delta > 0 : d(a,b) \leq \delta^2 + \delta d(a,b)\} \\ &= \inf\{\delta > 0 : d(a,b) - \delta^2 - \delta d(a,b) \leq 0\}. \end{aligned}$$

Now, solving $d(a,b) - \delta^2 - \delta d(a,b) \leq 0$ for $\delta > 0$, we get that

$$\delta = \frac{1}{2}\left(\sqrt{[d(a,b)]^2 + 4d(a,b)} - d(a,b)\right).$$

Therefore we have

$$d_m(a,b) = \frac{1}{2}\left(\sqrt{[d(a,b)]^2 + 4d(a,b)} - d(a,b)\right)$$

for all $a, b \in S$.

By definition of $d_m^*(a,b)$, we have

$$\begin{aligned} d_m^*(a,b) &= \inf\{\delta > 0 : m(\delta,a,b) \leq 1\} \\ &= \inf\left\{\delta > 0 : \frac{d(a,b)}{\delta + d(a,b)} \leq 1\right\} \\ &= \inf\{\delta > 0 : d(a,b) \leq \delta + d(a,b)\} \\ &= \inf\{\delta > 0 : 0 \leq \delta\} \\ &= 0. \end{aligned}$$

Therefore we have $d_m^*(a,b) = 0$. Furthermore

$$d_m^*(a,b) = 0 < 1 \quad \textit{and} \quad d_m(a,b) = \frac{1}{2}\left(\sqrt{[d(a,b)]^2 + 4d(a,b)} - d(a,b)\right).$$

Moreover, inequalities in Theorem 2.4.9 do not hold.
Thus inequalities $\sqrt{d_m^*(a,b)} \leq d_m(a,b) \leq d_m^*(a,b)$ do not hold. □

2.5. Convergence in a modular space

Our next study is convergence of sequences in modular metric spaces.

Definition 2.5.1. *([7]) Let S be a non-empty set and m a metric modular on S. Consider a sequence (a_n) on modular set S_m.*
The sequence (a_n) is said to converge to $a \in S_m$ denoted as $(a_n) \to a$ if $m(\delta, a_n, a) \to 0$ as $n \to \infty$ for all $\delta > 0$.

Theorem 2.5.2. *([7, Theorem 2.13.]) Let S be a non-empty set and m be a modular metric on S. If (a_n) is a sequence in S_m, and $a \in S_m$, then $d_m(a_n, a) \to 0$ as $n \to \infty$ if and only if $m(\delta, a_n, a) \to 0$ as $n \to \infty$ for all $\delta > 0$.*

Proof. Suppose $d_m(a_n, a) \to 0$ as $n \to \infty$. Given any arbitrary $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $d_m(a_n, a) < \epsilon$ for all $n \geq n(\epsilon)$. Since

$$d_m(a_n, a) = \inf\{\delta > 0 : m(\delta, a_n, a) \leq \delta\}$$

for each $\epsilon > 0$ and some $\delta > 0$, we have two cases, either (a) $0 < \epsilon < \delta$ or (b) $\epsilon \geq \delta$.

(a) If $0 < \epsilon < \delta$, then there exists $n(\epsilon) \in \mathbb{N}$ such that for any $n \leq n(\epsilon)$, $d_m(a_n, a) < \epsilon$. We get $m(\epsilon, a_n, a) \leq \epsilon$ for $n > n(\epsilon)$.

Moreover we have

$$m(\delta, a_n, a) \leq m(\epsilon, a_n, a) \leq \epsilon$$

for all $n > n(\epsilon)$ since $\epsilon < \delta$. Hence $m(\delta, a_n, a) \to 0$ as $n \to \infty$ for all $\delta > 0$.

(b) If $\epsilon \geq \delta$, let $\delta = \dfrac{\epsilon}{2}$ and we have

$$m(\delta, a_n, a) \leq m\left(\frac{\delta}{2}, a_n, a\right) \leq \epsilon$$

for $n > n(\epsilon)$ hence $m(\delta, a_n, a) \to 0$ as $n \to \infty$ for all $\delta > 0$.

Suppose $m(\delta, a_n, a) \to 0$ as $n \to \infty$ for all $\delta > 0$. Then there exists $n(\delta) \in \mathbb{N}$ such that $m(\delta, a_n, a) \leq \delta$ for all $n \geq n(\delta)$. Thus $d_m(\delta, a_n, a) \leq \delta$ whenever $n \geq n(\delta)$. Hence

$$d_m(a_n, a) \to 0 \quad as \quad n \to \infty.$$

□

3

Modular quasi-pseudometric spaces

In this chapter, we start our investigations on modular quasi-pseudometric spaces. Firstly, we define a modular quasi-pseudometric and give some interesting examples, and secondly we define a convex modular quasi-pseudometric. Thirdly, we then study convergence on modular quasi-pseudometric spaces.

3.1. Modular quasi-pseudometric space

In this section, we define a modular quasi-pseudometric and its conjugate.

Definition 3.1.1. *(Compare Definition 2.1.1) Consider a non-empty set* X. *A function* $w : (0,\infty) \times X \times X \to [0,\infty]$ *is said to be a modular quasi-pseudometric on* X *if it satisfies the following conditions:*

(i) $w(\lambda, x, x) = 0$ *whenever* $x \in X$ *and* $\lambda \in (0,\infty)$*,*

(ii) $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y)$ *whenever* $x, y, z \in X$ *and* $\lambda, \mu \in (0,\infty)$.

We shall say that w is a modular quasi-metric provided that w also satisfies the condition: whenever $x, y \in X$ and $\lambda \in (0,\infty)$,
$w(\lambda, x, y) = 0$ and $w(\lambda, y, x) = 0$ imply $x = y$.

Remark 3.1.2. *Let* w *be a modular quasi-pseudometric on a set* X. *Then the function* $w^{-1} : (0,\infty)\times X\times X \to [0,\infty]$ *defined by* $w^{-1}(\lambda, x, y) = w(\lambda, y, x)$ *whenever* $x, y \in X$ *and* $\lambda \in (0,\infty)$ *is also a modular quasi-pseudometric on* X *that we call the conjugate modular quasi-pseudometric of* w. *If* $w = w^{-1}$, *then* w *is a modular pseudometric on* X.

Remark 3.1.3. *For any quasi-(pseudo)metric modular* w *on* X, *the function* $w^s(\lambda, x, y) = \max\{w(\lambda, x, y), w^{-1}(\lambda, y, x)\}$ *whenever* $x, y \in X$ *and* $\lambda \in (0,\infty)$ *is a modular (pseudo)metric on* X *in the sense of Chistyakov (see Definition 2.1.1).*

For any modular quasi-pseudometric w on a set X, if $w = w^{-1}$, then w is a modular pseudometric on X.

The following result is an asymmetric version of Lemma 2.1.5.

Lemma 3.1.4. *If w is a modular quasi-pseudometric on a non-empty set X, then the function $\phi : (0, \infty) \to [0, \infty]$ defined by $\phi(\lambda) = w(\lambda, x, y)$ whenever $x, y \in X$ and $\lambda \in (0, \infty)$ is non-increasing.*

Proof. Let $x, y \in X$ and $\lambda, \mu \in (0, \infty)$ with $\mu < \lambda$. Since $\lambda - \mu > 0$, we have

$$\phi(\lambda) = w(\lambda, x, y) = w(\lambda - \mu + \mu, x, y) \leq w(\lambda - \mu, x, x) + w(\mu, x, y)$$

by Definition 3.1.1(ii). It follows that

$$\phi(\lambda) = w(\lambda, x, y) \leq w(\mu, x, y) = \phi(\mu)$$

since $w(\lambda - \mu, x, x) = 0$ as w is a modular quasi-pseudometric. Therefore ϕ is a non-increasing function. $\square$

Example 3.1.1. Let $X = \mathbb{R}$. We define $w : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty]$ by

$$w(\lambda, x, y) = \begin{cases} \infty & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

whenever $\lambda > 0$. Then w is a modular quasi-metric on $\mathbb{R}$. Furthermore, the symmetrized w^s of w is exactly the metric modular in Example 2.1.1.

Example 3.1.2. Let (X, d) be a quasi-(pseudo)metric space and $\psi : (0, \infty) \to (0, \infty)$ be a non-decreasing function. Then the function $w : (0, \infty) \times X \times X \to [0, \infty]$ defined by

$$w(\lambda, x, y) = \frac{d(x, y)}{\psi(\lambda)}$$

is a modular quasi-(pseudo)metric on X.

Proof. For any $\lambda > 0$ and for any $x \in X$, it is easy to see that $w(\lambda, x, x) = 0$ since $\psi(\lambda) \neq 0$ and d is a quasi-pseudometric on X.

Let us check that w satisfies condition (ii) of Definition 3.1.1. Let $x, y, z \in X$ and $\lambda, \mu > 0$. Since, ψ is a non-decreasing function on $(0, \infty)$, we have

$$\frac{1}{\psi(\lambda + \mu)} \leq \frac{1}{\psi(\lambda)} \quad \text{and} \quad \frac{1}{\psi(\lambda + \mu)} \leq \frac{1}{\psi(\mu)}$$

since $\lambda < \lambda + \mu$ and $\mu < \lambda + \mu$. By the triangle inequality of d, we have $d(x, y) \leq d(x, z) + d(z, y)$. It follows that

$$\frac{d(x, y)}{\psi(\lambda + \mu)} \leq \frac{d(x, z) + d(z, y)}{\psi(\lambda + \mu)} \leq \frac{d(x, z)}{\psi(\lambda)} + \frac{d(z, y)}{\psi(\mu)}.$$

Therefore $w(\lambda+\mu,x,y) \leq w(\lambda,x,z)+w(\mu,z,y)$ whenever $x,y,z \in X$ and $\lambda,\mu > 0$. Furthermore, if d is a quasi-metric and $\lambda > 0$, then $w(\lambda,x,y) = 0$ and $w(\lambda,y,x) = 0$ imply $d(x,y) = 0$ and $d(y,x) = 0$ which in turn imply that $x = y$. ☐

Remark 3.1.5. *Observe that the w^s of Example 3.1.2 is exactly the same as in Example 2.1.2. Let w be a modular quasi-pseudometric on a non-empty set X. We define the relation $\overset{w}{\approx}$ on X by*

$$x \overset{w}{\approx} y \quad \textit{if} \quad \textit{and} \quad \textit{only} \quad \textit{if} \quad \lim_{\lambda\to\infty} w(\lambda,x,y) = 0 = \lim_{\lambda\to\infty} w(\lambda,y,x).$$

Lemma 3.1.6. *If w is a modular quasi-pseudometric on a non-empty set X, then the relation $\overset{w}{\approx}$ defined above is an equivalence relation on X.*

Proof. Since w is a modular quasi-pseudometric, then $w(\lambda,x,x) = 0$ whenever $x \in X$ and $\lambda > 0$. It follows that

$$\lim_{\lambda\to\infty} w(\lambda,x,x) = 0.$$

So $x \overset{w}{\approx} x$.

