Equity factor timing framework using the Kalman Filter and Gaussian Hidden Markov Model

TP Mashamba
orcid.org/0009-0000-9140-1683

Thesis accepted in fulfilment of the requirements for the degree *Doctor of Philosophy* in Science with Risk Analytics at the North-West University

Promoter: Prof MB Seitshiro
Co-promoter: Dr I Takaidza

Graduation: October 2025

Acknowledgements

I would like to thank my supervisors, Prof MB Seitshiro and Dr I Takaidza, for their guidance and support throughout the course of this research. Their constructive suggestions and feedback have significantly improved the quality of this work.

I am also grateful to my colleagues for their valuable assistance and for creating a supportive working environment.

Finally, I extend my heartfelt thanks to my family for their unwavering support throughout this journey. I appreciate their understanding and encouragement. I could not have accomplished so much without their support.

Declaration

I, Tsumbedzo Pertunia Mashamba, the undersigned, declares that the work entitled, *Equity factor timing framework using the Kalman Filter and Gaussian Hidden Markov Model*, is my own original work and has not been submitted to any other university or institution for examination purposes. All the sources consulted in this work are fully acknowledged and are completely found in the reference section. Furthermore, this research work is submitted in fulfilment of the requirements for the degree, Doctor of Philosophy in Science with Risk Analytics at the Centre for Business Mathematics and Informatics, North-West University (Potchefstroom Campus).

Signature: ______________________ Date: 23 May 2025

Authorship

This thesis is written in article format consisting of three articles, which can be found in Chapters 2, 3 and 4. These articles were written for the purpose of this thesis. The work presented in this thesis is entirely my own. I was responsible for all aspects of the research, including the implementation of algorithms, data analysis and writing of the manuscripts (as the first author). Although I received valuable feedback and guidance from my supervisors, Prof MB Seitshiro and Dr I Takaidza, the work itself is a result of my independent efforts.

1. The first article, *A Comprehensive High Pure Momentum Equity Timing Framework using the Kalman filter and ARIMA Forecasting*, was published in the Data Science in Finance and Economics (DSFE) Journal, 2024, 4(4): 548-569. doi: 10.3934/DSFE.2024023.
   Authors: Tsumbedzo Mashamba, Modisane Seitshiro and Isaac Takaidza

2. The second article, *Evaluating portfolio diversification of large-capitalisation momentum stocks with measures of dependence*, was submitted to the Financial Markets, Institutions & Instruments Journal for publication consideration.
   Authors: Tsumbedzo Mashamba, Modisane Seitshiro and Isaac Takaidza

3. The third article, *A comparison of value and growth investment strategies using the Hidden Markov Model with switching and the Kaplan-Meier method*, was submitted to the Journal of Capital Market Studies for publication consideration.
   Authors: Tsumbedzo Mashamba, Modisane Seitshiro and Isaac Takaidza

4. Chapter 5 applies and compares the factor timing methodologies from the first article and the third article, for the five popular investment factors, namely momentum, value, growth, quality and size.

The promoters gave their consent for the use of these articles as part of the final thesis.

Statement of co-authors

Herewith is a statement of the co-authors giving permission that the articles may form part of this thesis.

I **Prof MB Seitshiro**, hereby approve the articles and give my consent that these articles may be published as part of the thesis for the degree Doctor of Philosophy in Science with Risk Analytics.

Signature: ______________________ Date: 23 May 2025

I **Dr I Takaidza**, hereby approve the articles and give my consent that these articles may be published as part of the thesis for the degree Doctor of Philosophy in Science with Risk Analytics.

Signature: ______________________ Date: 23 May 2025

Abstract

Factor investment strategies aim to capture persistent risk premia, enhancing risk-adjusted returns over the long term. Traditional approaches, however, provide static exposure and fail to adapt to changing market conditions. Factor timing introduces dynamic adjustments to factor exposures based on expected market trends, with the goal of limiting risk, minimising losses and maximising returns by favouring outperforming factors and avoiding underperforming ones. Although factor timing presents opportunities to exploit market anomalies and inefficiencies, existing literature offers mixed evidence on its effectiveness due to several challenges that includes, knowing when to time factor adjustments during market fluctuations, creating reliable trading signals, accounting for transaction costs, evaluating diversification in dynamic portfolios and limiting the influence of manager's skill when implementing a factor timing strategy.

This study addresses these challenges by using the Kalman filter, ARIMA forecasting, the Hidden Markov model, copula-based diversification analysis, and a hybrid fund of funds approach. The focus is mainly placed on the five most popular factors, namely momentum, value, growth, quality, and size. A momentum factor timing strategy entails a portfolio optimisation process for constructing a large-capitalisation pure momentum portfolio. The process includes a dynamic portfolio construction criterion for selecting stocks, estimated from historical data of the United States of America (US) large-capitalisation stocks from January 2013 to June 2023 and South African (SA) large-capitalisation stocks from January 2013 to June 2024. The Kalman filter is applied to assess historical performance, while ARIMA forecasting estimates expected returns and confidence intervals. The mixture copula models are utilised to determine the dependence structure of a pure momentum portfolio. The portfolio is constructed from a population of large-capitalisation stocks, in which the top 20 stocks with the highest average momentum scores are selected. The value versus growth factor investing framework is assessed and analysed using the US exchange-traded funds (ETFs) from January 2013 to December 2024. A fund of funds is composed of the three largest exchange-traded funds ranked by assets under management, listed in the US. The HMM is used for factor timing.

The findings reveal that a dynamic, large-capitalisation momentum portfolio constructed using historical criteria and optimised using a Kalman filter and ARIMA performed well when trading costs were low. Copula analysis revealed that SA momentum portfolios were more diversified than US equivalents. HMM showed value factors recovered faster after market downturns. A combined value and growth fund of funds achieved higher returns than either alone. Comparing Kalman filter and HMM revealed that shorter rebalancing periods improve factor timing effectiveness, suiting active management styles. The HMM outperformed the Kalman filter. It turns out that factor timing can enhance returns by exploiting cyclical factor performance, but effectiveness is constrained by transaction costs and the need for frequent rebalancing. It is recommended that factor timing should complement, rather than replace, static multifactor portfolios. Empirical results show that HMM and Kalman filter methods can theoretically generate profitable trading signals, but practical success depends on cost control and disciplined execution.

Keywords: ARIMA, factor timing equity, growth, hidden Markov model, Kalman filter, mixture copulas, momentum, quality, size, value

Table of abbreviations

A table containing a list of abbreviations that will be used throughout the text.

APT	Arbitrage pricing theory
ARIMA	Autoregressive integrated moving average
AUM	Assets under management
BMV	Book market value
BPS	Basis points
CAPM	Capital asset pricing model
CPI	Consumer price index
D/E	Debt-to-equity ratio
DPS	Dividends per share
DSFE	Data science in finance and economics journal
EMH	Efficient market hypothesis
EPS	Earnings per share
ESG	Environmental, social, and governance
ETF	Exchange-traded fund
FOF	Fund of Fund
GDP	Gross domestic product
HMM	Hidden Markov Model
IPO	Initial public offering
KF	Kalman filter
Large-cap	Large capitalisation companies
Market-cap	Market capitalisation
Mid-cap	Medium capitalisation companies
MMF	Money market fund
MPT	Modern portfolio theory
MSCI	Morgan Stanley capital international
NAV	Net asset value
NAVPS	Net Asset value per share
NYSE	New York stock exchange
P/B	Price-to-book ratio
P/E	Price-to-earnings ratio
PEG	Price/earnings to growth ratio
PM	Portfolio manager
ROA	Return on assets
ROE	Return on equity
SA	South Africa
Small-Cap	Small capitalisation companies
S&P	Standard & Poor’s
US	United States

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In every investment, there is a trade-off between risk and return. The returns over time can be somewhat predictable, even when the market conditions change and investors overreact to circumstances surrounding certain equities (Haddad et al., 2020). An investment is the allocation of funds or resources towards different assets for the future generation of profit or returns (Reilly and Brown, 2002). Investments can be broken down into different types of asset classes, namely equities, bonds, property, cash and others.

- **Equities** are shares in a company that give investors partial ownership to companies they are invested in (Ljungberg and Svedman, 2017). Investors that own shares in a company are referred to as shareholders.
- **Bonds** are issued by governments and corporations to raise capital and they are debt securities or shares, which pay investors monthly interest referred to as coupon rates. The invested capital is fully payable when the bond matures. Unlike equities, bonds do not give the shareholder partial ownership to the company (Andritzky, 2012).
- **Property** investments may include buying of property company shares or buying of physical property for income purposes.
- **Cash** investments include fixed term cash deposit, savings accounts, money market securities, and others.

An investment portfolio is a collection of assets, belonging to one or more asset classes, grouped together with the objective of maximising returns (Cooper et al, 2000). Therefore, asset classes are the basic building blocks for any investment portfolio. The process of selecting and constructing a suitable portfolio taking into account desired returns, risk and diversification, is crucial (Lee, 2011). Every portfolio is constructed to fit a certain risk tolerance level (Hue et al., 2019). This risk can be mitigated by diversifying the portfolio (Fosberg and Madura, 1990). For example, investing in stocks with different industries or asset classes helps reduce risk, further ensuring optimal favourable returns. It is believed that the higher the risk an investor is willing to tolerate the higher the returns and vice versa, the more risk averse the investor is, the lower the returns (Kapoor, 2014; Schoenmaker and Schramade, 2005).

Investors typically have different risk tolerance levels. In modern portfolio theory (MPT) if a risk averse investor has two options of portfolios that offer the same returns with different risk levels, the investor will pick a portfolio with the lower risk (Koedijk et al., 2014; Liu, 2022). Investors prefer efficient portfolios. An efficient portfolio is one that maximises the balance between risk and return (Popescu, 2022). Essentially, investors tend to prefer guaranteed outcomes rather than uncertainty in returns (Phelps, 2024). Risk-averse investors also require liquidity in the market. The liquidity effect is crucial and ensures smooth functionalities of the financial markets. Amihud and Mendelson (1991, 2006) show that expected returns are dependent on the liquidity effect, hence, higher risk aversion is associated with low liquidity. Liquid assets can be quickly sold or bought on the market which creates trust of efficiency, making a publicly traded asset desirable to investors. Naik and Reddy (2021) reviewed the literature on liquidity and found that regulatory policies, trading systems, volatility and corporate governance are some of the factors that play a role in market liquidity, impacting expected returns. Liquid stocks have higher expected returns as compared to their illiquid counterparts (Pástor and Stambaugh, 2003).

1.1 Investment factors

Factors are systematic quantitative characteristics shared among asset classes or among equities. Investment factors are the drivers of investment returns and capture different attributes of risk and return (Bender et al., 2013). Factor investing is a quantitative approach popular for its unique strategy that combines elements from both active and passive portfolio construction and management styles. The most attractive attribute of a factor investing strategy is that it is rule-based and aims to eliminate manager portfolio selection bias (Mainie, 2015).

There are a number of different factors that exist in the literature and for this research the focus will be placed on the following investment factors, namely, momentum, value, growth, quality and size.

- **Momentum** takes into consideration the historical trends of stocks, as documented by Jegadeesh and Titman (1993), where they examined a strategy of buying stocks that performed well in the past and selling stocks that performed poorly. It was found that past winners continued on a winning streak and past losers continued on a losing streak. This factor can be assumed in a sense of Newton's laws of motion. An equity in a state of uniform motion (upward or downward) tends to continue in that state of motion unless an external force (either positive or negative market information) that impacts the price trend is applied to it (Chang et al., 2014). Buying past winners was rewarded with outperformance, which was not attributable to the systematic risk as well as reactions to certain stock factors. The results also imply that there might be a compensation for investors' slow reaction to factor prices, also noting that short-term underreaction and long-term overreactions might play a role in the prices of stocks (Hong and Stein, 1999).
- **Value** is based on the potential returns that a low-priced equity might have from an estimated future value (Bender et al., 2013). Graham and Dodd (1934) laid the foundation for value investing, which involves the process of buying underpriced or undervalued equities. Value investing has evolved significantly but still draws on the foundations presented in that text. The value factor remains one of the most popular factors in investing today. A value strategy is characterised by the practice of buying seemingly under-priced stocks, i.e., trading below perceived intrinsic value (Stagnol et al., 2021).
- **Growth** is focused on the earnings growth potential of a company as opposed to value which is focused on stocks that are undervalued. The growth factor uses metrics that are essential for determining or valuing the growth potential of a business, such as profitability, competitive advantage and business strategy (Damodaran, 2012).

- **Quality** does not have a standard definition but is generally measured by a combination of fundamental characteristics such as company profitability, earnings stability, capital structure, growth, accounting quality, dividend yield, and many more (Hsu et al., 2019; Lepetit et al., 2021).

- **Size** maximises benefits during periods of economic expansion and quantifies the outperformance of small-cap and mid-cap stocks against large-cap stocks in the long-term (Brown, 2017). Banz (1981) looked at the relationship between total market value and returns of stocks. The sample consisted of data from NYSE over a period of 1926 – 1975, and it was found that smaller companies outperformed their larger counterparts, which is now known as the size effect.

1.2 Factor models

Factors have a long history and a number of models were developed to explain the risk and return trade-off relationship. These models capture the cross-section of returns and the influence of each factor within a static multifactor portfolio (Back and Ober, 2025). The most popular and fundamental models are the capital asset pricing model (CAPM), arbitrage pricing theory (APT), Carhart four-factor model and Fama-French five-factor model.

1.2.1 Capital asset pricing model

Sharpe (1964) and Lintner (1965) introduced the capital asset pricing model (CAPM). The CAPM focuses on one factor, the market factor, which drives investment returns. The CAPM is based on the notion that there is a mean-variance linear relationship between expected returns and the market variable, where other variables do not necessarily have an impact on returns.

The CAPM formula is given by

$$E(\delta_i) = \lambda + \beta_i \left(E(\varphi) - \lambda \right), \tag{1.1}$$

where $E(\delta_i)$ is the expected return for asset i, λ is the risk-free rate, β_i is the beta for asset i (A beta greater than one is considered to be riskier than the market), $E(\varphi)$ is the expected market factor return and $E(\varphi) - \lambda$ is the market risk premium.

The CAPM is a foundational model used for building advanced versions of factor models (Fama and French, 2004). As research began uncovering more factors, there was a need for models that could capture additional factors.

1.2.2 Arbitrage pricing theory

The CAPM set the foundation for the Arbitrage Pricing theory (APT). APT was developed by Ross (1976). The APT is an alternative model to the CAPM, and captures multiple systematic factors (Huberman and Wang, 2005).

$$E(\delta_i) = \lambda + \beta_{i1} F_1 + \beta_{i2} F_2 + \cdots + \beta_{in} F_n, \tag{1.2}$$

where $E(\delta_i)$is the expected return of asset i, λ is the risk-free rate, β_{ij} is the sensitivity of asset i to factor j, F_j is the risk premium associated with factor $j = 1, \cdots, n$.

Removing the risk free rate λ results in Equation 1.3 an alternative version of the model.

$$\delta_i - \lambda = \beta_{i1}(F_1 - \lambda) + \beta_{i2}(F_2 - \lambda) + \cdots + \beta_{in}(F_n - \lambda) + \epsilon_i, \tag{1.3}$$

where ϵ_i is the asset-specific risk or the error term.

1.2.3 Fama-French three-factor model

Fama and French (1992) expanded on the earlier CAPM Model with their three-factor model, an asset pricing model that includes size, value, and excess returns on the market.

$$\delta_{it} - \lambda = \alpha_i + \beta_{MKT}(\varphi_t - \lambda) + \beta_{SMB} \cdot SMB_t + \beta_{HML} \cdot HML_t + \epsilon_{it}, \tag{1.4}$$

where δ_{it} is the return of asset i at time t, λ is the risk-free rate, α_i is the intercept, φ_t is the return on the market portfolio at time t, SMB_t is the return spread of small-cap minus large-cap stocks (Size), HML_t is the return spread of high book-to-market minus low book-to-market stocks (Value), and ϵ_{it} is the error term. $\varphi_t - \lambda$ represents the excess returns of the market over the risk-free rate.

1.2.4 Carhart four-factor model

The Carhart (1997) model adds momentum factor onto the Fama-French three-factor model. Carhart introduces the fourth factor which is momentum, which can be used for forecasting future returns using both risk-based and behavioural-based explanations to determine returns.

$$\begin{aligned} \delta_{it} - \lambda = \alpha_i + \beta_{MKT}(\varphi_t - \lambda) + \beta_{SMB} \cdot SMB_t \\ + \beta_{HML} \cdot HML_t + \beta_{UMD} \cdot UMD_t + \epsilon_{it}, \end{aligned} \tag{1.5}$$

where φ_t is the return of the market portfolio and UMD_t is the return of past winners minus past losers (momentum).

1.2.5 Fama-French five-factor model

The Fama and French (2015) model also builds on the earlier three-factor model, which adds two factors: profitability (which uses operating profit where stocks with high operating profitability are subtracted from stocks with low operating profitability) and investment (which is the amount of capital invested in a company for its growth and maintenance).

$$\begin{aligned} \delta_{it} - \lambda = \alpha_i + \beta_{MKT}(\varphi_t - \lambda) + \beta_{SMB} \cdot SMB_t \quad + \beta_{HML} \cdot HML_t \\ + \beta_{RMW} \cdot RMW_t + \beta_{CMA} \cdot CMA_t + \epsilon_{it}, \end{aligned} \tag{1.6}$$

where RMW_t is the robust minus weak (profitability) and CMA_t is the conservative minus aggressive (investment).

Factor models explain returns of static multi-factor portfolios, however, they may not be able to capture the dynamics of returns when market conditions fluctuates (Lleo et al., 2024; Ahmed et al., 2019). Factor models are one dimensional they do not show the hidden observations and breakdown returns into different components. Hence, state space models will show the time-varying breakdown hidden observations of equity returns. Historical investment data constantly switch between recurring hidden states, indicating that some time variations in the parameters exists. State space models are better suited to explain the time varying relationships between observed and unobserved hidden variables.

1.3 State space models

A model of a collection of time-dependent random variables X_t, where $t \epsilon S$ is known as a stochastic process, where X_t is a random variable, t represents the time period and S is the time set of random variables which may be discrete or continuous. If the statistical properties of this process remain constant over time, then the process is known as stationary (Shumway and Stoffer, 2017).

State space models consist of observed random variables and unobserved counterparts known as hidden states. The hidden state within the state space model refers to the part of the data that is unobserved or not directly observable. Figure 1.1 illustrates an example of a general state space model with observed random variable X_t and the hidden states O_t.

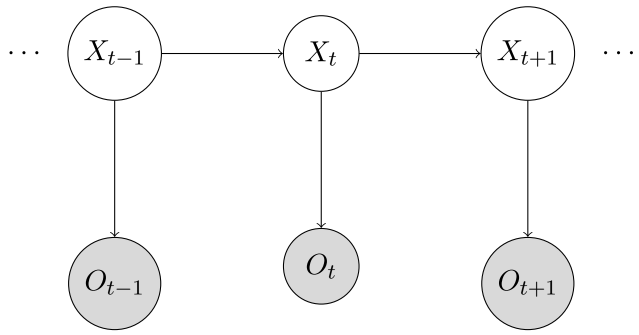


Figure 1.1: Diagram of a state-space model.

1.3.1 Kalman filter

The Kalman filter is a state-space model, which is used to estimate future values, where information is constantly coming into play (i.e. noisy sources), it is useful in filtering, as well as providing more accurate forecasts based on historical data. The Kalman filter is recursive in nature and iterative; it works by estimating the state from noisy measurements. The Kalman filter is explained in more detail in Section 2.3.2.

1.3.2 Hidden Markov Model

Asset prices are dependent on the latest information and as a result exhibit the Markov property, i.e., the probability of future asset prices is not known and may depend on currently available information, which can drive prices either up or down. A Markov process is one where the latest value in the process implies that there are enough details to attempt to forecast future values of the process. The hidden Markov model (HMM) has the Markovian property. The HMM is explained in more detail in Section 4.2.1.

The Kalman filter and the HMM are both state space models, their main difference lies in the nature of the unobserved hidden states. In the Kalman filter the hidden states are continuous and in the hidden Markov model they follow the discrete Markovian chain meaning that the current state only depends on the immediate previous state and not on the entire historical data. State space models are more suitable for generating factor timing signals.

1.4 Factor timing

Portfolios constructed based on factors tend to be superior, for example, Ang et al. (2009), Chow et al. (2011), Amenc et al. (2014) and Bessler et al. (2021) found that portfolios that are focused on factor exposures provide higher returns compared to sector allocated portfolios. A portfolio with multiple factors tends to perform better than a portfolio with just one factor, for example, Lester (2019) found that multifactor portfolios outperformed single factor portfolios, and the correlation between factors decreased as the number of factors increased.

Factor timing is a process in which factor exposure is adjusted for portfolio return enhancement, by identifying periods of expected outperformance or underperformance and adjusting the portfolio allocation of factors (Haddad et al., 2020). Aiming to limit risk, minimise losses and maximise returns by limiting exposures to underperforming factors. This process is done for the purpose of obtaining higher than average returns for both short and long-term horizons. This strategy is not easy especially when factoring in the uncertainty of forecasting returns (Metcalfe, 2018). Factor timing is a challenging endeavour, and while there are many ways to implement it, the strategies can be broadly categorised into three main types, namely predictive or active, reactive and defensive (Neo et al., 2024).

There are numerous factors documented in the literature commonly referred to as the "factor zoo" (Cochrane, 2011). While many of these factors meaningfully contribute to explaining expected returns, a large number of them are redundant (Feng et al., 2020). Therefore, when implementing a factor-based investment strategy, it is crucial to identify the most significant factors (Bartram et al., 2021). Neuhierl et al. (2024) applied a factor timing strategy on more than 300 equity factors by combining multiple predictors using partial least squares and found that factor timing produced significantly higher results compared to static multifactor portfolios. Li et al. (2023) highlights the role of economic policy in the Chinese market and found that economic policy uncertainty can significantly contribute to the success and effectiveness of timing the factor zoo. A high number of factors has lead to an increase in incomparable results across factor timing strategies, making it difficult to compare findings due to the sheer volume of factors involved. Cross-sectional research indicates that factors can interact in complex ways, and differences in factor selection and timing methods contribute to greater variability in factor timing results (Hodges et al., 2017; Blitz, 2023).

Research has produced mixed results on the effectiveness of a factor timing strategy. Some research studies show that factor timing can be an effective strategy. For example, Lehnherr et al. (2025) proposed a shrinkage-based framework for timing equity factors in a high-dimensional setting and found that it produced significant returns when using macroeconomic and factor-specific predictors. Factors tend to be cyclical in nature and Polk et al. (2020) found that multifactor rotation timing strategy performed better compared to a static multifactor strategy. Kagkadis et al. (2024) found that using portfolio characteristics with dimension-reduction techniques improved factor timing accuracy and investment performance compared to traditional predictor-based methods.

Bender et al. (2018) found that factor timing can offer marginal benefits when tailored to the right predictors and horizons, but its success is limited by time-varying relationships and risks of data mining.

While some studies highlight its effectiveness, others argue that it is not a successful strategy. Hence, there remains an ongoing debate about the true efficacy of factor timing. For example, Giamouridis (2017) found that factor timing is challenging and it may be more beneficial to diversify weakly correlated factors. Madhavan et al. (2020) found that factor timing has limited benefits. Ammann et al. (2020) found that the managers attempting factor timing are unsuccessful at timing in general. Boudoukh et al. (2019) found that returns might not be as predictable as indicated by the majority of the literature. Studies such as, Aiken and Kang (2023), Clare et al. (2022), Li (2025) and Chin and Gupta (2020) assessed a manager's skill contribution to factor timing and its implications on returns. When accounting for this skill, the results also vary, further highlighting the mixed evidence regarding the effectiveness of factor timing.

1.5 Problem statement

Factor timing research consists of varying findings and conclusions, leading to a lack of consensus on the effectiveness of a factor timing strategy (Hotze et al., 2024). The main challenges of factor timing include

1. Timing and adjustments of factors during market fluctuations.
2. Developing reliable trading signals.
3. Accounting for the effects of transaction costs on factor timing strategies.
4. Diversification assessment of dynamic factor portfolio construction.
5. Manager's skill to time factors can be a contributing factor to a successful factor timing strategy.

Equity factor timing strategies seek to enhance the traditional smart beta or factor investment strategies. Another significant challenge is the large number of factors documented in the literature, which limits the comparability of factor timing results. The focus is placed on the five most widely recognised factors, namely momentum, value, growth, quality, and size. These challenges are addressed by:

1. Using the Kalman filter to assess historical performance, developing trading signals as well as adding the ARIMA forecasting methodology to estimate the expected performance and the confidence intervals. An iterative process that can incorporate new information as it becomes available and further enhance the monitoring and rebalancing process of a factor portfolio. This adaptive approach enables the portfolio to capitalise on time-varying return anomalies as they occur.
2. Assessing the transaction costs when implementing a Kalman filter factor timing strategy.
3. Using the hidden Markov model with switching and the Kaplan-Meier method to identify the hidden states from historical daily returns and identify the persistence in days of the hidden states. This provides a length in time in which a factor will produce low returns, raising an opportunity to time factors during periods of underperformance.

4. Using Measures of association, namely mixture copula models to determine the association and dependence of stocks within a factor portfolio to assess the diversification. This shows the behaviour stocks within a factor portfolio in extreme conditions.

5. Using funds of funds to provide a hybrid approach to factor timing which reduces the effects and influence of a manager's skill to time factors.

1.6 Aim, objectives and research questions

1.6.1 Aim

To develop an effective equity factor timing framework using the Kalman filter and the HMM. Furthermore, to compare the Kalman filter and the HMM strategies and determine which state space model produces better results.

1.6.2 Objectives

The objectives of the study are:

i. To identify existing equity factor timing strategies in literature.

ii. To develop a factor timing framework using the Kalman filter and the HMM.

iii. To compare the Kalman filter and the HMM factor timing frameworks, for momentum, value, growth, quality and size factors.

iv. To determine which state space model factor timing strategy produces higher returns.

1.6.3 Research questions

i. What are existing factor timing strategies?

ii. What effects will a factor timing strategy have on portfolio returns using Kalman filter and the HMM state space models?

iii. Can factor timing be successfully done by state space models?

1.7 Data collection and analysis

This research used publicly available data collected from the Yahoo finance database. This data will be helpful in providing practical investment universes (Samples) to be analysed even further as compared to taking the data of the entire market (population).

Table 1.1: Summary of data sources and periods used per chapter

Chapter	Type and description of data	Data range for factor timing
Chapter 2	This chapter makes use of large capitalisation stocks listed on United States stock exchanges. No specific exchange (e.g., NYSE or NASDAQ) is isolated; rather, the focus is on the market capitalisation of the stocks found on the Yahoo database.	01 January 2013 to 30 June 2023
Chapter 3	A comparative study is conducted between large capitalisation stocks from the United States (No specific exchange) and those listed on the Johannesburg stock exchange (JSE) in South Africa.	01 January 2013 to 30 June 2024
Chapters 4 and 5	These chapters focus on US-listed exchange-traded funds (ETFs). These ETFs are chosen based on their assets under management (AUMs)	01 January 2013 to 31 December 2024

1.8 Ethical considerations

The research proposal study was submitted to the Faculty of Natural and Agricultural Sciences Research Ethics Committee (FNASREC) at the North-West University for clearance and compliance with the ethical standards of academic research and received clearance to proceed. This research will be conducted using secondary data, which is publicly available to develop an equity factor timing framework.

1.9 Outline of thesis

The thesis consists of the following chapters.

- **Chapter 1**: Introduction
  This chapter presents the introduction, factor definitions and outlines the aim and objectives.
- **Chapter 2**: A comprehensive high pure momentum equity timing framework using the Kalman filter and ARIMA forecasting.
  This chapter presents the momentum factor timing results using the Kalman filter and ARIMA forecasting
- **Chapter 3**: Evaluating portfolio diversification of large-cap momentum stocks with measures of dependence.
  This chapter evaluates the momentum large-cap portfolio diversification using the measures of association.
- **Chapter 4**: A comparison of value and growth investment strategies using the HMM with switching and the Kaplan-Meier method.
  This chapter compares the value and growth factors and factor timing using the HMM.

- **Chapter 5**
  This chapter compares the Kalman filter and HMM investment factor timing strategy results.
- **Chapter 6: Conclusion**
  This chapter provides the conclusions and recommendations.

References

[1] Ahmed, S., Bu, Z., and Tsvetanov, D. (2019). Best of the Best: A Comparison of Factor Models. The Journal of Financial and Quantitative Analysis, 54(4), 1713–1758.

[2] Aiken, A.L. and Kang, M. (2023). Hedge fund manager timing and selectivity skill over time, A holdings-based estimate. *Finance Research Letters*, 58.

[3] Ammann, M., Fischer, S., and Weigert, F. (2020). Factor Exposure Variation and Mutual Fund Performance. *Financial Analysts Journal*, 76(4), 101–118.

[4] Amenc, N., Goltz, F., Lodh, A., and Martellini, L. (2014). Towards Smart Equity Factor Indices: Harvesting Risk Premia without Taking Unrewarded Risks. *Journal of Portfolio Management*. 40(4), 106-22.

[5] Amihud, Y., and Mendelson, H. (1991). Liquidity, Asset Prices and Financial Policy. *Financial Analysts Journal*, 47(6): 56–66.

[6] Amihud, Y., and Mendelson, H. (2006). Stock and bond liquidity and its effect on prices and financial policies. *Financial Markets and Portfolio Management*, 20(1), 19–32.

[7] Andritzky, J.R. (2012). Government Bonds and Their Investors: What Are the Facts and Do They Matter? (I. F. Department, Ed.) *IMF Working Paper No. 2012/158*.

[8] Ang, A., Goetzmann, W. N., and Schaefer, S. M., (2009). Evaluation of Active Management of the Norwegian Government Pension Fund—Global. *Norwegian Ministry of Finance*.

[9] Back, K. and Ober, A. (2025). Performance of factor models in a simple economy. SSRN.

[10] Banz, R. W. (1981). The Relationship between Return and Market Value of Common Stocks. *Journal of Financial Economics*, 9, 3-18.

[11] Bender, J., Briand, R., Melas, D., and Subramanian, R. A. (2013). Foundations of Factor Investing. *MSCI Research Insight*, 1-33.

[12] Bender, J., Sun, X., Thomas, R. and Zdorovtsov, V. (2018). The promises and pitfalls of factor timing. The Journal of Portfolio Management, 44(4, Quantitative Special Issue), 79–92.

[13] Bessler, Wo., Taushanov, G., and Wolff, D. (2021). Factor investing and asset allocation strategies: a comparison of factor versus sector optimization. *Journal of Asset Management*, 22, 488-506.

[14] Blitz, D. (2023). The cross-section of factor returns. Robeco White Paper. SSRN.

[15] Boudoukh, J., Israel, R. and Richardson, M. (2019). Long-horizon predictability: a cautionary tale. Financial Analysts Journal. 75(1), 17–30.

[16] Brown, M. R. (2017). For Style Factors, One Size Does Not Fit All. *The Journal of Investing*, 26(4), 127-137.

[17] Carhart, M.M. (1997). On persistence in mutual fund performance. *Journal of Finance*, 52(1), 57-82.

[18] Chang, W., Bell, B., and Jones, A. (2014). Historical development of Newton's laws of motion and suggestions for teaching content . *Asia-Pacific Forum on Science Learning and Teaching*, 15(1).

[19] Chin, A. and Gupta, P. (2020). Timing is Not Everything—Assessing Manager Skill in Factor Timing. *Journal Of Investment Management*, 18(1), 34–51.

[20] Chow, T., Hsu, J., Kalesnik, V., and Little, B., (2011). A Survey of Alternative Equity Index Strategies. *Financial Analysts Journal*, 67(5), 37–57.

[21] Clare, A., Sherman, M., O'Sullivan, N., Gao, J. and Zhu, S. (2022). Manager characteristics: Predicting fund performance. *International Review of Financial Analysis*, 80.

[22] Cochrane, J. H. (2011). Discount rates. *National Bureau of Economic Research*, NBER Working Paper No. 16972

[23] Cooper, R.G., Edgett, S.J., and Kleinschmidt, E.J. (2000). New Problems, New Solutions: Making Portfolio Management More Effective. *Research Technology Management*, 43(2), 1-27.

[24] Damodaran, A. (2012). Growth Investing: Betting on the Future? *SSRN*.

[25] Fama, E.F. and French, K.R. (1992). The cross-section of expected stock returns. *The Journal of Finance*, 47(2), 427-465.

[26] Fama, E.F., and French, K.R. (2004). The Capital Asset Pricing Model. *Journal of Economic Perspectives*, 18(3), 25-46.

[27] Fama, E.F. and French, K.R. (2015). A five-factor asset pricing model. *Journal of Financial Economics*, 116(1), 1-22.

[28] Feng, G., Giglio, S., and Xiu, D. (2020). Taming the factor zoo: A test of new factors. The Journal of Finance, 75(3), 1327–1370.

[29] Fosberg, R. H., and Madura, J. (1990). Risk Reduction Benefits from International Diversification: Journal of Multinational Finance Management, 1(1), 35–42.

[30] Giamouridis, D. (2017). Systematic Investment Strategies. *Financial Analysts Journal*, 73(4), 10–14.

[31] Graham, B., and Dodd, D. L. (1934). Security Analysis: The Classic 1934 Edition. *McGraw-Hill*.

[32] Haddad, V., Kozak, S. and Santosh, S. (2020). Factor Timing. *National Bureau of Economic Research*, NBER Working Paper No. 26708.

[33] Hodges, P., Hogan, K., Peterson, J. R. and Ang, A. (2017). Factor timing with cross-sectional and time-series predictors. The Journal of Portfolio Management, 44(1), 30–43.

[34] Hong, H. and Stein, J. C., (1999). A Unified Theory of Underreaction, momentum trading, and overreaction in Asset markets. *The Journal of Finance*, 54(6), 2143-2184

[35] Hotze, S., Hachenberg, B. and Schiereck, D. (2024). Factor Timing in Asset Management: A Literature Review. Credit and Capital Markets, 57(1–4), 107–156.

[36] Hsu, J., Kalesnik, V., and Kose, E. (2019). What Is Quality? *Financial Analysts Journal*, 75(2), 44-61.

[37] Huberman, G., and Wang, Z. (2005). Arbitrage Pricing Theory. *Federal Reserve Bank of New York Staff Reports*(Staff Report no. 216).

[38] Hue, B., Jinks, A., Spain, J., Bora, M. and Siew, S. (2019). Investment risk for long-term investors: risk measurement approaches – Considerations for pension funds and insurers. *British Actuarial Journal*, 24(16), 1-52.

[39] Jegadeesh, N. and Titman, S. (1993). Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency. T*he Journal of Finance*, 48(1), 65-91.

[40] Kagkadis, A., Nolte, I., Nolte, S. and Vasilas, N. (2024). Factor timing with portfolio characteristics. The Review of Asset Pricing Studies, 14(1), 84–118.

[41] Kapoor, N. (2014). Financial Portfolio management: Overview and Decision Making in investment Process. *International Journal of Research*, 1(10), 1362-1370.

[42] Koedijk, C.G., Slager A.M.H., Stork, P.A., (2014). Factor Investing in Practice: A Trustees' Guide to Implementation . *Research Report for Robeco*.

[43] Lee, W. (2011). Risk-Based Asset Allocation. *The Journal of Portfolio Management*, 37(4), 11-28.

[44] Lehnherr, R., Mehta, M., and Nagel, S. (2025). Optimal Factor Timing in a High-Dimensional Setting. *Financial Analysts Journal*, 1–16.

[45] Lepetit, F., Cherief, A., Ly, Y., and Sekine, T. (2021). Revisiting Quality Investing, . *SSRN*.

[46] Lester, A. (2019). On the Theory and Practice of Multifactor Portfolios. *The Journal of Portfolio Management Quantitative Special Issue*, 45(3), 87-100.

[47] Li, K. (2025). Market timing and managerial talent. Asia-Pacific Financial Markets.

[48] Li, Z., Wan, Y., Wang, T., and Yu, M. (2023). Factor-timing in the Chinese factor zoo: The role of economic policy uncertainty. Journal of International Financial Markets, Institutions and Money, 85, 101782.

[49] Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. The Review of Economics and Statistics, 47(1), 13–37.

[50] Liu, Y. (2022). Application of modern portfolio theory in stock market. In Proceedings of the 2022 7th International Conference on Financial Innovation and Economic Development (ICFIED 2022), 2653-2658.

[51] Ljungberg K., and Svedman S., (2017). The Private Equity Investment Process - An Aggregated View on Information. *(D. o. Studies, Ed.) Uppsala University.*

[52] Lleo, S., Ziemba, W. T., and Li, J. (2024). The Capital Asset Pricing Model and its Modern Applications. The Journal of Portfolio Management, 50(3), 111–131.

