

Applications of conic finance on the South African financial markets

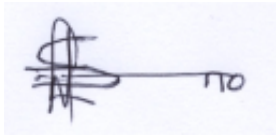
by
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Masimba Energy Sonono

16 November 2012

Date

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Dedication

To my late mother. I love you and will always cherish you.

“... Tomorrow, I will stand at the top of the hill, holding the staff of God in my hand.”

Exodus 17 vs 9.

Acknowledgements

My profound acknowledgements go to Professor Phillip Mashele, who supervised this thesis. I thank him for the commitment, good vision and guidance, without which this thesis would not have been a success. The work in this thesis was inspiring, often exciting, though at times challenging, but always interesting experience.

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My special appreciations extend to my family for their love, care, moral support and constant prayers. I cannot find a suitable phrase to express my appreciation, but thank you for always being there for me. My profound gratitude also extend to many other people that I have not managed to mention by their names, for making this phase a success.

Above all, I thank the Almighty God for the love, strength, wisdom and guidance. I have learnt to have faith, patience and hope in order to achieve all goals.

Executive Summary

Conic finance is a brand new quantitative finance theory. The thesis is on the applications of conic finance on South African Financial Markets. Conic finance gives a new perspective on the way people should perceive financial markets. Particularly in incomplete markets, where there are non-unique prices and the residual risk is rampant, conic finance plays a crucial role in providing prices that are acceptable at a stress level. The theory assumes that price depends on the direction of trade and there are two prices, one for buying from the market called the ask price and one for selling to the market called the bid price. The bid-ask spread reflects the substantial cost of the unhedgeable risk that is present in the market. The hypothesis being considered in this thesis is whether conic finance can reduce the residual risk?

Conic finance models bid-ask prices of cashflows by applying the theory of acceptability indices to cashflows. The theory of acceptability combines elements of arbitrage pricing theory and expected utility theory. Combining the two theories, set of arbitrage opportunities are extended to the set of all opportunities that a wide range of market participants are prepared to accept. The preferences of the market participants are captured by utility functions. The utility functions lead to the concepts of acceptance sets and the associated coherent risk measures. The acceptance sets (market preferences) are modeled using sets of probability measures. The set accepted by all market participants is the intersection of all the sets, which is convex. The size of this set is characterized by an index of acceptability. This index of acceptability allows one to speak of cashflows acceptable at a level γ , known as the stress level. The relevant set of probability measures that can value the cashflows properly is found through the use of distortion functions.

In the first chapter, we introduce the theory of conic finance and build a foundation that leads to the problem and objectives of the thesis. In chapter two, we build on the foundation built in the previous chapter, and we explain in depth the theory of

acceptability indices and coherent risk measures. A brief discussion on coherent risk measures is done here since the theory of acceptability indices builds on coherent risk measures. It is also in this chapter, that some new acceptability indices are introduced.

In chapter three, focus is shifted to mathematical tools for financial applications. The chapter can be seen as a prerequisite as it bridges the gap from mathematical tools in complete markets to incomplete markets, which is the market that conic finance theory is trying to exploit. As the chapter ends, models used for continuous time modeling and simulations of stochastic processes are presented.

In chapter four, the attention is focussed on the numerical methods that are relevant to the thesis. Details on obtaining parameters using the maximum likelihood method and calibrating the parameters to market prices are presented. Next, option pricing by Fourier transform methods is detailed. Finally a discussion on the bid-ask formulas relevant to the thesis is done. Most of the numerical implementations were carried out in Matlab.

Chapter five gives an introduction to the world of option trading strategies. Some illustrations are used to try and explain the option trading strategies. Explanations of the possible scenarios at the expiration date for the different option strategies are also included.

Chapter six is the apex of the thesis, where results from possible real market scenarios are presented and discussed. Only numerical results were reported on in the thesis. Empirical experiments could not be done due to limitations of availability of real market data. The findings from the numerical experiments showed that the spreads from conic finance are reduced. This results in reduced residual risk and reduced low cost of entering into the trading strategies. The thesis ends with formal discussions of the findings in the thesis and some possible directions for further research in chapter seven.

Keywords: conic finance, coherent risk measures, acceptability indices, incomplete markets, trading strategies, risk profiles, bid-ask prices, option pricing, Fourier transform method, calibration, maximum likelihood method

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1. Introduction

Conic finance is a brand new quantitative finance theory. This thesis brings the basic introduction of the cutting edge theory. Both the theory and applications on the South African financial markets are presented in this dissertation.

In an efficient market, a financial security trades at a unique price at a given time. The equilibrium conditions required to obtain a unique price depends on rapid arbitrage opportunities, which are immediately exploited, and on readily availability of trading counterparties. A broad spectrum of financial securities trade in diverse markets and few of them satisfy the equilibrium conditions mentioned above. Clearing of markets becomes problematic leading to non-unique prices for equivalent securities in different markets. When the conditions are not satisfied, the ‘law of one price’ fails to hold; and such state of markets are called incomplete. Incompleteness means that there is presence of residual risk which cannot be eliminated inspite of the best hedge (Eberlein, Gehrig & Madan 2012). Furthermore, the markets manifest a phenomenon not anticipated in the one price theory; illiquidity. Illiquidity can be described as the inability of the market to reach at unique price, that is, as the spread between bid-ask prices. This comes about when there is death of information and/or scarcity of interested parties. In such instances, the bid-ask prices are the only real market observables.

It follows from above that the conceptual framework used to describe the one price market is inadequate to deal with many situations encountered in practice. An expanded framework which is realistic enough to capture the essential features is advocated for. The framework, which is a minimal extension of the one price economy and which will be used in this thesis, is known as the theory of two price economies and has been branded the name ‘conic finance’. Some recent studies which apply the concepts of conic finance include Madan (2009), Madan (2010) , Cherny & Madan (2010*a*) , Eberlein et al. (2012), Eberlein & Madan (2012), Eberlein & Madan (2009),

Madan & Schoutens (2011), and Madan & Schoutens (2012a).

The significant difference of the theory of two price economies from the theory of one price economy is that price depends on the direction of trade. There are now two prices, one for buying from the market called the ask price and the other one for selling to the market called the bid price. In the one price economy, the market acts as the auctioneer, clearing trades and deciding the prices. In the two price economy the market acts as a passive counterparty to all transactions, buying at the ask price and selling at the bid price. The spread between bid-ask prices is a measure of illiquidity. Besides measuring illiquidity, it measures the capital required to support a position and the cost of unwinding a position.

1.1 Bid-Ask Spreads

There are variety of theoretical approaches that try to model bid-ask spreads. Cherny & Madan (2010b) gives some of the approaches that have been used to model bid-ask spreads. Copeland & D.Galai (1983), Stoll (1978), Glosten & Milgron (1985) and Ahimud & Mendelson (1980) focus on order processing and inventory costs of liquidity providers. Huang & Stoll (1997) decided to decompose the spread into order processing, inventory and adverse selection components. Constantinides & Lapied (1986) and Jouini & Kallal (1995) studied spreads which included transaction costs of trading in liquid markets. However, the above studies are relatively suited to liquid markets where a transaction can take place at a price at which a reversal of trade direction can possibly happen without price effect. The spreads which relate to the theory of two price economies in this thesis are the ones studied in Cochrane & Saá-Requejo (2000), Bernardo & Ledoit (2000), Jaschke & Küchler (2001) and Carr, Geman & Madan (2001), cited in Cherny & Madan (2010b). They are more related to finding genuine long term counterparties who are prepared to maintain a position for an extended period of time.

The bid-ask spread can be seen as a holding charge exerted when the market does not clear quickly, as finding a counterparty requires some time and effort since there is no possibility of trading in both directions at any observed transaction price. In other words, there is no possibility of complete replication and the bid-ask spread is a reflection of the cost of holding residual risk (Eberlein, Madan & Schoutens 2011). As a result, transactions take place either near or at the ask or near or at the bid, depending on the direction of trade. Conic finance tries to model the bid-ask spread by applying the notion of acceptability of cashflows to the market. The market is assumed to require a minimum level of acceptability for a position to be marketable. Due to competition, the bid price is raised and the ask price is lowered so as for the position to remain acceptable. Consequently, the spread is narrowed and the risk of a position is minimized. This spread can be viewed as the cost of unwinding a position. In this work, we shall concentrate on the spreads with an aim of reducing risks of trading positions.

1.2 Two Price Economy

In the two price economy a relatively classical view of markets compatible with its role in traditional competitive analysis, where markets serve as counterparties to transactions, is assumed. The only departure from the traditional perspective is that the terms of trade depend on the direction of trade, with the market buying at bid price and selling at ask price.

Consider the classical market, where trading is done in both directions at the going price. The market accepts to sell at a higher price or buy at a lower price and accepts all random cashflows at zero cost if they have a positive expectation under the equilibrium pricing kernel. This is a very large set of risks that are accepted by the classical market on a risk-neutral measure. The two price market is more restrictive as to which trades it will accept. The set of zero cost risks acceptable

by the market is a much smaller set. The modeling of this set of acceptable risks follows Artzner, Delbaen, Eber & Heath (1999), Carr et al. (2001) and was further developed in Cherny & Madan (2009) and Cherny & Madan (2010*a*). In particular, the zero cost risks acceptable to the market as a set of random variables is modeled as a convex cone containing the nonnegative random variables.

The conceptual framework required to support the two price economy has been given attention in the past few years. The theory was introduced into financial mathematics by Constantinides & Lapied (1986) and was popularized as coherent risk measures by Artzner et al. (1999). The work linking the theory of two price economies to concave distortions was done by Cherny & Madan (2009) and Cherny & Madan (2010*a*). It is in this work that gave an understanding of how to construct the bid-ask prices relevant to the theory of two prices. The theory of two prices was then named ‘conic finance’. In the following section, we present the theory of two prices in an abstract manner as set out in Carr, Madan & Alvarez (2011).

1.3 Modeling Conic Two Price Markets

In this section, the theory of two price markets is presented. In modeling the two price market, the market is viewed as a passive counterparty accepting zero cost trades proposed by opposite market participants. Cashflows to trades are considered as bounded random variables on a fixed probability space (Ω, \mathcal{F}, P) for a base probability measure (risk neutral measure) selected by the economy. The set of cashflows accepted at zero cost form a convex cone. The convex cone containing the cashflows is a special structure of cashflows acceptable to the market, which act as a counterparty.

The classical model with its law of one price asserts that if a cash flow X is acceptable to the market with $E^P[X] = 0$, then trade takes place in both directions at the same price and so $-X$ is also acceptable. The set of acceptable cash flows is represented

with the half space defined by the condition $E^P[X] \geq 0$. In the two price markets we stop asserting that the law of one price still holds. The set of cashflows acceptable at zero cost is now a proper convex cone containing nonnegative cashflows. Denote this set of cashflows acceptable to the market at zero cost by \mathcal{A} . The set is smaller than the classical set of the one price economy. Furthermore, if X is acceptable, then $-X$ will not be acceptable as direction of trade cannot be reversed on the same terms.

Artzner et al. (1999) provided a constructive characterization of the nonnegative cash flows in the convex cones. They showed that for any set \mathcal{A} of acceptable risks (cashflows), there exists a convex set \mathcal{M} of probability measures $Q \in \mathcal{M}$, Q equivalent to P with the property that $X \in \mathcal{A}$ (that is X is acceptable) if and only if:

$$E^Q[X] \geq 0, \quad \text{all } Q \in \mathcal{M}.$$

Acceptability of cashflows is linked to positive expectation via concave distortion. The preferred concave distortion is one defined on the unit interval. So, we take any concave distribution function on the unit interval $\Psi(u)$, $0 \leq u \leq 1$, and define a random variable X with distribution function $F(x)$ to be acceptable provided:

$$\int_{-\infty}^{\infty} x \, d\Psi(F(x)) \geq 0. \quad (1.3.1)$$

Models for markets are then constructed by specifying intersecting sets of supporting measures. However, this is not that simple a task. Cherny & Madan (2009) defined operational cones that depend only on the probability law of the cashflows being accessed. Each market is then defined by a convex cone of zero cost cashflows acceptable to the market, which has an associated convex set of probability measures $Q \in \mathcal{M}$.

The condition [1.3.1](#) defines a cone of acceptable cashflows that depend on only infor-

mation of the distribution function of the cashflow. The integral in condition 1.3.1, may be written as follows:

$$\int_{-\infty}^{\infty} x\Psi'(F(x))f(x)dx, \quad (1.3.2)$$

where $f(x) = F'(x)$.

This expectation under concave distortion is also an expectation under a measure change. Note that large losses with $F(x)$ near zero are reweighted upwards by $\Psi'(F(x))$ as Ψ' decreases for any concave distortion. The more concave the distortion the higher the upward reweighting of losses and the more difficult it is to be acceptable.

As shown in Cherny (2006), the set of supporting measures \mathcal{M} for this set of cashflows are all measures Q with density $Z = \frac{dQ}{dP}$ satisfying the condition:

$$E^P[(Z - a)^+] \leq \Phi(a), \quad \text{for all } a \geq 0 \quad (1.3.3)$$

where $\Psi(a)$ is the conjugate of Φ ,

$$\Psi = \sup_{u \in [0,1]} (\Psi(u) - ua).$$

Cherny & Madan (2009) proposed a sequence of concave distortions indexed by a real number γ that are increasingly more concave with a corresponding decreasing sequence of sets of acceptability. The level γ can be thought of as the stress level of distortion being applied to the cashflow X which is being tested for acceptability.

The cashflow is acceptable if the stressed expectation still remains positive. If we index the concave distortion function $\Psi(F(x))$ in 1.3.1 with a real number γ , we can compute it numerically once we have the distribution function of X . It becomes simple to compute the integral if we employ the empirical distribution function of a sample x_1, \dots, x_n . In this case,

$$\int_{-\infty}^{\infty} x d\Psi^\gamma(F_X(x)) = \sum_{n=1}^N x_{(n)} \left(\Psi^\gamma\left(\frac{n}{N}\right) - \Psi^\gamma\left(\frac{n-1}{N-1}\right) \right), \quad (1.3.4)$$

where $x_{(n)}$ are the values x_n sorted in increasing order.

We introduce the index of acceptability, also known as the measure of performance, that enables us to speak of cashflows acceptable at a level γ (see Cherny & Madan (2009)). The index of acceptability is a non-negative real number, and associated with each level of the index is a collection of terminal cashflows viewed as random variables acceptable at this level. In this work, we employ a static notion of acceptable cashflows. As suggested in Cherny & Madan (2009), the index of acceptability can be constructed from a particular family of distortions. We shall explore distortions introduced in Cherny & Madan (2009) which are MINVAR, MAXVAR, MAXMINVAR and MINMAXVAR. MINVAR constructs a worst case scenario by forming the expectation of the minimum of numerous draws from the cashflow distribution. MAXVAR constructs a distribution from which one draws numerous times and takes the maximum to get the cash flow distribution being evaluated. The last two measures MINMAXVAR and MAXMINVAR combine these approaches to constructing worst case scenarios.

Now the question we consider is: Can two markets be arbitrated by buying some cashflow at the bid price from one market and selling it to the other market at a higher ask price? Different markets are modeled using different convex cones of acceptable cashflows. For instance, two markets may be modeled with different cones

of acceptable zero cost cashflows $\mathcal{A}_1, \mathcal{A}_2$ with associated sets of measures $\mathcal{M}_1, \mathcal{M}_2$. For any cashflow X we determine the market ask price $a(X)$ by noting that:

$$a(X) - X \in \mathcal{A},$$

or equivalently that:

$$a(X) - E^Q[X] \geq 0, \text{ for all } Q \in \mathcal{M}, \quad (1.3.5)$$

and so,

$$\begin{aligned} a(X) &= \int_{-\infty}^{\infty} x d\Psi(1 - F(-x)) \\ &= \sup_{Q \in \mathcal{M}} E^Q[X]. \end{aligned} \quad (1.3.6)$$

Similarly for the bid price,

$$\begin{aligned} b(X) &= \int_{-\infty}^{\infty} x d\Psi(F(x)) \\ &= \inf_{Q \in \mathcal{M}} E^Q[X]. \end{aligned} \quad (1.3.7)$$

Provided the set of supporting measures $\mathcal{M}_1, \mathcal{M}_2$ have a common element \bar{Q} , then

$$a_1(X) \geq E^{\bar{Q}}[X] \geq b_2(X),$$

and the bid price of market two is never above the ask price of market one. Therefore, one may use different cones to define different markets provided the set of supporting measures have a nonempty intersection.

1.4 Problem Statement

The argument here is that many risks that affect option prices are manifested through market illiquidity. Illiquidity is manifested as the inability of the market to reach a unique price. Bid-ask prices become the only real market observables. The spread between bid-ask prices becomes the measure of illiquidity in the market. In such a scenario, the law of one price fails to hold and terms of trade depend on the direction of trade (Eberlein et al. 2012).

The illiquidity implies that the market is incomplete and using the law of one price leads to improper pricing of options. Incompleteness means that there is a presence of residual risk which cannot be totally eliminated as the best hedge (replication) leaves a market participant still exposed to residual risk (Eberlein et al. 2012). The bid-ask spread reflects the substantial cost of the unhedgeable residual risk (Cherny & Madan 2010*b*). The question is: What level of residual risk is considered to be acceptable by the market participants?

Artzner et al. (1999) axiomatized the acceptable risks as some convex cone containing nonnegative cash flows. Cherny & Madan (2009) further added on to the literature by suggesting cones of acceptability that depend only on probability law and provided way of computing bid and ask prices. Cherny & Madan (2010*b*) went on to derive closed form formulas for option prices, which are used in this project. In order to

check whether the acceptable residual risk can be reduced, analysis is done on the risk profiles of option strategies. Maximum risk, maximum reward and breakeven price are determined for each of the strategies. The theory of conic finance provides bid-ask prices, which depend on the risk appetite of investors. A comparison of the risk is done using two models, Black-Scholes model, which is commonly used by practitioners, and the Variance Gamma Scalable Self Decomposable (VGSSD) model.

1.5 Aim of the Study

To apply conic finance on South African financial markets.

1.6 Objectives of the Study

- To estimate bid-ask prices using conic finance.
- To investigate the risk profiles of option strategies using expected bid-ask prices in conic finance.

1.7 Significance of the Study

The theory of two price markets or conic finance, which yields closed forms for bid-ask option prices, has significant contributions to the financial derivatives markets globally. In fact, the theory can also be relevant to the South African financial market, which is an emerging market. The options derive their value from the prices of the underlying assets. The underlying assets can be shares, currencies, equity indices or fixed interest bearing securities. Most options are sold on single shares or equity indices. In South Africa, options are usually equity options that are based on

futures. They are future style options and allow investors to buy or sell a future of the underlying equity asset.

The theory has significant applications in product development. A segment of the financial markets where the theory is being used is over-the-counter structured products market. Structured products are tailor made products with specific risk profiles that suite investor needs. Transacting in the structured markets is infrequent such that there are two prices, prices for buying from or selling to the market. Therefore, the theory is needed for such a two price market. Structured products have begun to play a significant role in the South African investment landscape. They have become so popular internationally such that investors like including them in their investment portfolios.

Apart from product development, the theory serves as a pricing device at particular levels of acceptability. In otherwords, we can price our options using the theory of conic finance. The theory provides bid-ask quotes for over-the-counter options at different levels of acceptability. In the South African market the theory can be useful in providing quotes for warrants and single stocks futures which meet the risk appetite of investors. In their simplest form, warrants and single stock futures are similar to options and can be used to modify risk profiles of financial positions.

Conic finance can be a useful tool in risk management. Risk management is the process of identifying actual risk levels and altering it to reach a desired risk level and is crucial to the management of many levels of risks. This can be achieved by hedging and speculation activities. However, not all risks can be hedged. The risks which cannot be hedged can be controlled by restricting the trade prices. The theory of conic finance provides a set of prices that are acceptable to the market at certain levels of risk. The prices from conic finance can go a long way in assisting to manage risks of trades or risks of financial positions.

2. Theory of Acceptability Indices

Cherny & Madan (2009) coined the term *Acceptability Index* as a mathematical terminology for studying risk measures in a systematic way. This theory of acceptability indices builds on the theory of coherent risk measures and acceptable sets studied in Artzner et al. (1999) and Carr et al. (2001). The acceptable risk sets are a result of the axiomatic approach to risk measures introduced into financial literature by Artzner et al. (1999). The axiomatic approach to measuring risk included setting axioms on a random variable and then determining the mathematical function fitting to the set of axioms.

In this chapter we give an overview of the theory of acceptability indices as in Artzner et al. (1999). We present the axioms for the acceptability indices and also provide an overview of coherent risk measures, since they are naturally related to acceptability indices. After that we pass on to acceptability indices. We then look at a family of distortion functions, from which acceptability indices are constructed. Finally, we give examples of acceptability indices which are relevant to this work.

2.1 Basic Definitions and Theorems

In this section we lay down some basic definitions and theorems which are commonly used throughout the thesis. The definitions presented here are mainly taken from Föllmer & Schied (2004), Dunford & Schwartz (1958), Dudley (1989), Rockafeller (1970), and Körezlioğlu & Hayfavi (2001), unless otherwise stated.

2.1.1 Definition. Finitely Additive Set Function

A set function is a function defined on a family of sets, and having values either in a Banach Space, which may be the set of real or complex numbers, or in the extended

real number system, in which case its range contains one of the improper values ∞ and $-\infty$. A set function μ defined on a family τ of sets is said to be infinitely additive if τ contains the void set \emptyset , if $\mu(\emptyset) = 0$ and if $\mu(A_1 \cup A_2 \cup \dots \cup A_n) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_n)$ for every finite family $\{A_1, A_2, A_3, \dots\}$ of disjoint subsets of τ whose union is in τ . Thus all such sums must be defined, so that there can not be both $\mu(A_i) = -\infty$ and $\mu(A_j) = +\infty$ for some i and j .

2.1.2 Definition. Countably Additive Function

Let μ be a finitely additive, real valued function on an algebra \mathcal{A} . Then μ is countably additive iff μ is continuous at \emptyset , that is $\mu(A_n) \rightarrow 0$ whenever $A_n \downarrow \emptyset$ and $A_n \in \mathcal{A}$.

2.1.3 Definition. Sigma Algebra

Given a set X , a collection $\mathcal{A} \subset 2^X$ is called a ring iff $\emptyset \in \mathcal{A}$ and for all A and B in \mathcal{A} , we have $A \cup B \in \mathcal{A}$ and $B \setminus A \in \mathcal{A}$. A ring \mathcal{A} is called an algebra iff $X \in \mathcal{A}$. An algebra is called a σ -algebra if for any sequence $\{A_n\}$ of sets in \mathcal{A} , $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.

2.1.4 Definition. Measurable Space

A measurable space is a pair (X, \mathcal{Y}) where X is a set and \mathcal{Y} is a σ -algebra of subsets of X .