One observes that $\overset{w}{\approx}$ is symmetric from the definition.

Let $x,y,z \in X$, if $x \overset{w}{\approx} z$ and $z \overset{w}{\approx} y$, then we prove that $x \overset{w}{\approx} y$. Since $x \overset{w}{\approx} z$ then

$$\lim_{\frac{\lambda}{2}\to\infty} w\left(\frac{\lambda}{2},x,z\right) = 0 = \lim_{\frac{\lambda}{2}\to\infty} w\left(\frac{\lambda}{2},z,x\right)$$

and since $z \overset{w}{\approx} y$ then

$$\lim_{\frac{\lambda}{2}\to\infty} w\left(\frac{\lambda}{2},z,y\right) = 0 = \lim_{\frac{\lambda}{2}\to\infty} w\left(\frac{\lambda}{2},y,z\right).$$

We have $w(\lambda,x,y) = w\left(\frac{\lambda}{2}+\frac{\lambda}{2},x,y\right) \leq w\left(\frac{\lambda}{2},x,z\right) + \left(\frac{\lambda}{2},z,y\right)$ since w satisfies condition (ii) of Definition 3.1.1. Hence

$$\lim_{\frac{\lambda}{2}\to\infty} w(\lambda,x,y) = 0.$$

Similarly, one shows that

$$\lim_{\frac{\lambda}{2}\to\infty} w(\lambda,y,x) = 0.$$

Therefore $x \overset{w}{\approx} y$.

☐

Thus for any modular quasi-pseudometric w on a non-empty set X, the relation $\overset{w}{\approx}$ is an equivalence relation on X. Then for any $x \in X$, we denote by $X_w(x) = \{y \in X : y \overset{w}{\approx} x\}$ the equivalence class of x and the quotient set is denoted by $X/\overset{w}{\approx}$. Consider an element $x_0 \in X$, we set $X_w = X_w(x_0)$. Then the set X_w is called a modular set.

Example 3.1.3. Let (X, d) be a quasi-pseudometric space and w be the modular quasi-pseudometric in Example 3.1.2. Let $a \in X$. Then

$$X_w(a) = \left\{ x \in X : \lim_{\lambda \to \infty} \frac{d(x,a)}{\psi(\lambda)} = 0 = \lim_{\lambda \to \infty} \frac{d(a,x)}{\psi(\lambda)} \right\}.$$

(a) If $\psi(\lambda) \to \infty$ whenever $\lambda \to \infty$, then

$$\lim_{\lambda \to \infty} \frac{d(x,a)}{\psi(\lambda)} = 0$$

and

$$\lim_{\lambda \to \infty} \frac{d(x,a)}{\psi(\lambda)} = 0.$$

Therefore, $X_w(a) = X$.

(b) If ψ is bounded above, then $\lim\limits_{\lambda \to \infty} \psi(\lambda) < \infty$. It follows that $d(x,a) = 0$ and $d(a,x) = 0$. Thus $X_w(a) = \{a\}$ if d is T_0-quasimetric on X. Moreover, if d is not a T_0-quasi-pseudometric on X, then $X_w(a) \neq \{a\}$.

The following result can be compared with [7, Theorem 2.6].

Theorem 3.1.7. *Let X be a non-empty set and w be a modular quasi-metric on X. Then the function d_w on X_w defined by*

$$d_w(x,y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$$

whenever $x, y \in X_w$ is a T_0-quasi-metric on X_w.

Proof. For any $x, y \in X_w$, we have that $d_w(x,y) \in [0, \infty)$ since $x \overset{w}{\approx} x_0$ and $x_0 \overset{w}{\approx} y$, it follows that $x \overset{w}{\approx} y$ i.e. $\lim\limits_{\lambda \to \infty} w(\lambda, x, y) = 0 = \lim\limits_{\lambda \to \infty} w(\lambda, y, x)$. Moreover, there exists $\lambda_0 > 0$ such that $w(\lambda, x, y) \leq 1$ whenever $\lambda \geq \lambda_0$. By setting $\lambda_1 = \max\{1, \lambda_0\}$, it follows that $w(\lambda, x, y) \leq 1 \leq \lambda_1$. Then $d_w(x,y) = \inf \lambda : w(\lambda, x, y) \leq \lambda \leq \lambda_1 < \infty$. Therefore, the function d_w is well defined. Let $x \in X_w$. Then we have that $w(\lambda, x, x) = 0 < \lambda$ whenever $\lambda > 0$, hence $d_w(x,x) = 0$. Lastly we show that the triangle inequality holds. For all $x, y, z \in X_w$. By definition of $d_w(x,z)$, it follows that

$$\lambda > d_w(x,z) \qquad and \qquad w(\lambda, x, z) \leq \lambda \tag{3.1}$$

whenever $\lambda > 0$. Furthermore, $d_w(z,y)$ implies

$$\mu > d_w(z,y) \qquad and \qquad w(\mu, z, y) \leq \mu \tag{3.2}$$

whenever $\mu > 0$. Thus,

$$w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y) \leq \lambda + \mu$$

since w is a modular quasi-metric on X. Combining inequalities (3.1) and (3.2), we have $d_w(x,y) \leq \lambda + \mu$. It follows that

$$d_w(x,y) \leq d_w(x,z) + d_w(z,y) \qquad \textit{whenever} \qquad x,y,z \in X_w.$$

Now we prove that d_w is T_0. Suppose that $d_w(x,y) = 0$ and $d_w(y,x) = 0$. From $d_w(x,y) = 0$, it follows that $w(\mu,x,y) = 0$ for some $\mu > 0$. It follows that for any $\lambda > 0$ with $0 < \mu < \lambda$ and by Lemma 3.1.4, we have $w(\lambda,x,y) \leq w(\mu,x,y) = 0$. Thus

$$w(\lambda,x,y) = 0 \tag{3.3}$$

whenever $\lambda > 0$. From $d_w(y,x) = 0$, it follows by a similar argument that

$$w(\lambda,y,x) = 0 \tag{3.4}$$

whenever $\lambda > 0$. Therefore, by combining (3.3) and (3.4), $x = y$ since w is a modular quasi-metric on X. □

Example 3.1.4. Consider the modular quasi-pseudometric w in Example 3.1.2 on a quasi-pseudometric (X,d). Let $a \in X$. Since $X_w = X_w(a) = X$ if $\psi(\lambda) \to \infty$ whenever $\lambda \to \infty$ from Example 3.1.3(a). We have that

$$d_w(x,y) = \inf\{\lambda : w(\lambda,x,y) \leq \lambda\}.$$

We know that $w(\lambda,x,y) = \dfrac{d(x,a)}{\psi(\lambda)} \leq \lambda$. Then $\phi^{-1}(d(x,y)) \leq \lambda$ where ϕ is a non-decreasing function on $(0,\infty)$ into $(0,\infty)$ defined by $\phi(\lambda) = \lambda\psi(\lambda)$ whenever $\lambda > 0$. It follows that

$$d_w(x,y) = \phi^{-1}(d(x,y))$$

whenever $x,y \in X_w$.

Example 3.1.5. Let (X,d) be a T_0- quasi-metric space and w be the modular quasi-metric in Example 3.1.2 on (X,d). Then, $X_w = X_w(a) = \{a\}$ if ψ is bounded above. Then obviously, $d_w(x,y) = 0$ whenever $x,y \in X_w$.

Example 3.1.6. Consider the set of real numbers $\mathbb{R}$ equipped with its usual T_0-quasimetric u defined by $u(x,y) = \max\{x-y,0\}$. Let p be positive constant, observe that $w(\lambda,x,y) = \dfrac{u(x,y)}{\lambda^p}$ is a modular quasi-metric on $\mathbb{R}$. If $a \in \mathbb{R}$, then $\mathbb{R}_w(a) = \mathbb{R}$ and $d_w(x,y) = [u(x,y)]^{\frac{1}{p+1}}$. Moreover, $d_w^s(x,y) = |x-y|^{\frac{1}{p+1}}$, whenever $x,y \in \mathbb{R}$.

Let w be a modular quasi-pseudometric on a non-empty set X. For any $x,y \in X_w$ and $\lambda > 0$, the quasi-pseudometric d_w on X_w satisfies the following conditions.

(a) If $d_w(x,y) < \lambda$, then $w(\lambda,x,y) \leq d_w(x,y) < \lambda$;

(b) If $w(\lambda,x,y) = \lambda$, then $d_w(x,y) = \lambda$.

3.2. Convex modular quasi-pseudometric space

This section is devoted to convexity of a non-empty set endowed with a modular quasi-pseudometric.