[53] Madhavan, A., Sobczyk, A. and Ang, A. (2020). Alpha vs. alpha: selection, timing, and factor exposures from different factor models. *The Journal of Portfolio Management*, 46(5), 90–103.

[54] Mainie, S. (2015). The Story of Factor-Based Investing. *S&P Dow Jones Indices Research Paper.*

[55] Metcalfe, G. (2018). The mathematics of market timing. *PLoS ONE*, 13(7).

[56] Naik, P., and Reddy, Y. V. (2021). Stock market liquidity: A literature review. *SAGE Open*, 11(1), 1–15.

[57] Neo, P. L., Tee, C. W. and Kerkhof, J. (2024). Universal return and factor timing. Singapore Management University. 1–22.

[58] Neuhierl, A., Randl, O., Reschenhofer, C. and Zechner, J. (2024). Timing the factor zoo. SSRN.

[59] Pástor, L., and Stambaugh, R. F. (2003). Liquidity Risk and Expected Stock Returns. *Journal of Political Economy*, 111(3), 642–685.

[60] Phelps, C.E. (2024). A user's guide to economic utility functions. *Journal of Risk and Uncertainty*, 69, 235-280.

[61] Polk, C., Haghbin, M., and de Longis, A. (2020). Time-Series Variation in Factor Premia: The Influence of the Business Cycle. *Journal Of Investment Management*, 18(1), 69-89.

[62] Popescu, A.D., (2022). Efficient frontier in portfolios containing stock market and financial digital assets. *Annals of the University of Craiova*, 73, 229-240.

[63] Reilly, F.K. and Brown, K.C. (2002). Investment Analysis and Portfolio Management (10th ed.) *South-Western, Cengage Learning*.

[64] Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13(3), 341-360.

[65] Schoenmaker, D., Schramade, W. (2023). Risk-Return Analysis. In: Corporate Finance for Long-Term Value. Springer Texts in Business and Economics. *Springer*, 325-366

[66] Shumway, R. H. and Stoffer, D. S. (2017). Time Series Analysis and Its Applications: With R Examples (4th ed.). *Springer*.

[67] Sharpe, W. F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 19(3), 425–442.

[68] Stagnol, L., Lopez, C., Roncalli, T., and Taillardat, B., (2021). Understanding the Performance of the Equity Value Factor. *SSRN*.

Chapter 2

A Comprehensive High Pure Momentum Equity Timing Framework using the Kalman filter and ARIMA Forecasting

Abstract

The pursuit of higher returns has led to a growing interest in factor timing as a strategy to enhance portfolio returns. Momentum is a popular factor, which involves buying securities that have shown consistent price appreciation over the past 3 to 12 months or past few years, with the expectation that the trend will continue and reducing exposure to those that consistently declined. An important part of a factor timing strategy is in the portfolio optimisation process. This article aims to first construct a large capitalisation pure momentum portfolio, which includes a dynamic stringent portfolio construction process criteria for selecting stocks estimated from historical data. Secondly, as part of the portfolio's risk management strategy, adding the Kalman filter to the historical performance of this portfolio. Lastly, using the ARIMA forecast to estimate expected performance and the confidence intervals. The empirical results show that this pure equity momentum factor timing framework with the Kalman filter together with the ARIMA forecasting methodology is iterative and can incorporate new information as it becomes available and further enhance the monitoring and rebalancing process. This adaptive approach enables the portfolio to capitalise on time-varying return anomalies as they occur.

Keywords

Factor investing, factor timing, investment style factors, Kalman filter, momentum, return predictability.

2.1 Introduction

Momentum is a popular factor, which involves buying securities that have shown consistent price appreciation over the past 3 to 12 months or past few years, with the expectation that the trend will continue and reducing exposure to those that consistently declined. This factor takes into consideration the historical trends of stocks, as documented by Jegadeesh and Titman (1993), where they examined a strategy of buying stocks that performed well in the past using a sample that spanned over a period of 24 years (from 1965 to 1989) and found that past winners continued on a winning streak and past losers continued on a losing streak; buying past winners was rewarded with outperformance which was not attributable to systematic risk and investors' reactions to certain stock factors. Their results also imply that there is a compensation for investors' slow reaction to factor prices, where short-term underreaction and long-term overreactions might play a role in the prices of stocks, especially in the short-term (Hong and Stein, 1999).

Factors have a long history and have evolved since the introduction of the capital asset pricing model (CAPM) by William Sharpe (1964) and John Lintner (1965), where it was believed that only the market factor was the driver of investment returns. The CAPM set the groundwork for further developments of factor models, which later led to the arbitrage pricing theory (APT) by Stephen Ross in 1976. The APT is an alternative model to the CAPM, and it is one of the foundational models which showed that investment returns could be modelled by more than one factor, due to the existing linear relationship between expected investment returns and other response variables (factors) (Huberman and Wang, 2005). This further led to the Fama and French (1992) three-factor model, an expansion of the earlier CAPM, which included size, value, and market excess returns. Carhart (1997) further expanded on the Fama and French three-factor model by adding momentum as a fourth factor.

Timing strategies aim to limit risk, minimise losses and maximise returns by limiting exposures to certain stocks with the consideration of future market trends. Treynor and Mazuy (1966) conceptualised the idea of market timing, where they attribute a manager's skill to be a contributing factor to the success, or lack thereof, of obtaining favourable returns. The expectations of the market factor increasing or decreasing would lead to the manager adjusting the portfolio positions by either increasing exposure or decreasing exposure based on market expectations. They included 57 mutual funds in the study, and ultimately found that at the time, none of the funds were successful at outperforming the market. The main difference between market and factor timing is that a market timing strategy includes the overall market trend analysis, and factor timing is rule based and focuses on a portfolio's factor exposure.

Equity factor timing aims to potentially increase returns by buying or holding equity factors that are expected to outperform and selling those that are expected to underperform, essentially taking advantage of market anomalies and inefficiencies. This is an important part of factor investing, presenting unique opportunities to generate alpha. Though timing can be a challenging endeavour, in this article the Kalman filter is applied as an iterative aspect of the process.

This article starts with the introduction of momentum and factor timing, followed by factor timing overview in Section 2.2. Methodology is in Section 2.3, the framework is in Section 2.4, the empirical results are in Section 2.5, the transaction costs are in Section 2.6, and finally, the conclusion is in Section 2.7.

2.2 Factor timing overview

There is a growing interest in factor timing as a strategy to enhance portfolio returns. This is supported by a growing level of research consisting of varying opinions and findings, where some find factor timing to be an effective strategy and others find it does not produce abnormal returns, especially when faced with high transaction costs and market movements amongst other factors that impact returns. Traditional momentum strategies may not yield the desired returns, and after Gupta and Kelly (2019) studied momentum extensively to prove that factors can be timed based on recent historical performance, they found that adding factor timing as a strategy was beneficial with positive returns. Further comparing the returns from a traditional momentum strategy showed that adding time series momentum factor timing into a portfolio as a strategy also yielded better results. Momentum equity factor timing is based on the assumption that during market upswings this strategy will outperform, enhance portfolio returns, and reduce risk.

A portfolio manager's skill can be attributed to returns, leading to the assumption that skill is also an important aspect of a successful factor timing strategy. For example, Osinga et al. (2020), found that hedge fund managers are better suited for incorporating factor timing as a strategy, which is the main contributor to positive factor timing returns. Daniel et al. (1997) analysed returns from 125 passive portfolios in comparison to their respective benchmarks and found that factor timing was unsuccessful and it might not be a strategy that could be applied successfully to reach optimal results for passive funds, noting that the manager skill would be the main contributing factor if funds produce desired results. Aiken and Kang (2023) used a holdings-based approach to assess the impact a manager's skill might have on the performance of a fund. This was used for portfolio selection and factor timing skills, and found that the manager's skill plays an important role in the generation of alpha. Clare et al. (2022) found that experience may be a contributing factor to timing returns, as managers that combine experience with skill are better equipped to bring positive portfolio returns than managers who rely solely on skill. Another contributing factor is that a manager with a stake in the fund tends to be more careful when managing the fund, as this will directly affect their income. Hence, skill with experience is much better than skill without much experience.

However, an overreliance on a manager's skill might be detrimental and will have the opposite results, as shown by Drew et al. (2005), where the portfolio selection process, accuracy of forecasting future returns and their factor timing abilities were assessed. The reliance on managers' skills produced disappointing results, and might not be beneficial to the general investor. Forecasting based on past performance is not the best way to estimate future returns, especially if the time period is short. The relationship between portfolio construction and stock selection process does not result in a successful factor timing strategy, implying that the manager's stock selection ability is not directly linked to a successful factor timing strategy. Hedge funds are different from passive investments because they are actively managed, whereas, passive investment strategies have restrictive rules compared to active management, especially how frequently trading can occur within a portfolio. Davies et al. (2019) took a passive investment equity momentum and value factor strategies approach, which are widely popular especially for the ease of accessibility and affordability, and found that the Sharpe ratio decreased significantly for both momentum and value strategies within the passive investment space. Due to the rules in place, a passive factor investing strategy may perform differently and may even underperform as opposed to an active factor investing strategy for value and momentum, which may outperform and provide a higher Sharpe ratio. Fergis et al. (2019) suggested a framework which might be more beneficial for passive funds which intends to maintain and reserve capital in periods such as recessions and market downturns and use methods of factor diversification to minimise the risk of being exposed to certain factors. In a multifactor portfolio market, signals can be used for a successful defensive factor timing strategy, which is more passive as opposed to most proposed factor timing strategies. In order to successfully apply a defensive factor timing strategy, there has to be a number of measurements such as the level of risk that can be tolerated and a factor diversification strategy. Zheng et al. (2024) showed another approach to factor timing, which relies on sentiment, however, executing a momentum sentiment timing is challenging as momentum is rule based and requires more than sentiment for it to work as a strategy. A timing strategy works because of the predictability of factors, implying that factor timing can be added to factor exposed funds as a strategy, adding that expected factor premiums may be due to the reward of systematic risk (Souza, 2020).

Momentum outperformance can, for example, be attributed to the industry of stocks. Moskowitz and Grinblatt (1999) found that industry-specific momentum showed significant differences in returns and improved profitability, and this might be used to explain persistence in momentum strategies returns anomalies. George and Hwang (2004), on the other hand, found that momentum investing strategies' outperformance can be attributed to the 52-week high price that takes into account the performance of underlying stocks, which also plays a large role in the forecasting results of momentum returns. A momentum strategy can be impacted by behavioural biases and market sentiment, which persists during market downturns (Karki and Khadka, 2024). Short-term momentum can be subject to reversals and affect the short-term performance of a momentum investment strategy, overall, this strategy tends to persist (Huang et al., 2023).

There are many factor timing strategies and frameworks that can be used, such as a factor rotation strategy, which aims to maximise the benefits of factor exposure. Kwon (2022) found that there were exploitable return differences in equity factors when combined with economic factor analysis, which may be used as a factor timing strategy by rotating the factors in accordance with economic outlooks. Aked (2021) used three methods, namely, historical returns, economic cycle, and factor discount with momentum, and found that information within the economic cycle was already reflected in the factor discount and momentum. Factor's discount and momentum strategies should provide better limited returns, and there is a greater need for continuous improvement of forecasting methodologies of future returns which remains a great challenge in investment management. Chin and Gupta (2020) set out a framework that seeks to assess a factor timing strategy that can attribute and contribute to returns by taking the difference between long- and short-term factor investing strategies, where the significant contributor to returns was the stock selection process. However, ultimately a factor timing strategy failed to outperform.

The main question might be this: can factor timing be used in practice? Asness et al. (2018) attempted to answer this question using momentum, value, and style premia, which showed that though factors are expensive, they are not as expensive as they used to be, and that factor timing cannot not be justified as a strategy as there is still lack of substantial evidence to prove that it can outperform. The value factor showed some promise if it is within a single factor portfolio as opposed to a multifactor portfolio.

Time series predictors and cross-sectional tilting presents some benefits, however, the benefits of fundamental and technical time series predictability are not enough to offset the transaction costs required for a factor timing strategy, which makes cost effectiveness difficult (Dichtl et al., 2019). Asness et al. (2017) examined a value factor timing strategy, noted that a basic performance forecasting method might be enough to predict future returns and an attempt to outperform a traditional passive buy and hold strategy is not easy and furthermore attempted to answer the question of whether a factor timing strategy will be beneficial or detrimental to returns. There are challenges to value investing as it is not enough for forecasting and timing of the market. Further cautions against the simplicity of ex-post contrarian which might be applied in a factor timing strategy, and concludes that a value timing strategy will be improved by adding a momentum component into it, which produces minimal performance levels.

2.3 Methodology

2.3.1 Portfolio construction, diversification and monitoring

One of the most commonly used portfolio construction methods is the Markowitz (1952) mean-variance optimization model, in which a risk averse investor is inclined to expect compensation for the risk taken. Expected returns can be optimised by diversification, in a momentum equity portfolio construction process of selecting stocks and allocating a percentage to achieve the stipulated investment objectives with a specified risk tolerance. Diversification of a momentum equity portfolio involves spreading exposure across different industries for risk reduction.

Suppose a momentum portfolio only has two stocks with weights w_1 and w_2. The portfolio will have a variance as,

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2 \cdot w_1 w_2 \sigma_{12}, \tag{2.1}$$

where σ_p^2 is the variance of the portfolio and σ_{12} is the covariance between the returns of assets 1 and 2. This portfolio will be considered well-diversified if the two stocks have a low correlation coefficient estimated by:

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}, \tag{2.2}$$

where ρ_{12} is the correlation coefficient between the returns of assets 1 and 2, $-1 < \rho_{12} < 1$. This two stock portfolio will have a diversification benefit (DB), where there is a reduction in portfolio risk, given by:

$$DB = \sigma_1 + \sigma_2 - \sigma_p. \tag{2.3}$$

To monitor a momentum equity only portfolio, returns will have to be estimated on a regular basis, together with the standard deviation and Sharpe ratio (i.e., a measure for the risk-adjusted performance), as well as the tracking error which is the difference between the portfolio return and the benchmark return. Included as well is the information ratio to estimate the active return of the portfolio compared to the benchmark. The portfolio return is given by:

$$R_p = \sum_{i=1}^{n} w_i R_i, \tag{2.4}$$

where R_p is the portfolio return, w_i is the weight of the i-th asset in the portfolio, and R_i is the return of the i-th asset. The Sharpe ratio is defined as,

$$SR = \frac{R_p - R_f}{\sigma_p}, \tag{2.5}$$

where R_p is the portfolio return, R_f is the risk-free rate, and σ_p is the standard deviation of the portfolio return. The tracking error is defined by:

$$TE = \sqrt{\text{Var}(R_p - R_b)}, \tag{2.6}$$

where R_b is the benchmark return. The information ratio is defined by:

$$IR = \frac{R_p - R_b}{TE}. \tag{2.7}$$

The portfolio variance on its own is limited and there are a number of different ways in which risk can be estimated. Some of the ways to estimate risk includes the downside risk, shortfall probabilities, value at risk (VaR), conditional value at risk (CVaR), and the maximum drawdown.

Although investors are interested in the balance between risk and return in most cases, they are more interested in the maximum value they could lose, which is the downside risk. The downside risk is also known as the semi-variance defined by:

$$SV = \int_{-\infty}^{\mu} (r-\mu)^2 f(r)dr, \tag{2.8}$$

where SV is the semi-variance of return, μ is the expected return, r is the return on the investment, and $f(r)$ is the probability density function of the return. In some cases, the level of tolerable losses is specified to estimate the shortfall risk, which is the probability that the loss will fall below the specified benchmark level (L). In this case an investor will not accept a risk that the loss will fall below what has been specified. The shortfall probability is given by:

$$SP = P(r < L) = \int_{-\infty}^{L} f(r)dr, \tag{2.9}$$

where $f(r)$ is the probability density function of portfolio returns and r is the return on the investment. The VaR is the maximum loss of a portfolio at a particular confidence level and it is defined by:

$$\text{VaR}_\alpha = \inf\{r \in \mathbb{R} : F(r) \geq \alpha\}, \tag{2.10}$$

where α is the significance level. The CVaR is the expected value of losses exceeding the VaR defined by:

$$\text{CVaR}_\alpha = \frac{1}{1-\alpha} \int_{-\infty}^{-\text{VaR}_\alpha} f(r)\, dr, \tag{2.11}$$

The maximum drawdown (MD) is the difference between the peak and trough before the next peak occurs defined by:

$$MD = \frac{\text{Peak} - \text{Trough}}{\text{Peak}} \times 100, \tag{2.12}$$

where the peak is the highest point and trough is the lowest point.

2.3.2 Kalman filter and ARIMA forecasting

The Kalman filter was developed by Rudolf E. Kalman in 1960, where the Wiener problem was formulated and solved to obtain the attributes of a linear state of a system where vector spaces were taken into consideration. The results were a set of equations that are stochastic optimal estimators, iterative and recursive in nature, which are used to estimate the state of a system. New information can be incorporated into the model as it becomes available, making it better suited for a factor timing strategy. An investment in a state space dynamic may be affected by new information which affects returns causing them to significantly vary with time, where the state space evolves in line with the discrete time stochastic model. The autoregressive integrated moving average model (ARIMA) is a time series model, which uses historical data to forecast future values. The Kalman filter and ARIMA model are both used for prediction, and combining them improves prediction accuracy. Using the mean square error estimation method found in Shumway and Stoffer (2017), the state transition from time t to $t+1$ is given by:

$$A_{t+1} = \varphi A_t + \theta_t, \tag{2.13}$$

where A_t is the state vector A at time t i.e., the state vector A_t represents the current state of the momentum factor and can also be considered to be a simple autoregressive time series model (AR(1)). φ is the state transition matrix from time t to $t+1$. θ_t is the Gaussian white noise process with a known covariance matrix. The observation equation is given by

$$B_t = HA_t + \delta_t, \tag{2.14}$$

where B_t is the actual measurement of A at time t. H is stationary and does not contain noise, as well as shows the relationship between the state vector and the measurement vector. δ_t is the Gaussian white noise process with a known covariance matrix. θ_t and δ_t are independent of each other. B_t is the observed momentum factor values or returns, and H represents the relationship between the observed momentum factor and the underlying momentum state.

$$\zeta = \mathbb{E}[\theta\theta^T]_t. \tag{2.15}$$

$$\xi = \mathbb{E}[\delta\delta^T]_t, \tag{2.16}$$

where ζ and ξ are the uncertainty, i.e., volatility in the state transition, which captures the uncertainty in how momentum evolves over a period of time.

$$P_t = \mathbb{E}[\eta_t\eta_t^T] = \mathbb{E}[(A_t - \hat{A}_t)(A_t - \hat{A}_t)^T], \tag{2.17}$$

where the covariance matrix of the estimation error quantifies the uncertainty in the estimated state. This is related to the mean squared error, the expected value of the squared estimation error is equal to the error covariance matrix. The covariance matrix quantifies the uncertainty in the state estimates. The mean square error is essential in quantifying the uncertainty in the estimated state and is crucial in the update and prediction steps of the filter. The update step of the Kalman filter, where the prior estimate is combined with the measurement data is given by:

$$\hat{A}_t = \hat{A}_t' + K_t(HA_t + \delta_t - H\hat{A}_t'), \tag{2.18}$$

where $\hat{A}_t'$ is the prior estimate of $\hat{A}_t$ gained by knowledge of the system and K_t is the Kalman gain. Hence,

$$P_t = \mathbb{E}\left[\Big((I - K_tH)(A_t - \hat{A}_t') - K_t\delta_t\Big)\Big((I - K_tH)(A_t - \hat{A}_t') - K_t\delta_t\Big)^T\right]. \tag{2.19}$$

$$P_t = (I - K_tH)P_t'(I - K_tH)^T + K_t\xi(K_t)^T, \tag{2.20}$$

where P_t' is the prior estimate of P_t and $A_t - \hat{A}_t'$ is the prior error estimate at time t.

$$P_t = P_t' - K_tHP_t' - P_t'H^TK_t^T + K_t(HP_t'H^T + \xi)K_t^T. \tag{2.21}$$

The derivative of the trace matrix is

$$\frac{d\text{Tr}(P_t)}{dK_t} = -2(HP_t')^T + 2K_t(H^TP_t'H + \xi) = 0. \tag{2.22}$$

The Kalman gain is, therefore,

$$K_t = P_t'H^T(HP_t'H^T + \xi)^{-1}. \tag{2.23}$$

The update equation is, therefore,

$$P_t = P_t' - P_t'H^T(HP_t'H^T + \xi)^{-1}HP_t'. \tag{2.24}$$

$$P_t = (I - K_tH)P_t'. \tag{2.25}$$

Let $\eta^{'}_{t+1}$ be the prior error at time $t+1$ and defined by

$$\eta^{'}_{t+1} = \varphi(A_t - \hat{A}^{'}_t) + \theta_t. \tag{2.26}$$

$$\eta^{'}_{t+1} = \varphi\eta^{'}_t + \theta_t. \tag{2.27}$$

Prior error at the next time step $(t+1)$ is related to the prior error at the current time step (t) and the process noise θ_t. Let $P^{'}_{t+1}$ be the error covariance matrix at time $t+1$:

$$P^{'}_{t+1} = E[(\varphi\eta^{'}_t + \theta_t)(\varphi\eta^{'}_t + \theta_t)^T]. \tag{2.28}$$

$$P^{'}_{t+1} = \varphi^2 E[\eta^{'}_t\eta^{'T}_t] + \varphi E[\eta^{'}{}_t\theta^T_t] + \varphi E[\theta_t\eta^{'T}_t] + E[\theta_t\theta^T_t], \tag{2.29}$$

where $\eta^{'}_t$ and θ_t have no cross-correlation, $E[\eta^{'}_t\theta^T_t] = E[\theta_t e^{'T}_t] = 0$. Therefore,

$$P^{'}_{t+1} = \varphi P_t \varphi^T + \zeta. \tag{2.30}$$

The error covariance matrix at the next time step $(t+1)$ is related to the error covariance matrix at the current time step (t), the state transition matrix φ, and the process noise covariance matrix ζ. The final Kalman filter equations are illustrated by the following five steps, 1. State Prediction:

$$\hat{A}_{t+1|t} = \varphi\hat{A}_{t|t}, \tag{2.31}$$

predicts the state at time $t+1$ $(\hat{A}_{t+1|t})$ based on the state estimate at time t $(\hat{A}_{t|t})$ and the state transition matrix (φ). 2. State covariance prediction:

$$P_{t+1|t} = \varphi P_{t|t}\varphi^T + \zeta, \tag{2.32}$$

predicts the error covariance matrix at time $t+1$ $(P_{t+1|t})$ based on the error covariance matrix at time t $(P_{t|t})$, the state transition matrix (φ), and the process noise covariance matrix (ζ).
3. Kalman gain:

$$K_{t+1} = P_{t+1|t}H^T(HP_{t+1|t}H^T + \xi)^{-1}, \tag{2.33}$$

estimates the Kalman gain (K_{t+1}) based on the predicted error covariance matrix $(P_{t+1|t})$, the measurement matrix (H), and the measurement noise covariance matrix (ξ). 4. State update:

$$\hat{A}_{t+1|t+1} = \hat{A}_{t+1|t} + K_{t+1}(B_{t+1} - H\hat{A}_{t+1|t}), \tag{2.34}$$

Updates the state estimate at time $t+1$ $(\hat{A}_{t+1|t+1})$ based on the predicted state at time $t+1$ $(\hat{A}_{t+1|t})$, the Kalman gain (K_{t+1}), and the difference between the actual measurement (B_{t+1}) and the predicted measurement $(H\hat{A}_{t+1|t})$. 5. Error covariance update:

$$P_{t+1|t+1} = (I - K_{t+1}H)P_{t+1|t}, \tag{2.35}$$

Updates the error covariance matrix at time $t+1$ $(P_{t+1|t+1})$ based on the predicted error covariance at time $t+1$ $(P_{t+1|t})$ and the Kalman gain (K_{t+1}). The Kalman filter steps here do not include a multistep ahead predictor, thus, as a part of the continuous risk monitoring process, the following addition to the Kalman filter is added for forecasting and signal purposes. The state transition and observation equations are in a simple autoregressive model form (i.e., $AR(1)$), therefore, the proposal here is to add an $ARIMA(p,d,q)$ (where p - autoregressive term, d - integrated order required for stationarity, q - lagged forecast errors moving average term) process for a multistep prediction (forecast) after the Kalman filter has been applied, as well as estimate the confidence intervals. The general form of an ARIMA(p,d,q) process is

$$\varphi(E)(1-E)^d A_t = \theta(E)dW_t, \tag{2.36}$$

where E is the back shift operator in the form $EA_t = A_{t-1}$ and W_t white noise process with a known covariance matrix, $(1-E)^d$ is integrated of order d. $AR(p)$ is defined by:

$$\varphi(E) = 1 - \varphi_1 E - \varphi_2 E^2 - \ldots - \varphi_p E^p, \tag{2.37}$$

and $MA(q)$ is defined by:

$$\theta(E) = 1 + \theta_1 E + \theta_2 E^2 + \ldots + \theta_q E^q. \tag{2.38}$$

The Kalman filter starts off with the state transition and the observation equation as $AR(1)$ processes. Thus, the MA part of the model is set to be of order $q = 0$, and integrated of order $d = 1$ for stationarity of the returns. The historical prices only need to be differenced once to achieve stationarity, resulting in an $ARIMA(1,1,0)$. The ARIMA model will be applied to the Kalman filtered data. Estimating the multistep prediction interval is preceded by determining the forecast error and the prediction error. The multistep forecast error is defined by:

$$\eta^n_{n+m} = A_{n+m} - A^n_{n+m}. \tag{2.39}$$

$$A^n_{n+m} = \varphi^m A_n. \tag{2.40}$$

The multistep prediction error is defined by:

$$P^n_{n+m} = Var(A_{n+m} - A^n_{n+m}) = Var(\eta^n_{n+m}). \tag{2.41}$$

The multistep prediction interval is, therefore,

$$A^n_{n+m} \pm z_{\frac{\alpha}{2}} \sqrt{P^n_{n+m}}, \tag{2.42}$$

where n is the number of observations and m is the number of prediction steps.

2.4 Proposed framework

The proposed high pure momentum equity timing framework is illustrated by the flow chart in Figure 2.1. The process starts with the overall stock data, from which the population of large capitalization stocks will be selected. This process ends with the portfolio rebalancing, followed by the portfolio performance evaluation.

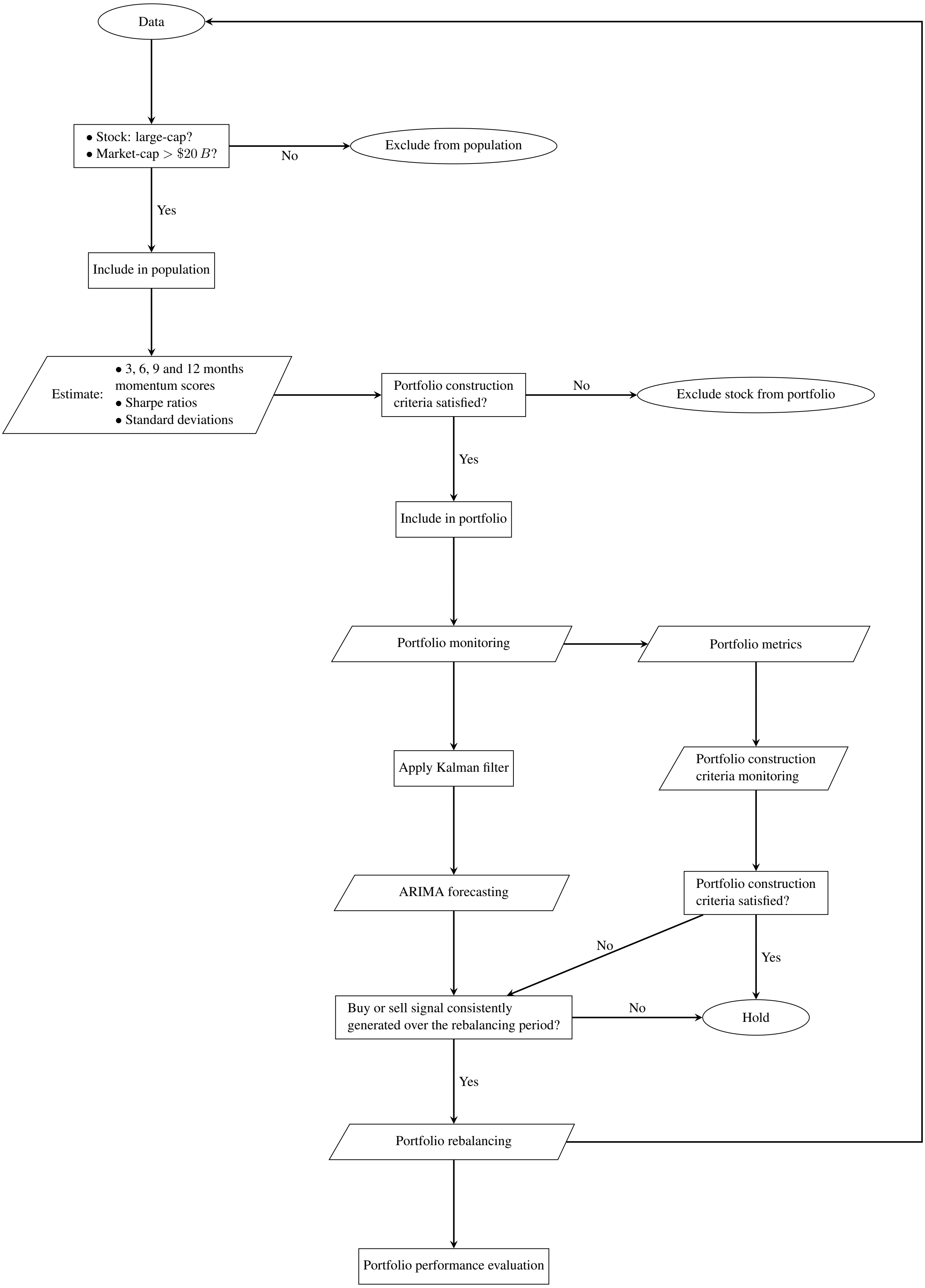


Figure 2.1: Momentum equity timing framework. Authors' own contribution

2.4.1 Data

The population consists of 540 large capitalization US listed stocks, with more than $20 billion dollars in market capitalization. The historical data period was set to be from 01 January 2013 to 31 December 2023, taken from the Yahoo finance database. The portfolio criteria was used to select momentum stocks. This resulted in an initial portfolio of 54 stocks.

2.4.2 Portfolio construction criteria

The portfolio construction process is an important part of a factor timing framework, and starts off with defining the investable universe population of stocks, followed by estimating the momentum scores, Sharpe ratio and standard deviation. The momentum score is estimated by:

$$MS = \frac{P_t - P_{t-s}}{P_{t-s}} \times 100, \tag{2.43}$$

where P_t is the current price and P_{t-s} is the price at the beginning of the period. The market capitalisation (MC) is estimated by:

$$MC = PS \times OS, \tag{2.44}$$

where PS is the current market price per share and OS is the total number of outstanding shares. After all the metrics above have been estimated, the next part is defining the portfolio selection criteria. The stocks that fit the below criteria are included in the portfolio.

1. 3-months momentum score $> 10\%$
2. 6-months momentum score $> 15\%$
3. 9-months momentum score $> 20\%$
4. 12-months momentum score $> 30\%$
5. Sharpe ratio > 0.7
6. Standard deviations < 0.25

Momentum gains are subject to short-term reversals, therefore the thresholds are conservative. A Sharpe ratio above 0.7 is considered good. The standard deviation of below 0.25 is low risk and low volatility. The above criteria is set to limit the risk of investing in momentum stocks.

2.4.3 Portfolio rebalancing and performance evaluation

Portfolio rebalancing is a process where some stocks will be bought (added into the portfolio) and sold, to ensure that the stocks are still within the initial criteria. The market upturns and downturns may change the metrics used in the construction process to change significantly; rebalancing maintains the asset allocation criteria (Kimball et al., 2020; Fischer et al., 2021). The rebalancing period is followed by the portfolio performance evaluation, which involves assessing returns earned within the rebalance period (Plastira, 2014).

2.5 Empirical results

Table 2.1 presents the initial portfolio constructed using the selection criteria.

Table 2.1: Initial portfolio constructed using the selection criteria.

Symbol	Description	3-Months	6-Months	9-Months	12-Months	Sharpe Ratio	Standard Deviation
NVDA	NVIDIA Corporation	10.59433136	16.78126421	80.43768882	246.0983373	1.26710737	0.028147058
AMZN	Amazon.com, Inc.	17.36443269	16.6794663	46.16643281	77.04498134	0.851726283	0.020567677
META	Meta Platforms, Inc.	15.36404764	23.75359013	64.84724951	183.758223	0.795465724	0.02412872
AVGO	Broadcom Inc.	34.30537794	28.64431864	78.66948701	106.2686067	1.187161527	0.021668064
NVO	Novo Nordisk A/S	12.69062621	30.39886221	31.09779796	53.00867092	0.839034754	0.016593725
COST	Costco Wholesale Corporation	18.38221012	25.24164456	36.42349095	50.06235101	1.028850178	0.013021139
AMD	Advanced Micro Devices, Inc.	42.74233413	27.27508552	53.76029982	130.2561877	0.926067692	0.036203874
ADBE	Adobe Inc.	14.4819853	22.95706727	54.90068416	77.07466218	0.950663157	0.019874772
INTU	Intuit Inc.	20.91908102	38.89376618	42.8613522	60.94612887	0.886367935	0.018753059
PDD	PDD Holdings Inc.	46.79442184	105.2321443	99.87705418	73.08647778	0.77998403	0.048813747
NOW	ServiceNow, Inc.	27.24276698	25.51566016	48.40668172	83.26588592	0.911687887	0.025500119
BX	Blackstone Inc.	23.24061869	40.44177126	65.45178543	78.09046953	0.887277843	0.0214946
MU	Micron Technology, Inc.	25.95177023	34.20639753	50.04574591	70.59995941	0.748027812	0.028166555
ETN	Eaton Corporation plc	15.00282665	20.99063704	50.37858267	55.43148966	0.718579387	0.017132361
LRCX	Lam Research Corporation	24.49205203	21.01097475	56.64416157	91.36082903	0.931538062	0.024530173
KLAC	KLA Corporation	26.54408107	20.55738631	49.28312541	56.24836143	0.920050579	0.022711055
SHOP	Shopify Inc.	44.25926208	20.12336678	63.68985349	118.3295988	1.018964952	0.039144939
PANW	Palo Alto Networks, Inc.	24.54280507	15.80270405	50.02035437	112.986646	0.867442583	0.024271038
KKR	KKR & Co. Inc.	36.18028349	47.40475551	63.81022106	79.47745348	0.730641813	0.021258249
DELL	Dell Technologies Inc.	13.63644415	41.85501394	91.18587756	92.59426258	0.901406608	0.022556047
SNPS	Synopsys, Inc.	11.0150414	18.64011464	33.76717402	61.07044074	1.065284995	0.017099892
ANET	Arista Networks, Inc.	25.39800733	47.28580581	41.24384476	94.7812318	0.897128435	0.027619942
CDNS	Cadence Design Systems, Inc.	15.17188413	15.70518359	28.77405427	70.61513165	1.05750787	0.01873592
SHW	The Sherwin-Williams Company	23.73412144	18.61901006	40.72000842	31.59449596	0.803611683	0.016106649
STLA	Stellantis N.V.	23.32099507	30.9376759	41.48762857	74.36241033	0.753285988	0.025164742
RELX	RELX PLC	18.84926884	20.73084076	24.18272439	45.77388631	0.809707056	0.012892264
CRWD	CrowdStrike Holdings, Inc.	50.59573991	74.92463386	86.55560879	147.211474	0.842925719	0.037694839
MAR	Marriott International, Inc.	16.68543983	23.02565505	38.30927896	54.23834213	0.701551537	0.019730663
MELI	MercadoLibre, Inc.	24.78382706	31.64733312	20.71126888	90.23375421	0.804798822	0.02995831
PH	Parker-Hannifin Corporation	20.01806558	18.68797821	45.39440952	60.39240001	0.71811907	0.018713831
APH	Amphenol Corporation	20.20587212	18.45162967	25.3927607	31.05243903	0.85351241	0.014524717
CTAS	Cintas Corporation	24.60678347	24.29478328	33.50394459	35.61233196	1.141890667	0.015809445
TDG	TransDigm Group Incorporated	27.2220435	16.99404634	42.56879154	67.60040166	0.902680183	0.020196651
TT	Trane Technologies plc	23.53579994	28.46180007	41.9914447	44.43744943	0.927943241	0.016294082
CEG	Constellation Energy Corporation	11.86941724	28.24692605	55.0810585	44.44362031	1.589334207	0.024262035
TEAM	Atlassian Corporation	20.45983618	41.92971288	42.86743514	88.03162104	0.840574422	0.032610153
DHI	D.R. Horton, Inc.	43.47039855	27.02567595	57.06760037	69.14983165	0.723872265	0.022227888
URI	United Rentals, Inc.	31.55034448	28.63989792	61.39941906	63.02588499	0.753044739	0.026398223
DDOG	Datadog, Inc.	32.16463643	23.49170427	75.10097777	68.37287199	0.744583765	0.040413051
IR	Ingersoll Rand Inc.	21.63842538	18.61625561	40.08534617	46.10885678	0.732573065	0.022311443
IT	Gartner, Inc.	30.37860848	29.97291771	43.62901319	33.66618223	0.86736747	0.017808524
MPWR	Monolithic Power Systems, Inc.	37.53918296	16.70525232	31.34530319	85.36402381	0.977600083	0.02517973
VRT	Vertiv Holdings Co	25.11485268	93.23124282	266.8443661	261.3279128	0.838094781	0.031530406
NET	Cloudflare, Inc.	33.19469584	26.49650945	35.97909972	93.53789226	0.850583626	0.046429274
FICO	Fair Isaac Corporation	34.78733815	46.41635343	68.87087392	96.89270877	1.06509056	0.020985282
ZS	Zscaler, Inc.	37.70899273	51.14264486	100.4886375	101.0708709	0.840145677	0.038594287
ICLR	ICON Public Limited Company	16.42264047	16.47533586	34.25821108	46.22140228	0.835171098	0.019419194
ARES	Ares Management Corporation	16.8772408	24.97193959	50.37064775	79.7513977	0.863940276	0.021670163
BR	Broadridge Financial Solutions, Inc.	16.73261031	26.60157385	45.2276038	55.73197355	1.071776412	0.014227325
BLDR	Builders FirstSource, Inc.	35.72357922	22.45287185	91.0943238	155.4552507	0.835582112	0.03276591
PTC	PTC Inc.	24.22608014	24.66866986	37.11599436	46.22649769	0.74578355	0.019649932
DECK	Deckers Outdoor Corporation	29.1527405	25.77239398	46.82378824	71.73136033	0.850109584	0.0252146
DKNG	DraftKings Inc.	21.38429369	34.28571429	84.74842841	219.0045194	0.765326057	0.043951352
CBOE	Cboe Global Markets, Inc.	14.22484439	30.83153957	33.41978594	43.38262867	0.889541723	0.015152917

The initial portfolio in Table 2.1 contains 54 stocks. The selection criteria determines the portfolio size. Hence, the maximum number of stocks that should be included in the portfolio will vary, after each rebalancing period.

