DESIGNS AND CODES FROM CERTAIN FINITE SIMPLE GROUPS

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Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

— Bertrand Russell (1872-1970), *The Study of Mathematics*

The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.

— J.H. Poincaré (1854-1912), cited in [47]

Abstract

In this dissertation, we study four methods for constructing codes and designs from finite groups. The first method was developed by Carmichael and Ernst Witt in the nineteen thirty's. The second method is a generalisation of the first one by D.R. Hughes in the nineteen sixty's. These first two methods use t-transitive groups to construct t-designs. The last methods are two recent techniques developed by J.D. Key and J.Moori (2002), they use primitive finite groups to build 1-designs. We will apply these methods to simple groups, and use the incidence matrix of the constructed designs to generate codes.

Preface

The work described in this dissertation was carried out under the supervision of Professor Jamshid Moori, Department of Mathematical Sciences, North-West University, Mafikeng, from October 2012 to November 2013.

The dissertation represents original work by the author and has not been submitted in any form for any degree or diploma to any university. Where use has been made of the work of others, it is duly acknowledged in the text.

G.F Randriafanomezantsoa-Radohery | Prof. J. Moori
(Student) | (Supervisor)

Dedication

This work is dedicated to my parents who have handed down to me the love of knowledge and hard work that has always driven me.

Acknowledgments

I offer my sincerest gratitude to my supervisor Professor Jamshid Moori for his invaluable guidance and all the help he provided at each level of this project. With his great passion and priceless experiences each obstacle had become the occasion of a precious learning moment.

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Contents

List of notations

$GF(q)$, $\mathbb{F}_q$	finite field with q elements
1_G, e	identity element of the group G
$\|X\|$	size of the set X
$\|G\|$	order of the group G
$G \cong H$	G is isomorphic to H
$\mathrm{Fix}(H)$	set of points fixed by H
$H < G$	H is a proper subgroup of G
$H \leq G$	H is a subgroup of G
$H \lhd G$	H is a normal subgroup of G
$N_G(H)$	normaliser of H in G
$[G : H]$	index of H in G
$G.H$	general extension
$G{:}H$	split extension
$G{}^{\cdot}H$	non-split extension
S_X	symmetric group on X
G_x	stabiliser of $x \in X$ when G acts on X
$\mathrm{Aut}(G)$	full automorphism group of G
$C_G(g)$	the centraliser in G of the element g
S_n, A_n	symmetric group, alternating group of degree n
$\mathbb{Z}_n$	Cyclic group of order n
H^n	direct product of n groups which are isomorphic to H
p^n	elementary abelian group of order p^n where p is prime
p^{m+n}	an extension of p^n by p^m
(m, n)	$\gcd(m, n)$
$p{:}q$	$\mathbb{Z}_p{:}\mathbb{Z}_q$
$PG(n, q)$	projective space of dimension n over $GF(q)$
$GL_n(q)$	general linear group of degree n over $GF(q)$
$P\Gamma L_n(q)$	projective semi-linear group of degree n over $GF(q)$
$L_n(q)$	projective special linear group of degree n over $GF(q)$

$U_n(q)$	projective special unitary group of degree n over $GF(q)$
$S_n(q)$	projective symplectic group of degree n over $GF(q)$
χ	character of a finite group
χ_ρ	character afforded by a representation ρ of a finite group
$\chi(G\|H)$	permutation character of G on the cosets of H in G
I_n	identity matrix of order n
$\mathcal{D}$	design
$\mathcal{C}_p(\mathcal{D})$	p-linear code from the incidence matrix of a design $\mathcal{D}$

1 Introduction

Groups, designs and codes are three independent structures in mathematics. Group theory finds its roots mainly in the works of mathematicians such as Lagrange [62], Ruffini [78], Cauchy [21,22], Abel [1] and Galois on the algebraic theory of equations, Gauss in number theory [33] and Klein in geometry [60]. Back then, a group was just seen as a tool to solve other problems in mathematics. A. Cayley gave the first definition of an abstract group [23] and mathematicians such as Sylow and Hölder started to study groups as independent mathematical objects.

Designs have been known and studied at least since the 18th century under different forms such as Latin square, Steiner triple system, Hadamard matrices, tactical configuration and so on. In the 1920's Ronald Fisher and his collaborators started to develop a mathematical theory of design for its application in statistics [16]. The Indian statistician R.C. Bose was the first to study designs using tools from finite field, finite geometry and combinatorics, hence he started an independent mathematical theory of designs [11,12]. He was also among the first to notice the fertile connection between design theory and coding theory [13,14].

Coding theory was initiated by C. Shannon with his 1948 seminal paper [80]. Motivated by problems from information transmission, he proved that when a message is transmitted through a noisy channel, it is possible to encode it in such a way that it is received with an arbitrarily low error rate. M. J. E. Golay and R. W. Hamming gave the first examples of such codes in 1949 and 1950, respectively [35,41].

In this work, we are going to see how groups, designs and codes can interact. Finite groups can be used to construct designs. Indeed, we can cite for instance the recent joint work of J.D. Key and J. Moori [53,54,68] in which they developed two methods for constructing designs from the conjugacy classes of maximal subgroups and from the conjugacy classes of elements of a finite primitive group. But the use of finite groups for the construction of designs can be traced all the way back to the 30's with Ernst Witt and Carmichael [20,88,89]. They developed independently a method to construct Steiner systems from t-transitive groups. In 1965, D.R. Hughes generalised this method to build t-designs [46]. The designs constructed from these four methods will be such that the initial group is an automorphism group of these designs. The adjacency matrix of these designs can be used to generate linear codes and the initial groups will also act as automorphism groups of these codes.

This situation is interesting from the coding theory point of view. Indeed, codes with large automorphism groups can be useful in the process of encoding and decoding (see for instance [52,83]) and we see that deriving codes from designs constructed from a large group using one of the methods mentioned earlier is a way of ensuring that such a code is obtained.

We concentrate mainly on finite simple groups because of the significant amount of information that we have on the structure of these groups (see for instance [4,19,25,37,87]). We use Carmichael-Witt's method, its Hughes's generalisation and the two methods of Key and Moori to construct designs and codes from these finite simple groups. Chapter 2 gives the background materials and results required from the theories of designs, groups and codes for the development and the application of the four methods discussed in this thesis. After describing these methods in Chapter 3, we apply them on the classical simple groups, the alternating groups and some sporadic simple groups in Chapter 4, 5 and 6, respectively. Finally, we discuss in Chapter 7 the construction of Hadamard designs using one of the methods of Key and Moori. In the Appendix, we list MAGMA [15] implementations of the four methods and some codes and designs constructed from some projective special linear groups.

2 Preliminaries

This chapter covers the background materials and results from group theory, representation theory, coding theory and design theory that are used throughout this dissertation. Section 2.1 deals with simple groups and group extension theory. Sections 2.2 and 2.3 discuss the fundamentals of group action, representations and character theory. Sections 2.4 and 2.5 talk about some elementary concepts from design theory and coding theory, respectively.

2.1 Group extensions and simple groups

Simple groups will be the raw materials of the different methods used in this work. Moreover, most of the groups used in this work will be described as extensions from simple groups. The notation related to the notion of *extension* and *simple group* will be as in $\mathbb{ATLAS}$ [25].
The *extension theory* allows us to construct a larger group from two smaller groups and is useful in describing group structures.

Definition 2.1.1. *Consider the groups K and Q. An* ***extension*** *of K by Q is any group G having a normal subgroup $N \cong K$, for which $G/N \cong Q$. In that case G is denoted by $K.Q$.*

Remark 2.1.2. If $K \trianglelefteq G$, $Q < G$ and $K \cap Q = \{e\}$, then the extension $G = K.Q$ is called a *split extension* or a *semi-direct product* and is denoted by $K{:}Q$. A *non-split extension* is any extension $K.Q$ which is not a split extension and is denoted by $K{}^{\cdot}Q$.

If the extension is considered as a *product* between two groups and the quotient a *division*, then a simple group can be seen as an irreducible finite group, irreducible in the sense that it is not an extension (or product) of any other finite groups.

Definition 2.1.3. *A group G is* ***simple*** *if it has no non-trivial normal subgroup.*

A finite group G can be *broken down* into a unique set of simple groups (Jordan-Hölder's theorem, see for instance [77, Theorem 5.12 p.100]) and it can be reconstructed as a particular successive extension by these simple groups. Therefore, simple groups can be considered as the building blocks of finite groups and a classification of all simple groups will theoretically imply a classification of all finite groups.
It took mathematicians almost 150 years to classify all finite simple groups. Hölder was among the first to start that classification. Although in 1892 he classified all simple groups of order less than 200 [45], we had to wait until the early 1980's and the first decade of the 21st century to see the final classification of all simple groups [4–6, 36].

Theorem 2.1.4 (CFSG). *Every finite simple group is isomorphic to one of the following groups*

- *A cyclic group $\mathbb{Z}_p$ of prime order p;*
- *An alternating group A_n of degree n at least 5;*
- *A simple group of Lie type:*
  - *the classical Lie groups, namely :*
    - *linear : $L_n(q)$, $n \geq 2$, $(n, q) \neq (2,2), (2,3)$,*
    - *unitary : $U_n(q)$, $n \geq 3$, $q \geq 3$.*
    - *symplectic : $S_{2n}(q)$, $n \geq 2$, $q \geq 3$,*
    - *orthogonal : $O^{\epsilon}_{2n}(q)$, $n \geq 4$, $\epsilon = \pm$; $O_{2n+1}(q)$, q odd,*

    where q is a power of a prime p,
  - *the exceptional groups of Lie type and the twisted groups of Lie type (including the Tits group which is not strictly a group of Lie type).*
- *One of 26 sporadic simple groups:*
  - *The Mathieu groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24}.*
  - *The groups related to the Leech lattice: the Conway groups Co_1, Co_2, Co_3; the MacLaughlin group McL; Suzuki group Suz; Higman-Sims group HS; Janko group J_2.*
  - *The Fischer groups Fi_{22}, Fi_{23}, Fi'_{24}.*
  - *The Monster M; the baby monster B; Thompson group Th; Held group He; Harada-Norton group HN.*
  - *The pariahs : Janko groups J_1, J_3, J_4; O'Nan group ON; Lyon group Ly; Rudvalis group Ru.*

2.2 Group action and permutation representation

Group action is an important notion in group theory. It gives to an abstract group a more concrete aspect and is a valuable tool to a better understanding of the set X on which the group is acting. That set can be the group itself or a set of subsets of that group. Group action constitutes the main tool used in the construction methods that will be discussed throughout the thesis.
The notion of group action is intimately in relation with the notion of permutation.

Definition 2.2.1. *Let X be a set. A **permutation** of X is a bijection of X onto X. The set of all permutations of X form a group called the **symmetric group** of X and denoted by S_X.*

Definition 2.2.2. *Let G be a finite group and X a non-empty set. We say that the group G acts on X if there exists a homomorphism $\phi : G \to S_X$. For $g \in G$ and $x \in X$, $\phi(g)$ is denoted by ϕ_g and $\phi_g(x)$ is denoted by x^g.*

Example 2.2.3 (Action by right/left multiplication)**.** The group G acts on itself by right/left multiplication. For $x, g \in G$ the action is defined by $x^g = xg$ ($x^g = gx$).

Example 2.2.4 (Action by conjugation)**.** The group G acts on itself by conjugation. For $x, g \in G$ that action is defined by $x^g = g^{-1}xg$.

Example 2.2.5 (Action on the conjugates of a subgroup)**.** If $H < G$, then G acts on the set X of all conjugates of H in G. For $x, g \in G$ and $xHx^{-1} \in X$, the action is defined by $(xHx^{-1})^g = gxH(gx)^{-1}$.

Example 2.2.6 (Action on right cosets)**.** Consider $H < G$ and $X = \{Hx \,|\, x \in G\}$ the set of right cosets of H in G. For $x, g \in G$ and $Hx, Hxg \in X$, G acts on X by right multiplication: $(Hx)^g = Hxg$.

Definitions 2.2.7. *Let G be a group acting on a non-empty set X.*

(a) We say that G acts transitively on X if for all x and $y \in X$, we can find a $g \in G$ such that $x^g = y$. In that case G is said to be ***transitive****.*

(b) We say that G acts k-transitively on X if for all $(x_1, x_2, \dots, x_k)$, $(y_1, y_2, \dots, y_k) \in X^k$ with $x_i \neq x_j$ and $y_i \neq y_j$ if $i \neq j$, we can find a $g \in G$ such that $x_i^g = y_i$. In that case G is said to be ***k-transitive****.*

(c) We say that G acts faithfully on X if for $g \in G$ and $x \in X$, $x^g = x$ if and only if $g = 1_G$. In that case G is said to be ***faithful****.*

Remark 2.2.8. By Burnside's theorem the subgroup generated by the set of all minimal normal subgroup of a finite 2-transitive group is either an elementary abelian or a *primitive* (see Definition 2.2.19) simple group [17, Theorem XIII]. Hence, by using CFSG (see Theorem 2.1.4) we can classify all 2-transitive finite groups. In particular, Table 2.2.1 gives the list of the 2-transitive simple groups.