Definition 3.2.1. *Let w be a modular quasi-pseudometric on a non-empty set X. Let $x_0 \in X$. We define $X^*_w(x_0)$ by*

$$X^*_w(x_0) = \{x \in X : \exists \lambda > 0,\ w(\lambda, x, x_0) < \infty \quad \text{and} \quad w(\lambda, x_0, x) < \infty\}.$$

For any $x_0 \in X$, we set $X^*_w = X^*_w(x_0)$.

Example 3.2.1. Consider the modular quasi-metric w in Example 3.1.2 on a T_0 -quasi-metric space (X, d). Let $x_0 \in X$. Then observe that $X^*_w = X$.

Remark 3.2.2. *Note that for any modular quasi-pseudometric w on a non-empty set X, $X_w \subseteq X^*_w$.*

Example 3.2.2. Let w be a modular quasi-pseudometric on a non-empty set X. We say that w is convex on X if w satisfies the inequality for any $\lambda, \mu > 0$,

$$w(\lambda + \mu, x, y) \leq \frac{\lambda}{\lambda + \mu} w(\lambda, x, z) + \frac{\mu}{\lambda + \mu} w(\mu, z, y)$$

whenever $x, y \in X$. If a modular quasi-(pseudo) metric w is convex on a set X, we say that w is a convex modular quasi-(pseudo)metric on X.

Remark 3.2.3. *Let w be a modular quasi-(pseudo) metric on X. If w is a convex modular quasi-(pseudo) metric on X, then its conjugate modular quasi- (pseudo) metric w^{-1} on X is convex too. Moreover, its symmetrized (pseudo) metric modular w^s is convex on X.*

Lemma 3.2.4. *Let w be a modular quasi-(pseudo) metric on X. Then w is convex if and only if the function $\hat{w}(\lambda, x, y) = \lambda w(\lambda, x, y)$ is a modular quasi- (pseudo) metric on X whenever $\lambda > 0$ and $x, y \in X$.*

Proof. We first suppose that w is convex. We prove that $\hat{w}$ is a modular quasi-pseudometric on X. Indeed, for any $x, y, z \in X$ and $\mu, \lambda > 0$ we obviously have that $\hat{w}(\lambda, x, x) = \lambda w(\lambda, x, x) = 0$. Moreover,

$$\hat{w}(\lambda + \mu, x, y) = (\lambda + \mu) w(\lambda + \mu, x, y) \leq (\lambda + \mu) \left[\frac{\lambda}{\lambda + \mu} w(\lambda, x, z) + \frac{\mu}{\lambda + \mu} w(\mu, z, y)\right].$$

Thus,

$$\hat{w}(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y) = \hat{w}(\lambda, x, z) + \hat{w}(\mu, z, y).$$

If $\hat{w}(\lambda, x, y) = 0 = \hat{w}(\lambda, y, x)$ it is evident that $x = y$.
Secondly, suppose that $\hat{w}$ is a modular quasi-pseudometric. Then we prove that w is convex on X. Let $x, y, z \in X$ and $\lambda, \mu > 0$. Since $\hat{w}$ is modular quasi-pseudometric on X, we have

$$\hat{w}(\lambda + \mu, x, y) \leq \hat{w}(\lambda, x, z) + \hat{w}(\mu, z, y)$$

which implies that

$$(\lambda+\mu)w(\lambda+\mu,x,y) \leq \lambda w(\lambda,x,z)+\mu w(\mu,z,y).$$

Thus

$$w(\lambda+\mu,x,y) \leq \frac{\lambda}{\lambda+\mu}w(\lambda,x,z)+\frac{\mu}{\lambda+\mu}w(\mu,z,y).$$

Therefore, w is convex on X. $\square$

The following result can be compared to Lemma 2.3.3.

Lemma 3.2.5. *If w is a convex modular quasi-pseudometric on a non-empty set X, then the function $\rho : (0,\infty) \to (0,\infty]$ defined by*

$$\rho(\lambda) = \hat{w}(\lambda,x,y) = \lambda w(\lambda,x,y)$$

whenever $\lambda > 0$ and $x, y \in X$ is non-increasing.

Proof. Consider $x, y \in X$ and $\lambda, \mu \in (0,\infty)$ with $\mu < \lambda$. Since w is a convex modular quasipseudometric on X, we have

$$w(\lambda\mu+\mu,x,y) \leq \frac{\lambda\mu}{\mu}w(\lambda\mu,x,x)+\frac{\mu}{\lambda}w(\mu,x,y).$$

Moreover, we have

$$w(\lambda,x,y) \leq \frac{\mu}{\lambda}w(\mu,x,y).$$

So $\rho(\lambda) = \lambda w(\lambda,x,y) \leq \mu w(\mu,x,y) = \rho(\mu)$. $\square$

Proposition 3.2.6. *Let w be a modular quasi-pseudometric on a non-empty set X. If w is convex on X, then $X_w(x_0) = X_w^*(x_0)$ whenever $x_0 \in X$.*

Proof. It is sufficient to prove that $X_w(x_0) \supseteq X_w^*(x_0)$ since $X_w(x_0) \subseteq X_w^*(x_0)$ from Remark 3.2.2. Let $x \in X_w^*(x_0)$. Then $w(\mu,x,x_0) < \infty$ and $w(\mu,x_0,x) < \infty$ for some $\mu > 0$. For any $\lambda > \mu$, we have that

$$w(\lambda,x,x_0) \leq \frac{\mu}{\lambda}w(\mu,x,x_0)$$

and

$$w(\lambda,x_0,x) \leq \frac{\mu}{\lambda}w(\mu,x_0,x),$$

since the function ρ is non-increasing from Lemma 3.1.4. Hence,

$$\lim_{\lambda\to\infty} w(\lambda,x,x_0) = \lim_{\lambda\to\infty}\frac{\mu}{\lambda}w(\lambda,x,x_0) = 0$$

and

$$\lim_{\lambda\to\infty} w(\lambda,x_0,x) = \lim_{\lambda\to\infty}\frac{\mu}{\lambda}w(\lambda,x_0,x) = 0.$$

Therefore $x \in X_w(x_0)$. $\square$

Remark 3.2.7. *If w is a convex modular quasi-pseudometric on a non-empty set X and $x_0 \in X$, then*

$$\begin{aligned} X^*_{\hat{w}}(x_0) &= \{x \in X : \exists\lambda > 0,\, \hat{w}(\lambda, x, x_0) < \infty \quad \textit{and} \quad \hat{w}(\lambda, x_0, x) < \infty\} \\ &= \{x \in X : \exists\lambda > 0,\, \lambda w(\lambda, x, x_0) < \infty \quad \textit{and} \quad \lambda w(\lambda, x_0, x) < \infty\} \\ &= \{x \in X : \lambda > 0,\, w(\lambda, x, x_0) < \infty \quad \textit{and} \quad w(\lambda, x_0, x) < \infty\}. \end{aligned}$$

So $X^*_{\hat{w}}(x_0) = X^*_w(x_0)$.
Moreover, whenever $x, y \in X^*_w(x_0)$

$$\begin{aligned} d_{\hat{w}}(x, y) &= \inf\{\lambda > 0 : \hat{w}(\lambda, x, y) \leq \lambda\} \\ &= \inf\{\lambda > 0 : \lambda w(\lambda, x, y) \leq \lambda\} \\ &= \inf\{\lambda > 0 : w(\lambda, x, y) \leq 1\}. \end{aligned}$$

In the following we denote $X^*_w(x_0)$ by X^*_w with $x_0 \in X$ and w is convex modular quasi-pseudometric on X.

Theorem 3.2.8. *Let w be a convex modular quasi-(pseudo)metric on a non-empty set X. Then the function d^*_w defined by*

$$d^*_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq 1\}$$

whenever $x, y \in X^*_w$ is a quasi-(pseudo)metric on X^*_w. Moreover, (X^*_w, d^*_w) is a quasi-(pseudo)metric space.

Proof. Let $x, y \in X^*_w$, then there exist $\lambda > 0$ and $\mu > 0$ such that

$$w(\lambda, x, x_0) < \infty \quad \textit{and} \quad w(\lambda, x_0, x) < \infty$$

and

$$w(\mu, y, x_0) < \infty \quad \textit{and} \quad w(\mu, x_0, y) < \infty.$$

Let $\varphi > 0$ such that $\lambda + \mu \leq \varphi$. Since w is convex, we have

$$\varphi w(\varphi, x, y) \leq (\lambda + \mu) w(\lambda + \mu, x, y) \tag{3.5}$$

and

$$w(\lambda + \mu, x, y) \leq \frac{\lambda}{\lambda + \mu} + w(\lambda, x, x_0) + \frac{\mu}{\lambda + \mu} w(\mu, x_0, y). \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we have

$$w(\varphi, x, y) \leq \frac{\lambda}{\varphi} w(\lambda, x, x_0) + \frac{\mu}{\varphi} w(\mu, x_0, y).$$

Furthermore,

$$\lim_{\varphi \to \infty} w(\varphi, x, y) \leq \lim_{\varphi \to \infty} \left[\frac{\lambda}{\varphi} w(\lambda, x, x_0) + \frac{\mu}{\varphi} w(\mu, x_0, y)\right] = 0.$$

Thus,

$$\lim_{\varphi\to\infty} w(\varphi, x, y) = 0.$$

It follows that there exists $\varphi_0 > 0$ such that $w(\varphi, x, y) \leq 1$ and $\varphi > \varphi_0$. So, $d^*_w(x, y) \leq \varphi_0 < \infty$. Therefore d^*_w is well defined. Let $x, y, z \in X^*_w$. Then whenever $\lambda > 0$ such that $w(\lambda, x, x) = 0 < 1$, $d^*_w(x, x) = 0$. Moreover, let $\lambda > d^*_w(x, z)$ and $\mu > d^*_w(z, y)$. Then by definition of d^*_w we have $w(\lambda, x, z) \leq 1$ and $w(\mu, z, y) \leq 1$ and since

$$w(\lambda + \mu, x, y) \leq \frac{\lambda}{\lambda+\mu} w(\lambda, x, z) + \frac{\mu}{\lambda+\mu} w(\mu, z, y)$$

we have

$$w(\lambda + \mu, x, y) \leq \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} = 1.$$