- 3 months momentum scores are mostly the lowest scores compared to other periods, and this might be subject to reversals.
- 6 months momentum scores will either be greater than the 3 months momentum scores or they will be on a reversal trajectory (i.e., decrease).
- 9 months momentum scores are higher compared to the 6 months time period.
- 12 months momentum scores indicate the persistence of the price momentum with high scores.
- Sharpe ratios are obtained by subtracting the risk-free-rate of 0.05, indicating that each stock had significant excess returns.
- Standard deviation is lower than the threshold of 0.25, indicating that the stocks within the portfolio have low volatility.

2.5.1 Population and portfolio comparison

Table 2.2 compares the metrics of both the population of stocks and the backtested portfolio.

Table 2.2: Population of Stocks and Portfolio Metrics Comparison.

Metric	Population	Portfolio
Total Return	4.5377	14.6402
Annualised Return	0.1686	0.2845
Volatility	0.0111	0.0146
Sharpe Ratio	10.6844	16.0229
Max Drawdown	-0.3608	-0.3699
Downside Risk	0.0092	0.0117
Shortfall Probability	0.4434	0.4257
VaR (95%)	-0.0176	-0.0230
CVaR (95%)	-0.0280	-0.0343

- Total and annualised returns: indicate that even though the population contained more stocks, this did not automatically result in higher returns over a period of 10 years (2013 - 2023).
- Volatility: was relatively stable when spread out through the 10 year period for both the constructed portfolio and the population of stocks.
- Sharpe ratio is relatively high, indicating better risk-adjusted performance.
- Max drawdown: the largest peak-to-trough decline was negative indicating that the population could lose -36.08% and portfolio -36.99% in value.
- Downside risk: the volatility of returns below zero was also relatively low.
- Shortfall probability: the likelihood of not achieving projected positive returns was 0.4424 and 0.4311.

- VaR (95%): the value at risk, which is the maximum value that could be lost for the population, is -0.0176 and for the portfolio it is -0.0230.
- CVaR (95%): the conditional value at risk is -0.0280 and -0.0343 for the population and portfolio, respectively.

2.5.2 Portfolio sector exposure

Sector exposure is a crucial aspect of portfolio risk management (Keisler and Linkov, 2011). A portfolio will have a wide range of sector exposures. Market sentiment is another driving force within the stock market and herd mentality can be an influential factor in the liquidity of stocks; these are the overall expectations from investors (Aggarwal, 2022).

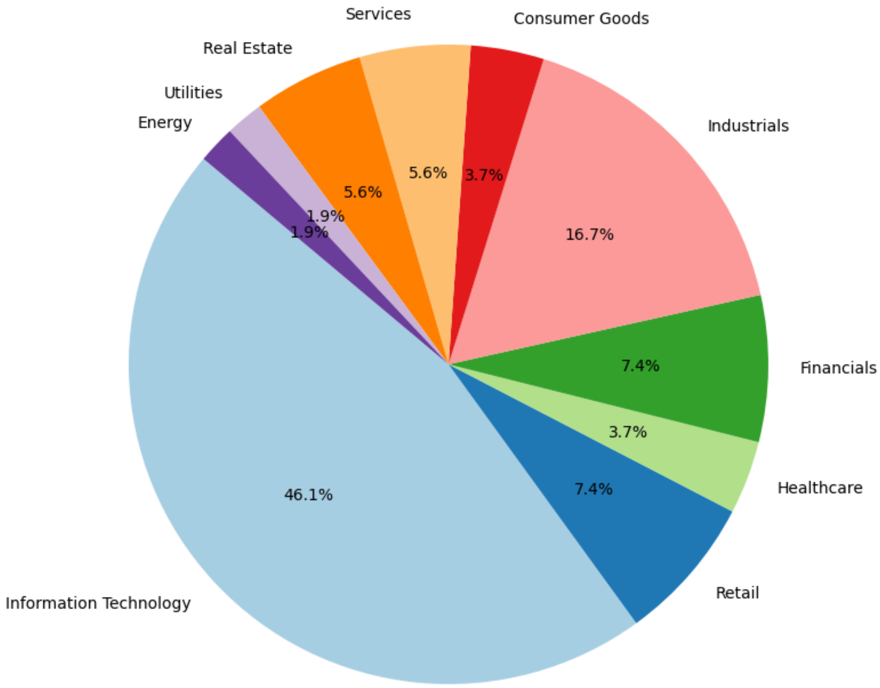


Figure 2.2: Portfolio industry exposure.

Figure 2.2 shows that the initial portfolio constructed by the selection criteria has high exposure of information technology stocks that tend to have high momentum due to their market sentiment, perceived future potential growth, and their liquidity. This is reflected by the overall portfolio exposure of 46.1%. This indicates that information technology stocks have been high performers and are one of the most popular stocks in the market. This drives up their value, further increasing their momentum scores at a higher rate as compared to other sectors.

The next largest exposure is industrial which includes manufacturing, construction and others. Financials and retail are tied with an exposure of 7.4% each, and finance stocks are popular due to their stability in the overall market. The rest of the sectors make up the remaining exposure.

2.5.3 Kalman filter, ARIMA forecasting and trading signal

The portfolio construction is followed by the risk management process. The Kalman filter is applied to the portfolio as a part of the risk management process. The Kalman filter incorporates information as it becomes available and is better equipped to handle missing data. This is then followed by the ARIMA forecasting and the confidence interval for each underlying stock.

Figure 2.3 illustrates the trading signals over a period of 6 months for the first four stocks within the initial portfolio. The trading signals are generated for each underlying stock. The Kalman filter, and ARIMA forecast with the confidence intervals of the forecast are also depicted in Figure 2.3.

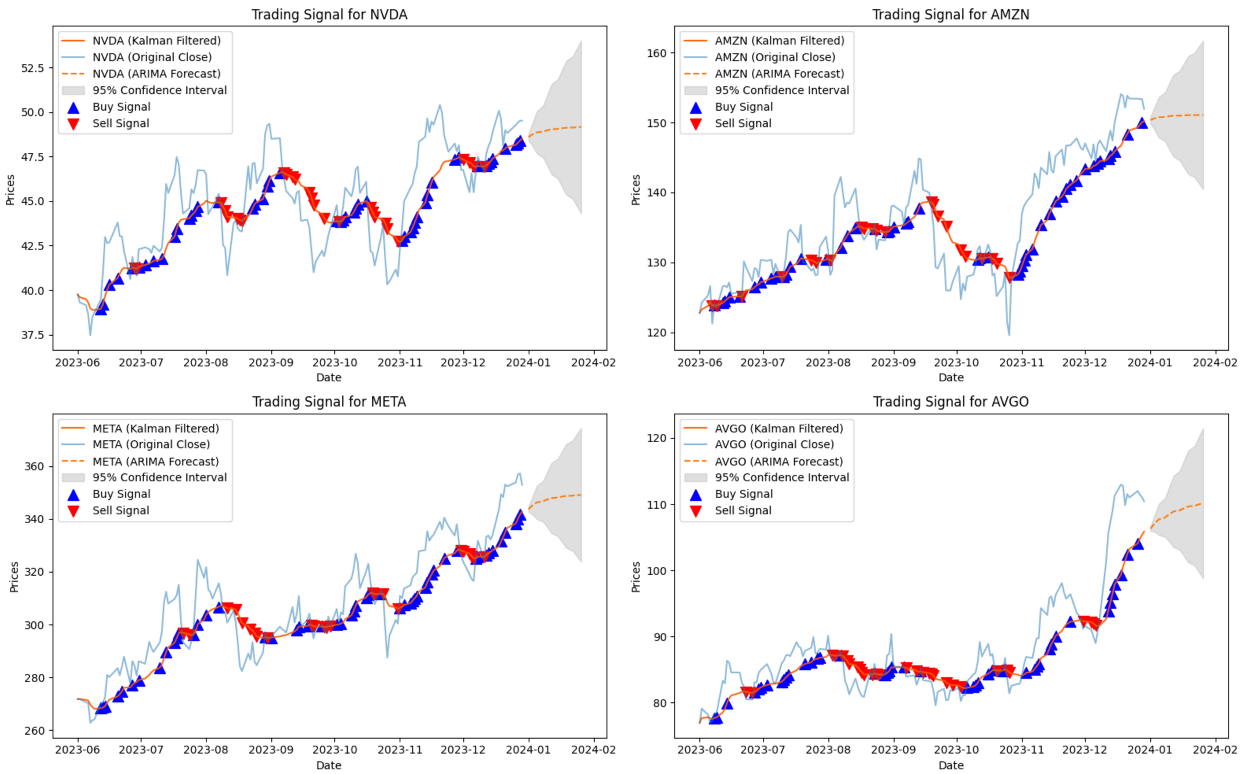


Figure 2.3: Trading signals for the first four stocks within the portfolio.

- The blue line shows the historical daily prices over the period of 6 months starting from 01 June 2023 to 31 December 2023 for the first four stocks within the portfolio on Table 2.1.
- The Kalman filter reduces the short-term fluctuations of the historical daily data, minimises noise, and provides a better view of the historical trends. The trend generated by the Kalman filter is used as an indicator for the buy and sell signal.
- The Kalman filtered historical daily data is used for the ARIMA forecast (forecast period: 20 days).
- The 95% confidence interval is estimated for the forecast, and this increases as the forecast period increases.
- The four stocks are currently showing an increasing upward trend, signaling an increase in the momentum score.
- The buy signal is generated when the Kalman filter trend line is below the historical prices and forecast.

- The sell signal is generated when the Kalman filter trend line is above the historical prices and forecast.

2.6 Transaction costs

There are a number of transaction costs that can be detrimental to portfolio returns and could be the main difference between the outperformance and underperformance of a factor timing strategy (Wang and Siu, 2024). An active factor timing strategy depends heavily on the costs involved during the implementation process. An equilibrium of transaction costs is essential, especially when trading occurs frequently and liquidity premium becomes an added cost (Isaenko, 2023). Figure 2.4 shows a breakdown of transaction costs.

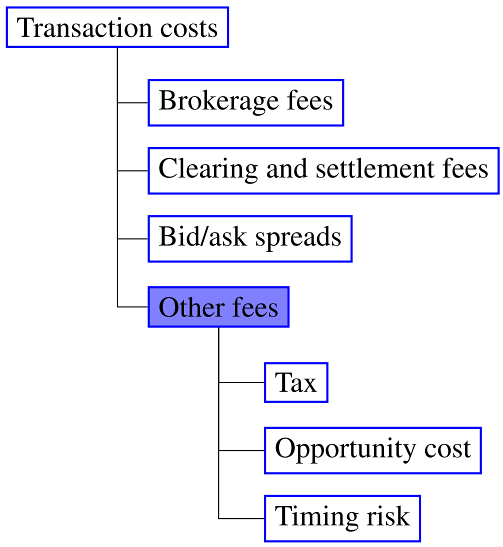


Figure 2.4: Transaction costs breakdown.

A brokerage fee is a commissions that a broker will charge on a trade; clearing and settlement fees are charged by the exchange for the execution of a trade; and bid/ask spreads are the price difference of the stock that is being traded (Galati, 2024). The breakdown of transaction costs can be broader and might include other costs depending on the market and country.
Figure 2.5 shows a comparison between the buy-and-hold strategy with the monthly, quarterly and annual rotations, illustrating the impact of transaction costs on each strategy. The transaction costs here are considered on a total basis.
The transaction costs make short-term timing rotation strategy more expensive and diminishes cumulative returns. A momentum portfolio is vulnerable to market downturns, which can negatively affect the portfolio's performance. The portfolio has large concentration of technology stocks, which could contribute to low returns when that industry is underperforming.

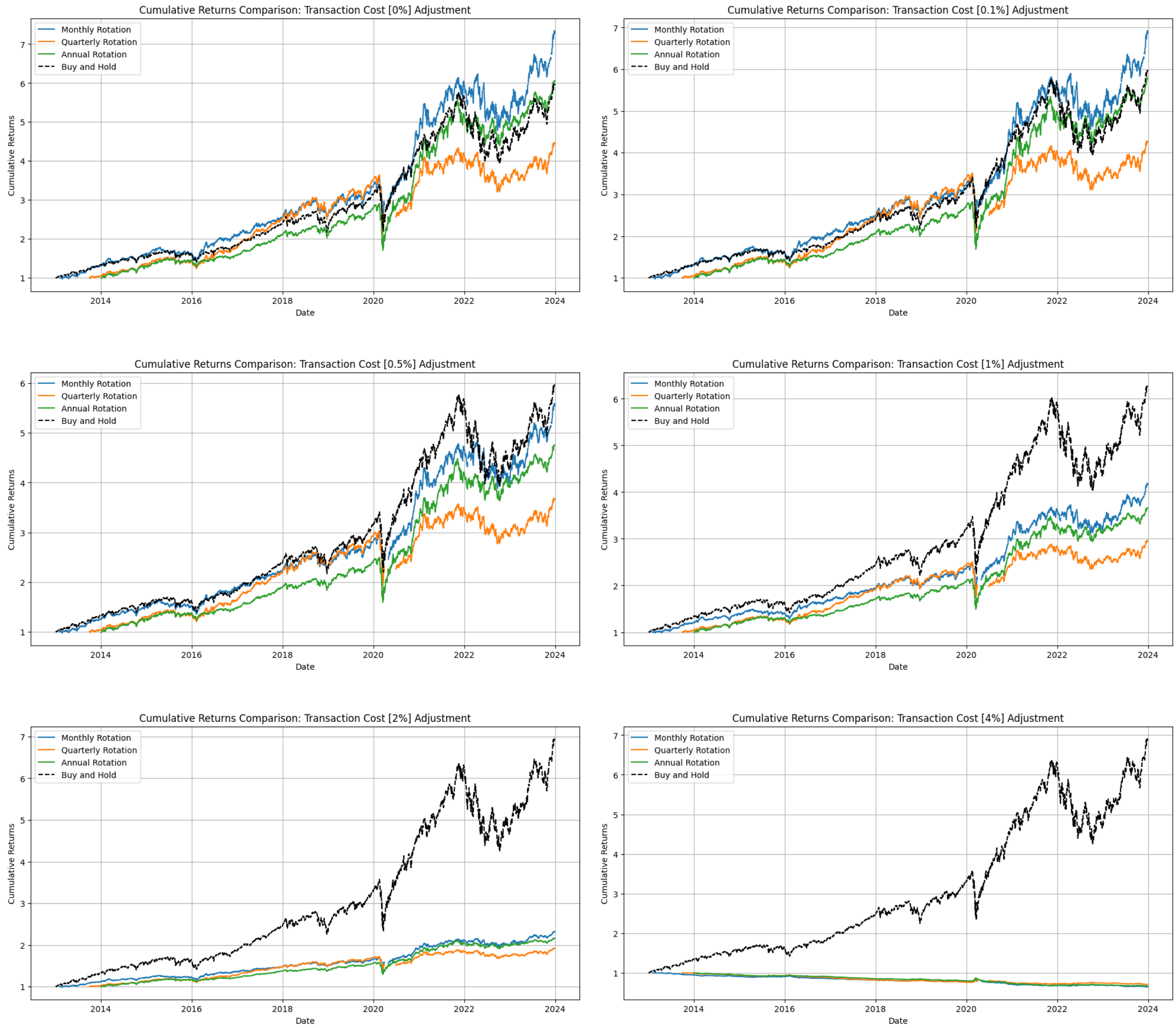


Figure 2.5: Buy-and-hold strategy, monthly, quarterly, and annual rotations' cumulative returns comparison with transaction costs [0%, 0.1%, 0.5%, 1%, 2% and 4%] adjustments.

- A 0% transaction cost in reality is not possible, as there are unavoidable costs when trading on the market. However, from the simulated results, the monthly rotation has the highest cumulative returns, and significantly outperforms other strategies.
- A total transaction costs of 0.1% is low. The monthly rotation closely matches that of the buy-and-hold strategy.
- At 0.5% total transaction cost adjustment, the buy-and-hold strategy has higher cumulative returns, followed by the monthly rotation.
- The 1% total transaction costs affects the returns significantly, and the rotation strategies have diminished cumulative returns compared to the lower 0.1% and 0.5% transaction costs.
- At 2% total transaction costs, the cumulative returns are still increasing, however, at a much lower rate. The buy-and-hold strategy is not affected as there are no significant trades occurring.
- The transaction costs can in some cases, be as high as 4% and the strategy with the least amount of trades, will have the highest cumulative returns. The buy-and-hold strategy

significantly outperforms other rotation strategies, which have decreasing cumulative returns, closer to 0.

Relatively, low transaction costs should have a limited effect on the rotation strategy. The monthly rotation process will have the best results, followed by the annual and lastly quarterly rotation, when the costs are at 0% and 0.1%. Surprisingly, the rotation that is below the buy-and-hold strategy at 0% costs is the quarterly rotation. 3 months momentum is subject to reversals and, as such, at the 3 month point this reversal might affect the bid/ask spread, and make the stocks more costly. 12 months momentum in some cases will be higher than 3 months momentum and, as a result, the annual rotation performed better than the quarterly rotation. The monthly rotation takes advantage of the short-term increases, and this has resulted in the highest cumulative returns when transaction costs are 0%.

2.7 Conclusion

This article presents a practical pure momentum factor timing strategy summarised below:

- Investment objectives and portfolio strategy defined.
- Portfolio construction process using momentum scores, Sharpe ratio, and standard deviation.
- Implementation of the Kalman filter for signal processing.
- Implementation of the ARIMA and confidence intervals.
- Signal assessments of both historical trends and the forecasts trend.
- Rebalancing (adjusting portfolio exposure) procedure based on the results of the Kalman filter, ARIMA and confidence intervals.

A dynamic portfolio construction process, together with the Kalman filter, was implemented for the state dynamic system estimation using daily historical data. Additionally, the ARIMA forecasting and confidence interval enhanced the forecast accuracy. Momentum strategies can be susceptible to sharp downturns, reversals, market volatility, and sentiment arising from unforeseen events. However, returns are somewhat predictable, and this predictability plays a pivotal role in the way a pure momentum strategy is implemented.

A factor timing strategy does rely on this predictability and the forecasts of future returns, even though past returns do not guarantee future returns. A Kalman filter approach addresses this past returns limitation by iteratively incorporating new information as it becomes available. The momentum definition does assert that if a fund or stock has been on a recent increasing trajectory, it tends to continue unless a significant event disrupts the trend.

A high pure momentum only equity factor timing framework presented in this article, offers a comprehensive approach starting with the selection of stocks, dynamic monitoring, and rebalancing, together with the forecasting procedure to achieve superior returns. The costs involved with implementing this strategy can be minimised by choosing a rebalancing period that limits the trading frequency.

References

[1] Aked, M. (2021). Factor Timing: Keep It Simple. *Research Affiliates.*

[2] Aiken, A.L. and Kang, M. (2023). Hedge fund manager timing and selectivity skill over time, A holdings-based estimate. *Finance Research Letters*, 58.

[3] Aggarwal, D. (2022). Defining and measuring market sentiments: a review of the literature. *Qualitative Research in Financial Markets*, 14(2), 270-288.

[4] Asness, C., Chandra, S., Ilmanen, A., and Israel, R. (2018). Contrarian Factor Timing Is Deceptively Difficult. *The Journal of Portfolio Management*, Special Issue, 43(5).

[5] Asness, C., Ilmanen, A., and Maloney, T. (2017). Market Timing: Sin a Little, Resolving the Valuation Timing Puzzle. *Journal of Investment Management*, 15(3), 23–40.

[6] Carhart, M. M., (1997). On Persistence in Mutual Fund Performance. *The Journal of Finance, 52(1)*, 57-82.

[7] Chin, A. and Gupta, P. (2020). Timing is Not Everything—Assessing Manager Skill in Factor Timing. *Journal Of Investment Management*, 18(1), 34–51.

[8] Clare, A., Sherman, M., O'Sullivan, N., Gao, J. and Zhu, S. (2022). Manager characteristics: Predicting fund performance. *International Review of Financial Analysis*, 80.

[9] Daniel, K., Grinblatt, M., Titman, S., and Wermers, R. (1997). Measuring Mutual Fund Performance with Characteristic-Based Benchmarks. *The Journal of Finance*, 52(3), 1035-1058.

[10] Davies, J., Gibbon, D., Shores, S., and Smith, J. (2019). Implementation Matters: Relaxing Constraints Can Improve the Potential Returns of Factor Strategies. *The Journal of Portfolio Management Quantitative Special Issue* , 45(3), 101-114.

[11] Dichtl, H., Drobetz, W., Lohre, H., Rother, C., and Vosskamp, P. (2019). Optimal Timing and Tilting of Equity Factors. *Financial Analysts Journal*, 75(4), 84-102.

[12] Drew, M. E., Veeraraghavan, M., and Wilson, V. (2005). Market Timing, Selectivity and Alpha Generation: Evidence from Australian Equity Superannuation Funds.*Investment Management and Financial Innovations*, 2(2), 111-127.

[13] Fama, E. F. and French, K. R. (1992). The cross-section of expected stock returns. *The Journal of Finance*, 47(2), 427-465.

[14] Fergis, K., Gallagher, K., Hodges, P., and Hogan, K. (2019). Defensive Factor Timing. *The Journal of Portfolio Management Quantitative Special Issue*, 45(3), 50-68.

[15] Fischer, A.M., Greminger, R.P., Grisse, C. and Kaufmann, S. (2021). Portfolio rebalancing in times of stress. *Journal of International Money and Finance*, 113.

[16] Galati, L. (2024). Exchange market share, market makers, and murky behavior: The impact of no-fee trading on cryptocurrency market quality. *Journal of Banking and Finance*, 165.

[17] George, T. J. and Hwang, C. (2004). The 52-Week High and Momentum Investing. *The Journal of Finance*, 59(5), 2145-2176.

[18] Gupta, T. and Kelly, B. (2019). Factor Momentum Everywhere. *The Journal of Portfolio Management Quantitative Special Issue*, 45(3), 13-36.

[19] Huang, J., Zhang, P. and Zhang, J. (2024). Understanding Momentum and Reversal Investing Strategies. *Journal of Economics, Finance and Accounting Studies*, 5(1), 106-112.

[20] Huberman, G. and Wang, Z. (2005). Arbitrage Pricing Theory. *Federal Reserve Bank of New York Staff Reports* (Staff Report no. 216).

[21] Hong, H. and Stein, J. C. (1999). A Unified Theory of Underreaction, momentum trading, and overreaction in Asset markets. *The Journal of Finance*, 54(6), 2143-2184.

[22] Isaenko, S. (2023). Transaction costs, frequent trading, and stock prices. *Journal of Financial Markets*, 64.

[23] Jegadeesh, N. and Titman, S. (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of Finance*, 48(1), 65-91.

[24] Kálmán, R. E. (1960). A New Approach to Linear Filtering and Prediction Problems. *Journal of Basic Engineering*, 82(1), 35-45.

[25] Karki, D. and Khadka, P.B. (2023). Momentum Investment Strategies across Time and Trends: A Review and Preview. *Nepal Journal of Multidisciplinary Research*, 7(1), 62-83.

[26] Keisler, J. and Linkov, I. (2011). Managing a portfolio of risks. *Management Science and Information Systems Faculty Publication Series*, 33.

[27] Kimball, M.S., Shapiro, M.D., Shumway, T. and Zhang, J. (2020). Portfolio rebalancing in general equilibrium. *Journal of Financial Economics*, 135(3), 816-834.

[28] Kwon, D. (2022). Dynamic Factor Rotation Strategy: A Business Cycle Approach. *International Journal of Financial Studies*, 10(2), 46.

[29] Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics*, 47(1), 13-37.

[30] Markowitz, H. (1952). Portfolio Selection . *The Journal of Finance*, 7(1), 77-91.

[31] Moskowitz, T. J. and Grinblatt, M. (1999). Do industries explain momentum?. *The Journal of Finance*, 54(4), 1249-1290.

[32] Osinga, B., Schauten, M., and Zwinkels, R. C. J. (2020). Timing is Money: The Factor Timing Ability of Hedge Fund Managers. Available at SSRN 2811163.

[33] Plastira, S. (2014). Performance evaluation of size, book-to-market and momentum portfolios. *Procedia Economics and Finance*, 14, 481–490.

[34] Ross, S. A. (1976). The Arbitrage Theory of Asset Pricing. *Journal of Economic Theory*, 13, 341-360.

[35] Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance*, 19(3), 425-442.

[36] Shumway, R. H. and Stoffer, D. S. (2017). Time Series Analysis and Its Applications: With R Examples (4th ed.). *Springer*.

[37] Souza, T. O. (2020). Macro-Finance and Factor Timing: Time-Varying Factor Risk and Price of Risk Premiums. *Available at SSRN 3296418.*

[38] Treynor, J. and Mazuy, K. (1966). Can Mutual Funds Outguess the Market? *Harvard Business Review*, 44, 131-136.

[39] Wang, N. and Siu, T.K. (2024). Investment-consumption optimization with transaction cost and learning about return predictability. *European Journal of Operational Research*, 318, 877-891.

[40] Zheng, Y., Osmer, E. and Zu, D. (2024). Timing sentiment with style: Evidence from mutual funds. *Journal of Banking and Finance*, 164.

Chapter 3

Evaluating portfolio diversification of large-cap momentum stocks with measures of dependence

Abstract

The aim of this study was to use mixture copula models to determine the dependence structure of a pure momentum portfolio. First, the portfolio was constructed from a population of large market capitalisation stocks, where the 3, 6, 9 and 12 months momentum scores were estimated and the top 20 stocks with the highest average scores were selected. Second, Kendall's tau and Spearman's rho were used to assess the portfolio's diversification. Third, a combination of three copula mixture models were evaluated for both the large-cap population of stocks and the momentum portfolios. The mixture copula models are essential for the measurement of complex dependencies that are found within a portfolio of stocks. Lastly, the AIC and BIC were estimated for each mixture combination to determine the copula mixture model that best fits the portfolio. This was done for both the stocks listed in South Africa and those that are listed in the United States. The SA momentum portfolio was more diversified than the US momentum portfolio. The SA and US momentum portfolios had the same top three copula mixture models with the lowest AIC and BIC.

Keywords

Copula selection, copulas, factor investing, Kendall's tau, mixture copulas, momentum,portfolio diversification, Spearman's rho.

3.1 Introduction

An important aspect of portfolio management is diversification, a way in which an investor will allocate capital to limit the risk associated with the exposure to an asset class. This allows for a reduction of risk without compromising the anticipated portfolio returns and managing the effects of volatility of a portfolio. In modern portfolio theory, risk averse investors will typically have preferred levels of risk that they are willing to take on. The risk–return relationship is linear in nature and investors normally select their portfolios on this risk–return balance and merit (Mao, 1970).

There are two types of risks, namely systematic and idiosyncratic risk. Systematic risk is the type of risk that cannot be diversified and idiosyncratic which can be reduced by diversification. As shown by Frahm and Wiechers (2011), who compare the smallest portfolio returns variance to the overall portfolio return variance, there is an aspect of portfolio risk that is undiversifiable.

Diversification is qualitatively intuitive and easy to understand, however, it is not a simple task to measure. Extreme events such as the 2008 global financial crisis and the outbreak of COVID-19 exposed the inefficiencies of quantitative diversification measures. In these events, stocks react the same, resulting in a correlation convergence, global shocks, and a reduction in the liquidity of the stock market (Koumou, 2020). The core principle of diversification is clear, when one stock experiences a shock, the shock will be well absorbed by the portfolio, and the portfolio will grow and remain stable even in highly volatile periods. In practice it is quite a challenging endeavour, Goetzmann and Kumar (2008) and Flint et al. (2020) agree that there is a lack of a generally widely accepted quantitative diversification measure. They explored methods which attempt to answer questions related to the number of assets, weights of the underlying portfolio holdings, expected returns, and volatility for risk management.

Some studies have focused on the number of stocks required for an optimally diversified portfolio. Zaimovic et al. (2021) found that determining the number of stocks required is complex, especially when faced with challenges of market movements and the effects of black swan events such as COVID-19. Sankaran and Patil (1999) explored issues faced during the portfolio construction process to maximise the ratio of excess returns, by constructing a portfolio with a large number of stocks. The diversification benefit decreased as the number of stocks increased, and this gives precedence that, diversification does not lie only in the number of stocks but in the quality of stocks being selected and included within the portfolio. Therefore, when attempting to diversify, simply having a large number of stocks within a portfolio does not automatically guarantee that a portfolio will be diversified.

Martínez-Nieto et al. (2021) found that equally weighted diversification models performed poorly, with stability index performing better, the cost function was at the top of the list and with global maximum returns results in the competitive range. Diversification and dependence are related and Chollete et al. (2010) used a correlation and copula approach in different regions, and found that dependence has increased over the years, and assert that the cost and degree of dependence can affect the diversification benefit. An accurate measurement of dependence is paramount. Ji et al. (2018) uses the tail dependence clustering with the aim of selecting unique assets into the portfolio. Assets were separated into their dependence cluster categories and an asset was selected based on the cluster without overlapping. The authors combined cluster analysis with portfolio selection process.

In extreme events the interactions between the underlying stocks within a portfolio can be explained by dependence structures. The joint distributions of random variables can be written in terms of their marginal distributions and a copula function. A copula function can be used to capture the dependence structure of the portfolio (Abraj et al., 2022). Two or more copulas can be combined to form a copula mixture model to capture the complex dependence structures within a portfolio by essentially using two or three copulas in one model with multiple parameters and density functions (Kasa et al., 2020; Tewari, 2023; Alzaid and Alhadlaq, 2024).

The aim of this study is divided into four objectives as follows:

1. To construct a momentum portfolio, by selecting the top 20 stocks with the highest average of the 3, 6, 9 and 12 months momentum scores.
2. Assess the portfolio's diversification using the Kendall's tau and the Spearman's rho.
3. Use mixture copula models to estimate the dependence structure of the portfolio.
4. Estimate the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), for the best fit mixture copula model.

This article begins with Section 3.1, which outlines diversification and copulas, followed by portfolio construction methodology in Section 3.2. Measures of dependence are outlined in Section 3.3, and empirical analysis in Section 3.4 provides supporting evidence for the study and concludes with Section 3.5.

3.2 Portfolio construction

3.2.1 Momentum portfolio construction

One of the most popular factors in factor investing is momentum, which captures excess returns in stocks that have performed well in the past. This factor takes into account the historical trends of stocks, or rate at which the price changes over time (Jegadeesh and Titman, 1993). A simple reason why momentum works as a strategy is that, if a stock is performing well it tends to garner attention from investors and as a result the demand for that stock will grow, causing the price to continue increasing and vice versa.

The 3, 6, 9 and 12 months momentum score for each stock is estimated by

$$MS = \frac{P_t - P_{t-s}}{P_{t-s}} \times 100, \tag{3.1}$$

where MS is the momentum score, P_t is the current price and $P_{(t-s)}$ is the price at the beginning of the period.

The portfolio will include the top 20 stocks with the highest average momentum scores. The average is estimated by

$$\overline{MS} = \frac{1}{n}\sum_{i=1}^{n} MS_i, \tag{3.2}$$

where $n = 4$ is the number of scores estimated for each stock. As part of the portfolio monitoring process, the portfolio metrics are estimated to evaluate the effectiveness of the strategy. The standard deviation is estimated by first estimating the variance

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}, \\ \sqrt{\sigma_p^2} &= \sigma_p, \end{aligned} \tag{3.3}$$

where σ_p^2 is the variance of return of the portfolio, w_i and w_j are the weights of the i^{th} and j^{th} assets in the portfolio, and σ_{ij} is the covariance between the returns of the i^{th} and j^{th} assets.

The most popular measure of performance is the Sharpe ratio which is used to estimate the risk-adjusted returns. The performance of the portfolio compared to the risk-free asset is measured by the Sharpe ratio (SR_p)

$$SR_p = \frac{R_p - R_f}{\sigma_p}, \tag{3.4}$$

where R_p is the portfolio return and R_f is the risk-free rate.

The Sortino ratio is another performance measure which is similar to the Sharpe ratio. This only uses the negative standard deviation or the downside risk.

$$ST = \frac{R_f - R_p}{\sigma_p(DR)}, \tag{3.5}$$

Skewness measures the asymmetry of the return distribution and is the 3^{rd} moment.

$$Skewenes = \frac{\frac{1}{n}\sum_{p=1}^{n}(R_p - \bar{R_p})^3}{\sigma_p^3}. \tag{3.6}$$

Kurtosis measures tail of the return distribution and is the $4th$ moment.

$$Kurtosis = \frac{\frac{1}{n}\sum_{p=1}^{n}(R_p - \bar{R_p})^4}{\sigma_p^4}. \tag{3.7}$$

3.3 Measures of dependence

There are a number of different measures of association which can be used to measure diversification of a portfolio. These measures work by assessing the association structure of the underlying stocks within the portfolio. The Pearson's correlation coefficient measured in terms of the variance and standard deviations is one way to measure the association between the underlying stocks within a portfolio. Negatively correlated or low positively correlated stocks will not react the same way to market volatility and inefficiencies.

3.3.1 Kendall's tau and Spearman's rho

The Pearson's correlation coefficient is mostly revered for its simplicity and ease of use, however, there are some drawbacks. For example, Bhatti and Nguyen (2012) shows that Pearson's correlation does not capture the dependence structure, especially when dealing with heavy tailed distributions.

In this study, the focus will be on the measures of association, namely the Kendall's tau and the Spearman's rho, both of these measures are non-parametric, thus there is no initial assumption of the distribution (Bolbolian, 2020). They both measure the association between the underlying stocks. A positive and negative value indicates an increasing and a decreasing trend respectively. Kendall's tau (τ) estimates slightly lower correlation values compared to the Spearman's rho (ρ) (Taylor, 1987). Kendall's tau measures the concordance and discordance between the underlying stocks, where a set of stocks would be considered concordant if they react the same way and discordant if they react differently, within the same uncertain conditions. Schweizer and Wolff (1981), were the main contributor to defining the copula Kendall's tau and Spearman's rho. Kendall's tau measure is define as follows:

$$\tau = 4 \int_0^1 \int_0^1 C(u,v)\, dC(u,v) - 1. \tag{3.8}$$

Spearman's rho measure is define as follows:

$$\rho = 12 \int_0^1 \int_0^1 C(u,v)\, du\, dv - 3. \tag{3.9}$$

where $C(u,v)$ is the copula function, u and v are the marginal distributions and θ is the parameter of the copula.