2.1.5 Definition. Measure, Measure Space

A countably additive function μ from σ -algebra (\mathcal{Y}) of subsets of X into $[0, \infty]$ is called a measure. Then (X, \mathcal{Y}, μ) is called a measure space.

2.1.6 Definition. Atom

If (X, \mathcal{Y}, μ) is a measure space, a set $A \in \mathcal{Y}$ is called an atom of μ iff $0 < \mu(A) < \infty$ and for every $C \subset A$ with $C \in \mathcal{Y}$, either $\mu(C) = 0$ or $\mu(C) = \mu(A)$.

2.1.7 Definition. Probability Measure, Probability Space

A measurable space (Ω, \mathcal{Y}) is a set Ω with a σ -algebra \mathcal{Y} of subsets of Ω . A probability measure P is a measure on \mathcal{Y} with $P(\Omega) = 1$. Then (Ω, \mathcal{Y}, P) is called a probability space. Members of \mathcal{Y} are called events in a probability space.

2.1.8 Definition. Random Variable

If (Ω, \mathcal{A}, P) is a probability space and (S, \mathcal{B}) is any measurable space, a measurable function X from Ω into S is called a random variable. Then the image measure $P \circ X^{-1}$ defined on \mathcal{B} is the probability measure which is called the law of X .

2.1.9 Definition. Absolutely Continuous Probability Measure

Let P and Q be two probability measures on measurable space (Ω, \mathcal{F}) . Q is said to be absolutely continuous with respect to P iff for $A \in \mathcal{F}$:

$$P(A) = 0 \implies Q(A) = 0$$

.

2.1.10 Theorem. Radon-Nikodym

Q is absolutely continuous with respect to P on \mathcal{F} if and only if there exists an \mathcal{F} measurable function $\varphi \geq 0$ such that

$$\int F dQ = \int F \varphi dP$$

,

for all \mathcal{F} -measurable functions $F \geq 0$.

The function φ is called the Radon-Nikodym derivative of Q with respect to P and we write $\frac{dQ}{dP} := \varphi$.

2.1.11 Definition. Seminorm, Norm

Let X be a real vector space. A seminorm on X is a function $\| \cdot \|$ from X into $[0, \infty]$ such that

1. $\| cx \| = |c| \| x \|$ for all $c \in \mathbb{R}$ and $x \in X$.
2. $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$.

A norm is called a seminorm iff $\| x \| = 0$ only for $x = 0$.

2.1.12 Definition. \mathcal{L}^p Spaces

Suppose that (X, \mathcal{F}, μ) is a measure space. If $1 < p < \infty$ (p need not to be an integer), then $\mathcal{L}^p(S, \mathcal{F}, \mu)$ is defined to be the set of a \mathcal{F} -measurable functions $S \xrightarrow{f} \mathbb{R}$ such that $|f|^p$ is μ -integrable.

A function $S \xrightarrow{f} \mathbb{R}$ is said to be essentially bounded iff there is a real number M such that $|f| \leq M$ μ -a.e. $\mathcal{L}^\infty(S, \mathcal{F}, \mu)$ is defined to be the set of essentially bounded \mathcal{F} -measurable $S \xrightarrow{f} \mathbb{R}$. For each $1 < p < \infty$, we define a map $\| \cdot \|_p: \mathcal{L}^p(S, \mathcal{F}, \mu) \rightarrow \mathbb{R}$ by

$$\| f \|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

\mathcal{L}^1 is just the family of μ -integrable functions. The maps $\| \cdot \|_p$ are called \mathcal{L}^p -norms, or just p -norms.

2.1.13 Definition. Convex Set

A subset S of a given vector space X is called a convex set if $x \in S$, $y \in S$, and $\lambda \in [0, 1]$ always imply that $\lambda x + (1 - \lambda)y \in S$.

So for any two given points in the set, the line segment connecting these two points lies entirely in the set. Given a convex set S , a function $f: S \rightarrow \mathbb{R}$ is called a convex function if $\forall x \in S, y \in S$ and $\lambda \in [0, 1]$ the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

We say that f is strictly convex function if $x \in S$, $y \in S$ and $\lambda \in [0, 1]$ implies the following strict inequality:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

A function f is concave if $-f$ is convex. Equivalently, f is concave if, $\forall x \in S$, $\forall y \in S$ and $\lambda \in [0, 1]$ the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

A function f is strictly concave if $-f$ is strictly convex.

2.1.14 Definition. Cone, Convex Cone

(i) A set $K \subset \mathbb{R}^m$ is called a cone if:

$$\forall x \in K, \forall \lambda \in \mathbb{R}, \lambda \geq 0 \implies \lambda x \in K.$$

(ii) A set $K \subset \mathbb{R}^m$ is a convex cone, if it convex and a cone.

(iii) A cone $K \subset \mathbb{R}^m$ is called a proper cone or ordering cone if it closed and convex, has non-empty interior and is pointed, meaning that:

$$x \in K, -x \in K \implies x = 0.$$

2.1.15 Definition. Risk Measure

Let \mathcal{G} represent the set of all positions, that is the set of all real valued functions on Ω . Then a risk measure ρ is any mapping from the set of all random variables onto the real number line, that is

$$\rho : \mathcal{G} \longrightarrow \mathbb{R}.$$

2.2 Axioms for Acceptability Indices

In this section, we look at the axiomatic structure for acceptability indices as proposed by Artzner et al. (1999). They restricted attention to the class of bounded variables given by $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$ so as to avoid technicalities associated with finiteness of moment. An acceptability index was defined as a map from \mathcal{L}^∞ to the extended positive reals $[0, \infty]$, with $\alpha(X)$ being the level of acceptability of the random variable $X \in \mathcal{L}^\infty$. We discuss the eight properties an acceptability index should satisfy. The first four properties define what is termed to be the coherent acceptability index. The remainder of the properties are additional ones which enable to make further comparisons between acceptability indices.

1. Quasi-Concavity

The set of cashflows acceptable at level x is defined as:

$$A_x = \{X : \alpha(X) \geq X\}, \quad x \in \mathbb{R}_+.$$

The quasi-concavity property requires the sets to be convex. Along with the scale invariance property to be described below means that A_x are convex cones. By the quasi-convexity condition, A_x is a convex set for all x :

if $\alpha(X) \geq x$ and $\alpha(Y) \geq x$,
then $\alpha(\lambda X + (1 - \lambda)Y) \geq x$ for any $\lambda \in [0, 1]$.

2. Monotonicity

Monotonicity is a basic property of acceptability where there is a general preference for more over less. Technically, this is the condition that if X is acceptable at a level and Y dominates X as a random variable, then Y is acceptable at the same level. The monotonicity property should be satisfied for all levels, that is

if $X \leq Y$ a.s., then $\alpha(X) \leq \alpha(Y)$.

3. Scale Invariance

Scale invariance requires that the level of acceptability of X does not change under scaling. The sets of acceptability are required to be convex at all levels. Thus we require

$$\alpha(\lambda X) = \alpha(X), \text{ for } \lambda > 0.$$

Interest is devoted to determining the direction of trades and not their scale, hence the significance of the scale invariance property. The scale maybe determined by other considerations such as market impact, liquidity or depth of the market.

4. Fatou Property

This is a continuity or closure property. It states that: For any countable collection of random variables X_n with $|X_n| \leq 1$ such that $\alpha(X_n) \geq x$, we require that if X_n converges to X in probability, then $\alpha(X) \geq x$.

5. Law Invariance

We require that the index of acceptability depend on just the probability law of the random variable. Formally,

$$\text{if } X \stackrel{\text{law}}{=} Y, \text{ then } \alpha(X) = \alpha(Y),$$

where $X \stackrel{\text{law}}{=} Y$ means that X and Y have the same probability distribution. In other words, when two cashflows have the same probability distribution, they should have the same level of acceptability.

6. Second Order Monotonicity

Acceptability indices are consistent with expected utility theory. If participants' preferences are described by expected utility theory, we have the property that says Y dominates X in the second order ($X \preceq^2 Y$) if $E[f(X)] \leq E[f(Y)]$ for any increasing concave function f . For the index to be consistent with expected utility theory, we must have that:

$$\text{if } X \preceq^2 Y, \text{ then } \alpha(X) \leq \alpha(Y).$$

The last two properties are related to the extreme values of the index.

7. Arbitrage Consistency

Arbitrage consistency deals with high values of the index. In the setting of acceptability indices, an arbitrage is a positive random variable X with $P(X > 0) > 0$. As arbitrages are universally acceptable, it is desirable that the level of acceptability for such outcomes be set at infinity, and so for arbitrage consistency we require that:

$$X \geq 0 \text{ a.s. if and only if } \alpha(X) = \infty.$$

Note that acceptability indices depart from traditional preference orderings here as we are converting the entire positive orthant to a bliss point at infinity and we do not rank two positive cashflows from an acceptability perspective.

8. Expectation Consistency

Expectation consistency deals with low values of the index and requires that

$$\begin{aligned} \text{if } E[X] < 0, \text{ then } \alpha(X) &= 0; \\ \text{if } E[X] > 0, \text{ then } \alpha(X) &> 0. \end{aligned}$$

In the next section, we review coherent risk measures since they are naturally related to acceptability indices.

2.3 Coherent Risk Measures

The underlying idea of the theory of coherent risk measures is that an appropriate measure of risk should be consistent with finance theory. Previous studies defined financial risk as a change in value of a position between two dates. However, Artzner et al. (1999) in their paper argue that since risk is related to the variability of the future value of a position, it is better to consider future values only. Risk is thus considered as a random variable and there is no need to take the initial costs into consideration. For an unacceptable risk (that is a position with unacceptable future net worth) there are two remedies that can be implemented. The first remedy may be to alter the position and the second remedy might be looking for some commonly accepted instruments so that when added to the current position, makes the future value of the initial position acceptable.

Basic Framework

Let Ω be the set of states of nature and assume it is finite. Suppose that the set of all possible states of the world at the end of the period are known. This assumption implies that we know all the possible events that may occur in future and they are finite. However, the probabilities of the various states occurring may be unknown. Now consider a one period economy starting at time 0 and ending at a date T . Let the networth of a portfolio be denoted as a random variable X , which has the value $X(\omega)$ as the state of the nature ω occurs. Also assume that markets at date T are liquid. Let \mathcal{G} represent the set of all risks, that is, the set of all real valued functions on ω . Since ω is assumed to be finite, \mathcal{G} can be identified with \mathbb{R}^n , where $n = \text{card}(\Omega)$. L_+ denotes the cone of non-negative elements in \mathcal{G} and its negative by L_- . A coherent risk measure is formally defined as follows:

2.3.1 Definition. Coherent Risk Measure

A risk measure ρ is coherent if it satisfies the following axioms:

1. Translation Invariance: $\rho(X + \alpha r) = \rho(X) - \alpha$ for all $X \in \mathcal{G}, \alpha \in \mathbb{R}$.
2. Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \geq Y$ a.s.
3. Positive Homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$.
4. Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{G}$.
5. Relevance: $\rho(X) > 0$ if $X \leq 0$ and $X \neq 0$.

Artzner et al. (1999) included the last property although it is not a direct determinant of coherency. Translation invariance axiom implies that adding a fixed amount α to the initial position and investing it in a reference instrument, the risk $\rho(X)$ decreases by α . This property ensures that the risk measure and returns are in the same unit. The monotonicity axiom implies that if $X(\omega) \geq Y(\omega)$ for every state of nature ω , Y is more risky because it has higher risk potential. Risk assessment of a financial position

appears as a numerical representation of preferences from an investor's viewpoint. The risk measure is viewed as a capital requirement from a regulator's viewpoint.

The positive homogeneity axiom implies that risk linearly increases with size of the position. This property may not be satisfied in the real world because markets may be illiquid. Illiquidity of markets implies that increasing the amount of position may create extra risk. The subadditivity axiom postulates that the risk of a portfolio is always less than or equal to the sum of the risks of its subcomponents. This axiom ensures that diversification decreases the risk. Relevance tells us that a position having zero or negative (at least for some state of nature ω) future net worth has a positive risk. This axiom ensures that the risk measure identifies a random portfolio as risky (Jarrow & Purnanandam 2005).

According to the basic representation theorem proved by Artzner et al. (1999) for a finite Ω , any coherent risk measure admits a representation of the form:

$$\rho(X) = - \inf_{Q \in \mathcal{D}} E^Q[X], \quad (2.3.1)$$

with a certain set \mathcal{D} of probability measures with respect to Q . A cashflow X is acceptable if it has negative risk, that is $\rho(X) \leq 0$. The measures from \mathcal{D} are called generalized scenarios in (Artzner et al. 1999) and are called test measures in (Carr et al. 2001).

The supporting set \mathcal{D} defining a coherent risk measure or equivalently acceptability through 2.3.1 is not unique. For example, if it is not convex, then \mathcal{D} and its convex combinations define the same ρ . However, there exists the largest set given by

$$\mathcal{D} = \{Q \in \mathcal{P} : E^Q[X] \geq -\rho(X) \quad \forall x \in \mathcal{L}^\infty\}, \quad (2.3.2)$$

where \mathcal{P} denotes the set of probability measures absolutely continuous with respect to Q . The set \mathcal{D} is called the set of supporting kernels of ρ^* . Supporting kernels play a significant role in applications of coherent risks to pricing (see Carr et al. (2001)).

An important aspect associated with a set of kernels supporting a cone of acceptability is the extreme measure.

2.3.2 Definition. Extreme Measure

The set of extreme measures corresponding to a random variable X denoted by $Q^*(X)$ is defined as the set of supporting kernels Q , at which the minimum of expectations $E^Q[X]$ is attained.

In typical situations, the measure exists and is unique. The understanding of extreme measures is of essence because it embeds the idea of obtaining the set of supporting kernels from the set of extreme measures by taking convex combinations.

Next, we look at the acceptability set associated with a coherent risk measure which is formally defined below (see Artzner et al. (1999)).

2.3.3 Definition. Acceptability Set

An acceptability set basically represents the set of acceptable future net worths. Artzner et al. (1999) argue that all sensible risk measures should be associated with an acceptability set that satisfies the following conditions:

1. The acceptability set \mathcal{A} contains \mathcal{L}_+ .
2. The acceptance set \mathcal{A} does not intersect \mathcal{L}_- where

$$\mathcal{L}_{--} = \{X \mid \text{for each } \omega \in \Omega, X(\omega) < 0\}.$$

A stronger axiom would be

- 2'. The acceptance set \mathcal{A} satisfies $\mathcal{A} \cap \mathcal{L}_{--} = \{0\}$.
3. The acceptance set \mathcal{A} is convex.
4. The acceptance set \mathcal{A} is a positively homogeneous cone.

The above properties tell us that a reasonable acceptability set accepts any portfolio which has positive return \mathcal{L}_+ and does not contain a portfolio with sure loss \mathcal{L}_{--} . The more stronger axiom 2' states that the intersection of the non positive orthant and acceptability set contains the origin only. Convexity of the acceptability set indicates that linear combinations of acceptability portfolios are also acceptable, and are contained in the acceptability set. The fourth property tells us that an acceptable position can be scaled up or down in size without losing its acceptability.

2.3.4 Definition. Acceptability set associated with a risk measure

The acceptability set associated with a risk measure ρ is the set \mathcal{A}_ρ defined by:

$$\mathcal{A}_\rho = \{X \in \mathcal{G} : \rho(X) \leq 0\} \tag{2.3.3}$$

This is the set of positions that have a positive expectation under each measure from the set of supporting kernels, that is, the positions supported by all the measures. The risk measure associated with the acceptability set is:

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}. \quad (2.3.4)$$

2.4 Acceptability Indices

In conic finance, the aim is to come up with pricing models for financial securities, whilst at the same time not making strict assumptions about preferences of market participants. The notion is to extend the set of arbitrage opportunities to the set of all opportunities that wide range of risk-averse participants are willing to accept. So if a market participant starts from a position with zero cost, any positions that will increase expected utility are acceptable to the market participants. These positions form a convex set that contains nonnegative terminal cashflows. Every market participant has an acceptable set depending on his/her preferences. The preferences of the market participants are modeled using a set of probability measures. When a wider range of market participants are willing to accept a certain position, it is awarded a higher level of acceptability. The set accepted by all market participants is the intersection of all the sets, which is a convex set. It is called the acceptance set. The acceptance sets have been studied by Artzner et al. (1999) and Carr et al. (2001) and have been defined in the previous section.

2.4.1 Definition. Index of Acceptability

According to (Cherny & Madan 2009), they define the index of acceptability as a mapping α from the set of bounded random variables to the extended half-line $[0, \infty]$. The index satisfies the following four properties:

1. Monotonicity

If Y dominates X , that is $X \leq Y$, then $\alpha(X) \leq \alpha(Y)$.

2. Scale invariance

$\alpha(X)$ stays the same when X is scaled by a positive number, that is $\alpha(cX) = \alpha(X)$ for $c > 0$.

3. Quasi-concavity

If $\alpha(X) \geq Y$ and $\alpha(Y) \geq Y$, then $\alpha(\lambda X + (1 - \lambda)Y) \geq Y$ for any $\lambda \in [0, 1]$.

4. Fatou Property (Convergence)

Let $\{X_n\}$ be a sequence of random variable. $|X_n| \leq 1$ and X_n converges in probability to a random variable X . If $\alpha(X_n) \geq x$, then $\alpha(X) \geq x$.

$\alpha(X)$ can be considered as the degree of measure of the quality of terminal cashflow X . A higher value of $\alpha(X)$ means a higher level of acceptability. $\alpha(X) = +\infty$ represents arbitrage and all random variables in the acceptance cone(set) are nonnegative.

These four properties which define an index of acceptability also provide a useful representation which connects the indices $\alpha(X)$ to the family of probability measures.

2.4.2 Theorem. *Representation Theorem of Acceptability Indices*

Let $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega, \mathcal{F}, \tilde{P})$ be the probability space of bounded random variables X . $\alpha(X)$ is an index of acceptability, that is, a map $\alpha : \mathcal{L}^\infty \rightarrow [0, \infty]$ and satisfies the conditions 1 – 4 if and only if there exists a family of subset $\mathcal{D}_\gamma : \gamma > 0$ of \tilde{P} such that:

$$\alpha(X) = \sup\{\gamma \in \mathbb{R}^+ : \inf E^Q[X] \geq 0\}, \quad (2.4.1)$$

and $\mathcal{D}_\gamma : \gamma > 0$ is an increasing family of sets of probability measures, that is, $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ for $\gamma \leq \gamma'$.

The intuition here is that the set \mathcal{D}_γ contains a range of probability measures representing the different risk-preferences of market participants. The more market participants have a positive expectation of zero cost cashflow X , the larger the size of the set of probability measures in D_γ supporting X , and hence, the higher the level of acceptability of the cashflow X . Thus the acceptability level of a cashflow represents the largest possible size of a set of probability measures that all have a positive expectation of X . In other words, $\alpha(X) = \gamma$ is the highest value that makes the expectation of X positive under all probability measures in D_γ . The acceptability index $\alpha(X)$ is linked to $\rho(X)$ by the following relationship:

$$\alpha(X) = \sup\{\gamma \in \mathbb{R}^+ : \rho_\gamma(X) \leq 0\}. \quad (2.4.2)$$

Thus $\alpha(X)$ is the largest risk level that the cashflow X is acceptable at a risk level γ . The proof and parallels between acceptability indices and coherent risk measures are found in Cherny & Madan (2009).

2.5 Distortion Functions

We want to determine the relevant set of probability measures that value X positively. From the previous section, we know that D_γ is an increasing set, and its size is determined by γ . Furthermore, the only information needed to determine the acceptability level of X is its cumulative distribution function F_X . The distortion functions along with the importance of ideas stated above will be handy in representing the set of supporting probability measures.

2.5.1 Definition. Distortion Function

Let $g : [0, 1] \rightarrow [0, 1]$ be an increasing function with $g(0) = 0$ and $g(1) = 1$. For a

random variable X with cumulative distribution function F_X , the transform:

$$F^* = g(F(x)), \quad (2.5.1)$$

defines a distorted probability measure where “ g ” is called the distortion function.

When applied to F_X , it distorts F_X at a rate specified by the parameter γ . Since the distortion function is concave, lower outcomes of the random variable have higher weighting and higher outcomes have lower weighting. The distorted expectation is defined as:

$$E^{Q_\gamma}(X) = \int_{-\infty}^{\infty} x \, d^\gamma(F_X(x)). \quad (2.5.2)$$

The distortion parameter γ is seen as a measure of market price risk. We can write [2.5.2](#) as the following operational acceptability level.

$$\alpha(X) = \sup\{\gamma \geq 0 : E^{Q_\gamma}[X] \geq 0\} = \sup\left\{\gamma \geq 0 : \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_X(x)) \geq 0\right\}. \quad (2.5.3)$$

From this relationship we can say that X is in the set of cashflows acceptable at a level γ if and only if $E^{Q_\gamma}(X) \geq 0$. The level of acceptability of a cash flow can be considered as the maximum level of distortion that the cashflow can withstand such that its distorted expectation remains positive. It can be seen as the maximum level of stress that a random cashflow can withstand while remaining attractive to a range of market participants.

Cherny & Madan (2010*b*) and Cherny & Madan (2010*a*) then use a fixed acceptability level γ for a fixed acceptability index α . When the market sells a cashflow X it charges a minimal price a , perpetuated by competition. The residual cashflow $a - X$ must be α -acceptable at level γ , or $a_\gamma(X) - X \in \mathcal{D}_\gamma$. Using 2.5.2, the ask price is derived as follows:

$$\alpha(a - X) \geq \gamma \iff \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_{a-X}(x)) \geq 0 \quad (2.5.4)$$

$$\iff a + \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_{-X}(x)) \geq 0, \quad (2.5.5)$$

so that the minimum value of a leads to the ask price:

$$a_\gamma(X) = - \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_{-X}(x)). \quad (2.5.6)$$

When the market buys X for a price b , $X - b$ must be acceptable at a level γ or $X - b \in \mathcal{D}_\gamma$. Similarly, the bid price is derived as follows:

$$\alpha(X - b) \geq \gamma \iff \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_{X-b}(x)) \geq 0 \quad (2.5.7)$$

$$\iff -b + \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_{-X}(x)) \geq 0, \quad (2.5.8)$$

so that the maximum of b leads to the bid price:

$$b_\gamma(X) = \int_{-\infty}^{\infty} x \, d\Psi^\gamma(F_X(x)). \quad (2.5.9)$$

The concavity of the distortion ensures that the ask price is greater than the bid price.