It implies that $d^*_w(x, y) \leq \lambda + \mu$. Hence $d^*_w(x, z) \leq d^*_w(y, z) + d^*_w(x, y)$. Let w be a convex modular quasi-metric and $d^*_w(x, y) = 0 = d^*_w(y, x)$. We prove that $x = y$. If $d^*_w(x, y) = 0$, then $w(\mu, x, y) \leq 1$ for some $\mu > 0$. Let $\lambda > 0$ such that $0 < \mu < \lambda$. If follows that

$$w(\lambda, x, y) \leq \frac{\mu}{\lambda} w(\mu, x, y) \leq \frac{\mu}{\lambda}$$

since the function $\rho(\lambda) = \lambda w(\lambda, x, y)$ is non-increasing. We have

$$\lim_{\mu\to 0} w(\lambda, x, y) \leq \lim_{\mu\to 0} \frac{\mu}{\lambda} = 0.$$

Thus $w(\lambda, x, y) = 0$ whenever $\lambda > 0$. Similarly, one shows that $w(\lambda, y, x) = 0$ whenever $\lambda > 0$. It follows that $x = y$ since w is a modular quasi-metric on X. □

3.3. Convergence in a modular quasi-pseudometric space

Definition 3.3.1. *Consider a modular quasi-pseudometric w on a non-empty set X. Let X_w be a modular set corresponding to w and (x_n) be a sequence on X_w.*

- *(a) The sequence (x_n) is said to be w-convergent to $x \in X_w$ denoted by $(x_n) \to x$ if $w(\lambda, x_n, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$.*
- *(b) The sequence (x_n) is said to be w^{-1}-convergent to $x \in X_w$ denoted by $(x_n) \to x$ if $w(\lambda, x, x_n) \to 0$ as $n \to \infty$ whenever $\lambda > 0$*
- *(c) The sequence (x_n) is said to be left K-Cauchy if $w(\lambda, x_k, x_n) \to 0$ as $k, n \to \infty$ whenever $\lambda > 0$.*
- *(d) The sequence (x_n) is said to be right K-Cauchy if $w(\lambda, x_n, x_k) \to 0$ as $k, n \to \infty$ whenever $\lambda > 0$.*
- *(e) The sequence (x_n) is said to be w^s-Cauchy if $w^s(\lambda, x_k, x_n) \to 0$ as $n, k \to \infty$ whenever $\lambda > 0$.*

Proposition 3.3.2. *Let w be a modular quasi-pseudometric on a set X and (x_n) a sequence in X_w.*

(a) If (x_n) is w-convergent to $x \in X_w$ and (x_n) is w^{-1}-convergent to $y \in X_w$, then $w(\lambda, y, x) = 0$ whenever $\lambda > 0$.

(b) If (x_n) is w-convergent to $x \in X_w$ and $w(\lambda, y, x) = 0$ whenever $\lambda > 0$, then (x_n) is also w-convergent to $y \in X_w$.

Proof. We prove (a) and leave (b) to the reader. Suppose that (x_n) is w-convergent to x. Then for any $\mu > 0$, we have $w(\mu, x_n, x) \to 0$ as $n \to \infty$. Moreover, for any $\varphi > 0$, we have $w(\varphi, y, x_n) \to 0$ as $n \to \infty$ since (x_n) is w^{-1}-convergent to y. Then

$$w(\varphi + \mu, y, x) \leq w(\varphi, y, x_n) + w(\mu, x_n, x).$$

For any $\lambda > 0$ such that $\lambda > \mu + \varphi$, we have

$$w(\lambda, y, x) \leq w(\varphi + \mu, y, x) \leq w(\varphi, y, x_n) + w(\mu, x_n, x).$$

Thus $w(\lambda, y, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$. $\square$

Corollary 3.3.3. *If (x_n) is w^{-1}-convergent to $x \in X_w$ and $w(\lambda, x, y) = 0$ whenever $\lambda > 0$, then (x_n) is also w^{-1}-convergent to $y \in X_w$.*

Theorem 3.3.4. *Let w be a modular quasi-pseudometric on a set X. If (x_n) is a sequence in X_w and $x \in X_w$, then $d_w(x_n, x) \to 0$ as $n \to \infty$ if and only if $w(\lambda, x_n, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$.*

Proof. Suppose that $d_w(x_n, x) \to 0$ as $n \to \infty$. Then for any $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $d_w(x_n, x) < \epsilon$ whenever $n \geq n(\epsilon)$. Since

$$d_w(x_n, x) = \inf\{\lambda : w(\lambda, x_n, x) \leq \lambda\},$$

then for some $\lambda > 0$, we have two cases ($0 < \epsilon < \lambda$ or $\epsilon \geq \lambda$).

Case 1. If $0 < \epsilon < \lambda$, then there exist $n(\epsilon) \in \mathbb{N}$ such that for any $n \geq n(\epsilon)$ such that $d_w(x_n, x) < \epsilon$. Hence $w(\epsilon, x_n, x) \leq \epsilon$ whenever $n \geq n(\epsilon)$. Furthermore, we have

$$w(\lambda, x_n, x) \leq w(\epsilon, x_n, x) < \epsilon$$

whenever $n \geq n(\epsilon)$, since $\epsilon < \lambda$. Thus $w(\lambda, x_n, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$.

Case 2. Suppose $\epsilon \geq \lambda$. Let $\lambda = \frac{\epsilon}{2}$. It follows that

$$w(\lambda, x_n, x) \leq w\left(\frac{\lambda}{2}, x_n, x\right) \leq \frac{\lambda}{2} < \lambda \leq \epsilon$$

whenever $n \geq n(\epsilon)$. Hence $w(\lambda, x_n, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$.

Conversely, suppose that $w(\lambda, x_n, x) \to 0$ as $n \to \infty$ whenever $\lambda > 0$. Then there exists $n(\lambda) \in \mathbb{N}$ such that $w(\lambda, x_n, x) \leq \lambda$ whenever $n \geq n(\lambda)$. Thus $d_w(x_n, x) \leq \lambda$ whenever $n \geq n(\lambda)$. Hence

$$d_w(x_n, x) \to 0 \quad as \quad n \to \infty.$$

$\square$

4

Nonsymmetric modulars

In [14], Herda studied the equivalence between a nonsymmetric modular and symmetrized modular from the perspective of [18] on a real linear space. In this chapter, we study nonsymmetric modulars. Then we investigate connections between nonsymmetric modulars and modular quasi-metric on a real linear space.

4.1. Nonsymmetric modular

Definition 4.1.1. *[27] Let V be a vector space over $\mathbb{R}$. A function $\rho : V \to [0,\infty]$ is said to be a nonsymmetric modular on V if :*

(i) $\rho(0) = 0$*,*

(ii) $\rho(\alpha a + \beta b) \leq \rho(a) + \rho(b)$ *for all* $a, b \in V$ *and* $\alpha, \beta \in [0,\infty)$ *with* $\alpha + \beta = 1$*.*

We shall say that ρ is a T_0-nonsymmetric modular provided that ρ satisfies the condition that

(iii) if $\rho(a) = 0$ then $a = 0$ for all $a, b \in V$ and $\alpha, \beta \in [0,\infty)$ with $\alpha + \beta = 1$.

We say that ρ is a convex nonsymmetric modular if condition (ii) in Definition 4.1.1 is replaced by

(iv) $\rho(\alpha a + \beta b) \leq \alpha\rho(a) + \beta\rho(b)$ for all $a, b \in V$ and $\alpha, \beta \in [0,\infty)$ with $\alpha + \beta = 1$.

Remark 4.1.2. *Let ρ be a nonsymmetric modular for a vector space V over $\mathbb{R}$. Then the function $\rho^{-1} : V \to [0,\infty]$ defined by $\rho^{-1}(a) = \rho(-a)$ for $a \in V$ is a nonsymmetric modular on V. We shall call ρ^{-1} the conjugate of the nonsymmetric modular ρ on V. Furthermore, the function ρ^s defined by $\rho^s(a) = \max\{\rho(a), \rho^{-1}(a)\}$ for $a \in V$ is a modular on V. From the perspective of [18], we shall call ρ^s a symmetrized modular on V ([27]).*

Example 4.1.1. Let $(V, ||.|)$ be a nonsymmetric normed linear space. Then the function $\rho(a) = ||a|$ for $a \in V$ is a nonsymmetric modular on V. Therefore the conjugate nonsymmetric modular ρ^{-1} on V and the symmetrized modular on V is given by

$$\rho^{-1}(a) = |a|| \qquad \textit{and} \qquad \rho^s(a) = ||a|| \qquad \text{respectively} \quad for \quad a \in V.$$

Example 4.1.2. (Compare [14, Example 2.1]) Consider the convex real continuous function

$$f(a) = \begin{cases} a^2 & \text{if } a \geq 0; \\ -a & \text{if } a < 0. \end{cases}$$

Let $S = \{(a_n) : (a_n) \text{ is a sequence of real numbers}\}$. Then the function $\rho_f : S \to [0, \infty]$ defined by

$$\rho_f((a_n)) = \sum_{n=1}^{\infty} f(a_n)$$

is a nonsymmetric modular on S. The conjugate nonsymmetric modular of ρ_f is given by

$$\rho_f^{-1}((a_n)) = \sum_{n=1}^{\infty} f(-a_n)$$

and the symmetrized modular on S is the function

$$\rho_f^s((a_n)) = \sum_{n=1}^{\infty} f^s(a)$$

where

$$f^s(a) = \begin{cases} a^2 & \text{if } a < -1 \quad \textit{or} \quad a > 1; \\ a & \text{if } 0 \leq a \leq 1; \\ -a & \text{if } -1 \leq a < 0 \end{cases}$$

for all $a \in \mathbb{R}$.