3.3.2 Copulas

The copula information and definitions in this section can be found in (Nelsen, 2006; Jaworski et al., 2010; Durante and Sempi, 2015; Charfeddine et al., 2016). A bivariate copula couples the joint distributions F and G of the random variables X and Y into a copula C as below:

$$H(X,Y) = C(F(X), G(Y)), \tag{3.10}$$

where $H(X,Y)$ is the joint distribution of random variables X and Y, $F(X)$ and $G(Y)$ are the marginal distributions of X and Y, C is the copula which measures the dependence structure and has standard uniform margins. Let $F(X) = u$, $G(Y) = v$, where $u, v \in [0,1]$ are the uniform marginals and θ measures strength of dependence. The copulas to be used in this study are namely, Archimedean copulas, Galambos copula and the Farlie-Gumbel-Morgenstern copula (FGM). These copulas model different temporal dependence structures, and are defined below.

Archimedean copula

An Archimedean copula is constructed using a generator function, which is continuous and strictly decreasing (φ). The copula function is defined by:

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)), \tag{3.11}$$

where $u, v \in [0,1]$ are the uniform marginals, and φ^{-1} is the inverse of the generator function. The most commonly used Archimedean copulas are the Clayton, Frank and Gumbel copulas which are briefly described below.

Clayton copula

The Clayton copula is used to measure lower tail dependence. It models the joint extreme lower tail values. The Clayton copula has a generator function:

$$\varphi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right), \quad \theta \geq -1, \tag{3.12}$$

and is defined by:

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \tag{3.13}$$

where $u, v \in [0, 1]$ are the uniform marginals.

Frank copula

The Frank copula measures symmetric dependence and has a generator function:

$$\varphi(t) = -\log\left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right), \quad -\infty < \theta < \infty, \tag{3.14}$$

and is defined by:

$$C(u, v) = -\frac{1}{\theta}\log\left[1 + \frac{(\exp(-\theta u) - 1)(\exp(-\theta v) - 1)}{\exp(-\theta) - 1}\right]. \tag{3.15}$$

Gumbel copula

The Gumbel copula models upper tail dependence. It captures joint extreme values in the upper tail. The Gumbel copula has a generator function:

$$\varphi(t) = (-\log t)^{\theta}, \quad \theta \geq 1, \tag{3.16}$$

and is defined by:

$$C(u, v) = \exp\left[-\left((-\log u)^{\theta} + (-\log v)^{\theta}\right)^{1/\theta}\right]. \tag{3.17}$$

Galambos copula

The Galambos copula is similar to the Gumbel copula in that it also measures upper tail dependence. This is a bivariate extreme value copula (Loko et al., 2023). The Galambos copula has a generator function:

$$\varphi(t) = \exp(-t^{-\theta}), \quad \theta > 0, \tag{3.18}$$

and is defined by:

$$C(u, v) = uv \cdot \exp\left[-\left((-\ln u)^{-\theta} + (-\ln v)^{-\theta}\right)^{-1/\theta}\right]. \tag{3.19}$$

Farlie-Gumbel-Morgenstern copula (FGM)

The FGM copula measures symmetric dependence. Therefore, this copula does not capture tail dependence and is suitable only for modelling relatively weak dependence. The FGM copula has a generator function:

$$\varphi(t) = \theta t(1-t), \quad -1 \leq \theta \leq 1, \tag{3.20}$$

and is defined by:

$$C(u, v) = uv + \theta uv(1-u)(1-v). \tag{3.21}$$

The copula mixture model will be created using the above individual copulas. Each mixture model will have three copula models and estimation is repeated for the number of combinations. The number of individual copulas is $n = 5$ and each copula mixture model will include $k = 3$ individual copulas.

$$\begin{aligned} C(n,k) &= \binom{n}{k} = \frac{n!}{k!(n-k)!} \\ &= \frac{5!}{3!(5-3)!} \\ &= 10 \end{aligned} \tag{3.22}$$

Table 3.1 shows a list of the 10 combinations of the copula mixtures to be evaluated.

Table 3.1: Combinations of copula mixtures ($k = 3$)

Combination	**Mixture of 3 Copulas**
1.	Clayton-Frank-Gumbel
2.	Clayton-Frank-FGM
3.	Clayton-Frank-Galambos
4.	Clayton-Gumbel-FGM
5.	Clayton-Gumbel-Galambos
6.	Clayton-FGM-Galambos
7.	Frank-Gumbel-FGM
8.	Frank-Gumbel-Galambos
9.	Frank-FGM-Galambos
10.	Gumbel-FGM-Galambos

The Akaike information criterion (AIC) and Bayesian information criterion (BIC) was used to select the best copula mixture. The AIC and BIC evaluate the quality of models and predictive accuracy (Bai et al., 2022).

$$\text{AIC} = 2k - 2\ln(\hat{L}), \tag{3.23}$$

$$\text{BIC} = -2\log(\hat{L}) + k\log(b), \tag{3.24}$$

where k is the number of parameters, L is the maximum likelihood estimation, and b is the number of observations.

3.4 Empirical results

3.4.1 Data

The population for South Africa was the top 100 stocks and for the United States it was the top 300 stocks ordered by market capitalisation. The historical data period was set to be from 01 January 2013 to 30 June 2024, taken from the Yahoo Finance database. The momentum portfolios below consist of the top 20 stocks with the highest average momentum scores for both the SA and US listed stocks.

3.4.2 Momentum portfolio

Table 3.2: US momentum portfolio

Symbol	Description	Momentum Scores				Average Momentum Scores
		3-Months	6-Months	9-Months	12-Months	
PLTR	Palantir Technologies Inc.	157.3301246	192.7941168	270.6648957	354.5661964	243.8388334
COIN	Coinbase Global, Inc.	33.71050298	36.08117527	251.5774054	337.6659407	164.7587561
SPOT	Spotify Technology S.A.	49.39967829	91.35405698	192.2456217	201.2559917	133.5638372
NVDA	NVIDIA Corporation	24.0360578	54.83924825	201.2552576	227.2972322	126.856949
KKR	KKR & Co. Inc.	33.40987643	65.51319997	155.315876	184.9992076	109.80954
IBKR	Interactive Brokers Group, Inc.	56.90807055	75.00490626	130.9724545	127.1647467	97.51254452
CEG	Constellation Energy Corporation	44.92066163	45.36947831	109.9824278	175.6735468	93.98652864
APO	Apollo Global Management, Inc.	42.64746566	57.8629053	109.07043	130.830358	85.10278973
DASH	DoorDash, Inc.	66.30342763	34.72827098	98.25103153	130.6529752	82.48392632
ANET	Arista Networks Inc	28.12042367	43.13745407	93.17497208	153.4184923	79.46283553
NFLX	Netflix, Inc.	42.3471977	43.77356639	105.34733497	102.606198	73.51857425
BX	Blackstone Inc.	33.02091086	53.16969538	93.4163885	109.2154505	72.20561132
SHOP	Shopify Inc.	86.64607439	44.13130982	81.1395773	72.9798845	71.2242115
TSM	Taiwan Semiconductor Manufacturing Company Limited	21.73696357	44.88458465	113.3145638	96.34073778	69.06921246
GS	The Goldman Sachs Group, Inc.	23.95273243	58.09085011	92.58258079	95.0373288	67.41587303
TT	Trane Technologies plc	23.48385227	39.25786595	88.22131133	116.9701602	66.98329742
CRWD	CrowdStrike Holdings, Inc.	34.45385196	8.452292861	80.90040023	136.9306343	65.18429483
MRVL	Marvell Technology, Inc.	51.47453997	50.37537662	91.35941792	64.17960582	64.34723508
BKNG	Booking Holdings Inc.	39.16645515	49.39931924	74.35222528	94.46934335	64.34683576
META	Meta Platforms, Inc.	30.31468713	22.41880844	90.74575061	111.9087216	63.84699193

Table 3.2 shows the portfolio constructed from the top 300 large-cap US stocks. PLTR has the highest momentum score of 243.84. This implies that the stock has seen significant growth over the past 12 months. There is currently a boom around AI (artificial intelligence), and some of the company's operations include an AI offering. This stock was recently added to the S&P500. The lowest average within the portfolio is another technology stock, META, with an average of 63.85. The US momentum portfolio consists mostly of technology stocks. As a result, diversification might be reduced.

Table 3.3: SA momentum portfolio

Symbol	Description	Momentum Scores				Average Momentum Scores
		3-Months	6-Months	9-Months	12-Months	
PAN.JO	Pan African Resources PLC	29.47673667	64.41101808	143.3184301	207.4361285	111.1605783
MRP.JO	Mr Price Group Limited	39.8192601	73.43291762	119.060321	125.0605565	89.34326381
OUT.JO	OUTsurance Group Limited	39.07623351	72.9979009	60.63199577	103.4555054	69.04040891
CPI.JO	Capitec Bank Holdings Limited	17.03126693	57.81828908	75.3050685	117.2920751	66.86167491
HYP.JO	Hyprop Investments Limited	50.81475405	56.40228356	81.41730123	77.66040029	66.57368478
TFG.JO	The Foschini Group Limited	28.61250369	69.86830026	62.49812042	98.02086439	64.74994719
QLT.JO	Quilter plc	17.84180054	52.93010741	84.96534903	99.39475786	63.78300371
WBO.JO	Wilson Bayly Holmes-Ovcon Limited	15.37695345	56.05442202	65.72696007	105.4553552	60.65342268
KRO.JO	Karooooo Ltd.	23.26656853	66.42912805	65.47953011	76.64939626	57.95615574
HAR.JO	Harmony Gold Mining Company Limited	2.223585481	17.97495003	83.67364899	114.1143513	54.49663396
FFB.JO	Fortress Real Estate Investments Limited	22.02137221	42.36374563	68.09874817	70.06098384	50.63621246
MTM.JO	Momentum Group Limited	20.23356151	48.91840501	45.68692383	87.03937129	50.46956541
TRU.JO	Truworths International Limited	19.70478371	37.81713038	35.02218451	106.9738844	49.87949576
PPH.JO	Pepkor Holdings Limited	35.51775888	42.42902208	50.15058951	71.03307404	49.78261113
RES.JO	Resilient REIT Limited	24.71544749	41.11883365	64.2886617	63.05965982	48.29565066
AVI.JO	AVI Limited	17.68827897	25.38070512	63.83877184	80.82696204	46.93367949
AIL.JO	African Rainbow Capital Investments Limited	22.80701754	79.06976744	54	18.46153846	43.58458086
DSY.JO	Discovery Limited	42.88147916	47.29187433	45.85909264	37.65622929	43.42216886
LHC.JO	Life Healthcare Group Holdings Limited	30.8801214	58.4794976	48.0006527	35.16905314	43.13233121
RDF.JO	Redefine Properties Limited	17.05237747	26.26495992	53.51123584	74.95406014	42.94565834

Table 3.3 shows the portfolio constructed from the top 100 large-cap SA stocks. The stock with the highest average momentum score for the SA momentum portfolio is not a technology stock. PAN is an industrial stock with an average of 111.16. This is lower compared to the PLTR stock in the US momentum stock. The SA momentum portfolio is much more balanced in terms of the industry exposure and this implies that it might be a well-diversified portfolio.

Table 3.4: Large-cap and momentum portfolio metrics comparison

Metric	US		SA	
	Large-Cap	Portfolio	Large-Cap	Portfolio
Standard Deviation	0.1785	0.3422	0.1408	0.1456
Sharpe Ratio	0.8187	0.9737	0.8185	1.3006
Sortino Ratio	1.2180	1.5150	1.3530	2.1628
Spearman's Rho	0.34561	0.4932	0.19400	0.1881
Kendall's Tau	0.24087	0.34937	0.1330	0.1296
Downside Risk	-0.1867	-0.3734	-0.1366	-0.1538
Skewness	-0.0373	0.0150	0.2573	0.2476
Kurtosis	1.6882	0.9814	1.2162	2.1445

Table 3.4 shows the back tested metrics of both the US and SA population of large-cap stocks and the constructed momentum portfolios.

- The US momentum portfolio is the most volatile with a standard deviation of 0.3422. The SA Large-cap population of stocks is least volatile of the 4, with a standard deviation of 0.1408.
- The returns are not normally distributed and are not heavily skewed. The kurtosis is below 3 for both the US and SA portfolios, indicating thin tailed return distributions.
- The kurtosis for both the large-cap population of stocks and the momentum portfolios are all less than 3.

- The Spearman's rho shows that there is a minimal positive monotonic dependence between the stocks in the portfolio, which indicates that the portfolio is well- diversified.
- The Kendall's tau implies minimal positive dependence between the stocks. Britt and Napoli (2005) show that the Kendall's tau and Spearman's rho are non-parametric monotonic measures of association that assesses the monotonic relationship between variables.
- The Sharpe ratio indicates that the momentum portfolio in both countries are higher than their large-cap population, and furthermore, the SA momentum portfolio has the highest Sharpe ratio.
- The SA momentum portfolio has the highest Sortino ratio. Therefore, considering only negative deviations, for each unit of the downside risk there are approximately two times the returns.

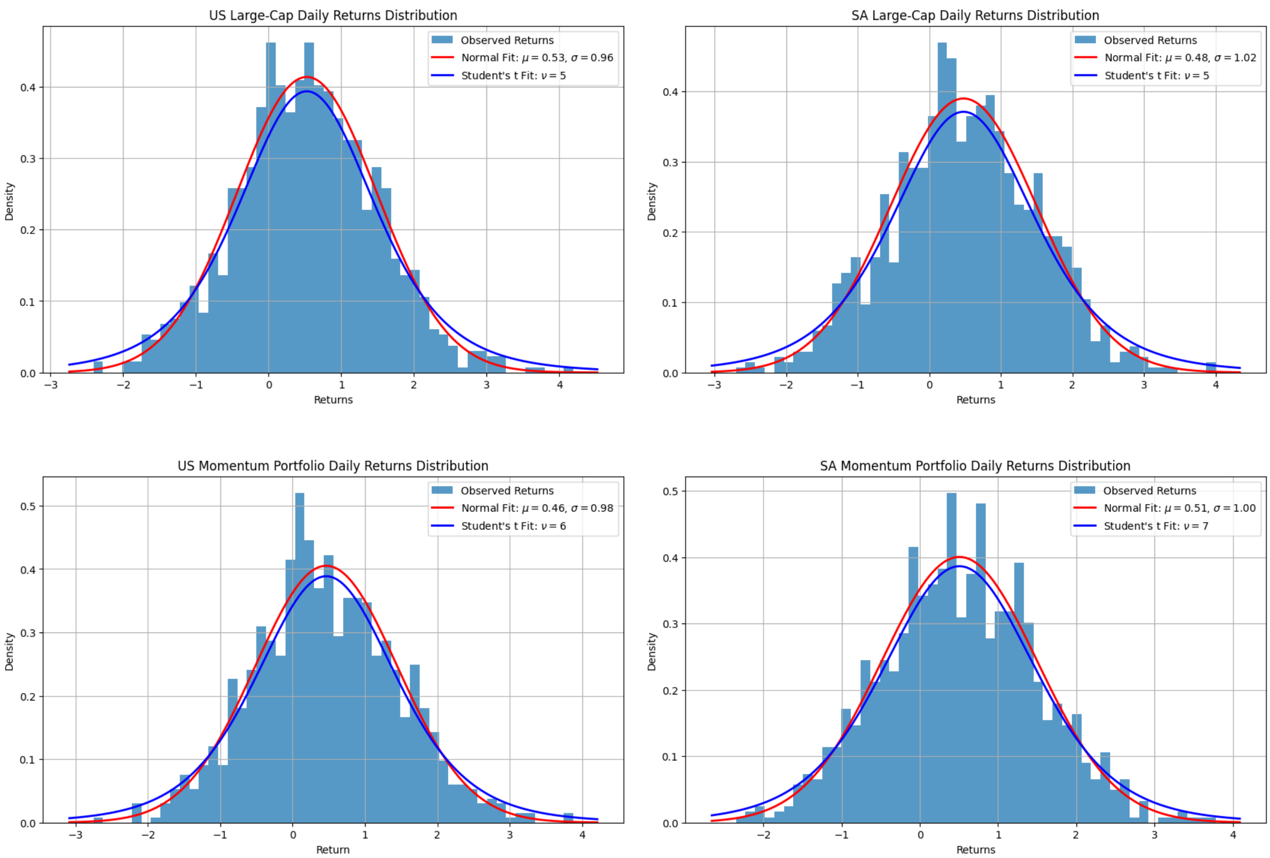


Figure 3.1: Large-cap and momentum portfolio daily returns distribution comparison for both the US and SA

Figure 3.1 shows four plots consisting of the US large-cap and momentum portfolio together with the SA Large-cap and momentum portfolio. Fitted are the normal and student-t distributions on the historical daily returns. The returns are not normally distributed and this is supported by the kurtosis that is less than 3 for all four portfolios and skewness that is less than 0. Therefore, the Gaussian and student-t copulas will not be included in the mixture model.

3.4.3 Portfolio's pairwise matrix Kendall's tau and Spearman's rho

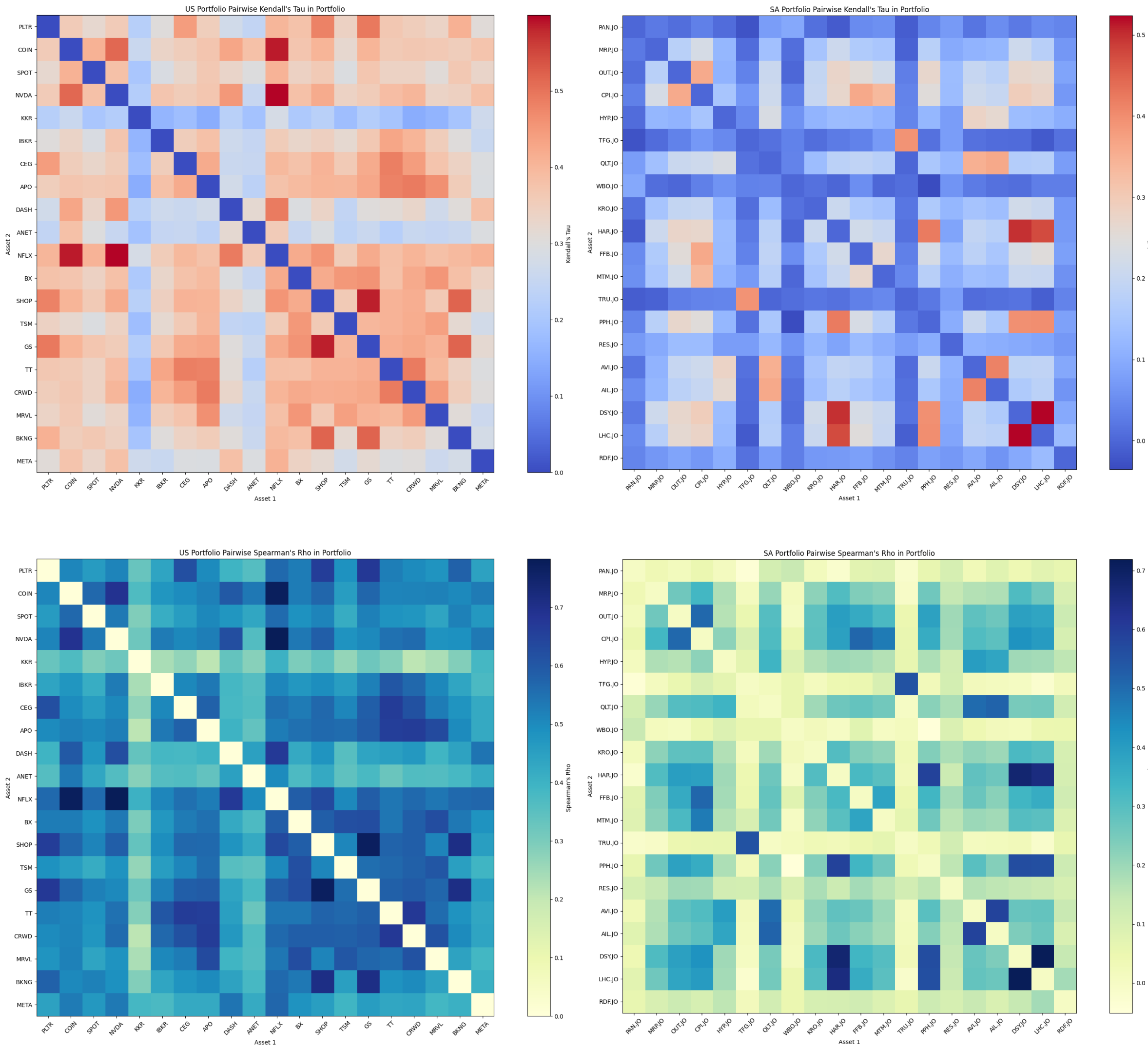


Figure 3.2: Kendall tau and Spearman's rho comparison

Figure 3.2 shows the matrices of the Kendall's tau and the Spearman's rho for the US and SA momentum portfolio. The Kendall's tau is an ordinal rank correlation which measures the strength and direction of association of the underlying stocks. This is measured in terms of concordance and discordance, and directly makes a pairwise comparison. The Spearman's rho is a nominal rank measure that estimates the Pearson correlation coefficient between the pairwise underlying stocks and does not directly make a pairwise comparison between the underlying stocks, but rather provides pairwise numeric values. As a result, the Spearman's rho will have slightly higher values compared to that of the Kendall's tau. The overall US momentum portfolio Kendall's Tau is 0.34937 , which is higher than the SA momentum Kendall's Tau of 0.1296. The red indicates a strong positive association and the navy blue indicate little to no association between the bivariate stocks. The overall US momentum portfolio Spearman's rho is 0.4932 which is higher than the SA momentum Spearman's rho of 0.1881. The Spearman's rho values above show that there are no negative correlation values, the dark blue indicates strong positive association and the yellow indicate little to no association between the bivariate pair. The SA momentum portfolio is more diversified than the US momentum portfolio as a result of the lower Kendall's tau and Spearman's rho.

3.4.4 Copula mixture models

Table 3.5: Mixture of 3 copulas: US and SA Comparison

Copula Mixture			US		SA	
			Large-Cap	Portfolio	Large-Cap	Portfolio
Clayton-Frank-Gumbel	Clayton	Percentage	43.56 %	11.62 %	14.89 %	0.00 %
		Parameter	0.2579	2.1132	0.4374	2.1230
	Frank	Percentage	4.76 %	50.98 %	0.00 %	7.06 %
		Parameter	-12.0242	3.7593	-5.8454	-1.7563
	Gumbel	Percentage	51.69 %	37.40 %	85.11 %	92.94 %
		Parameter	1.1390	1.4227	1.0507	1.0604
Clayton-Frank-FGM	Clayton	Percentage	34.70 %	8.52 %	13.50 %	3.07 %
		Parameter	0.2698	2.1366	0.4828	1.00e-06
	Frank	Percentage	33.36 %	86.30 %	39.83 %	22.82 %
		Parameter	3.3307	3.8387	-0.0039	3.7143
	FGM	Percentage	31.94 %	5.18 %	46.67 %	74.11 %
		Parameter	-1.00	-1.00	0.3754	-0.2591
Clayton-Frank-Galambos	Clayton	Percentage	44.33 %	8.73 %	12.45 %	0.00 %
		Parameter	0.3449	2.1284	0.5853	0.5165
	Frank	Percentage	3.14 %	79.31 %	0.00 %	0.00 %
		Parameter	-15.8522	3.9987	-0.1936	-0.0146
	Galambos	Percentage	52.53 %	11.96 %	87.55 %	100 %
		Parameter	0.2539	0.2356	0.2370	0.2249
Clayton-Gumbel-FGM	Clayton	Percentage	36.84 %	19.13 %	14.88 %	0.00 %
		Parameter	0.3346	2.0694	0.4376	0.0018
	Gumbel	Percentage	39.22 %	54.89 %	85.12 %	93.68 %
		Parameter	1.2665	1.5902	1.0507	1.0611
	FGM	Percentage	23.94 %	25.98 %	0.00 %	6.32 %
		Parameter	-1.00	0.8873	0.0180	-1.00
Clayton-Gumbel-Galambos	Clayton	Percentage	26.91 %	25.38 %	14.89 %	0.00 %
		Parameter	0.3157	1.7356	0.4372	0.0102
	Gumbel	Percentage	22.29 %	69.46 %	85.11 %	100 %
		Parameter	1.3118	1.5056	1.0507	1.0460
	Galambos	Percentage	50.80 %	5.16 %	0.00 %	0.00 %
		Parameter	0.0042	0.0467	0.0544	0.0005
Clayton-FGM-Galambos	Clayton	Percentage	40.49 %	28.61 %	12.46 %	0.00 %
		Parameter	0.3595	2.4150	0.5852	0.00
	FGM	Percentage	3.69 %	71.39 %	0.00 %	0.00 %
		Parameter	-1.00	1.00	0.1292	-0.0094
	Galambos	Percentage	55.82 %	0.00 %	87.54 %	100 %
		Parameter	0.2296	0.5733	0.2370	0.2249
Frank-Gumbel-FGM	Frank	Percentage	33.62 %	66.45 %	3.48 %	7.05 %
		Parameter	2.8263	4.2302	-20.0554	-1.7579
	Gumbel	Percentage	30.55 %	29.35 %	73.57 %	92.95 %
		Parameter	1.2687	1.5021	1.0515	1.0604
	FGM	Percentage	35.83 %	4.20 %	23.04 %	0.00 %
		Parameter	-1.00	-1.00	1.00	-1.00
Frank-Gumbel-Galambos	Frank	Percentage	2.63 %	62.96 %	9.61 %	5.97 %
		Parameter	-15.6827	4.3335	1.2235	-1.9333
	Gumbel	Percentage	39.73 %	29.74 %	19.12 %	67.09 %
		Parameter	1.3197	1.4979	1.1131	1.0755
	Galambos	Percentage	57.64 %	7.30 %	71.28 %	26.94%
		Parameter	0.0013	0.00	0.2315	0.1853
Frank-FGM-Galambos	Frank	Percentage	50.15 %	86.92 %	0.00 %	0.00 %
		Parameter	3.0871	4.2481	-2.3051	0.1764
	FGM	Percentage	36.06 %	0.00 %	12.18 %	0.00 %
		Parameter	-1.00	0.4611	1.00	0.1810
	Galambos	Percentage	13.79 %	13.08 %	87.82 %	100 %
		Parameter	0.2156	0.2136	0.2364	0.2249
Gumbel-FGM-Galambos	Gumbel	Percentage	36.30 %	55.07 %	11.18 %	93.69 %
		Parameter	1.3625	1.7962	1.1099	1.0611
	FGM	Percentage	14.25 %	44.93 %	10.59 %	6.31 %
		Parameter	-0.4480	0.9913	1.00	-1.00
	Galambos	Percentage	49.45 %	0.00 %	78.23 %	0.00 %
		Parameter	0.0311	0.4343	0.2290	0.0083

Table 3.5 shows the percentage and the parameters for each copula mix. The maximum likelihood estimation method was applied in the estimation of the parameters.

- The percentages explain the dependence that can be attributed to that specific copula within the copula mixture model. The data that has a copula with 100% attributed to it, implies that the copula mixture is not the best choice or fit.
- The Clayton copula has a parameter property ($\theta \geq -1$) and in all the copula mixtures ($\theta > 0$) this indicates positive dependence. with the exception of the Clayton-FGM-Galambos mixture. The copula is valid within all the mixture models.
- The Frank parameter ($-\infty < \theta < \infty$) is satisfied within all the mixture copulas.
- The Gumbel copula parameter property (≥ 1), is satisfied within all the mixture copulas.
- The Galambos copula parameter ($\theta > 0$), are mostly valid, The variation of the results for a mixture with this copula is in the data.
- The FGM copula ($-1 \leq \theta \leq 1$) is also satisfied within the copula mixture models, however when in mixture with the Frank it tends to either be ± 1, indicating that combining the Frank and the FGM copulas in a mixture will not provide any useful information on the dependence structure.
- Extreme events that affect the stock market do not occur regularly and the returns are not heavy tailed, as seen in Figure 3.1. This is also supported by the skewness and kurtosis in Table 3.4.

Table 3.6: AIC and BIC Copula mixture

Copula Mixture		**US**		**SA**	
		Large-Cap	**Portfolio**	**Large-Cap**	**Portfolio**
Clayton-Frank-Gumbel	AIC	-3.4111	-178.4974	4.1632	6.7555
	BIC	23.0991	-151.9873	29.9679	34.6026
Clayton-Frank-FGM	AIC	-1.1120	-175.6154	6.0688	8.1068
	BIC	25.3982	-149.1053	31.8735	35.9538
Clayton-Frank-Galambos	AIC	-1.1625	-175.5239	3.7802	7.5466
	BIC	25.3477	-149.0137	29.5849	35.3937
Clayton-Gumbel-FGM	AIC	-2.8078	-174.7013	4.1632	6.7755
	BIC	23.7024	-148.1911	29.96790	34.6226
Clayton-Gumbel-Galambos	AIC	-1.7139	-173.6316	4.16319	6.9012
	BIC	24.7963	-147.1214	29.9679	34.7483
Clayton-FGM-Galambos	AIC	0.2329	-157.3101	3.7802	7.5466
	BIC	26.7431	-130.7999	29.5849	35.3937
Frank-Gumbel-FGM	AIC	-0.8818	-176.4907	3.9649	6.7555
	BIC	25.6284	-149.9805	29.7697	34.6026
Frank-Gumbel-Galambos	AIC	-1.0844	-176.3635	5.0483	6.7395
	BIC	25.4257	-149.8533	30.8530	34.5866
Frank-FGM-Galambos	AIC	0.2598	-173.9274	4.8809	7.5466
	BIC	26.7700	-147.4172	30.6856	35.3937
Gumbel-FGM-Galambos	AIC	-0.2446	-167.6904	4.8530	6.7755
	BIC	26.2656	-141.1802	30.6578	34.6226

Table 3.6 shows the results of the AIC and BIC.

- The Clayton-Frank-Gumbel copula mixture has the lowest AIC and BIC for both the US large-cap stocks and the US momentum portfolios.
- The Clayton-Frank-Galambos and Clayton-FGM-Galambos copula mixtures have identical AIC and BIC values, which are also the lowest for the SA large-cap stocks.
- The Frank-Gumbel-Galambos copula mixture has the lowest AIC and BIC for the SA momentum portfolio.
- The US and SA momentum portfolios have the same top three copula mixtures, i.e., Clayton-Frank-Gumbel, Frank-Gumbel-FGM, and the Frank-Gumbel-Galambos. Although they contain stocks from different markets and have various levels of diversification, both are momentum portfolios that behave similarly. Hence, they have dependence structures can be modelled using the same copula mixtures.

3.5 Conclusion

This study commences with the construction of a momentum portfolio, using the historical data from large-cap stocks. The top 20 stocks with the highest average momentum scores over 3, 6, 9 and 12 months were selected and included in the portfolio. This construction was done for the stocks listed in SA, as well as those that are listed in the US. The resulting portfolios included stocks that significantly outperformed expectations. The SA momentum portfolio had a more varied industry exposure, compared to the US momentum portfolio which consisted mainly of technology stocks.

The Kendall's tau and Spearman's rho were employed on both momentum portfolios to assess whether they were diversified. The SA Momentum portfolio had a lower Kendall's tau (0.1296) and Spearman's rho (0.1881) compared to the US momentum portfolio, which had a higher Kendall's tau (0.3494) and Spearman's rho (0.4932). Although both portfolios can be considered diversified, the SA momentum portfolio is more diversified with lower volatility ($\sigma_p = 0.1456$) compared to the volatility ($\sigma_p = 0.3422$) from the US momentum portfolio.

Furthermore, a combination of 10 copula mixtures were evaluated for both countries, using the populations from which the portfolios were selected (i.e., large-cap stocks) and the momentum portfolios. The copula mixtures are all valid satisfying their individual parameter boundaries within the mixture. Mixture copula models were applied to the population of the large-cap stocks, and the momentum portfolios. The US and SA momentum portfolios have the same top three copula mixture models. They have dependence structures that can be modelled by the same copula mixture models. The results signify that a pure momentum portfolio is well diversified and although the stocks are listed in different countries, they behave the same for momentum factor.

References

[1] Abraj, M., Wang, Y.G. and Thompson, M.H. (2022). A new mixture copula model for spatially correlated multiple variables with an environmental application. *Scientific Reports*, 12, 13867.

[2] Alzaid, A.A. and Alhadlaq, W.M. (2024). A New Family of Archimedean Copulas: The Half-Logistic Family of Copulas. *Mathematics*, 12(1), 101.

[3] Bai, Z., Choi, K.P., Fujikoshi, Y. and Hu, J. (2022). Asymptotics of AIC, BIC and C_p model selection rules in high-dimensional regression. *Bernoulli*, 28(4), 2375-2403.

[4] Bhatti, M. I., and Nguyen, C. C. (2012). Diversification evidence from international equity markets using extreme values and stochastic copulas. *Journal of International Financial Markets, Institutions and Money*, 22, 622–646.

[5] Bolbolian G., M. (2020). Relationship Between Kendall's tau Correlation and Mutual Information [Relación entre la correlación tau de Kendall e información mutua]. *Revista Colombiana de Estadística*, 43(1), 3-20.

[6] Britt, S. and Napoli, A. (2005). Linear correlation as a measure of dependency. Presented at the *Institute of Actuaries of Australia XVth General Insurance Seminar*, 16-19 October 2005.

[7] Charfeddine, L., and Benlagha, N. (2016). A time-varying copula approach for modelling dependency: New evidence from commodity and stock markets. *Journal of Multinational Financial Management*, 37–38, 168–189.

[8] Chollete, L., de la Peña, V., and Lu, C. C. (2011). International diversification: A copula approach. *Journal of Banking and Finance*, 35, 403–417.

[9] Durante, F. and Sempi, C. (2015). Principles of Copula Theory (1st ed.), New York: *Chapman and Hall/CRC*.

[10] Flint, E., Seymour, A., and Chikurunhe, F. (2020). Defining and measuring portfolio diversification. *South African Actuarial Journal*, 20(1), 17–48.

[11] Frahm, G., and Wiechers, C. (2011). On the diversification of portfolios of risky assets. Discussion Papers in Statistics and Econometrics, No. 2/11, *University of Cologne, Institute of Econometrics and Statistics.*

[12] Goetzmann, W. N., and Kumar, A. (2008). Equity Portfolio Diversification. *Review of Finance*, 12, 433–463.

[13] Jaworski, P., Durante, F., Härdle, W.K. and Rychlik, T. (2010). Copula Theory and Its Applications: Proceedings of the Workshop Held in Warsaw, 25–26 September 2009. (1st ed.), Berlin: *Springer Science and Business Media.*

[14] Jegadeesh, N. and Titman, S. (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of Finance*, 48(1), 65-91.

[15] Ji, H., Wang, H., and Liseo, B. (2018). Portfolio diversification strategy via tail dependence clustering and ARMA-GARCH vine copula approach. *Australian Economic Papers*.

[16] Kasa, S.R., Bhattacharyya, S., and Rajan, V. (2020). Gaussian mixture copulas for high-dimensional clustering and dependency-based subtyping. *Bioinformatics*, 36(2), 621–628.

[17] Koumou, G. B. (2020). Diversification and portfolio theory: a review. *Financial Markets and Portfolio Management*, 34(3), 267–312.

[18] Loko, R., Vouvoungui, C., Koukouatikissa, R., Bidounga, R., and Barro, D. (2023). A Singular Property of Galambos Copula. *Advances in Mathematics: Scientific Journal*, 12(7), 673–683. DOI: 10.37418/amsj.12.7.3

[19] Mao, J. C. T. (1970). Essentials of Portfolio Diversification Strategy. *The Journal of Finance*, 25(5), 1109-1121.

[20] Martínez-Nieto, L., Fernández-Navarro, F., Carbonero-Ruz, M., and Montero-Romero, T. (2021). An experimental study on diversification in portfolio optimization. *Expert Systems With Applications*, 181, 115203.

[21] Nelsen, R. B. (2006). An Introduction to Copulas (2nd ed.). *Springer*.

[22] Sankaran, J. K., and Patil, A. A. (1999). On the optimal selection of portfolios under limited diversification. *Journal of Banking and Finance*, 23(1999), 1655-1666.

[23] Schweizer, B. and Wolff E.F . (1981). On nonparametric measures of dependence for random variables. *The annals of statistics*, 9(4), 879-885.