2.6 New Acceptability Index Measures

In this section we present the new acceptability index measures, motivated by axioms and distortions functions introduced earlier in the chapter. Extensive work on the performance measures was carried out by Cherny & Madan (2009). These acceptability indices are developed from the weighted VAR (WVAR) which has the form:

$$\text{WVAR}_\gamma(X) = - \int_{\mathbf{R}} x \, d \Psi_\gamma(F_X(x)), \quad (2.6.1)$$

where $F_X(x)$ is the cumulative distribution function of the random variable X . $\{\Psi_\gamma : \gamma \geq 0\}$ is a set of increasing concave continuous functions with mapping $\Psi : [0, 1] \rightarrow [0, 1]$, where $\Psi(0) = 0$ and $\Psi(1) = 1$. In addition, for a fixed value y , $\Psi_\gamma(y)$ increases in γ . Therefore $\Psi_\gamma(y)$ can be seen as a function that distorts the cumulative distribution function $y = F_X(x)$ by adding more weight to losses in the area where $F_X(x)$ is close to x . Using the Representation Theorem, the WVAR acceptability index is defined as:

$$\alpha(X) = \sup\{\gamma \in \mathbb{R}^+ : \int_{\mathbf{R}} x. d \Psi_{\gamma}(F_X(x)) \geq 0\}, \quad (2.6.2)$$

where $\alpha(X)$ is the biggest value of γ such that the distorted expectation is still positive. The expectation of X is taken under a new probability measure $Q_{\gamma} \in \mathcal{D}_{\gamma}$ by a measure change $\frac{dQ_{\gamma}}{dP} = \Psi'_{\gamma}(F_X(x))$ where P is the original probability measure of X .

The new acceptability indices which cropped from the WVAR introduced in Cherny & Madan (2009) are:

1. MINVAR Acceptability Index - AIMIN(X)

AIMIN is the largest number x such that the expectation of the minimum of $x + 1$ draws from cashflow distribution is still positive. Let $Y \stackrel{law}{=} \{\min X_1, \dots, X_{x+1}\}$, where X_1, \dots, X_{x+1} are independent draws from X . The concave distortion function is given by:

$$\Psi_x(y) = 1 - (1 - y)^{x+1}, \quad x \in \mathbb{R}^+, y \in [0, 1]. \quad (2.6.3)$$

Figure 2.1 shows several plots of the MINVAR.

For a continuous distribution X , the extreme measure density is given by:

$$\frac{dQ_x^*(X)}{dP} = (x + 1)(1 - F_X(x))^x, \quad x \in \mathbb{R}^+. \quad (2.6.4)$$

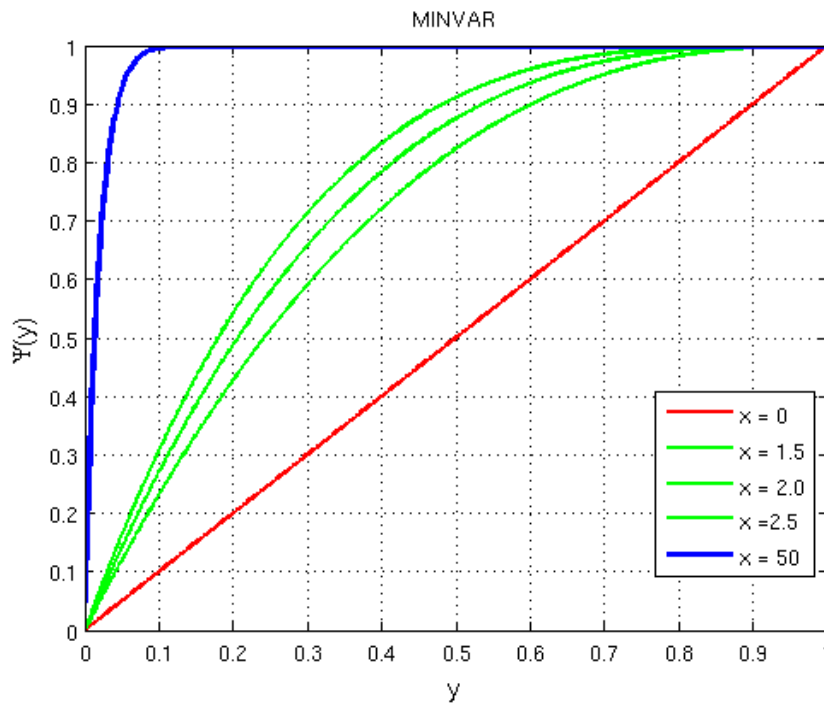


Figure 2.1: Plots of the MINVAR

The above derivative shows that AIMIN adds more weight to large losses (where $F_X(x)$ is close to zero) and reduces more weight to large gains (where $F_X(x)$ is close to one). However, AIMIN density tends to a finite value $x + 1$ at negative infinity. The attained values of x are not very large and so the maximal weight on large losses is small.

2. MAXVAR Acceptability Index - AIMAX(X)

AIMAX constructs a distribution from which one draws numerous times and takes the maximum to get the cashflow distribution being evaluated. The distortion for this index is known as proportional hazards transform in insurance. Let $\max\{Y_1, \dots, Y_{x+1}\} \stackrel{law}{=} X$, where Y_1, \dots, Y_{x+1} are independent draws of Y . The concave distortion function is given by:

$$\Psi_x(y) = y^{\frac{1}{x+1}}, \quad x \in \mathbb{R}^+, y \in [0, 1]. \quad (2.6.5)$$

Figure 2.2 shows several plots of the MAXVAR.

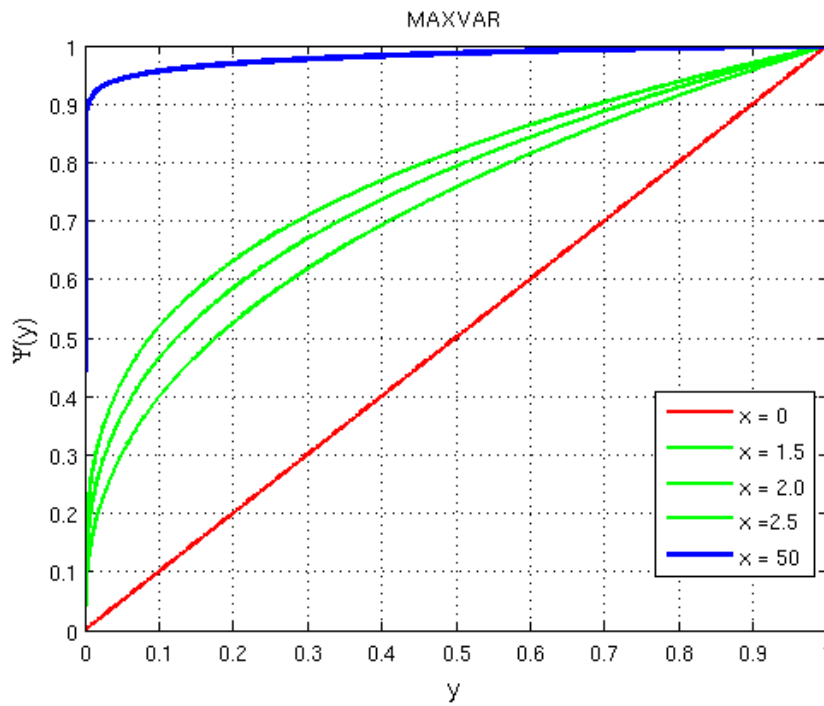


Figure 2.2: Plots of the MAXVAR

The extreme measure density for a continuous distribution X is given by:

$$\frac{dQ_x^*(X)}{dP} = \frac{1}{x+1} (F_X(x))^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}^+. \quad (2.6.6)$$

The density tends to infinity at negative infinity and tends to a positive value $1/(x+1)$

at positive infinity. This is an asymptotically linear weighting for large gains and is potentially realistic.

3. MAXMINVAR Acceptability Index - AIMAXMIN(X)

AIMAXMIN is constructed by first using the MINVAR and then followed by the MAXVAR to create worst case scenarios. Let $\max\{Y_1, \dots, Y_{x+1}\} \stackrel{law}{=} \min\{X_1, \dots, X_{x+1}\}$ where X_1, \dots, X_{x+1} are independent draws of X and Y_1, \dots, Y_{x+1} are independent draws of Y . Combining the MINVAR and MAXVAR, we have the distortion function:

$$\Psi_x(y) = (1 - (1 - y)^{x+1})^{\frac{1}{x+1}}, \quad x \in \mathbb{R}^+ \quad y \in [0, 1]. \quad (2.6.7)$$

Figure 2.3 shows several plots of the MAXVAR. The density tends to infinity at negative infinity and to zero at positive infinity.

$$\frac{dQ_x^*(X)}{dP} = (1 - F_X(x))^x (1 - (1 - F_X(x))^{x+1})^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}^+ \quad y \in [0, 1]. \quad (2.6.8)$$

4. MINMAXVAR Acceptability Index - AIMINMAX(X)

AIMAXMIN is constructed by first using the MAXVAR and then followed by the MINVAR to create worst case scenarios. Let

$$\begin{aligned} Y &\stackrel{law}{=} \min Z_1, \dots, Z_{x+1}, \\ \max Z_1, \dots, Z_{x+1} &\stackrel{law}{=} X, \end{aligned}$$

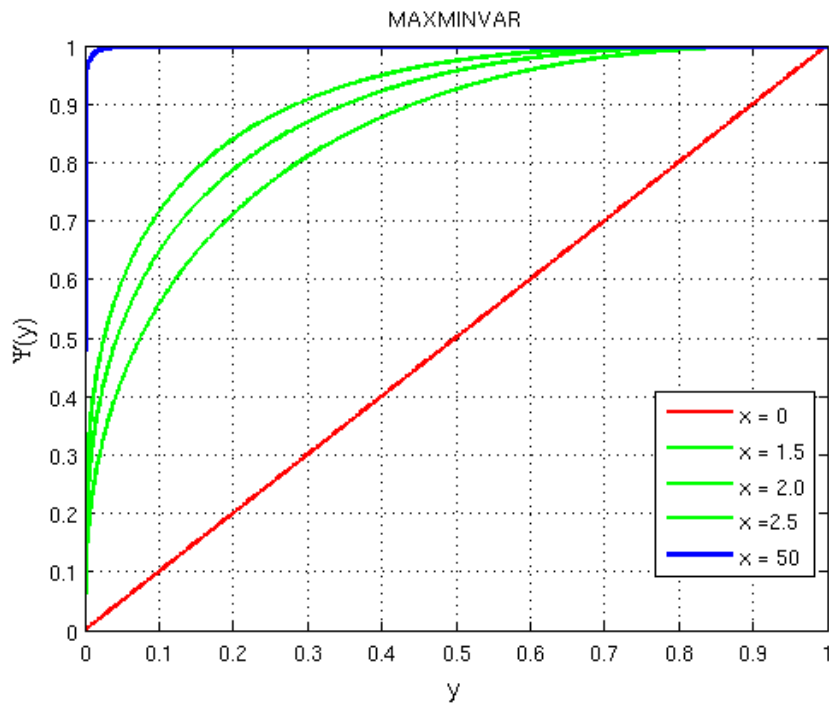


Figure 2.3: Plots of the MAXMINVAR

where Z_1, \dots, Z_{x+1} are independent draws of Z . Combining the MINVAR and MAXVAR, we have the distortion function:

$$\Psi_x(y) = 1 - (1 - y^{\frac{1}{x+1}})^{x+1}, \quad x \in \mathbb{R}^+ \quad y \in [0, 1]. \quad (2.6.9)$$

Figure 2.4 shows several plots of the MAXVAR.

For a continuous distribution of X , the extreme measure density is given by:

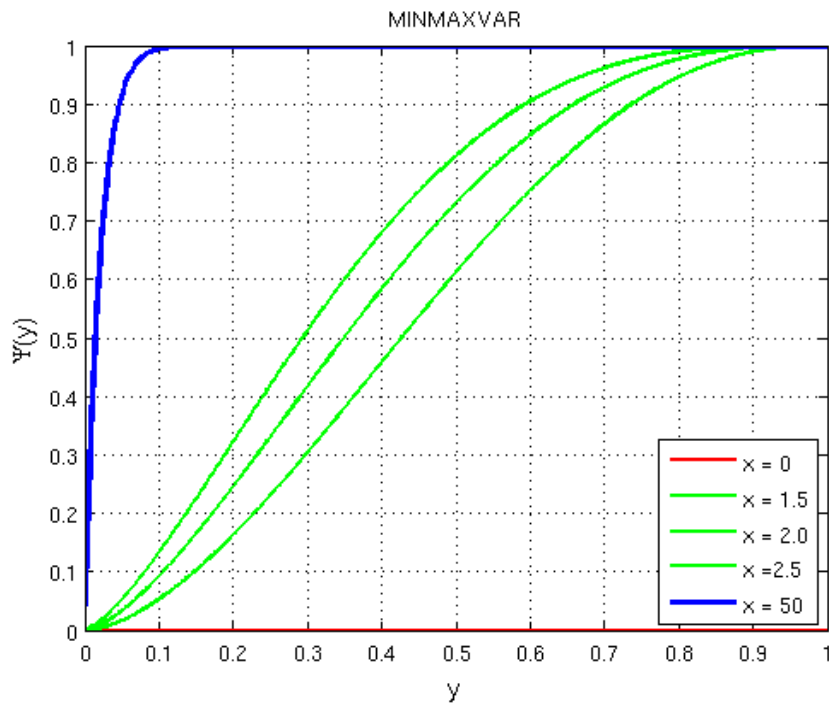


Figure 2.4: Plots of the MINMAXVAR

$$\frac{dQ_x^*(X)}{dP} = \left(1 - F_X(x)^{\frac{1}{x+1}}\right)^x F_X(x)^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}^+. \quad (2.6.10)$$

The density tends to infinity at negative infinity and to zero at positive infinity.

The MINMAXVAR can be generalized to a parameter family of cones termed MINMAXVAR2 introduced by Madan & Schoutens (2011). The concave distortion function is given by:

$$\Psi^{\lambda,\gamma}(u) = (1 - u^{\frac{1}{1+\lambda}})^{\gamma}, \quad \lambda, \gamma \geq 0, \quad u \in [0, 1]. \quad (2.6.11)$$

λ is referred to as a measure of risk aversion while γ is a measure of the absence of gain enticement. The parameter λ controls the rate at which $\Psi^{\lambda,\gamma}(u)$ approaches infinity as u tends to zero while γ controls the rate at which the density approaches zero as u tends to infinity.

3. A Primer of Mathematical Tools for Financial Applications

In this chapter mathematical tools for option pricing are discussed. This chapter can be seen as a prerequisite, as it bridges the gap from mathematical tools currently being applied up to the mathematical tools relevant to conic finance. Mainly, stochastic processes, mathematics of finance in continuous time, relevant models for continuous time modeling and simulation of stochastic processes are presented in this chapter. For more details on concepts in this chapter, the reader is referred to Björk (2004), Kopp (2005) or Schoutens (2003).

3.1 Stochastic Processes

3.1.1 Definition. Stochastic Process

A stochastic process $(X_t)_{t \in [0, T]}$ is a family of random variables indexed by time, defined on a filtered probability space (Ω, \mathcal{Y}, P) .

The time parameter t maybe either discrete or continuous. The trajectory $X(\omega) : t \rightarrow X_t(\omega)$ defines a sample path of the process for each realization, ω , of the random process.

3.1.2 Definition. C adl ag function

A function $f : [0, T] \rightarrow \mathbb{R}$ is said to be c adl ag if it is right continuous with left limits. If the process is c adl ag, one should be able to “predict” the value at t - “see it coming” - knowing the values before t .

3.1.3 Definition. Adapted Process

A stochastic process $(X_t)_{t \in [0, T]}$ is said to be \mathcal{F}_t -adapted if for each $t \in [0, T]$, the value of X_t is revealed at time t .

3.2 Classes of Processes

3.2.1 Markov Process

3.2.2 Definition. Markov Process

Let (Ω, \mathcal{F}, P) be a probability space, let T be a fixed positive number, and let $(\mathcal{F}_t)_{t \in [0, T]}$ be a filtration. Consider an adapted stochastic process $(X_t)_{t \in [0, T]}$. If for a well-behaved function f ,

$$E[f(X_t) | \mathcal{F}_s] = E[f(X_t) | X_s], \quad (3.2.1)$$

the process $(X_t)_{t \in [0, T]}$ is a Markov process.

In a Markov process, the present value of a variable is the one that is only relevant for predicting the future. The past history is said to be integrated in the present value.

3.2.3 Martingales

3.2.4 Definition. Martingale

A càdlàg stochastic process $X = (X_t)_{t \in [0, T]}$ is a martingale relative to (P, \mathcal{F}) if:

- (i) X is \mathcal{F}_t -adapted,

(ii) $E[|X_t|] < \infty$ for any $t \in [0, T]$, and

(iii) For all $s < t$,

$$E[X_t | \mathcal{F}_s] = X_s. \quad (3.2.2)$$

X is a submartingale if

$$E[X_t | \mathcal{F}_s] \geq X_s \quad \text{for all } s < t. \quad (3.2.3)$$

X is a supermartingale if

$$E[X_t | \mathcal{F}_s] \leq X_s \quad \text{for all } s < t. \quad (3.2.4)$$

A martingale is a process such that the best prediction of a future value is its present value. In other words, a martingale represents a process with zero drift. A martingale is used to model a fair game and is “constant on average”, a submartingale models a favourable game and is “increasing on average”, and a supermartingale models an unfavourable game and is “decreasing on average”.

3.3 Brownian Motion

The Brownian motion is one of the simplest stochastic processes and is a dynamic counterpart of the Normal distribution. It was first introduced by Robert Brown to

describe the movement of particles contained in the pollen grains of plants. Since then, it has been widely used in many domains of physics such as diffusion of fluid particles, fractal theory and statistical physics, among others. The Brownian motion was introduced into finance by Louis Bachelier in 1900 but was first proved mathematically by Nobert Weiner in 1923. Hence in honor of this, the Brownian motion is also known as the Weiner process.

3.3.1 Definition. Brownian Motion

A stochastic process $X = (X_t)_{t \geq 0}$ is a standard (one-dimensional) Brownian motion, W , on some probability space (Ω, \mathcal{F}, P) if:

- (i) $X(0) = 0$ almost surely,
- (ii) X has independent increments, that is, $X(t + u) - X(t)$ is independent of $\{X(s), s \leq t\}$, for $u \geq 0$,
- (iii) X has stationary increments, that is, the distribution of $X(t+u) - X(u)$ depends only on u ,
- (iv) X has Gaussian increments, that is $X(t + u) - X(t) \sim N(0, u)$, and
- (v) X has continuous sample paths $t \rightarrow X(t, \omega)$ for all $\omega \in \Omega$. This means that the graph of $X(t, \omega)$ as a function of t does not have any breaks in it.

3.4 Itô Calculus - Stochastic Calculus

Stochastic calculus was introduced by K. Itô in 1944, hence the name Itô calculus. The Itô calculus is an answer to calculus for Brownian motion and other diffusions, which have sample paths that are nowhere differentiable.

3.4.1 Itô's Lemma

Suppose that b is adapted and locally integrable, and σ is adapted and measurable so that $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral. Then

$$X(t) = x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s), \quad (3.4.1)$$

is an Itô (stochastic) process. The above equation has a stochastic differential representation of the form

$$dX_t = b(t)dt + \sigma(t)dW_t, \quad X(0) = x_0. \quad (3.4.2)$$

Now, let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $C^{1,2}$ continuously differentiable function once in its first argument (usually time) and twice in its second argument (usually space). The following theorem summarizes the Itô's Lemma.

3.4.2 Theorem. Itô's Lemma

If a stochastic process X_t has a stochastic differential of the form $dX_t = b(t)dt + \sigma(t)dW_t$, then $f = f(t, X_t)$ has a stochastic differential:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2. \quad (3.4.3)$$

3.4.3 Geometric Brownian Motion

The geometric Brownian Motion is used to model the evolution of a stock price $S(t)$, which is represented by the following differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S(0) > 0 \quad (3.4.4)$$

where μ is the drift of the stock, σ is the volatility of the stock and W_t is a Brownian process. The differential equation has a unique solution:

$$S(t) = S(0) \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right] \quad (3.4.5)$$

The proof is derived using the Itô's lemma as follows.

Proof

Let $(W_t)_{t \geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ be an associated filtration. Also, let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the process

$$X_t = \int_0^t \sigma(s) dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2} \sigma^2(s) \right) ds. \quad (3.4.6)$$

Then

$$dX_t = \sigma(t)dW_t + \left(\alpha(t) - \frac{1}{2}\sigma^2(t) \right),$$

and

$$(dX_t)^2 = \sigma^2(t)(dW_t)^2 = \sigma^2(t)dt,$$

since $(dW_t)^2 = dt$.

Now, consider an asset price process given by:

$$S_t = S(0) \exp(X_t) = S(0) \exp \left\{ \int_0^t \sigma(s)dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \right\}, \quad (3.4.7)$$

where $S(0) > 0$. Let $S_t = f(X_t)$, where $f(x) = S(0) \exp(x)$ so that $f' = S(0) \exp(x)$ and $f'' = S(0) \exp(x)$. According to the Itô formula,

$$\begin{aligned} dS_t &= df(X_t) = f' dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= S(0) \exp(x)dX_t + \frac{1}{2}S(0) \exp(x)(dX_t)^2 \\ &= S_t dX_t + \frac{1}{2}S_t(dX_t)^2 \\ &\text{substituting } dX_t \text{ and } (dX_t)^2 \\ &= \alpha(t)S_t dt + \sigma(t)S_t dW_t. \end{aligned} \quad (3.4.8)$$

The asset price S_t has instantaneous mean rate of return $\alpha(t)$ and volatility $\sigma(t)$.

If α and σ are constant, we have the regular geometric Brownian Motion model $dS_t = S_t(\alpha dt + \sigma dW_t)$, and S_t is log-normally distributed as follows

$$S_t = S(0) \exp \left\{ \left(\alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \quad (3.4.9)$$

3.5 Continuous Time Mathematics of Finance

3.5.1 Trading Strategy

Suppose a market has d assets whose prices are described by a stochastic process $S_t = (S_t^1, \dots, S_t^d)$ and there is a portfolio $\phi = (\phi^1, \dots, \phi^d)$ composed of certain amounts of each asset. The value of the portfolio at different dates is given by

$$V_t(\phi) = \sum_{i=1}^d \phi^i S_t^i = \phi \cdot S_t. \quad (3.5.1)$$

A trading strategy consists of creating a dynamic portfolio at different dates $T_0 = 0 < T_1 < T_2, \dots, T_n < T$ through buying and selling of the assets. The portfolio ϕ_t held at time t may be expressed as:

$$\phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^{n-1} \phi_i \mathbf{1}_{[T_i, T_{i+1}]}(t). \quad (3.5.2)$$

The capital gain of the portfolio up to time t is then defined as:

$$\begin{aligned}
G_t(\phi) &= \phi_0 S_0 + \sum_{i=0}^{j-1} \phi_i (S_{T_{i+1}} - S_{T_i}) + \phi_j (S_t - S_{T_j}) \text{ for } T_j < t \leq T_{j+1} \\
G_t(\phi) &= \int_0^t \phi_u \, dS_u.
\end{aligned} \tag{3.5.3}$$

The difference between the value of the portfolio and the capital gain gives the cost of the strategy up to time t

$$\begin{aligned}
C_t(\phi) &= V_t(\phi) - G_t(\phi) \\
&= \phi S_t - \int_0^t \phi_u \, dS_u.
\end{aligned} \tag{3.5.4}$$

A strategy ϕ is said to be self-financing if the cost is equal to zero, that is, the value of the portfolio is equal to its initial value plus the capital gain accumulated between 0 and t .