A 6-transitive finite permutation group is symmetric or alternating [19, Theorem 5.3, Corollary 5.4] and apart from the symmetric and alternating groups the Mathieu groups are the only faithful 4-transitive and 5-transitive groups .

Definitions 2.2.9. *Consider the group G acting on a set X and $x_0, x_1, \dots, x_t \in X$.*

(a) The ***stabiliser*** *of x_0, denoted by G_{x_0} is the subgroup*

$$G_{x_0} = \{g \in G : {x_0}^g = x_0\}.$$

Group	Degree
A_n, $n \geq 5$	n
$L_n(q)$, $n \geq 2$, $(n,q) \neq (2,2)$, $(2,3)$	$(q^n-1)/(q-1)$
$S_{2n}(2)$, $n \geq 3$	$2^{2n-1}+2^{n-1}$
$S_{2n}(2)$, $n \geq 3$	$2^{2n-1}-2^{n-1}$
$U_3(q)$, $q \geq 3$	q^3+1
$Sz(q)$, $q = 2^{2d+1} > 2$	q^2+1
$L_2(11)$	11
M_{11}	11
M_{12}	12
A_7	15
M_{22}	22
M_{23}	23
M_{24}	24
$L_2(8)$	28
HS	176
Co_3	276

Table 2.2.1: 2-transitive simple groups [19]

(b) The stabiliser of the t points $x_1, x_2 \ldots, x_t$ is the subgroup

$$G_{x_1,x_2\ldots,x_t} = \{g \in G : x_i{}^g = x_i, 1 \leq i \leq t\}.$$

(c) If $Y \subseteq X$, then the stabiliser of Y is the subgroup

$$G_Y = \{g \in G : Y^g = Y\}.$$

Remark 2.2.10. If $Y = \{x_1, x_2 \ldots, x_t\}$, then $G_{x_1,x_2\ldots,x_t} < G_Y$.

Definition 2.2.11. *For $x \in X$ **the orbit** of x, denoted by x^G, is the set of images of x under the action of G, that is $x^G = \{x^g : g \in G\}$.*

Example 2.2.12. Consider a group G, $H < G$ and X the set of all conjugates of H in G. When G acts on X, the point stabilizer G_H is $N_G(H)$, since $H^g = H$ if and only if $g \in N_G(H)$.

Example 2.2.13. When the group G acts on itself by conjugation, the orbit of an element of g is its conjugacy class $[g]$ and the point stabilizer G_g is $C_G(g)$.

Theorem 2.2.14. *If G acts on X and $x \in X$, then $|x^G| = [G : G_x]$.*

Proof. See [77, Theorem 3.19]. □

Definition 2.2.15. *The actions of the group G on the sets X and Y are said to be isomorphic if there is a bijection $\phi : X \to Y$ such that $\phi(x^g) = \phi(x)^g$ for all $g \in G$ and $x \in X$.*

Any transitive action of a group G can be considered as the action of G on the cosets of one of its subgroup. More precisely:

Theorem 2.2.16. *If the group G acts transitively on the set X, then for all $x \in X$, the action of G on X is isomorphic to the action of G on G/G_x the set of left cosets of G_x.*

Proof. Since the action of G on X is transitive, for any $x, y \in X$ we can find an element $g \in G$ such that $x^g = y$. Consider $\phi : X \to G/G_x$, $y \mapsto gG_x$ with $x^g = y$. Then ϕ is well-defined since if $x^g = x^h$, then $gh^{-1} \in G_x$. If $gG_x = hG_x$, then $h^{-1}g \in G_x$ and $x^g = x^h$. Hence ϕ is one-to-one and ϕ is obviously onto. Now for $y \in X$ we have $\phi(y^h) = \phi(x^{gh}) = hgG_x = \phi(y)^h$ □

Definition 2.2.17. *If G is a group acting on a set X, then a subset B of X is said to be a* **block** *for G, if for all $g \in G$, either $B^g = B$ or $B^g \cap B = \emptyset$ where $B^g = \{b^g : b \in B\}$.*

Example 2.2.18. A singleton, $\emptyset$, and X are trivial blocks for G, any other block is called a *non-trivial* block.

Definition 2.2.19. *A group G acting on a set X is said to be* **primitive** *if G has no non-trivial blocks on X.*

Theorem 2.2.20. *Any 2-transitive group is primitive.*

Proof. See [77, Theorem 9.12]. □

A group G has as many primitive actions as maximal subgroups.

Theorem 2.2.21. *Let G be a transitive permutation group on a set X. Then G is primitive if and only if G_x is a maximal subgroup of G for all $x \in X$.*

Proof. See [77, Theorem 9.15]. □

Definition 2.2.22. *If G is a group and X a set, then a* **permutation representation** *of G on X is a homomorphism $\rho : G \to S_X$.*

Remark 2.2.23. The action of a group G on a set X gives a permutation representation of G on X.

Example 2.2.24. The permutation representation of G corresponding to the action by right multiplication is called the *right regular representation.*

Example 2.2.25. The permutation representation of G corresponding to its action on the conjugates of one of its subgroup H is called the *permutation representation of G on the conjugates of H.*

2.3 Elementary representation theory

The previous section has shown that a group action allows to represent an abstract group as a group of permutation of a certain object X. In this section, X will be a vector space of dimension n over the complex field $\mathbb{C}$. Hence each element of the group will be seen as a linear transformation of $\mathbb{C}^n$.

Definition 2.3.1. *Let G be a group. A* ***matrix representation*** *or simply a representation of G is an homomorphism $\rho : G \to GL_n(\mathbb{C})$. The integer n is called the* degree *of ρ.*

Example 2.3.2. Let us consider the group homomorphism $P : S_n \to GL_n(\mathbb{C})$, $\sigma \mapsto (\delta_{i,\sigma(j)})_{1\leq i,j\leq n}$ where $\delta_{i,j}$ is the Kronecker's symbol. If ρ is a permutation representation of G on a set X of size n, then $\pi = P \circ \rho$ is a matrix representation of G of degree n.

Example 2.3.3 (Trivial representation)**.** For a group G the representation $\rho : G \to GL_1(\mathbb{C}), g \mapsto \rho(g) = \mathrm{id}_{\mathbb{C}}$ is called the *trivial representation* of G.

Definition 2.3.4 (Induced representation)**.** *Let $H < G$ and ρ a representation of H of degree n. Define $\rho^0 : G \to GL_n(\mathbb{C})$ by $\rho^0(g) = \rho(g)$ if $g \in H$ and $\rho^0(g) = 0$ if $g \notin H$. Let $\{x_1, x_2, \ldots, x_r\}$ be a right transversal of H in G. Define $\rho\uparrow G$ by $\rho\uparrow G(g) = (\rho^0(x_i g x_j{}^{-1}))_{1\leq i,j\leq r}$ for all $g \in G$. It can be proven that $\rho\uparrow G$ is a representation of degree nr of G (see for instance* [69, Theorem 5.5.1]*). The representation $\rho\uparrow G$ is said to be* ***induced*** *from the representation ρ of H.*

Definition 2.3.5. *Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of G. The* ***character*** *afforded by the representation ρ of G is the map χ_ρ defined by $\chi_\rho : G \to \mathbb{C}$, $g \mapsto \chi_\rho(g) = trace(\rho(g))$. The degree of χ_ρ is the degree of ρ.*

Example 2.3.6 (Permutation character)**.** The character χ_π afforded by the representation π defined in Example 2.3.2 is called a *permutation character* of G and is denoted by $\chi(G|X)$. If H is a subgroup of G and X the set of cosets of H or the set of its conjugates in G, then $\chi(G|X)$ is denoted by $\chi(G|H)$ or simply χ_H.
For $g \in G$, if the permutation representation ρ associates to g the permutation σ with the cycle shape $1^s 2^t \ldots$, then $\pi(g) = P \circ \rho(g) = P(\sigma)$ will have s non zero diagonal entries and $\chi_\pi(g) = \mathrm{trace}(\pi(g)) = s$. Thus $\chi_\pi(g)$ is the number of points of X fixed by g. In particular for any $g \in G$ the degree of χ_π is $\chi_\pi(1_G) = |X|$ and for all $g \in G$, $\chi_\pi(g) \in \mathbb{N} \cup \{0\}$.

Example 2.3.7 (Trivial character)**.** The character χ_ρ afforded by the trivial representation of a group G is called the *trivial character* of G and $\chi_\rho(g) = 1$ for all $g \in G$. The trivial character of G is denoted by $\mathbf{1}_G$.

Example 2.3.8 (Induced character)**.** Let $H < G$ and let χ_ρ be the character afforded by a representation ρ of H. The character afforded by the representation $\rho\uparrow G$ of G defined as in Definition

2.3.4 is called *induced character* from χ_ρ and is denoted by $(\chi_\rho)^G$.
If χ_ρ is extended to G by $\chi_\rho^0(g) = \chi_\rho(g)$ if $g \in H$ and $\chi_\rho^0(g) = 0$ if $g \notin H$, then

$$(\chi_\rho)^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi_\rho^0(xgx^{-1}) \tag{1}$$

and

$$(\chi_\rho)^G(g) = |C_G(g)| \sum_{i=1}^{r} \frac{\chi_\rho(x_i)}{|C_H(x_i)|}, \tag{2}$$

where $x_1, x_2, \ldots, x_r$ are representatives of the conjugacy classes of H that fuse to $[g]$ in G. Indeed,

$$(\chi_\rho)^G(g) = \mathrm{trace}(\rho\uparrow G(g)) = \sum_{i=1}^{r} \mathrm{trace}(\rho^0(x_i g x_i^{-1})) = \sum_{i=1}^{r} \chi_\rho^0(x_i g x_i^{-1}).$$

The value of $\chi_\rho^0(x_i g x_i^{-1})$ does not depend on the right transversal x_i of H, so $\chi_\rho^0(x_i g x_i^{-1}) = \chi_\rho^0(hx_i g(hx_i)^{-1})$, for all $h \in H$. Hence, when h ranges over H and i ranges over $\{1, 2, \ldots, r\}$, hx_i ranges over G. Therefore

$$\sum_{h \in H} (\chi_\rho)^G(g) = \sum_{h \in H} \sum_{i=1}^{r} \chi_\rho^0(hx_i g(hx_i)^{-1}) = \sum_{x \in G} \chi_\rho^0(xgx^{-1}) = |H|(\chi_\rho)^G(g)$$

and (1) comes from the last equality.
Now if $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$ and therefore $\chi_\rho^0(xgx^{-1}) = 0$. Assume $H \cap [g] \neq \emptyset$. As x ranges G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned}
(\chi_\rho)^G(g) &= \frac{|C_G(g)|}{|H|} \sum_{x \in [g]} \chi_\rho^0(x) \\
&= \frac{|C_G(g)|}{|H|} \sum_{x \in H \cap [g]} \chi_\rho(x) \\
&= \frac{|C_G(g)|}{|H|} \sum_{i=1}^{r} [H : C_H(x_i)] \chi_\rho(x_i) \\
&= |C_G(g)| \sum_{i=1}^{r} \frac{\chi_\rho(x_i)}{|C_H(x_i)|},
\end{aligned}$$

and we obtain (2) .

Example 2.3.9 (Induction of the trivial character). In the previous example, if $\chi_\rho = \mathbf{1}_H$, then $(\mathbf{1}_H)^G = \chi(G|H)$. Indeed, for $g \in G$

$$(\mathbf{1}_H)^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi_\rho^0(xgx^{-1}) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1}} \mathbf{1}_H(xgx^{-1}) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} 1.$$

Now for $x \in G$, $xgx^{-1} \in H$ if and only if $xg \in Hx$, that is $Hxg = Hx$ and g fixes Hx. Hence the summation is taken over the $x \in G$ such that $Hxg = Hx$. For x and $y \in G$, if Hx is fixed by g, then for all $y \in Hx$, Hy is also fixed by g. Hence

$$\sum_{x \in G, xgx^{-1} \in H} 1 = \sum_{x \in G, Hxg = Hx} 1 = |H||\{Hx_i | Hx_i g = Hx_i\}|,$$

where the x_i are the right transversals of H in G and

$$(\mathbf{1}_H)^G(g) = \frac{1}{|H|}|H||\{Hx_i|Hx_ig = Hx_i\}| = |\{Hx_i|Hx_ig = Hx_i\}| = \chi(G|H)(g).$$

Proposition 2.3.10. *If G is a group acting transitively on a set X, then for $x \in X$, $\chi(G|X) = \chi(G|G_x)$.*

Proof. By Theorem 2.2.16 the action of G on the set of left cosets of G_x is isomorphic to the action of G on X. Therefore the number of points fixed by an element of G in one of the action will be the same as in the other action and we have $\chi(G|X) = \chi(G|G_x)$. □

Example 2.3.11. If H is a subgroup of the group G and X the set of all conjugates of H in G, then by Example 2.2.12 the point stabilizer G_H is $N_G(H)$ and $\chi(G|X) = \chi(G|N_G(H))$.