[24] Taylor, J. M. (1987). Kendall's and Spearman's correlation coefficients in the presence of a blocking variable. *Biometrics*, 43(2), 409-416.

[25] Tewari, A. (2023). On the Estimation of Gaussian Mixture Copula Models. Proceedings of the *40th International Conference on Machine Learning*, Honolulu, Hawaii, USA, PMLR 202.

[26] Zaimovic, A., Omanovic, A., and Arnaut-Berilo, A. (2021). How Many Stocks Are Sufficient for Equity Portfolio Diversification? A Review of the Literature. *Journal of Risk and Financial Management*, 14(11), 551. https://doi.org/10.3390/jrfm14110551

Chapter 4

A comparison of value and growth investment strategies using the Hidden Markov Model with switching and the Kaplan-Meier method

Abstract

The value versus growth debate has been going on among financial practitioners and academics alike for a very long time. In the past, value was superior, outperforming both growth and the overall market. However, the dynamics surrounding these two investment strategies have since shifted with growth now outperforming both value and the overall market. The purpose of this article is to assess and compare the dynamics surrounding these two factor investment strategies. First, a fund of funds composed of the three largest exchange-traded funds ranked by assets under management listed in the United States, were constructed for both value and growth. Second, the hidden Markov Model with switching was used to identify the hidden states from historical daily returns. Third, the Kaplan-Meier method, predominantly used in survival analysis, was adjusted to fit the premise of the hidden Markov Model to identify the persistence in days of the hidden states. Lastly, the Hidden Markov Model was used for factor timing. The results show that value recovered faster than growth after a negative impact on the markets. The Hidden Markov Model factor timing strategy had higher returns for a fund of funds consisting of both value and growth.

Keywords

Exchange-traded funds, factor investing, factor timing, fund of funds, Markov switching autoregressive model.

4.1 Introduction

Value and growth are complementary popular investment factors, heavily embedded in literature and highly regarded by investors (Siegel and Alexander, 2000). Value investing has been around for a very long time and was first conceptualised by Graham and Dodd (1934) who introduced the idea of value, which has evolved significantly since then, but still draws on the foundations presented in that text. Basu (1983) and Fama and French (1992) further developed value as a strategy, by stating the relationship that exist between price-to-earnings ratios and returns. Growth, on the other hand, was conceptualised by Price (1950), who stated that growth stocks are those that have consistently grown previously without an indication of slowing down. Value is a fundamental factor, and long-term investors tend to prefer value as opposed to growth (Roca, 2021). A value strategy is characterised by the practice of buying seemingly under-priced stocks trading below their perceived intrinsic value, using fundamental analysis (Stagnol et al., 2021). Growth focuses on the earnings growth potential of a company. The growth factor uses metrics that are essential for determining or valuing the growth potential of a business, such as profitability, competitive advantage and business strategy.

Although value is measured by metrics such as price-to-book ratios and price-earnings ratios it can be quite objective and also depends on the view that an investor might have on these metrics and stocks (Kok et al., 2017). The ratios used to identify value stocks may be misleading, as they might understate the challenges within the company. For example, internal turmoil and financial challenges (such as being on the verge of bankruptcy), which is the reason why they are priced below their intrinsic values and as a result this can make them riskier (Chen and Zhang, 1998). The financial ratios in this case will be low, with too many variables affecting their future earnings and profitability.

In the past, value was deemed superior to growth, for example, Capaul et al. (1993), Fama and French (1998), Chan and Lakonishok (2004), Elze (2010), Beukes (2011), Asness et al. (2015) and Teti et al. (2019) evaluated a value vs growth strategy and concluded that value outperforms growth. This outperformance was due to value stocks being more established trusted brands with a solid history and strong fundamental analysis results.

There used to be a general overconfidence in the metrics of these stocks, as they had seemingly great and dependable business management strategies. Perez (2017) showed that classifying stocks as value or growth required more than just low price-to-earnings ratios, with the portfolio optimisation process being significant in how the strategy performs. However, for value stocks, it really did not matter historically which metrics were used as long as the they had low price-to-book ratios, they would still manage to outperform growth stocks (Athanassakos, 2009).

Asness et al. (2000) noted that value investing may result in long-term periods of low returns. Even though in the past value outperformed growth, value was susceptible to market downturns and resulted in underperformance against the overall market, Penman and Reggiani (2018) called this the value trap.

The consensus on the value vs growth debate has shifted, and the dynamics have changed, which shows that a strategy that worked well previously might not be as effective in the future (Abhyankar et al., 2008). Value is a strategy that is currently not favoured by investors, (i.e. value is currently out of favour while growth is preferred). The value factor is heavily linked to economic conditions, for over a decade this factor has not been performing as well as expected. Its underperformance since 2007 has received criticism ranging from "value is dead" to "value is not an effective strategy for higher returns" (Israel et al., 2022). Kakinuma (2017) found that in the long run it does perform, value investing requires a long-term patient approach, portfolios with low price-to-earnings (P/E) and price-to-book (P/B) ratios produce higher returns and is an effective long-term strategy with better risk-return results.

In this article, value and growth fund of funds will be constructed, using the top three exchange-traded funds with the highest assets under management (AUM). A Hidden Markov Model (HMM) will be applied to the value and growth fund of funds with the aim of discovering the hidden states that impact value and growth returns. The hidden states will show a period of low variance (stable returns) and high variance (volatile returns). The Kaplan-Meier method will also be applied to measure persistence (number of days) of the hidden states, further showing how long a portfolio will have stable returns and volatile returns.

This article starts with the introduction of value and growth in Section 4.1, followed by the methodology in Section 4.2. The empirical results are found in Section 4.3. Finally, the conclusion is in Section 4.4.

4.2 Methodology

4.2.1 Hidden Markov Model

Value and growth have historical returns with hidden unobserved events in the market. The returns will transition from a state of low variance (stable and predictable) to a state of high variance (highly volatile and unstable) and vice versa. In the Hidden Markov Model these transitions follow a Markov process. The Hidden Markov Model can be found in Rabiner and Juang (1986), Rabiner (1989), Shumway and Stoffer (2017), Netzer et al. (2017), Cole (2019), Mor et al. (2021), Hassan et al. (2021) and Ravari et al. (2024).

Let $\{X_t\}$ be a time series of historical returns indexed by time period $\{t = 1, 2, 3 \ldots\}$ consisting of historical returns $\{X_{t-1}, X_{t-2}, \ldots, X_1, X_0\}$, current returns $\{X_t\}$ and future returns $\{X_{t+1}, X_{t+2}, \ldots, X_T\}$ characterised by the Markov property Baum and Petrie (1966). The Markov property is a memoryless characteristic of a stochastic process, where $\{X_{t+1}, X_{t+2}, \ldots, X_T\}$ depends on $\{X_t\}$ and $\{X_{t-1}, X_{t-2}, \ldots, X_1, X_0\}$ is of no significance.

The probability of predicting a series of outcomes based on the most current information is known as the Markov chain . Hidden Markov Models are a type of state space models where the system being modelled is assumed to follow a Markov process with unobserved hidden states $s_t = \{s_1, s_2, \ldots, s_m\}$. The Markov property is defined by:

$$P(X_{t+1} = s \mid X_t = s_t, X_{t-1} = s_{t-1}, \ldots, X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t), \tag{4.1}$$

for all $t = 1, 2, 3, \ldots$ and for all states $s_0, s_1, \ldots, s_t, s$. Hence,

$$P(X_{t+1} = s \mid X_t = s_t, X_{t-1} = s_{t-1}, \ldots, X_1, X_0 = s_0). \tag{4.2}$$

A Markov chain consisting of two states, 0 and 1, has the transition matrix $A = [a_{ij}]$ that describes the probabilities of moving between these states at each time step.

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \tag{4.3}$$

$$a_{00} + a_{01} = 1, \quad a_{10} + a_{11} = 1. \tag{4.4}$$

where a_{00} is the probability of staying in state 0, a_{01} is the probability of transitioning from state 0 to state 1, a_{10} is the probability of transitioning from state 1 to state 0 and a_{11} is the probability of staying in state 1. Let s_t be hidden states at time $t = 0, 1, \ldots, m$, which follows a Markov process with m possible states. The initial state distribution $\pi = [\pi_i]$, is

$$\pi_i = P(s_1 = i), \tag{4.5}$$

which is the probability that the system starts in state i, with some state $j = 0$. The probability of a state j at time t is given by

$$a_j(t) = P(s_t = j). \tag{4.6}$$

The probability of transitioning between states depends only on the current state. The state transition probability from state i at time t to state j at time $t + 1$ is

$$a_{ij} = P(s_{t+1} = j \mid s_t = i), \tag{4.7}$$

where

$$\sum_{j=1}^{m} a_{ij} = 1, \quad \forall ij. \tag{4.8}$$

Let $\{X_t\}$ be the observed data at time $t = 0, 1, \ldots$ independent of the hidden states, with parameters that depend on the current state. The observation probability of starting in state j at time t is $B = [b_j(X_t)]$, which is the probability of observing a particular output given the current state. $b_j(X_t) = P(X_t|s_t = j)$ is the probability of observing X_t given that the system is in state j at time t.

$$B = b_j(X_t) = P(X_t \mid s_t = j). \tag{4.9}$$

Let θ be a HMM automaton, such that

$$\theta = (A, B, \pi). \tag{4.10}$$

The probability of an observed sequence can be computed by the forward algorithm. Suppose $\alpha_t(j)$ is the probability of the partial observation sequence $\{X_1, X_2, \ldots, X_t\}$ up to time t, with the system being in state j at time t. Then,

$$\alpha_t(j) = P(X_1, X_2, \ldots, X_t, s_t = j|\theta), \tag{4.11}$$

where $\alpha_t(j)$ is the joint probability of the observed sequence up to time t. At time $t = 1$, the initial probability of being in state j and observing X_1 is

$$\alpha_1(j) = \pi_j b_j(X_1), \quad 1 \leq j \leq m. \tag{4.12}$$

The probability of reaching state j at time $t + 1$ given the sequence of observations up to time t is the sum over all possible prior states i:

$$\alpha_{t+1}(j) = \sum_{i=1}^{m} \alpha_t(i) a_{ij} b_j(X_{t+1}), \quad 1 \leq j \leq m, 1 < t \leq T. \tag{4.13}$$

After the final time step T, the probability of observing the entire sequence is the sum of the probabilities of being in any of the states at time T:

$$P(X_t|\theta) = \sum_{i=1}^{m} \alpha_T(i). \tag{4.14}$$

The most probable sequence of states given the observed sequence can be found by the Viterbi algorithm. Suppose $\delta_t(i)$ is the probability of the most likely state sequence up to time t that ends in state i at time t, given the observation sequence.

$$\delta_t(i) = \max_{s_1,\ldots,s_{t-1}} P(s_1, s_2, \ldots, s_t = s_i, X_1, X_2, ..., X_t|\theta). \tag{4.15}$$

At time $t = 1$, the probability is given by

$$\delta_1(j) = \pi_j b_i(X_1), \quad 1 \leq j \leq m, \tag{4.16}$$

$$b_1(j) = 0, \quad 1 \leq j \leq m. \tag{4.17}$$

The most likely state sequence ending at state j at time $t + 1$ is determined by considering the most likely sequence ending in all possible previous states i.

$$\delta_t(j) = \max_{i=1,\ldots,m} \delta_{t-1}(i) a_{ij} b_j(X_t), \quad 1 \leq j \leq m, 1 < t \leq T, \tag{4.18}$$

$$b_t(j) = \arg \max_{i=1,\ldots,m} \delta_{t-1}(i) a_{ij} b_j(X_t), \quad 1 \leq j \leq m, 1 < t \leq T, \tag{4.19}$$

$$\delta_{t+1}(j) = \max_{i=1,\ldots,m} \delta_t(i) a_{ij} b_j(X_{t+1}). \tag{4.20}$$

Once the final probabilities are calculated, the most probable state sequence is determined by backtracking through the states with the highest probabilities. Let $\beta_t(i)$ be the probability of observing the observations from time $t + 1$ to the end,
$\{X_{t+1}, X_{t+2}, \ldots, X_T\}$, starting from state i at time t

$$\beta_t(i) = P(X_{t+1}, X_{t+2}, \ldots, X_T|s_t = s_i, \theta). \tag{4.21}$$

At the final time step T, the probability of future observations is 1.

$$\beta_T(i) = 1, \quad 1 \leq i \leq m. \tag{4.22}$$

The probability of the future observations given that the system is in state i at time t is computed by summing over all possible next states j.

$$\beta_t(i) = \sum_{j=1}^{m} a_{ij} b_j(X_{t+1}) \beta_{t+1}(j), \quad 1 \leq i \leq m, 1 \leq t < T. \tag{4.23}$$

$$P(X_t|\theta) = \sum_{j=1}^{m} \pi_j b_j(X_1) \beta_1(j). \tag{4.24}$$

The Baum-Welch algorithm optimises the transition and observation probabilities and is used to learn the parameters of an HMM from observed data.

$$\gamma_t(j) = P(s_t = j|X_t, \theta) = \frac{\alpha_t(j)\beta_t(j)}{P(X_t|\theta)} \tag{4.25}$$

.

$$\xi_t(i, j) = P(s_t = i, s_{t+1} = j|X_t, \theta) = \frac{\alpha_t(i) a_{ij} b_j(X_{t+1}) \beta_{t+1}(j)}{P(X_t|\theta)}. \tag{4.26}$$

Estimates the expected number of transitions from state i to state j,

$$a_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i, j)}{\sum_{t=1}^{T-1} \gamma_t(i)}. \tag{4.27}$$

where $\xi_t(i,j)$ represents the probability of being in state i at time t and transitioning to state j at time $t+1$,

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(X_{t+1}) \beta_{t+1}(j)}{P(X_t|\theta)}, \tag{4.28}$$

$$b_j(\delta_k) = \frac{\sum_{t=1, X_t=\delta_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}, \tag{4.29}$$

where $\sum_{t=1, X_t=\delta_k}^{T}$ is the sum of all t for which the observation at time t is δ_k.

4.2.2 Markov switching autoregressive model

The Markov switching regression model shows the probability of transitioning and switching from state 0 to state 1, and the probability of remaining within the same state. Historical investment data constantly switch between recurring hidden states (as illustrated in Figure 4.3), indicating that some time variations in the parameters exists. The Markov switching regression model will be applied to the funds of funds to estimate the switching probabilities of the fund of fund returns. The estimation method presented here can be found in Quandt (1972), Hamilton (1989), Nielsen and Overgaard (2000), Shumway and Stoffer (2017), Wang et al. (2020) and Adejumo et al. (2020). Let (X_t) be modeled by an autoregressive model of order p ($AR(p)$), such that,

$$X_t = \phi_0^{(s_t)} + \sum_{i=1}^{p} \phi_i^{(s_t)} X_{t-i} + \epsilon_t^{(s_t)}, \tag{4.30}$$

where X_t is the random variable at time t, $\phi_i^{(s_t)}$ are the coefficients that depend on the state s_t, and $\epsilon_t^{(s_t)} \sim N(0, \sigma^2)$ is the normally distributed error term with a variance that depends on the state (s_t). A simple switching stationary $AR(1)$ process with mean (μ), that switches from state 0 $(s_t = 0)$ to state 1 $(s_t = 1)$ is given by

$$X_t = \begin{cases} \phi_0 + \phi_1 X_{t-1} + \varepsilon_t, \quad |\phi_1| < 1, \quad \mu = \frac{\phi_0}{1-\phi_1} & s_t = 0, \\ \phi_0 + \phi_1 + \phi_2 X_{t-1} + \varepsilon_t, \quad \phi_1 \neq 0, \quad \mu = \frac{\phi_0+\phi_1}{1-\phi_2} = \frac{\phi_{01}}{1-\phi_2} & s_t = 1. \end{cases} \tag{4.31}$$

The conditional distribution of the observations is given by:

$$b_j(X_t) = \mathbb{G}\left(X_t; \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} X_{t-i}, \sigma^{2(j)}\right), \tag{4.32}$$

where $\mathbb{G}$ represents the normal density function and $\sigma^{2(j)}$ is the variance associated with state j. The likelihood of the observations is given by:

$$\ln \mathcal{L}(\theta|X_{1:p}) = \sum_{t=p+1}^{T} \ln\left(\sum_{j=1}^{m} \pi_j(t|t-1) b_j(X_t)\right), \tag{4.33}$$

where $\mathcal{L}(\theta|X_{1:p})$ is the likelihood of the observations and $\pi_j(t|t-1)$ is the transition probability to state j at time t.

Let $(\mathcal{L})$ be the log likelihood, (k) be the number of parameters, and (n) be the number of observations. The AIC is used to select the best model based on the prediction accuracy (Weakliem (2016); Portet (2020)).

$$AIC = -2\ln(\mathcal{L}) + 2k. \tag{4.34}$$

The BIC is more conservative than the AIC and seeks a true model (Watanabe, 2013; Nguefack-Tsague and Bulla, 2014).

$$BIC = -2\ln(\mathcal{L}) + k\ln(n). \tag{4.35}$$

The HQIC is used when the datasets are large and provides a balance between AIC and BIC in the way a model is selected. It is more stringent than the AIC but less conservative than the BIC (Maïnassara and Kokonendji, 2016; Iheagwara et al., 2018).

$$HQIC = -2\ln(\mathcal{L}) + 2k\ln(\ln(n)). \tag{4.36}$$

4.2.3 Kaplan-Meier method

The Kaplan-Meier method is predominantly used in survival analysis, for studies that deal with life or death cases. This method will be applied on a financial time series, thus needs to be adjusted to fit the premise of the Hidden Markov Model. The hidden states may persist over a period of time before a transition from one state to the other occurs. The Kaplan-Meier method can be used to assess the time period of each state's persistence (Kaplan and Meier, 1958; Stalpers and Kaplan, 2018; Sun and Wang, 2024). Let s_t be a hidden state at time t, suppose there are observed hidden states associated with i and j, i.e., low and high variance states, respectively. The historical data, i.e., time series X_t at time t will have the transition probabilities as follows, the probability that the portfolio will be in a state of low variance and remain in a state of low variance,

$$P(s_t = i \mid s_{t-1} = i) \quad \forall t. \tag{4.37}$$

The probability that the portfolio will transition from a state of low variance to a state of high variance at time t,

$$P(s_t = i \mid s_{t-1} = j). \tag{4.38}$$

The probability that the portfolio will be in a state of high variance and remain in a state of high variance,

$$P(s_t = j \mid s_{t-1} = j) \quad \forall t. \tag{4.39}$$

The probability that the portfolio will transition from a state of high variance to a state of low variance at time t,

$$P(s_t = j \mid s_{t-1} = i). \tag{4.40}$$

Let λ_t be the transition rate, therefore the transition probabilities will be as below. The probability of remaining in one state either i or j is

$$P(s_t = i \mid s_{t-1} = i) = P(s_t = j \mid s_{t-1} = j) = \lambda_t \quad \forall t. \tag{4.41}$$

The probability of transitioning from one state to another (i.e., from i to j or from j to i) is

$$P(s_t = i \mid s_{t-1} = j) = P(s_t = j \mid s_{t-1} = i) = 1 - \lambda_t. \tag{4.42}$$

Let $\hat{Z}(t)$ denote the survival probability at time t, of either remaining in a state or transitioning from one state to another. Transition probabilities will be

$$\hat{Z}(t) = \prod_{i=0}^{j\leq t} P(s_t = j \mid s_{t-1} = i) = \prod_{i=0}^{j\leq t}(1 - \lambda_t). \tag{4.43}$$

Let d_i^j be number of data points that transition from state i to state j at time t and n_i^j be the number of data at risk of transitioning from state i to state j at time t. The likelihood function for survival data

$$\mathcal{L}(\lambda_t; d_i^j, n_i^j) = \prod_{i=0}^{j \leq t} \lambda_t^{d_i^j} (1-\lambda_t)^{n_i^j - d_i^j} \binom{n_i^j}{d_i^j}, \tag{4.44}$$

$$\log(\mathcal{L}) = \sum_{i=0}^{j} \left(d_i^j \log(\lambda_t) + (n_i^j - d_i^j) \log(1-\lambda_t) + \log \binom{n_i^j}{d_i^j} \right), \tag{4.45}$$

$$\frac{\partial \log(\mathcal{L})}{\partial \lambda_t} = \frac{d_i^j}{\hat{\lambda}_t} - \frac{n_i^j - d_i^j}{1-\hat{\lambda}_t} = 0, \tag{4.46}$$

$$\hat{\lambda}_t = \frac{d_i^j}{n_i^j}. \tag{4.47}$$

Thus, the survival probability is:

$$\hat{Z}(t) = \prod_{i=0}^{j \leq t} (1-\hat{\lambda}_t) = \prod_{i=0}^{j \leq t} \left(1 - \frac{d_i^j}{n_i^j} \right). \tag{4.48}$$

4.2.4 Portfolio metrics

The portfolio metrics includes the portfolio returns, maximum drawdown, standard deviation and Sharpe ratio.

Portfolio return

The portfolio returns are estimated by,

$$R_p = \sum_{i=1}^{n} w_i R_i \tag{4.49}$$

where R_p is the portfolio return, w_i is the weight of the i-th asset in the portfolio and R_i is the return of the i-th asset.

Max drawdown

The maximum drawdown (MD) is the difference between the peak and trough before the next peak occurs defined by

$$MD = \frac{\text{Peak} - \text{Trough}}{\text{Peak}} \times 100\%, \tag{4.50}$$

where the peak is the highest point and trough is the lowest point.

Standard deviation

Standard deviation

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j \neq i} w_i w_j \sigma_{ij} \tag{4.51}$$

Sharpe ratio

The Sharpe ratio is defined as,

$$SR = \frac{R_p - R_f}{\sigma_p}, \tag{4.52}$$

where R_p is the portfolio return, R_f is the risk-free rate and σ_p is the standard deviation of the portfolio return.

4.3 Empirical results

4.3.1 Data

In investment management there are two prominent strategies often compared with each other, i.e., active and passive investing. In active investing, the individual stocks are selected to construct a portfolio and there is regular trading that occurs within the portfolio (Carneiro et al., 2022). A passive strategy can be considered a buy-and-hold strategy with equidistant rebalancing periods to reposition the portfolio to ensure that the fund is holding the same underlying stocks, with similar percentages compared to the benchmark it tracks (Grégoire, 2020).

The majority of portfolios within the passive investment space are factor-based exchange-traded funds, known as smart beta funds (Millo et al., 2023). Exchange-traded funds are index tracking funds that invest in a basket of shares, which seeks to replicate the index's underlying share exposure, diversification and industry exposure (Conlon et al., 2023). A fund of funds aims to capitalise on the management styles unique to each investment strategy and portfolio.

Passive investing has seen a rise in popularity, fuelled by their ease of accessibility, affordability and diversification prowess. As a result, passive funds have garnered more investment flows than active investments in recent years, as investors benefit from getting a diversified portfolio at lower overall fees compared to active portfolios (Cong et al., 2024).

The historical data used in this study is publicly available on the Yahoo finance database. Value and growth funds of funds were constructed, by selecting the largest three US listed ETFs ranked by assets under management (AUMs). The historical data consists of closing daily prices from January 2013 to December 2024. Table 4.1 shows a list of the top three ETFs for both value and growth, with the overall market consisting of two indexes (denoted by **) and an ETF.

Table 4.1: Funds of Funds

Factor	Exchange-Traded Fund (ETF)	Yahoo Finance Symbol	Assets Under Management (AUM) [US Dollar ($), Billion]
Value	Vanguard Value Index Fund ETF iShares Russell 1000 Value ETF iShares S&P 500 Value ETF	VTV IWD IVE	135 61 36
Growth	Vanguard Growth ETF iShares Russell 1000 Growth ETF iShares S&P 500 Growth ETF	VUG IWF IVW	160 105 57
Overall Market	**Russell 3000 index **S&P 500 index iShares MSCI World Index ETF	RUA GSPC XWD.TO	- - 1.1

Vanguard and Blackrock are two of the largest ETFs providers and as a result the selected ETFs are managed by both investment management firms. A fund of funds, is a portfolio that invests in other funds, combining different strategies from the underlying funds to achieve higher returns (Ang et al., 2008). Four funds of funds will be constructed for value, growth, value-growth and the overall market using the information in Table 4.1.

A comparison of the two factors' historical results has shown that value has not been performing as well as expected by investors, for example, the impact of the global financial crisis hit value the hardest compared to growth (Bevanda et al., 2021). Zhou and Liu (2021) found that the effectiveness of a value investment strategy after COVID-19 has declined, and although the effectiveness of this strategy is in question, value investing remains relatively low risk compared to a growth strategy. Value has been facing some significant challenges and has had a long-range series of underperformance, whereas growth has been consistently outperforming value. Growth, on the other hand, has been thriving and outperforming value. Investors are willing to pay more for growth stocks as there is a prospect of higher investment growth and returns in the future.

Figure 4.1 illustrates a comparison of annual total returns of the value, growth, value-growth, and the overall market funds of funds starting in 2001.

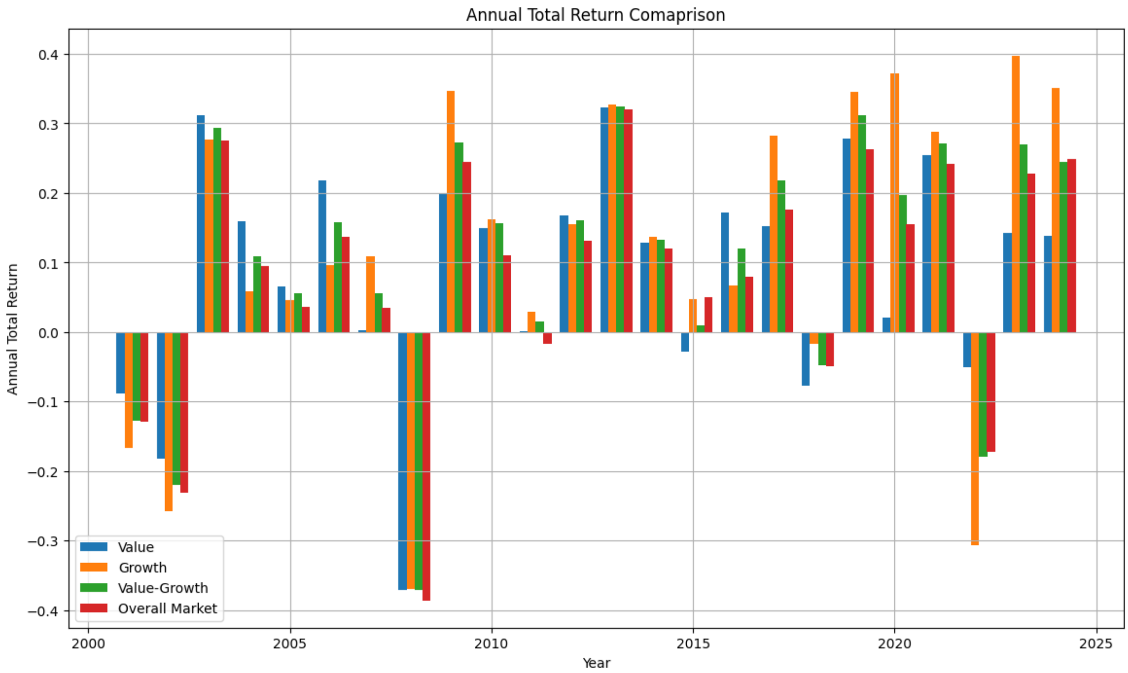


Figure 4.1: Value and growth annual total returns comparison.

A shift in returns for growth occurred in 2007, with an outperformance of growth stocks compared to value. After 2007, value has outperformed growth three times, i.e., in 2012, 2016 and in 2022. Table 4.2 shows a brief overview of events that had a negative impact on the annual total returns for each fund of funds.

Table 4.2: Financial crisis affecting the stock market returns

Year	**Event**
2001	Dot-com recession, impact continued to 2002
2008	Global financial crisis, which began in the US
2011	Black Monday 2011 - Debt ceiling crisis followed by the downgrade of the US Standard & Poor's credit rating from AAA to AA+
2015	Chinese economy crisis, collapse in oil prices
2016	Flash crashes, frantic selloffs by investors fuelled by fears of economic slowdowns
2018	China-US tariff war, Big tech scrutiny from regulators and general slowdown in Chinese economy
2020	Worldwide COVID-19 lockdowns
2022	Invasion of Ukraine by Russia and sharp rise in inflation

4.3.2 Hidden Markov Model results

Applying the HMM on a portfolio that includes all the ETFs and indexes from Table 4.1, results in Figure 4.2.

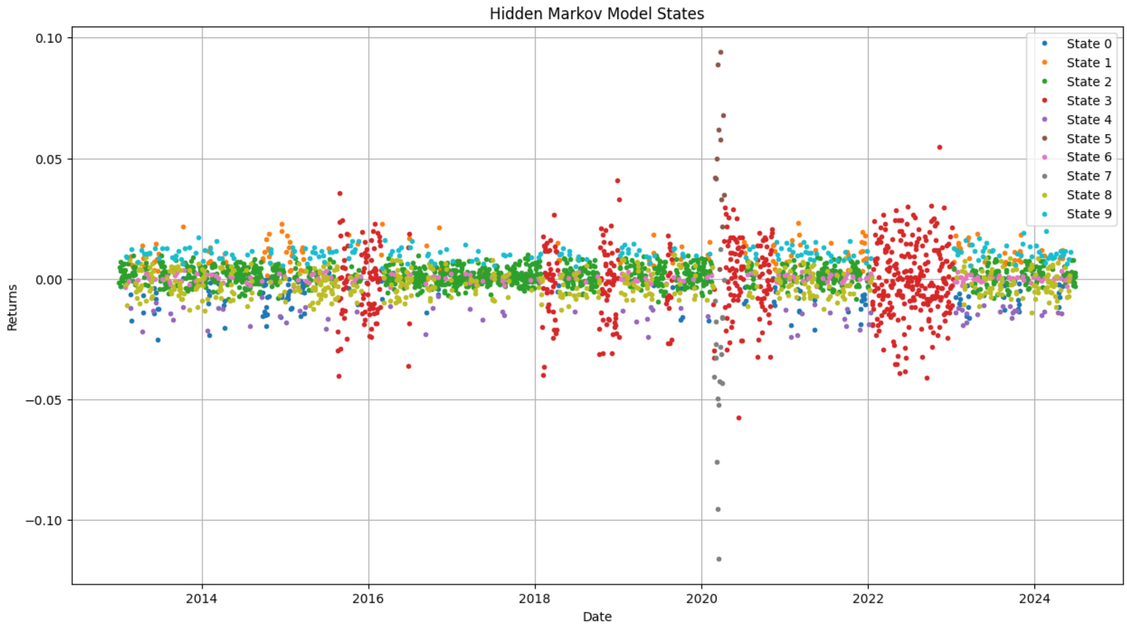


Figure 4.2: Hidden Markov Model with 9 states

Figure 4.2 presents the historical returns of the overall portfolio with nine hidden states. A higher number of hidden states can increase the complexity of the model. Table 4.3 shows the distribution of occurrence of three states (i.e. state 0, 1 and 2). State 2 has a distribution below 2.2% for the value and growth funds of funds. Therefore, only state 0 (Low variance) and state 1 (high variance) will be considered for subsequent results in this study.

Table 4.3: Distribution of occurrence

Factor	**Hidden State**	**Distribution of occurrence (%)**
Value	0	78.263751
	1	21.736249
	2	—
Growth	0	33.499006
	1	65.374420
	2	1.126574
Value-Growth	0	53.611663
	1	2.186879
	2	44.201458
Overall Market	0	61.172685
	1	1.966393
	2	36.860922

Applying the Hidden Markov Model to the value, growth, value-growth, and the overall market fund of funds, respectively, resulted in Figure 4.3. State 0 shows the period of stable predictable returns and state 1 shows a wider disparity between the highest and lowest returns.

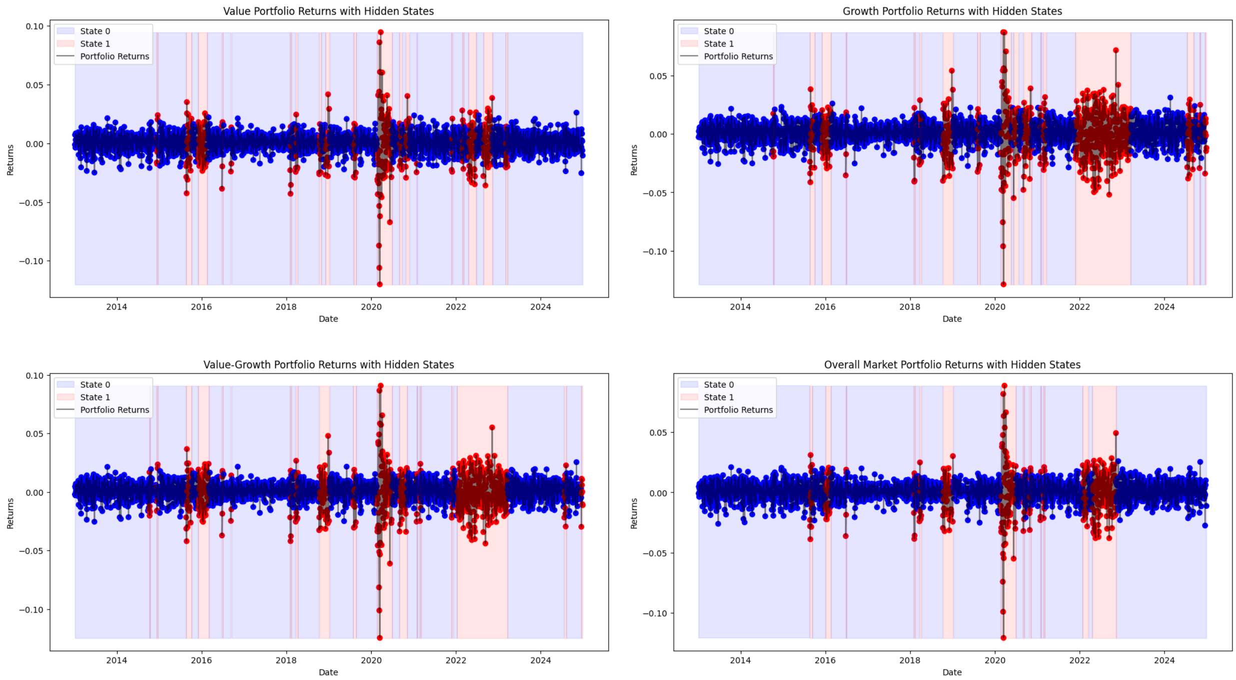


Figure 4.3: Historical daily returns hidden states identification.

The returns remain stable unless a market shock occurs, or events that negatively affect the stock market. Table 4.2 provides a limited summary of events that can cause a portfolio to transition from state 0 to state 1. Value is more stable than growth. There is a greater frequency of transitions from state 0 to state 1 within growth and value-growth fund.

4.3.3 Markov switching regression model results

The results of fitting the Markov switching regression model to the funds of funds for value, growth, value-growth and the overall market are shown in Tables 4.4, 4.5, 4.6 and 4.7.

Table 4.4: Log likelihood, AIC, BIC and HQIC results

Factor	Log Likelihood	AIC	BIC	HQIC
Value	-3658.342	7332.685	7380.784	7349.981
Growth	-4210.823	8437.646	8485.745	8454.942
Value-Growth	-3771.771	7559.542	7607.640	7576.837
Overall Market	-3585.912	7187.824	7235.846	7205.101

Table 4.4 shows the best combination of the log likelihood, AIC, BIC and HQIC for the four fund of funds. The log likelihood is negative with the highest being the overall market. The fit and complexity of the model measured by the AIC is lower for the overall market fund. The BIC, which favours simpler models is also the lowest for the overall market. The HQIC is consistent for large datasets and therefore is the lowest for the overall market fund.

Tables 4.5, 4.6 and 4.7, show the state parameters as well as the state transition parameters.

Table 4.5: State 0 parameters

	Factor	Coefficient	Standard Error	z-value	$p > \|z\|$	[0.025	0.975]
	Value	-1.5516	0.118	-13.196	<0.0001	-1.782	-1.321
ϕ_0	Growth	1.5755	0.164	9.612	<0.0001	1.254	1.897
	Value-Growth	0.1213	0.013	9.364	<0.0001	0.096	0.147
	Overall Market	0.1077	0.013	8.487	<0.0001	0.083	0.133
	Value	1.6448	0.118	13.920	<0.0001	1.413	1.876
ϕ_1	Growth	-1.4416	0.165	-8.751	<0.0001	-1.764	-1.119
	Value-Growth	-1.3069	0.160	-8.187	<0.0001	-1.620	-0.994
	Overall Market	-1.6872	0.215	-7.858	<0.0001	-2.108	-1.266
	Value	0.3330	0.014	24.141	<0.0001	0.306	0.360
σ^2	Growth	0.4016	0.018	22.088	<0.0001	0.366	0.437
	Value-Growth	0.3031	0.013	23.768	<0.0001	0.278	0.328
	Overall Market	0.3048	0.013	24.090	<0.0001	0.280	0.330

Table 4.5 shows the state 0 parameters. In state 0 the intercept ϕ_0 will be significant at 1% level of significance if the p-value <0.0001. Hence, when ϕ_1 is zero, the mean returns in state 0 will be ϕ_0. A value of ϕ_1 will be significant at 1% level of significance, and an increase of ϕ_1 will significantly reduce the mean returns for a fund if it is negative. State 0 is characterised by low variance σ^2, therefore, the mean returns in this state will be more stable.