3.5.2 Risk-Neutral Pricing

There are two important concepts in mathematics of finance which are namely no-arbitrage theory and risk-neutral pricing. No-arbitrage theory imposes constraints on the way instruments are priced in the market. The risk-neutral pricing theory tries to represent the price of an instrument in an arbitrage free market as its discounted expected payoff under an appropriate probability measure called the “risk-neutral” measure. The two concepts use the important notion of equivalent martingale measure.

Pricing Rule

Here, we try to understand the equivalent martingale measure and its relation to arbitrage pricing and market completeness.

Now, consider a market between times 0 and T and its possible evolutions are described by a scenario space (Ω, \mathcal{F}) . \mathcal{F} contains all information about behaviour of prices between times 0 and T . Let $S_t(\omega)$ denote the value of asset i at time t in the market scenario ω and S_t^0 is a numéraire.

3.5.3 Definition. Numéraire

A numéraire is a price process S_t^0 which is almost surely strictly positive for each $t \in [0, T]$.

The numéraire is used for discounting purposes. For any portfolio with a value V_t , the discounted value is defined as

$$\hat{V}_t = \frac{V_t}{S_t^0}, \quad (3.5.5)$$

and $B(t, T) = S_t^0/S_T^0$ is called the discount factor. Suppose the numéraire $S_t^0 = e^{rt}$, then $S_T^0 = e^{rT}$ and the discount factor $B(t, T) = e^{-r(T-t)}$.

3.5.4 Definition. Contingent Claim

A contingent claim is a random variable H representing a payoff at time T . If the state of the world turns out to be ω , then the writer of the contingent claim will pay the holder an amount $H(\omega)$.

The question now is how we can attribute a value to each contingent claim H ?

3.5.5 Definition. Pricing Rule

A pricing rule is a procedure which attributes to each contingent claim a value $\Pi_t(H)$ at each point of time, using the information given at time t .

So a pricing rule assists in attributing a value to each contingent claim. For any random payoff:

$$\Pi_0(H) = e^{-rT} E^Q[H], \quad (3.5.6)$$

where Q is the risk-neutral measure. It is crucial to understand that the risk-neutral measure Q has nothing to do with the objective probability of occurrences of scenarios. It is just convenient representation of a price rule Π . H is obtained by looking at prices of contingent claims at $t = 0$. In addition, the pricing rule Π must be time consistent, that is, the value at time $t = 0$ of the payoff H at T is the same as the value at $t = 0$ of the payoff $\Pi_t(H)$ at t . In other words, $\Pi_t(H)$ is given by the discounted conditional expectation with respect to Q :

$$\Pi_t(H) = e^{-rT} E^Q[H|\mathcal{F}_t]. \quad (3.5.7)$$

Next, the problem is which measure Q should be used? To understand this, we examine the no-arbitrage theory and see the restrictions that can be imposed on Q .

No-Arbitrage Pricing

Consider a market represented by a probability space, (Ω, \mathcal{F}, P) , where in addition to market scenarios and information flow \mathcal{F} , the probability of occurrences of scenarios are known and are represented by the measure Q . Absence of arbitrage opportunities is a fundamental requirement for a pricing rule.

Let S^i be an asset traded at price S_t^i . Suppose the asset is held until time T , generating a terminal payoff S_T^i , or suppose the asset is sold for S_t^i , the resulting sum is invested at an interest rate r and generating a terminal wealth of $e^{r(T-t)}S_t^i$. These buy-and-hold strategies are self financing and have the same terminal payoff. According to the law of one price, they should have the same value at time t ,

$$E^Q[S_T^i|\mathcal{F}_t] = E^Q[e^{r(T-t)}S_t^i|\mathcal{F}_t] = e^{r(T-t)}S_t^i. \quad (3.5.8)$$

Dividing by $S_T^0 = e^{rT}$, we get:

$$E^Q \left[\frac{S_T^i}{S_T^0} | \mathcal{F}_t \right] = \frac{S_t^i}{S_t^0}, \quad i.e. E^Q[\hat{S}_T^i | \mathcal{F}_t] = \hat{S}_t^i. \quad (3.5.9)$$

Absence of arbitrage therefore implies that discounted values, $\hat{S}_t^i = e^{-rt}S_t^i$, of all traded assets are martingales with respect to the probability measure Q . A probability measure that verifies these two equations is called an equivalent martingale measure. Thus a no-arbitrage pricing rule is given by an equivalent martingale measure. There is a one-to-one correspondence between no-arbitrage pricing rule and equivalent martingale measure. Formally, an equivalent martingale measure is defined as follows:

3.5.6 Definition. Equivalent Martingale Measure

A probability measure Q is an equivalent martingale measure if:

- (i) Q is equivalent to P , that is, they define the same set of possible/impossible events (null set),

$$P \sim Q : \forall A \in \mathcal{F} \quad Q(A) = 0 \iff P(A) = 0. \quad (3.5.10)$$

- (ii) The discounted stock price process, $\hat{S}_t^i = e^{-rt} S_t^i$ for $t \geq 0$, is a martingale under measure Q if:

$$E^Q[\hat{S}_T^i | \mathcal{F}_t] = \hat{S}_t^i. \quad (3.5.11)$$

3.5.7 Definition. Risk-Neutral Pricing

Any no-arbitrage pricing rule, Π , represented as

$$\Pi_t(H) = e^{-r(T-t)} E^Q[H | \mathcal{F}_t], \quad (3.5.12)$$

where Q is an equivalent martingale measure on a market described by a measure P on market scenarios is a risk-neutral pricing rule.

The risk-neutral pricing theory is reiterated in the First Fundamental Theorem of Asset Pricing.

3.5.8 Theorem. *First Fundamental Theorem of Asset Pricing*

The market model defined by a probability space, (Ω, \mathcal{F}, P) , and asset prices, $(S_t)_{t \in [0, T]}$, is arbitrage free if and only if there exists a probability measure, $Q \sim P$, such that the discounted assets $(\hat{S}_t)_{t \in [0, T]}$ are martingales with respect to Q .

The economic interpretation of the law is that today's asset prices are obtained as the expected value of tomorrow's asset prices, discounted with the risk free rate.

3.5.9 Complete and Incomplete Markets

A market is said to be complete if any contingent claim admits a replicating portfolio. A self financing strategy (ϕ_t^0, ϕ_t) is said to be a replication (perfect hedge) for a contingent claim if

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0. \quad (3.5.13)$$

The value of any contingent claim is given by the initial capital needed to set up a replication strategy for H . All equivalent martingale measures give the same pricing rules. The Second Fundamental Theorem of Asset Pricing connects the market completeness to the martingale measure.

3.5.10 Theorem. *Second Fundamental Theorem of Asset Pricing*

A market defined by the assets $(S_t^0, S_t^1, \dots, S_t^d)_{t \in [0, T]}$, described as a stochastic process on the probability space, (Ω, \mathcal{F}, P) , is complete if and only if there exists a unique martingale measure Q equivalent to P .

In a complete market, we have seen that the price of a contingent claim equals the cost of the self-financing replication portfolio. However, in an incomplete market not every portfolio is attainable. In many instances, attempting to replicate a portfolio of assets is impracticable because trading cannot be done continuously and is also costly. In addition, this might be challenging as the portfolio may contain assets that cannot affect the price of the contingent claim.

An equally economic interpretation which distinguishes between complete and incomplete markets is interpreting this in terms of an Arrow Debreu security. The payoff of the security is associated with a particular state of the world. If this state oc-

curs, the holder of the Arrow security will be paid R_1 , and nothing otherwise. The risk-neutral probability of a state is the price of the Arrow security associated with that state, assuming the risk-free rate is zero. In a complete market, all Arrow securities can be constructed. In an incomplete market, not all Arrow securities can be constructed. The implication of this in an incomplete market is that there is a range of arbitrage-free prices for certain securities. In other words, there is a range of risk-neutral probabilities that exist with respect to the states for which no Arrow securities exist yet.

Therefore, there exist different risk-neutral measures or equivalent martingale measures and hence different arbitrage-free prices. The arbitrage-free approach is only vital in providing the bounds on the value of a contingent claim. In technical terms, there are multiple elements in the set of admissible martingale measures that correctly price the assets in the portfolio. The assets in the portfolio may not hedge the contingent claim perfectly. The hedges on the portfolio do not eliminate all risk, and instead we require that the remaining risk be an acceptable opportunity.

The pricing in the incomplete market with remaining risk (residual risk) is determined by conic finance, which was explained in depth in the first two chapters of this work.

3.6 Continuous Time Models for Stock Returns and Option Prices

In this section, we introduce the models that are used for modeling stock returns as well as option prices.

3.6.1 Log-Normal Model

The log-normal process models continuously compounded log returns using the general Brownian motion so that

$$X(t) = \nu t + \sigma W(t), \quad (3.6.1)$$

where $W(t)$ is a standard Weiner process, ν is the instantaneous drift and σ is the instantaneous volatility of returns. The stochastic differential equation of the stock price is:

$$dS(t) = S(t) (\mu dt + \sigma dW(t)), \quad (3.6.2)$$

where μ is the growth rate of the stock and is related to ν as follows $\nu = \mu - 1/2\sigma^2$. The stochastic differential equation can be solved to give the following dynamics of the stock price:

$$S(t) = S(0) \exp \left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right\} \quad (3.6.3)$$

The characteristic function for the logarithm of the stock price is:

$$E[e^{iu \ln(S(t))}] = \exp \left\{ iu \left[\ln S(0) + \left(\mu - \frac{1}{2}\sigma^2 \right) t - \frac{1}{2}\sigma^2 u^2 t \right] \right\} \quad (3.6.4)$$

3.6.2 Variance Gamma Model

Madan, Carr & Chang (1998) define a Variance Gamma process, $X(t, \nu, \sigma, \theta)$, as a time changed Brownian motion as follows:

$$X(t, \nu, \sigma, \theta) = \theta G(t) + \sigma W(G(t)), \quad (3.6.5)$$

where $G(t)$ is a Gamma process with parameters a and b , that is, $G(t) \sim \text{Gamma}(at, b)$ where the gamma probability density function $\text{Gamma}(a, b)$ is given by

$$f_G(x, a, b) = \frac{b^a x^{a-1}}{\Gamma(a)} e^{-bx}, \quad x > 0. \quad (3.6.6)$$

θ and σ are respectively the instantaneous drift and volatility and $W(t)$ is a standard Brownian motion. The Variance Gamma process uses a gamma process to time change a Brownian motion. The density function of a Variance Gamma process is known in closed form and requires the computation of the modified Bessel function of the second kind which can be time consuming. Thus we resort to using the characteristic function, which is found by the conditioning on the jump $G(t)$ as in many Lévy processes and is given by:

$$\Phi_{X(t)}(u) = \left(1 - iu\nu\theta + \frac{1}{2}u^2\nu\sigma^2 \right)^{-t/\nu}. \quad (3.6.7)$$

The dynamics of the stock price are given by:

$$S(t) = S(0) \exp\{(\mu + \omega)t + X(t, \nu, \sigma, \theta)\}, \quad (3.6.8)$$

where μ is the instantaneous expected return of the stock evaluated at calendar time and ω is a compensator to ensure that

$$E^P[S(t)] = S(0) \exp(\mu t). \quad (3.6.9)$$

The characteristic function for the logarithm of stock price is

$$E[e^{iu \ln(S(t))}] = \exp\{iu [\ln S(0) + (\mu + \omega)t]\} \Phi_{X(t)}(u). \quad (3.6.10)$$

The compensator term can be found from the characteristic function and is given by $\omega = \frac{-1}{t} \ln(\Phi_{X(t)}(-i))$.

3.6.3 Variance Gamma Scalable Self Decomposable (VGSSD)

Evidence from options markets indicate that higher moments implied from options prices are constant or are slightly increasing over time, that is, skewness and kurtosis are constant over time. We seek a suitable model that captures the features of the higher moments. A suitable model is the Sato process model first introduced by Carr, Geman, Madan & Yor (2007). The Sato process was shown to be effective in synthesizing many options on numerous underliers at the same time. The idea behind the model was to construct stochastic processes that had inhomogeneous independent

increments from Lévy processes with homogeneous independent increments such that the higher moments are constant over the time horizon.

The starting point for the construction of the Sato model is the self decomposable law. Loosely speaking, the self decomposable law describes random variables that decompose into the sum of a scaled down version of themselves and an independent residual term. The scaling property means the distribution of increments of various time scales can be obtained from that of other time scale by rescaling the random variable at that time scale. Thus the distribution at larger time scales are derived from those at smaller time scales, which are easier to estimate as the data is sufficient. Sato (1991) proposed that the self decomposable law is associated with the unit time distribution of self-similar additive process whose increments are independent, but not necessarily stationary.

It is known that stock prices are moved by many pieces of information. If the pieces of information are considered as a sequence of independent random variables $(Z_i : i = 1, 2, \dots)$, then the price changes are consequences of the impacts from all Z_i . Now, let $S_n = \sum_{i=0} Z_i$ denote their sum. Suppose that there exist centering constants c_n and scaling constants b_n such that the distribution of $b_n S_n + c_n$ converges to the distribution of the random variable X , which belongs to a family law 'class L'. Then the random variable X is said to have the class L property. So, the price change over the time horizon is the outcome of many independent random variables which can be approximated as a random variable X that has the law of 'class L'. Sato (1999) define the self decomposable law as follows

3.6.4 Definition. Self Decomposable Law

A random variable X is self decomposable if for all $c \in (0, 1)$,

$$X \stackrel{law}{=} cX + X^c, \tag{3.6.11}$$

where X^c is a random variable independent of X .

The self decomposable random variable X can be decomposed into a partial of itself and another independent random variable. Sato (1999) also shows that one may associate with such a self decomposable law at unit time a process with independent but inhomogeneous increments by defining the marginal law of the process at time points t upon scaling the law at unit time. Therefore we have that:

$$X(t) = t^\gamma X, \quad t > 0. \quad (3.6.12)$$

Thus we can study the price changes easily using self decomposable laws, which are easier to handle than class L.

Self decomposable laws are an important sub-class of the class of infinitely divisible laws Carr, Geman, Madan & Yor (2005). The characteristic function of the self decomposable laws has the form (see (Sato 1999)):

$$E[e^{iux}] = \exp \left\{ iru - \frac{1}{2}\sigma^2 u^2 + \int_R \left(e^{iux} - 1 - iux 1_{|x| \leq 1} \frac{g(x)}{|x|} dx \right) \right\}, \quad (3.6.13)$$

where r, σ are constants, $\sigma^2 \geq 0$, $\int_R (|x|^2 \wedge 1) \frac{g(x)}{|x|} dx < \infty$, and $g(x)$ is an increasing function when $x < 0$ and decreasing function when $x > 0$. An infinitely divisible law is self decomposable if the corresponding Lévy density has the form $\frac{g(x)}{|x|}$, where $g(x)$ is increasing for negative x and decreasing for positive x .

The dynamics of the stock price is defined as:

$$S(t) = S(0) \exp \{r(t) + X(t) + \omega(t)\}, \quad (3.6.14)$$

where $\omega(t)$ is a compensator term chosen to ensure that:

$$\exp(-\omega(t)) = E_0[\exp(X(t))]. \quad (3.6.15)$$

The Sato process used in this work is the one constructed from the variance gamma process and is known as the Variance Gamma Scalable Self Decomposable (VGSSD) process. The variance gamma process is defined by time changing an arithmetic Brownian motion with drift θ and volatility σ by an independent gamma process with unit mean rate and variance rate ν . Let $G(t; \nu)$ be the gamma process, then the variance gamma process is written as:

$$X_{VG}(t, \sigma, \nu, \theta) = \theta G(t; \nu) + \sigma W(G(t; \nu)), \quad (3.6.16)$$

where $W(t)$ is an independent standard Brownian motion.

The gamma process is an increasing pure jump Lévy process with independent identically distributed increments over regular nonoverlapping intervals of length h that are gamma distributed with density $f_h(g)$ where:

$$f_h(g) = \frac{g^{\frac{h}{\nu}-1} e^{-\frac{g}{\nu}}}{\nu^{\frac{h}{\nu}} \Gamma\left(\frac{h}{\nu}\right)}, \quad g > 0. \quad (3.6.17)$$

The VGSSD is constructed from the variance gamma process by defining the scaled stochastic process $X(t)$ such that it is equal in law to $t^\gamma X_{VG}(1)$ where $X_{VG}(1)$ is a variance gamma random variable at unit time. It follows that the characteristic function of $X(t)$ is given by Carr et al. (2005):

$$\Phi_{X(t)}(u) = \Phi_{X_{VG}(1)} = \left(1 - iut^\gamma\nu\theta + \frac{1}{2}u^2t^{2\gamma}\nu\sigma \right) \quad (3.6.18)$$

Since the VGSSD is a scaled stochastic process, its higher moments remain constant over time.

3.7 Simulation Techniques and Monte Carlo Method

In this section, an overview on how to simulate some of the processes that are encountered in this work is detailed here. A general reference for the simulations is Schoutens (2003). The Monte Carlo method is used to simulate the processes. For the Monte Carlo method, one is referred to Glasserman (2003). Before an overview on how to simulate the processes, first lets introduce the Monte Carlo method as it is crucial to the simulation process.

3.7.1 Monte Carlo Method

The method was named after a city in Monaco that had fame because of its casinos and games of chances. The main idea behind the method is to make use of random samples of parameters or inputs to simulate behaviour of complex systems or processes that involve uncertainty. The Monte Carlo method is based on the analogy between probability and volume. The mathematics of measure theory formalizes the intuitive

notion of probability. The measure theory associates an event with a set of outcomes and defines the probability of the event to be its volume or measure relative to that of the universe of possible outcomes. In the Monte Carlo method, this identity from measure theory is used in a reverse way and the volume of a set is calculated by interpreting the volume as a probability. In other words, this means sampling randomly from a universe of possible outcomes and then take the fraction of random draws that fall in a given set as an estimate of the set's volume. The estimate converges to the real value as the number of draws increases due to the law of large numbers (Glasserman 2003).

In general, if there is an integral like:

$$I = \int_{\mathcal{A}} \phi(x) dx, \quad (3.7.1)$$

where $\mathcal{A} \in \mathbb{R}^n$, I is estimated by randomly sampling a sequence of points $x \in \mathcal{A}$, $i = 1, \dots, m$, and building the estimator

$$\hat{I} = \frac{vol(\mathcal{A})}{m} \sum_{i=1}^m \phi(x_i), \quad (3.7.2)$$

where $vol(\mathcal{A})$ denotes the volume of the region \mathcal{A} . The ratio $(1/m) \sum_{i=1}^m \phi(x_i)$ estimates the average value of the function, which must be multiplied by the volume of the integration region in order to get the integral. The Monte Carlo becomes highly attractive in evaluating integrals in high dimensions, where analytical solutions might be too complex or may not be available at all.

The Monte Carlo method is relevant to valuing of financial derivatives, which typically

involve simulating paths of stochastic processes used to describe the evolution of the underlying asset prices. Instead of simply drawing points at random from $[0, 1]$ or $[0, 1]^d$, samples are done from a space of paths. However, this is dependent on the problem at hand such that the dimension of the relevant space maybe very large or infinite. This can have a bearing on the speed of convergence of the Monte Carlo method.

In the crude form, Monte Carlo simulations are computationally inefficient. A large number of simulations are generally required so as to achieve a high degree of accuracy. However, the efficiency can be improved by either using variance reduction methods or quasi-Monte Carlo method (Corwin, Boyle & Tan 1996).

Next, we move to the simulating of the paths of the processes in this work.

3.7.2 Simulation of Standard Brownian Motion

To simulate a standard Brownian motion, discretize time by taking time steps of size Δt and then simulate the value of the Brownian motion at time points $\{n\Delta t, n = 0, 1, \dots\}$. The discretized equation is:

$$W_0 = 0, \quad W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t} v_n, \quad n \geq 1, \quad (3.7.3)$$

where $\{v_n, v = 1, 2, \dots\}$ is a series of standard Normal random numbers. Figure 3.1 shows a path drawn from the standard Brownian motion.

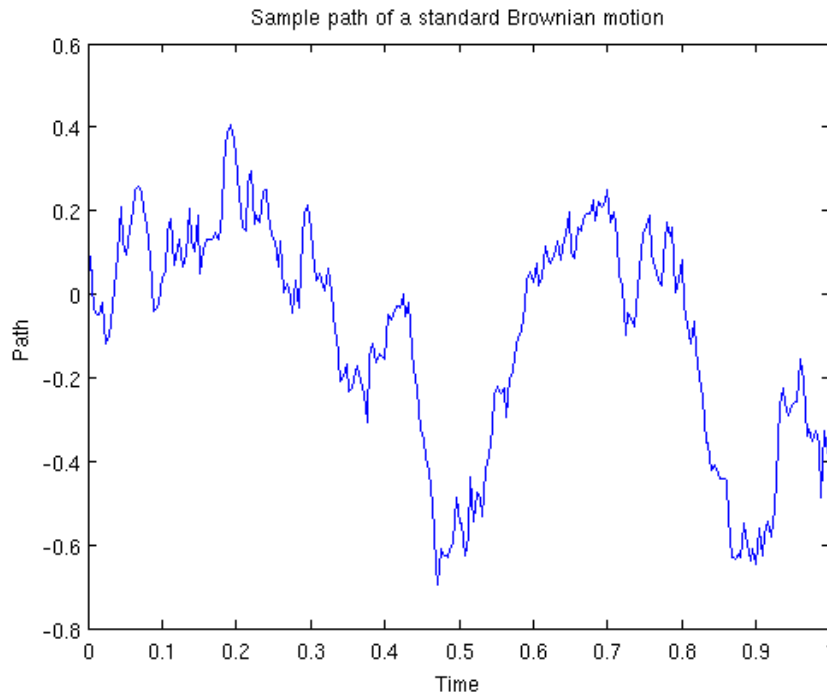


Figure 3.1: A sample path of the standard Brownian Motion

3.7.3 Simulation of the Gamma Process

To simulate paths of a Gamma process $X = \{X_t, t \geq 0\}$, where X_t follows a $\text{Gamma}(at, b)$ law at time points $\{n\Delta t, n = 0, 1, \dots\}$:

- (i) Generate independent $\text{Gamma}(a\Delta t, b)$ random numbers $\{g_n, n \geq 1\}$.
- (ii) Set $X_0 = 0$ and

$$X_{n\Delta t} = X_{(n-1)\Delta t} + g_n, \quad n \geq 1. \quad (3.7.4)$$

Figure 3.2 shows a path of the Gamma process.