We recall the following important lemma that can be found in [14].

Lemma 4.1.3. *[14, Lemma 2.1] Let ρ be a nonsymmetric modular on a real linear space S. Then we have*

(a) for all $a \in S$ and $\beta \in [0, \infty)$, $\rho(\beta a)$ is a non-decreasing function.

(b) If for $i \in \{1, ..., n\}$, $a_i \in S$ and $\beta_i \geq 0$ with

$$\sum_{i=1}^{n} \beta_i = 1,$$

then

$$\rho\left(\sum_{i=1}^{n} \beta_i a_i\right) \leq \sum_{i=1}^{n} \rho(\beta_i).$$

Let ρ be a nonsymmetric modular on a real linear space S. We define the set S_ρ by

$$S_\rho = \left\{ a \in S : \lim_{\beta \to 0} \rho(\beta a) = 0 \qquad \textit{and} \qquad \lim_{\beta \to 0} \rho^{-1}(\beta a) = 0 \right\}.$$

We shall call the set S_ρ the modular set with respect to a nonsymmetric modular ρ and it is a subset of S. Furthermore we define the set S^*_ρ by

$$S^*_\rho = \{a \in S : \exists \beta > 0,\, \rho(\beta a) < \infty \quad and \quad \rho^{-1}(\beta a) < \infty\}$$

and we call the set S^*_ρ a convex modular set and it is a subset of S [14, Lemma 2.1 (c)].

Remark 4.1.4. *It is important for the reader to notice that the modular set* $S_\rho = \left\{a \in S : \lim_{\beta \to 0} \rho(\beta a) = 0 \quad and \quad \lim_{\beta \to 0} \rho^{-1}(\beta a) = 0\right\}$ *is not the same as the subset* $\overline{S}_\rho = \{a \in S : \rho(a) < \infty \quad and \quad \rho(-a) < \infty\}$.

The same observation goes for the convex modular set $S^*_\rho = \{a \in S : \exists \beta > 0,\, \rho(\beta a) < \infty \quad and \quad \rho^{-1}(\beta a) < \infty\}$ *and the subset* $\overline{S}^*_\rho = \{a \in S_\rho : \beta a \in S_\rho \quad for \quad some\, \beta \geq 0\}$ *of* S *in the setting of [14].*

We next prove the convexity of the modular set with respect to nonsymmetric modular and show that the convexity of S^*_ρ and S_ρ do not rely on the symmetry property of the modular.

Remark 4.1.5. *For any nonsymmetric modular* ρ *on a real linear space* S*, we have that* S_ρ *and* S^*_ρ *are convex subsets of* S*. Let* $a, b \in S_\rho$ *and real numbers* β, α *such that* $0 \leq \alpha \leq 1$*. Then*

$$\rho(\beta(\alpha a + (1-\alpha)b)) \leq \rho(\beta a) + \rho(\beta b).$$

Thus

$$\lim_{\beta \to 0} \rho(\beta(\alpha a + \beta b)) = 0.$$

Therefore $\alpha a + (1-\alpha)b \in S_\rho$.

Example 4.1.3. Consider a real linear space $\mathbb{R}$ equipped with the nonsymmetric modular ρ where

$$\rho(a) = \begin{cases} \infty & \text{if } a > 0; \\ 0 & \text{if } a \leq 0 \end{cases}$$

for $a \in \mathbb{R}$. It is obvious that in this case $S_\rho = \{0\} \subseteq \mathbb{R}$.

Example 4.1.4. Let $S = \{(a_n) : (a_n)$ is a sequence of real numbers $\}$ be a real linear space. Consider the nonsymmetric modular ρ_f on Example 4.1.2 in S. Then its corresponding convex subset $S_\rho = S$.

Remark 4.1.6. *We notice that for any nonsymmetric modular* ρ *on a real linear space* S*, the convex subset* S_ρ *of* S *is a subset of the convex subset* S^*_ρ *of* S*.*

The next lemma extends the observation in Remark 4.1.6 (see [7, p.9]) when ρ is convex on S.

Lemma 4.1.7. *If ρ is a convex nonsymmetric modular on a real linear space S, then $S_\rho = S^*_\rho$.*

Proof. By observations from Remark 4.1.6, we have that $S_\rho \subset S^*_\rho$. So we only need to prove that $S_\rho \supseteq S^*_\rho$ and this will be sufficient to prove that $S_\rho = S^*_\rho$.
Let $a \in S^*_\rho$. Then there exists $\beta > 0$ such that $\rho(\beta a) < \infty$ and $\rho^{-1}(\beta a) < \infty$ by the definition of S^*_ρ. Hence

$$\begin{aligned}\rho^{-1}(\beta a) &= \rho[\beta(-a) + (1-\beta)0]\\ &\leq \beta\rho(-a).\end{aligned}$$

Therefore

$$\lim_{\beta\to 0} \rho^{-1}(\beta a) = 0.$$

Similarly we can show that

$$\lim_{\beta\to 0} \rho(\beta a) = 0.$$

Hence $a \in S_\rho$, and so $S^*_\rho \subseteq S_\rho$.
Therefore

$$S_\rho = S^*_\rho.$$

□

Remark 4.1.8. *We notice that for any nonsymmetric modular ρ on a real linear space S, we have $S_\rho = S_{\rho^{-1}}$ and $S_{\rho^s} = S_\rho$. Furthermore, we have $S^*_{\rho^s} = S^*_\rho$ and $S^*_\rho = S^*_{\rho^{-1}}$.*

The following observation is important to prove since it has some variations from symmetric modular to nonsymmetric modular point of view. This result can be compared to [14, Lemma 2.1 (d)] since we have seen that by Remark 4.1.4 S^*_ρ and S_ρ are different.

Lemma 4.1.9. *If ρ is a convex nonsymmetric modular on a real linear space S, then S^*_ρ is a real linear subspace of S.*

Proof. We have $0 \in S^*_\rho$ since $\rho(0) = 0 < \infty$ and $\rho^{-1}(0) = 0 < \infty$. Let $a, b \in S^*_\rho$. Then there exists some $\lambda_1 > 0$ such that

$$\rho(\lambda_1 a) < \infty \quad \textit{and} \quad \rho^{-1}(\lambda_1 a) < \infty.$$

Furthermore, there exists some $\lambda_2 > 0$ such that

$$\rho(\lambda_2 b) < \infty \quad \textit{and} \quad \rho^{-1}(\lambda_2 b) < \infty.$$

Suppose $\lambda = \min\{\lambda_1, \lambda_2\}$. This means $\frac{\lambda}{2} < \lambda_1$ and $\frac{\lambda}{2} < \lambda_2$. Then we have

$$\begin{aligned}\rho(\frac{\lambda}{2}a + \frac{\lambda}{2}b) &\leq \rho(\lambda a) + \rho(\lambda b)\\ &\leq \rho(\lambda_1 a) + \rho(\lambda_2 b)\\ &< \infty.\end{aligned}$$

Similarly we have

$$\rho^{-1}\left(\frac{\lambda}{2}a + \frac{\lambda}{2}b\right) < \infty.$$

Hence $a + b \in S_\rho^*$.
Let $\delta \in \mathbb{R}$ and $a \in S_\rho^*$. Then there exists $\lambda > 0$ such that $\rho(\lambda a) < \infty$ and $\rho^{-1}(\lambda a) < \infty$. We want to show that $\delta a \in S_\rho^*$. If $\delta = 0$, it is obvious that $0 \in S_\rho^*$. So we have two cases to prove.

Case 1. If $\delta > 0$, then

$$\begin{aligned}\rho(\lambda(\delta a)) &= \rho(\delta(\lambda a)) \\ &\leq \rho(\lambda a) \\ &< \infty\end{aligned}$$

and

$$\begin{aligned}\rho^{-1}(\lambda(\delta a)) &= \rho^{-1}(\delta(\lambda a)) \\ &\leq \rho^{-1}(\lambda a) \\ &= \rho(\lambda(-a)) \\ &< \infty.\end{aligned}$$

Case 2. If $\delta < 0$ then we have $-\delta = \beta > 0$. Hence we have

$$\begin{aligned}\rho(\lambda(\delta a)) &= \rho(\lambda(-\beta a)) \\ &= \rho(\beta(-\lambda a)) \\ &\leq \rho(-\lambda a) \\ &= \rho(\lambda(-a)) \\ &= \rho^{-1}(\lambda a) \\ &< \infty\end{aligned}$$

and

$$\begin{aligned}\rho^{-1}(\lambda(\delta a)) &= \rho^{-1}(\lambda(-\beta a)) \\ &= \rho^{-1}(\beta(-\lambda a)) \\ &\leq \rho^{-1}(-\lambda a) \\ &= \rho^{-1}(\lambda(-a)) \\ &= \rho(\lambda a) \\ &< \infty.\end{aligned}$$

Thus $\delta a \in S_\rho^*$.
Therefore S_ρ^* is a real linear subspace of S. □

We want to look at the relationship between a nonsymmetric modular and a modular quasi-metric on a real linear space. We want to prove that [7, Theorem 3.11] does not rely on the symmetry property of modular and modular metric. That is, Theorem 3.11 [7] still hold on nonsymmetric setting.

We firstly need to recall some results that have been proven by Herda [14] which can be compared to [18, 1.21,p.52] for symmetric modulars.

