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Goodness–of–fit tests based on the min–characteristic function

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ABSTRACT

Tests of fit for classes of distributions that include the Weibull, the Pareto and the Fréchet families are proposed. The new tests employ the novel tool of the min–characteristic function and are based on an L^2–type weighted distance between this function and its empirical counterpart applied on suitably standardized data. If data–standardization is performed using the MLE of the distributional parameters then the method reduces to testing for the standard member of the family, with parameter values known and set equal to one. Asymptotic properties of the tests are investigated. A Monte Carlo study is presented that includes the new procedure as well as competitors for the purpose of specification testing with three extreme value distributions. The new tests are also applied on a few real–data sets.

1. Introduction

The min–characteristic function (minCF) of a univariate random variable $X > 0$ is defined as $\Psi_X(t) = \mathsf{E}(\min\{1, tX\}),\ t > 0$. The minCF is a novel tool introduced recently by Falk and Stupfler (2021), who show uniqueness, boundedness, and other interesting properties of $\Psi_X(\cdot)$. The same authors introduce and prove uniform strong consistency and weak convergence over compact intervals of the associated empirical minCF,

$$\Psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \min\{1, tX_j\}, \tag{1}$$

where $(X_j,\ j = 1, \ldots, n)$ denote independent copies of X.

Note in this connection that for certain random variables, the minCF is often easier to compute than the classical characteristic function. This feature combined with the uniqueness of $\Psi_X(\cdot)$ for positively supported distributions, motivates goodness–of–fit (GOF) tests based on some distance between the population minCF and the empirical minCF, $\Psi_n(\cdot)$. However, as with tests based on the characteristic function, the drawback with many such tests is with composite null hypotheses involving parameters that actually occur in the population minCF of the distribution under test. These parameters are considered unknown and must be estimated from the data $(X_j,\ j = 1, \ldots, n)$ and it is often the case that the finite–sample as well as the asymptotic distribution of the resulting tests depends on the unknown true values of these parameters. If so, then GOF tests must be performed by bootstrap resampling

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methods that incur a certain computational cost on the implementation of the test procedure (see e.g. Jiménez-Gamero et al. (2009); Jiménez-Gamero and Kim (2015)).

In this paper we suggest minCF–based GOF tests for a certain classes of composite null hypotheses that do not have the aforementioned computational drawback. The paper is organized as follows. In Section 2 we introduce the null hypothesis to be tested and the corresponding tests statistics. Then in Section 3 we prove that the resulting tests are parameter–free if the distributional parameters are replaced by maximum likelihood estimators (MLE), while in Section 4 computational formulae are provided for the test statistic with specific instances of distributions under test. In Section 5 we present weak convergence of the test statistic under the null hypothesis and its behavior under alternatives. Section 6 presents the results of a Monte Carlo study on the power of the new tests, as well as comparisons with other GOF tests. Real–data examples are included in Section 7, and the paper concludes in Section 8 with discussion. All proofs are deferred to the Appendix.

2. Goodness–of–fit tests

Consider the null hypothesis that the law of X belongs to $\mathcal{F}$, for some $\vartheta \in \Theta$, where $\mathcal{F} = \{F_\vartheta; \vartheta \in \Theta\}$ is the particular class of distribution functions indexed by a parameter ϑ assumed to take values in Θ, the parameter space. The standard approach to GOF testing is by means of a distance measure between the population distribution function (DF) and its empirical counterpart, the empirical DF; a classic reference for empirical DF–based tests is D'Agostino and Stephens (1986). Another approach which has become popular recently is the one based on the empirical characteristic function, while in the context of positive random variables the empirical Laplace transform has also attracted attention. The interested reader is referred to the papers of Ndwandwe et al. (2023b), Ebner et al. (2022), Cuparić et al. (2022), Bothma et al. (2021), Lee et al. (2019), Meintanis and Ushakov (2016), Milošević and Obradović (2016), Meintanis et al. (2015), Meintanis and Iliopoulos (2003), among others. Meintanis (2016) gives a review of such GOF tests, while some earlier contributions for tests based on the characteristic function may be found in (Ushakov, 1999, §3.9-3.10).

We will introduce our test in $L^2(w)$, the (separable) Hilbert space setting of measurable functions $f \in \mathcal{H}$, $f : \mathbb{R}^+ \to \mathbb{R}$, which are squared integrable with respect to a weight function $w(\cdot)$. In this setting the inner product of $f, g \in \mathcal{H}$ is defined as

$$< f, g >_w = \left(\int_0^\infty f(t)g(t)w(t)\,\mathrm{d}t \right)^{1/2},$$

with corresponding norm

$$\| f \|_w = (< f, f >)^{1/2} = \left(\int_0^\infty f^2(t)w(t)\,\mathrm{d}t \right)^{1/2}.$$

In complete analogy with the aforementioned tests based on the empirical DF (or the empirical characteristic function or the empirical Laplace transform), a reasonable test procedure would be to reject the null hypothesis for large values of the test statistic

$$T_{n,w} = \| \sqrt{n} \left(\Psi_n - \Psi_{\widehat{\vartheta}_n} \right) \|_w^2, \tag{2}$$

where $\Psi_\vartheta(\cdot)$ is the minCF corresponding to F_ϑ, $\Psi_n(\cdot)$ is the empirical minCF defined in (1), and $\widehat{\vartheta}_n := \widehat{\vartheta}_n(X_1, ..., X_n)$ denotes some consistent estimator of ϑ.

Here however we will consider families of distributions with parameters $\vartheta = (c, \varphi)$, and such that the corresponding DF may be written as

$$F_{c,\varphi}(x) = F_0\left(\left(\frac{x}{c}\right)^\varphi \right), \tag{3}$$

for some $c, \varphi > 0$, and some fixed DF $F_0(\cdot)$. Clearly the corresponding family, say $\mathcal{F}_0$, is completely determined by F_0, and then the null hypothesis may be restated as

$$\Upsilon_0 : \text{ The law of } X \in \mathcal{F}_0, \text{ for some } c, \varphi > 0. \tag{4}$$

Specific instances of $\mathcal{F}_0$ to be considered in this work are the Weibull, the Pareto type I, and the Fréchet distributions, all belonging to the class of extreme–value distributions; see Kotz and Nadarajah (2000) for an account of extreme–value distributions.

3. Data transformations based on the MLE

In this section we investigate certain equivariance properties of the MLE under the class $\mathcal{F}_0$ with DF as in (3). These properties will have a direct impact on the proposed test, and specifically it will be shown that the test statistic will be rendered free of parameter values if the empirical minCF is applied on suitably standardized data. To this end, let $(\widehat{c}_n, \widehat{\varphi}_n)$ be the MLE of (c, φ), and consider the standardized observations

$$\hat{Y}_j = \left(\frac{X_j}{\hat{c}_n}\right)^{\hat{\varphi}_n}, \; j = 1, ..., n. \tag{5}$$

In the next result we investigate the aforementioned equivariance properties of the MLE; for a different proof we refer to Matsui and Meintanis (2023). A general discussion of equivariant estimators is provided by (Borovkov, 1998, §29), and (Romano and Lehmann, 2005, Chapter 6).

Proposition 3.1. *Let $(X_j,\; j = 1, ..., n)$ be independent copies of $X > 0$, and assume that the law of $X \in \mathcal{F}_0$, for some $c, \varphi > 0$. Assume further that the density of X exists, and that the* MLEs *$\hat{c}_n(X_1, ..., X_n)$ and $\hat{\varphi}_n(X_1, ..., X_n)$ of the parameters c and φ also exist, and that they are unique. Then the* MLE *corresponding to $(\hat{Y}_j,\; j = 1, ..., n)$ satisfy $\hat{c}_n(\hat{Y}_1, ..., \hat{Y}_n) = 1$ and $\hat{\varphi}_n(\hat{Y}_1, ..., \hat{Y}_n) = 1$.*

An important consequence of Proposition 3.1 is that any GOF procedure for families satisfying (3) that depends on $(X_j,\; j = 1, ..., n)$ only via the observations $(\hat{Y}_j,\; j = 1, ..., n)$ defined by (5) satisfies

$$T_n(aX_1^{1/b}, ..., aX_n^{1/b}) = T_n(X_1, ..., X_n), \tag{6}$$

for each $a, b > 0$, and consequently and without loss of generality, we may perform the test by assuming that c and φ are fixed at $c = \varphi = 1$. In this connection we suggest to apply the test defined by (2) by replacing the empirical minCF by

$$\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \min\{1, t\hat{Y}_j\}, \tag{7}$$

and the estimated population minCF $\Psi_{\hat{\vartheta}_n}(\cdot) := \Psi_{\hat{c}_n, \hat{\varphi}_n}(\cdot)$ figuring in $T_{n,w}$ by $\psi_0(t) := \Psi_{1,1}(t)$. As a consequence, the finite–sample as well as the asymptotic distribution of the resulting test statistic, say $\mathcal{T}_{n,w}$, will be independent of the true but unknown values of (c, φ). On a practical level, (6) implies that a potentially much simpler test may be invoked for testing families satisfying (3), such as in the case of the Weibull distribution whereby any test for exponentiality applied on $(\hat{Y}_j,\; j = 1, ..., n)$ may be used. Of course the idea of data–transformation in the context of GOF testing is not new. By way of example, for Pareto, generalized Pareto, and Weibull, distributions, log–transformations may be applied that have a similar effect, i.e. they reduce to exponentiality tests on the log–transformed data or to tests for the extreme–value distribution; the reader is referred to Ndwandwe et al. (2023a), Chu et al. (2019), Meintanis and Bassiakos (2007), and Cabaña and Quiroz (2005) for such tests.