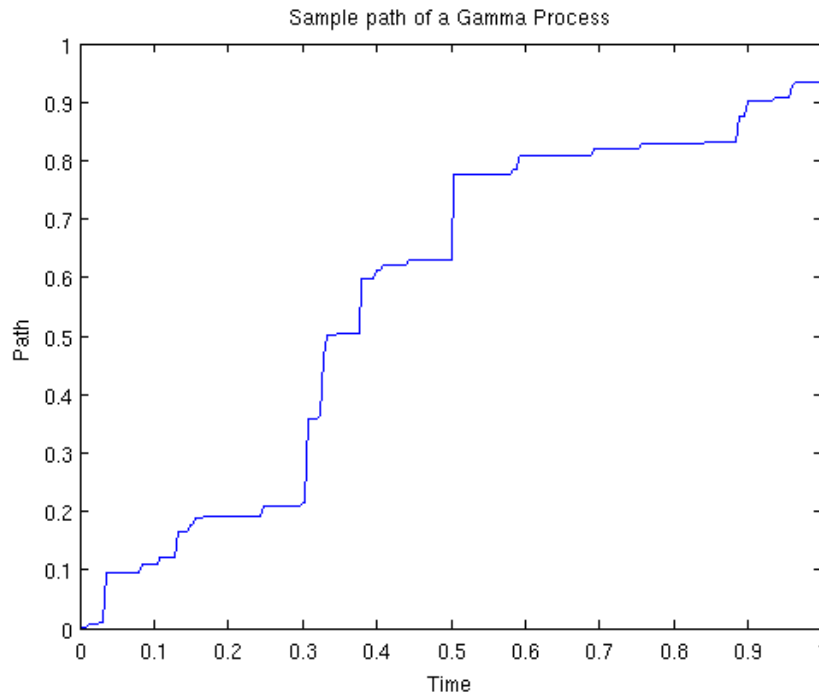


Figure 3.2: A sample path of the Gamma Process ($a = 10$, $b = 20$)

3.7.4 Simulation of a Variance Gamma Process

To simulate a Variance Gamma process as a time changed Brownian motion, the following procedure is implemented:

- (i) Simulate n Gamma variables γ_t with parameters (ah, b) , where h is the discretization step.
- (ii) Simulate n iid $N(0, 1)$ random variables.
- (iii) Simulate the process at

$$X_t^{VG} = \theta G_t + \sigma W_{G_t}, \quad (3.7.5)$$

where W is the Brownian motion and G_t is a Gamma process.

Figure 3.3 shows a path of the Variance Gamma process.

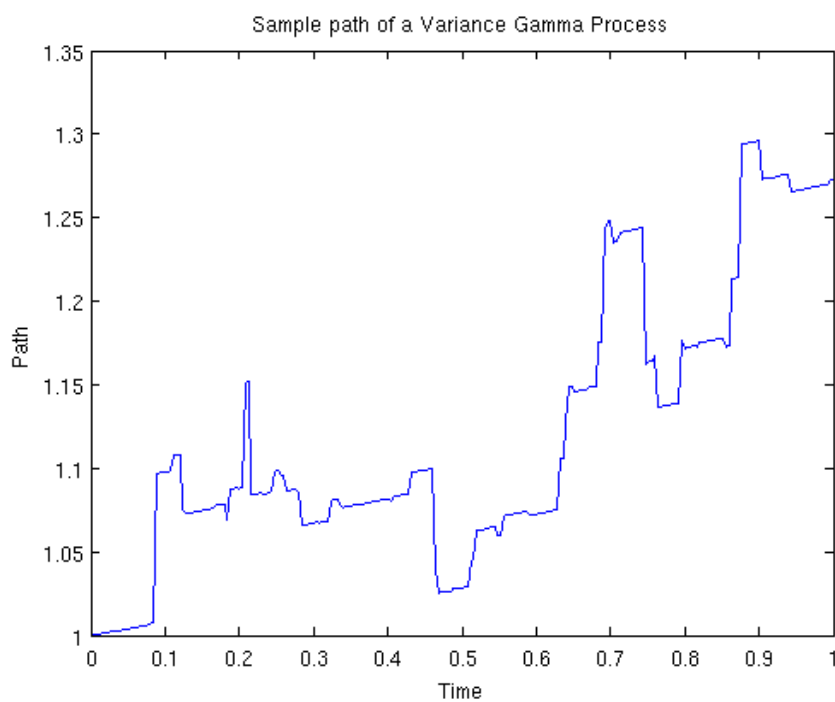


Figure 3.3: A sample path of the Variance Gamma Process ($\nu = 0.15, \sigma = 0.25, \theta = -0.10$)

3.7.5 Simulation of a VGSSD Process

The unit law comes from the variance gamma Lévy process

$$X_{VG}(t, \sigma, \nu, \theta) = \theta G(t; \nu) + \sigma W(G(t; \nu)), \quad (3.7.6)$$

where $W(t)$ is an independent standard Brownian motion and $G(t; \nu)$ is the gamma process with unit mean.

To generate the random number $X(t)$ at time t , simulate two independent random variables separately: a standard normal random variable $W(1)$ and a gamma random variable $G(t; \nu)$. $W(G(t; \nu))$ is a Gaussian random variable, so it can be written as

$$W(G(t; \nu)) = \sqrt{G(t; \nu)} W(1). \quad (3.7.7)$$

As $W(1)$ and $G(t; \nu)$ are independent, the product of these two simulated numbers gives $W(G(t; \nu))$. In summary to generate a VGSSD process

- (i) Generate a standard normal variable $z \sim N(0, 1)$.
- (ii) Generate a gamma random number $g \sim \text{Gamma}(t/\nu, \nu)$.
- (iii) $X(t)$ at time t is given by

$$X_t = \theta g + \sigma \sqrt{g} z. \quad (3.7.8)$$

The distribution at time points t is obtained by scaling the unit law by the scaling factor t^γ . Figure 3.4 shows a path of the VGSSD process at unit time.

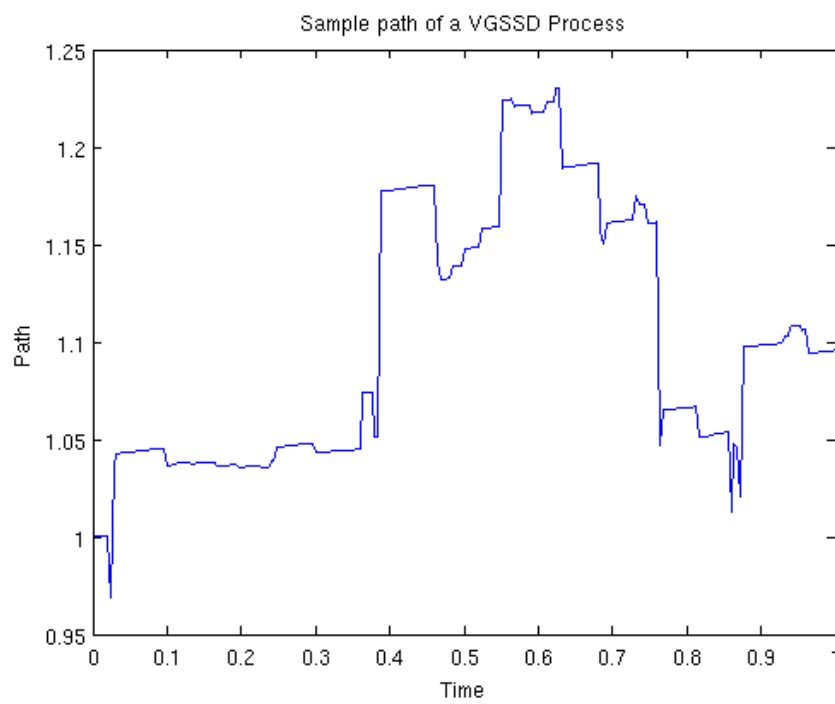


Figure 3.4: A sample path of the VGSSD Process ($t = 1, \nu = 0.20, \sigma = 0.20, \theta = -0.05$)

4. Numerical Implementation

In the previous chapter having reviewed the mathematical tools useful for financial applications, the focus in this chapter is shifted to the numerical methods that are relevant to the thesis. We start by explaining the procedure used to obtain the parameters for the models and the calibration to market prices. Then the method used for option pricing is explained in detail.

4.1 Parameter Estimation and the Method of Maximum Likelihood

The parameters in the models are initially estimated using the maximum likelihood estimation (MLE) method. Let x_1, x_2, \dots, x_n be n iid sample data points. The probability density function of the distribution is $f(x; \vec{\theta})$, where $\vec{\theta} = \{\theta_1, \theta_2, \dots, \theta_k\}$ is the parameter set. The likelihood function for the parameter set is defined as:

$$\mathcal{L}(\vec{\theta} | x) = \prod_{i=1}^n f(x_i; \vec{\theta}). \quad (4.1.1)$$

The above function can be written in logarithmic form as:

$$\log \mathcal{L}(\vec{\theta} | x) = \sum_{i=1}^n \log f(x_i; \vec{\theta}). \quad (4.1.2)$$

MLE method estimates $\vec{\theta}$ by finding values of a parameter set θ which minimizes $\mathcal{L}(\vec{\theta}|x)$. Maximization is often conducted on the logarithmic equation above.

Now, the closed-form of the probability density function might not be known or maybe too complicated to use in the MLE. The characteristic function is used instead. In this work, the characteristic function is used as the pdf might not be known in closed form. So the values of the pdf are obtained from the characteristic function via the Fourier transform method formula:

$$f(x, \vec{\theta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \Phi_X(u) du. \quad (4.1.3)$$

Since $f(x, \vec{\theta})$ is a real-valued function, we restrict 4.1.3 to the real part and equation 4.1.3 becomes:

$$f(x, \vec{\theta}) = \frac{1}{2\pi} \int_0^{\infty} e^{-iux} \Phi_X(u) du. \quad (4.1.4)$$

Numerically, $f(x, \vec{\theta})$ can be discretized as follows:

$$f(x_k, \vec{\theta}) = \frac{1}{\pi} \sum_{j=0}^{N-1} e^{-iu_j x_k} \Phi_X(u_j) \cdot \eta \quad (4.1.5)$$

The fast Fourier Transform (FFT) is an efficient algorithm to compute the discretized Fourier Transform (DFT) in equation 4.1.5. The FFT algorithm is explained in Section 4.4 below. Applying the FFT algorithm and writing $f(x, \vec{\theta})$ in a standard

discrete Fourier Transform becomes

$$f(x_k, \vec{\theta}) = \frac{1}{\pi} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} j k} \cdot Z(j), \quad (4.1.6)$$

where

$$Z(j) = \frac{1}{\pi} e^{i b u_j} \Phi_X(u_j) \cdot \eta$$

Carr, Geman, Madan & Yor (2002) suggested a method, which is used in this work, to approximate more efficiently the likelihood function using the fast Fourier Transform. The fast Fourier Transform was used to invert the characteristic as explained above and the density was evaluated at some prespecified points. For m observed data points, Carr et al. (2002) arranged the observed data x_i for $i = 1, \dots, m$ into their corresponding intervals $x_i \in [y_j, y_{j+1}]$ for $j = 1, \dots, N - 1$ and counted the number of observed data points that fell into each interval. The intervals for the bins are formed using the density evaluated at some prespecified points. The likelihood of observing this binned data is then maximized by appropriate choice of the parameter vector.

4.2 Calibration

After initial estimation of the parameters by MLE method, we then calibrate the parameters to market prices. Calibration is an optimization problem, where we seek the ‘best parameters’ that give a model that is consistent with option prices observed in the market. The calibration done in this work is along the lines suggested by Cont

& Tankov (2004). In other words, we search for a risk-neutral model which correctly prices the options for a set of call options with strikes K_i and maturities T_i :

$$C_i^{\text{market}} \approx C_i^{\text{model}} = e^{-rt} E^Q[S_{T_i} - K_i | S_t = s]. \quad (4.2.1)$$

Contrary to when model parameters are given and one wants to price options, we would like to obtain model parameters from observed prices. The calibration can be implemented using a nonlinear least squares technique:

$$\theta^Q = \arg \min_{\theta^Q} \sum_{i=1}^N |C_i^{\text{model}}(\theta^Q) - C_i^{\text{market}}|^2. \quad (4.2.2)$$

The parameters for the model are found by minimizing the sum of squared differences (errors) between market prices and model prices.

4.3 Option Pricing by Fourier Transform

In this section, we turn to the pricing of options using the Fourier transforms. The probability densities of some models are not known in closed forms but the characteristic functions always do, and this makes Fourier based pricing more suitable.

4.3.1 Definition. Fourier Transform

The Fourier transform of a square integrable function, $g(x)$, is given by:

$$\hat{g}(u) = \int_{-\infty}^{\infty} \exp(iux) g(x) dx. \quad (4.3.1)$$

4.3.2 Definition. Inverse Fourier Transform

The inverse Fourier transform of a square integrable function, $\hat{g}(u)$, is given by:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-iux) \hat{g}(u) du. \quad (4.3.2)$$

The Fourier transform of a function is simply change of the function, $g(x)$, from the real number domain to the frequency domain. In order to recover the function, $g(x)$, from the frequency domain we apply the inverse Fourier transform.

4.3.3 Definition. Characteristic Function

The characteristic function of a random variable X_T is given by

$$\begin{aligned} \Phi_T(u) &= E[e^{iuX_T}] \\ &= \int_{-\infty}^{\infty} \exp(iuX_T) q_T(X_T) dX, \end{aligned} \quad (4.3.3)$$

where q_T is the probability density function of X_T .

In this work, we adopt the modified Fourier transform method of pricing options by Carr & Madan (1998). Now consider pricing a European call option with maturity T written on an underlying with terminal price S_T . Let s_T denote the log of S_T , that is, $s_T = \log(S_T)$. Let k denote the log of the strike price K , and let $C_T(k)$ be the

desired value of a call option with strike $\exp(k)$ and maturity T . Let the risk-neutral density of the log price s_T be $q_T(s)$. The characteristic function of this density is

$$\Phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T ds. \quad (4.3.4)$$

The modified call option price $c_T(k)$ is defined as:

$$c_T(k) = e^{\alpha k} C_T(k) \quad \alpha > 0. \quad (4.3.5)$$

The Fourier transform and inverse Fourier transform of $c_T(k)$ are:

$$\mathbf{F}_{c_T}(v) = \psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk \quad (4.3.6)$$

$$c_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \mathbf{F}_{c_T}(v) dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v), \quad (4.3.7)$$

and

$$\begin{aligned} C_T(k) &= e^{-\alpha k} c_T(k) \\ &= e^{-\alpha k} \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_T(v). \end{aligned} \quad (4.3.8)$$

since

$$\begin{aligned}
\int_{-\infty}^{+\infty} \psi_T(v) &= \int_0^{+\infty} e^{-ivk} \psi_T(v) + \int_{-\infty}^0 e^{-ivk} \psi_T(v) \\
&= \int_0^{+\infty} e^{-ivk} \psi_T(v) + \int_0^{+\infty} e^{-ivk} \psi_T(-v). \tag{4.3.9}
\end{aligned}$$

Taking symmetry into account, $\psi_T(v) = \psi_T(-v)$. Then,

$$\int_{-\infty}^{+\infty} e^{-ivk} \psi_T(v) = 2\mathbf{Re} \int_0^{+\infty} e^{-ivk} \psi_T(v), \tag{4.3.10}$$

and

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \mathbf{Re} \int_0^{+\infty} e^{-ivk} \psi_T(v). \tag{4.3.11}$$

Since $C_T(k)$ is real,

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-ivk} \psi_T(v). \tag{4.3.12}$$

4.4 Fast Fourier Transform (FFT)

The FFT is an efficient method for computing the sum (see Cooley & Tukey (1965) and Carr & Madan (1998) for the method)

$$\int_0^{\infty} e^{-ixu} h(u) du \approx \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} h(j) \quad k = 1, 2, \dots, N. \quad (4.4.1)$$

It consists of approximating the continuous Fourier transform by the discrete counterpart. In this thesis the trapezoidal rule is used for integration. The approximation for the integral in formula 4.3.12 is

$$\int_0^{\infty} e^{-ivk} \psi(v) dv \approx \sum_{j=1}^N e^{-iv_j k} \psi'(v_j) \eta, \quad (4.4.2)$$

where η is the discretization step (analogue to Δx in the trapezoidal rule). The points v_j are equidistant with spacing of η such that, $v_j = \eta j$. ψ' is just ψ weighted by the integration rule. The effective upper limit of integration is $a = \eta N$.

The interest here is in strikes near the spot price since such options are frequently traded. Equally spaced log-strikes are considered with

$$k_u = -b + \lambda u \quad \text{for } u = 1, 2, \dots, N, \quad (4.4.3)$$

where $b = \frac{N\lambda}{2}$. Our log-strikes range from $-b$ to b . We can rewrite formula 4.4.2 as

$$\begin{aligned}
\sum_{j=1}^N e^{-iv_j k u} \psi'(v_j) \eta &= \sum_{j=1}^N e^{-i(-\frac{1}{2}n\lambda + \lambda u)v_j} \psi'(v_j) \eta \\
&= \sum_{j=1}^N e^{-i(-\frac{1}{2}n\lambda + \lambda u)\eta j} \psi'(v_j) \eta \\
&= \sum_{j=1}^N e^{-i\lambda u \eta j} e^{i\frac{N\lambda}{2}\eta j} \psi'(v_j) \eta.
\end{aligned}$$

Setting $\lambda\eta = \frac{2\pi}{N}$ and $h_j = e^{i\frac{N\lambda}{2}\eta j} \psi'(v_j) \eta$, the summation above becomes

$$\sum_{j=1}^N e^{-i\frac{2\pi}{N}uj} h_j. \tag{4.4.4}$$

We then apply the FFT algorithm on the vector $h_j = e^{i\frac{N\lambda}{2}\eta j} \psi'(v_j) \eta$ provided that

$$\lambda\eta = \frac{2\pi}{N}. \tag{4.4.5}$$

The challenge with the FFT algorithm is that [4.4.5](#) must be met so as to ensure a good integral approximation. To achieve this, N is increased in such a way that dense log-strikes are obtained.

4.5 Bid-Ask Prices in Conic Finance

The bid-ask prices used in this work are found using Fourier based option pricing. In this section, the bid-ask formulas used in this work are presented. For the call and put bid-ask prices, the closed form formulas proposed in Cherny & Madan (2010b) are used. Let S be the random variable at time T of an underlying asset. The price at time T of a call option C is given by $(S - K)^+$ and put option P is given by $(K - S)^+$, where K is the strike price. The following are the closed bid-ask prices expressions:

$$a_\gamma(C) = \int_K^\infty \Psi^\gamma(1 - F_S(x))dx, \quad (4.5.1)$$

$$b_\gamma(C) = \int_K^\infty (1 - \Psi^\gamma(F_S(x)))dx, \quad (4.5.2)$$

$$a_\gamma(P) = \int_0^K \Psi^\gamma(F_S(x))dx, \quad (4.5.3)$$

$$b_\gamma(P) = \int_0^K \Psi^\gamma(1 - \Psi^\gamma(1 - F_S(x)))dx. \quad (4.5.4)$$

The derivation of the expressions is in Appendix A. Fourier inversions for the above formulas are found using the procedure of Carr & Madan (1998) discussed in the previous sections. F_s is the distribution function of the logarithm of the stock price S and is important since the bid-ask prices are determined completely by this distribution. The distribution function is found using the following relationship:

$$F_t(x) = P(X \leq x) = \int_{-\infty}^x f_t(t)dt. \quad (4.5.5)$$

The above relationship tells us that in order to obtain the distribution of a function,

the probability density function should be known. Now, the probability density functions of some models are not known in closed-form, but their characteristic functions do. Schoutens (2003) says that a distribution can be uniquely determined by its characteristic function since there is a one-to-one relationship between a distribution function and its characteristic function. In other words, the characteristic function of a distribution or equivalently of a random variable X , is the Fourier transform of the distribution function, that is,

$$\phi_x(u) = E[\exp(iuX)] = \int_{-\infty}^{\infty} \exp(iux) dF(x) = \int_{-\infty}^{\infty} \exp(iux) f_x df(x). \quad (4.5.6)$$

Instead of using the density function, the characteristic function is used to determine completely the distribution of the stock price. The characteristic functions for the models were covered in Chapter 3.

Now, the challenge in mathematics of finance is access to market data. This work was not spared from the challenge. However, the market bid-ask prices were generated by the distorted expectations approach using the Wang Transform approach on a log-normal distribution of the underlying asset. The approach yields option prices that are similar to the Black-Scholes model, which is used in the market, but is able to explain bid-ask prices at different levels of acceptability. In a bivariate case, the Wang Transform approach tells us that the distortion function has the following expression:

$$\Psi^\gamma(F_X(x)) = \Phi(\Phi^{-1}(F_X(x)) + \gamma\rho) = \Phi\left(\frac{x - \mu_x + \gamma\rho\sigma_x}{\sigma_x}\right), \quad (4.5.7)$$

where ρ is the correlation coefficient, μ_x and σ_x are mean and variance of X respectively, γ is the level of acceptability and $F_X(x)$ is the distribution function of X . Under

the Black-Scholes model,

$$\Psi^\gamma(F_{S_T}(x)) = \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right). \quad (4.5.8)$$

The following are the theoretical bid-ask prices for European call and put options.

Table 4.1: Wang-Transform Bid-Ask Formulas

| Option Type | Formula | d_1 | d_2 |
|---------------|--|---|------------------------|
| $b_\gamma(C)$ | $S_0 e^{-\gamma\rho\sigma\sqrt{T}} \Phi(d_1) - K e^{-rT}(d_2)$ | $\frac{\ln(S_0/K) + (r + 0.5\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}$ | $d_1 - \sigma\sqrt{T}$ |
| $a_\gamma(C)$ | $S_0 e^{\gamma\rho\sigma\sqrt{T}} \Phi(d_1) - K e^{-rT}(d_2)$ | $\frac{\ln(S_0/K) + (r + 0.5\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}$ | $d_1 - \sigma\sqrt{T}$ |
| $b_\gamma(P)$ | $K e^{-rT}(d_2) - S_0 e^{\gamma\rho\sigma\sqrt{T}} \Phi(d_1)$ | $\frac{\ln(K/S_0) - (r + 0.5\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}$ | $d_1 + \sigma\sqrt{T}$ |
| $a_\gamma(P)$ | $K e^{-rT}(d_2) - S_0 e^{-\gamma\rho\sigma\sqrt{T}} \Phi(d_1)$ | $\frac{\ln(K/S_0) - (r + 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}$ | $d_1 + \sigma\sqrt{T}$ |

When $\gamma = 0$, the expressions reduce to the regular Black-Scholes option prices. Appendix A shows the derivation of the theoretic bid-ask prices using the Wang Transform approach.