Proposition 2.3.12. *For any $g \in G$, the number of conjugates of H in G containing g is given by*

$$\chi(G|N_G(H))(g) = [N_G(H) : H]^{-1} \sum_{i=1}^{k} \frac{|C_G(g)|}{|C_H(h_i)|},$$

where h_i are representatives of the conjugacy classes of H that fuse to $[g]$ in G.

Proof. See [30, Lemma 1] and [32, Corollary 3.1.3] □

2.4 Design theory

Combinatorial design theory deals with the problem of existence of arrangement of objects into subsets of the same size such that any t of these objects will belong to the same number of common subsets.

Definition 2.4.1. *An **incidence structure** is a triple $I = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, $\mathcal{P}$ is called the point set, $\mathcal{B}$ is called the block set and $\mathcal{I}$ is an incidence relation between $\mathcal{P}$ and $\mathcal{B}$. The elements of $\mathcal{I}$ are called flags.*

Definition 2.4.2. *A **t-design** or more precisely a t-(v, k, λ) design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ such that $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points and every t distinct points are together incident with precisely λ blocks. A symmetric design is a design with the same number of points and blocks.*

Example 2.4.3. Consider $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{B} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{1, 5, 6\}, \{2, 6, 7\}, \{1, 3, 7\}\}.$$

The incidence relation $\mathcal{I}$ is defined by the membership relation, that is $(p, B) \in \mathcal{I}$ if and only if $p \in B$ and we have

$$\mathcal{I} = \left\{ \begin{array}{l} (1,\{1,2,4\});\ (2,\ \{2,3,5\});\ (3,\{3,4,6\});\ (4,\{4,5,7\});\ (1,\{1,5,6\}); \\ (2,\{2,6,7\})\ ;\ (1,\{1,3,7\});\ (2,\{1,2,4\});\ (3,\ \{2,3,5\});\ (4,\{3,4,6\}); \\ (5,\{4,5,7\});\ (5,\{1,5,6\});\ (6,\{2,6,7\});\ (3,\{1,3,7\});\ (4,\{1,2,4\}); \\ (5,\ \{2,3,5\});\ (6,\{3,4,6\});\ (7,\{4,5,7\});\ (6,\{1,5,6\});\ (7,\{2,6,7\}); \\ (7,\{1,3,7\}) \end{array} \right\}$$

Each block of $\mathcal{B}$ has 3 points and any point of $\mathcal{P}$ belongs to exactly 3 blocks of $\mathcal{B}$. Thus $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a symmetric 1-(7,3,3) design. Any 2 points of $\mathcal{P}$ belongs to exactly 1 block, thus $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is also a 2-(7,3,1) design. Indeed, any 2-design is also a 1-design and generally we have the following result:

Theorem 2.4.4. *If $\mathcal{D}$ is a t-(v, k, λ) design and $1 \le s \le t$, then $\mathcal{D}$ is also a s-(v, k, λ_s) design where*

$$\lambda_s = \lambda \frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)}.$$

Proof. [7, Theorem 1.3.1] □

Remark 2.4.5.

- For s=0, λ_0 is the number of blocks of the design and is denoted by b.
- For s=1, λ_1 is called the replication number and is denoted by r.
- The parameters v, k, r, b satisfy the identity $bk = vr$.

Definition 2.4.6. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a design in which $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. Then the **incidence matrix** of $\mathcal{D}$ is defined to be a $b \times v$ matrix $A = (a_{ij})$ such that*

$$a_{ij} = \begin{cases} 1 & if (p_i, B_j) \in \mathcal{I} \\ 0 & if (p_i, B_j) \notin \mathcal{I} \end{cases}.$$

Example 2.4.7. The incidence matrix of the design $\mathcal{D}$ defined in Example 2.4.3 is

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} \{1,2,4\} & \{2,3,5\} & \{3,4,6\} & \{4,5,7\} & \{1,5,6\} & \{2,6,7\} & \{1,3,7\} \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right) \end{array}$$

Example 2.4.8 (Incidence matrix of a design from a Hadamard matrix). A Hadamard matrix H of order n is a matrix with entries $+1$ and -1 such that $HH^T = nI_n$. A Hadamard matrix could be changed into another Hadamard matrix by row and column permutation and by multiplication of some rows and columns by -1. Using these operations, any Hadamard matrix can be changed into a normalized Hadamard matrix.

A normalized Hadamard matrix is a Hadamard matrix in which the entries of the first column and the first row are equal to $+1$. It can be proven that the order of a Hadamard matrix is 1, 2 or a multiple of 4 (see [81, Theorem 4.4]). From a normalized Hadamard matrix of order $4t$, if the first row and the first column are removed and the -1 is changed into 0, then we obtain the incidence matrix of a 2-$(4t-1, 2t-1, t-1)$ design [81, Theorem 4.5].

Definition 2.4.9. *Let $1 < k < v$ be integers. A* ***Steiner system*** *of type $S(t, k, v)$ is a t-$(v, k, 1)$ design.*

Example 2.4.10 (Projective plane). Let V be a vector space of dimension $n+1$ over a field F. We define the following equivalence relation on $V - \{0\}$: for $u, v \in V - \{0\}$, $u \equiv v$ if u and v are collinear, in other words all the points of the line passing through the point v and the origin are identified to the point v. The equivalence class associated with this equivalence relation is denoted by $[v]$. The set of equivalence classes $PG(V) = \{[v] : v \in V - \{0\}\}$ is called a projective n-space and n is called the projective dimension of $PG(V)$.

If W is a subspace of dimension $m+1$ of V, then $PG(W)$ is called a projective m-subspace of $PG(V)$. A projective $(n-1)$-space is called a projective hyperplane, a projective 0-space is called a projective point, a projective 1-space is called projective line and a projective 2-space is called a projective plane.

If $F = \mathbb{F}_q$, then $PG(V)$ is denoted by $PG(n, q)$. The projective plane $PG(2, q)$ contains q^2+q+1 projective points and q^2+q+1 projective lines and each projective line contains $q+1$ points [77, Theorem 9.40] . In particular, any two projective points determine a line and any two projective lines determine a point (direct application of [77, Theorem 9.39]). Thus, if the lines of $PG(2, q)$ are considered as blocks, then a projective plane is a 2-$(q^2+q+1, q+1, 1)$ design that is a $S(2, q+1, q^2+q+1)$ Steiner system.

Example 2.4.11 (Steiner triple system). A *Steiner triple system* is a S(2,3,v) Steiner system, in that case v is called the order of the Steiner triple system. Such systems exist if and only if $v \equiv 1$ or 3 (mod 6) [58].

A *subsystem* of a Steiner triple system $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a Steiner triple system $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{I}' \subseteq \mathcal{I}$. If $\mathcal{S}' \neq \mathcal{S}$, then $\mathcal{S}'$ is called a proper subsystem. A proper subsystem $\mathcal{S}'$ is called a projective hyperplane if every block of $\mathcal{S}$ intersects $\mathcal{P}'$. The set of all projective hyperplanes of a Steiner triple system has the structure of a projective space over $GF(2)$. The dimension of this projective space is called the *projective dimension* d_p of the Steiner triple system.

A proper subsystem $\mathcal{S}'$ is called an *affine hyperplane* if for every $x \notin \mathcal{S}'$ the union of $\{x\}$ and all blocks containing x disjoint from $\mathcal{S}'$ is a subsystem $\mathcal{S}''$ and if moreover any block having exactly one point in $\mathcal{S}'$ has a point in $\mathcal{S}''$. The set of all affine hyperplanes has the same structure as the set of hyperplanes of an affine space over $GF(3)$. The dimension of this affine space is called the *affine dimension* d_A of the Steiner triple system.

Definition 2.4.12. *Two designs $\mathcal{D}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ and $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ are isomorphic if there is a bijection ϕ from $\mathcal{P}'$ to $\mathcal{P}$ so that if $(p', B') \in \mathcal{I}'$, then $(\phi(p'), \phi(B')) \in \mathcal{I}$. A bijection from a design $\mathcal{D}$ to itself is called an automorphism. The group of all automorphism of $\mathcal{D}$ is denoted by $Aut(\mathcal{D})$.*

Example 2.4.13. Up to isomorphism the design $\mathcal{D}$ defined in Example 2.4.3 is the unique 2-(7,3,1) design and there are 10 non isomorphic 2-(7,3,3) designs [24, II.1.2]. The design $\mathcal{D}$ is also a S(2,3,7) Steiner system , it is the projective plane $PG(2,2)$ and $\mathrm{Aut}(\mathcal{D}) = L_3(2)$.

Definition 2.4.14. *An automorphism group of a design is **flag-transitive** if it is transitive on the flags that is the group is transitive on the blocks and a block stabilizer is transitive on the points of that block.*

Definition 2.4.15. *An automorphism group of a design is **t-flag-transitive** if it is transitive on the blocks and a block stabilizer is t-transitive on the point of that block.*

Remark 2.4.16. In the following the incidence $\mathcal{I}$ will always be defined by the membership relation between points and block and will be omitted.

2.5 Coding theory

Coding theory deals with the problem of protecting information against noises during a transmission. To make sure that a message will be properly received at the other end of a noisy channel, only precise elements from a set called *code* are sent through that channel. The message is first transformed into an element of the code (encoding) and then sent through the channel. If the received message is not an element of the code then errors have occurred during transmission, otherwise the original message is extracted from the received message (decoding).

In 1948 Claude E. Shannon proved the existence of codes that allow transmission with small error frequency [80]. Coding theory consists of the different techniques of construction and study of such codes.

Definition 2.5.1. *A q-**ary block code** is a set $\mathcal{C} = \{w_1, \dots, w_t\} \subseteq (F_q)^n$ where F_q is a set of q symbols and n is the length of each element of the code. An element of $\mathcal{C}$ is called* codeword.

The set $(F_q)^n$ is endowed with the *Hamming distance* defined as follows:

Definition 2.5.2. *Let $u=(u_1, u_2, \dots, u_n)$ and $v=(v_1, v_2, \dots, v_n)$ be two elements of $(F_q)^n$. The **Hamming distance**, $d(u, v)$ between u and v is the number of entries in which they differ that is $d(u, v) = |\{i : u_i \neq v_i\}|$.*

Definition 2.5.3. *The minimum distance $d(\mathcal{C})$ of a code $\mathcal{C}$ is the smallest of the Hamming distances between distinct codewords, that is*

$$d(\mathcal{C}) = \min\{d(u, v)|u, v \in \mathcal{C}, u \neq v\}.$$

The minimum distance of a code is an important parameter that measures the capacity of the code to correct and to detect errors.

Proposition 2.5.4. *Let $\mathcal{C}$ be a code with minimum distance d.*

- *If $d \geq s + 1 \geq 2$, then $\mathcal{C}$ can be used to detect up to s errors.*
- *If $d \geq 2t + 1$, then $\mathcal{C}$ can be used to correct up to t errors.*

Proof. [7, Theorem 2.1.1] □

In this work we are interested in a particular kind of q-ary block code called *q-linear code*.

Definition 2.5.5. *A q-**linear code** of length n and dimension k is a subspace of dimension k of F^n where $F = GF(q)$ is the finite field with q elements. If $d(\mathcal{C}) = d$, then we write $[n, k, d]_q$ for $\mathcal{C}$.*

Example 2.5.6 (Codes from designs). The code $\mathcal{C}_F(\mathcal{D})$ of the design $\mathcal{D}$ over the field F is the space spanned by the rows of the incidence matrix of $\mathcal{D}$ over F. If F is a finite field $GF(p)$, then $\mathcal{C}_F(\mathcal{D})$ is denoted by $\mathcal{C}_p(\mathcal{D})$ and the dimension of $\mathcal{C}_p(\mathcal{D})$ is the p-rank of the incidence matrix of $\mathcal{D}$.