4. Calculation of the test statistic

In this section we particularize the families to be tested as per the null hypothesis in (4) and the functional form in $F_0(x)$ figuring in (3). In this connection note that in the context of testing the null hypothesis Υ_0 and in view of the results of the previous section, we may perform the GOF test based on the test statistic

$$\mathcal{T}_{n,w} := \| \sqrt{n}\left(\psi_n - \psi_0\right) \|_w^2 = n\left(\|\psi_n\|_w^2 + \|\psi_0\|_w^2 - 2 < \psi_n, \psi_0 >_w\right), \tag{8}$$

for any given family $\mathcal{F}_0$. In fact if we use $w(t) = \mathrm{e}^{-\gamma t},\; \gamma > 0$, as weight function, the resulting test statistic, say $\mathcal{T}_{n,\gamma}$, may be written as

$$\mathcal{T}_{n,\gamma} = \frac{1}{n}\sum_{j,k=1}^{n} \Lambda_\gamma(\hat{Y}_j, \hat{Y}_k) + n\mathcal{L}_\gamma - 2\sum_{j=1}^{n} \lambda_\gamma(\hat{Y}_j), \tag{9}$$

where

$$\Lambda_\gamma(z_1, z_2) = \int_0^\infty \min\{1, tz_1\}\min\{1, tz_2\}\mathrm{e}^{-\gamma t}\mathrm{d}t,$$

$$\mathcal{L}_\gamma = \int_0^\infty \psi_0^2(t)\mathrm{e}^{-\gamma t}\mathrm{d}t,$$

and

$$\lambda_\gamma(z) = \int_0^\infty \min\{1, tz\}\psi_0(t)\mathrm{e}^{-\gamma t}\mathrm{d}t.$$

The families to be considered are:

- the Weibull with $F_0(x) = 1 - \mathrm{e}^{-x},\; x > 0$, and $\psi_0(t) = t(1 - \mathrm{e}^{-1/t})$,
- the Pareto with $F_0(x) = 1 - x^{-1},\; x > 1$, and $\psi_0(t) = t(1 - \log t),\; t \le 1$, and $\psi_0(t) = 1,\; t > 1$,
- the Fréchet with $F_0(x) = \mathrm{e}^{-x^{-1}},\; x > 0$, and $\psi_0(t) = 1 - \mathrm{e}^{-t} + t\Gamma[t]$, where $\Gamma[z] = \int_z^\infty u^{-1}\mathrm{e}^{-u}\mathrm{d}u$.

Clearly, the first integral $\Lambda_\gamma(z_1, z_2)$ does not depend on the family $\mathcal{F}_0$ being tested and depends solely on the data $(\widehat{Y}_j,\ j = 1, \dots, n)$ and the value of the weight parameter γ. This integral is straightforward to compute. Specifically after some straightforward algebra we obtain

$$\Lambda_\gamma(z_1, z_2) = \frac{z_1}{z_2} \frac{2z_2^2 - \mathrm{e}^{-\gamma/z_2}(2z_2^2 + 2\gamma z_2 + \gamma^2)}{\gamma^3} + \frac{z_1}{z_2} \frac{\mathrm{e}^{-\gamma/z_2}(\gamma + z_2)}{\gamma^2} - z_1 \frac{\mathrm{e}^{-\gamma/z_1}}{\gamma^2},\ z_1 \le z_2.$$

For $z_1 > z_2$ we simply replace z_1 (resp. z_2) with z_2 (resp. z_1).

On the other hand, the second and third integrals depend on the family under test, with $\mathcal{L}_\gamma$ been independent of the observations whereas the third integral $\lambda_\gamma(\cdot)$, is the only one that involves both the family $\mathcal{F}_0$ as well as the observations.

Specifically for the Weibull distribution it follows after some long but otherwise routine calculations that the test statistic may be written as

$$\mathcal{T}_{n,\gamma}^{\mathrm{W}} = \frac{1}{n} \sum_{j,k=1}^{n} \Lambda_\gamma(\widehat{Y}_j, \widehat{Y}_k) + n\mathcal{L}_\gamma^{\mathrm{W}} - 2 \sum_{j=1}^{n} \lambda_\gamma^{\mathrm{W}}(\widehat{Y}_j), \tag{10}$$

where

$$\mathcal{L}_\gamma^{\mathrm{W}} = \frac{2}{\gamma^3} + \frac{4\sqrt{2}K_3[\sqrt{8\gamma}] - 4K_3[\sqrt{4\gamma}]}{\gamma^{3/2}},$$

$$\lambda_\gamma^{\mathrm{W}}(z) = \frac{2z - \mathrm{e}^{-\gamma/z}(\gamma + 2z)}{\gamma^3} - (zM_{\gamma,1}^{\mathrm{W}}(z) + M_{\gamma,2}^{\mathrm{W}}(z)),$$

$$M_{\gamma,1}^{\mathrm{W}}(z) = \int_0^{1/z} t^2 \mathrm{e}^{-(t^{-1}+\gamma t)} \mathrm{d}t,$$

and

$$M_{\gamma,2}^{\mathrm{W}}(z) = \int_{1/z}^{\infty} t\, \mathrm{e}^{-(t^{-1}+\gamma t)} \mathrm{d}t,$$

where $K_\nu[z] = (1/2)(z/2)^\nu \int_0^\infty \mathrm{e}^{-t-(z^2/4t)} t^{-(\nu+1)} \mathrm{d}t$, stands for the modified Bessel function of the second kind of order ν.

By analogous calculations the test statistic for the Pareto distribution may be written as

$$\mathcal{T}_{n,\gamma}^{\mathrm{P}} = \frac{1}{n} \sum_{j,k=1}^{n} \Lambda_\gamma(\widehat{Y}_j, \widehat{Y}_k) + n\mathcal{L}_\gamma^{\mathrm{P}} - 2 \sum_{j=1}^{n} \lambda_\gamma^{\mathrm{P}}(\widehat{Y}_j), \tag{11}$$

where

$$\mathcal{L}_\gamma^{\mathrm{P}} = \frac{\mathrm{e}^{-\gamma}}{\gamma} + \frac{\mathrm{e}^{-\gamma}(4 - \gamma^2) + 4(\log\gamma + \Gamma[\gamma] + \gamma_{\mathrm{E}} - 1)}{\gamma^3} + I_\gamma^{\mathrm{P}},$$

$$\lambda_\gamma^{\mathrm{P}}(z) = z\left(\frac{2 - \mathrm{e}^{-\gamma}(2 + 2\gamma + \gamma^2)}{\gamma^3} - \frac{3 - 2\gamma_{\mathrm{E}} - \mathrm{e}^{-\gamma}(3 + \gamma) - 2\Gamma[\gamma] - 2\log\gamma}{\gamma^3}\right) + \frac{z}{\gamma^2}(\mathrm{e}^{-\gamma}(1 + \gamma) - \mathrm{e}^{-\gamma/z}),\ z \le 1,$$

$$\lambda_\gamma^{\mathrm{P}}(z) = \frac{2z}{\gamma^3} - \frac{\mathrm{e}^{-\gamma/z}(2z^2 + 2\gamma z + \gamma^2)}{z\gamma^3} - zM_{\gamma,1}^{\mathrm{P}}(z) + \frac{\mathrm{e}^{-\gamma/z}(\gamma + z)}{\gamma^2 z} - \frac{\mathrm{e}^{-\gamma}}{\gamma^2} - M_{\gamma,2}^{\mathrm{P}}(z),\ z > 1,$$

$$I_\gamma^{\mathrm{P}} = \int_0^1 t^2 \log^2 t\, \mathrm{e}^{-\gamma t} \mathrm{d}t,$$

$$M_{\gamma,1}^{\mathrm{P}}(z) = \int_0^{1/z} t^2 \log t\, \mathrm{e}^{-\gamma t} \mathrm{d}t,$$

and

$$M_{\gamma,2}^{\mathrm{P}}(z) = \int_{1/z}^{1} t \log t\, \mathrm{e}^{-\gamma t} \mathrm{d}t,$$

where $\gamma_{\mathrm{E}}(\approx 0.57721)$ denotes the Euler constant.