- Value has $\phi_0 = -1.5516$, which indicates persistent negative returns. Therefore, even in the state of low variance ($\sigma^2 = 0.3330$), value produces negative returns.
- Growth has the highest ϕ_0 value and the highest mean returns, with the highest variance among the four funds.
- Value-Growth is a combination of the value ETFs and the growth ETFs and heavily impacts the returns generated by growth, as shown by $\phi_0 = 0.1213$. The variance of this 2-factor fund is the lowest, which in this case corresponds to the lowest returns.
- The overall market has a lower ϕ_0 value than the value and growth funds. Its ϕ_1 is negative and its variance is slightly higher than that of the value-growth fund's σ^2.

Table 4.6: State 1 parameters

	Factor	Coefficient	Standard Error	z-value	$p > \|z\|$	[0.025	0.975]
	Value	0.0162	0.074	0.220	0.826	-0.128	0.161
ϕ_{01}	Growth	-0.1247	0.060	-2.074	0.038	-0.242	-0.007
	Value-Growth	0.0988	0.180	0.550	0.582	-0.253	0.451
	Overall Market	0.0397	0.149	0.266	0.790	-0.252	0.332
	Value	0.0813	0.185	0.440	0.660	-0.281	0.443
ϕ_2	Growth	0.1886	0.175	1.079	0.281	-0.154	0.531
	Value-Growth	-0.1110	0.189	-0.587	0.557	-0.482	0.260
	Overall Market	-0.0719	0.160	-0.448	0.654	-0.386	0.243
	Value	3.0621	0.187	16.409	0.000	2.696	3.428
σ^2	Growth	3.2245	0.159	20.234	0.000	2.912	3.537
	Value-Growth	2.7513	0.138	19.948	0.000	2.481	3.022
	Overall Market	2.5289	0.135	18.724	0.000	2.264	2.794

Table 4.6 shows the state 1 parameters. In state 1 the intercept ϕ_{01} will be insignificant at 1% level of significance if the p-value > 0.000. Hence, when ϕ_2 is zero, the mean returns in state 1 will be ϕ_{01}. A value of ϕ_2 will also be insignificant at 1% level of significance. State 1 is characterised by high variance (σ^2), therefore, the mean returns in this state will be more volatile.

- Value has more stable returns in state 1 compared to growth. Therefore, value benefits from a high variance state 1 compared to state 0 returns. Although the returns are higher, they are not significant and thus will not improve the overall results of the value factor fund.
- Growth has the lowest mean returns with $\phi_{01} = -0.1247$, the mean returns for value are negative within this state. The mean returns are not significant. Growth has the highest variance within state 1.
- Value-growth shows the highest mean returns $\phi_{01} = 0.016$, though it is statistically insignificant.
- The overall market shows but statistically insignificant mean returns, with the lowest variance.

Table 4.7: State transition parameters

	Factor	**Coefficient**	**Standard Error**	z**-value**	$p > \|z\|$	**[0.025**	**0.975]**
$p[0->0]$	Value	0.9786	0.004	239.252	<0.0001	0.971	0.987
	Growth	0.9803	0.004	235.962	<0.0001	0.972	0.988
	Value-Growth	0.9786	0.004	237.743	<0.0001	0.971	0.987
	Overall Market	0.9832	0.004	280.809	<0.0001	0.976	0.990
$p[1->0]$	Value	0.0679	0.012	5.905	<0.0001	0.045	0.090
	Growth	0.0360	0.007	5.056	<0.0001	0.022	0.050
	Value-Growth	0.0453	0.008	5.501	<0.0001	0.029	0.061
	Overall Market	0.0399	0.008	5.092	<0.0001	0.025	0.055

Table 4.7 shows the transition probabilities of the value, growth, value-growth, and the overall market portfolios. $p[0->0]$ shows the transition probabilities of remaining in the same state and $p[1->0]$ shows the transition from state 1 to state 0. The probability of staying in state 0 is higher than 97% for all four fund of funds.

4.3.4 Kaplan-Meier method results

Figure 4.4 illustrates the results of applying the Kaplan-Meier method on a financial time series to estimate persistence, measured in days of the hidden states.

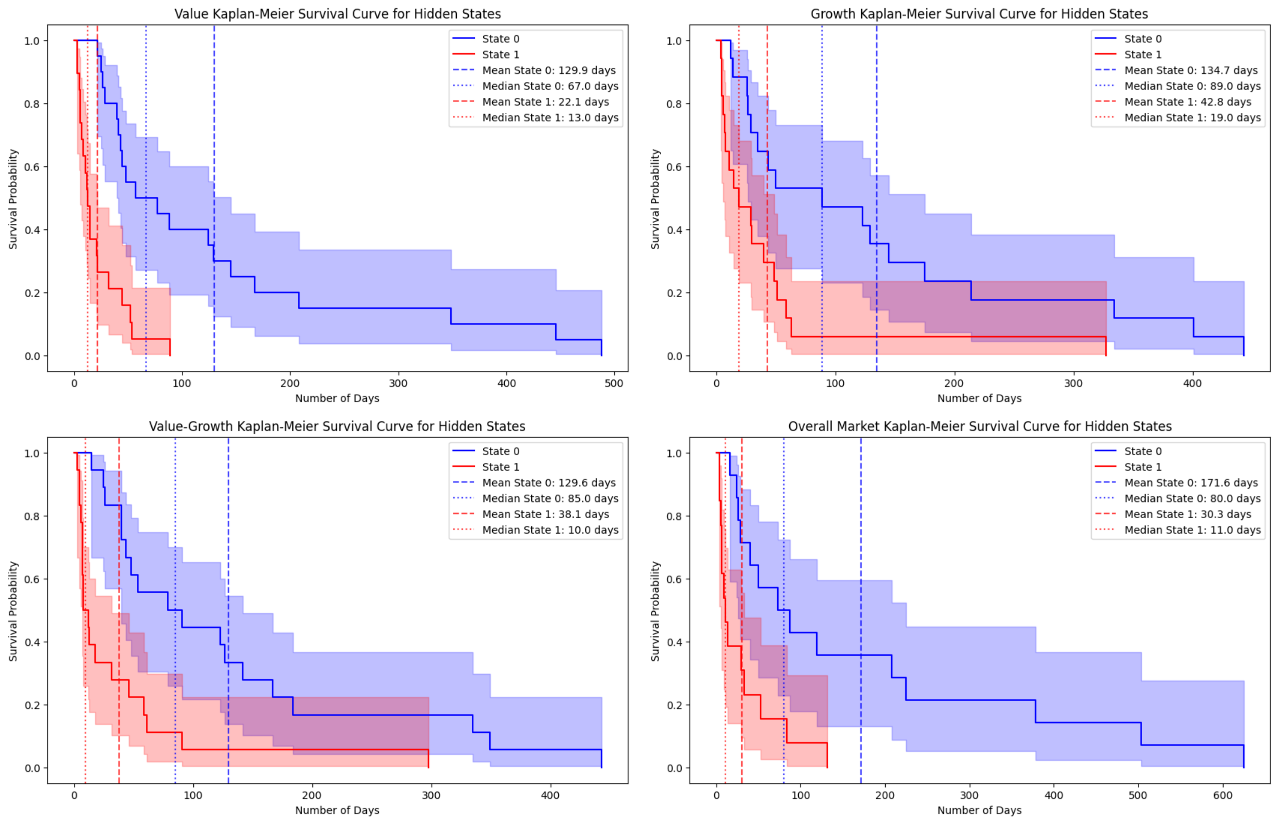


Figure 4.4: Kaplan-Meier survival curves

Figure 4.4 shows the Kaplan-Meier results for value, growth, value-growth and the overall market.

- Value stays on average 129.9 days in state 0 and when it does transition from state 0 to state 1, it remains on average 22.1 days in state 1.
- The overall market has the highest average number of days of remaining within state 0. Growth remains in state 1 for the longest average duration.

4.3.5 Factor timing

Factor timing is an investment strategy that seeks to adjust exposure to certain stocks that belong to the same factor in order to increase and generate superior returns. HMM presents a unique opportunity where a portfolio's exposure can be adjusted to improve returns and lower volatility. A HMM factor timing strategy will involve regular dynamic portfolio exposure adjustment during the transition of hidden states, as factors may be cyclical (Haddad et al., 2020). The adjustments here will adhere to the state transitions probabilities, Li (2024) documents that factor timing results differ by time period.

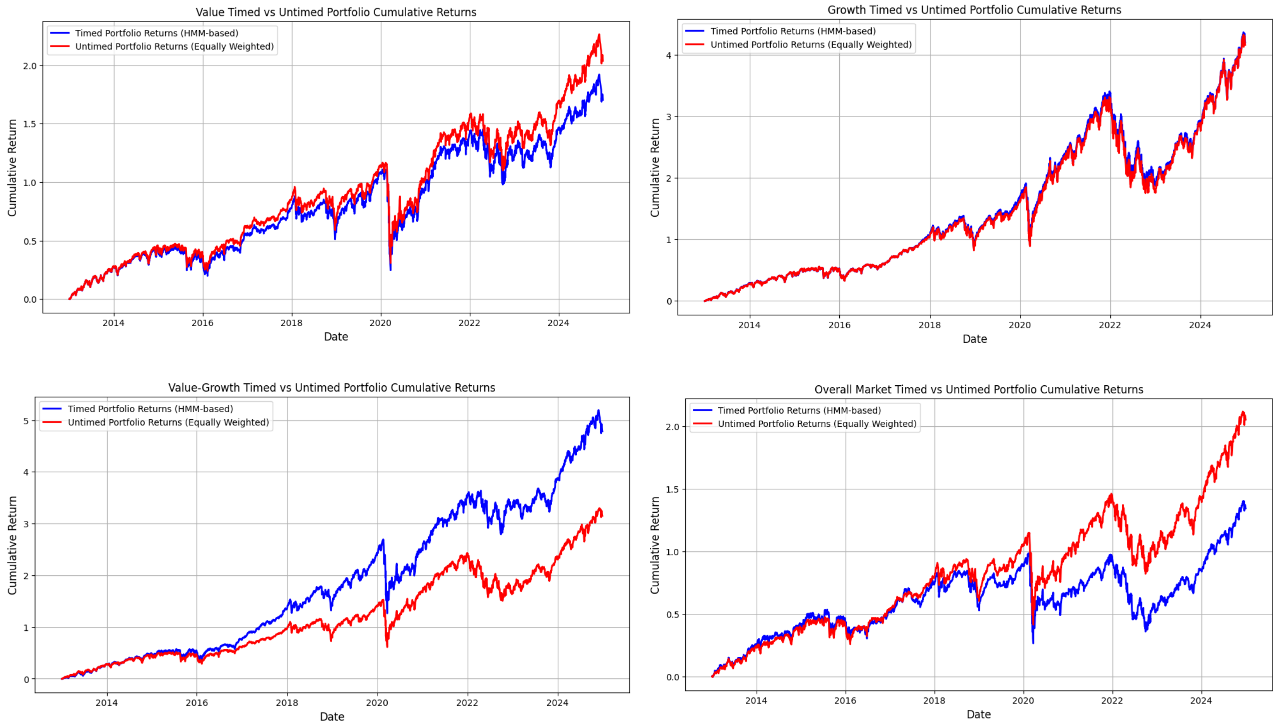


Figure 4.5: HMM factor timing cumulative returns comparison

Figure 4.5 shows the historical results of the HMM factor timing strategy.

- Value has a higher chance of remaining in state 0 and therefore the HMM factor timing method did not have impactful results for the value fund of funds.
- Growth had similar returns to the untimed fund. Therefore, timing growth did not produce significant returns.
- Value combined with growth within the same fund produced significant results after applying the HMM factor timing. The balance that each factor provides to the 2 factor portfolio makes it suitable for a timing strategy.
- The overall market performed dismally. The timed portfolio had a large gap after the 2020 drop in returns.

The portfolio metrics comparison of the timed and untimed portfolios split by the hidden states found in Table 4.8.

Table 4.8 shows the results of a timed portfolio and an untimed portfolio for each hidden state.

Table 4.8: Hidden states metrics: timed vs untimed comparison

Factor	**State**	**Average Return**		**Cumulative Return**		**Max Drawdown**		**Standard Deviation**		**Distribution of Occurrence**		**Sharpe Ratio**	
		Timed	**Untimed**	**Timed**	**Untimed**	**Timed**	**Untimed**	**Timed**	**Untimed**	**Timed**	**Untimed**	**Timed**	**Untimed**
Value	0	0.190966	0.195456	4.957359	5.212621	0.094377	0.090189	0.107224	0.106519	78.031809	78.031809	1.781000	1.834937
	1	-0.237801	-0.211597	-0.465082	-0.426904	0.880345	0.845477	0.287628	0.281962	21.968191	21.968191	-0.826765	-0.750445
Growth	0	0.304677	0.301829	15.951466	15.508855	0.095163	0.100120	0.120766	0.121206	77.567926	77.567926	2.522880	2.490226
	1	-0.360420	-0.350899	-0.620261	-0.610423	1.052321	1.062973	0.324758	0.332130	22.432074	22.432074	-1.109811	-1.056512
Value-Growth	0	0.279103	0.243547	6.358171	4.706198	0.068934	0.059848	0.106318	0.100757	59.708416	59.708416	2.625172	2.417163
	1	-0.016413	-0.030303	-0.076143	-0.136037	0.802332	0.908864	0.221900	0.235966	40.291584	40.291584	-0.073965	-0.128422
Overall Market	0	0.209801	0.228687	6.701395	8.254980	0.092623	0.109390	0.105288	0.109294	87.665356	87.665356	1.992649	2.092412
	1	-0.760027	-0.704954	-0.646727	-0.619061	1.084148	0.998116	0.366652	0.348007	12.334644	12.334644	-2.072884	-2.025687

Table 4.8 shows the timed and untimed metrics split by hidden states 0 and 1. Timing value and Growth using the HMM to dynamically adjust the portfolio when a portfolio transitions between states does not significantly increase the average returns. Combining value and growth and applying the HMM factor timing method produces the best results. Timing the overall market using the HMM does not produce meaningful results.

4.4 Conclusion

Factor investing strategies are popular and two factors have long been compared with each other. This study analyses the historical returns of value vs growth, by comparing them to the overall market. A fund of funds was constructed consisting of three ETFs each. The historical returns of value, growth, value-growth and overall market fund of funds were compared, and the results show that growth outperformed value. Value underperformed, while growth demonstrated strong returns.

The HMM was applied to the fund of funds to identify the hidden states of the historical daily returns. Growth tends to be more susceptible to market condition changes, which causes more hidden states transitions.

The Markov switching regression model was applied to the fund of funds to identify time variant parameters, including the state transition parameters showing the probability of staying within a state or transitioning between the states. Each fund of funds exhibited a probability greater than 97% of remaining in state 0. The Kaplan-Meier method, predominantly used in survival analysis, was adjusted to fit the premise of the HMM dynamics to identify the persistence in days of the hidden states.

The overall market fund of funds had the highest average in days of staying within state 0. Growth had the highest average in days of staying in state 1. These results imply that growth is riskier factor than value. However, when a factor timing using the HMM was applied to the fund of funds, growth did not significantly deviate from the original untimed equally weighted portfolio. The fund of funds with both value and growth produced significant results from a factor timing HMM approach. The results signify that a hybrid approach to factor timing that limits the manager's skill can be an effective strategy to a passive investment portfolio.

References

[1] Abhyankar, A., Ho, K.-Y. and Zhao, H. (2008). Value versus Growth: Stochastic Dominance Criteria. *Quantitative Finance*, 8(7), 693-704.

[2] Adejumo, O.A., Albert, S. and Asemota, O.J. (2020). Markov Regime-Switching Autoregressive Model of Stock Market Returns in Nigeria. *CBN Journal of Applied Statistics*, 11(2), 65-83.

[3] Ang, A., Rhodes-Kropf, M. and Zhao, R. (2008). Do funds-of-funds deserve their fees-on-fees?.*National Bureau of Economic Research*, NBER Working Paper No. 13944.

[4] Asness, C.S, Frazzini, A., Israel, R. and Moskowitz, T. (2015). Fact, fiction, and value investing. *The Journal of Portfolio Management*, 42(1), 33-52.

[5] Asness, C.S., Friedman, J.A., Krail, R.J., and Liew, J.M. (2000). Style Timing: Value versus Growth. *The Journal of Portfolio Management*, 26(3), 50-60

[6] Athanassakos, G. (2009). Value versus growth stock returns and the value premium: The Canadian experience 1985-2005. *Canadian Journal of Administrative Sciences/Revue canadienne des sciences de l'administration*, 26, 109-121.

[7] Basu, S. (1977). Investment Performance of Common Stocks in Relation to Their Price-Earnings Ratios: A Test of the Efficient Market Hypothesis. The *Journal of Finance*, 32(3), 663-682.

[8] Baum, L. E. and Petrie, T. (1966). Statistical inference for probabilistic functions of finite-state Markov chains. *Annals of Mathematical Statistics*, 37(6), 1554-1563.

[9] Beukes, A. (2011). Value investing: International comparison. *International Business and Economics Research Journal*, 10(5), 1-10.

[10] Bevanda, L.-M., Zaimović, A. and Arnaut-Berilo, A. (2021). Performance of value and growth stocks in the aftermath of the global financial crisis. *Business Systems Research*, 12(2), 268-283.

[11] Capaul, C., Rowley, I., and Sharpe, W.F. (1993). International value and growth stock returns. *Financial Analysts Journal*, 49(1), 27-36.

[12] Carneiro, L.M., Junior, W.E. and Yoshinaga, C.E. (2022). The implications of passive investments for active fund management: International evidence. *Global Finance Journal*, 53, 100623.

[13] Chan, L.K.C. and Lakonishok, J. (2004). Value and growth investing: Review and update. *Financial Analysts Journal*, 60(1), 71-86.

[14] Chen, N. and Zhang, F. (1998). Risk and return of value stocks. *The Journal of Business*, 71(4), 501-535.

[15] Cole, D.J. (2019). Parameter redundancy and identifiability in hidden Markov models. *METRON*, 77, 105-118.

[16] Cong, L.W., Huang, S., and Xu, D. (2024). The rise of factor investing: "Passive" security design and market implications.*National Bureau of Economic Research*, NBER Working Paper No. 32016.

[17] Conlon, T., Cotter, J., Kovalenko, I., and Post, T. (2023). A financial modeling approach to industry exchange-traded funds selection. *Journal of Empirical Finance*, 74, 101441.

[18] Elze, G. (2010). Value investing anomalies in the European stock market: Multiple value, consistent earner, and recognized value. *The Quarterly Review of Economics and Finance*, 50, 527-537.

[19] Fama, E. F., and French, K. R. (1992). The Cross-Section of Expected Stock Returns. *The Journal of Finance*, 47, 427-465.

[20] Fama, E.F. and French, K.R. (1998) Value versus growth: The international evidence. *The Journal of Finance*, 53(6), 1975-1999.

[21] Graham, B., and Dodd, D. L. (1934). Security Analysis: The Classic 1934 Edition. *McGraw-Hill*.

[22] Grégoire, V. (2020). The rise of passive investing and index-linked comovement. *North American Journal of Economics and Finance*, 51, 101059.

[23] Haddad, Valentin and Kozak, Serhiy and Santosh, Shrihari, Factor Timing (2020). *National Bureau of Economic Research*, NBER Working Paper No. w26708.

[24] Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57, 357-384.

[25] Hassan, S. S., Särkkä, S., and García-Fernández, Á. (2021). Temporal Parallelization of Inference in Hidden Markov Models. *IEEE Transactions on Signal Processing*, 69, 4875-4887.

[26] Iheagwara, A. I., Okenwe, I. and Ihekuna, S. O. (2018). Model selection in multiple regression models using budgeted profit, production and sales variables. *IJRDO - Journal of Applied Science*, 4(8), 27-38.

[27] Israel, R., Laursen, K., and Richardson, S. (2022). Is (Systematic) Value Investing Dead?. *The Journal of Portfolio Management Quantitative Special Issue*, 47(2), 38-62.

[28] Kakinuma, Y. (2017). Time reward of value investing: Evidence from the Southeast Asia stock markets. *Journal of Economics, Finance and Accounting (JEFA)*, 4(2), 70-86.

[29] Kaplan, E.L. and Meier, P. (1958) Nonparametric Estimation from Incomplete Observations. *Journal of the American Statistical Association*, 53, 457-481.

[30] Kok, U-W., Ribando, J. and Sloan, R. (2017). Facts about formulaic value investing. *Financial Analysts Journal*, 73(2), 81-99.

[31] Li, Q. (2024). Actively managed stock mutual funds and market efficiency: Evidence from China. *Journal of Fintech and Business Analysis*, 1, Published online: 14 October.

[32] Maïnassara, Y. B. and Kokonendji, C. C. (2016). Modified Schwarz and Hannan-Quinn information criteria for weak VARMA models. *Statistical Inference for Stochastic Processes*, 19, 199-217.

[33] Millo, Y., Spence, C. and Valentine, J.J. (2023). Active fund managers and the rise of passive investing: Epistemic opportunism in financial markets. *Economy and Society*, 52(2), 227-249.

[34] Mor, B., Garhwal, S. and Kumar, A. (2021). A systematic review of hidden Markov models and their applications. *Archives of Computational Methods in Engineering*, 28, 1429-1448.

[35] Netzer, O., Ebbes, P. and Bijmolt, T.H. (2017). Hidden Markov models in marketing. In Advanced Methods for Modeling Markets. *Springer*, 405-449.

[36] Nguefack-Tsague, G. and Bulla, I. (2014). A Focused Bayesian Information Criterion. *Advances in Statistics*, 2014, Article ID 504325, 1-8.

[37] Nielsen, S. and Overgaard Olesen, J. (2000). Regime-Switching Stock Returns and Mean Reversion. *Copenhagen Business School*.

[38] Penman, S. and Reggiani, F. (2018). Fundamentals of value versus growth investing and an explanation for the value trap. *Financial Analysts Journal*, 74(4), 103-119.

[39] Perez, G.A., (2017). Value investing in the stock market of Thailand. *International Journal of Financial Studies*, 5(4), 30.

[40] Portet, S. (2020). A primer on model selection using the Akaike Information Criterion. *Infectious Disease Modelling*, 5, 111-128.

[41] Price, T. R. (1950). Choosing Growth Stocks for the 1950s. *Barron's National Business and Financial Weekly*, 13.

[42] Quandt, R. E. (1972). A New Approach to Estimating Switching Regressions. *Journal of the American Statistical Association*, 67(338), 306-310.

[43] Rabiner, L. R. (1989). A tutorial on hidden Markov models and selected applications in speech recognition. *Proceedings of the IEEE*, 77(2):257-286.

[44] Rabiner, L.R. and Juang, B.H. (1986). An introduction to hidden Markov models. IEEE Acoust. *Speech Signal Process*, ASSP-34, 4-16.

[45] Ravari, A., Ghoreishi, S.F. and Imani, M. (2024). Optimal inference of hidden Markov models through expert-acquired data. *IEEE Transactions on Artificial Intelligence*, 5(8), 3985.

[46] Roca, F. (2021). What is new in value investing? A systematic literature review. *Journal of New Finance*, 2(2), 1-44.

[47] Shumway, R. H. and Stoffer, D. S. (2017). Time Series Analysis and Its Applications: With R Examples (4th ed.). *Springer*, 336-346.

[48] Siegel, L.B. and Alexander, J.G. (2000). The future of value investing. *The Journal of Investing*, 9(4), 33-45.

[49] Stagnol, L., Lopez, C., Roncalli, T., and Taillardat, B. (2021). Understanding the Performance of the Equity Value Factor. *SSRN*.

[50] Stalpers, L.J.A. and Kaplan, E.L. (2018). Edward L. Kaplan and the Kaplan-Meier Survival Curve. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 33(2), 109-135.

[51] Sun, X. and Wang, Y. (2024). Conformal prediction with censored data using Kaplan-Meier method. *Journal of Physics: Conference Series*, 2898(1), 012030.

[52] Teti, E., Dallocchio, M. and Tamburnotti, T. (2019). Value vs. growth investment strategies: An empirical analysis in the European context. *European Journal of Economics, Finance and Administrative Sciences*, (102), 44-64.

[53] Wang, M., Lin, Y.-H. and Mikhelson, I. (2020). Regime-switching factor investing with hidden Markov models. *Journal of Risk and Financial Management*, 13(12), 311.

[54] Watanabe, S. (2013). A Widely Applicable Bayesian Information Criterion. *Journal of Machine Learning Research*, 14, 867-897.

[55] Weakliem, D. L. (2016). Hypothesis testing and model selection in the social sciences. *Guilford*.

[56] Zhou, Q. and Liu, Y. (2021). The effectiveness of the value investment theory during the COVID-19 pandemic: Using the heavy asset industry as an example. Advances in Economics, Business and Management Research, 178, *Proceedings of the 2021 International Conference on Enterprise Management and Economic Development (ICEMED 2021).*

Chapter 5

The Kalman filter and Hidden Markov model investment factor timing strategy comparison

A fund of funds is an investment instrument that invests in other funds, such as, exchange-traded funds, mutual funds, equity funds and others. On the surface a fund of funds can be expensive with a double layer of fees, i.e., an investor will pay the fees of the fund of funds, as well as, the fees of the underlying funds (Brown et al. (2004); Ang et al. (2008)). This makes them more expensive than buying the individual funds. However, not all investors are well versed in choosing the right investments and thus the fund of funds can provide an opportunity to invest in multiple funds, with different management styles, and can be invaluable to these investors.

In this chapter:

1. The ETFs were selected based on their AUMs and the factor they track.
2. A fund of funds for each factor was constructed using the ETFs.
3. A comparison of the backtested historical performance of the fund of funds and the individual funds was conducted.
4. The Kalman filter was applied to each fund of funds for factor timing.
5. The hidden Markov model was applied to each fund of funds for factor timing.
6. The factor timing results of the Kalman filter and HMM were compared.
7. The model with the best results was selected as the superior model based on the results.

This chapter provides a comparison of the factor timing methodology used in Chapter 2, namely the Kalman filter (Section 2.3.2) and the factor timing methodology in Chapter 4, i.e., the hidden Markov model (Section 4.2.1). This chapter aims to compare the factor timing strategies of two state space models' for the following popular investment factors, namely, momentum, value, growth, quality, size (i.e., small-cap, mid-cap and large-cap), and the multifactor fund of funds.

5.1 Data

The method of constructing the fund of funds can be found in Section 4.3.1. The fund of funds for each factor consists of the three top exchange-traded funds that track each specific factor. The multifactor portfolio is constructed by including all the ETFs from each factor's fund of funds portfolio. The factors are explained in Section 1.1.

Table 5.1: Fund of Funds

Factor	**Yahoo Finance Symbol**	**Exchange-Traded Fund (ETF)**	**Assets Under Management (AUM) [US Dollar ($), Billion (B), Trillion (T)]**
Momentum	MTUM	iShares MSCI USA Momentum Factor ETF	13.8B
	SPMO	Invesco S&P 500 Momentum ETF	5.37B
	XMMO	Invesco S&P MidCap Momentum ETF	3.45B
Value	VTV	Vanguard Value Index Fund ETF Shares	192.07B
	IWD	iShares Russell 1000 Value ETF	61.86B
	IVE	iShares S&P 500 Value ETF	37.26B
Growth	VUG	Vanguard Growth Index Fund ETF Shares	272.06B
	IWF	iShares Russell 1000 Growth ETF	96.25B
	IVW	iShares S&P 500 Growth ETF	51.52B
Quality	QUAL	iShares MSCI USA Quality Factor ETF	48.44B
	SPHQ	Invesco S&P 500 Quality ETF	11.65B
	IQLT	iShares MSCI Intl Quality Factor ETF	8.83B
Size	VOO	Vanguard S&P 500 ETF	1.37T
	VO	Vanguard Mid-Cap Index Fund ETF Shares	179.91B
	VB	Vanguard Small-Cap Index Fund ETF Shares	149.29B
Size: Small Cap	VB	Vanguard Small-Cap Index Fund ETF Shares	149.29B
	IJR	iShares Core S&P Small-Cap ETF	78.38B
	IWM	iShares Russell 2000 ETF	63.88B
Size: Mid Cap	VO	Vanguard Mid-Cap Index Fund ETF Shares	179.91B
	IJH	iShares Core S&P Mid-Cap ETF	89.44B
	IWR	iShares Russell Mid-Cap ETF	38.27B
Size: Large Cap	VOO	Vanguard S&P 500 ETF	1.37T
	IVV	iShares Core S&P 500 ETF	579.87B
	SPY	SPDR S&P 500 ETF Trust	576.01B

5.2 Fund of funds performance

Figure 5.1 presents the backtested historical returns of the fund of funds. Each ETF has an inception date and therefore, the backtested returns will go back as far as the underlying ETFs have been listed.

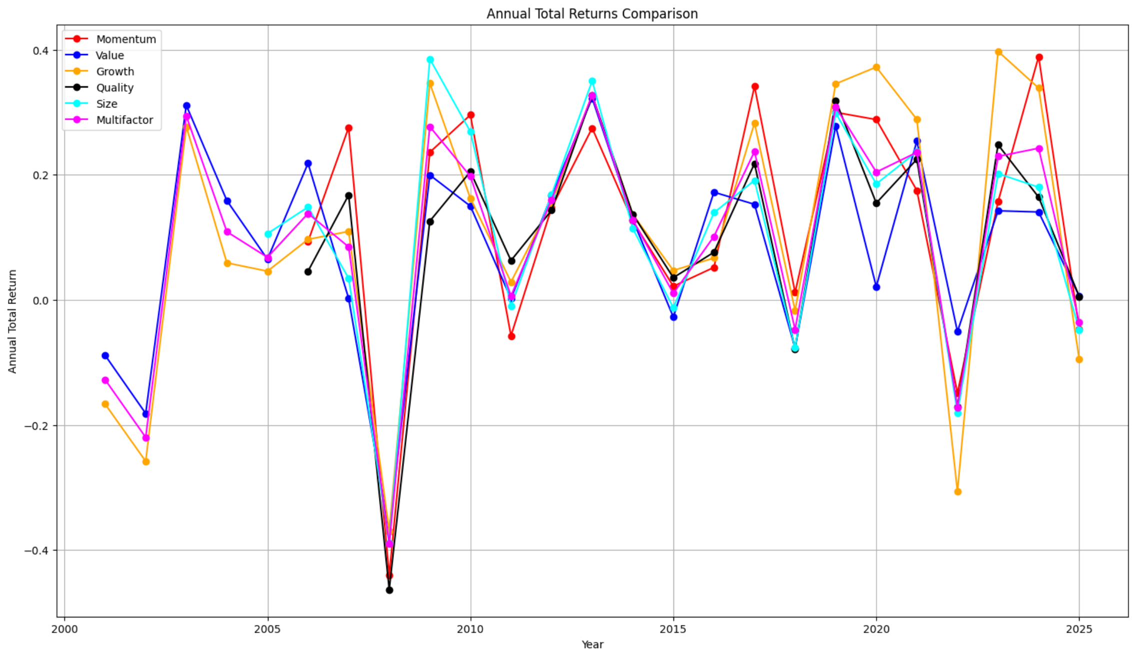


Figure 5.1: Fund of funds annual total returns comparison

Table 4.2 presents a list of financial crises that could impact historical returns, while Figure 5.1 illustrates historical annual total returns comparison of the momentum, value, growth, quality and size factors.

- Momentum fund of funds outperformed other factor funds in 2007, just before the financial crisis, and achieved the highest annual total returns in 2017.
- Value previously outperformed other factors before 2007; however, it has struggled to recover and has underperformed compared to other factors since 2008.
- Growth performed poorly before the 2008 financial crisis. However, after 2008 it has outperformed most other factors, showing similar results to of the momentum factor.
- Quality depends heavily on how the ETF manager defines the quality factor. Since the metrics for constructing a quality ETF can vary by manager, this factor tends to exhibit marginal average historical annual total returns.
- The size fund of funds includes the small-cap, mid-cap and large-cap ETFs. The large-cap stocks typically outperform and while the size factor outperforms value, it performs poorly compared to the other factors.
- The multifactor fund of funds, which includes all the ETFs listed in Table 5.1, does not outperform the single-factor fund of funds, except in the cases of momentum and growth.

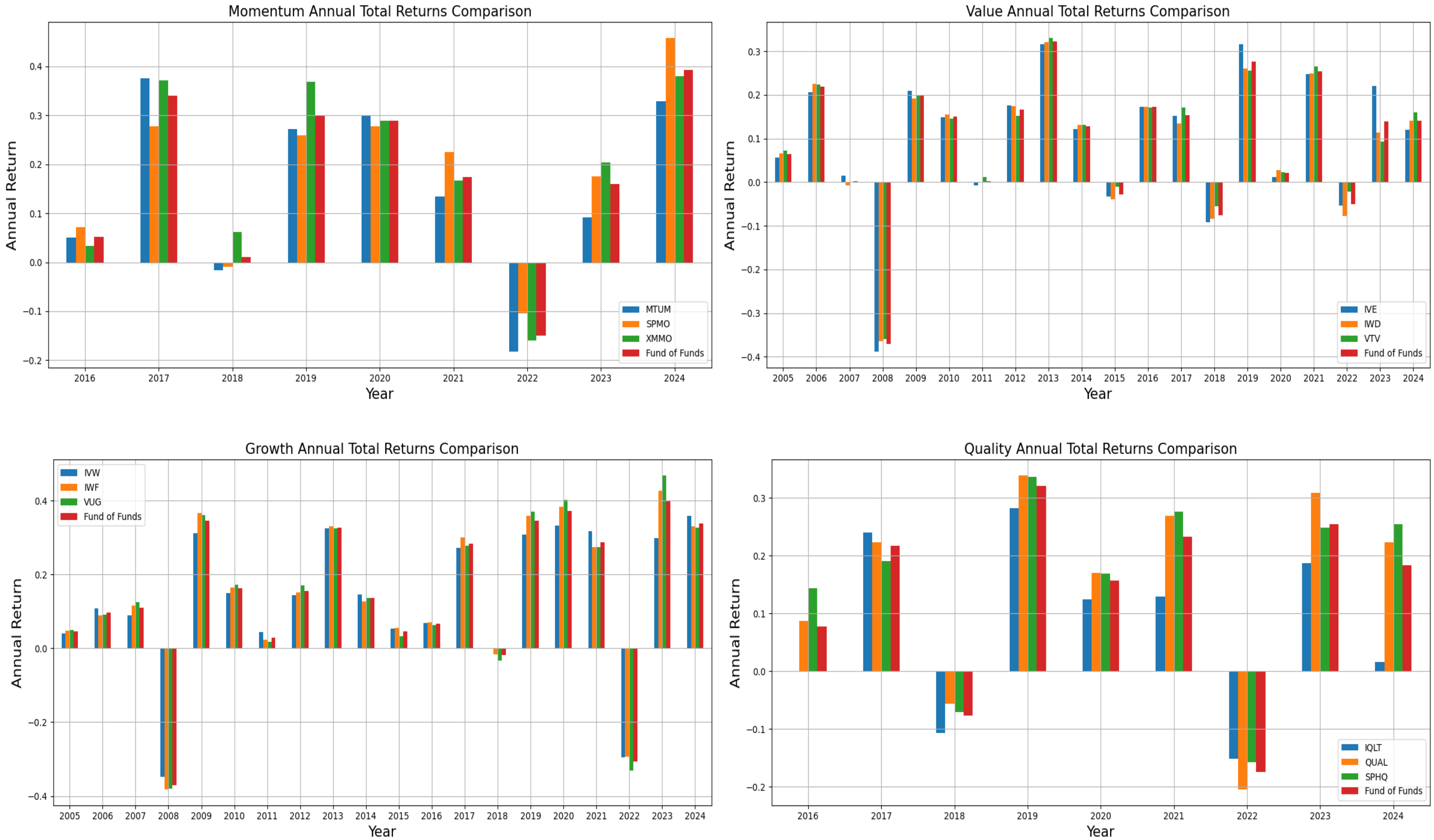


Figure 5.2: Annual total returns comparison of the individual funds with the fund of funds

Although, the ETFs track the same factor, the portfolio construction and management style will differ. Figure 5.2 illustrates the comparison of the annual total returns for momentum, value, growth and quality.