Definition 2.5.7. *Two linear codes of F^n are said to be isomorphic if each can be obtained from the other by permuting coordinate positions.*

Definition 2.5.8. *An automorphism group of a linear code $\mathcal{C} \subset F^n$ is a group acting on F^n by permuting coordinate positions and preserving $\mathcal{C}$. The full automorphism group of a code is denoted by Aut($\mathcal{C}$).*

Remark 2.5.9. The parameter k is the length of the real information contained in a codeword, $n - k$ is called the redundancy and k/n is the information rate.

A word of length k can be wrapped inside a codeword of length n (the encoding) by using a generator matrix of the code .

Definition 2.5.10. *If $\mathcal{C}$ is a q-ary code of length n and dimension k, a **generator matrix** E of $\mathcal{C}$ is a $k \times n$ matrices composed by any k linearly independent vectors of $\mathcal{C}$.*

Remark 2.5.11. The reduced row echelon form E'=$[I_k|B]$ of E is called the standard-form of E, it is the generator matrix of a code $\mathcal{C}'$ isomorphic to $\mathcal{C}$. If E' is multiplied to the left by a row vector of F^k, then we obtain a codeword of $\mathcal{C}'$ for which the k first entries is the word of length k represented by the row vector of F^k.

The space F^n is endowed with the standard inner product. For $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n) \in F^n$ the inner product of u and v is denoted by (u, v) where $(u, v) = \sum_{i=1}^{n} u_i v_i$.

Definition 2.5.12. *If $\mathcal{C}$ is a q-ary linear code of dimension k of F^n, then the **dual** code of $\mathcal{C}$ denoted by $\mathcal{C}^\perp$ is the orthogonal complement of $\mathcal{C}$ in F^n, that is*

$$\mathcal{C}^\perp = \{v \in F^n | (u, v) = 0 \text{ for all } u \in \mathcal{C}\}.$$

*The code $\mathcal{C}$ is called **self-orthogonal** if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and **self-dual** if $\mathcal{C} = \mathcal{C}^\perp$.*

Definition 2.5.13. *The intersection of $\mathcal{C}^\perp$ and $\mathcal{C}$ is called the **Hull** of $\mathcal{C}$ and denoted by Hull($\mathcal{C}$).*

Definition 2.5.14. *If $\mathcal{C}$ is a linear code, then any generator matrix for $\mathcal{C}^\perp$ is said to be a **parity check matrix** for $\mathcal{C}$.*

The parity check matrix can be used in the decoding process since it can be exploited to test if a word of F^n is a codeword or not. In fact a parity check matrix can be used to characterise the code.

Proposition 2.5.15. *If E and D are generator and parity-check matrices of a code $\mathcal{C}$, then we have $ED^t = 0_{k\times(n-k)}$ and $c \in \mathcal{C}$ if and only if $cD^t = 0$ if and only if $Dc^t = 0$.*

Proof. Follows from the fact that the rows of D are codewords of $\mathcal{C}^\perp$. □

Example 2.5.16 (Hamming code). A binary Hamming code of length $n = 2^r - 1$ $(r \geq 2)$, is a code that has for parity check-matrix the $r \times (2^r - 1)$ matrix H of all non zero binary vectors of length r.

Definition 2.5.17. *For $v = (v_1, v_2, \ldots, v_n) \in F^n$, the **support** of v is the set of index $S = \{i | v_i \neq 0\}$. The **weight** of v is $|S|$. The **minimum weight** of a code is the minimum of the weights of the non-zero codewords.*

Remark 2.5.18. For a linear code the minimum distance is equal to the minimum weight.

Definition 2.5.19. *Let $\mathcal{C}$ be a linear code of length n. The **weight enumerator** of $\mathcal{C}$ is the polynomial*

$$W_{\mathcal{C}}(z) = \sum_{i=0}^{n} A_i z^i,$$

where A_i is the number of codewords of weight i.

3 Designs from finite groups

This chapter provides the techniques that are used throughout this work to construct t-designs from finite groups. Section 3.1 gives two methods that exploit the t-transitive ($t \geq 2$) actions of finite groups. Section 3.2 discusses two other methods to construct 1-designs from primitive actions of finite groups. From these four methods we extracted algorithms that were implemented with the software package MAGMA [15] (see Appendix A).

3.1 Designs from t-transitive groups

The methods discussed in this section (Methods 3.1.2 and 3.1.4) together with Methods 3.2.1 and 3.2.5 in the next section are based on the following result:

Theorem 3.1.1. *Let G be a group acting t-transitively on a set X of size v and let B be a k-subset of X with $1 < k < v-1$. Then the set $\mathcal{B} = \{B^g | g \in G\}$ is the set of blocks of a t-(v, k, λ) design with $[G : G_B]$ blocks. Furthermore G is a group of automorphisms of the design acting transitively on $\mathcal{B}$.*

Proof. Consider T and T' two t-subsets of X. There exists a $g \in G$ such that $T' = T^g$ and if B is a block containing T, then B^g will be a block containing T'. Thus g is a bijection between the set of blocks containing T and the set of blocks containing T'. Hence any t points will belong to the same number of blocks λ. The number of blocks of the design is the size of the orbit B^G which is $[G : G_B]$. □

Each method will differ on the nature of the k-subset B. For the first two methods, namely Methods 3.1.2 and 3.1.4, B will be the subset of X fixed by a Sylow p-subgroup of the t-point stabiliser.

Notation. For a group G acting on a set X and $H < G$, the elements of X fixed by all elements of H is denoted by Fix(H), that is Fix(H) $= \{x \in X : x^h = x, \text{ for all } h \in H\}$.

The first method (Method 3.1.2 below) was developed by Ernst Witt and Carmichael in the 1930's [20, 88, 89].

Method 3.1.2 (Carmichael-Witt's method). *Let G be a group acting faithfully and t-transitively on a set X, where $t \geq 2$. Let H be the point-wise stabiliser of t points $x_1, x_2, \ldots, x_t$ in X, and let U be a Sylow p-subgroup of H for some prime p.*

(i) $N_G(U)$ acts t-transitively on $Fix(U)$.

(ii) *If* $k = |Fix(U)| > t$ *and* U *is a nontrivial normal subgroup of* H, *then* $(X, \mathcal{B})$ *is a* $S(t, k, v)$ *Steiner system, where* $v = |X|$ *and* $\mathcal{B} = \{Fix(U^g): g \in G\}$.

Proof. We follow the same reasoning as in [77, Therorem 9.66].
(i) First, notice that $N_G(U) < G_{\text{Fix}(U)}$. Indeed for $g \in N_G(U)$, $\text{Fix}(U)^g = \text{Fix}(U^g) = \text{Fix}(U)$ and thus $N_G(U)$ is acting on $\text{Fix}(U)$. Consider a t-tuple $(y_1, y_2, \ldots, y_t)$ of $\text{Fix}(U)$. Since $U < H$, $x_1, x_2, \ldots, x_t$ is also fixed by U, therefore $(x_1, x_2, \ldots, x_t)$ is a t-tuple of $\text{Fix}(U)$ and $|\text{Fix}(U)| \geq t$. Consider $i \in \{1, 2, \ldots, t\}$. By the t-transitivity of G there exists a $g \in G$ such that $y_i^g = x_i$. For $u \in U$, $x_i^{g^{-1}ug} = (y_i)^{ug} = y_i^g = x_i$, that is $g^{-1}ug \in H$ and $U^g < H$ and by Sylow's theorem there exists a $h \in H$ such that $U^g = U^h$. Hence $gh^{-1} \in N_G(U)$ and $y_i^{gh^{-1}} = x_i^{h^{-1}} = x_i$.
(ii) For any $g \in G$, it is obvious that $|\text{Fix}(U)| = |\text{Fix}(U^g)| = k$. It remains to show that a set of t points belongs to a unique block. Consider $i \in \{1, 2, \ldots, t\}$. For $g \in G$, consider $y_i \in \text{Fix}(U^g) = \text{Fix}(U)^g$, then we can find $z_i \in \text{Fix}(U)$ such that $y_i = z_i^g$. Suppose that there is an $h \neq g$ such that $y_i \in \text{Fix}(U^h) = \text{Fix}(U)^h$, that is there exists $w_i \in \text{Fix}(U)$ such that $y_i = w_i^h$. According to (i) we can find $a, b \in N_G(U)$ such that $z_i = x_i^a$ and $w_i = x_i^b$. Now $x_i^{ag} = x_i^{bh}$ implies that $x_i = x_i^{bh(ag)^{-1}}$, that is $bh(ag)^{-1} \in H$. As $U \lhd H$, we have $H < N_G(U)$, hence $bh(ag)^{-1} \in N_G(U)$ and $bhg^{-1} \in N_G(U)$. Now $U^{bhg^{-1}} = U^{hg^{-1}} = U$, implies $U^h = U^g$, therefore $\text{Fix}(U^h) = \text{Fix}(U^g)$. □

William M. Kantor et al. [51] have proved that in their 2-transitive permutation representation, the 2-point stabilisers of $L_2(q)$, $Sz(q)$, $U_3(q)$ and the Ree groups are cyclic. So Carmichael-Witt's method can be applied at least to these simple groups, since in these cases any Sylow p-subgroup of the 2-point stabiliser will be normal. That being said, the hypothesis of normality of the Sylow p-subgroup of the t-point stabiliser is too strong and limit considerably the number of groups on which Carmichael-Witt's method apply. In 1965 Hughes alleviated the normality condition and hence generalized Carmichael-Witt's method to work on any Sylow p-subgroup of the t-point stabiliser [46]. To obtain Hughes's result we need the following theorem:

Theorem 3.1.3. *Let* G *be a* t*-transitive automorphism group of a* t-(v, k, λ) *design. For* $s \leq t$, *let* H *be the point-wise stabiliser of* s *points. Then* G *is* s*-flag-transitive if and only if* $[H : H_B] = \lambda_s$ *where* B *is a block of the design.*

Proof. We refer to [46, Theorem 3.3] □

Method 3.1.4 (Hughes's method). *Let* G *be a* t*-transitive permutation group on a set of* v *points* X, H *the stabiliser of* t *points and* U *a Sylow* p*-subgroup of* H *such that* $k = |Fix(U)| > t$ *then*

(i) $N_G(U)$ *acts* t*-transitively on* $Fix(U)$.

(ii) $(X, \mathcal{B})$ *is a* t-(v, k, λ) *design, where* $\lambda = [H : H_{Fix(U)}]$, $\mathcal{B} = \{Fix(U^g): g \in G\}$.

Proof. (i) The proof is the same as for Method 3.1.2 (i).
(ii) Since $N_G(U) < G_{\text{Fix}(U)}$ and $N_G(U)$ acts t-transitively on $\text{Fix}(U)$, $G_{\text{Fix}(U)}$ is also t-transitive on $\text{Fix}(U)$. Therefore, G is t-flag-transitive and the result follows from Theorem 3.1.3. □

Remark 3.1.5. In Method 3.1.4, if U is the unique Sylow p-subgroup of H then $N_G(U) = G_{\mathrm{Fix}(U)}$. Indeed if $g \in G_{\mathrm{Fix}(U)}$, then $\mathrm{Fix}(U^g) = \mathrm{Fix}(U)$ that is U^g fixes the t points fixed by H and $U^g < H$. Therefore $U^g = U$, that is $g \in N_G(U)$ and $G_{\mathrm{Fix}(U)} < N_G(U)$. In any case $N_G(U) < G_{\mathrm{Fix}(U)}$. Now

$$H_{\mathrm{Fix}(U)} = H \cap G_{\mathrm{Fix}(U)} = H \cap N_G(U) = N_H(U) = H.$$

Thus if U is a normal Sylow p-subgroup of H, then $\lambda = 1$ and we get back to Method 3.1.2.

Method 3.1.4 will be applied to some 2-transitive simple groups to construct 2-designs. Table 2.2.1 in Chapter 1 gives the list of all 2-transitive simple groups.
All symmetric 2-designs with 2-transitive automorphism groups have been classified by William M. Kantor :

Theorem 3.1.6. *Let $\mathcal{D}$ be a symmetric design with $v > 2k$ such that $Aut(\mathcal{D})$ is 2-transitive on points. Then $\mathcal{D}$ is one of the following:*

(i) a projective space;

(ii) the unique Hadamard design with $v = 11$, and $k = 5$;

(iii) a unique design with $v = 176$, $k = 50$ and $\lambda = 14$; or

(iv) a design with $v = 2^{2m}$, $k = 2^{m-1}(2^m - 1)$ and $\lambda = 2^{m-1}(2^{m-1} - 1)$, of which there is exactly one for each $m \geq 2$.