Finally, for the Fréchet distribution we have

$$\mathcal{T}^{\mathrm{F}}_{n,\gamma} = \frac{1}{n}\sum_{j,k=1}^{n} \Lambda_\gamma(\widehat{Y}_j, \widehat{Y}_k) + n\mathcal{L}^{\mathrm{F}}_\gamma - 2\sum_{j=1}^{n} \lambda^{\mathrm{F}}_\gamma(\widehat{Y}_j), \tag{12}$$

where

$$\mathcal{L}^{\mathrm{F}}_\gamma = \frac{1}{2+\gamma} + \frac{\gamma + 2(\log(1+\gamma) + (1+\gamma)^{-1} - 1)}{\gamma^2} - \frac{2(1+\gamma+\log(2+\gamma)+(2+\gamma)^{-1}-1)}{(1+\gamma)^2} + I^{\mathrm{F}}_\gamma,$$

$$\lambda^{\mathrm{F}}_\gamma(z) = \frac{z(1-\mathrm{e}^{-\gamma/z})}{\gamma^2} - \frac{z(1-\mathrm{e}^{-(1+\gamma)/z})}{(1+\gamma)^2} + zM^{\mathrm{F}}_{\gamma,1}(z) + M^{\mathrm{F}}_{\gamma,2}(z),$$

$$I^{\mathrm{F}}_\gamma = \int_0^\infty t^2\Gamma^2[t]\,\mathrm{e}^{-\gamma t}\mathrm{d}t,$$

$$M^{\mathrm{F}}_{\gamma,1}(z) = \int_0^{1/z} t^2\Gamma[t]\mathrm{e}^{-\gamma t}\mathrm{d}t,$$

and

$$M^{\mathrm{F}}_{\gamma,2}(z) = \int_{1/z}^\infty t\Gamma[t]\,\mathrm{e}^{-\gamma t}\mathrm{d}t.$$

In the next section we investigate the asymptotic behavior of the test $\mathcal{T}_{n,w}$ defined by (8), both under the null hypothesis Υ_0 as well as under alternatives.

Remark 4.1. More general distributions with extra parameters may be included in our framework of testing GOF. For instance the Burr type XII distribution belongs to the class of distributions defined by (3), with DF $F_0(x) = 1 - (1+x)^{-\xi}$, $\xi > 0$, and thus also satisfies Proposition 3.1. Specifically if X follows a Burr type XII distribution with parameters (c, φ, ξ) then $aX^{1/b}$ follows a Burr type XII distribution with parameters $(ac^{1/b}, b\varphi, \xi)$. Other classes include members of the exponentiated class of distributions, like the exponentiated Weibull distribution with $F_0(x) = (1-\mathrm{e}^{-x})^\xi$, $\xi > 0$ (see Mudholkar et al. (1995)), and the generalized gamma distribution with density $f_{c,\varphi,\xi}(x) = (\varphi x^{\varphi\xi-1})/(c^{\varphi\xi}\Gamma(\xi))\mathrm{e}^{-(x/c)^\varphi}$, $\xi > 0$.

5. Limit null distribution and consistency of the test

Here and in what follows, the notation $\xrightarrow{\mathcal{D}}$ and $\stackrel{\mathcal{D}}{=}$ mean convergence and equality in distribution of random elements and random variables, respectively, $\xrightarrow{\mathrm{P}}$ means convergence in probability, $\xrightarrow{\text{a.s.}}$ means almost sure convergence, $\mathrm{o_P}(1)$ stands for convergence in probability to 0, and $\mathrm{O_P}(1)$ denotes a bounded in probability random variable or process. All limits are understood as $n \to \infty$, where n stands for the sample size.

While for computational convenience we have suggested to actually perform the test based on the weight function $w(t) = \mathrm{e}^{-\gamma t}$, $\gamma > 0$, nevertheless the limit behavior of the statistic will be studied for arbitrary weight functions satisfying the following assumption.

Assumption 5.1. $w : [0,\infty) \mapsto (0,\infty)$, is continuous on $(0,\infty)$ and satisfies $\int_0^\infty w(t)\,\mathrm{d}t < \infty$.

Recall also that the test figuring in (8) implicitly involves the MLE via the transformed observations $(\widehat{Y}_j,\ j = 1,\ldots,n)$ defined in (5). In this connection we assume that $\widehat{c}_n$ and $\widehat{\varphi}_n$ admit the asymptotic representations in the next Assumption.

Assumption 5.2.

$$\sqrt{n}(\widehat{c}_n - c_0) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \mathrm{C}(X_j) + \mathrm{o_P}(1), \tag{13}$$

and

$$\sqrt{n}(\widehat{\varphi}_n - \varphi_0) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \Phi(X_j) + \mathrm{o_P}(1), \tag{14}$$

where $\mathsf{E}_0(\mathrm{C}(X_1)) = 0 = \mathsf{E}_0(\Phi(X_1))$, $\mathsf{E}_0(\mathrm{C}^2(X_1)) < \infty$, $\mathsf{E}_0(\Phi^2(X_1)) < \infty$, and $\mathsf{E}_0(\cdot)$ indicating expectation taken under F_{c_0,φ_0}, with (c_0, φ_0) being the true parameter values.

Remark 5.3. If the family of distributions satisfy certain regularity conditions (see, for example Theorem 5.1 in Lehmann and Casella (1998)), then the MLEs of the parameters c and φ satisfy Assumption 5.2 with

$$\begin{pmatrix} \mathrm{C}(x) \\ \Phi(x) \end{pmatrix} = I^{-1}(c_0, \varphi_0) \begin{pmatrix} l_c(x; c_0, \varphi_0) \\ l_\varphi(x; c_0, \varphi_0) \end{pmatrix},$$

where $I(c, \varphi)$ is the information matrix and $l_c(x; c, \varphi) = \frac{\partial}{\partial c} \log f(x; c, \varphi)$, $l_\varepsilon(x; c, \varphi) = \frac{\partial}{\partial \varphi} \log f(x; c, \varphi)$. This is the case for the families Weibull and Fréchet. Moreover, the likelihood equations associated with these two families have a unique solution (see Ramos et al. (2020) for the Fréchet family and Balakrishnan and Kateri (2008) for the Weibull family). The Pareto family does not satisfy the above cited regularity conditions. Nevertheless, the MLE of c and φ still fulfills Assumption 5.2. Moreover, in this family, the MLE of the parameter c is $X_{(1)} = \min\{X_1, \dots, X_n\}$, and it may easily be checked that

$$\sqrt{n}\left(X_{(1)} - c\right) = o_P(1),$$

and thus (13) trivially holds with $\mathrm{C}(x) = 0, \forall x$.

As already emphasized and due to test invariance with respect to (c, φ), the study of the asymptotic null distribution of the test statistic $\mathcal{T}_{n,w}$ provided by Theorem 5.5 below, is carried out by setting $(c_0, \varphi_0) = (1, 1)$.

It will be also assumed that F_0 has a density (with respect to the Lebesgue measure) as expressed in the next assumption.

Assumption 5.4. Let $a = \inf\{x > 0 : F_0(x) > 0\}$, and assume that F_0 has a density f_0 such that $f_0(a) = \lim_{x \to a^+} f_0(x)$ exists and is continuous on $[a, b]$, $\forall b > a$. Moreover suppose that

$$\int \log^\kappa(u) u^{\pm\rho} f_0^2(u)\, du < \infty,$$

for $\kappa = 0$ and $\kappa = 2$, and for some $0 < \rho < 1$.

Theorem 5.5. *Let $(X_j,\ j = 1, \dots, n)$ be independent copies of $X > 0$, and assume that the law of $X \in \mathcal{F}_0$, for some $c, \varphi > 0$. Suppose that Assumptions 5.1, 5.2 and 5.4 hold, that $\mathsf{E}_0(\log^4(X)) < \infty$ and that there exists $0 < \varepsilon < 1$ such that*

$$\int_1^\infty t^{8\varepsilon/(1-\varepsilon)} w(t)\, dt < \infty. \tag{15}$$

Then,

$$\mathcal{T}_{n,w} - \mathcal{T}_{n,w,0} = o_P(1), \tag{16}$$

and

$$\mathcal{T}_{n,w} \xrightarrow{\mathcal{D}} \|\mathcal{Z}\|_w^2, \tag{17}$$

where

$$\mathcal{T}_{n,w,0} = \|\sqrt{n} V_n\|_w^2, \quad V_n(t) = \frac{1}{n} \sum_{j=1}^n g(X_j; t), \tag{18}$$

$$g(x; t) = \min(1, tx) - \psi_0(t) - \mu(t)\mathrm{C}(x),$$

$$\mu(t) = -t \int_0^{1/t} u f_0(u) du,$$

and $\mathcal{Z}$ is a centered Gaussian random element of $\mathcal{H}$ having covariance kernel

$$c(t, s) = \mathsf{E}_0(g(X; t) g(X; s)). \tag{19}$$

Remark 5.6. We have that $\|\mathcal{Z}\|_w^2 \overset{\mathcal{D}}{=} \sum_{j\ge1} \lambda_j \chi_{1j}^2$, where $\chi_{11}^2, \chi_{12}^2, \dots$ are independent chi-squared random variables each with one degree of freedom and $\{\lambda_j\}_{j\ge1}$ are the eigenvalues of the operator A defined on $L^2(w)$ through

$$A\phi(x) = \int_0^\infty h_*(x, y)\phi(y) f_0(y) dy,$$

with

$$h_*(x,y) = \int_0^\infty g(x;t)g(y;t)w(t)\,\mathrm{d}t.$$

Explicit expressions of $h_*(x,y)$ can be calculated for the Weibull, Pareto and Fréchet families of distributions. The easiest case is obtained for the Pareto family because, as observed in Remark 5.3, in this case $\mathrm{C}(x) = 0$, $\forall x$, and thus

$$h_*(x,y)^{\mathrm{P}} = \Lambda_\gamma^{\mathrm{P}}(x,y) - \lambda_\gamma^{\mathrm{P}}(x) - \lambda_\gamma^{\mathrm{P}}(y) + \mathcal{L}_\gamma^{\mathrm{P}},$$

where $\Lambda_\gamma^{\mathrm{P}}$, $\lambda_\gamma^{\mathrm{P}}$ and $\mathcal{L}_\gamma^{\mathrm{P}}$ are as defined in Section 4. We did not obtain the eigenvalues of the operator associated to the considered families. Nevertheless, they can be approximated by using a result in Dehling and Mikosch (1994) (see Remark (c) after Corollary 3.3), which states that if $(X_j,\ j=1,\dots,n)$ are independent copies of $X>0$, and the law of $X \in \mathcal{F}_0$, then the eigenvalues of the random matrix $M = \left(n^{-1}h_*(X_j,X_k)\right)_{1\le j,k\le n}$ converge a.s. to those of the operator A; a related work is Ngatchou-Wandji (2009). Nevertheless approximating these eigenvalues is a non–trivial numerical exercise that, in the context of goodness–of–fit testing, has been carried out in only a few selective cases. We refer to Cuparić et al. (2022), Ebner and Henze (2023) and Meintanis et al. (2023).