Lemma 4.1.10. *[14, Lemma 3.1] Let ρ be a nonsymmetric modular on a real linear space. Then the function*

$$||a|_\rho = \inf\left\{\delta > 0 : \rho\left(\frac{a}{\delta}\right) \leq \delta\right\}$$

for $a \in S_\rho$ satisfies the following properties :

- *(i)* $||a|_\rho \geq 0$ *for all* $a \in S_\rho$,
- *(ii)* $||a|_\rho = 0$ *if and only if* $a = 0$ *for* $a \in S_\rho$,
- *(iii)* $||a + b|_\rho \leq ||a|_\rho + ||b|_\rho$ *for* $a, b \in S_\rho$,
- *(iv) If* $a \in S_\rho$, *then the function* $\psi(\beta) = ||\beta a|_\rho$ *is non-decreasing for all* $\beta \geq 0$,
- *(v) If a real valued sequence* α_n *converges to* α *and the sequence* (a_n) *in* S_ρ *converges to* $a \in S_\rho$ $(i.e||a_n - a|_\rho \to 0)$, *then*
$$||\alpha_n a_n - \alpha a|_\rho \to 0.$$

Definition 4.1.11. *Let ρ be a nonsymmetric modular on a real linear space. The function*

$$||a|_\rho = \inf\{\delta > 0 : \rho(\frac{a}{\delta}) \leq \delta\}$$

for all $a \in S_\rho$ is called a nonsymmetric ρ-norm on the modular set S_ρ.

Remark 4.1.12. *Note that*

$$||a|_{\rho^{-1}} = |a||_\rho \quad for\, all \quad a \in S_\rho$$

and

$$||a||_\rho = ||a|_{\rho^s} \quad for\, all \quad a \in S_{\rho^s}.$$

Remark 4.1.13. *Let ρ be a nonsymmetric modular on a real linear space S. Then the symmetrized F-norm $||.||_\rho$ defined by*

$$||a||_\rho = \max\{||a|_\rho, |a||_\rho\}$$

for all $a \in S_{\rho^s}$ is equivalent to the F-norm in the sense of Orliz (see [7, 12]).

Definition 4.1.14. *Let ρ be a convex nonsymmetric modular on a real linear space. Then the function $||.||^*_\rho$ defined by*

$$||a|^*_\rho = \inf\left\{\delta > 0 : \rho\left(\frac{a}{\delta}\right) \leq 1\right\}$$

for all $a \in S^*_\rho$ is called a nonsymmetric ρ-norm on the set $S^*_\rho = S_\rho$.

4.2. Nonsymmetric modular and modular quasi-metric space

We now study connections between nonsymmetric modulars and modular quasi-metrics on a real linear space. This study is inspired by the work of Chistyakov [8] on modulars on a real linear space.

The following result generalises [7, Theorem 3.11] to an asymmetric context.

Theorem 4.2.1. *(compare [7, Theorem 3.11]) Let S be a real linear space. Let a function $\rho : S \to [0, \infty]$ be a nonsymmetric modular on S. The function $m : (0, \infty) \times S \times S \to [0, \infty]$ is given by*

$$m(\delta, a, b) = \rho\left(\frac{a-b}{\delta}\right).$$

Then the function ρ is a (convex) nonsymmetric modular on S if and only if the function m is a (convex) modular quasi-metric on S.

Proof. Suppose that the function ρ is a nonsymmetric modular on a real linear space S. So we want to prove that the function m is a modular quasi-metric on real linear space S.

(i) If $\delta > 0$ and $a \in S$, then

$$\begin{aligned} m(\delta, a, a) &= \rho\left(\frac{a-a}{\delta}\right) \\ &= \rho(0) \\ &= 0. \end{aligned}$$

(ii) If $a, b, \in S$ such that $m(\delta, a, b) = 0 = m(\delta, b, a)$ for $\delta > 0$, then

$$\rho\left(\frac{a-b}{\delta}\right) = 0 = \rho\left(\frac{b-a}{\delta}\right)$$

for $\delta > 0$. Therefore $\dfrac{a-b}{\delta} = 0 = \dfrac{b-a}{\delta}$. Hence $a = b$.

(iii) Let $a, b \in S$ and $\delta, \alpha > 0$. Then

$$\begin{aligned} m(\delta + \alpha, a, b) &= \rho\left(\frac{a-b}{\delta+\alpha}\right) \\ &= \rho\left[\frac{\delta}{\delta+\alpha}\left(\frac{a-c}{\delta}\right) + \frac{\alpha}{\delta+\alpha}\left(\frac{c-b}{\alpha}\right)\right]. \end{aligned}$$

By property (ii) of Definition 4.1.1, we have

$$\frac{\delta}{\delta+\alpha} + \frac{\alpha}{\delta+\alpha} = 1.$$

So we have

$$\begin{aligned} m(\delta+\alpha, a, b) &\leq \rho\left(\frac{a-c}{\delta}\right) + \rho\left(\frac{c-b}{\alpha}\right) \\ &= m(\delta, a, c) + m(\alpha, c, b). \end{aligned}$$

Therefore m is a modular quasi-metric on S.

(iv) If ρ is a convex nonsymmetric modular on S, then for $a, b, c \in S$ and $\delta, \alpha > 0$, we have

$$\begin{aligned} m(\delta+\alpha, a, b) &= \rho\left(\frac{a-b}{\delta+\alpha}\right) \\ &= \rho\left[\frac{\delta}{\delta+\alpha}\left(\frac{a-c}{\delta}\right) + \frac{\alpha}{\delta+\alpha}\left(\frac{c-b}{\alpha}\right)\right]. \end{aligned}$$

By convexity of ρ, we have

$$\begin{aligned} m(\delta+\alpha, a, b) &\leq \frac{\delta}{\delta+\alpha}\rho\left(\frac{a-c}{\delta}\right) + \frac{\alpha}{\delta+\alpha}\rho\left(\frac{c-b}{\alpha}\right) \\ &= \frac{\delta}{\delta+\alpha}m(\delta, a, c) + \frac{\alpha}{\delta+\alpha}m(\alpha, c, b). \end{aligned}$$

Therefore m is a convex modular quasi-metric on S.

Conversely suppose that the function m is a modular quasi-metric on the real linear space S. We want to show that the function ρ is a nonsymmetric modular on the real linear space S.

(i) Consider $a \in S$ and $\delta > 0$, then

$$\begin{aligned} \rho(a) &= \rho\left(\frac{a-a}{\delta}\right) \\ &= m(\delta, a, a) \\ &= 0. \end{aligned}$$

Furthermore, if $a \neq 0$, then

$$\begin{aligned} \rho(a) &= \rho\left(\frac{a-0}{\delta}\right) \\ &= m(\delta, a, 0) \\ &\neq 0. \end{aligned}$$

(ii) Suppose $a, b \in S$ and $\delta, \alpha > 0$ such that $\delta + \alpha = 1$. Then we have

$$\begin{aligned} \rho(\delta a + \alpha b) &= \rho\left(\frac{\delta a - (-\alpha b)}{1}\right) \\ &= m(1, \delta a, -\alpha b) \\ &= m(\delta+\alpha, \delta a, -\alpha b). \end{aligned}$$

Since m is a modular quasi-metric on S, then

$$\begin{aligned}\rho(\delta a+\alpha b) &\leq m(\delta,\delta a,0)+m(\alpha,0,-\alpha a)\\ &=\rho\Big(\frac{\delta a-0}{\delta}\Big)+\rho\Big(\frac{0-(\alpha b)}{\alpha}\Big)\\ &=\rho(a)+\rho(b).\end{aligned}$$

Therefore ρ is a nonsymmetric modular on S.

(iii) If m is a convex modular quasi-metric on S, then for $a,b\in S$ and $\delta,\alpha>0$ such that $\delta+\alpha=1$, we have

$$\begin{aligned}\rho(\delta a+\alpha b) &=\rho\Big(\frac{\delta a-(-\alpha b)}{1}\Big)\\ &=m(1,\delta a,-\alpha b)\\ &=m(\delta+\alpha,\delta a,-\alpha b)\\ &\leq\frac{\delta}{\delta+\alpha}m(\delta,\delta a,0)+\frac{\alpha}{\delta+\alpha}m(\alpha,0,-\alpha b)\\ &=\frac{\delta}{\delta+\alpha}\rho\Big(\frac{\delta a-0}{\delta}\Big)+\frac{\alpha}{\delta+\alpha}\rho\Big(\frac{0-(\alpha b)}{\alpha}\Big).\end{aligned}$$

Since $\delta+\alpha=1$, then we have

$$\rho(\delta a+\alpha b)\leq\delta\rho(a)+\alpha\rho(b).$$

Therefore ρ is a convex nonsymmetric modular on S.

□

Corollary 4.2.2. *Let S be a real linear space. If m and ρ are functions defined in Theorem 4.2.1 and satisfy Theorem 4.2.1, then the set $S_\rho=S_m$ and is a real linear subspace of S and the nonsymmetric ρ-norm $||a|_\rho=d_m(a,0)$ for all $a\in S_\rho$.*
Moreover, if m is a convex modular quasi-metric on S, then the set $S^*_\rho=S^*_m$ and the nonsymmetric ρ-norm $||a||^*_\rho=d^*_m(a,0)$ for all $a\in S^*_\rho$.

Remark 4.2.3. *We observe that $\rho(-a)=m(1,-a,0)=m^{-1}(1,0,-a)$ for all $a\in S$ where m is a modular quasi-metric on a real linear space S and ρ is a nonsymmetric modular on a real linear space S.*

Theorem 4.2.4. *Consider a real linear space S. If the function $m:(0,\infty)\times S\times S\to[0,\infty]$ satisfies the conditions :*

(i) $m(\delta,\alpha a,0)=m(\frac{\delta}{\alpha},a,0)$ and $m(\delta,0,\alpha b)=m(\frac{\delta}{\alpha},0,-b)$ for all $a,b\in S$ and $\delta,\alpha>0$,

(ii) $m(\delta,a+c,c+b)=m(\delta,a,b)$ for any $a,b,c\in S$ and $\delta>0$

and the function $\rho:S\to[0,\infty]$ satisfies the condition

$$\rho(a)=m(1,a,0)\quad for\ all\quad a\in S,$$

then m is a (convex) modular quasi-metric on S if and only if ρ is a (convex) nonsymmetric modular on S.

Proof. Suppose m is a (convex) modular quasi-metric on S. We only prove that ρ is a nonsymmetric modular on S and leave the convexity proof as it is the same as the convexity proof in Theorem 4.2.1.