Proof. See [50]. □

We will see that the designs in (i) can be explicitly constructed from the simple groups $L_3(q)$ using Method 3.1.2. Using Method 3.1.1, the design in (iii) can be constructed indirectly from the Higman-Sims group and the unique Hadamard design in (ii) can be constructed from $L_2(11)$ using a modification of the same method. Using Method 3.1.4, we couldn't find any 2-transitive simple group giving the designs in (iv).

3.2 Designs from primitive groups

The class of highly t-transitive simple groups is rather restricted and the two methods developed in the previous section do not allow to exploit a large number of simple groups. The next two methods have the advantage that they can be applied to any simple groups. For the first method the subset B of Theorem 3.1.1 will be an orbit of the point stabiliser of the group G in one of its primitive representation. More precisely:

Method 3.2.1 (Key-Moori Method 1). *Let G be a finite primitive permutation group acting on a set X of size n. Let $x \in X$, and let $\Delta \neq \{x\}$ be an orbit of G_x, the stabiliser of x. If $\mathcal{B} = \{\Delta^g : g \in G\}$,* *then $\mathcal{D} = (X, \mathcal{B})$ is a symmetric 1-$(n, |\Delta|, |\Delta|)$ design, with G acting as an automorphism group, primitive on points and blocks of the design.*

Proof. The main idea of the proof is from [53, 54]. The group G acts transitively on X, so by Theorem 3.1.1 the design $\mathcal{D}$ is a 1-design with $[G : G_\Delta]$ blocks. Since Δ is an orbit of G_x, $G_x \subseteq G_\Delta$ and given that G is primitive on X, we have G_x is maximal in G and hence $G_x = G_\Delta$. Therefore the number of blocks is $|G|/|G_x| = |x^G| = |X| = n$.
Let $g, g' \in G$. Consider Δ^g and $\Delta^{g'}$ two blocks of $\mathcal{D}$. We have $(\Delta^g)^{g^{-1}g'} = \Delta^{g'}$, hence G is transitive on blocks. If $x \in \Delta^g$, then $x^{g'} \in \Delta^{gg'}$ and therefore $G \subseteq \mathrm{Aut}(\mathcal{D})$. □

Theorem 3.2.2. *Let G be a primitive group acting on a set X of size n. Let Δ be a union of some orbits of the stabiliser including the orbit of size one. If $\mathcal{B} = \{\Delta^g : g \in G\}$, then $\mathcal{D} = (X, \mathcal{B})$ is a symmetric 1-$(n, |\Delta|, |\Delta|)$ design, with G acting as an automorphism group, primitive on points and blocks of the design. Moreover, the blocks of any symmetric 1-design with an automorphism group G acting primitively on points can be constructed as union of orbits of the G-stabiliser.*

Proof. If $\Delta \subset X$ is an union of orbits of the stabiliser G_x , then Δ is G_x invariant and $G_x < G_\Delta$. By the maximality of G_x, $G_\Delta = G$ or $G_\Delta = G_x$. The first case is impossible since $\Delta \subset X$. Thus $G_\Delta = G_x$ and the design $\mathcal{D}$ has $[G : G_x] = |X| = n$ blocks.
For the proof of the second part of the result we refer to [53, Lemma 2]. □

Remark 3.2.3. Any t-design is a 1-design (see Theorem 2.4.4), but the converse is not always true. In Chapter 7, we will see that some of the 1-designs constructed from Key-Moori Method 1 are 2-designs.

For the next method, X will be a conjugacy class of element of G and the subset B will be a subset of X included in one of the maximal subgroups of G.

Lemma 3.2.4. *Consider $H < G$ and nX a conjugacy class of G. If $H \cap nX \neq \emptyset$, then $nX = \cup_{g \in G}(H^g \cap nX)$.*

Proof. It is clear that $\cup_{g \in G}(H^g \cap nX) \subseteq nX$. Let $x \in H \cap nX$. Then for all $g \in G$, $x^g \in (H \cap nX)^g = H^g \cap nX$. Since $nX = [x] = \{x^g | g \in G\}$, we have $nX \subset \cup_{g \in G} H^g \cap nX$. Therefore $nX = \cup_{g \in G}(H^g \cap nX)$. □

Method 3.2.5 (Key-Moori Method 2). *Let G be a finite simple group, M a maximal subgroup of G and nX a conjugacy class of elements of order n in G such that $M \cap nX \neq \emptyset$. If $\mathcal{B} = \{(M \cap nX)^g | g \in G\}$, then $\mathcal{D} = (nX, \mathcal{B})$ is a 1-$(|nX|, |M \cap nX|, \chi_M(h))$ design, where $h \in nX$. The group G acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive (not necessarily primitive) on points of $\mathcal{D}$.*

Proof. The main idea of the proof comes from [68, Theorem 12]. The action of G on its conjugacy classes nX is clearly transitive, so according to Theorem 3.1.1 the obtained structure will be a 1-design.
The blocks are the orbits of $B = M \cap nX$ under G and we have

$$k = |M \cap nX| = \sum_{i=1}^{\ell} |[x_i]_M| = |M| \sum_{i=1}^{\ell} \frac{1}{|C_M(x_i)|},$$

where the x_i are the representatives of the conjugacy classes of M that fuse to $[h]$.
Now assume that $nX \neq \{1_G\}$. We want to prove that $G_B = M$. For $g \in N_G(M)$, $(M \cap nX)^g = M^g \cap nX = M \cap nX$. Thus, $N_G(M) < G_B$. The maximality of M and the simplicity of G imply $N_G(M) = M$ and therefore $G_B = M$ or $G_B = G$. Suppose that $G_B = G$, that is $M^g \cap nX = M \cap nX$ for all $g \in G$. By Lemma 3.2.1, $nX = \cup_{g \in G}(M^g \cap nX) = \cup_{g \in G}(M \cap nX) = M \cap nX$, that is $nX \subseteq M$. Hence $\langle nX \rangle < M$ and $\langle nX \rangle < G$. Since $\langle nX \rangle \triangleleft G$ and G is simple , $nX = \{1_G\}$ which is impossible. Therefore $M = G_B$ and $b = [G : M]$.
By Proposition 2.3.12 the number of blocks containing an element h of nX is

$$\lambda = \chi(G|N_G(M))(h) = \chi(G|M)(h) = \chi_M(h).$$

The action of G on the blocks comes from the action of G on the conjugates of M and hence the maximality of M implies the primitivity. The action of G on nX is equivalent to the action of G on the cosets of $C_G(g)$. So the action on the points is primitive if and only if $C_G(g)$ is maximal in G. □

Remark 3.2.6.

- By Remark 2.4.5 $bk = v\lambda$ so

$$k = |M \cap nX| = \frac{\chi_M(h) \times |nX|}{[G : M]}.$$

- The design obtained from Method 3.2.5 is symmetric if and only if

$$v = b \Leftrightarrow [G : C_G(g)] = [G : M] \Leftrightarrow |M| = |C_G(g)|.$$

Theorem 3.2.7. *If $\lambda = 1$, then $\mathcal{D}$ is a 1-$(|nX|, k, 1)$ design with $Aut(\mathcal{D}) = (S_k)^b{:}S_b$ where b is the number of blocks and for all p, $\mathcal{C}_p(\mathcal{D}) = [|nX|, b, k]_p$ with $Aut(\mathcal{C}) = Aut(\mathcal{D})$.*

Proof. If $\lambda = 1$, then the b blocks of $\mathcal{D}$ form a partition of nX. Let A be Aut($\mathcal{D}$). We act A on $\mathcal{B}$ and since $\mathcal{B}$ is a partition of nX, the kernel of that action will be $(S_k)^b$. Now it is easy to see that $A/(S_k)^b \cong S_{\mathcal{B}} \cong S_b$. Thus $A \cong (S_k)^b{:}S_b$. The incidence matrix of $\mathcal{B}$ is of full rank and the vectors of minimum weight are exactly the incidence vectors of the blocks of $\mathcal{D}$. □

4 Designs and codes from classical simple groups

In this chapter, using the four methods developed in the previous chapter, we will construct designs from the classical simple groups and generate the linear codes associated with these designs. The first section defines briefly the classical groups. Some of the notations and results used in that section come from [59]. The rest of the chapter is devoted to the exploration of designs and codes built from the classical simple groups. The notation that we use for the permutation characters is as in $\mathbb{ATLAS}$.

4.1 Classical groups

In the following, V and V' are vector spaces of dimension n over the finite field $F = \mathbb{F}_q$.

Definition 4.1.1. *A function $g : V \to V'$ is a* **semi-linear transformation** *if there exist $\sigma \in Aut(F)$ and $\lambda \in F$ such that, for all $x,\ y \in V$,*

$$g(x + y) = g(x) + g(y) \text{ and } g(\lambda x) = \lambda^{\sigma} g(x).$$

Remark 4.1.2. All linear transformations are semi-linear ($\sigma = \mathrm{id}_F$).

Definition 4.1.3. *The* **general semi-linear group** *of V denoted by $\Gamma L(n, q)$ is the group of all invertible semi-linear transformations from V to V.*

Definition 4.1.4. *The* **general linear group** *of V denoted by $GL(n, q)$ is the group of all invertible linear transformations from V to V.*

Definition 4.1.5. *The* **special semi-linear** *group of V denoted by $\Sigma L(n, q)$is the kernel of the group homomorphism* $\det : \Gamma L(n, q) \to F^*$.

Definition 4.1.6. *The* **special linear group** *of V denoted by $SL(n, q)$ is the kernel of the group homomorphism* $\det : GL(n, q) \to F^*$.

Remark 4.1.7. $GL(n, q) \lhd \Gamma L(n, q)$, $SL(n, q) \lhd \Gamma L(n, q)$ and $SL(n, q) \lhd \Sigma L(n, q)$.

Let G be a group, we denote by $Z(G)$ the set of elements of G commuting with all elements of G, $Z(G) \lhd G$. The center of $GL(n, q)$ is

$$Z(GL(n, q)) = \{\mathrm{diag}(\lambda, \cdots, \lambda),\ \lambda \in F^*\} \cong F^*,$$

$Z(GL(n, q)) \lhd \Gamma L(n, q)$ and $Z(SL(n, q)) \lhd \Sigma L(n, q)$.

Definition 4.1.8. *The **projective semi-linear group** of V denoted by $P\Gamma L(n,q)$ is the quotient $\Gamma L(n,q)/Z(GL(n,q))$.*

Definition 4.1.9. *The **projective special semi-linear group** of V denoted by $P\Sigma L(n,q)$ is the quotient $\Sigma L(n,q)/Z(SL(n,q))$.*

Definition 4.1.10. *The **projective general linear group** of V denoted by $PGL(n,q)$ is the quotient $GL(n,q)/Z(GL(n,q))$.*

Definition 4.1.11. *The **projective special linear group** of V denoted by $PSL(n,q)$ is the quotient $SL(n,q)/Z(SL(n,q))$.*

Definitions 4.1.12. *Consider a map $f : V \times V \to F$. The map f is said to be **non-degenerate** if for each $v \in V - \{0\}$, the maps $V \to F$ given by $x \mapsto f(x,v)$ and $x \mapsto f(v,x)$ are non zero.*

$$\text{The map } f \text{ is } \begin{cases} \textbf{symmetric} & \text{if } \; f(v,w) = f(w,v) \\ \textbf{skew-symmetric} & \text{if } \; f(v,w) = -f(w,v) \\ \textbf{alternating} & \text{if } \; f(v,v) = 0 \end{cases},$$

for all $v, w \in V$.

*The map f is called a **bilinear form** if for each $v \in V$ the two applications $V \to F$ defined by $x \mapsto f(x,v)$ and $y \mapsto f(v,y)$ are linear. In that case, for $v \in V$, $f(v,v)$ is called the **f-norm** of v.*

*The map f is said to be **symplectic** if f is bilinear, skew-symmetric and alternating.*

*The map f is said to be **unitary** if f is left-linear and if there exist an involutory filed automorphism σ such that $f(v,w) = f(w,v)^{\sigma}$ for all $u, w \in V$.*

Definition 4.1.13. *Consider the bilinear form f. A **quadratic form** associated to f is a map $Q : V \to F$ satisfying*

$$Q(\lambda u + v) = \lambda^2 Q(u) + \lambda f(u,v) + Q(v), \text{ for all } u, v \in V \text{ and } \lambda \in F.$$

In that case f is called the bilinear form associated to Q and Q is non-degenerate if f is non-degenerate.

Definition 4.1.14. *Let V be a vector space endowed with a bilinear form f with an associate quadratic form Q and let W be a subspace of V. The subspace W is called **non-degenerate** if $f|_W$ or $Q|_W$ is non-degenerate otherwise W is called totally **singular**. The subspace W is called **isotropic** if f vanishes on W. For $v \in V$, v is isotropic if $f(v,v) = 0$ and v is singular if $Q(v) = 0$.*

Consider V and V', two n-dimensional vector spaces over F equipped with the bilinear forms f and f', respectively.