Remark 5.7. Notice that the weight function $w(t) = \mathrm{e}^{-\gamma t}$ adopted in Section 4 satisfies Assumption 5.1 as well as (15), $\forall\, 0<\varepsilon<1$, for each $\gamma > 0$.

Remark 5.8. In Section 4 we considered three special instances of distributions F_0. It can be easily checked that each of these distributions satisfies Assumption 5.4, $\forall\, \rho \in (0,1)$, and that $\mathsf{E}_0(\log^4(X)) < \infty$.

We now consider the asymptotic behavior of $\mathcal{T}_{n,w}$ under fixed alternatives to the null hypothesis Υ_0. The following result implies the consistency of the test that rejects Υ_0 for large values of $\mathcal{T}_{n,w}$.

Theorem 5.9. *Let $(X_j,\ j=1,\dots,n)$ be independent copies of $X>0$. Assume that $(\widehat{c}_n,\widehat{\varphi}_n) \xrightarrow{\mathrm{P}} (c_X,\varphi_X)$, for some $c_X \in (0,\infty)$, $\varphi_X \in (0,\infty)$, and that $\mathsf{E}(|\log(X)|) < \infty$. Then,*

$$\frac{\mathcal{T}_{n,w}}{n} \xrightarrow{\mathrm{P}} \|\psi_X - \psi_0\|_w^2 := \Delta_w, \tag{20}$$

where $\psi_X(t) = \mathsf{E}(\min\{1, t(X/c_X)^{\varphi_X}\})$.

Remark 5.10. If in the assumptions of Theorem 5.9, convergence in probability is replaced by almost sure convergence, i.e. if $(\widehat{c}_n,\widehat{\varphi}_n) \xrightarrow{\text{a.s.}} (c_X,\varphi_X)$, then we may write $\xrightarrow{\text{a.s.}}$ in equation (20).

Under the standing assumptions, Δ_w is positive unless $\psi_X(t)$ is equal to $\psi_0(t)$ identically in t, i.e. if the minCF of $(X/c_X)^{\varphi_X}$ is equal to $\psi_0(\cdot)$. In turn by the uniqueness of the minCF this implies that $(X/c_X)^{\varphi_X} \in \mathcal{F}_0$, which, since the transformation $X \mapsto (X/c_X)^{\varphi_X}$ is one–to–one, entails $X \sim F_{c,\varphi}$ with $(c,\varphi) = (c_X,\varphi_X)$. Thus $\Delta_w > 0$ unless Υ_0 is true, and consequently the test that rejects the null hypothesis Υ_0 for large values of $\mathcal{T}_{n,w}$ is consistent against each fixed alternative distribution satisfying the assumptions stated in Theorem 5.9.

Finally, the next result gives the asymptotic behavior of the test statistic under contiguous alternatives that approach Υ_0 at the rate $n^{-1/2}$. To simplify notation, we identify the distribution function of a random variable with the associated probability measure. It will be assumed that the data come from a variable having distribution function F_n that meets the following:

$$\begin{gathered} F_n \text{ is a measure dominated by some fixed } F_0 \in \mathcal{F}_0, \text{ with Radon-Nikodym derivative } \mathrm{d}F_n/\mathrm{d}F_0 = 1 + n^{-1/2}a, \text{ for} \\ \text{some measurable bounded function } a : [0,\infty) \mapsto \mathbb{R} \text{ satisfying } \int_0^\infty a(u)f_0(u)\mathrm{d}u < \infty. \end{gathered} \tag{21}$$

The boundedness of g implies $dF_n/dF_0 \ge 0$ for large enough n.

Theorem 5.11. *Let $(X_{n,j},\ j=1,\dots,n)$ be independent copies of $X_n > 0$ with distribution function F_n satisfying (21). Suppose that Assumptions 5.1, 5.2 and 5.4 hold, that $\mathsf{E}_0(\log^4(X)) < \infty$ and (15) also hold. Then,*

$$\mathcal{T}_{n,w} \xrightarrow{\mathcal{D}} \|\mathcal{Z} + d\|_w^2,$$

where $\mathcal{Z}$ is the Gaussian random element figuring in Theorem 5.5, and $d(t) = \mathsf{E}_0(g(X;t)a(X))$.

As a consequence of Theorem 5.5 and Theorem 5.11, the test that rejects the null hypothesis Υ_0 for large values of $\mathcal{T}_{n,w}$ is able to detect contiguous alternatives satisfying (21).

Table 1
Distributions included in the power study.

Distribution	Notation	Density function
Weibull	$W(\varphi, c)$	$\frac{\varphi x^{\varphi-1}}{c^{\varphi}} e^{-(\frac{x}{c})^{\varepsilon}}$, $x > 0;\ \varphi > 0, c > 0;$
Pareto	$P(\varphi, c)$	$\frac{\varphi c^{\varphi}}{x^{\varphi+1}}$ $x > c;\ \varphi > 0, c > 0;$
Gamma	$\Gamma(\varphi, c)$	$\frac{x^{\varphi-1}}{c^{\varphi}\Gamma(\varphi)} e^{-\frac{x}{c}}$, $x > 0;\ \varphi > 0, c > 0;$
Lognormal	$LN(\mu, \sigma)$	$\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$, $x > 0;\ \mu \in \mathbf{R}, \sigma > 0;$
Halfnormal	$HN(\theta)$	$\frac{\sqrt{2}}{\theta\sqrt{\pi}} e^{-\frac{x^2}{2\theta^2}}$, $x > 0;\ \theta > 0;$
Linear failure rate	$LFR(\theta)$	$(1+\theta x)e^{-x-\frac{1}{2}\theta x^2}$, $x > 0;\ \theta > 0;$
Chen	$CH(\theta)$	$2\theta x^{\theta-1} e^{x^{\theta}-2(1-e^{x^{\theta}})}$, $x > 0;\ \theta > 0;$
Fréchet	$F(\varphi, c)$	$\frac{\varphi}{c}\left(\frac{x}{c}\right)^{-1-\varphi} e^{-(\frac{x}{c})^{-\varphi}}$, $x > 0;\ \varphi > 0, c > 0;$

6. Monte Carlo study

In this section we study the finite–sample performance of the new GOF tests, simply denoted it by $\mathcal{T}_\gamma$, in comparison with other tests. In order to perform comparisons, all tests are applied on standardized data (5) with sample size $n = 20$ and $n = 50$. In this connection and since the null distributions of tests statistics do not depend on unknown parameters, the tests' powers are obtained using a Monte Carlo procedure with $N = 10,000$ replicates, with nominal level of significance $\alpha = 0.05$. The study includes the following distributions: Weibull, Pareto, Gamma, Lognormal, Halfnormal, Linear failure rate, Fréchet, as well as Chen distributions, for some specific choices of distributional parameters. These particular distributions occur in analogous Monte Carlo studies; see e.g. Cuparić et al. (2020), Allison et al. (2022), and Ndwandwe et al. (2023b). Their density functions are given in Table 1.

For the Weibull distribution we include the Anderson–Darling test (often the best test amongst the classical tests based on the empirical DF), and the test of Öztürk and Korukogu (1988), which are labeled as AD and OK, respectively. We also include a recent test based on a Stein-type characterization. This test is denoted by S_a, and we adopt the value $a = 1$ for the value of tuning parameter following the authors' recommendation; see Ebner et al. (2022). We have also included an empirical–Laplace–transform based test that utilizes transformation of a Weibull variate to extreme value distribution of type I, labeled as $\widehat{LT}^2$ (see Krit (2014)). Another test which is related to the test based on the minCF utilizes the max–characteristic function defined as $\Xi_X(t) = \mathsf{E}(\max\{1, tX\})$; see Falk and Stupfler (2017). The maxCF exists provided that $\mathsf{E}(X) < \infty$, and then the corresponding test, denoted by FS_γ, has a direct connection to a test based on the minCF, since $\Xi_X(t) = 1 + t\mathsf{E}(X) - \Psi_X(t)$. Here we consider the maxCF test FS_γ constructed analogously to our test with weight function $w(t) = e^{-\gamma t}$, with $\gamma = 1$. We highlight that tests based on the maxCF are also considered for the first time in the literature.