5. Option Trading Strategies

In conic finance, one buys from the market at the ask price and sells to the market at the bid price. As mentioned earlier before, there is no complete replication since the bid-ask spread is a reflection of the cost of holding residual risk (Eberlein et al. 2011). So, the spread is very important as it modifies the risk positions of transactions. It significantly reduces risks while allowing positions to be acceptable to the market. In this section, focus is shifted to basic option trading strategies, where one buys from the market at the ask price and sells to the market at the bid price. The strategies given in brief here are a covered call, a protective call (synthetic put), a bull call spread, a bear call spread and a butterfly call spread. All the strategies are European type call options. For detailed information and illustrations on the strategies, one is referred to Schap (2005) and Cohen (2002).

5.1 Covered Call

The combination of buying a stock and selling a call option is known as a covered call. The strategy is not riskless, but it reduces the risk of holding the stock outright. Writing a call option against the stock leads to an inflow of premium if the stock price falls substantially. The profit increases as the stock price increases, but gets capped at the strike price. The following illustration is a simple graphical representation of a covered call.

Illustration

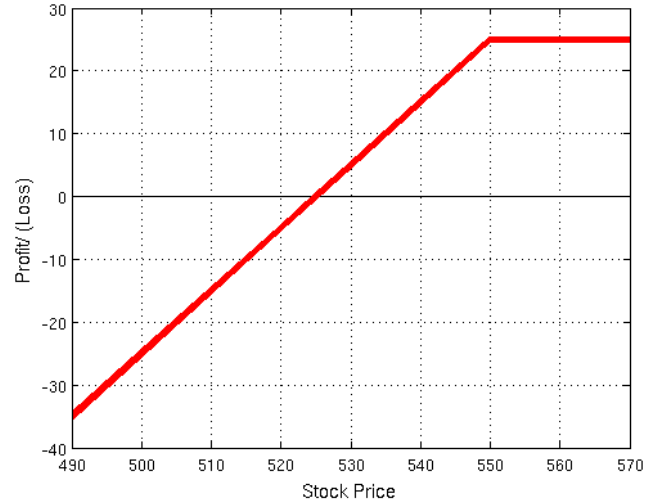
XYZ is currently trading at R54. An investor buys an the option on the stock XYZ at R53. The investor does not anticipate the stock to go beyond R55. A covered call strategy can be initiated by selling a call option strike R55 @ R 0.50. The profit/loss

and the chart for the covered call are show in Tables 5.1 and 5.2.

Table 5.1: Covered Call Profit/Loss

| XYZ @ expiry | Profit/Loss |
|--------------|-------------|
| 490 | -35 |
| 500 | -25 |
| 510 | -15 |
| 520 | -5 |
| 525 | 0 |
| 530 | 5 |
| 540 | 15 |
| 550 | 25 |
| 560 | 25 |
| 570 | 25 |

Table 5.2: Covered Call Profit/Loss Chart



The breakeven point occurs the moment the stock crosses R525.

Now, consider a case of one share of the stock and one short call option on the stock. Let C be the premium of the option and X be the strike price of the option. The profit equation is:

$$\Pi = S_T - S_0 - \max(0, S_T - X) + C. \quad (5.1.1)$$

The possible scenarios at the expiration date are:

$$\begin{aligned}\Pi &= S_T - S_0 + C && \text{if } S_T \leq X \\ \Pi &= S_T - S_0 - S_T + X + C \\ &= X - S_0 + C && \text{if } S_T > X\end{aligned}\tag{5.1.2}$$

If the option ends up out-of-the-money, the loss on the stock will be reduced by the call premium. If the option ends up in-the-money, it will be exercised and the stock has to be delivered (Chance (1991)). The breakeven stock price is found when the stock price occurs where the profit is zero. Let the breakeven stock price be S_T^* , then:

$$\Pi = S_T^* - S_0 + C,$$

and solving for S_T^* gives a breakeven price

$$S_T^* = S_0 - C.\tag{5.1.3}$$

5.2 Synthetic Put (Protective Call)

The combination of taking a long position in a call option and short selling the stock is known as a synthetic put(protective call). A synthetic put enables an investor to protect an investment against a bearish market. Also, a synthetic put enables an investor to take advantage of mispricing in the relationship between put and call options. The following illustration is a simple graphical representation of a synthetic put.

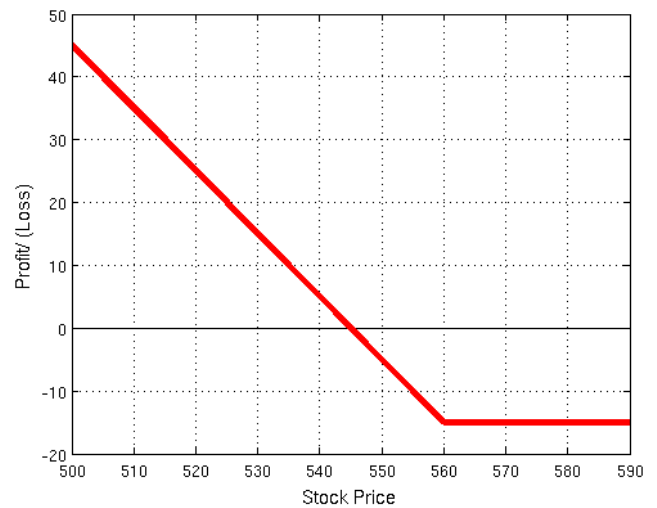
Illustration

XYZ stock is currently trading at R540. An investor anticipates the market to be bearish but does not expect it to drop below R520. The investor buys an option on the XYZ stock at R550. A synthetic put strategy can be done by buying a call option strike R560 @ R5 and shorting the stock at the current trading price. The profit/loss and chart for the synthetic put are shown in Tables 5.3 and 5.4.

Table 5.3: Synthetic Put Profit/Loss

| XYZ @ expiry | Profit/Loss |
|--------------|-------------|
| 500 | 45 |
| 510 | 35 |
| 520 | 25 |
| 530 | 15 |
| 540 | 5 |
| 545 | 0 |
| 550 | -5 |
| 560 | -15 |
| 570 | -15 |
| 580 | -15 |
| 590 | -15 |

Table 5.4: Synthetic Put Profit/Loss Chart



The breakeven point occurs the point the moment the stock crosses R545.

Now, consider a scenario of being long in one call option and shorting a share of the same underlying stock. Let C be the premium of the option and let X be the strike price of the option. The profit equation is:

$$\Pi = S_T - S_0 - \max(0, S_T - X) - C. \quad (5.2.1)$$

The possible scenarios at the expiration date are:

$$\begin{aligned} \Pi &= -C - S_T + S_0 && \text{if } S_T \leq X \\ \Pi &= S_T - X - C - S_T + S_0 \\ &= S_0 - X - C && \text{if } S_T > X \end{aligned} \quad (5.2.2)$$

If the stock price at expiration is less than or equal the exercise price, X , the profit will vary inversely with the stock price at expiration. If the stock price at expiration is above the exercise price, the profit will not be affected by the stock price at expiration (Chance (1991)). The breakeven stock price is found when the stock price occurs where the profit is zero. Let the breakeven stock price be S_T^* , then:

$$\Pi = -C - S_T^* + S_0,$$

and solving for S_T^* gives a breakeven price

$$S_T^* = S_0 - C. \quad (5.2.3)$$

5.3 Bull Call Spread

A bull call spread is a bullish strategy formed by buying an ‘in-the-money call option’¹ (lower strike) and selling ‘out-of-the-money’² (higher strike). Both call options must

¹in-the-money option leads to a positive cash flow to the holder if it is exercised immediately.

²out-of-the-money option leads to negative cash flow if it is exercised immediately.

be on the same underlying and expiration date. The strategy's net effect is to bring down the cost and breakeven (long call strike price + net debt) on a buy call (long call) strategy. A simple graphical representation is demonstrated in the following illustration.

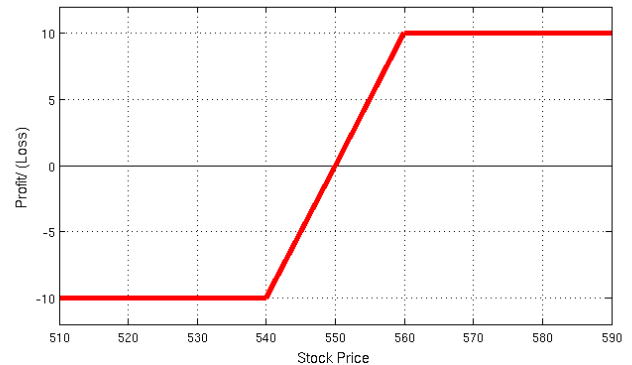
Illustration

XYZ stock is currently priced at R50. Suppose an investor anticipates the markets to rise upwards from the current levels. Buying a call option with a strike R540 @ R15 and selling a call option with a strike of R560 @ R5 with the same expiration date will benefit an investor if XYZ goes above R550. The profit/loss and chart for the strategy are shown in Tables 5.5 and 5.6.

Table 5.5: Bull Spread Profit/Loss

| XYZ @ expiry | Profit/Loss |
|--------------|-------------|
| 510 | -10 |
| 520 | -10 |
| 530 | -10 |
| 540 | -10 |
| 550 | 0 |
| 560 | 10 |
| 570 | 10 |
| 580 | 10 |
| 590 | 10 |

Table 5.6: Bull Profit/Loss Chart



The risk is limited to a maximum of R500. The breakeven point occurs at the moment the stock crosses R550.

Now, consider two call options with exercise prices and premiums (E_1, C_1) and (E_2, C_2) , respectively, where $E_1 < E_2$ and $C_1 > C_2$. The net payoff from the strategy will be:

$$\Pi = \max(0, S_T - E_1) - C_1 - \max(0, S_T - E_2) + C_2. \quad (5.3.1)$$

The stock price at expiration can be in any one of the ranges: less than or equal to E_1 , greater than E_1 but less than or equal to E_2 , or greater than E_2 (Chance (1991)). The payoffs for the ranges will be:

$$\begin{aligned} \Pi &= -C_1 + C_2 && \text{if } S_T \leq E_1 < E_2 \\ \Pi &= S_T - E_1 - C_1 + C_2 && \text{if } E_1 < S_T \leq E_2 \\ \Pi &= S_T - E_1 - C_1 - S_T + E_2 + C_2 \\ &= E_2 - E_1 - C_1 + C_2 && \text{if } E_1 < E_2 < S_T \end{aligned} \quad (5.3.2)$$

The breakeven stock price at expiration is found when it exceeds the lower exercise price by the the difference in premiums. Let the breakeven stock price be S_T^* , then:

$$S_T^* = E_1 + C_1 - C_2. \quad (5.3.3)$$

5.4 Bear Call Spread

A bear call spread is a bearish strategy formed by buying an ‘out-of-the-money’ call option (higher strike) and selling an ‘in-the-money’ call option (lower strike). Both call options must be on the same underlying security and expiration date. The strategy’s concept is to protect the downside of the sold call option by buying a call option of

higher strike price. Then, the investor receives a net credit since the call option which has been bought has a higher strike price than the sold option. The breakeven will be the sum of the strike price of the short call option plus the premium received. A graphical representation of the bear call spread is shown in the following illustration.

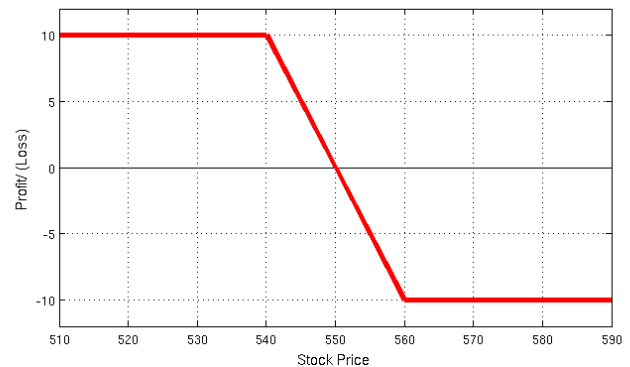
Illustration

XYZ stock is currently priced at R550. Suppose an investor anticipates the markets to trend down from the current levels. Selling a call option with a strike R540 @ R15 and buying a call option with a strike R560 @ R5 with the same expiration date will benefit an investor if stock XYZ stays below R500. The profit/loss and chart for the strategy are shown in Tables 5.7 and 5.8.

Table 5.7: Bear Spread Profit/Loss

| XYZ @ expiry | Profit/Loss |
|--------------|-------------|
| 510 | 10 |
| 520 | 10 |
| 530 | 10 |
| 540 | 10 |
| 550 | 0 |
| 560 | -10 |
| 570 | -10 |
| 580 | -10 |
| 590 | -10 |

Table 5.8: Bear Profit/Loss Chart



The risk is limited to a maximum of R500. The breakeven point occurs the moment the stock crosses R550.

Now, consider two call options differing only by their exercise prices which are E_1 and E_2 , where $E_1 < E_2$. The call options have premiums C_1 and C_2 . A bear call spread can be considered as the mirror image of the bull spread and so the trader is short in the low exercise price option and long in the high exercise price option (Chance (1991)). The net payoff will be:

$$\Pi = \max(0, S_T - E_1) + C_1 - \max(0, S_T - E_2) - C_2. \quad (5.4.1)$$

The possible outcomes at the expiration date will be:

$$\begin{aligned} \Pi &= C_1 - C_2 && \text{if } S_T \leq E_1 < E_2 \\ \Pi &= -S_T + E_1 + C_1 - C_2 && \text{if } E_1 < S_T \leq E_2 \\ \Pi &= -S_T + E_1 + C_1 + S_T - E_2 - C_2 \\ &= E_1 - E_2 + C_1 - C_2 && \text{if } E_1 < E_2 < S_T \end{aligned} \quad (5.4.2)$$

The spread generates a positive payoff if the stock prices fall. The breakeven stock price is still:

$$S_T^* = E_1 + C_1 - C_2. \quad (5.4.3)$$

5.5 Long Call Butterfly Spread

A butterfly spread is a combination of a bull spread and a bear spread. A long call butterfly spread is formed by selling two ‘at-the-money’³ call options, buying one ‘in-the-money’ call option (lower strike) and one ‘out-of-the-money’ call option (higher strike). An investor stands to gain if the price of the underlying remains around the middle strike price upon expiration. The risk of the strategy is limited to the

³at-the-money option leads to zero cash flow if it is exercised immediately.

premium paid. The following illustration depicts a long call butterfly spread.

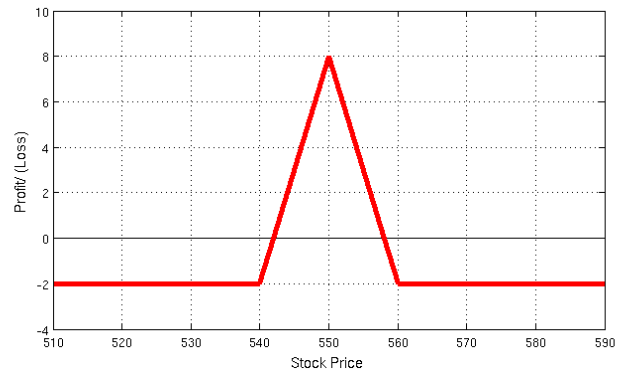
Illustration

XYZ is currently priced at R550. Buying a call option with a strike R540 @ R20, strike R560 @ R8 and selling two call options with strike R550 @ R13 will result in a butterfly spread which benefit the investor if stock XYZ expires at R550. The payoff schedule and chart for the strategy are shown in Tables 5.9 and 5.10.

Table 5.9: Butterfly Spread Profit/Loss

| XYZ @ expiry | Profit/Loss |
|--------------|-------------|
| 510 | -2 |
| 520 | -2 |
| 530 | -2 |
| 540 | -2 |
| 542 | 0 |
| 550 | 8 |
| 558 | 0 |
| 560 | -2 |
| 570 | -2 |
| 580 | -2 |
| 590 | -2 |

Table 5.10: Butterfly Profit/Loss Chart



Suppose there are three exercise prices: E_1 , E_2 and E_3 , where $E_1 < E_2 < E_3$, with premiums C_1 , C_2 , C_3 , respectively. A call bull spread is constructed by purchasing the call option with the low exercise price E_1 , and write the call with the middle exercise price, E_1 . A call bear spread is constructed by purchasing the call option with the high exercise price, E_3 , and write the call with middle exercise price, E_2 (Chance (1991)). A combination of these positions form a butterfly spread and the net payoff is:

$$\begin{aligned} \Pi = & \max(0, S_T - E_1) - C_1 - 2 \max(0, S_T - E_2) \\ & + 2C_2 + \max(0, S_T - E_3) - C_3. \end{aligned} \quad (5.5.1)$$

The four possible outcomes at the expiration are:

$$\begin{aligned}
 \Pi &= -C_1 + 2C_2 - C_3 && \text{if } S_T \leq E_1 < E_2 < E_3 \\
 \Pi &= S_T - E_1 - C_1 + 2C_2 - C_3 && \text{if } E_1 < S_T \leq E_2 < E_3 \\
 \Pi &= S_T - E_1 - C_1 - 2S_T + 2E_2 + 2C_2 - C_3 \\
 &= -S_T + 2E_2 - E_1 - C_1 + 2C_2 - C_3 && \text{if } E_1 < E_2 < S_T \leq E_3 \\
 \Pi &= S_T - E_1 - C_1 - 2S_T + 2E_2 + 2C_2 + S_T - E_3 - C_3 \\
 &= -E_1 + 2E_2 - E_3 - C_1 + 2C_2 - C_3 && \text{if } E_1 < E_2 < E_3 < S_T \quad (5.5.2)
 \end{aligned}$$

Since an investor stands to gain when the price of the underlying lies within a certain range, there are two breakeven points namely lower breakeven point and higher breakeven point. Lower breakeven point is the sum of the strike price of lower strike long call plus net premium paid. If S_T^* denotes the breakeven stock price, then:

$$S_T^* = E_1 + C_1 - 2C_2 + C_3. \quad (5.5.3)$$

The higher breakeven point consists of strike price of higher long call minus net premium paid. In this case,

$$S_T^* = 2E_2 - E_2 - C_1 + 2C_2 - C_3. \quad (5.5.4)$$

6. Numerical Results of Trading Strategies

In this chapter, numerical results obtained using the bid-ask prices from conic finance of some option strategies discussed in the last chapter are presented. The numerical results presented in this chapter are only for spread strategies. Proper two price markets include structured products, which have become a significant part of financial markets. Structured products are traded over the counter, where there are typically two prices either for buying from the market or selling to the market (Eberlein et al. (2011)). The prices are determined by pricing models since they are not quoted in the market. However, structured products markets data is not easily accessible. In line with Cherny & Madan (2010*b*) and Eberlein et al. (2011), we also employ option markets as proxy for two price markets and ignore the little liquidity that might be available.

The risk profiles of the option strategies are analyzed at different stress (risk appetite) levels. In particular for each risk profile; maximum risk, maximum reward and breakeven points are determined. The theory of conic finance provides bid-ask prices, which depend on the risk appetite of investors. For evaluation of bid-ask prices, we use acceptability indices based on the MAXMINVAR. The options used in the strategies are of European type and are applied to Single Stocks Futures options and warrants offered in the South African financial markets. Comparison of risk profiles is done between the VGSSD model and the Black-Scholes model. The Black-Scholes model is considered here since it is the one that is mostly used by industrial practitioners. So, the Black-Scholes is a proxy for market prices. All the numerical implementations were carried out in Matlab R2009b.

In summary, the main numerical implementations in this work are:

- Estimation of parameters (see Sections 4.1 and 4.4)

The genetic algorithm (ga) in Matlab was used for this implementation. The genetic algorithm was appropriate for this task as it has high potential of obtaining a global minima within the bounded constraints, despite the lengthy time it takes to achieve this.

- Calibration of the models to market prices (see Section 4.2)

Again, the genetic algorithm (ga) in Matlab was used to calibrate the models to market prices.

- Estimation of bid-ask prices using conic finance theory (see Sections 4.3, 4.4 and 4.5)

The FFT algorithm in Matlab was used to obtain the bid-ask prices.

6.1 Data Description

For numerical illustration purposes, we used two large South African banks - ABSA and Standard Bank, for option strategies. Note that, the illustrations do not pertain to any real positions on the banks. The bid-ask prices were computed at various theoretical prices of the underlying on the expiration date. One difficulty that often arise in practice is on the choice of the proper interest rate to use as an input. The 3-month JIBAR is used as a proxy for the risk-free interest rate, as it is widely used in practice. To do model calibration, we need market prices. Simulated data set of bid-ask options at different strikes maturing on the same date were generated using the models introduced in Chapter 3. The calibration is implemented as explained in Chapter 4.

Next, we look at the results of the risk profiles of the option strategies. We present results for the option spread strategies here. The illustrations are implemented at

stress (risk) levels of 0.01, 0.05 and 0.10.

6.2 Bull Call Spread Risk Profile

Scenario

An investor owns 100 shares in ABSA Bank (ASAQ), which in early July are trading at a Single Stock Future (SSF) fair value of R140. The investor believes the market will be bullish in the coming 6 months and decided to create a bull call spread. So the investor buys a DEC ASAQ 140 call option and sells a DEC ASAQ call option with a higher strike price, so as to create the bull call spread strategy. The concern for the investor is on the appropriate higher strike which can create an attractive strategy.

1. At different stress (risk) levels, the investor determines the bid-ask prices for the range of strike prices.
2. The investor analyzes the risk profiles at each strike price choice so as to create an appropriate trade.
3. The investor finally assesses the performance of the strategy, given a range of possible values of the underlying at expiration for the appropriate strike price from step 2 at a stress level of 0.01.

In addition, the investor gathers the following information:

3month JIBAR rate = 5.01%

Time to expiration = 6/12 yr

Dividend yield = 0% (assumption).

In order to create the strategy appropriately, the investor implemented the following steps.

Step 1 - Bid-Ask Prices at Different Stress (γ) Levels

The calibrated parameters used for this strategy are $\sigma = 0.240$ in the Black-Scholes model and $\sigma = 0.226, \theta = -0.131, \nu = 0.08, \gamma = 0.480$ in the VGSSD model. An attractive bull call spread is created when an investor buys a lower strike call and sells a higher strike call. In the scenario presented above, the investor has the following choices:

| | | |
|---------------|-------------------------|-----------------------|
| Step 1 | Long Call | Buy R140 Strike Call |
| Step 2 | (i) Short Call | Sell R141 Strike Call |
| Or | (ii) Short Call | Sell R142 Strike Call |
| Or | (iii) Short Call | Sell R143 Strike Call |
| Or | (iv) Short Call | Sell R144 Strike Call |
| Or | (v) Short Call | Sell R145 Strike Call |

Table B.1 shows the bid-ask prices for the options using both the Black-Scholes model and VGSSD model.