Definition 4.1.15. *An **isometry** is an invertible element $g \in Hom(V,V')$ which satisfies $f'((u,v)^g) = f(u,v)$, for all $(u,v) \in V \times V'$.*

Definition 4.1.16. *A **similarity** is an invertible element $g \in Hom(V, V')$ which satisfies $f'((u,v)^g) = \lambda f(u,v)$ for a $\lambda \in F$, for all $(u,v) \in V \times V'$.*

Remark 4.1.17. The previous two definitions hold if the bilinear forms are replaced with the associated quadratic forms. In the case $V = V'$ and $f = f'$, we talk about f-isometry and f-similarity.

The set of f-isometries and f-similarities are subgroups of $GL(n,q)$ denoted by $I(n,q,f)$ and $\Delta(n,q,f)$, respectively. The subgroup $I(n,q,f) \cap SL(n,q)$ is denoted by $S(n,q,f)$. An element $g \in \Gamma L(n,q)$ is a f-semi-similarity if there exist $\sigma \in \mathrm{Aut}(F)$ and $\lambda \in F$ such that $f((u,v)^g) = \lambda f(u,v)^\sigma$. The set of all f-semi-similarity from a subgroup of $\Gamma L(n,q)$ denoted by $\Gamma(n,q,f)$. Since any f-similarity is also an f-semi-similarity ($\sigma = \mathrm{id}_F$), $\Delta(n,q,f) \leq \Gamma(n,q,f)$. It is also clear that $S(n,q,f) \leq I(n,q,f)$ and it can be proven that $I(n,q,f)$ is a subgroup of $\Delta(n,q,f)$ (see [59, Lemma 2.1.2]). Thus we have the chain of subgroups

$$S(n,q,f) \leq I(n,q,f) \leq \Delta(n,q,f) \leq \Gamma(n,q,f). \tag{3}$$

When the bilinear form f is a degenerate form, a symplectic form or a non-degenerate unitary form, the groups in that chain are called *general linear groups, symplectic groups* or *unitary groups*, respectively. If instead of a non-degenerate bilinear form f we use its associate quadratic form Q to define $I(n,q,Q)$, $S(n,q,Q)$, $\Delta(n,q,Q)$ and $\Gamma(n,q,Q)$, then they are called *orthogonal groups*. Along with these groups we have also the corresponding chain of projective groups when each group is reduced modulo F^*.

Consider V a vector space of dimension n over F endowed with a non-degenerate quadratic form Q.

Definition 4.1.18. *A subspace W of V is called **anisotropic** if $W - \{0\}$ is isotropic.*

Lemma 4.1.19. *If W is a non-degenerate anisotropic subspace of V, then*

(i) $\dim(W) \leq 2$.

(ii) If $\dim(W) = 2$, then W has a basis u, v such that $(Q(u), Q(v), f(u,v)) = (1, \zeta, 1)$, where f is the binomial form associated to Q and the polynomial $x^2 + x + \zeta$ is irreducible over F.

Proof. We refer to [59, Lemma 2.5.2]. □

Proposition 4.1.20. *The space (V, Q) has a basis of one of the following forms:*

(i) $\{e_1, \ldots e_m, f_1, \ldots, f_m\}$, *where $Q(e_i) = Q(f_i) = 0$ and $f(e_i, f_j) = \delta_{ij}$ for all i, j;*

(ii) $\{e_1, \ldots e_{m-1}, f_1, \ldots, f_{m-}, u, v\}$, *where $Q(e_i) = Q(f_i) = 0$ and $f(e_i, f_j) = \delta_{ij}$ and $f(e_i, u) = f(e_i, v) = f(f_i, u) = f(f_i, v) = 0$ for all i, j and $\{u, v\}$ as in Lemma 4.1.19;*

(iii) $\{e_1, \ldots e_m, f_1, \ldots, f_m, u\}$, *where $Q(e_i) = Q(f_i) = 0$ and $f(e_i, f_j) = \delta_{ij}$ and $f(e_i, u) = f(f_i, u)$ for all i, j and u is non-singular.*

Remark 4.1.21. Hence, in the case n even there exist two types of orthogonal groups denoted by $X^{\epsilon}(n,q,Q)$, with $\epsilon=+$ in the case of Proposition 4.1.20 (i) and $\epsilon=-$ in the case of Proposition 4.1.20 (ii) and where X ranges over I, S, Δ and Γ. For $n\geq 2$, the group $S(n,q,Q)$ has a unique subgroup of index 2 that we denote by $\Omega(n,q,Q)$ [3, 22.9].
In the case of the general linear group, when $n\geq 3$, S has an inverse transpose automorphism group ι. Hence, in addition to I, S, Δ and Γ we define two other groups $A=\Gamma\langle\iota\rangle$ in the case of the general linear group, $A=\Gamma$ otherwise. In the orthogonal case, Ω is the subgroup of index 2 in S, otherwise $\Omega=S$.

Definition 4.1.22. *A classical group is any group G satisfying*

$$\Omega\leq G\leq A \text{ or } P\Omega\leq G\leq PA.$$

Where $P\Omega$ and PA are the projective groups corresponding to Ω and A, respectively.

Classical groups	Notation
Linear	$S=SL_n(q)$ $I=\Delta=GL_n(q)$ $\Gamma=\Gamma L_n(q)$
Symplectic	$I=S=Sp_n(q)$ $\Delta=GSp_n(q)$ $\Gamma=\Gamma Sp_n(q)$
Unitary	$S=SU_n(q)$ $I=GU_n(q)$ $\Gamma=\Gamma U_n(q)$
Orthogonal	$\Omega\in\{\Omega_n^{\pm}(q),\Omega_n(q)\}$ $S\in\{SO_n^{\pm}(q),SO_n(q)\}$ $\Delta\in\{GO_n^{\pm}(q),GO_n(q)\}$ $\Gamma\in\{\Gamma O_n^{\pm}(q),\Gamma O_n(q)\}$

Table 4.1.1: Some classical groups

Table 4.1.1 gives the notation corresponding to Ω, I, S, Δ and Γ for the different types of classical groups. For the corresponding projective groups, it suffices to add a 'P' before the notation of each group except for $SL_n(q)$, $Sp_n(q)$, $SU_n(q)$, $\Omega_n^{\pm}(q)$ and $\Omega_n(q)$. Indeed $PSL_n(q)$, $PSp_n(q)$, $PSU_n(q)$, $P\Omega_n^{\pm}(q)$ and $P\Omega_n(q)$ will be denoted by $L_n(q)$, $S_n(q)$, $U_n(q)$, $O_n^{\pm}(q)$ and $O_n(q)$, respectively.

Theorem 4.1.23. *The classical groups $L_n(q)$ $(n\geq 2)$, $S_n(q)$ $(n\geq 4)$, $U_n(q)$ $(n\geq 3)$, $O_n^{\pm}(q)$ and $O_n(q)$ $(n\geq 7)$ are non abelian simple expect for $L_2(2)$, $L_2(3)$, $U_3(2)$ and $S_4(2)$.*

Proof. See for instance [87, 3.3.2, 3.5.2, 3.6.1 and 3.7.3]. □

4.2 Designs and codes from $L_2(7)$

In this section we will explore some designs and codes that we can construct from the projective special linear group $L_2(7)$. The designs are constructed from the primitive permutation representations of degree 7 and 8 of $L_2(7)$ and from the action of $L_2(7)$ on the conjugacy classes of elements which are inside its maximal subgroups. The Fano plane is constructed from the primitive representation of degree 7 of $\mathbb{Z}_7{:}\mathbb{Z}_3$.

4.2.1 Designs and codes from maximal subgroups

We applied Method 3.2.1 to the group $G = L_2(7)$ and X the set of all cosets of one of its maximal subgroup. Using MAGMA [15] we considered a primitive representation of $L_2(7)$ and we formed the orbits of the point stabiliser. For each of the non-trivial orbits, we constructed a symmetric 1-design according to Method 3.2.1.

The group $L_2(7)$ has 3 primitive permutation representations corresponding to its action on the cosets of its maximal subgroups. Table 4.2.1 gives the list of the maximal subgroups of $L_2(7)$, the degree of the representation corresponding to each of them, the number of orbits of the point stabilisers and the lengths of the non trivial orbits of the point stabilisers.

Max. sub.	Degree	#Orbits	Length
S_4	7	2	6
S_4	7	2	6
7:3	8	2	7

Table 4.2.1: Maximal subgroups and orbits of the point stabilisers of $L_2(7)$

Proposition 4.2.1. *By taking the images under $L_2(7)$ of the non-trivial orbits of the point stabiliser we can construct up to isomorphism a 1-(7,6,6) symmetric design $\mathcal{D}_1$ from the two primitive permutation representations of degree 7 of $L_2(7)$ and a 1-(8,7,7) symmetric design $\mathcal{D}_2$ from its primitive permutation representation of degree 8. The group $L_2(7)$ acts primitively on the points and the blocks of these designs.*

Proof. Direct application of Method 3.2.1. □

Proposition 4.2.2. *The full automorphism group of the 1-(7,6,6) and 1-(8,7,7) designs are the symmetric groups S_7 and S_8, respectively.*

Proof. We can construct an automorphism ϕ of the 1-(7,6,6) design $\mathcal{D}$ as follows: let us choose a point p and let $\{b_1, \ldots, b_6\}$ and b_7 be respectively the six blocks intersecting on p and the block which does not contain p. We send p to a point p', $\{b_1, \ldots, b_6\}$ to $\{b'_1, \ldots, b'_6\}$ such that $\{b'_1, \ldots, b'_6\}$ intersect in p' and b_7 to b'_7 the block which doesn't contain p'. For $i \in \{1, \ldots, 6\}$, let p'_i be the intersection of b'_7 with the blocks containing p' except the i-th block, p_i are also defined in the same way with respect to $\{b_1, \dots, b_7\}$. For $i, j \in \{1, \dots, 6\}$, p'_i and p'_j are different from p' since they belong to b'_7 and $p'_i \neq p'_j$ otherwise p'_i would belong to $\{b'_1, \dots, b'_7\}$ which is not possible, the same argument applies to the p_i. Thus for $i \in \{1, \dots, 6\}$, p and p_i give us the 7 points of the design, p' and $\phi(p_i) = p'_i$ are pairwise different and ϕ is a bijection which sends block to block. Now we have $7 \cdot 6! = 7!$ ways to choose $(p, b_1, \dots, b_6)$ that is $|\mathrm{Aut}(\mathcal{D})| \geq |S_7|$. Since $\mathrm{Aut}(\mathcal{D})$ is isomorphic to a subgroup of S_7, we have $\mathrm{Aut}(\mathcal{D}) \cong S_7$. The same arguments apply to the 1-$(8, 7, 7)$ symmetric design. □

Remark 4.2.3. The arguments used in the above proof can be generalized to prove that the automorphism group of a symmetric 1-$(n, n-1, n-1)$ design is the symmetric group S_n.
According to Method 3.2.1 , $L_2(7)$ is a subgroup of $\mathrm{Aut}(\mathcal{D}_1) \cong S_7$ and $L_2(7)$ is a subgroup of $\mathrm{Aut}(\mathcal{D}_2) \cong S_8$. We would like to know if $\mathrm{Aut}(L_2(7))$ is a subgroup of $\mathrm{Aut}(\mathcal{D}_1)$ and $\mathrm{Aut}(\mathcal{D}_2)$.

Proposition 4.2.4. *For the 1-(7,6,6) design $\mathcal{D}_1$, $Aut(L_2(7)) = L_2(7){:}2 \not< Aut(\mathcal{D}_1)$.*

Proof. Indeed, inside $L_2(7)$ there are 14 copies of S_4 that split into 2 conjugacy classes of size 7, the point set of $\mathcal{D}_1$ is one of these conjugacy classes. The elements of $\mathrm{Aut}(\mathcal{D}_1)$ are supposed to permute these 7 subgroups among themselves only, now $\mathrm{Aut}(L_2(7))$ has an outer automorphism of order 2 which will send some groups from the first conjugacy class to some groups inside the second conjugacy class and hence it will fuse the two conjugacy classes. Therefore that outer automorphism can not be inside $\mathrm{Aut}(\mathcal{D}_1)$ and hence $\mathrm{Aut}(L_2(7)) \not< \mathrm{Aut}(\mathcal{D}_1) \cong S_7$. □

Remark 4.2.5. For the 1-$(8, 7, 7)$ design $\mathcal{D}_2$, $L_2(7){:}2 < \mathrm{Aut}(\mathcal{D}_2)$.