The figures in Table 2 show percentage of rejection of the null hypothesis of a Weibull distribution for each test considered rounded to the nearest integer. Overall, the best performing test appears to be the OK test. For sample size $n = 50$ our test with $\gamma = 5$ is a good contender of OK, and preferable to the next best test, i.e. the AD test. For $n = 20$, our test $\mathcal{T}_5$ is less of a contender, but still ranks close to the AD test which is second best. In this connection we wish to point out that since the best performing $\mathcal{T}_\gamma$–test across γ is a formidable opponent of the OK test, data–driven selections of the weight parameter γ such as the ones suggested by Allison and Santana (2015) and Tenreiro (2019), seem to hold further promise for the new test. However more research is needed in this direction both in the actual implementation of such a test as well as for its asymptotic.

For the Pareto type I distribution we compare our test with the M_a–test that is based on the difference between the empirical characteristic function of transformed data and that of the standard uniform distribution (see Meintanis et al. (2014) and Ndwandwe et al. (2023b) for details), and the test based on the Mellin transform proposed by Meintanis (2009), denoted by G_a, which is amongst the best performers in the Monte Carlo study of Ndwandwe et al. (2023a). We also consider a recent characteristic-function based test proposed in Ndwandwe et al. (2023b), denoted by $S^{(1)}_{3,2}$. Additionally, we include the likelihood ratio–based test Z_C proposed by Zhang (2002) and the test I_2 which is one of the best performers among the recent characterization–based tests utilizing the V-empirical distribution approach (see Allison et al. (2022)). The figures in Table 3 show clearly that the new tests based on $\mathcal{T}_\gamma$ are the most powerful or second best nearly uniformly over alternatives and for all values of γ with both sample sizes, followed by the M_1–based test.

Finally, for the Fréchet distribution the proposed GOF test is compared against the likelihood ratio–based test Z_A of Zhang (2002) and the AD test. The figures shown in Table 4 show a similar picture to that of Table 2, i.e., the new tests rank second best after the Z_A test, being mostly more powerful than the AD–test, but not by a wide margin.

7. Real–data applications

The Weibull distribution is often used in the context of reliability analysis. In this connection, Wu et al. (2021) consider the Weibull distribution for strength–modeling of two glass fibers, of length 1.5 cm and 15 cm, respectively. The datasets are originally taken from Smith and Naylor (1987). The main goal of their study was to consider different methods for parameter estimation, while the fit of the Weibull distribution has been justified using the classical Kolmogorov-Smirnov test. However, it is not clear how this test was used, i.e. it is not known whether the fact that it is not distribution free in the presence of estimation of unknown parameters was taken into account. Here we apply the tests utilized in our power study to check goodness–of–fit to the Weibull distribution.

Table 2
GOF tests for the Weibull distribution: Empirical percentage of rejection.

	$n=20$								$n=50$							
Dist	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	OK	S_1	$\widehat{LT}^2$	FS_1	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	OK	S_1	$\widehat{LT}^2$	FS_1
$W(1,0.5)$	5	5	5	5	5	5	5	4	5	6	5	5	5	5	5	5
$W(0.5,1)$	5	5	5	6	5	5	5	5	5	5	5	5	5	5	5	5
$\Gamma(0.8,1)$	5	5	5	6	5	6	5	5	5	6	6	6	5	6	5	6
$\Gamma(2,1)$	6	6	7	6	7	5	7	5	8	9	12	9	11	7	11	8
$\Gamma(3,1)$	8	7	9	7	9	5	9	7	13	13	18	13	13	10	13	12
$LN(1)$	23	20	26	23	30	17	27	20	56	53	63	56	70	47	62	52
$LN(2.5)$	24	21	26	22	30	18	27	19	57	54	65	56	70	47	61	53
$HN(1)$	7	8	8	9	6	10	8	7	11	13	16	13	13	13	13	12
$LFR(0.2)$	55	47	43	55	56	39	43	47	95	92	89	97	96	94	86	92
$LFR(0.5)$	49	41	39	48	52	32	38	41	92	88	85	94	93	85	81	87
$LFR(0.8)$	45	37	37	43	47	31	35	38	89	85	82	93	92	82	77	85
$LFR(1)$	44	36	36	43	46	29	35	36	89	83	81	91	91	80	77	84
$CH(0.8)$	7	7	8	8	5	9	8	7	11	12	16	12	13	11	12	12
$CH(1)$	7	8	8	8	6	9	8	7	11	12	15	12	12	11	12	12
$CH(1.5)$	8	8	8	8	6	9	7	7	11	12	16	12	12	11	21	12

Table 3
GOF tests for the Pareto distribution: Empirical percentage of rejection.

	$n=20$									$n=50$								
Dist	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	$S^{(1)}_{3,2}$	M_1	G_2	Z_C	I_2	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	$S^{(1)}_{3,2}$	M_1	G_2	Z_C	I_2
$P(1,1)$	5	5	5	5	5	5	5	5	4	5	5	5	5	5	5	5	5	5
$P(2,1)$	5	5	5	5	5	5	5	5	4	5	5	5	5	5	5	5	5	5
$W(0.5,1)+1$	29	27	19	55	25	34	34	51	39	59	57	43	91	72	66	89	89	79
$W(0.8,1)+1$	6	5	7	5	13	3	2	6	4	16	17	24	13	25	12	7	13	6
$W(1.2,1)+1$	38	36	38	13	49	27	17	5	22	94	94	95	84	93	91	88	61	80
$W(1.5,1)+1$	62	61	63	39	73	53	38	12	42	100	100	100	98	99	99	98	93	97
$\Gamma(0.8,1)+1$	7	7	8	4	15	4	2	6	4	22	23	29	15	29	16	10	12	8
$\Gamma(1,1)+1$	18	18	19	5	29	12	7	4	9	66	66	70	44	67	59	51	22	40
$\Gamma(1.2,1)+1$	34	32	34	24	46	24	15	4	18	91	91	93	78	90	88	83	53	76
$HN(1)+1$	47	51	55	20	64	37	25	9	29	98	98	99	91	97	95	92	75	87
$LN(1)+1$	15	14	13	4	24	11	7	2	9	63	64	56	51	66	65	61	23	55
$LN(1.2)+1$	7	7	6	2	13	5	3	2	5	32	31	28	21	35	32	29	7	24
$LN(1.5)+1$	4	4	4	2	7	3	2	3	4	6	6	7	4	9	6	4	3	4
$LN(2.5)+1$	24	26	18	42	19	31	31	34	39	55	52	38	79	58	64	65	66	72
$LFR(0.2)+1$	12	12	14	4	22	8	4	3	5	50	48	53	30	52	4	35	13	27
$LFR(0.5)+1$	18	18	19	6	29	13	6	4	5	65	65	69	45	67	58	49	20	41
$LFR(0.8)+1$	22	22	23	6	33	15	8	4	8	76	75	79	54	75	68	59	27	50
$LFR(1)+1$	24	23	25	7	36	17	4	9	11	79	80	81	59	80	62	64	32	54
$CH(0.8)+1$	6	6	8	5	14	4	2	8	3	19	19	26	13	28	13	5	13	10
$CH(1)+1$	22	22	25	7	35	16	4	5	7	77	78	82	57	78	67	59	33	48
$CH(1.5)+1$	73	74	76	41	82	64	47	19	50	100	100	100	100	100	99	99	98	99

The results are presented in Table 5. The p–values reported suggest that for Dataset 2 the Weibull distribution may be a reasonable model supported by all tests considered. On the other hand, the fit of the Weibull distribution to Dataset 1 is questionable, with the exception of the OK test, $\widehat{LT}^2$ test and marginally so by the new test for $\gamma=5$.

Next we turn to the fit to the Fréchet distribution. In this connection, Ramos et al. (2020) present an overview of the properties of this distribution and, as an illustration, they considered five real data sets related to minimum monthly flows of water (m^3/s) in the Piracicaba River, located in São Paulo state, Brazil. They concluded, using several information criteria, that in all cases the Fréchet distribution is the best of the models considered. The p–values reported in Table 6 seem to corroborate the conclusions of Ramos et al. (2020) by means of the formal GOF tests applied herein.

As an extremal event with a potentially good Pareto fit, we apply our tests to the monetary expenses incurred as a result of wind related catastrophes in 40 separate instances during 1977 (after the de-grouping algorithm used in Allison et al. (2022)). The results presented in Table 7 indicate that the Pareto distribution may be one of the possible choices for modeling these data.

Another case of extremity is top income earnings, and as such, a typical example is the earnings of top athletes. In this connection, we tested the earnings of the top 20 tennis players as of January 29, 2024, sourced from https://www.spotrac.com/atp/. The results presented in Table 8 confirm the suitability of the Pareto distribution for this dataset.

Table 4
GOF tests for the Fréchet distribution: Empirical percentage of rejection.

	$n = 20$					$n = 50$				
Dist	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	Z_A	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	Z_A
$F(2,1)$	5	5	5	5	5	5	5	5	5	5
$F(1,0.5)$	5	5	5	5	5	5	5	5	4	5
$W(0.8,1)$	72	73	73	69	79	99	99	99	98	82
$W(1.5,1)$	72	72	73	68	80	99	99	99	99	99
$\Gamma(0.8,1)$	76	77	78	78	82	99	99	100	99	99
$\Gamma(1,1)$	72	72	73	74	78	99	99	99	99	99
$\Gamma(2,1)$	60	58	58	54	66	96	96	97	95	98
$\Gamma(5,1)$	48	46	46	43	46	88	89	89	87	94
$LFR(0.5)$	7	7	8	9	5	21	18	23	22	23
$LFR(0.8)$	8	7	8	8	5	22	19	23	22	23
$LFR(1)$	7	7	9	8	5	20	18	24	22	24
$LN(0.8)$	25	24	22	20	29	60	60	58	54	69
$LN(1.5)$	26	23	23	21	28	61	69	59	54	70
$HN(1)$	82	83	86	81	89	99	99	100	99	100

Table 5
Glass fiber strength: p-values for tests for the Weibull distribution.