Step 2 - Risk Profile Analysis

Next, we look at the risk profiles for each of the choices using bid-ask prices provided in Step 1 at a stress level of 0.01.

From Table 6.1, an attractive strategy can be created by choice (iv). The reason being that the risk and breakeven point are lower whilst maximum reward and maximum Return on Investment (ROI) are high enough to be attractive.

Also under the VGSSD model, an attractive strategy can be created using choice (iv) as shown in Table 6.2. The reason being again that the risk and breakeven are lower whilst the maximum reward and maximum Return on Investment (ROI) are high

Table 6.1: Bull Call Spread Risk Profile using the Black-Scholes Model

| Step 1 | Long Call | | | | | |
|---------------|--------------------------------|-------------|---------------|------------------|----------------|--|
| | Buy R140 Strike Call @ R11.83 | | | | | |
| Step 2 | Short Call | Risk | Reward | Breakeven | Max ROI | |
| (i) | Sell R141 Strike Call @ R10.56 | R1.27 | -R0.27 | R141.27 | -21.26 % | |
| (ii) | Sell R142 Strike Call @ R10.07 | R1.76 | R0.24 | R141.76 | 13.64% | |
| (iii) | Sell R143 Strike Call @ R9.62 | R2.21 | R0.79 | R142.21 | 35.75% | |
| (iv) | Sell R144 Strike Call @ R9.16 | R2.67 | R1.33 | R142.67 | 49.81% | |
| (v) | Sell R145 Strike Call @ R8.71 | R3.12 | R1.88 | R143.12 | 60.26% | |

Table 6.2: Bull Call Spread Risk Profile using the VGSSD Model

| Step 1 | Long Call | | | | | |
|---------------|--------------------------------|-------------|---------------|------------------|----------------|--|
| | Buy R140 Strike Call @ R11.77 | | | | | |
| Step 2 | Short Call | Risk | Reward | Breakeven | Max ROI | |
| (i) | Sell R141 Strike Call @ R10.54 | R1.23 | -R0.23 | R141.23 | -18.70 % | |
| (ii) | Sell R142 Strike Call @ R10.04 | R1.73 | R0.27 | R141.73 | 15.61% | |
| (iii) | Sell R143 Strike Call @ R9.59 | R2.18 | R0.82 | R142.18 | 37.61% | |
| (iv) | Sell R144 Strike Call @ R9.15 | R2.62 | R1.38 | R142.62 | 52.67% | |
| (v) | Sell R145 Strike Call @ R8.70 | R3.07 | R1.93 | R143.07 | 62.87% | |

enough to be attractive.

Step 3 - Scenario Analysis at the Expiration Date

After choosing an attractive choice from Step 2, we now look at the profit/loss of the strategy at expiration for a range of prices for the underlying. We compare the profit/loss under the two models - Black-Scholes model and VGSSD model. Table 6.3 shows the profit/loss of the strategy under the two models. Figure 6.1 shows the plot of the profit/loss of the strategy for a range of prices of the underlying at expiration.

Table 6.3: Bull Call Spread Profit/Loss under the Black-Scholes and VGSSD Models

| Black-Scholes | Model | | VGSSD | Model |
|----------------------|--------------|-----------|---------------|--------------|
| ASAP @ expiry | Profit/Loss | | ASAP @ expiry | Profit/Loss |
| 135 | -2.67 | | 135 | -2.62 |
| 136 | -2.67 | | 136 | -2.62 |
| 137 | -2.67 | | 137 | -2.62 |
| 138 | -2.67 | | 138 | -2.62 |
| 139 | -2.67 | | 139 | -2.62 |
| 140 | -2.67 | | 140 | -2.62 |
| 141 | -1.67 | | 141 | -1.62 |
| 142 | -0.67 | | 142 | -0.62 |
| 142.67 | 0.00 | | 142.62 | 0 |
| 143 | 0.33 | | 143 | 0.38 |
| 144 | 1.33 | | 144 | 1.38 |
| 145 | 1.33 | | 145 | 1.38 |
| 146 | 1.33 | | 146 | 1.38 |
| 147 | 1.33 | | 147 | 1.38 |
| 148 | 1.33 | | 148 | 1.38 |
| 149 | 1.33 | | 149 | 1.38 |
| 150 | 1.33 | | 150 | 1.38 |

From Figure 6.1, it can be observed that the breakeven point is lower using the

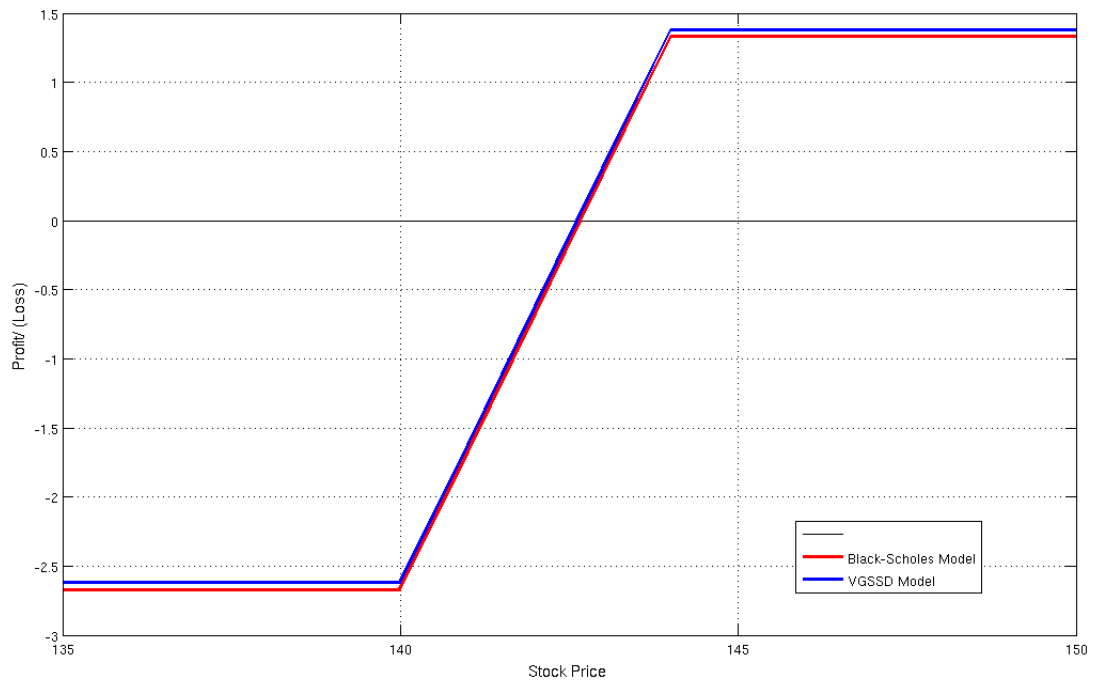


Figure 6.1: Plot of Bull Call Spread Profit/Loss

VGSSD model than the Black-Scholes model. A lower breakeven point is ideal for a strategy which intends to reduce risk.

Comment on the Strategy

The spread was observed to be lower in the VGSSD model than in the Black-Scholes model. Reduced spread can minimize the unhedgeable risk, which can be a major boost for option trading strategies. As a result, the cost of trade is lowered as the sold options can offset the cost of the bought option. In conclusion, the strategy becomes less risky in terms of lower risk and lower breakeven point but offers limited potential reward, which can still be highly attractive.

6.3 Bear Call Spread Risk Profile

In early July an investor believes the SSF fair price of Standard Bank (SBKQ) is going to fall from the current levels of R120 to around R117.50. The investor wants to create an attractive bear call spread. So the investor writes a SEP SBKQ 119 call option and buys a higher SEP SBKQ strike call, so as to create a bear call strategy. A little bit of concern to the investor is on the appropriate higher strike to choose so as to create an attractive strategy.

1. Now at different stress (risk) levels, the investor determines the bid-ask prices for the range of higher strike prices.
2. The investor analyzes the risk profiles at each of strike price choices so as to create an appropriate trade.
3. Finally, the investor accesses the performance of the strategy for the appropriate strike price in Step 2 at a stress level of 0.01 given a range of possible values of the underlying at expiration.

The investor also gathers the following information:

3month JIBAR rate = 5.01%

Time to expiration = 2/12 yr

Dividend yield = 0% (assumption).

In order to create the strategy appropriately, the investor implemented the following steps.

Step 1 - Bid-Ask Prices at Different Stress (γ) Levels

The calibrated parameters used for this strategy are $\sigma = 0.306$ in the Black-Scholes model and $\sigma = 0.285, \theta = -0.070, \nu = 0.060, \gamma = 0.510$ in the VGSSD model. An attractive bear call spread is created when an investor sells a lower strike call and buys a higher strike call. In the scenario presented here, the investor has the following choices:

| | | |
|---------------|------------------------|-----------------------|
| Step 1 | Short Call | Sell R119 Strike Call |
| Step 2 | (i) Long Call | Buy R120 Strike Call |
| Or | (ii) Long Call | Buy R121 Strike Call |
| Or | (iii) Long Call | Buy R122 Strike Call |
| Or | (iv) Long Call | Buy R123 Strike Call |
| Or | (v) Long Call | Buy R124 Strike Call |

Table B.2 shows the bid-ask prices for the options using both the Black-Scholes model and VGSSD model.

Step 2 - Risk Profile Analysis

We now look at the risk profiles for each of the choices using bid-ask prices provided in Step 1 at a stress level of 0.01.

In Table 6.4 a potential strategy can be created using a strike which provides reduced risk and a lower breakeven point. In addition, the gain on this strategy is the net credit received upon entering the trade. As a result choice (iv) is attractive to create the strategy since the net credit is fairly high, and the breakeven point as well as the risk are reduced.

In Table 6.5 a potential strategy again can be created using a strike which provides reduced risk and lower breakeven. Also, the gain on this strategy is the net credit received upon entering the trade. As a result choice (iv) is attractive to create the

Table 6.4: Bear Call Spread Risk Profile using the Black-Scholes Model

| Step 1 | Short Call | | | | |
|---------------|-------------------------------|-------------|---------------|------------------|----------------|
| | Sell R119 Strike Call @ R6.91 | | | | |
| Step 2 | Long Call | Risk | Reward | Breakeven | Max ROI |
| (i) | Buy R120 Strike Call @ R6.90 | R0.99 | R0.01 | R119.01 | 1.01% |
| (ii) | Buy R121 Strike Call @ R6.40 | R1.49 | R0.51 | R119.51 | 34.23% |
| (iii) | Buy R122 Strike Call @ R5.93 | R2.02 | R0.98 | R119.98 | 48.51% |
| (iv) | Buy R123 Strike Call @ R5.45 | R2.54 | R1.46 | R120.46 | 57.48% |
| (v) | Buy R124 Strike Call @ R5.03 | R3.12 | R1.88 | R120.88 | 60.26% |

Table 6.5: Bear Call Spread Risk Profile using the VGSSD Model

| Step 1 | Short Call | | | | |
|---------------|-------------------------------|-------------|---------------|------------------|----------------|
| | Sell R119 Strike Call @ R6.87 | | | | |
| Step 2 | Long Call | Risk | Reward | Breakeven | Max ROI |
| (i) | Buy R120 Strike Call @ R6.86 | R0.99 | R0.01 | R119.01 | 100.00% |
| (ii) | Buy R121 Strike Call @ R6.35 | R1.48 | R0.52 | R119.52 | 100.00% |
| (iii) | Buy R122 Strike Call @ R5.89 | R2.02 | R0.98 | R119.98 | 100.00% |
| (iv) | Buy R123 Strike Call @ R5.43 | R2.56 | R1.44 | R120.44 | 100.00% |
| (v) | Buy R124 Strike Call @ R5.02 | R3.15 | R1.85 | R120.85 | 100.00% |

strategy since the net credit is fairly high, and the breakeven point is reduced and the risk is fairly low.

Step 3 - Scenario Analysis at the Expiration Date

Next, we look at the profit/loss of the strategy at expiration for a range of prices for the underlying using the choice selected in Step 2. We compare the profit/loss under the two models - Black-Scholes model and VGSSD model. Table 6.6 shows the profit/loss of the strategy under the two models. Figure 6.2 shows the plot of the profit/loss of the strategy for a range of prices of the underlying at expiration.

Table 6.6: Bear Call Spread Profit/Loss under the Black-Scholes and VGSSD Models

| Black-Scholes | Model | VGSSD | Model |
|----------------------|--------------|---------------|--------------|
| SBKQ @ expiry | Profit/Loss | SBKQ @ expiry | Profit/Loss |
| 125 | -2.54 | 125 | -2.56 |
| 124 | -2.54 | 124 | -2.56 |
| 123 | -2.54 | 123 | -2.56 |
| 122 | -1.54 | 122 | -1.56 |
| 121 | -0.54 | 121 | -0.56 |
| 120.46 | 0.00 | 120.44 | 0.00 |
| 120 | 0.46 | 120 | 0.44 |
| 119 | 1.46 | 119 | 1.44 |
| 118 | 1.46 | 118 | 1.44 |
| 117 | 1.46 | 117 | 1.44 |
| 116 | 1.46 | 144 | 1.44 |
| 115 | 1.46 | 115 | 1.44 |

The breakeven point is lower in the VGSSD model than the Black-Scholes model, which is ideal in creating a strategy with reduced risk.

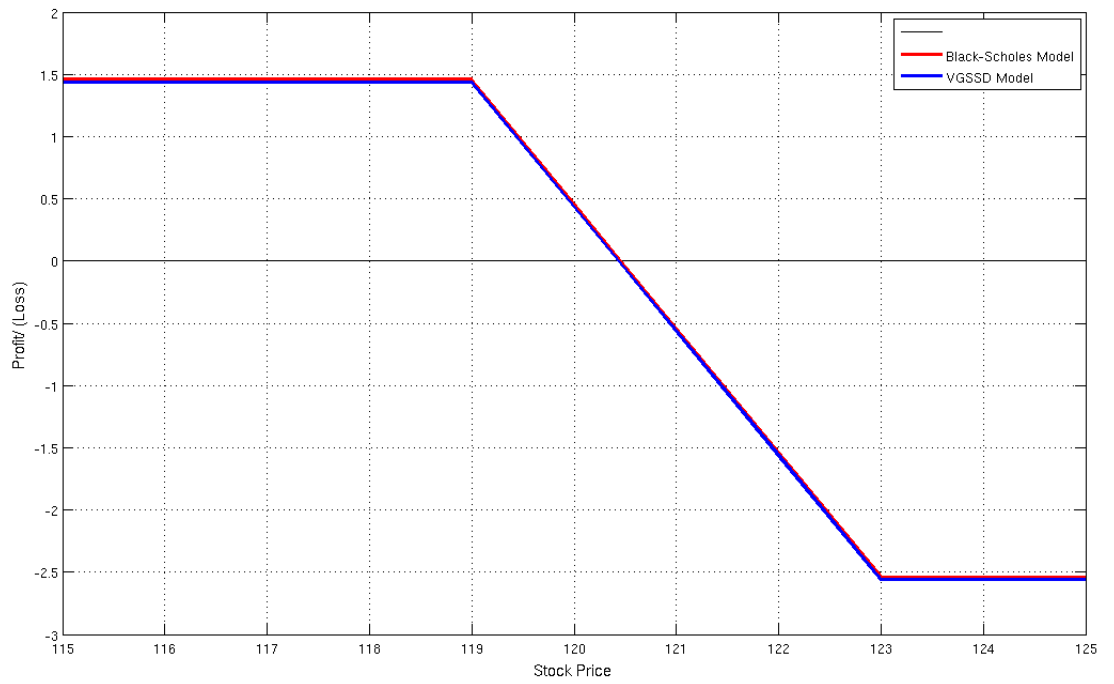


Figure 6.2: Plot of Bear Call Spread Profit/Loss

Comment on the Strategy

Under this strategy, the spread is reduced under the VGSSD model than the Black-Scholes model. The lower spread implies reduced cost of risk and lower breakeven point. However, in this strategy reducing the risk impacts on the potential reward under the VGSSD model as compared to the Black-Scholes model.

6.4 Long Call Butterfly Spread Risk Profile

An investor's perception about the market is neutral for the coming two months. The investor decides to create an attractive long call butterfly spread in the commodity reference warrant (CoRW) market. The current rand gold price is trading at R8.00. To create the strategy, the investor buys a call GOLSBA warrant at R7.50 strike, a call GOLSBA warrant at R8.50 and sells two call GOLSBA warrants at R8.00 each.

1. At different stress (risk) levels, the investor analyzes the risk profile of getting into the trade.
2. The investor assesses the performance of the strategy, given a range of possible rand gold prices at expiration at a stress level of 0.01.

The investor also gathers the following information:

3month JIBAR rate = 5.01%

Time to expiration = 2/12 yr

Dividend yield = 0% (assumption).

In order to create the strategy appropriately, the investor implemented the following steps.

Step 1 - Bid-Ask Prices at Different Stress (γ) Levels

The calibrated parameters used for this strategy are $\sigma = 0.299$ in the Black-Scholes model and $\sigma = 0.250, \theta = -0.060, \nu = 0.141, \gamma = 0.493$ in the VGSSD model. An attractive bear call spread is created by buying 1 lower strike ITM call, 1 higher strike OTM call and selling 2 ATM calls. In the scenario presented here, the investor buys a GOLSBA call warrant at R7.50 strike, GOLSBA call warrant at R8.50 strike and sells two GOLSBA call warrants at R8.00 strike each, so as to create a butterfly spread strategy. Table B.3 shows the bid-ask prices for the warrants using both the Black-Scholes model and VGSSD model.

Step 2 - Risk Profile Analysis

We now analyze the risk profiles for the potential butterfly spread under the two models using the bid-ask prices from Step 1 at a stress level of 0.01. We can form an attractive strategy using either of the two models as the maximum risk is reduced and the maximum return on investment is fairly high. However, the spread from the VGSSD model is reduced as shown in Table 6.7, hence the maximum return on investment using the bid-ask prices from the model outperform those from the Black-Scholes model.

Step 3 - Scenario Analysis at the Expiration Date

Next, we look at the profit/loss of the strategy at expiration for a range of prices for the underlying under the two models - Black-Scholes model and VGSSD model. Table 6.8 shows the profit/loss of the strategy under the two models. Figure 6.3 shows the plot of the profit/loss of the strategy for a range of prices of the underlying at expiration.

Table 6.7: Risk Profile Analysis of the Long Call Butterfly Spread

| | Black-Scholes Model | VGSSD Model |
|--------------------------------------|----------------------------|--------------------------|
| Buy R7.50 Strike warrant | R0.78 | R0.76 |
| Sell $2 \times$ R8.00 Strike warrant | $R0.42 \times 2 = R0.84$ | $R0.41 \times 2 = R0.82$ |
| Buy R8.50 Strike warrant | R0.25 | R0.24 |
| Net debit | R0.19 | R0.18 |
| Maximum risk | R0.19 | R0.18 |
| Maximum reward | R0.31 | R0.32 |
| Maximum return on Investement | 163.16% | 177.78% |
| Breakeven(downside) | R7.69 | R7.68 |
| Breakeven(upside) | R8.31 | R8.32 |
| Maximum risk on net debit | 100% | 100% |

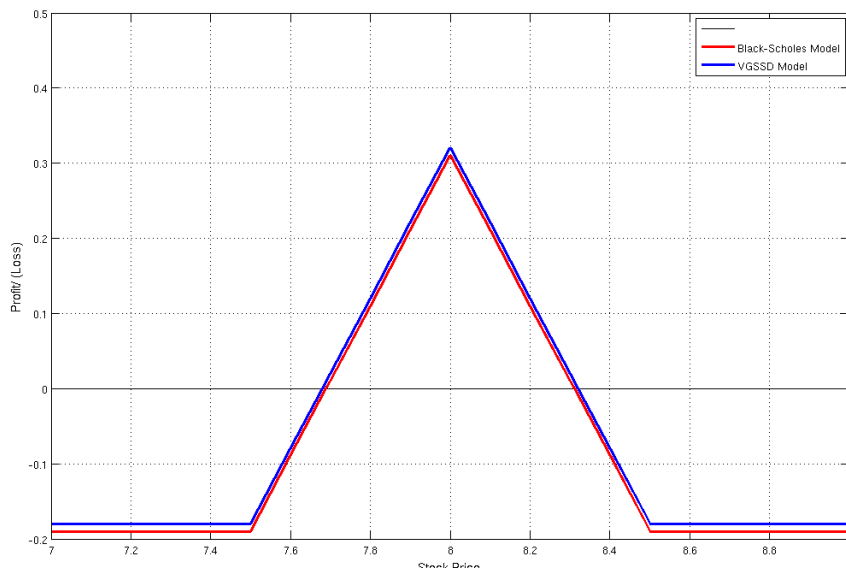


Figure 6.3: Plot of Butterfly Call Spread Profit/Loss

Table 6.8: Butterfly Call Spread Profit/Loss under the Black-Scholes and VGSSD Models

| Black-Scholes | Model | VGSSD | Model |
|----------------------|--------------|-----------------|--------------|
| GOLSBA @ expiry | Profit/Loss | GOLSBA @ expiry | Profit/Loss |
| 7.00 | -0.19 | 7.00 | -0.18 |
| 7.10 | -0.19 | 7.10 | -0.18 |
| 7.20 | -0.19 | 7.20 | -0.18 |
| 7.30 | -0.19 | 7.30 | -0.18 |
| 7.40 | -0.19 | 7.40 | -0.18 |
| 7.50 | -0.19 | 7.50 | -0.18 |
| 7.60 | -0.09 | 7.60 | -0.08 |
| 7.69 | 0.00 | 7.68 | 0.00 |
| 7.70 | 0.01 | 7.70 | 0.02 |
| 7.80 | 0.11 | 7.80 | 0.12 |
| 7.90 | 0.21 | 7.90 | 0.22 |
| 8.00 | 0.31 | 8.00 | 0.32 |
| 8.10 | 0.21 | 8.10 | 0.22 |
| 8.20 | 0.11 | 8.20 | 0.12 |
| 8.30 | 0.01 | 8.30 | 0.02 |
| 8.31 | 0.00 | 8.32 | 0.00 |
| 8.40 | -0.09 | 8.40 | -0.08 |
| 8.50 | -0.19 | 8.50 | -0.18 |
| 8.60 | -0.19 | 8.60 | -0.18 |
| 8.70 | -0.19 | 8.70 | -0.18 |
| 8.80 | -0.19 | 8.80 | -0.18 |
| 8.90 | -0.19 | 8.90 | -0.18 |
| 9.00 | -0.19 | 9.00 | -0.18 |

Comment on the Strategy

The VGSSD model is preferable for this strategy as the spread under former model is reduced as compared to the latter model. As a result, the risk is reduced and the reward is high for the VGSSD model in comparison to the Black-Scholes model. However, take note that the maximum risk on the strategy is 100 % if the market moves against the investor.