- The momentum fund of funds' annual total returns comparison with the underlying funds shows that the fund with the highest AUM does not necessarily outperform funds with lower AUMs. The fund of funds provides average returns to investors, balancing out the losses from one fund to the other.
- Value has been around for a very long time, therefore the backtesting goes back to 2005. Value is constructed using fundamental ratios and metrics, and hence each manager will have a different construction method and as a result the funds will perform differently. The fund of funds provide average returns, does not outperform the individual underlying funds.
- Growth has significantly underperformed in four instances, in 2006, 2011, 2018 and 2022. The growth fund of funds does not outperform the underlying individual funds.
- Quality similarly underperformed in 2018 and 2022. The fund of funds also provide marginal returns compared to the underlying individual funds.

Each factor's fund of funds does not outperform the underlying ETFs and as a results only provides marginal average returns of the underlying funds. The fund of funds does not outperform the individual underlying funds.

Figure 5.3 shows the annual total returns comparison of the individual funds with the fund of funds for the size factor. The overall size funds are sensitive to market conditions (downturns); whenever something negative occurs in the market, the returns get adversely affected. Small-cap and mid-cap underperform compared to the large-cap funds. Large-cap is the least complicated portfolio to construct and as a result, all the ETFs will have similar stocks, with almost identical returns. This is not significantly influenced by management styles. Large-cap funds outperform both small-cap and mid-cap funds. The fund of funds does not outperform the underlying funds.

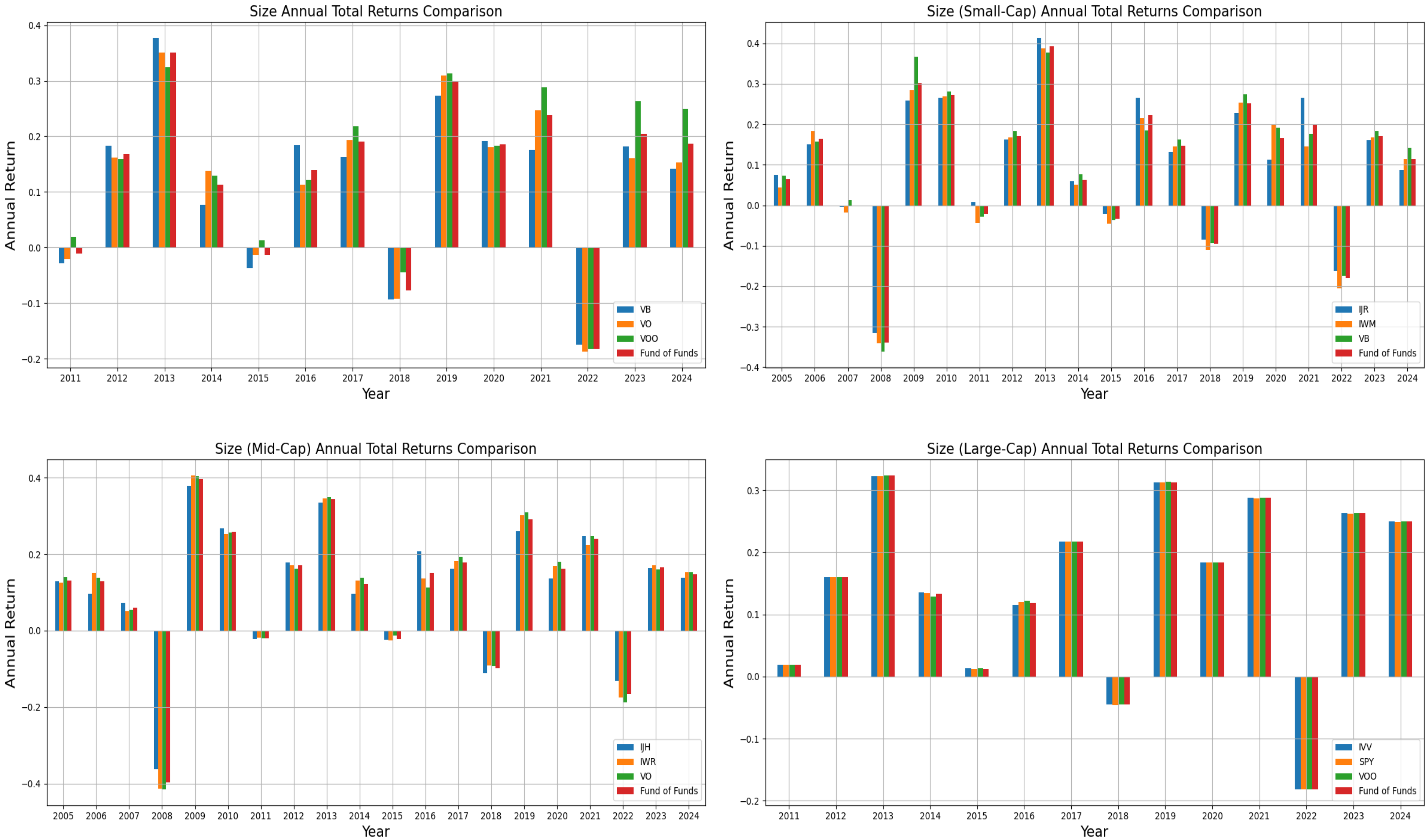


Figure 5.3: Annual total returns comparison of the individual funds with the fund of funds (Size factor)

5.3 Kalman filter versus Hidden Markov Model factor timing

The Kalman filter and the HMM are both state space models, but they differ in key ways.

- In the HMM, the hidden states are discrete, whereas in the Kalman filter they are continuous.
- The HMM can handle nonlinear measurements, while the Kalman filter measures linear changes. The HMM can only move from one state to another one at a time, whereas the Kalman filter can move between multiple states.
- Both the HMM and the Kalman filter contain observed variables and hidden unobserved states that change over time. In the Kalman filter, the hidden states move continuously according to linear dynamics and in the HMM the states follow a discrete Markov chain.

Tables 5.2, 5.3, 5.4, 5.5 and 5.6 show the results of the Kalman filter and the hidden Markov model factor timing strategies, with different rebalancing period, namely, weekly, monthly, quarterly, bi-annually and annually.

5.4 Weekly Kalman filter and HMM factor timing metrics

Table 5.2: Weekly Kalman filter and HMM factor timing metrics

Factor	State space model	State	Average Return	Cumulative Return	Max Drawdown	Standard Deviation	Distribution of Occurrence	Sharpe Ratio
Momentum	HMM	0	0.121606758	1.842142299	0.059464078	0.063457209	53.21563682	1.91635842
		1	0.009322512	1.039531168	0.160533035	0.083527846	44.05212274	0.111609628
		2	-0.059135971	0.984862417	0.067291249	0.091308029	2.732240437	-0.64765357
	Kalman filter	–	0.051284438	1.622794267	0.178122657	0.089012876	–	0.576146294
Value	HMM	0	0.103331981	1.236648166	0.044851205	0.080032037	16.82910981	1.291132717
		1	0.072941262	1.878401044	0.124171125	0.063343753	70.76023392	1.151514691
		2	-0.054853653	0.920212026	0.124951953	0.105923274	12.41065627	-0.517862135
	Kalman filter	–	0.043084244	1.692561854	0.219329504	0.070958077	–	0.607178862
Growth	HMM	0	0.056779281	1.275214576	0.193865235	0.097512375	35.05523067	0.582277689
		1	0.09004481	1.012584753	0.126732476	0.45856214	1.137102014	0.196363375
		2	0.075807904	1.805468894	0.217584613	0.084822506	63.80766732	0.893723942
	Kalman filter	–	0.060780031	2.100939579	0.190163083	0.084281158	–	0.721157994
Quality	HMM	0	-0.098197771	0.899783076	0.130018478	0.060124501	10.57767369	-1.633240526
		1	-0.10165319	0.891757066	0.182656226	0.130917182	11.08508977	-0.776469432
		2	0.049433126	1.48245506	0.093531782	0.057203208	78.33723653	0.864167023
	Kalman filter	–	-0.007712655	0.924583422	0.316222971	0.08300363	–	-0.092919496
Size	HMM	0	0.097481273	2.542213204	0.2337535	0.085535418	78.3625731	1.139659745
		1	-0.009542399	0.997843928	0.00948457	0.024269844	1.851851852	-0.393179244
		2	0.023831326	1.059283113	0.121885618	0.114896437	19.78557505	0.207415707
	Kalman filter	–	0.057987815	2.030495269	0.232352846	0.080831706	–	0.717389476
Size Small-Cap	HMM	0	-0.002108373	0.998344793	0.167800088	0.211354673	6.432748538	-0.00997552
		1	0.066177365	1.592977697	0.283996409	0.087255981	57.60233918	0.758427832
		2	0.081824581	1.432532271	0.109880901	0.097493255	35.96491228	0.83928454
	Kalman filter	–	0.053679565	1.926408794	0.288268468	0.093353119	–	0.575016295
Size Mid-Cap	HMM	0	0.071761345	1.721250613	0.191339836	0.074355737	61.95581546	0.965108379
		1	-0.020468156	0.997080249	0.043242473	0.105127166	1.169590643	-0.194699014
		2	0.057473385	1.295450227	0.152538686	0.099675203	36.87459389	0.576606647
	Kalman filter	–	0.055943378	1.980418926	0.228626299	0.082124434	–	0.681202598
Size Large-Cap	HMM	0	0.088930416	2.128042917	0.086946194	0.065848882	69.52566602	1.350522792
		1	0.012917593	1.047000302	0.158732506	0.082529107	29.10981157	0.156521664
		2	0.032956598	1.005507879	0.030334922	0.112495617	1.364522417	0.292958951
	Kalman filter	–	0.050582312	1.854892633	0.185979956	0.074252413	–	0.681221127
Multifactor	HMM	0	-0.054375066	0.74697277	0.314422904	0.094138797	45.95513256	-0.577605283
		1	-0.119971885	0.96262985	0.094669046	0.132007781	2.719238613	-0.908824342
		2	0.04649773	1.32130165	0.136277377	0.079109988	51.32562882	0.587760544
	Kalman filter	–	0.001579257	1.018608219	0.317821459	0.088152861	–	0.017914988

5.4.1 Weekly momentum factor timing

The most favourable hidden state for the momentum factor is state 0.

- Momentum is in this state 53% of the time. This state occurs more frequently compared to the other states.
- Average return is 12.16%, with the lowest volatility of 6.35%.
- Maximum drawdown is at the lowest at 5.95%, hence the downside risk is limited in this state.
- Sharpe ratio is the highest at 1.92, indicating that this the state with the highest returns per unit of risk.
- All the metrics within this state are greater than the Kalman filter's metrics, therefore, the HMM produces superior results for momentum.

5.4.2 Weekly value factor timing

The most favourable hidden state for the value factor is state 1.

- Average return in state 0 is 10.33% and in state 1 it is 7.2%.
- State 1 has the lowest volatility of 6.33%.
- Maximum drawdown is at the lowest at 4.49% for state 0, hence the downside risk is limited in this state.
- State 0 has the highest Sharpe ratio of 1.29, indicating that this is the state with the highest returns per unit of risk.
- Value is in this state 16% of the time. This state occurs less frequently compared to the other state 1 which occurs 70% of the time.
- State 1 occurs more frequently, has the lowest volatility, although it has a higher downside risk, the Sharpe ratio is greater than 1. Therefore the best state for value is state 1.
- All the metrics within state 1 are greater than the Kalman filter's metrics therefore the HMM produces superior results for momentum.

5.4.3 Weekly growth factor timing

The most favourable hidden state for the growth factor is state 2.

- Although the average return in state 2 is 7.58%, the cumulative return is 1.8055, which is higher compared to state 0 and 1.
- State 1 has the lowest volatility of 6.33%.

- Maximum drawdown is at the highest at 8.28%, which is comparable to that of the Kalman filter.
- State 0 has the highest Sharpe ratio of 1.29, indicating that this is the state with the highest returns per unit of risk.
- Growth is in this state 63% of the time. This state occurs more frequently compared to the other states.
- The Sharpe ratio is higher for state 2 at 0.89, it is closer to one.
- Therefore, for growth, the higher the risk, the higher the compensation. The Kalman filter does produce higher cumulative returns for this factor, however, the Sharpe ratio is lower for this model. Hence, the HMM produces better results.

5.4.4 Weekly quality factor timing

The most favourable hidden state for the quality factor is state 2.

- Quality is in this state 78% of the time. This state occurs more frequently compared to the other states.
- Average return is moderately low at 4.94%, this is the only state with positive returns.
- The volatility is at the lowest (5.72%) for this state.
- Maximum drawdown is at the lowest at 9.35%, hence the downside risk is limited in this state.
- Sharpe ratio is the highest at 0.86, and is the only positive value, indicating that this is the state with the highest returns per unit of risk.
- All the metrics within state 2 are greater than the Kalman filter's metrics therefore the HMM model produces superior results for Quality.

5.4.5 Weekly size factor timing

The most favourable hidden state for the size factor is state 0. The size factor includes small-cap, mid-cap and large-cap.

- Size is in state 0, 78% of the time. This state occurs more frequently compared to the other state 1 and 2.
- Average return is the highest at 9.75%.
- The volatility is similar to the Kalman filter at 8.56%.
- Maximum drawdown is higher at 23.37%, thus, this state is riskier.
- Sharpe ratio is the highest at 1.14, which is greater than 1, indicating that this is the state with the highest returns per unit of risk.

- Although this state is comparable to the Kalman factor results, the HMM still outperforms if for the size factor.

5.4.6 Weekly multifactor timing

The multifactor portfolio has mixed results, with state 2 having the highest Sharpe ratio at 0.59. Although this is a positive value, it is accompanied by a low average return. Therefore, the multifactor portfolio performs poorly for both state space models.

Overall, a factor timing strategy using the Kalman filter and the HMM with a weekly rebalancing results produces more favourable results for the HMM for the factor fund of funds. Therefore, a discrete list of hidden states is easier to time than the continuous hidden states found within the Kalman filter.

5.5 Monthly Kalman filter and HMM factor timing metrics

Table 5.3: Monthly Kalman filter and HMM factor timing metrics

Factor	State space model	State	Average Return	Cumulative Return	Max Drawdown	Standard Deviation	Distribution of Occurrence	Sharpe Ratio
Momentum	HMM	0	0.02714778	1.146123384	0.037623607	0.024970697	53.21563682	1.08718554
		1	0.007870381	1.033272335	0.086377455	0.042935013	44.05212274	0.183309154
		2	-0.008253152	0.997873475	0.048450394	0.113985305	2.732240437	-0.072405401
	Kalman filter	–	0.013874819	1.139950568	0.116339376	0.0452951	–	0.306320535
Value	HMM	0	0.024051177	1.050681006	0.038344821	0.037419669	16.82910981	0.642741577
		1	0.011714375	1.106548455	0.055367732	0.02329691	70.76023392	0.502829569
		2	-0.075725499	0.891553136	0.131612032	0.058495925	12.41065627	-1.29454316
	Kalman filter	–	0.0015144	1.018669452	0.118809016	0.032375779	–	0.046775716
Growth	HMM	0	-0.005594295	0.976331256	0.166776411	0.050949509	35.05523067	-0.109800755
		1	-0.276968916	0.962262588	0.038467905	0.10173496	1.137102014	-2.722455639
		2	0.022378429	1.190543182	0.039882715	0.027200732	63.80766732	0.822714202
	Kalman filter	–	0.008644612	1.111363636	0.17208069	0.038231165	–	0.226114274
Quality	HMM	0	-0.025155738	0.973310238	0.070027531	0.040511692	10.57767369	-0.620950065
		1	-0.078694435	0.915131618	0.126262945	0.072868589	11.08508977	-1.079950027
		2	0.007205678	1.059066729	0.047394506	0.021972993	78.33723653	0.327933374
	Kalman filter	–	-0.00384282	0.961684673	0.172470981	0.035503889	–	-0.108236581
Size	HMM	0	0.010967962	1.110687362	0.09451561	0.031838811	78.3625731	0.34448405
		1	-0.02101271	0.995258402	0.022290832	0.05227886	1.851851852	-0.401935119
		2	-0.03886391	0.910354572	0.134178917	0.05196564	19.78557505	-0.747877064
	Kalman filter	–	0.005984712	1.075837025	0.136721768	0.03714939	–	0.161098534
Size Small-Cap	HMM	0	-0.034163945	0.973513974	0.0579905	0.103445859	6.432748538	-0.330259187
		1	0.011140022	1.081531414	0.132386306	0.037594937	57.60233918	0.296317083
		2	-0.004977123	0.97837349	0.098423014	0.042022693	35.96491228	-0.118438934
	Kalman filter	–	0.002910519	1.036189361	0.149301175	0.044077543	–	0.066031785
Size Mid-Cap	HMM	0	0.012347929	1.09794743	0.087412374	0.029845344	61.95581546	0.413730499
		1	-0.221351905	0.968873037	0.031621701	0.063490394	1.169590643	-3.486384153
		2	-0.00479775	0.978622885	0.138219176	0.047363721	36.87459389	-0.101295893
	Kalman filter	–	0.003895538	1.048731373	0.143170314	0.037991501	–	0.102537098
Size Large-Cap	HMM	0	0.015665739	1.142289347	0.039242696	0.02293726	69.52566602	0.682982142
		1	-0.008238287	0.971133156	0.133615532	0.048818682	29.10981157	-0.168752754
		2	-0.151595159	0.975050651	0.029623376	0.062082956	1.364522417	-2.441816075
	Kalman filter	–	0.0052916	1.066767575	0.138338469	0.033774857	–	0.156672766
Multifactor	HMM	0	-0.027823882	0.861330203	0.175115445	0.045262191	45.95513256	-0.614726811
		1	-0.051929935	0.983649452	0.040478604	0.069376479	2.719238613	-0.748523643
		2	0.015010736	1.094114417	0.064400162	0.030791626	51.32562882	0.487494112
	Kalman filter	–	0.000282103	1.003298868	0.174678333	0.037181533	–	0.007587177

- The least frequent hidden state has a negative Sharpe ratio for all the factors, when rebalancing occurs monthly.
- The Kalman filter produces underwhelming results, with extremely low Sharpe ratios. Momentum manages a Sharpe ratio of 1.09 for state 0, however, that is the only factor to benefit from a factor timing strategy that rebalances once a month.
- The HMM produces higher returns than the Kalman filter for all the factors.

5.6 Quarterly Kalman filter and HMM factor timing metrics

Table 5.4: Quarterly Kalman filter and HMM factor timing metrics

Factor	State space model	State	Average Return	Cumulative Return	Max Drawdown	Standard Deviation	Distribution of Occurrence	Sharpe Ratio
Momentum	HMM	0	0.000408559	1.002054629	0.021608994	0.012501758	53.21563682	0.032680099
		1	0.007626795	1.032226153	0.023670563	0.022232167	44.05212274	0.343052267
		2	0	1	0	0	2.732240437	–
	Kalman filter	–	0.001301902	1.012366415	0.033024836	0.02125455	–	0.061252862
Value	HMM	0	0.004328096	1.008936335	0.01696267	0.021991011	16.82910981	0.196812064
		1	-0.003313303	0.971769733	0.039207409	0.012425075	70.76023392	-0.266662599
		2	-0.001908679	0.997110866	0.02130594	0.026614884	12.41065627	-0.071714735
	Kalman filter	–	-0.001709344	0.979338034	0.043783546	0.016539446	–	-0.103349504
Growth	HMM	0	0.010656781	1.046686678	0.029639337	0.021314854	35.05523067	0.499969695
		1	0	1	0	0	1.137102014	–
		2	-0.002939015	0.977354686	0.047587102	0.011714866	63.80766732	-0.250879114
	Kalman filter	–	0.002607104	1.032356353	0.039039096	0.015688002	–	0.166184565
Quality	HMM	0	0.001487557	1.001600994	0.044801805	0.033282478	10.57767369	0.044694892
		1	-0.038426359	0.957618413	0.060949969	0.050301514	11.08508977	-0.763920528
		2	0.003363632	1.027150978	0.029497403	0.012629539	78.33723653	0.266330581
	Kalman filter	–	-0.00063559	0.993558996	0.087962619	0.023515138	–	-0.027028992
Size	HMM	0	-0.010524479	0.90417324	0.11295761	0.015313966	78.3625731	-0.687247146
		1	0.029858336	1.006776529	0.013473968	0.050961381	1.851851852	0.585901245
		2	0.011650101	1.028554493	0.00612359	0.013609647	19.78557505	0.856017888
	Kalman filter	–	-0.001051346	0.987240665	0.049857335	0.016792636	–	-0.062607536
Size Small-Cap	HMM	0	0.003741746	1.002944269	0.021380477	0.054589131	6.432748538	0.068543783
		1	-0.010504097	0.928761016	0.082644454	0.016182098	57.60233918	-0.649118347
		2	0.004519668	1.020052662	0.029718662	0.022628751	35.96491228	0.199731218
	Kalman filter	–	-0.004456419	0.947022929	0.080334308	0.021466181	–	-0.207601846
Size Mid-Cap	HMM	0	-0.008976886	0.934323788	0.086552933	0.014781133	61.95581546	-0.607320578
		1	-0.070614075	0.989962985	0.010087725	0.026316312	1.169590643	-2.683281573
		2	0.011256563	1.052006403	0.019845497	0.021521542	36.87459389	0.523036986
	Kalman filter	–	-0.001477763	0.982112106	0.051665896	0.018349219	–	-0.080535453
Size Large-Cap	HMM	0	-0.00526078	0.956308348	0.068194087	0.010733571	69.52566602	-0.490123918
		1	0.016484999	1.060365151	0.013964074	0.020469125	29.10981157	0.805359262
		2	-0.033656717	0.994406251	0.005609453	0.013575737	1.364522417	-2.479181612
	Kalman filter	–	0.000805705	1.009889693	0.031699338	0.014400448	–	0.055949991
Multifactor	HMM	0	0.003468724	1.018784225	0.024493566	0.020577683	45.95513256	0.16856728
		1	-0.081543467	0.974445376	0.036688583	0.07286839	2.719238613	-1.119051293
		2	0.006018538	1.036721639	0.025562919	0.012099959	51.32562882	0.497401561
	Kalman filter	–	0.003333261	1.039681584	0.049984324	0.020763192	–	0.160537015

- The average returns are either negative or not significantly different from 0.
- A quarterly rebalancing period will not be beneficial.
- Size has a Sharpe ratio of 0.856 for state 2. Large-cap has a Sharpe ratio of 0.81. Size includes small-cap, mid-cap and a large-cap funds.
- Other factors produce significantly underwhelming results

Table 5.4 shows the results for the quarterly rebalancing period. This time period is too passive for a factor timing strategy. This is indicated by extremely low average returns, and low Sharpe ratios. Therefore, a comparison of the Kalman filter and the HMM is not feasible for the rebalancing period.

5.7 Bi-annual Kalman filter and HMM factor timing metrics

Table 5.5: Bi-annual Kalman filter and HMM factor timing metrics

Factor	State space model	State	Average Return	Cumulative Return	Max Drawdown	Standard Deviation	Distribution of Occurrence	Sharpe Ratio
Momentum	HMM	0	0.003074977	1.015568037	0.017104278	0.009486751	53.21563682	0.324133829
		1	0.001187388	1.004950241	0.011619014	0.008432869	44.05212274	0.140804809
		2	0	1	0	0	2.732240437	–
	Kalman filter	–	0.003202667	1.030696412	0.016937929	0.00981457	–	0.326317606
Value	HMM	0	0.0159078	1.033239866	0	0.017276533	16.82910981	0.920775028
		1	-0.000506348	0.995633265	0.020495145	0.009227184	70.76023392	-0.054875728
		2	-0.006935037	0.98954243	0.021640129	0.015053149	12.41065627	-0.460703383
	Kalman filter	–	0.001204221	1.014817402	0.021266231	0.011374897	–	0.105866519
Growth	HMM	0	0.013505345	1.059531107	0.009345594	0.015958554	35.05523067	0.846276241
		1	0	1	0	0	1.137102014	–
		2	-0.000784534	0.993904272	0.025092257	0.008537735	63.80766732	-0.091890156
	Kalman filter	–	0.004548943	1.057134653	0.012840219	0.011645148	–	0.390629905
Quality	HMM	0	0.008728414	1.0094307	0.029841488	0.029289202	10.57767369	0.298007921
		1	-0.000635883	0.999283627	0.043500191	0.040726514	11.08508977	-0.015613488
		2	0.003124676	1.025198046	0.015705671	0.0078695	78.33723653	0.397061518
	Kalman filter	–	0.003136376	1.032400311	0.063652448	0.018870983	–	0.166200994
Size	HMM	0	-0.002418122	0.977120906	0.043146312	0.010846017	78.3625731	-0.222950233
		1	0.095458907	1.021826687	0	0.044818287	1.851851852	2.129909748
		2	-0.000731613	0.998233497	0.006021288	0.004741919	19.78557505	-0.154286309
	Kalman filter	–	0.002203641	1.027281401	0.026714021	0.011647947	–	0.189187036
Size Small-Cap	HMM	0	-0.020414071	0.984088323	0.016039627	0.018049387	6.432748538	-1.131011855
		1	-0.005115634	0.964647874	0.050421796	0.013397502	57.60233918	-0.381834942
		2	0.009777431	1.043886595	0.008523399	0.01550607	35.96491228	0.630555076
	Kalman filter	–	-0.000337353	0.995887947	0.048464475	0.014531305	–	-0.023215631
Size Mid-Cap	HMM	0	0.001181639	1.008982106	0.030727827	0.007861318	61.95581546	0.15031056
		1	-0.073602494	0.989540444	0.010514642	0.02743003	1.169590643	-2.683281573
		2	0.00494706	1.022531484	0.019857577	0.015971821	36.87459389	0.309736726
	Kalman filter	–	0.001752208	1.021632634	0.02711373	0.012218849	–	0.143402054
Size Large-Cap	HMM	0	-0.000305286	0.997410847	0.016978369	0.005834605	69.52566602	-0.052323393
		1	0.012594468	1.045798104	0.013990656	0.017755957	29.10981157	0.709309445
		2	-0.035027972	0.994179013	0.005837995	0.014128845	1.364522417	-2.479181612
	Kalman filter	–	0.002964198	1.036868968	0.015149973	0.010945893	–	0.270804608
Multifactor	HMM	0	0.005513966	1.030024783	0.016103789	0.01469093	45.95513256	0.37533128
		1	-0.085992341	0.973070098	0.03805559	0.075583341	2.719238613	-1.137715522
		2	0.001255217	1.007549694	0.018698049	0.009609354	51.32562882	0.130624463
	Kalman filter	–	0.003269191	1.038904206	0.040994946	0.01559088	–	0.209686122

- The average returns are low, indicating that this rebalancing period is not suitable for a state space model factor timing strategy.
- Momentum is the only factor that has a positive Sharpe ratio for all the hidden states and the Kalman filter. All the other factors have at least one negative Sharpe ratio.

Table 5.5 shows results for the bi-annual rebalancing period. The bi-annual rebalancing period has mostly negative Sharpe ratios, indicating that this rebalancing period is not a good fit for a factor timing strategy and it would be difficult to make a fair comparison of the Kalman filter and the HMM.

5.8 Annual Kalman filter and HMM factor timing metrics

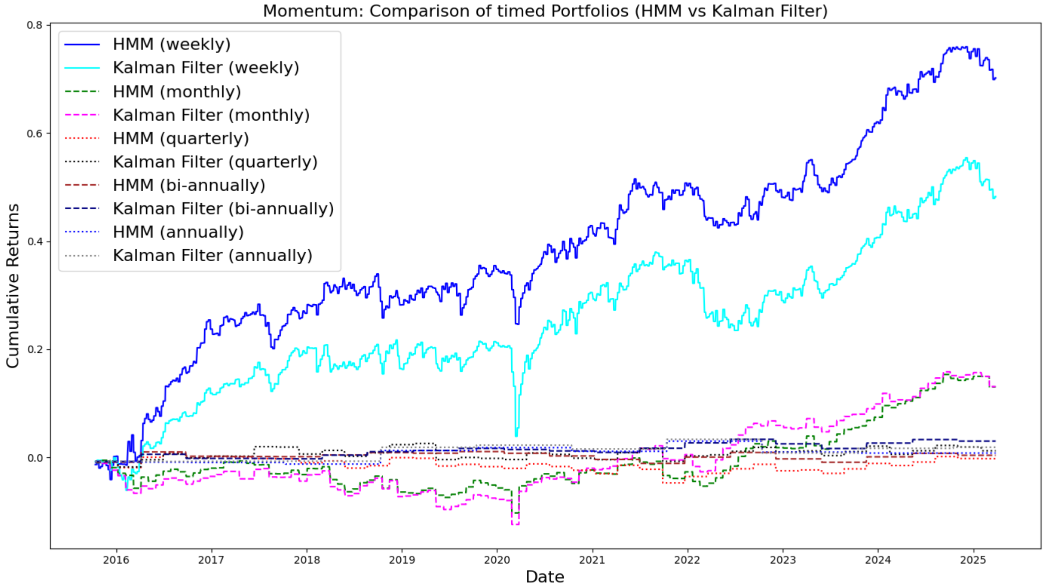
Table 5.6: Annual Kalman filter and HMM factor timing metrics

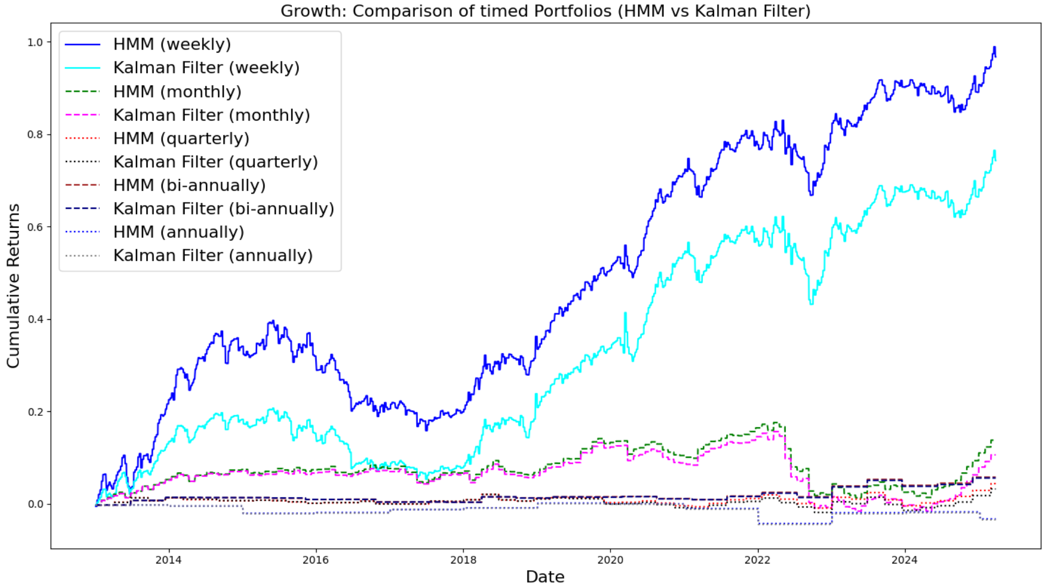
Factor	State space model	State	Average Return	Cumulative Return	Max Drawdown	Standard Deviation	Distribution of Occurrence	Sharpe Ratio
Momentum	HMM	0	0.002721206	1.013764692	0.007588795	0.009433385	53.21563682	0.288465456
		1	0.000110862	1.000461152	0.025168125	0.016240243	44.05212274	0.006826385
		2	0	1	0	0	2.732240437	–
	Kalman filter	–	0.002002519	1.01908456	0.025096564	0.013813714	–	0.144966024
Value	HMM	0	0.006960038	1.014409576	0.015706391	0.019911406	16.82910981	0.349550306
		1	0.0002092	1.001809724	0.018622465	0.011412525	70.76023392	0.018330766
		2	0	1	0	0	12.41065627	–
	Kalman filter	–	0.001219429	1.015005934	0.020686506	0.012197342	–	0.099975008
Growth	HMM	0	-0.003733468	0.984141332	0.030846351	0.020110708	35.05523067	-0.185645774
		1	0	1	0	0	1.137102014	
		2	-0.002551287	0.980312541	0.020024555	0.008122086	63.80766732	-0.314117173
	Kalman filter	–	-0.002839748	0.965909159	0.044831789	0.014045785	–	-0.20217792
Quality	HMM	0	0	1	0	0	10.57767369	–
		1	0.03557685	1.040909183	0	0.024011275	11.08508977	1.481672696
		2	-0.000441107	0.996493061	0.006939152	0.003789777	78.33723653	-0.11639395
	Kalman filter	–	0.003768579	1.039057328	0.008154493	0.008019341	–	0.469936266
Size	HMM	0	0.001042338	1.010026593	0.024852877	0.008757373	78.3625731	0.119024016
		1	0.030911671	1.007016426	0.015176134	0.056278595	1.851851852	0.549261599
		2	-0.005989803	0.985628907	0.020514979	0.013751449	19.78557505	-0.435576133
	Kalman filter	–	0.001129472	1.013891295	0.024950426	0.014632379	–	0.077189917
Size Small-Cap	HMM	0	0.060384054	1.048588123	0	0.053389409	6.432748538	1.131011855
		1	-0.00038895	0.997267198	0.01356077	0.009022349	57.60233918	-0.043109639
		2	-0.001102335	0.995169307	0.028325711	0.020433539	35.96491228	-0.053947315
	Kalman filter	–	0.00191781	1.023701193	0.03780428	0.019056644	–	0.100637348
Size Mid-Cap	HMM	0	-0.00187492	0.985911797	0.022920689	0.010107852	61.95581546	-0.185491456
		1	0.137329613	1.019812224	0	0.051179725	1.169590643	2.683281573
		2	0.002552173	1.011561227	0.025340309	0.017237072	36.87459389	0.148063039
	Kalman filter	–	0.00175094	1.021616814	0.024582611	0.015393576	–	0.113744866
Size Large-Cap	HMM	0	-0.000163628	0.998611428	0.015105413	0.006216289	69.52566602	-0.026322416
		1	-0.002613562	0.99075038	0.028130698	0.019017387	29.10981157	-0.137430115
		2	0	1	0	0	1.364522417	–
	Kalman filter	–	-0.000915376	0.988881609	0.019472417	0.011455059	–	-0.07991018
Multifactor	HMM	0	-0.000712194	0.996186314	0.010929242	0.00617604	45.95513256	-0.115315632
		1	-0.023709138	0.992501544	0	0.013274832	2.719238613	-1.786021673
		2	0.000391539	1.002348879	0.008962504	0.00523581	51.32562882	0.074780929
	Kalman filter	–	-0.000544889	0.993658824	0.016088494	0.00535254	–	-0.101800135

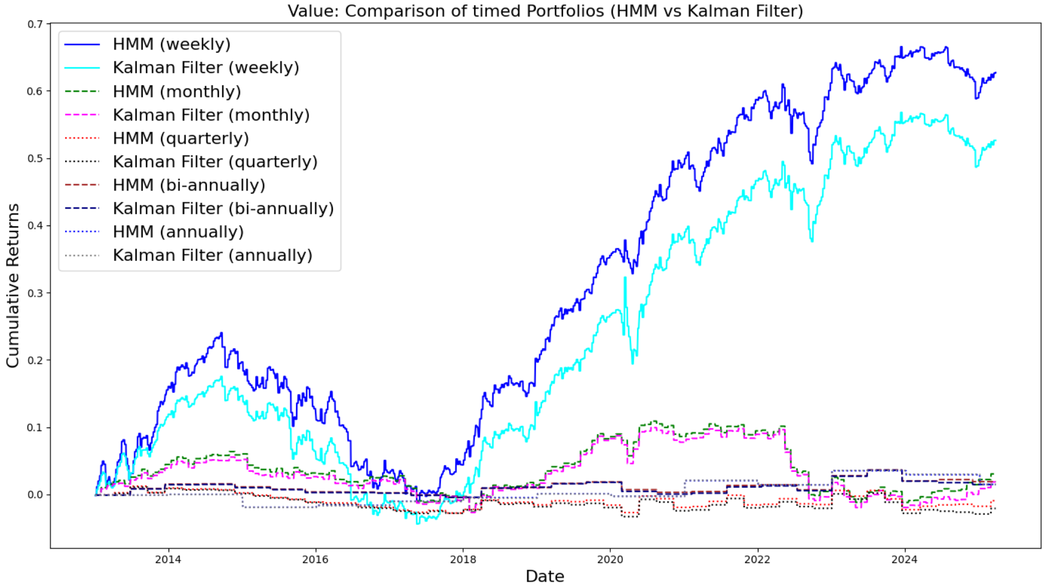
- Momentum and value are the only two factors with positive average returns for all the HMM hidden states and Kalman filter. The average returns are positive, although they are not significantly different from 0, with the exception of mid-cap portfolio having the highest average returns at 13.73%.
- The cumulative returns for both the HMM and the Kalman filter are all not significantly different from 1, implying that there is no loss to the initial principal investment.
- The maximum drawdowns and the standard deviation (volatility) are all lower than 2.5%, therefore there are no significant losses in the annual rotations and adjustments based on hidden states.
- Other factors have at least one hidden state with a negative average return.
- An annual factor timing period is too long for positive results; a shorter rebalancing frequency might be more effective.