		Test statistics							
	n	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	OK	S_1	$\widehat{LT}^2$	FS_1
Dataset 1	63	0.0066	0.0019	0.0693	0.0021	0.2676	0.0021	0.2599	0.019
Dataset 2	46	0.5508	0.3566	0.1666	0.526	0.1268	0.2649	0.2571	0.3602

Table 6
Minimum monthly flows of water: p-values for tests for the Frechét distribution.

		Test statistics				
	n	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	Z_A
May	40	0.996	0.9945	0.9607	0.9955	0.9780
June	39	0.2533	0.3376	0.5746	0.1163	0.1961
July	39	0.0666	0.0618	0.1057	0.1139	0.2947
August	41	0.3216	0.2968	0.1981	0.4738	0.5863
September	41	0.3216	0.2968	0.1981	0.4768	0.5854

Table 7
Wind catastrophes: p-values for tests for the Pareto distribution.

	Test statistics								
n	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	$S^{(1)}_{3,2}$	M_1	G_2	Z_C	I_2
40	0.1958	0.1916	0.1036	0.4011	0.3037	0.2926	0.2794	0.2686	0.9376

Table 8
ATP Career Earnings: p-values for tests for the Pareto distribution.

	Test statistics								
n	$\mathcal{T}_{0.5}$	$\mathcal{T}_1$	$\mathcal{T}_5$	AD	$S^{(1)}_{3,2}$	M_1	G_2	Z_C	I_2
20	0.1744	0.1732	0.1273	0.3021	0.2196	0.2443	0.2228	0.6137	0.3852

8. Conclusion and outlook

We propose for the first time GOF tests based on the minCF. The tests are computationally convenient and may be applied to arbitrary positively supported distributions with unknown parameters. It is shown that for certain families of such laws the implementation of the new tests on the basis of MLE–standardized data lead to simplified procedures that mimic simple, rather than composite, GOF tests with fixed parameter values. The asymptotic null distribution of the test statistic is obtained under general conditions, and moreover it was shown that the suggested tests are consistent against arbitrary deviations from the null hypothesis. A Monte Carlo study is conducted in order to assess the performance of the new tests against competitors for the cases of Weibull, Pareto type I and Fréchet, distributions. Our findings show that the minCF–based tests could in certain cases be the best option, but most importantly, and in view of the fact that there can not be any "true" best test that outperforms competitors uniformly over all alternatives (see Janssen (2000) and Escanciano (2009)), that even in the cases that the new tests are not most powerful, they often rank near the top. We conclude with a few real–data applications that complement earlier findings.

We close by pointing out that our tests may be applied with general families of distributions with positive support, and with consistent, but otherwise arbitrary, estimators of the distributional parameters. In such cases though, the parameter–free property implied in Theorem 5.5 by Proposition 3.1 is no longer necessarily true. In fact outside families satisfying eqn. (3) or if the tests are not implemented based on ML estimation, then the limit null distribution would depend on the value of shape parameter, and consequently simple Monte Carlo resampling should be replaced by the bootstrap in order to account for parameter uncertainty. This point has been made already in the Introduction and references are provided. For a detailed study of bootstrap resampling in the context of empirical–transform–based tests we also refer to Meintanis and Swanepoel (2007).

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Appendix A

Proof of Proposition 3.1. We will show that the MLE of (c,φ) based on $Y_j = aX_j^{1/b},\ j=1,\dots,n$, satisfy

$$\widehat{c}_n(Y_1,\dots,Y_n) = a\,\widehat{c}_n^{1/b}(X_1,\dots,X_n), \tag{22}$$

and

$$\widehat{\varphi}_n(Y_1,\dots,Y_n) = b\,\widehat{\varphi}_n(X_1,\dots,X_n), \tag{23}$$

for each $a,b>0$, and then the result trivially holds by applying (22) and (23) on $(\widehat{Y}_j,\ j=1,\dots,n)$, with (a,b) set equal to $(\widehat{c}_n^{-\widehat{\varphi}_n}, \widehat{\varphi}_n^{-1})$.

In order to proceed with the proof we first compute from (3) the density corresponding to $F_{c,\varphi}$ as

$$f_{c,\varphi}(x) = \frac{\mathrm{d}F_{c,\varphi}(x)}{\mathrm{d}x} = \frac{\mathrm{d}F_0((\frac{x}{c})^{\varphi})}{\mathrm{d}x} = \frac{\varphi}{c}\left(\frac{x}{c}\right)^{\varphi-1} f_0\left(\left(\frac{x}{c}\right)^{\varphi}\right), \tag{24}$$

where $f_0(x) := f_{1,1}(x) = \frac{\mathrm{d}F_0(x)}{\mathrm{d}x}$. Also note that if $X \sim F_{c,\varphi}$, then $aX^{1/b} \sim F_{\sigma,\kappa}$, where $(\sigma,\kappa) = (ac^{1/b}, b\varphi)$ for each $a,b>0$.

Then we calculate the likelihood function corresponding to the transformed parameters (σ,κ) and the transformed observations $(Y_j,\ j=1,\dots,n)$ as

$$\mathcal{L}(Y_1,\dots,Y_n;\sigma,\kappa) = \frac{\kappa^n}{\prod_{j=1}^n Y_j}\prod_{j=1}^n\left(\frac{Y_j}{\sigma}\right)^{\kappa} f_0\left(\left(\frac{Y_j}{\sigma}\right)^{\kappa}\right). \tag{25}$$

After some further algebra we obtain

$$\mathcal{L}(Y_1,\dots,Y_n;\sigma,\kappa) = D(X_1,\dots,X_n;a,b)\mathcal{L}(X_1,\dots,X_n;c,\varphi), \tag{26}$$

where

$$\mathcal{L}(X_1,\dots,X_n;c,\varphi) = \frac{\varphi^n}{\prod_{j=1}^n X_j}\prod_{j=1}^n\left(\frac{X_j}{c}\right)^{\varphi} f_0\left(\left(\frac{X_j}{c}\right)^{\varphi}\right),$$

is the likelihood function corresponding to the original parameters (c,φ) and the original observations $(X_j,\ j=1,\dots,n)$, and

$$D(X_1,\dots,X_n;a,b) = \left(\frac{b}{a}\right)^n\prod_{j=1}^n X_j^{1-(1/b)}.$$

Equation (26) implies that the two likelihoods are equivalent within a proportionality constant and thus are both maximized by setting (c,φ) equal to the MLE $(\widehat{c}_n(X_1,\dots,X_n), \widehat{\varphi}_n(X_1,\dots,X_n))$. In order to complete the proof we should show that

$$\mathcal{L}(Y_1,\dots,Y_n;\widehat{\sigma}_n,\widehat{\kappa}_n) = D(X_1,\dots,X_n;a,b)\mathcal{L}(X_1,\dots,X_n;\widehat{c}_n,\widehat{\varphi}_n),$$

where $(\widehat{\sigma}_n,\widehat{\kappa}_n) = (a\widehat{c}_n^{1/b}(X_1,\dots,X_n), b\widehat{\varphi}_n(X_1,\dots,X_n))$. This is done by replacing in (25), (σ,κ) by $(\widehat{\sigma}_n,\widehat{\kappa}_n)$ and Y_j by $aX_j^{1/b},\ j=1,\dots,n$, and by performing some straightforward algebra. □

Proof of Theorem 5.5. Notice that

$$\mathcal{T}_{n,w} = \frac{1}{n}\sum_{j,k=1}^{n} h(X_j, X_k; \hat{c}_n, \hat{\varphi}_n),$$

with

$$h(X_j, X_k; c, \varphi) = \int_0^\infty g(t, X_j; c, \varphi) g(t, X_k; c, \varphi) w(t)\,\mathrm{d}t,$$
$$g(t, x; c, \varphi) = \min\{1, t(x/c)^\varphi\} - \psi_0(t),$$

that is, $\mathcal{T}_{n,w}$ is n times a degree 2 V-statistic, where certain parameters have been replaced with estimators. A useful result to derive the asymptotic null distribution of these types of statistics is Theorem 2.16 in de Wet and Randles (1987). The proof will check that the conditions in such theorem hold true in our setting, implying that the claimed result is true. Specifically, we will check that Condition 2.9, Condition 2.10, Condition 2.11 and display (2.17) in de Wet and Randles (1987) are met.