7. Conclusion

The thesis has shed some light on the applications of conic finance on South African Financial Markets. The theory of conic finance can be seen as a major advancement to determining bid-ask prices, especially in incomplete markets, where residual risk cannot be totally eliminated as the best hedge can still leave a market participant still exposed to residual risk. The thesis gave an insight on how the residual risk can be reduced for some option trading strategies using bid-ask prices in conic finance. The spread between bid-ask prices was vital in assessing the reduction of the residual risk. The theory of conic finance tries to model explicitly the cone of acceptable risks for a cashflow. The level of acceptability of a cashflow can be considered as the maximum level of distortion that a cashflow can withstand such that its distorted expectation remains positive.

Cherny & Madan (2009) proposed operational cones for determining bid-ask prices for options, which were used heavily in this thesis. These bid-ask prices were determined numerically. The Fourier transform method was used to obtain the bid-ask prices. The distributions used for the bid-ask equations came from the Black-Scholes model and the VGSSD model. The parameters in the models were found by maximum likelihood estimation method and were then calibrated to market prices.

After obtaining the bid-ask prices, analysis on several option strategies - Bull Call Spread, Bear Call Spread and Long Butterfly Call Spread, were implemented. In the strategies, effort was made to try and establish whether the spread could be reduced and resultantly the residual risk could also be reduced. The results of the analysis showed that the spread was reduced, especially using the VGSSD model in comparison to the Black-Scholes model. Ultimately, the findings showed that the residual risk was reduced and the reward from the strategies had a potential of increasing.

In conclusion, whilst the residual risk was reduced considerably in the presence of

unhedged cashflows, as was the case in this thesis, how much more can the residual risk be reduced when a hedged cashflow is used? The question can be a potential area of future research, where one seeks for a hedged cashflow that can minimize the residual risk.

Appendix A. Derivation of Bid-Ask prices

A.1 Conic Theoretical Bid-Ask Prices

A.1.1 Call Bid Formula

$$b_\gamma(C) = \int_K^\infty x d\Psi^\gamma(F_C(x)) dx. \quad (\text{A.1.1})$$

We know that $C = S - K$ and we require $b_\gamma(c) \geq 0$. We recognize that

$$F_C(x) = F_S(S - K) \quad (\text{A.1.2})$$

Substitute in [A.1.1](#) and have a change of variable $x = S - K$

$$b_\gamma(C) = \int_0^\infty (S - K) d\Psi^\gamma(F_S(x)). \quad (\text{A.1.3})$$

Lets denote the random variable S by x to get

$$\begin{aligned}
b_\gamma(C) &= \int_0^\infty (x - K) d\Psi^\gamma(F_S(x)) \\
&= \underbrace{\int_0^K (x - K) d\Psi^\gamma(F_S(x))}_0 \text{ for a call option} + \int_K^\infty (x - K) d\Psi^\gamma(F_S(x)) \\
&= \int_K^\infty (x - K) d\Psi^\gamma(F_S(x)) \\
&\text{Integrate by parts} \\
&= (\Psi^\gamma(F_S(x)) - 1)(x - K)|_K^\infty + \int_K^\infty (1 - \Psi^\gamma(F_S(x))) dx \\
&= \int_K^\infty (1 - \Psi^\gamma(F_S(x))) dx \tag{A.1.4}
\end{aligned}$$

A.1.2 Call Ask Formula

$$a_\gamma(C) = \int_{-\infty}^0 x d\Psi^\gamma(F_{-C}(x)) dx. \tag{A.1.5}$$

From probability theory,

$$F_C(x) = \Pr(-(S - K) < x) = 1 - F_S(K - x), \quad x > 0. \tag{A.1.6}$$

We know that $C = S - K$ and we require that $a_\gamma(C) \geq 0$. We recognize that

$$F_{-C}(x) = F_S(-(S - K)) = 1 - F_S(K - S). \tag{A.1.7}$$

Substitute in [A.1.5](#)

$$\begin{aligned}
a_\gamma(C) &= - \int_{-\infty}^0 x d\Psi^\gamma(1 - F_S(K - x)) \\
&= - \int_0^\infty x d\Psi^\gamma(1 - F_S(K + x)) \\
&= - \left\{ \underbrace{\int_0^K x d\Psi^\gamma(1 - F_S(K + x))}_{0 \text{ for a call option}} + \int_K^\infty x d\Psi^\gamma(1 - F_S(K + x)) \right\} \\
&= - \int_K^\infty x d\Psi^\gamma(1 - F_S(K + x)) \\
&= - \int_K^\infty (x - K) d\Psi^\gamma(1 - F_S(x))
\end{aligned}$$

Integrate by parts

$$\begin{aligned}
&= -\Psi^\gamma(1 - F_S(x))(x - K)|_K^\infty + \int_K^\infty \Psi^\gamma(1 - F_S(x)) dx \\
&= \int_K^\infty \Psi^\gamma(1 - F_S(x)) dx.
\end{aligned} \tag{A.1.8}$$

A.1.3 Put Bid Formula

$$b_\gamma(P) = \int_0^\infty x d\Psi^\gamma(F_p(x)). \tag{A.1.9}$$

We know that $P = K - S$. We recognize that

$$F_p(x) = 1 - F_S(K - S) \quad (\text{A.1.10})$$

Substitute in [A.1.9](#) and have a change of variable $x = K - S$,

$$b_\gamma(P) = \int_0^\infty (K - S) d\Psi^\gamma(1 - F_S(x)). \quad (\text{A.1.11})$$

Denote the random variable S by x .

$$\begin{aligned} b_\gamma(P) &= \int_0^\infty (K - x) d\Psi^\gamma(1 - F_S(x)) \\ &= \int_0^K (K - x) d\Psi^\gamma(1 - F_S(x)) + \underbrace{\int_K^\infty (K - x) d\Psi^\gamma(1 - F_S(x))}_{0 \text{ for a put option}} \\ &= \int_0^K (K - x) d\Psi^\gamma(1 - F_S(x)) \end{aligned}$$

Integrate by parts

$$\begin{aligned} &= -(\Psi^\gamma(1 - F_S(x)) - 1)(K - S)|_0^K + \int_0^K (1 - \Psi^\gamma(1 - F_S(x))) dx \\ &= \int_0^K (1 - \Psi^\gamma(1 - F_S(x))) dx. \end{aligned} \quad (\text{A.1.12})$$

A.1.4 Put Ask Formula

$$a_\gamma(P) = - \int_{-\infty}^0 x d\Psi^\gamma(F_{-P}(x)). \quad (\text{A.1.13})$$

We know that $P = K - S$ and also recognize that

$$F_{-P}(x) = 1 - F_S(-(K - S)) = F_S(K + S). \quad (\text{A.1.14})$$

Substitute in [A.1.13](#) to get

$$\begin{aligned} a_\gamma(P) &= \int_{-\infty}^0 x d\Psi^\gamma(F_S(K + x)) \\ &= - \int_0^\infty x d\Psi^\gamma(F_S(K - x)) \\ &= \int_0^\infty (K - x) d\Psi^\gamma(F_S(x)) \\ &= \int_0^K (K - x) d\Psi^\gamma(F_S(x)) + \underbrace{\int_K^\infty (K - x) d\Psi^\gamma(F_S(x))}_{0 \text{ for a put option}} \\ &= \int_0^K (K - x) d\Psi^\gamma(F_S(x)) \end{aligned}$$

Integrate by parts

$$\begin{aligned} &= \Psi^\gamma(F_S(x))(x - S)|_0^K + \int_0^K \Psi^\gamma(F_S(x)) dx \\ &= \int_0^K \Psi^\gamma(F_S(x)) dx \end{aligned} \quad (\text{A.1.15})$$

A.2 Wang-Transform Theoretical Bid-Ask Prices

A.2.1 Call Bid Formula

$$\begin{aligned}
b_\gamma(C) &= \int_0^\infty x d\Psi^\gamma(F_C(x)) \\
&= \int_K^\infty (x - K) d\Psi^\gamma(F_S(x)) \\
&= \int_K^\infty (x - K) d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= \underbrace{\int_K^\infty x d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right)}_{(i)} \\
&\quad - \underbrace{\int_K^\infty K d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right)}_{(ii)} \tag{A.2.1}
\end{aligned}$$

Now, if $X \sim \text{Lognormal}(\mu, \sigma^2)$ with CDF $F_X(x)$, then

$$\int_K^\infty x dF(x) = e^{\mu+1/2\sigma^2} \Phi \left(\frac{\mu + \sigma^2 - \ln K}{\sigma} \right) \tag{A.2.2}$$

Integral (i) in equation [A.2.1](#) becomes

$$\begin{aligned}
& \int_K^\infty x d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= e^{\ln S_0 + (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T} + 1/2\sigma^2 T} \Phi \left(\frac{\ln S_0 + (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T} + \sigma^2 T - \ln K}{\sigma\sqrt{T}} \right) \\
&= S_0 e^{rT - \gamma\rho\sigma\sqrt{T}} \Phi \left(\frac{\ln(S_0/K) + (r + 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \tag{A.2.3}
\end{aligned}$$

Integral (ii) in equation A.2.1 is the partial integral of the lognormal density and is given by,

$$\begin{aligned}
& \int_K^\infty K d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= K \left(1 - \Phi \left(\frac{\ln(K/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \right). \tag{A.2.4}
\end{aligned}$$

Substituting the results of the two integrals in A.2.1 and multiplying by a discount factor gives:

$$b_\gamma(C) = S_0 e^{-\gamma\rho\sigma\sqrt{T}} \Phi(d_1) - K e^{-rT} (d_2), \tag{A.2.5}$$

$$\text{where } d_1 = \frac{\ln(S_0/K) + (r + 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

A.2.2 Call Ask Formula

$$\begin{aligned}
a_\gamma(C) &= - \int_0^\infty x d\Psi^\gamma(F_{-C_T}(x)) \\
&= - \int_K^\infty (x - K) d\Psi^\gamma(1 - F_{S_T}(x)) \\
&= - \int_K^\infty (x - K) d\Psi^\gamma \left(1 - \Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T}{\sigma\sqrt{T}} \right) \right) \\
&= - \int_K^\infty (x - K) d\Psi^\gamma \Phi \left(\frac{\ln(x/S_0) + (r - 1/2\sigma^2)T}{\sigma\sqrt{T}} \right) \\
&= - \int_K^\infty (x - K) d\Psi^\gamma \Phi \left(\frac{\ln(x/S_0) + (r - 1/2\sigma^2)T}{\sigma\sqrt{T}} \right) \\
&= - \int_K^\infty (x - K) d\Phi \left(\frac{\ln(S_0/x) + (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= \int_K^\infty (x - K) d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= \underbrace{\int_K^\infty x d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right)}_{(i)} \\
&= - \underbrace{\int_K^\infty K d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right)}_{(ii)} \tag{A.2.6}
\end{aligned}$$

Steps as in the derivation of the call bid can be applied here. After applying the continuous discount factor, we get:

$$a_\gamma(C) = S_0 e^{\gamma \rho \sigma \sqrt{T}} \Phi(d_1) - K e^{-rT} (d_2), \quad (\text{A.2.7})$$

$$\text{where } d_1 = \frac{\ln(S_0/K) + (r + 1/2\sigma^2)T + \gamma \rho \sigma \sqrt{T}}{\sigma \sqrt{T}} \text{ and } d_2 = d_1 - \sigma \sqrt{T}.$$

A.2.3 Bid Put Formula

$$\begin{aligned} b_\gamma(P) &= \int_0^\infty x d\Psi^\gamma(F_{P_T}(x)) \\ &= - \int_0^K (K-x) d\Psi^\gamma(1 - F_{S_T}(x)) \\ &= - \int_0^K (K-x) d\Psi^\gamma\left(\Phi\left(\frac{\ln(x/S_0) + (r - 1/2\sigma^2)T}{\sigma \sqrt{T}}\right)\right) \\ &= - \int_0^K (K-x) d\Phi\left(\frac{\ln(S_0/x) + (r - 1/2\sigma^2)T + \gamma \rho \sigma \sqrt{T}}{\sigma \sqrt{T}}\right) \\ &= \int_0^K (K-x) d\Phi\left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma \rho \sigma \sqrt{T}}{\sigma \sqrt{T}}\right) \\ &= \underbrace{\int_0^K K d\Phi\left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma \rho \sigma \sqrt{T}}{\sigma \sqrt{T}}\right)}_{(i)} \\ &\quad - \underbrace{\int_0^K x d\Phi\left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma \rho \sigma \sqrt{T}}{\sigma \sqrt{T}}\right)}_{(ii)} \end{aligned} \quad (\text{A.2.8})$$

Integral (i) in equation A.2.8 gives:

$$\begin{aligned}
& \int_0^K K d\Phi \left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= K \left[\Phi \left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \right]_0^K \\
&= K \Phi \left(\frac{\ln(K/S_0) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right). \tag{A.2.9}
\end{aligned}$$

Integral (ii) in equation A.2.8 gives:

$$\begin{aligned}
& \int_0^K x d\Phi \left(\frac{\ln(S_0/x) - (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= e^{\ln S_0 + (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T} + 1/2\sigma^2 T} \left(1 - \Phi \left(\frac{\ln S_0 + (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T} + \sigma^2 T - \ln K}{\sigma\sqrt{T}} \right) \right) \\
&= S_0 e^{rT + \gamma\rho\sigma\sqrt{T}} \Phi \left(\frac{\ln(K/S_0) - (r + 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \tag{A.2.10}
\end{aligned}$$

Substituting the results of the two integrals in A.2.8 and multiplying by a discount factor gives:

$$b_\gamma(P) = K e^{-rT} (d_2) - S_0 e^{\gamma\rho\sigma\sqrt{T}} \Phi(d_1), \tag{A.2.11}$$

$$\text{where } d_1 = \frac{\ln(K/S_0) - (r + 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 + \sigma\sqrt{T}.$$

A.2.4 Ask Put Formula

$$\begin{aligned}
a_\gamma(P) &= - \int_{-\infty}^0 x d\Psi^\gamma(F_{-P_T}(x)) \\
&= \int_0^K (K-x) d\Psi^\gamma(F_{S_T}(x)) \\
&= \int_0^K (K-x) d\Psi^\gamma\left(\Phi\left(\frac{\ln(x/S_0) + (r - 1/2\sigma^2)T}{\sigma\sqrt{T}}\right)\right) \\
&= \int_0^K (K-x) d\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right) \\
&= \underbrace{\int_0^K K d\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right)}_{(i)} \\
&\quad - \underbrace{\int_0^K x d\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right)}_{(ii)}
\end{aligned} \tag{A.2.12}$$

Integral (i) in equation A.2.12 gives:

$$\begin{aligned}
&\int_0^K K d\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right) \\
&= K \left[\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right) \right]_0^K \\
&= K \Phi\left(\frac{\ln(K/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right).
\end{aligned} \tag{A.2.13}$$

Integral (ii) in equation A.2.12 gives:

$$\begin{aligned}
& \int_0^K x d\Phi \left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \\
&= e^{\ln S_0 + (r - 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T} + 1/2\sigma^2 T} \left(1 - \Phi \left(\frac{\ln S_0 + (r + 1/2\sigma^2)T - \gamma\rho\sigma\sqrt{T} + \sigma^2 T - \ln K}{\sigma\sqrt{T}} \right) \right) \\
&= S_0 e^{rT - \gamma\rho\sigma\sqrt{T}} \Phi \left(\frac{\ln(K/S_0) - (r + 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \right) \tag{A.2.14}
\end{aligned}$$

Substituting the results of the two integrals in [A.2.12](#) and multiplying by a discount factor gives:

$$\begin{aligned}
b_\gamma(P) &= K e^{-rT} \Phi(d_2) - S_0 e^{-\gamma\rho\sigma\sqrt{T}} \Phi(d_1), \tag{A.2.15} \\
\text{where } d_1 &= \frac{\ln(K/S_0) - (r + 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 + \sigma\sqrt{T}.
\end{aligned}$$

Appendix B. Bid-Ask Prices

Numerical Results

B.1 Bull Call Spread Bid-Ask Prices

Table B.1: Bull Call Spread Bid-Ask Prices at Different Stress Levels

| Stess Level | Black-Scholes Model | | | | | VGSSD Model | | | | | |
|-------------|---------------------|-----|-------|-------|--------|-------------|-----|-------|-------|--------|------|
| | S | K | Bid | Ask | Spread | S | K | Bid | Ask | Spread | |
| 0.01 | 140 | 140 | 11.05 | 11.83 | 0.78 | 140 | 140 | 11.04 | 11.77 | 0.73 | |
| | | 141 | 10.56 | 11.32 | 0.76 | | | 141 | 10.54 | 11.25 | 0.71 |
| | | 142 | 10.07 | 10.81 | 0.74 | | | 142 | 10.04 | 10.73 | 0.69 |
| | | 143 | 9.62 | 10.33 | 0.71 | | | 143 | 9.59 | 10.26 | 0.67 |
| | | 144 | 9.16 | 9.85 | 0.69 | | | 144 | 9.15 | 9.79 | 0.64 |
| | | 145 | 8.71 | 9.37 | 0.66 | | | 145 | 8.70 | 9.32 | 0.62 |
| 0.05 | 140 | 140 | 10.50 | 14.59 | 4.09 | 140 | 140 | 10.49 | 14.28 | 3.79 | |
| | | 141 | 10.03 | 14.00 | 3.79 | | | 141 | 10.01 | 13.68 | 0.67 |
| | | 142 | 9.56 | 13.40 | 3.84 | | | 142 | 9.53 | 13.09 | 3.56 |
| | | 143 | 9.09 | 12.80 | 3.71 | | | 143 | 9.07 | 12.52 | 3.45 |
| | | 144 | 8.67 | 12.28 | 3.61 | | | 144 | 8.66 | 12.01 | 3.35 |
| | | 145 | 8.24 | 11.72 | 3.48 | | | 145 | 8.23 | 11.47 | 3.24 |
| 0.10 | 140 | 140 | 9.85 | 18.45 | 8.60 | 140 | 140 | 9.81 | 17.67 | 7.86 | |
| | | 141 | 9.38 | 17.72 | 8.34 | | | 141 | 9.35 | 16.99 | 7.64 |
| | | 142 | 8.93 | 17.03 | 8.10 | | | 142 | 8.92 | 16.35 | 7.43 |
| | | 143 | 8.49 | 16.33 | 7.84 | | | 143 | 8.46 | 15.67 | 7.21 |
| | | 144 | 8.07 | 15.67 | 7.60 | | | 144 | 8.05 | 15.05 | 7.00 |
| | | 145 | 7.68 | 15.05 | 7.37 | | | 145 | 7.67 | 14.46 | 6.79 |

B.2 Bear Call Spread Bid-Ask Prices

Table B.2: Bear Call Spread Bid-Ask Prices at Different Stress Levels

| Stess Level | Black-Scholes Model | | | | | VGSSD Model | | | | |
|-------------|---------------------|-----|------|-------|--------|-------------|-----|------|-------|--------|
| | S | K | Bid | Ask | Spread | S | K | Bid | Ask | Spread |
| 0.01 | 120 | 119 | 6.91 | 7.45 | 0.54 | 120 | 119 | 6.87 | 7.40 | 0.53 |
| | | 120 | 6.40 | 6.90 | 0.50 | | 120 | 6.37 | 6.86 | 0.49 |
| | | 121 | 5.92 | 6.40 | 0.48 | | 121 | 5.88 | 6.35 | 0.47 |
| | | 122 | 5.47 | 5.93 | 0.46 | | 122 | 5.43 | 5.89 | 0.46 |
| | | 123 | 5.02 | 5.45 | 0.43 | | 123 | 5.01 | 5.43 | 0.42 |
| | | 124 | 4.62 | 5.03 | 0.41 | | 124 | 4.61 | 4.61 | 0.41 |
| 0.05 | 120 | 119 | 6.57 | 9.42 | 2.85 | 120 | 119 | 6.55 | 9.34 | 2.79 |
| | | 120 | 6.06 | 8.76 | 2.70 | | 120 | 6.04 | 8.67 | 2.63 |
| | | 121 | 5.61 | 8.16 | 2.55 | | 121 | 5.59 | 8.09 | 2.50 |
| | | 122 | 5.18 | 7.61 | 2.43 | | 122 | 5.17 | 7.53 | 2.36 |
| | | 123 | 4.75 | 7.03 | 2.28 | | 123 | 4.73 | 6.95 | 2.22 |
| | | 124 | 4.35 | 6.50 | 2.15 | | 124 | 4.33 | 6.42 | 2.09 |
| 0.10 | 120 | 119 | 6.16 | 12.25 | 6.09 | 120 | 119 | 6.14 | 12.05 | 5.91 |
| | | 120 | 5.67 | 11.44 | 5.77 | | 120 | 5.66 | 11.28 | 5.62 |
| | | 121 | 5.25 | 10.74 | 5.49 | | 121 | 5.23 | 10.56 | 5.33 |
| | | 122 | 4.81 | 10.00 | 5.19 | | 122 | 4.80 | 9.85 | 5.05 |
| | | 123 | 4.42 | 9.33 | 4.91 | | 123 | 4.41 | 9.19 | 4.78 |
| | | 124 | 3.71 | 8.08 | 4.37 | | 124 | 3.69 | 7.94 | 4.25 |

B.3 Long Call Butterfly Spread Bid-Ask Prices

Table B.3: Butterfly Call Spread Bid-Ask Prices at Different Stress Levels

| Stess Level | Black-Scholes Model | | | | | VGSSD Model | | | | |
|-------------|---------------------|------|------|------|--------|-------------|------|------|------|--------|
| | S | K | Bid | Ask | Spread | S | K | Bid | Ask | Spread |
| 0.01 | 8.00 | 7.50 | 0.73 | 0.78 | 0.05 | 8.00 | 7.50 | 0.71 | 0.76 | 0.05 |
| | | 8.00 | 0.42 | 0.47 | 0.05 | | 8.00 | 0.41 | 0.45 | 0.04 |
| | | 8.50 | 0.23 | 0.25 | 0.02 | | 8.50 | 0.22 | 0.24 | 0.02 |
| 0.05 | 8.00 | 7.50 | 0.69 | 0.95 | 0.26 | 8.00 | 7.50 | 0.68 | 0.93 | 0.25 |
| | | 8.00 | 0.40 | 0.57 | 0.17 | | 8.00 | 0.39 | 0.55 | 0.16 |
| | | 8.50 | 0.21 | 0.32 | 0.11 | | 8.50 | 0.20 | 0.29 | 0.09 |
| 0.10 | 8.00 | 7.50 | 0.65 | 1.19 | 0.54 | 8.00 | 7.50 | 0.64 | 1.17 | 0.53 |
| | | 8.00 | 0.38 | 0.76 | 0.38 | | 8.00 | 0.37 | 0.75 | 0.38 |
| | | 8.50 | 0.20 | 0.45 | 0.25 | | 8.50 | 0.18 | 0.41 | 0.23 |

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