(i) For $a = 0$, we have

$$\rho(0) = m(1, 0, 0) = 0$$

and if $a \neq 0$, then we have

$$\rho(a) = m(1, a, 0) \neq 0.$$

(ii) Suppose $a, b, c \in S$ and $\delta, \alpha > 0$ such that $\delta + \alpha = 1$. Then

$$\begin{aligned}\rho(\delta a + \alpha b) &= m(1, \delta a + \alpha b, 0)\\ &= m(\delta + \alpha, \delta a + \alpha b, 0).\end{aligned}$$

Since m satisfy condition (ii), then we have

$$\begin{aligned}\rho(\delta a + \alpha b) &= m(\delta + \alpha, \delta a + \alpha b, \alpha b - \alpha b)\\ &= m(\delta + \alpha, \delta a, -\alpha b)\\ &\leq m(\delta, \delta a, 0) + m(\alpha, 0, -\alpha b).\end{aligned}$$

By condition (i), then

$$\begin{aligned}\rho(\delta a + \alpha b) &\leq m(\delta, \delta a, 0) + m(\alpha, 0, -\alpha b)\\ &= m(1, a, 0) + m(1, 0, -b)\\ &= \rho\left(\frac{a - 0}{1}\right) + \rho\left(\frac{0 - (-b)}{1}\right)\\ &= \rho(a) + \rho(b).\end{aligned}$$

Therefore $\rho(\delta a + \alpha b) \leq \rho(a) + \rho(b)$ for all $a, b \in S$ and $\delta, \alpha > 0$ such that $\delta + \alpha = 1$. Hence ρ is a nonsymmetric modular on S.

Conversely suppose ρ is a nonsymmetric modular on S. We want to show that m is a modular metric on S.

(i) $m(\delta, a, a) = \rho\left(\dfrac{a - a}{\delta}\right) = 0$ for all $a \in S$ and $\delta > 0$.
Furthermore, if $m(\delta, a, b) = 0 = m(\delta, b, a)$ for all $a, b \in S$ and $\delta > 0$, then

$$\rho\left(\frac{a - b}{\delta}\right) = 0 = \rho\left(\frac{b - a}{\delta}\right).$$

Since ρ is a nonsymmetric modular on S, then $a - b = 0 = b - a$ which implies $a = b$.

(ii) Let $a, b, c \in S$ and $\delta, \alpha > 0$. If $c = -b$, then by conditions (i) and (ii), we have

$$\begin{aligned} m(\delta+\alpha, a, b) &= m(\delta+\alpha, a+c, c+b) \\ &= m(\delta+\alpha, a-b, 0) \\ &= m\left(\delta+\alpha, (\delta+\alpha)\frac{a-b}{\delta+\alpha}, 0\right) \\ &= m\left(\frac{\delta+\alpha}{\delta+\alpha}, \frac{a-b}{\delta+\alpha}, 0\right) \\ &= \rho\left(\frac{a-b}{\delta+\alpha}\right) \\ &\leq \rho\left(\frac{a-c}{\delta}\right) + \rho\left(\frac{c-b}{\alpha}\right) \\ &= m(\delta, a, c) + m(\alpha, c, b). \end{aligned}$$

Hence $m(\delta+\alpha, a, b) \leq m(\delta, a, c) + m(\alpha, c, b)$.

Therefore m is a modular quasi-metric on S. □

4.3. Convergence in a modular set

Herda [14], introduced the theory of convergence in a nonsymmetric modular setting in 1967, and the concept of convergence with respect to nonsymmetric settings has developed since then. We know that a nonsymmetric distance function has its dual nonsymmetric function and its symmetrized distance function and it is important to approach convergence of a sequence with respect to all three distance functions. Hence the convergence of sequences in a modular set needs to be revisited and improved. Therefore in this section we will be revisiting and improving the convergence of sequences in a nonsymmetric modular setting.

Definition 4.3.1. *(compare [14, Definition 2.2]) Consider a nonsymmetric modular ρ on a real linear space S.*

(i) A sequence (a_n) in S_ρ is said to be ρ-convergent to $a \in S_\rho$ (that we denote by $(a_n \xrightarrow{\rho} a)$ if there exists $k > 0$ such that

$$\lim_{n\to\infty} \rho(k(a_n - a)) = 0.$$

(ii) A sequence (a_n) in S_ρ is said to be ρ^{-1}-convergent to $a \in S_\rho$ (that we denote by $(a_n \xrightarrow{\rho^{-1}} a)$ if there exists $k > 0$ such that

$$\lim_{n\to\infty} \rho(k(a - a_n)) = 0.$$

(iii) A sequence (a_n) in S_ρ is said to be ρ^s-convergent to $a \in S_\rho$ (that we denote by $(a_n \xrightarrow{\rho^s} a)$ if

there exists $k > 0$ such that

$$\lim_{n\to\infty} \rho^s(k(a_n - a)) = 0.$$

Example 4.3.1. (compare [14, Example 2.1]) Consider the real linear space S of Example 4.1.2 equipped with its convex nonsymmetric modular ρ_f. Consider the sequence $(a_k)_{k\in\mathbb{N}}$ where

$$(a_k)_{k\in\mathbb{N}} = \Big(\underbrace{0, 0, 0, ..., 0}_{k\ terms}, \frac{1}{k+1}, \frac{1}{k+2}, ...\Big) \in S_{\rho_f}.$$

It follows that the sequence $(a_k)_{k\in\mathbb{N}} \overset{\rho_f}{\to} (0) \in S_{\rho_f}$, since

$$\rho_f((a_k)) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < 2$$

and

$$\lim_{k\to 0} \rho_f(s((a_k) - (0))) = 0 \quad with \quad s = 1.$$

Furthermore,

$$\rho_f^{-1}((a_k)) = \rho_f((a_k)) = -\infty.$$

Hence $(a_k)_{k\in\mathbb{N}}$ is not ρ_f^{-1}-convergent.

Example 4.3.2. Let $S = l$, the space of sequences in $\mathbb{R}$ whose series is absolutely convergent. The function $||.| : S \to [0, \infty)$ defined by

$$||(a_n)_{n\in\mathbb{N}}| = \sum_{n=1}^{\infty} \max\{a_n, 0\}$$

for all $(a_n)_{n\in\mathbb{N}} \in S$ is a nonsymmetric norm on S.
Moreover, its conjugate nonsymmetric norm $|.||$ is given by

$$|(a_n)_{n\in\mathbb{N}}|| = \sum_{n=1}^{\infty} \max\{-a_n, 0\}$$

and its symmetrized norm $||.||$ is given by

$$\begin{aligned} ||(a_n)_{n\in\mathbb{N}}|| &= \max\{||(a_n)_{n\in\mathbb{N}}|, |(a_n)_{n\in\mathbb{N}}||\} \\ &= \sum_{n=1}^{\infty} |a_n|. \end{aligned}$$

It is readily checked that the function $\rho((a_n)_{n\in\mathbb{N}}) = ||(a_n)_{n\in\mathbb{N}}||$ is a nonsymmetric modular on S.

Then

$$(a_n)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}, 0, 0, ...\right)_{n\in\mathbb{N}} \in S_\rho$$

and

$$(a) = \left(1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}, \frac{1}{2^{n+1}}, ...\right)_{n\in\mathbb{N}}.$$

Observe that

$$\begin{aligned}\lim_{n\to\infty}(\rho(a_n - a)) &= \lim_{n\to\infty} ||a_n - a| \\ &= \lim_{n\to\infty}\sum_{j=n+1}^{\infty}\max\{-\frac{1}{2^j}, 0\} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}(\rho(a - a_n)) &= \lim_{n\to\infty} ||a - a_n| \\ &= \lim_{n\to\infty}\sum_{j=n+1}^{\infty}\max\{\frac{1}{2^j}, 0\} \\ &= \lim_{n\to\infty}\sum_{j=n+1}^{\infty}\frac{1}{2^j} \\ &= 0.\end{aligned}$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is ρ-convergent to (a) and $(a_n)_{n\in\mathbb{N}}$ is ρ^{-1}-convergent to (a).

The following result can be compared to [14, Lemma 2.2].

Proposition 4.3.2. *Consider a nonsymmetric modular ρ on a real linear space S and $(a_n)_{n\in\mathbb{N}}$ be a sequence in S_ρ. Then $(a_n)_{n\in\mathbb{N}}$ is ρ^s-convergent to a if and only if $(a_n)_{n\in\mathbb{N}}$ is ρ-convergent to a and is ρ^{-1}-convergent to a.*

Proof. Observe that the necessary condition is obvious and we prove the sufficient condition. Suppose that $(a_n)_{n\in\mathbb{N}}$ is ρ-convergent to a and $(a_n)_{n\in\mathbb{N}}$ is ρ^{-1}-convergent to a. Then there exists $n_\epsilon^1, n_\epsilon^2 \in \mathbb{N}$ for any $\epsilon > 0$ such that $n > n_\epsilon^1$, we have

$$\rho(k_1(a_n - a)) < \frac{\epsilon}{2}$$

for some $k_1 > 0$ and

$$\rho^{-1}(k_2(a_n - a)) < \frac{\epsilon}{2}$$

for some $k_2 > 0$. Let $k = \min\{k_1, k_2\}$ and $n_\epsilon = \max\{n_\epsilon^1, n_\epsilon^2\}$, then if $n > n_\epsilon$ we have

$$\rho\left(\frac{k}{2}(a_n - a)\right) \le \rho(k_1(a_n - a)) < \frac{\epsilon}{2}$$

and

$$\rho^{-1}\left(\frac{k}{2}(a_n - a)\right) \le \rho^{-1}(k_2(a_n - a)) < \frac{\epsilon}{2}.$$

Hence $\rho^s\left(\dfrac{k}{2}(a_n - a)\right) < \epsilon$ for some $\dfrac{k}{2} > 0$. Therefore $(a_n)_{n\in\mathbb{N}}$ is ρ^s-convergent to a.