Condition 2.9 in de Wet and Randles (1987) Recall that $\mathsf{E}_0(\cdot)$ stands for the expectation under the null hypothesis with $(c, \varphi) = (1, 1)$. Let

$$\mu(t; c, \varphi) = \mathsf{E}_0[g(t, X; c, \varphi)].$$

Since

$$|g(t, X; c, \varphi)| \le 2, \tag{27}$$

$\mu(t; c, \varphi)$ exits $\forall c, \varphi, t > 0$ and satisfies $\mu(t; 1, 1) = 0$. We must show that $\mu(t; c, \varphi)$ has an $L_2(w)$ differential at $(c, \varphi) = (1, 1)$, call it $d_1\mu(t; 1, 1) = (d_{1c}\mu(t; 1, 1), d_{1\varphi}\mu(t; 1, 1))$, with $d_{1c}\mu(t; 1, 1)$, $d_{1\varphi}\mu(t; 1, 1) \in L^2(w)$. The existence of an $L_2(w)$ differential means that for any $\eta > 0$, there is a ball centered at $(c, \varphi) = (1, 1)$, say B, such that $(c, \varphi) \in B$ implies that

$$\|(c, \varphi) - (1, 1)\|^{-2} \int_0^\infty \left\{\mu(t; c, \varphi) - d_{1c}\mu(t; 1, 1)(c - 1) - d_{1\varphi}\mu(t; 1, 1)(\varphi - 1)\right\}^2 w(t)\,\mathrm{d}t < \eta. \tag{28}$$

We have that,

$$\begin{aligned}
\mu(t; c, \varphi) &= \frac{t}{c^\varphi}\int_0^{c(1/t)^{1/\varphi}} x^\varphi f_0(x)\,dx + 1 - F_0(c(1/t)^{1/\varphi}) - \psi_0(t) \\
&:= I(t; c, \varphi) + 1 - F_0(c(1/t)^{1/\varphi}) - \psi_0(t), \\
\frac{\partial}{\partial c}\mu(t; c, \varphi) &= -\frac{\varphi}{c} I(t; c, \varphi), \\
\frac{\partial}{\partial \varphi}\mu(t; c, \varphi) &= -\log(c) I(t; c, \varphi), \\
\frac{\partial^2}{\partial c^2}\mu(t; c, \varphi) &= \frac{\varphi(\varphi + 1)}{c^2} I(t; c, \varphi) - \frac{\varphi}{c}(1/t)^{1/\varphi} f_0\left(c\,(1/t)^{1/\varphi}\right), \\
\frac{\partial^2}{\partial c\partial \varphi}\mu(t; c, \varphi) &= \frac{\varphi\log(c) - 1}{c} I(t; c, \varphi) - \frac{\log(t)}{\varphi}(1/t)^{1/\varphi} f_0\left(c\,(1/t)^{1/\varphi}\right), \\
\frac{\partial^2}{\partial \varphi^2}\mu(t; c, \varphi) &= \log^2(c) I(t; c, \varphi) - c\frac{\log(c)}{\varphi}\log(t)\,(1/t)^{1/\varphi} f_0\left(c\,(1/t)^{1/\varphi}\right).
\end{aligned}$$

Now,

$$d_{1c}\mu(t; 1, 1) = \frac{\partial}{\partial c}\mu(t; c, \varphi)\bigg|_{(c,\varphi)=(1,1)} = -I(t; 1, 1) := \mu(t), \tag{29}$$
$$d_{1\varphi}\mu(t; 1, 1) = \frac{\partial}{\partial \varphi}\mu(t; c, \varphi)\bigg|_{(c,\varphi)=(1,1)} = 0.$$

As

$$0 \le I(t; c, \varphi) \le 1, \quad \forall c, \varphi, t > 0, \tag{30}$$

it readily follows that $d_{1c}\mu(t; 1, 1), d_{1\varphi}\mu(t; 1, 1) \in L^2(w)$.

Let $H(t; c, \varphi)$ be the 2×2 matrix of second order partial derivatives of $\mu(t; c, \varphi)$ with respect to μ and φ, and let $\|\cdot\|_F$ denote the Frobenius norm. Taking into account (30), and Assumption 5.4, routine calculations show that for any (c, φ) such that $\|(c, \varphi) - (1, 1)\| \le \rho$,

$$I_H(c,\varphi)=\int \|H(t;c,\varphi)\|_F^2 w(t)\,\mathrm{d}t \le M, \tag{31}$$

for certain positive constant M.

By Taylor expansion,

$$\mu(t;c,\varphi)-\mu(t;1,1)-d_{1c}\mu(t;1,1)(c-1)=0.5(c-1,\varphi-1)H(t;\widetilde{c},\widetilde{\varphi})(c-1,\varphi-1)^\top,$$

where $(\widetilde{c},\widetilde{\varphi})$ is between (c,φ) and $(1,1)$, for each $t>0$, and hence

$$\left\{\mu(t;c,\varphi)-\mu(t;1,1)-d_{1c}\mu(t;1,1)(c-1)\right\}^2 \le 0.25\|(c-1,\varphi-1)\|^4\|H(t;\widetilde{c},\widetilde{\varphi})\|_F^2. \tag{32}$$

From (31) and (32), it readily follows that (28) holds true for B the ball centered at $(c,\varphi)=(1,1)$ with radius ρ.

Condition 2.10 in de Wet and Randles (1987). This is our Assumption 5.2.

Condition 2.11 (a) in de Wet and Randles (1987). We must show that there is a number $M>0$ and a neighborhood $\mathcal{N}$ of $(c,\varphi)=(1,1)$ such that if $(c,\varphi)\in\mathcal{N}$ and the open ball centered at (c,φ) with radius δ, $B((c,\varphi),\delta)$, is such that $B((c,\varphi),\delta)\subset\mathcal{N}$, then

$$\int_0^\infty \mathsf{E}^2\left[\sup_{(c_1,\varphi_1)\in B((c,\varphi),\delta)}\left|g(t,X;c_1,\varphi_1)-g(t,X;c,\varphi)\right|\right] w(t)\,\mathrm{d}t \le M\delta^2. \tag{33}$$

We have that

$$\left|g(t,x;c_1,\varphi_1)-g(t,x;c,\varphi)\right| \le t\left|\left(\frac{x}{c_1}\right)^{\varphi_1}-\left(\frac{x}{c}\right)^{\varphi}\right|,$$

if $0<x<\max\left\{c\,(1/t)^{1/\varphi},c_1\,(1/t)^{1/\varphi_1}\right\}$, and $g(t,x;c_1,\varphi_1)-g(t,x;c,\varphi)=0$ otherwise. Taking into account that

$$\frac{\partial}{\partial c}\left(\frac{x}{c}\right)^{\varphi}=-\frac{\varphi}{c}\left(\frac{x}{c}\right)^{\varphi},\quad \frac{\partial}{\partial\varphi}\left(\frac{x}{c}\right)^{\varphi}=\log\left(\frac{x}{c}\right)\left(\frac{x}{c}\right)^{\varphi},$$

applying the mean value theorem and the Cauchy–Schwarz inequality, one gets that if $(c_1,\varphi_1)\in B((c,\varphi),\delta)$ then

$$\left|g(t,x;c_1,\varphi_1)-g(t,x;c,\varphi)\right| \le \delta t\left(\frac{x}{\widetilde{c}}\right)^{\widetilde{\varphi}}\left\|\left(-\frac{\widetilde{\varphi}}{\widetilde{c}},\log\left(\frac{x}{\widetilde{c}}\right)\right)\right\|, \tag{34}$$

$0<x<\max\left\{c\,(1/t)^{1/\varphi},c_1\,(1/t)^{1/\varphi_1}\right\}$, where $(\widetilde{c},\widetilde{\varphi})=\alpha(c,\varphi)+(1-\alpha)(c_1,\varphi_1)$, for some $0<\alpha<1$ (the value of α may depend on t and on x). Let $\mathcal{N}=B((1,1),\varepsilon/2)$, which implies that $\delta<\varepsilon/2$,

$$1-\varepsilon<c,\ c_1,\ \widetilde{c}<1+\varepsilon,\quad 1-\varepsilon<\varphi,\ \varphi_1,\ \widetilde{\varphi}<1+\varepsilon,$$

and

$$\frac{1-\varepsilon}{1+\varepsilon}\le\frac{\widetilde{\varphi}}{\varphi}=\alpha+(1-\alpha)\frac{\varphi_1}{\varphi},\ \frac{\widetilde{\varphi}}{\varphi_1}=1-\alpha+\alpha\frac{\varphi}{\varphi_1}\le\frac{1+\varepsilon}{1-\varepsilon}.$$

Thus,

$$0<tx^{\widetilde{\varphi}}<t\max\left\{c\,(1/t)^{1/\varphi},c_1\,(1/t)^{1/\varphi_1}\right\}^{\widetilde{\varphi}}\le(1+\varepsilon)^{1+\varepsilon}\max\left\{t^{2\varepsilon/(1-\varepsilon)},t^{2\varepsilon/(1+\varepsilon)}\right\}. \tag{35}$$

From (34) and (35),

$$\left|g(t,x;c_1,\varphi_1)-g(t,x;c,\varphi)\right|^2\le \delta^2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(1+\varepsilon)}\left\{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2+2\log^2(x)+2\log^2(1+\varepsilon)\right\}\max\left\{t^{4\varepsilon/(1-\varepsilon)},t^{4\varepsilon/(1+\varepsilon)}\right\}. \tag{36}$$

Notice that

$$I=\int_0^\infty \max\left\{t^{4\varepsilon/(1-\varepsilon)},t^{4\varepsilon/(1+\varepsilon)}\right\}w(t)\,\mathrm{d}t\le\int_0^1 t^{4\varepsilon/(1+\varepsilon)}w(t)\,\mathrm{d}t+\int_0^\infty t^{4\varepsilon/(1-\varepsilon)}w(t)\,\mathrm{d}t:=I_1+I_2.$$

Clearly $0<I_1<\infty$ as it is the integral of a continuous function on a bounded interval; from (15) we also have that $0<I_2<\infty$. Since $0<I<\infty$ and $\mathsf{E}_0(\log^4(X))<\infty$, from (36) it readily follows that (33) holds true.