The Fano plane is a 2-(7,3,1) design and it can also be considered as a 1-(7,3,3) symmetric design. The question of the possibility of the construction of the Fano plane from Method 3.2.1 arises naturally. A first observation is that according to Theorem 3.2.2 , the group from which the Fano plane would be constructed must be a group with a permutation representation of degree 7 that sits inside the group $L_2(7)$ since it is the automorphism group of the Fano plane. That group can't be $L_2(7)$ since according to Proposition 4.2.1, the only symmetric designs that we can construct from that group using Method 3.2.1 is a 1-(7,6,6) or a 1-(8,7,7) design. However, we notice that the maximal subgroup 7:3 of $L_2(7)$ has a primitive representation of degree seven in its action on the cosets of its maximal subgroup $\mathbb{Z}_3$. This leads to the following result:

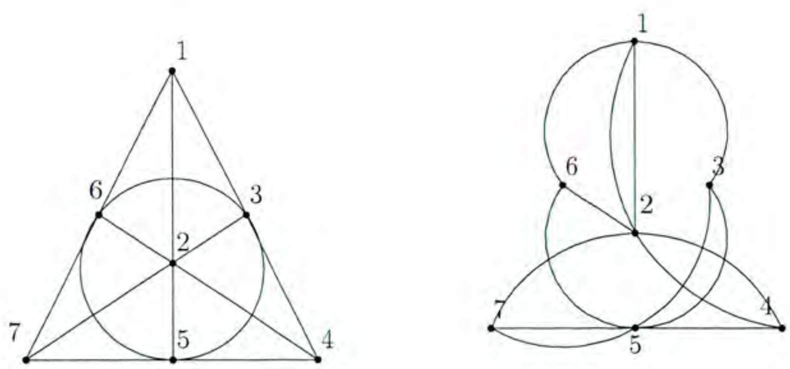
Proposition 4.2.6. *The Fano plane 1-(7,3,3) can be constructed from the action of 7:3 on a non-trivial orbit of the point stabiliser in its primitive representation of degree 7.*

Proof. The group $\mathbb{Z}_3$ is a subgroup of index 7 of 7:3 so it is maximal inside that group and the action of 7:3 on the set of 7 cosets of $\mathbb{Z}_3$ gives us a primitive permutation representation of degree 7 of 7:3 with point stabiliser isomorphic to $\mathbb{Z}_3$. Using the character table of 7:3 (cf. [85, page 58] for example) we can obtain the permutation character χ of 7:3 on the 7 cosets of $\mathbb{Z}_3$ and for g an element of order 3 we have $\chi(g) = 1$. That is an element of order 3 fixes exactly one point and the point stabiliser $\mathbb{Z}_3$ will have one orbit of length 1 and two orbits of length 3. Now by applying Method 3.2.1 to an orbit of length 3 we get a 1-(7,3,3) symmetric design that we denote by $\mathcal{F}$. Using MAGMA [15] we can construct the incidence matrix D of $\mathcal{F}$. If we denote a row of D by its index then the point set of $\mathcal{F}$ is $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$ and its block set is $\mathcal{B} = \{\{4,5,7\}, \{1,6,7\}, \{1,2,5\}, \{2,3,7\}, \{1,3,4\}, \{2,4,6\}, \{3,5,6\}\}$ and we can easily check that $\mathcal{F} = (\mathcal{P}, \mathcal{B})$ is a 2-(7,3,1) design and hence a Fano plane. □

Remark 4.2.7. The dihedral group $D_{14} \cong 7{:}2$ has also a primitive permutation representation of degree 7 in its action on the 7 cosets of its maximal subgroup $\mathbb{Z}_2$. In that representation, the point stabiliser has one orbit of length 1 and 3 orbits of length 2. By acting D_{14} on the union of the orbit of length 1 and one orbit of length 2, we obtain a 1-(7,3,3) design which is not isomorphic to the Fano plane. Indeed, consider $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{B}' = \{\{1,2,6\}, \{1,2,4\}, \{2,4,7\}, \{4,5,7\}, \{3,5,7\}, \{3,5,6\}, \{1,3,6\}\}$$

the point set and the block set of the design $\mathcal{D}$ constructed form D_{14}. The design $\mathcal{D}$ is a 1-(7,3,3) design but it is not a 2-(7,3,1) design since we have sets of two points that belong to more than one block: 1 and 2 belong to $\{1,2,6\}$ and $\{1,2,4\}$, 2 and 4 belong to $\{2,4,7\}$ and $\{1,2,4\}$, and 3 and 5 belong to $\{3,5,7\}$ and $\{3,5,6\}$.



(a) The Fano plane $\mathcal{F}$, a 1-(7,3,3) and a 2-(7,3,1) design

(b) a 1-(7,3,3) design non isomorphic to the Fano plane

Figure 4.2.1: Two non isomorphic 1-(7,3,3) designs

The group 7:2 and 7:3 are Frobenius groups and for particular values of the parameters p and q the design constructed from the Frobenius group $\mathbb{F}_{p,q}$ using Key-Moori Method 1 is a 2-design (see Chapter 7).

For $q \in \{2, 3, 7\}$, Table 4.2.2 and Table 4.2.3 gather the q-ary linear codes that we can construct from the symmetric designs obtained in Proposition 4.2.1 and Proposition 4.2.6.

Max. Sub.	**Design ($\mathcal{D}$)**	**Code (C)**	$C^\perp$	$C\cap C^\perp$	**Aut(C)**	**Aut($C^\perp$)**	**Aut($\mathcal{D}$)**
S_4	1-(7,6,6)	$[7,6,2]_2$	$[7,1,7]_2$	$[7,0,7]_2$	S_7	S_7	S_7
		$[7,6,2]_3$	$[7,1,7]_3$	$[7,0,7]_3$	$2\times S_7$	$2\times S_7$	
		$[7,7,1]_7$	$[7,0,7]_7$	$[7,0,7]_7$	$6^7{:}S_7$	$6^7{:}S_7$	
7:3	1-(8,7,7)	$[8,8,1]_2$	$[8,0,8]_2$	$[8,0,8]_2$	S_8	S_8	S_8
		$[8,8,1]_3$	$[8,0,8]_3$	$[8,0,8]_3$	$2^8:S_8$	$2^8:S_8$	
		$[8,7,2]_7$	$[8,1,8]_7$	$[8,0,8]_7$	$6\times S_8$	$6\times S_8$	

Table 4.2.2: Some codes from $L_2(7)$ using Method 3.2.1

Max. Sub.	**Design ($\mathcal{D}$)**	**Code (C)**	$C^\perp$	$C\cap C^\perp$	**Aut(C)**	**Aut($C^\perp$)**	**Aut($\mathcal{D}$)**
$\mathbb{Z}_3$	1-(7,3,3)	$[7,4,3]_2$	$[7,3,4]_2$	$[7,3,4]_2$	$L_2(7)$	$L_2(7)$	$L_2(7)$
		$[7,6,2]_3$	$[7,1,7]_3$	$[7,0,7]_3$	$L_2(7)$	$2\times S_7$	
		$[7,7,1]_7$	$[7,0,7]_7$	$[7,0,7]_7$	$6^7{:}S_7$	$6^7{:}S_7$	
	1-(7,4,4)	$[7,3,4]_2$	$[7,4,3]_2$	$[7,3,4]_2$	$L_2(7)$	$L_2(7)$	$L_2(7)$
		$[7,7,1]_3$	$[7,0,7]_3$	$[7,0,7]_3$	$2^7{:}S_7$	$2^7{:}S_7$	
		$[7,7,1]_7$	$[7,0,7]_7$	$[7,0,7]_7$	$6^7{:}S_7$	$6^7{:}S_7$	
	1-(7,6,6)	$[7,6,2]_2$	$[7,1,7]_2$	$[7,0,7]_2$	S_7	S_7	S_7
		$[7,6,2]_3$	$[7,1,7]_3$	$[7,0,7]_3$	$2\times S_7$	$2\times S_7$	
		$[7,7,1]_7$	$[7,0,7]_7$	$[7,0,7]_7$	$6^7{:}S_7$	$6^7{:}S_7$	

Table 4.2.3: Some codes from 7:3

Remark 4.2.8. In Table 4.2.3, the 1-(7,4,4) design comes from the action of 7:3 on the union of the trivial orbit and one orbit of length 3. The action of 7:3 on the union of the two orbits of length 3 gives a 1-(7,6,6) symmetric design which is isomorphic to the design constructed from the action of $L_2(7)$ on the non trivial orbit of the point stabiliser in its permutation representation of degree 6.

For F a finite filed $\mathbb{F}_p$ ($p\in\{2,3,7\}$) and n the degree of the permutation representation of $G\in\{7{:}3,\ L_2(7)\}$ we decomposed F^n into G-modules, using the codes in Table 4.2.2 and 4.2.3.

- For $L_2(7)$ of degree 7 over $\mathbb{F}_2$, the full space can be decomposed into $L_2(7)$-modules of degree 1 and 6, that is ${\mathbb{F}_2}^7=\mathbb{F}_2\jmath\oplus C_6$. The factor C_6 is the binary code $[7,6,2]_2$, C_6 is indecomposable but it has two irreducible constituents of dimension 3. Also $\mathrm{Soc}({\mathbb{F}_2}^7)=\mathbb{F}_2\jmath\oplus C_3$.
- For $L_2(7)$ of degree 7 over $\mathbb{F}_3$, the full space can be decomposed into irreducible $L_2(7)$-modules of dimensions 1 and 6, that is ${\mathbb{F}_3}^7=C_6\oplus\mathbb{F}_3\jmath$. The factor C_6 is the code $[7,6,2]_3$ and $\mathrm{Soc}({\mathbb{F}_3}^7)=C_6\oplus\mathbb{F}_3\jmath$.
- For $L_2(7)$ of degree 8 over $\mathbb{F}_2$, we have ${\mathbb{F}_2}^8=C_8$ is indecomposable but it has two irreducible constituents of dimension 3 and $\mathrm{Soc}({\mathbb{F}_2}^8)=\mathbb{F}_2\jmath$.

- For $L_2(7)$ of degree 8 over $\mathbb{F}_7$, the full space is decomposable into $L_2(7)$-modules of dimensions 1 and 7, that is $\mathbb{F}_7{}^8 = C_7 \oplus \mathbb{F}_{7J}$, the constituent C_7 is irreducible and corresponds to the code $[8,7,2]_7$ and $\mathrm{Soc}(\mathbb{F}_7{}^8) = C_7 \oplus \mathbb{F}_{7J}$.

- For 7:3 of degree 7 over $\mathbb{F}_2$, the full space can be completely decomposed into irreducible 7:3-modules that is $\mathbb{F}_2^7 = C_3 \oplus C_3 \oplus \mathbb{F}_{2J}$. The component $C_3 \oplus \mathbb{F}_{2J}$ is the binary code $[7,4,3]_2$, $C_3 \subset (C_3 \oplus \mathbb{F}_{2J})$ is the binary code $[7,3,4]_2$ and $\mathrm{Soc}(\mathbb{F}_2^7) = \mathbb{F}_{2J} \oplus C_3 \oplus C_3$.

4.2.2 Codes and designs from the conjugacy classes of elements

The group $L_2(7)$ has 6 conjugacy classes of elements listed in Table 4.2.4. Method 3.2.5 is applied to the maximal subgroups of $L_2(7)$ and its conjugacy classes of elements to construct some non-symmetric 1-designs. The corresponding binary codes are also constructed. The permutation character afforded by the action of $L_2(7)$ on the cosets of its maximal subgroups are listed in Table 4.2.5.

nX	# nX	$C_G(g)$	Maximal Centralizer
1A	1	G	Yes
2A	21	D_8	No
3A	56	3	No
4A	42	4	No
7AB	24	7	No

Table 4.2.4: Conjugacy classes of $L_2(7)$

Max. sub.	Degree	χ_M
S_4	7	$\chi_1 + \chi_4$
7:3	8	$\chi_1 + \chi_5$

Table 4.2.5: Permutation characters

$M = S_4,\ nX = 2A$

We have $\chi_M = \chi_1 + \chi_4$ and hence for g in $2A$

$$b = [G : M] = 7,\ v = |2A| = 21,\ k = |M \cap 2A| = 9,\ \lambda = \chi_M(g) = 1 + 2 = 3.$$

Thus we obtain a non symmetric 1-(21,9,3) design $\mathcal{D}$ with 7 blocks. The group $L_2(7)$ acts primitively on the blocks of $\mathcal{D}$.