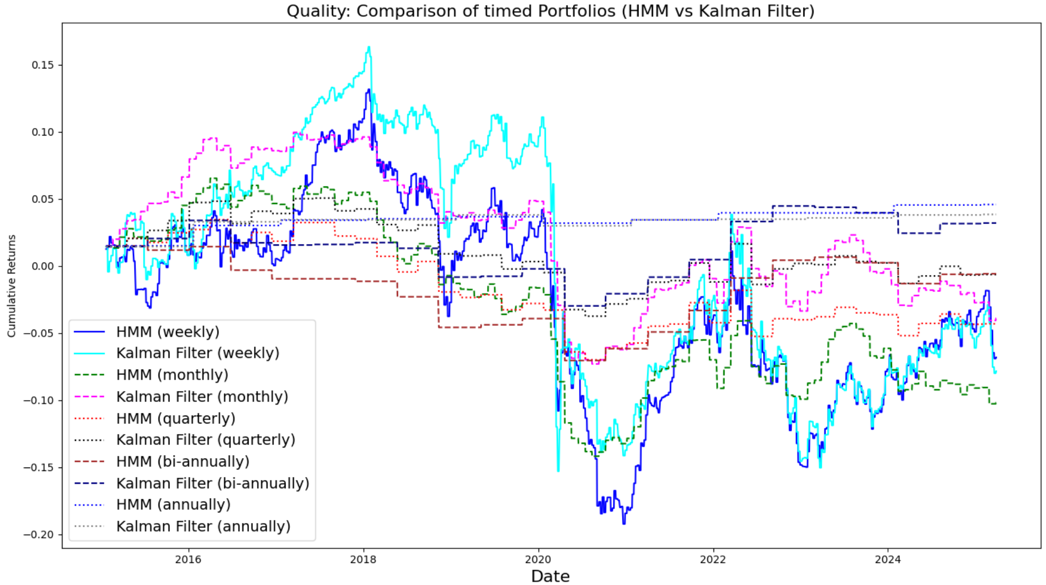
Table 5.6 shows results for the annual rebalancing period, which does not produce strong results for a factor timing strategy; as a result, whether the Kalman filter or the HMM is used, rebalancing once a year will not yield better results.

5.9 Kalman filter and HMM factor timing results (graphical)









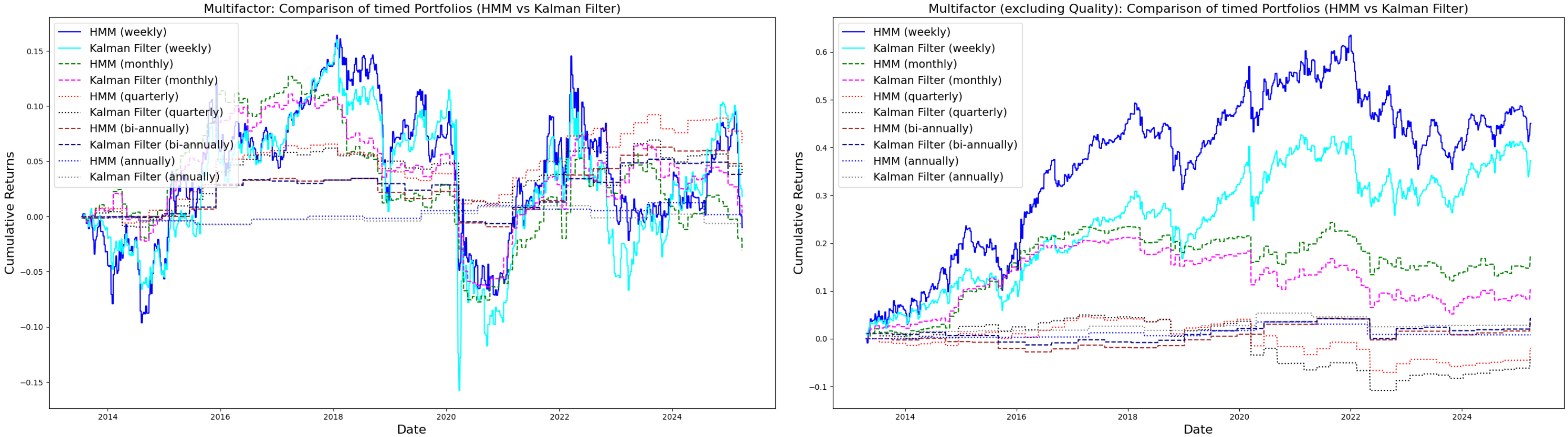


Figure 5.4: Factor timing comparison: Kalman filter versus the HMM

Figure 5.4 illustrates the graphical results for the Kalman filter and the HMM.

- The HMM performs well for momentum, value, growth and size. The results are mixed for quality and the multifactor portfolio.
- Momentum and growth have the highest returns for the weekly factor timing strategy, however the other time periods produce underwhelming results.
- Quality is the worst performing fund of funds, and the factor timing results improves when quality is removed from the multifactor fund of funds.
- Multifactor returns improved significantly after the quality factor was excluded. Implying that factor timing the quality factor is not beneficial.

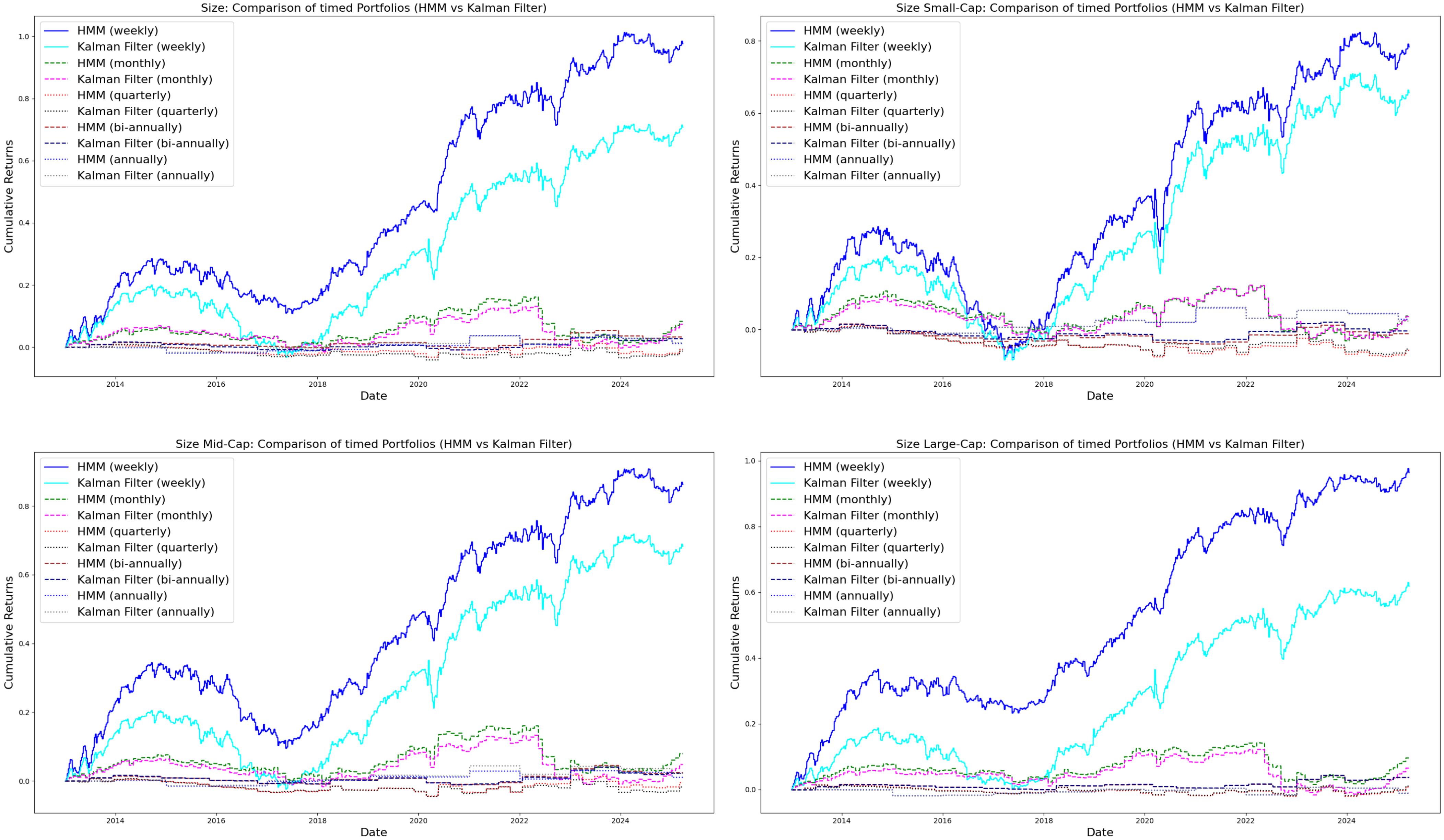


Figure 5.5: Size factor timing comparison: Kalman filter versus the HMM

Figure 5.5 shows the size factor comparison. The HMM outperforms the Kalman filter, for size factors.

- The large-cap fund of funds has the best results for the weekly factor timing strategy, compared to the small-cap and mid-cap for both the HMM and Kalman filter.

- The size fund of funds which includes the small-cap, mid-cap and the large-cap, has the best results compared to the individual size fund of funds.

References

[1] Ang, A., Rhodes-Kropf, M., and Zhao, R. (2008). Do funds-of-funds deserve their fees-on-fees? *National Bureau of Economic Research*, NBER Working Paper No. 13944.

[2] Brown, S., Goetzmann, W. and Liang, B. (2004). Fees on Fees in Funds of Funds. Yale ICF Working Paper No. 02-33, *SSRN*.

Chapter 6

Conclusions and recommendations

6.1 Summary of findings

This research study examined a factor timing strategy using the Kalman filter and the Hidden Markov model, two of the most popular state space models. This was done to assess the relationship that exists between factor timing and higher returns. The data used in this study are publicly available historical investment data, which can be found on the Yahoo finance database. The study mainly used equities listed in the US, while equities from South Africa were included in Chapter 3. Analysis of the data applied the Kalman filter, hidden Markov model, and measures of dependence.

In Chapter 2, a high pure momentum-only equity factor timing framework was assessed, offering a comprehensive approach starting with the selection of stocks, dynamic monitoring, and rebalancing, together with the forecasting procedure to achieve superior returns. A dynamic portfolio construction process, together with the Kalman filter, was implemented for the state dynamic system estimation using daily historical data. Additionally, the ARIMA forecasting and confidence intervals enhanced the forecast accuracy. Momentum strategies can be susceptible to sharp downturns, reversals, market volatility, and sentiment arising from unforeseen events. However, returns are somewhat predictable, and this predictability plays a pivotal role in the way a pure momentum strategy is implemented. A factor timing strategy relies on this predictability and the forecasting of future performance, even though past returns do not guarantee future outcomes. A Kalman filter approach addresses this past returns limitation by iteratively incorporating new information as it becomes available. Although factor timing for momentum in an active portfolio was successful, in practice, the associated costs can significantly reduce the returns achieved.

In Chapter 3, a momentum portfolio was constructed using the historical data from large-cap stocks to assess diversification of a factor portfolio. The top 20 stocks with the highest average momentum scores over 3, 6, 9 and 12 months were selected and included in the portfolio. This construction was done for the stocks listed in SA, as well as those that are listed in the US. The resulting portfolios included stocks that have significantly outperformed expectations. The SA momentum portfolio had a more varied industry exposure, compared to the US momentum portfolio, which consisted mainly of technology stocks. The Kendall's tau and Spearman's rho were employed on both momentum portfolios to assess whether they were diversified. The SA momentum portfolio had a lower Kendall's tau (0.1296) and Spearman's rho (0.1881) compared to the US momentum portfolio, which had a higher Kendall's tau (0.3494) and Spearman's rho (0.4932). Although both portfolios can be considered diversified, the SA momentum portfolio is more diversified with lower volatility ($\sigma_p = 0.1456$) compared to the volatility ($\sigma_p = 0.3422$) from the US momentum portfolio.

Furthermore, a combination of 10 copula mixtures were evaluated for both countries for the population from which the portfolio were selected (i.e. large-cap stocks) and the momentum portfolios. The copula mixtures are all valid satisfying their individual parameter boundaries within the mixture. Mixture copula models were applied to the population of large-cap stocks, and the momentum portfolios. The US and SA momentum portfolios have the same top three copula mixture models. This suggests that their dependence structures can be modelled by the same copula mixture models.

In Chapter 4, a fund of funds was constructed consisting of three ETFs each. The historical returns of value, growth, value-growth and the overall market funds of funds were compared, and the results show that growth outperformed value. Value underperformed while growth demonstrated strong returns. The HMM was applied to the fund of funds to identify the hidden states of the historical daily returns. Growth tends to be more susceptible to market condition changes, which causes more hidden states transitions. The Markov switching regression model was applied to the funds of funds to identify time variant parameters, including the state transition parameters showing the probability of staying within a state or transitioning between the states. Each fund of funds had a probability greater than 97% of staying within state 0. The Kaplan-Meier method, predominantly used in survival analysis, was adjusted to fit the premise of the HMM dynamics to identify the persistence in days of the hidden states. The overall market fund of funds had the highest average in days of staying within state 0. Growth had the highest average in days of staying in state 1. This result implies that growth is the riskier factor compared to value. However, when a factor timing strategy using the HMM was applied to the funds of funds, growth did not significantly deviate from the original untimed equally weighted portfolio. The fund of funds with both value and growth produced significant results from a factor timing HMM approach.

In Chapter 5, the Kalman filter and the HMM were compared, illustrating the comparative nature of state space models and their implications on a factor timing strategy. The factor timing strategy performs well when the rebalancing period is short and when the rebalancing period is increased the effectiveness of the strategy decreases. Thus, a weekly factor timing strategy has higher returns compared to an annual rebalancing period. These findings imply that, factor timing is a strategy that is best suited for an active management style and for it to be effective the time period of adjustments has to be short. The HMM performed better than the Kalman filter. This could be due to the fact that the hidden states within the HMM are discrete and provide a better opportunity for adjusting a portfolio when a state transition occurs.

6.2 Theoretical and practical implications of findings

The findings support the hypothesis that factor timing, although a challenging endeavour, is not impossible and can be done to produce moderate higher returns. In conclusion, this research study has demonstrated that factor timing is a viable option for higher returns. This study contributes to the growing literature on factor timing by:

- Providing empirical evidence that the Kalman filter and the HMM can in theory be used to create trading signals that can be used for factor timing.

- that factors do exhibit cyclicality in their returns.
- Confirming factor returns are cyclical and can be used as an advantage when implementing a factor timing strategy.
- Showing that transaction costs can diminish and limit the returns from a factor timing strategy, posing a challenge to practical implementation. Factor timing can improve portfolio returns if the transaction costs are kept at a minimum.

Therefore, although factor timing should not be a complete replacement for static multifactor portfolios, it can be used to enhance traditional factor investment strategies.

6.3 Limitations of the study

There are some limitations that must be acknowledged for this research, i.e.

- The analysis was limited to US equities, an in Chapter 2 there was a look at SA equities. However, other financial markets were not considered.
- The funds of funds were constructed using exchange-traded funds that are listed in the US only. other markets were excluded.
- Only five of the most popular factors were considered, namely momentum, value, growth, quality, size and other factors were excluded.
- Large capitalisation stocks were used, excluding small and medium capitalisation stocks.
- Historical data from 01 January 2013 to 31 December 2024 were used in the analysis and backtesting. However, historical performance might not be a true indicator of future performance.

6.4 Suggestions for future research

- The momentum dynamic allocation strategy can be expanded to include small and medium capitalisation stocks.
- This research can be extended to include multi-asset portfolios which may yield broader insights.
- The strategies in this research can be implemented for other financial markets beyond the US to enhance generalisability.

Appendix

A.1 DSFE author guidelines

1. Requirements

Before you submit, please check below information has been confirmed: One Author designated as corresponding Author:

- E-mail address
- Full postal address
- Telephone number

All necessary information has been provided

- Abstract, Keywords and Abbreviations
- All figures have captions and are mentioned in the text.
- All tables (including title, description, footnotes) and are mentioned in the text.
- Ethic declaration, Conflicts and Interests, Acknowledgement.

Further considerations

- Manuscript has been "spellchecked" and "grammar-checked"
- References are in the correct format for this journal
- All references mentioned in the Reference list are cited in the text
- Permission has been obtained for use of copyrighted material from other sources (including the Web)

All submissions should be prepared with the following files, and submitted via JAMS, our online submission and peer-review system.

1.1. Cover letter, Which may Include:

- Concise summary of why your paper is a valuable addition to the scientific literature
- Brief relation of your study to previously published work
- Any recommended or opposed reviewers. The authors can provide a list of five qualified, independent, prospective reviewers who could perform quality peer reviews of your submission. (Including their full names, affiliations and their current E-mail addresses.)
- Confirmation from all authors that the manuscript will be considered to be published in AIMS Press journal in Open Access Format, and the submission has not been published in another journal. The submission may be checked by the originality detection service CrossCheck.
- Confirmation from all authors that the manuscript will be considered for publication in AIMS Press journal in Open Access Format. It has not been published nor is it under consideration for publication elsewhere. Also confirm you have got the permission to reproduce the published materials in your manuscript.

1.2. Full Text, Including Figures and Tables

2. Manuscript Organization

2.1. File Format

Authors may submit their manuscript files in Word (as .doc or .docx) or LaTeX (as .pdf), format. Word files must not be protected.

- Microsoft Word
- Tex Template

Submissions with equations: Please follow the instructions below to make sure that your equations are editable when the manuscript enters production.

Ensure that the equations in your .docx file remain editable and not as images. In principle, Equations should be centered on the page. If equations are too wide to fit in a single column, indicate appropriate breaks. All equations must be numbered, type the number in parentheses flush with the left margin. Please avoid using Equation Editor for simple in-line mathematical copy, symbols, and equations. Type these in Word instead, using the "Symbol" function when necessary.

- Enable "Compatibility Mode" before you compose your article
- MathType to create the equation or
- Go to Insert > Object > Microsoft Equation 3.0 and create the equation

2.2. Manuscript Length

AIMS does not impose a limit on the length of manuscripts so authors can provide as many details of their research results as possible. However, we encourage authors to write an article not less than 6000 words in order to fully present your valuable research results.

2.3. Title

- Titles should be concise and informative.
- Titles are to be in the sentence case. Meaning, only the first word and proper nouns are capitalized. If a theory, formula, or specialized equation contains names (like Navier-Stokes) they are capitalized, otherwise leave them lower case.
- Capitalization after a colon is required.
- Ensure the correct usage of articles. (a, an, the)
- Ensure the appropriate use of grammar and spelling.

More tips and suggestions can be found here.

Examples of titles done correctly:

- "Optimal control of a stationary Navier-Stokes hemivariational inequality with numerical approximation"
- "Global attractor of the Euler-Bernoulli equations with a localized nonlinear damping"

2.4. Authors and Affiliations

All authors' full names (the middle name can be abbreviated) should be listed together and separated by commas. Link affiliations to the author's name with superscript numbers and list as follows: Laboratory, Department, Organization, City, State (in abbreviation if from USA , Canada , or Australia), and Country.

The Corresponding Author should be marked with an asterisk, and their exact contact address, email address and telephone number should be listed in a separate paragraph. This information will be published with the article if accepted.

Any addition, deletion or rearrangement of author names in the authorship list should be made only before the manuscript has been accepted and only if approved by the journal Editor. To request such a change, the Editor must receive the following from the corresponding author:

- the reason for the change in author list and
- written confirmation (e-mail, letter) from all authors that they agree with the addition, removal or rearrangement.

Any change of affiliation requests will not be allowed after publication.

If the article has been submitted on behalf of a consortium, all author names and affiliations should be listed at the end of the manuscript.

2.5. Headings and Subheadings

There should be no more than 4 levels of headings. Subsections should be numbered 1.1 (then 1.1.1, 1.1.2, ...), 1.2, etc. (the abstract is not included in section numbering). Any subsection may be given a brief heading. Each heading should appear on its own separate line. The font of headings and subheadings should be 12 point normal Times New Roman, and only the first word should be capitalized.

2.6. Abstract, Keywords and Abbreviations

The abstract should:

- Describe the the context and purpose of the study
- Explain how the study was performed, including any model organisms used, without methodological detail
- Summarize the main findings and their significance
- Be less than 300 words

Please minimize the use of abbreviations (if possible) and do not cite references in the abstract.

5 to 10 **keywords** should be provided after the abstract in a separate paragraph.

Abbreviations

Define abbreviations that are not standard in this field below the keywords of the article. Such abbreviations that are unavoidable in the abstract must be defined at their first mention there, as well as in the footnote. Ensure consistency of abbreviations throughout the article.

2.7. Main Text

The body text must be in 12 point normal Times New Roman font with a line space of at least 15 point. Any abbreviations should be listed before the introduction section. Standard International Units should be used throughout the manuscript.

The main text should include:

- **Introduction**

The Introduction section should provide a brief statement of the research background and whether the aim of the article was achieved.

- **Materials and methods**

  The Materials and methods section should provide sufficient detail to allow suitably skilled investigators to repeat your study. This section should include the design of the study and the type of materials involved, a clear description of all interventions and comparisons, and the type of analysis used, including a power calculation if appropriate. Generic drug names should generally be used. When proprietary brands are used in research, include the brand names in parentheses in the Materials and methods section.

  If materials, methods, and protocols are well established, authors may cite articles where those protocols are described in detail, but the submission should include sufficient information to be understood independent of these.

  For studies involving human participants, a statement detailing ethical approval and consent should be included in the methods section. For further details of the journal's editorial policies and ethical guidelines see "Specific Reporting Guidelines."

- **Results, Discussion, Conclusions**

  These sections may all be separate, or they may be combined to create a mixed Results/Discussion section (commonly labeled "Results and Discussion") or a mixed Discussion/Conclusions section (commonly labeled "Conclusions").

  Authors should describe and explain the results of the experiments in these sections; they should explain how the results relate to the hypothesis presented as the basis of the study and provide a concise explanation of the implications of the findings, particularly in relation to previous related studies and potential future directions for research.

- **Use of Generative-AI tools declaration**

  Here is our Guidelines for the Use of AI Tools in Writing and Research.

  Instructions for declaring Generative-AI tools are at the end of the document.

- **Acknowledgments (All sources of funding of the study must be disclosed)**

  If the research was funded, the author should list the funding information and grant number in the Acknowledgments section, for example, "This work was supported by the National Institutes of Health [grant numbers xxxx, yyyy]; the Bill & Melinda Gates Foundation, Seattle, WA [grant number zzzz]; and the United States Institutes" of Peace [grant number aaaa]". If no funding has been provided for the research, please include the following sentence: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. If there are any contributions from other institutions or people, the author should acknowledge them in this section as well.

  Authors should obtain permission to acknowledge from all those mentioned in the Acknowledgements section.

- **Conflict of interest**

  The author should declare all relationships, financial, commercial or otherwise, that might be perceived by the academic community as representing a potential conflict of interest. If there are no such relationships, the author can state "All authors declare no conflicts of interest in this paper" in this section.

- **References**

  References are to be listed alphabetically, last name first, followed by publication date in parentheses. Each reference should be cited in text at the appropriate place.

  All citations in the text should refer to:

  1. Single author: the author's last name and the year ofpublication;
  2. Two authors: both authors' last names and the year of publication;
  3. Three or more authors: first author's name followed by 'et al.' and the year of publication.

  Citations may be made directly (or parenthetically). Groups of references should be listed first alphabetically, then chronologically. Examples: 'as demonstrated (Allan, 2000a, 2000b, 1999; Allan and Jones, 1999). Kramer et al.(2010) have recently shown ....'

  References should be arranged first alphabetically and then further sorted chronologically if necessary. More than one reference from the same author(s) in the same year must be identified by the letters 'a', 'b', 'c', etc., placed after the year of publication. For material intended for publication but not yet accepted, use "unpublished work" or "submitted for publication". Unpublished data or personal communications should be cited within the text only and not listed in the references.

  References should be formatted as follows:

  – **Journal article style:** Benoist Y, Foulon P, Labourie F, et al. (Year) Anosov flows with stable and unstable differentiable distributions. J Amer Math Soc Volume: Starting Page-Ending Page. https://doi.org/10.1090/S0894-0347-1992-1124979-1
  – **Accepted, unpublished papers:** Same as above, but "In press" appears instead of the page numbers.
  – **Book style:** Serrin J (1971) Gradient estimates for solutions of nonlinear elliptic and parabolic equations , In: Zarantonello, E.Z. Author, Contributions to Nonlinear Functional Analysis , 2 Eds., New York : Academic Press , 35-75. https://doi.org/10.1016/B978-0-12-775850-3.50017-0
  – **Online content:** SARS Expert Committee, SARS in Hong Kong : From Experience to Action. Hong Kong SARS Expert Committee, 2003. Available from: http://www.sars-expertcom.gov.hk/english/reports/reports.html.

  Cited journals should be abbreviated according to ISO 4 rules. For examples, see http://www.issn.org/services/onli services/access-to-the-ltwa/.

  For more questions regarding reference style, please refer to Chicago-Style Citation Quick Guide.

- **Appendices**

  If there is more than one appendix, they should be identified as A, B, etc. Formulae and equations inappendices should be given separate numbering: Eq. (A.1), Eq. (A.2), etc.; in a subsequent appendix, Eq. (B.1) and so on. Similarly for tables and figures: Table A.1; Fig. A.1, etc.

- **Supplementary (if available and necessary)**

  We encourage authors to submit detailed supplementary, including dataset, document, image, video, software code, protocol, supporting information, table etc, but some large datasets

(>100 MB) should be deposited in specialized service providers by author.The supplementary should be submitted in a separated file.

AIMS Press Open Data Policy is at here.

- **Figures and tables**

  General Guidelines:

  - Make sure you use uniform lettering and sizing of your original artwork.
  - Embed the used fonts if the application provides that option.
  - Aim to use the following fonts in your illustrations: Arial, Times New Roman, Symbol.
  - Number the illustrations according to their sequence in the text.
  - Use a logical naming convention for your artwork files.
  - Provide captions to illustrations separately.
  - Size the illustrations close to the desired dimensions of the published version.

  If your electronic artwork is created in a Microsoft Office application (Word, PowerPoint, Excel) thenplease supply 'as is' in the native document format.

  Regardless of the application used other than Microsoft Office, when your electronic artwork isfinalized, please 'Save as' or convert the images to one of the following formats (note the resolutionrequirements for line drawings, halftones, and line/halftone combinations given below):

  - EPS (or PDF): Vector drawings, embed all used fonts.
  - TIFF (or JPEG): Color or grayscale photographs (halftones), keep to a minimum of 300 dpi.
  - TIFF (or JPEG): Bitmapped (pure black & white pixels) line drawings, keep to a minimum of 1000 dpi.
  - TIFF (or JPEG): Combinations bitmapped line/half-tone (color or grayscale), keep to a minimum of500 dpi.

  Please do not:

  - Supply files that are optimized for screen use (e.g., GIF, BMP, PICT, WPG); these typically have alow number of pixels and limited set of colors;
  - Supply files that are too low in resolution;
  - Submit graphics that are disproportionately large for the content.

  Figure captions

  Ensure that each illustration has a caption. Supply captions separately, not attached to the figure. A caption should comprise a brief title (**not** on the figure itself) and a description of the illustration. Keep text in the illustrations themselves to a minimum but explain all symbols and abbreviations used.

  Tables

  Please submit tables as editable text and not as images. Tables can be placed either next to the relevant text in the article, or on separate page(s). Number tables consecutively in accordance with their appearance in the text and place any table notes below the table body.

All the tables should be prepared in **three-line format**. Please avoid using vertical rules and shading in table cells.The font size in tables should be from 9 to 12 pt according to the table size.

Authors should insert the Figures and Tables into the main text of the manuscript and call out all figures and tables in numerical order. There must be a caption under figure, and above table. Authors must obtain permission for the reuse of published materials from other sources.

2.8 License term and copyright

All articles published by AIMS Press are Open Access under the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0). Under this license, authors retain ownership of the copyright for their content, and anyone can copy, distribute, or reuse these articles as long as the author and original source are properly cited.

3. English Editing

3.1 English editing by AI language tools

Enhancing manuscript quality with Assistive-Artificial Intelligence (AI) language tools

At AIMS, we are committed to maintaining high language quality standards in all submitted manuscripts. To support our authors, we recommend the use of Assistive-AI language tools. These tools can significantly enhance the clarity, coherence, and overall quality of your writing. Below are three Assistive-AI tools that you may find particularly useful:

- **Grammarly**
  Grammarly is an advanced writing assistant that helps improve your writing by providing suggestions on grammar, spelling, punctuation, style, and tone. It also includes a plagiarism detection feature and vocabulary enhancement suggestions. Grammarly integrates seamlessly with various platforms, including Microsoft Word and Google Docs, making it a convenient tool for refining your manuscripts.
  - Key features:
    * Grammar and spelling check
    * Style and tone suggestions
    * Plagiarism detection
    * Integration with multiple platforms
- **LanguageTool**
  LanguageTool is an open-source grammar and spell checker that supports over 20 languages. It identifies grammatical errors, spelling mistakes, and punctuation issues, offering suggestions for corrections. LanguageTool is especially beneficial for non-native English speakers and integrates with word processors like Microsoft Word and Google Docs.

- **Key features:**
    * Grammar and spelling check
    * Style suggestions
    * Multilingual support
    * Integration with multiple platforms

- **Curie**
  Curie is an AI writing assistant designed to enhance your writing by providing contextual grammar and style suggestions. It helps improve sentence structure, coherence, and readability. Curie also offers personalized feedback based on your writing style and integrates with various writing platforms.
  - **Key features:**
    * Contextual grammar and style suggestions
    * Sentence structure and readability improvement
    * Personalized feedback
    * Integration with multiple writing platforms

We follow COPE's guidelines and policies regarding the use of AI tools: COPE Policy on AI tools. Please disclose any Generative-AI use in your manuscript's "Use of Generative-AI tools declaration" portion at the end of your manuscript before the Acknowledgments section. (The tools listed above are not considered Generative-AI tools) We encourage our authors to utilize these Assistive-AI tools to enhance the language quality of their manuscripts. Doing so will help ensure clarity, coherence, and a professional standard in your writing. This ultimately facilitates a smoother review and publication process and may increase the possibility of acceptance.

3.2 English editing by service providers

In order to speed up the peer review procedure, we encourage non-native English speaking authors to send their manuscript to a native English speaker or an English editing company for polishing before submitting to our journal. There are some English editing companies below for your consideration. Please provide the certification from the company you used along with the submission to our journal. The code "AIMS" is active and can be used immediately. This will provide an ongoing 10% discount any time the author uses AJE services: English Language Editing, Manuscript Formatting, Figure Services, Translation. It will provide a 10% discount for using Charlesworth services by entering the code "AIMSPress".

4. Ethics approval of research

Methods sections of papers on research using human subjects must include ethics statements that specify:

- The name of the approving institutional review board or equivalent committee(s). If approval was not obtained, the authors must provide a detailed statement explaining why it was not needed.

- Whether informed consent was written or oral. If informed consent was oral, it must be stated in the manuscript:
  - Why written consent could not be obtained,
  - That the Institutional Review Board (IRB) approved use of oral consent,
  - How oral consent was documented.

For studies involving humans categorized by race/ethnicity, age, disease/disabilities, religion, sex/gender, sexual orientation, or other socially constructed groupings, authors should:

- Explicitly describe their methods of categorizing human populations,
- Define categories in as much detail as the study protocol allows,
- Justify their choices of definitions and categories, including, for example, whether any rules of human categorization were required by their funding agency.

Methods sections of manuscripts reporting results of animal research must include required ethics statements that specify:

- The full name of the relevant ethics committee that approved the work, and the associated permit number(s) (where ethical approval is not required, the manuscript should include a clear statement of this and the reason why).

5. Clinical trial registration

Clinical trials must be pre-registered in a public trial registry. A list of acceptable registries can be found at www.who.int/clinical-trials-registry-platform and www.icmje.org. Authors can cite a reference to the registration in the Materials and methods section.

6. Cell line research

Methods sections for submissions reporting on research with cell lines should state the origin of any cell lines. For established cell lines, the provenance should be stated and references must also be given to either a published paper or to a commercial source. If previously unpublished *de novo* cell lines were used, including those gifted from another laboratory, details of institutional review board or ethics committee approval must be given, and confirmation of written informed consent must be provided if the line is of human origin.

7. Article Processing Charge (APC)

The Article Processing Charge will be covered by Guangzhou University. The author does not need to pay the fees.

A.2 Ethics approval

NWU®
NORTH-WEST UNIVERSITY
NOORDWES-UNIVERSITEIT
YUNIBESITI YA BOKONE-BOPHIRIMA

Private Bag X1290, Potchefstroom
South Africa 2520

Tel: 018 299-1111/2222
Fax: 018 299-4910
Web: http://www.nwu.ac.za

Senate Committee for Research Ethics
Tel: 016 910 3446
Email: Feziwe.Mseleni@nwu.ac.za

ETHICS APPROVAL LETTER OF STUDY

Based on the review by the **Faculty of Natural and Agricultural Sciences Ethics Committee (FNASREC)**, the Committee hereby clears your study as no ethical risk. This implies that the FNASREC grants permission that, provided the general conditions specified below are met, the study may be initiated, using the ethics number below.

Study title: Equity factor timing framework development using Kalman Filter
Study Leader/Supervisor: Dr MB Seitshiro
Student: TP Mashamba

Ethics number:	N	W	U	-	0	1	4	0	6	-	2	3	-	A	9
	Institution				Study Number						Year			Status	

Status: S = Submission; R = Re-Submission; P = Provisional Authorisation; A = Authorisation

Application type: Single **Risk Category:** No Risk
Commencement date: 26/10/2023
Expiry date: 31/01/2025

General conditions:
The following general terms and conditions apply:

- *The commencement date indicates the date when the study may be started.*
- *In the interest of ethical responsibility, the NWU-SCRE and FNASREC reserves the right to:*
  - *request access to any information or data at any time during the course or after completion of the study;*
  - *to ask further questions, seek additional information, require further modification or monitor the conduct of your research or the informed consent process;*
  - *withdraw or postpone approval if:*
    - *any unethical principles or practices of the study are revealed or suspected;*
    - *it becomes apparent that any relevant information was withheld from the FNASREC or that information has been false or misrepresented;*
    - *submission of the annual (or otherwise stipulated) monitoring report, the required amendments, or reporting of adverse events or incidents was not done in a timely manner and accurately; and / or*
    - *new institutional rules, national legislation or international conventions deem it necessary.*
- *FNASREC can be contacted for further information or any report templates via Roelof.Burger@nwu.ac.za 018 299 4269*

The FNASREC would like to remain at your service as scientist and researcher, and wishes you well with your study. Please do not hesitate to contact the FNASREC or the NWU-SCRE for any further enquiries or requests for assistance.

Yours sincerely,

Prof Roelof Burger
Chairperson Faculty of Natural and Agricultural Sciences Ethics Committee (FNASREC)

File reference: 9.1.5.4.2 Page 1 of 1

A.3 Editing Certificate

Dr Linda Scott
083 654 4156
lindascott1984@gmail.com
www.justedit.co.za

15 Shannon Close
The Circus
Three Rivers
1929

LINDA SCOTT DEVELOPMENTS
PROMOTING LINGUISTIC & CULTURAL INCLUSIVITY

English language editing
Professional Editors' Guild No: SCO005

9 May 2025

To whom it may concern

This is to confirm that I, the undersigned, have language edited the thesis of

TP Mashamba

Entitled:

Equity factor timing framework using the Kalman filter and Gaussian Hidden Markov Model

The responsibility of implementing the recommended language changes rests with the author of the document.

Yours truly,

Dr Linda Scott

SATI membership number: 1002595
SACE membership number: 1096643
ORCID ID: 0000-0002-2049-0732
ETDP SETA Assessor/Moderator: SOR number: 468097

PhD: Language Practice
MA: Intercultural Communication
BA (Hons): Language Practice
NPDE and ACE