Condition 2.11 (b) in de Wet and Randles (1987) We must show that for any $\eta>0$, there is a δ^* such that $0<\delta<\delta^*$, $(c,\varphi)\in\mathcal{N}$ and $B((c,\varphi),\delta)\subset\mathcal{N}$ imply that

$$\int_0^\infty \mathsf{E}\left[\sup_{(c_1,\varphi_1)\in B((c,\varphi),\delta)}\left|g(t,X;c_1,\varphi_1)-g(t,X;c,\varphi)\right|^4\right] w(t)\,\mathrm{d}t\le\eta. \tag{37}$$

Proceeding as in the proof of Condition 2.11 (a), it can be seen that $\int_0^\infty \max\left\{t^{8\varepsilon/(1-\varepsilon)}, t^{8\varepsilon/(1+\varepsilon)}\right\} w(t)\,\mathrm{d}t < \infty$. Taking $\delta^* = \varepsilon/2$, from (36) and $\mathsf{E}_0(\log^4(X)) < \infty$ it readily follows that (37) holds true.

(2.17) in de Wet and Randles (1987) We must check that

$$\mathsf{E}_0[h_*^2(X_1, X_2)] < \infty, \quad \mathsf{E}_0[|h_*(X_1, X_1)|] < \infty. \tag{38}$$

With this end, notice that

$$h_*(x,y) = \int_0^\infty \{g(x,t;1,1) - \mu(t)\mathrm{C}(x)\}\{g(y,t;1,1) - \mu(t)\mathrm{C}(y)\}w(t)\,\mathrm{d}t.$$

Now, using (27), (29) and (30), we get that

$$|h_*(x,y)| \le M_1 + M_2\mathrm{C}(x) + M_3\mathrm{C}(y) + M_4\mathrm{C}(x)\mathrm{C}(y), \tag{39}$$

for certain positive constants M_1, M_2, M_3, M_4. Finally, from Assumption 5.2 and (39), it is concluded that (38) holds true □

Proof of Theorem 5.9. Sharing the notation in the proof of Theorem 5.5, we can write

$$\frac{\mathcal{T}_{n,w}}{n} = \frac{1}{n}U_{1,n} + \frac{n-1}{n}U_{2,n},$$

where

$$U_{1,n} = \frac{1}{n}\sum_{j,k=1}^{n} h(X_j, X_j; \widehat{c}_n, \widehat{\varphi}_n),$$

$$U_{2,n} = \frac{1}{n(n-1)}\sum_{1\le j\ne k\le n} h(X_j, X_k; \widehat{c}_n, \widehat{\varphi}_n).$$

As $|h(X_j, X_k; \widehat{c}_n, \widehat{\varphi}_n)| \le 4$, it readily follows that $(1/n)U_{1,n} \to 0$ a.s. To prove the result we show that $U_{2,n} \xrightarrow{\mathrm{P}} \Delta_w$. With this aim, we will show that

$$\lim_{d\to 0} \mathsf{E}\left[\sup_{\|(c,\varphi)-(c_X,\varphi_X)\|\le d} \left|h(X_1, X_2; c, \varphi) - h(X_1, X_2; c_X, \varphi_X)\right|\right] \to 0. \tag{40}$$

Then, applying Theorem 2.9 of Iverson and Randles (1987), the desired result follows.

Notice that $h(x,y;c,\varphi) = h_2(x,y;c,\varphi) - h_1(x;c,\varphi) - h_1(y;c,\varphi) + h_0$, with (assuming that $x \le y$)

$$\begin{aligned}
h_2(x,y;c,\varphi) &= \int_0^\infty \min\{1, t(x/c)^\varphi\}\min\{1, t(y/c)^\varphi\}w(y)\,\mathrm{d}t \\
&= \left(\frac{x}{c}\right)^\varphi \left(\frac{y}{c}\right)^\varphi \int_0^{(c/y)^\varphi} t^2 w(t)\,\mathrm{d}t + \left(\frac{x}{c}\right)^\varphi \int_{(c/y)^\varphi}^{(c/x)^\varphi} t w(t)\,\mathrm{d}t + \int_{(c/x)^\varphi}^\infty w(t)\,\mathrm{d}t \\
&:= h_{21}(x,y;c,\varphi) + h_{22}(x,y;c,\varphi) + h_{23}(x,y;c,\varphi), \\
h_1(x;c,\varphi) &= \int_0^\infty \min\{1, t(x/c)^\varphi\}\psi_0(t)w(y)\,\mathrm{d}t \\
&= \left(\frac{x}{c}\right)^\varphi \int_0^{(c/x)^\varphi} t\psi_0(t)w(t)\,\mathrm{d}t + \int_{(c/x)^\varphi}^\infty \psi_0(t)w(t)\,\mathrm{d}t \\
&:= h_{11}(x;c,\varphi) + h_{12}(x;c,\varphi), \\
h_0 &= \int_0^\infty \psi_0(t)^2 w(y)\,\mathrm{d}t.
\end{aligned}$$

We have that

$$\frac{\partial}{\partial c}h_2(x,y;c,\varphi) = -\frac{2\varphi}{c}h_{21}(x,y;c,\varphi) - \frac{\varphi}{c}h_{22}(x,y;c,\varphi),$$

$$\frac{\partial}{\partial c}h_1(x;c,\varphi) = -\frac{\varphi}{c}h_{11}(x;c,\varphi),$$

$$\frac{\partial}{\partial\varphi}h_2(x,y;c,\varphi)=\log\left(\frac{xy}{c}\right)h_{21}(x,y;c,\varphi)+\log\left(\frac{x}{c}\right)h_{22}(x,y;c,\varphi),$$
$$\frac{\partial}{\partial\varphi}h_1(x;c,\varphi)=\log\left(\frac{x}{c}\right)h_{11}(x;c,\varphi).$$

For each fixed $x, y > 0$, taking into account that $0 \le h_{21}, h_{22}, h_{11}, h_{12} \le 1$, it readily follows that the above derivatives are bounded when $\|(c,\varphi)-(c_X,\varphi_X)\| \le d$. Therefore, applying the mean value theorem and recalling that $\mathsf{E}(|\log(X)|) < \infty$, one gets

$$\mathsf{E}\left[\sup_{\|(c,\varphi)-(c_X,\varphi_X)\|\le d}\left|h(X_1,X_2;c,\varphi)-h(X_1,X_2;c_X,\varphi_X)\right|\right]\le Md,$$

for some positive constant $M > 0$, and (40) holds. □

Proof of Theorem 5.11. We write $Q^{(n)}$ and $P^{(n)}$ for the n-fold product measures of F_n and F_0, respectively, and we denote $L_n = \mathrm{d}Q^{(n)}/\mathrm{d}P^{(n)}$. Since the function a is bounded, a Taylor expansion of the function $t \mapsto \log(1+t)$ yields

$$\begin{aligned}\log L_n(X_1,\dots,X_n) &= \sum_{j=1}^{n}\log\left(1+\frac{1}{\sqrt{n}}a(X_j)\right)\\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}a(X_j)-\frac{1}{2n}\sum_{j=1}^{n}a^2(X_j)+r_n,\end{aligned}$$

where, under $P^{(n)}$, $r_n = o_P(1)$. By the central limit theorem and the law of large numbers, it follows that, under $P^{(n)}$, $\log L_n \xrightarrow{\mathcal{D}} \mathrm{N}(-\tau^2/2, \tau^2)$ with $\tau^2 = \int_0^\infty a^2(u) f_0(u)\mathrm{d}u$. Hence, by Le Cam's first lemma, see Li and Babu (2019) p. 297, the sequence $Q^{(n)}$ is contiguous to $P^{(n)}$. Recall the definition of $V_n(t)$ in (18). For fixed $k \ge 1$ and $t_1,\dots,t_k \in [0,\infty)$, the joint limiting distribution of $V_n(t_1),\dots,V_n(t_k)$ and $\log L_n$ under $P^{(n)}$ is the $(k+1)$-variate normal distribution with expectation vector $(0,\dots,0,-\tau^2/2)^\top$ and covariance matrix

$$\begin{pmatrix}\Sigma & \mathbf{d}\\ \mathbf{d}^\top & \tau^2\end{pmatrix},$$

where $\mathbf{d} = (d(t_1),\dots,d(t_k))^\top$ and Σ has entries $c(t_v, t_j)$, $1 \le v, j \le k$, with $c(t,s)$ as defined in (19). From Le Cam's third lemma, see Li and Babu (2019) p. 300, we thus obtain that, under $Q^{(n)}$, the finite-dimensional distributions of V_n converge to the finite-dimensional distributions of the shifted Gaussian random element $\mathcal{Z} + d$. Tightness of V_n under $P^{(n)}$ and the contiguity of $Q^{(n)}$ to $P^{(n)}$ entail tightness of V_n under $Q^{(n)}$. Therefore, $V_n \xrightarrow{\mathcal{D}} \mathcal{V} + d$ under $Q^{(n)}$. Because of contiguity, (16) also holds when data come from X_n. Finally, the assertion follows from the continuous mapping theorem. □

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