☐

Proposition 4.3.3. *[14, Lemma 2.5] Let ρ be a nonsymmetric modular on a real linear space S. Consider two sequences (a_n) and (b_n) in S_ρ. If (a_n) is ρ-convergent to a and (b_n) is ρ-convergent to b, then the sequence $(\delta a_n + \alpha b_n)$ is ρ-convergent to $\delta a + \alpha b$ where $\delta, \alpha \geq 0$.*

The next result follows by duality from the above proposition.

Corollary 4.3.4. *Let ρ be a nonsymmetric modular on a real linear space S. Consider two sequences (a_n) and (b_n) in S_ρ. If (a_n) is ρ^{-1}-convergent to a and (b_n) is ρ^{-1}-convergent to b, then the sequence $(\delta a_n + \alpha b_n)$ is ρ^{-1}-convergent to $\delta a + \alpha b$ where $\delta, \alpha \geq 0$.*

4.4. Nonsymmetric modular on a normed lattice

In this final section we provide a method of generating a modular in the sense of [18] from a given nonsymmetric modular on a real linear space and we study connections between norms induced by this nonsymmetric modular. These ideas were used for the first time by Ferrer et al. [12]. Furthermore, we introduce a nonsymmetric modular on normed lattices and show that some results due to Mabula and Conradie [10] still hold in the context of a nonsymmetric modular.

The following result can be compared to [12, Proposition 2.2].

Proposition 4.4.1. *Let ρ be a nonsymmetric modular on a real linear space S. Then the functions ρ^s_L and ρ^s_E defined by*

$$\rho^s_L(a) = \rho(a) + \rho(-a) \qquad \text{and} \qquad \rho^s_E = \sqrt{\rho(a)^2 + \rho(-a)^2}$$

for all $a \in S$ are (symmetric) modulars on S.

Proof. It can be easily seen that ρ^s_L and ρ^s_E satisfy the axiom (i) in Definition 4.1.1. So we prove the non-trivial axiom. Let $a, b \in S$ and $\delta, \alpha \geq 0$ with $\delta + \alpha = 1$. Then

$$\begin{aligned}\rho^s_L(\delta a + \alpha b) &= \rho(\delta a + \alpha b) + \rho(-(\delta a + \alpha b)) \\ &\leq \rho(a) + \rho(b) + \rho(-a) + \rho(-b) \\ &= \rho^s_L(a) + \rho^s_L(b)\end{aligned}$$

$$\begin{aligned}\rho^s_E(\delta a+\alpha b) &= \Big(\rho(\delta a+\alpha b)^2+\rho(-(\delta a+\alpha b))^2\Big)^{\frac{1}{2}}\\ &\leq \Big([\rho(a)+\rho(b)]^2+[\rho(-a)+\rho(-b)]^2\Big)^{\frac{1}{2}}\\ &\leq [\rho(a)^2+\rho(b)^2]^{\frac{1}{2}}+[\rho(-a)^2+\rho(-b)^2]^{\frac{1}{2}}\\ &= \rho^s_E(a)+\rho^s_E(b)\end{aligned}$$

Hence ρ^s_L and ρ^s_E are nonsymmetric modulars on S. □

Proposition 4.4.2. *Let ρ be a nonsymmetric modular on a real linear space S. We have that norms $||.||_{\rho^s}$, $||.||_{\rho^s_L}$ and $||.||_{\rho^s_E}$ induced respectively by ρ^s, ρ^s_L and ρ^s_E are equivalent norms on the modular set S_ρ.*

Proof. Let $a \in S_\rho$ and $\delta > 0$. Then we have

$$\rho^s\Big(\frac{a}{\delta}\Big) \leq \rho^s_E\Big(\frac{a}{\delta}\Big) \leq \rho^s_L\Big(\frac{a}{\delta}\Big). \tag{4.1}$$

If $||a||_{\rho^s_L} = \delta_1$ and $||a||_{\rho^s_E} = \delta_2$, then $\delta_1, \delta_2 > 0$ and $\rho^s_E(\dfrac{a}{\delta_2}) \leq \delta_2$ and from (4.1) we have

$$\rho^s_E\Big(\frac{a}{\delta_1}\Big) \leq \rho^s_L\Big(\frac{a}{\delta_1}\Big) \leq \delta_1.$$

It follows that

$$\begin{aligned}||a||_{\rho^s_E} &= \inf\Big\{\delta > 0 : \rho^s_E\Big(\frac{a}{\delta}\Big) \leq \delta\Big\}\\ &= \delta_2\\ &\leq \delta_1\\ &= ||a||_{\rho^s_L}.\end{aligned}$$

Hence $||a||_{\rho^s_E} \leq ||a||_{\rho^s_L}$.
Similarly one proves that

$$||a||_{\rho^s} \leq ||a||_{\rho^s_E}$$

for $a \in S_\rho$. □

Corollary 4.4.3. *If ρ is a convex nonsymmetric modular on a real linear space S, then we have*

$$||a||^*_{\rho^s} \leq ||a||^*_{\rho^s_E} \leq ||a||^*_{\rho^s_L}$$

for $a \in S^*_\rho$.

Lemma 4.4.4. *Let* $(S, ||.||)$ *be a normed lattice. Then the function* $\rho_{||.||}$ *defined by*

$$\rho_{||.||}(a) = ||a^+||$$

for all $a \in S$ *where* $a^+ = a \vee 0$ *is a nonsymmetric modular on* S.

Proof. For $a \in S$, the function $\rho_{||.||}(a) = ||a^+||$ is a nonsymmetric norm on S (see [12, Theorem 3.1]). Therefore $\rho_{||.||}(a)$, $a \in S$ is a nonsymmetric modular from Example 4.1.1. $\square$

Let $(S, ||.||)$ be a normed lattice and (a_n) be a sequence on the modular set $S_{\rho_{||.||}}$. We write $R(a_n) = \{a \in S_{\rho_{||.||}} : (a_n) \overset{\rho_{||.||}}{\to} a\}$.

If T is a Riesz space, then the subset A of T is called increasing whenever $a \in A$ and $t \in T$ such that $a \le t$, it follows that $t \in A$.

Proposition 4.4.5. *Let* $(S, ||.||)$ *be a normed lattice. If* (a_n) *is a* $\rho_{||.||}$*-convergent sequence in* $S_{\rho_{||.||}}$*, then* $R(a_n)$ *is increasing.*

Proof. Let $a \in R(a_n)$. Then there exists $k > 0$ such that

$$\lim_{n\to\infty} (k(a_n - a)) = 0.$$

If $b \in S$ such that $a \le b$, then $a - b \le 0$. For any $s > 0$

$$\begin{aligned} \rho_{||.||}(s(a-b)) &= ||s(a-b)^+|| \\ &= s||(a-b)^+|| \\ &= s.0 \\ &= 0. \end{aligned}$$

Furthermore,

$$\rho_{||.||}\left(\frac{k}{2}(a_n - b)\right) = \rho_{||.||}\left(\frac{k}{2}(a_n - a) + \frac{k}{2}(a-b)\right).$$

Moreover,

$$\rho_{||.||}\left(\frac{k}{2}(a_n - b)\right) \le \rho_{||.||}((k(a_n - a) + k(a-b)).$$

Hence $\rho_{||.||}\left(\frac{k}{2}(a_n - b)\right) \le \rho_{||.||}((k(a_n - a))$.
Thus for $s = \frac{k}{2}$, we have

$$\lim_{n\to\infty} (s(a_n - b)) = 0.$$

Therefore $b \in R(a_n)$. $\square$

5

Conclusion

In this MSc dissertation, we have successfully generalized and extended the concept of a modular metric on an arbitrary set S from the work of Chistyakov [7] in asymmetric settings. We have successfully proven that Chistyakov's results hold in asymmetric framework. Moreover, we have shown that Chistyakov's results do not use the symmetric effects of a modular metric, and we have proven that the results still hold even if the symmetric property is assumed not to be satisfied. Therefore Chistyakov's results hold in an asymmetric structure. In the second part of our investigation, we have revealed that many classical results about a symmetric modular on real linear space do not require the use of the symmetry property of the modular. Hence the results hold for nonsymmetric modular, although sometimes in a roughly different form.

In the first part of our investigations, we generalised the work of Chistyakov [7] in quasi-pseudometric setting. We defined a modular quasi-pseudometric and showed that it is a non-increasing function. We also defined an equivalence relation and equivalence class in the sense of modular quasi-pseudometric. We studied how a modular quasi-pseudometric changes in convex set up and gave examples. In the last section of the first part of our investigation, we looked at convergence and Cauchy convergence of sequences in modular quasi-pseudometric space.

The second part of our investigation is focused on the nonsymmetric modular. We proved that the modular set with respect to a nonsymmetric modular is convex and confirmed that the convexity of S_ρ^* and S_ρ do not depend on the symmetric property of the modular. We also confirmed that [7, Theorem 3.11] does not depend on the symmetry property of symmetric modular and modular metric but still hold in nonsymmetric setting. Convergence of sequences on a modular set was revisited and improved in an asymmetric framework. We finally introduced a nonsymmetric modular on normed lattices and showed that some results due to Mabula and Conradie [10] still hold in the context of a nonsymmetric modular.

Our conclusion leads us to list some open problems that we came across throughout our present investigations. We would like to study these problems in future work.

Problem 1. It seems natural to study the completion of modular quasi-metric spaces.

Problem 2. To investigate one-local retract sets in the framework of modular quasi-metric spaces.

Problem 3. Under what conditions on the quasi-metric induced by a modular quasi-metric space is q-hyperconvex?

Problem 4. To investigate the compactness and connectedness of modular quasi-metric spaces.

Problem 5. Is there any connection between modular quasi-metric spaces and Fuzzy quasi-metric spaces?

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