Lemma 4.2.9. *Two blocks of $\mathcal{D}$ intersect on 3 points and there are exactly 3 blocks containing those 3 points together.*

Proof. We consider $A = (a_{ij})_{1\leq i\leq 21,\,1\leq j\leq 7}$ the incidence matrix of $\mathcal{D}$ with the rows indexed by the 21 points and the columns indexed by the 7 blocks. The block intersection numbers of $\mathcal{D}$ is given by the entries of the symmetric matrix A^tA and using MAGMA we can find that any two distinct blocks of $\mathcal{D}$ intersect exactly on 3 points.
Let B_1 and B_2 be two blocks of $\mathcal{D}$. The three points of $B_1 \cap B_2$ are denoted by p_1, p_2 and p_3. Let us consider p_1, since there are two blocks containing p_1 there is a third block B containing p_1 that is $p_1 \in B \cap B_1$ and $p_1 \in B \cap B_2$. Apart from p_1, p_2 and p_3 we have 6 points $\{p_4, p_5, p_6, p_7, p_8, p_9\}$ inside B_1 which are different from the 6 remaining points $\{p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}\}$ of B_2 and 6 other points $\{p_{16}, p_{17}, p_{18}, p_{19}, p_{20}, p_{21}\}$ which are neither in B_1 nor B_2. Let us suppose that p_1 and p_2 are not in B, since B intersects B_1 and B_2 on three points, we have to take two points p_4, p_5 from B_1 and p_{10}, p_{11} from B_2 as points of B. To complete B we have to take 4 other points $p_{16}, p_{17}, p_{18}, p_{19}$ which are not inside B_1 or B_2. Thus we have $\binom{6}{2}^3 > 7$ possibilities for B which is absurd. Hence the two other points in $B \cap B_1$ and $B \cap B_2$ must be p_2 and p_3 and the three blocks containing p_1 are B, B_1 and B_2. The same arguments apply to p_2 and p_3, hence $B \cap B_1 \cap B_2 = \{p_1, p_2, p_3\}$ and B_1, B_2 and B are the only blocks containing these three points together. □

Lemma 4.2.10. *$2A$ is the disjoint union of seven 3-subsets and each block of $\mathcal{D}$ is a union of three of them.*

Proof. By the previous lemma two blocks B_i and B_j intersect exactly on three points and apart from those two blocks there is only a third block B_k containing those three points together and these 3 blocks are the only blocks containing each of these 3 points. We denote by P_{ijk} that set of three points, we have :

- B_1, B_2 and a third block B_3 intersect on P_{123}; B_1, B_4 and a third block B_5 intersect on P_{145} and $P_{145} \cap P_{123} = \emptyset$; B_1 and the two last blocks B_6 and B_7 intersect on P_{167}. Thus $B_1 = P_{123} \cup P_{145} \cup P_{167}$.

- $B_2 \cap B_1$ and $B_2 \cap B_3$ is P_{123}, $B_2 \cap B_4$ is contained inside B_5 or B_6 or B_7, it can not be contained inside B_5 since $B_4 \cap B_5$ is P_{145} and it remains two choices B_6 or B_7, we take B_6 as the third block and $B_2 \cap B_4 \cap B_6$ is P_{246}; the intersection of B_2 with B_5 is contained inside B_7 and $B_2 \cap B_5 \cap B_7$ is P_{257}. Thus $B_2 = P_{123} \cup P_{246} \cup P_{257}$.

- $B_3 \cap B_1 = B_3 \cap B_2 = P_{123}$, $B_3 \cap B_4 = B_3 \cap B_7 = P_{347}$, $B_3 \cap B_5 = B_3 \cap B_6 = P_{356}$ and $B_3 = P_{123} \cup P_{347} \cup P_{356}$.

- $B_4 \cap B_1 = B_4 \cap B_5 = P_{145}$, $B_4 \cap B_2 = B_4 \cap B_6 = P_{246}$, $B_4 \cap B_3 = B_4 \cap B_7 = P_{347}$ and $B_4 = P_{145} \cup P_{246} \cup P_{347}$.

- $B_5 \cap B_1 = B_5 \cap B_4 = P_{145}$, $B_5 \cap B_2 = B_5 \cap B_7 = P_{257}$, $B_5 \cap B_3 = B_5 \cap B_6 = P_{356}$ and $B_5 = P_{145} \cup P_{257} \cup P_{356}$.

- $B_6 \cap B_1 = B_6 \cap B_7 = P_{167}$, $B_6 \cap B_2 = B_6 \cap B_4 = P_{246}$, $B_6 \cap B_3 = B_6 \cap B_5 = P_{356}$ and $B_6 = P_{167} \cup P_{246} \cup P_{356}$.

- $B_7 \cap B_1 = B_7 \cap B_6 = P_{167}$, $B_7 \cap B_2 = B_7 \cap B_5 = P_{257}$, $B_7 \cap B_3 = B_7 \cap B_4 = P_{347}$ and $B_7 = P_{167} \cup P_{257} \cup P_{347}$.

Observe that $2A = P_{123} \cup P_{145} \cup P_{167} \cup P_{246} \cup P_{257} \cup P_{347} \cup P_{356}$. □

Proposition 4.2.11. *$2A$ is the disjoint union of 7 subsets of size 3 which are the points of a 2-(7,3,1) design and $Aut(\mathcal{D}) = {S_3}^7{:}L_2(7)$.*

Proof. The first part of the proposition is proved by the previous lemma. For the second part, if we consider the point set $\mathcal{P} = \{P_{123}, P_{145}, P_{167}, P_{246}, P_{257}, P_{347}, P_{356}\}$ and the block set

$$\mathcal{B} = \{(P_{123}, P_{145}, P_{167}), (P_{123}, P_{246}, P_{257}), (P_{123}, P_{347}, P_{356}), (P_{145}, P_{246}, P_{347}),$$
$$(P_{145}, P_{257}, P_{356}), (P_{167}, P_{246}, P_{356}), (P_{167}, P_{257}, P_{347})\},$$

then $(\mathcal{P}, \mathcal{B})$ is a 2-(7,3,1) design. □

Computation with MAGMA shows that the binary code C of the design $\mathcal{D}$ is a $[21, 4, 9]_2$ code. The weight enumerator of C is $W(x) = 1 + 7x^9 + 7x^{12} + x^{21}$. Further computation gives that $|\mathrm{Aut}(C)| = |\mathrm{Aut}(\mathcal{D})|$, thus we have $\mathrm{Aut}(C) \cong \mathrm{Aut}(\mathcal{D}) \cong {S_3}^7{:}L_2(7)$. The dual $C^\perp$ of C is a $[21, 17, 2]_2$ code and $\mathrm{Hull}(C)$ is a $[21, 3, 12]_2$ code. The full space can be decomposed into irreducible $L_2(7)$-modules as follows: ${\mathbb{F}_2}^{21} = C_8 \oplus C_3 \oplus C_3 \oplus C_3 \oplus \mathbb{F}_2\jmath \oplus C_3$ with $C^\perp = C_8 \oplus C_3 \oplus C_3 \oplus C_3$, $C = C_3 \oplus \mathbb{F}_2\jmath$ and $\mathrm{Hull}(C) = C_3$; $\dim(\mathrm{Soc}(V)) = 15$ and $\mathrm{Soc}(V) = C_3 \oplus C_3 \oplus C_8 \oplus \mathbb{F}_2\jmath$.

$M = S_4$, $nX \in \{3A, 4A\}$

We have $\chi_M = \chi_1 + \chi_4$ and for $g \in 3A$ or $4A$, $\lambda = \chi_M(g) = 1$. Hence the parameters of the designs and the codes are completely determined by Theorem 3.2.7. In both cases $b = [G : M] = 7$.

- For $nX = 3A$, $v = |3A| = 56$ and $k = |M \cap 3A| = 8$. Hence we obtain a non-symmetric 1-(56, 8, 1) design $\mathcal{D}$ with 7 blocks and with $\mathrm{Aut}(\mathcal{D}) = {S_8}^7{:}S_7$. The binary code of $\mathcal{D}$ is a $[56, 7, 8]_2$ doubly even code C with $\mathrm{Aut}(C) \cong \mathrm{Aut}(\mathcal{D})$ and with the weight enumerator

$$W(x) = 1 + 7x^8 + 21x^{16} + 35x^{24} + 35x^{32} + 21x^{40} + 7x^{48} + x^{56}.$$

  We have $C^\perp = [56, 49, 2]_2$ and $\mathrm{Hull}(C) = C$.
  The full space $V = {\mathbb{F}_2}^{56}$ is decomposable into $L_2(7)$-modules of dimensions 7 and 49, that is $V = C_7 \oplus C_{49}$, where C_7 is our code C. The code C is indecomposable but it has irreducible factors of dimension 1, 3 and 3. We have $\dim(\mathrm{Soc}(V)) = 23$ and $\mathrm{Soc}(V) = C_8 \oplus C_3 \oplus \mathbb{F}_2\jmath \oplus C_8 \oplus C_3$.

- For $nX = 4A$, $v = |4A| = 42$ and $k = |M \cap 3A| = 6$. Hence we obtain a non-symmetric 1-(42,6,1) design $\mathcal{D}$ with 7 blocks and with $\mathrm{Aut}(\mathcal{D}) = {S_6}^7{:}S_7$. The binary code of $\mathcal{D}$ is a $[42, 7, 6]_2$ code C with $\mathrm{Aut}(C) \cong \mathrm{Aut}(\mathcal{D})$ and with the weight enumerator

$$W(x) = 1 + 7x^6 + 21x^{12} + 35x^{18} + 35x^{24} + 21x^{30} + 7x^{36} + x^{42}.$$

The full space $V = {\mathbb{F}_2}^{42}$ is decomposable into $L_2(7)$-modules of dimensions 7 and 35 that is $V = C_7 \oplus C_{35}$, where C_7 is our code C. The code C is decomposable into $L_2(7)$-modules of dimensions 6 and 1, that is $C = C_6 \oplus \mathbb{F}_2\jmath$ and C_6 is not decomposable but it has two irreducible factors of dimension 3. We have $\dim(\mathrm{Soc}(V)) = 26$ and $\mathrm{Soc}(V) = C_3 \oplus C_8 \oplus \mathbb{F}_2\jmath \oplus C_3 \oplus C_8 \oplus C_3$.

$M = 7{:}3$, $nX \in \{3A, 7A\}$

First we notice that M is a maximal subgroup of index $7 + 1$ of $L_2(7)$. In [68], Moori has given the general parameters of the designs and codes obtained by using Method 3.2.5 on $L_2(q)$ and its maximal subgroup of degree $q + 1$.

Theorem 4.2.12. *For $G = L_2(q)$, let $M = G_1$ be the stabiliser of a point in the natural action of degree $q + 1$ on the set of all conjugate of $M \in G$.*

1. *Suppose $g \in nX \subseteq G$ is an element fixing exactly one point, and without loss of generality, assume $g \in M$. Then the replication number for the associated design is $r = \lambda = 1$. We also have*

   (i) If q is odd then $|g^G| = \frac{1}{2}(q^2 - 1)$, $|M \cap g^G| = \frac{1}{2}(q - 1)$, and $\mathcal{D}$ is a $(\frac{1}{2}(q^2 - 1), \frac{1}{2}(q - 1), 1)$ designs with $q + 1$ blocks and

   $$Aut(\mathcal{D}) = S_{\frac{1}{2}(q-1)} \wr S_{q+1} = (S_{\frac{1}{2}(q-1)})^{q+1}{:}S_{q+1}.$$

   For all p, $C = C_p(\mathcal{D}) = [\frac{1}{2}(q^2 - 1), q + 1, \frac{1}{2}(q - 1)]_p$, with $Aut(\mathcal{D}) = Aut(C)$.

   (ii) If q is even then $|g^G| = (q^2 - 1)$, $|M \cap g^G| = (q - 1)$, and $\mathcal{D}$ is a $((q^2 - 1), (q - 1), 1)$ design with $q + 1$ blocks and

   $$Aut(\mathcal{D}) = S_{(q-1)} \wr S_{q+1} = (S_{(q-1)})^{q+1}{:}S_{q+1}.$$

   For all p, $C = C_p(\mathcal{D}) = [(q^2 - 1), q + 1, (q - 1)]_p$, with $Aut(\mathcal{D}) = Aut(C)$.

2. *Now if we suppose that $g \in nX \subseteq G$ is an element fixing exactly two points, and without loss of generality, assume $g \in M = G_1$ and that $g \in G_2$. Then the replication number for the associated design is $r = \lambda = 2$. Moreover*

   (iii) If g is an involution, so that $q \equiv 1 \pmod 4$, the design $\mathcal{D}$ is a $1 - (\frac{1}{2}q(q + 1), q, 2)$ design with $q + 1$ blocks and $Aut(\mathcal{D}) = S_{q+1}$. Furthermore $C_2(\mathcal{D}) = [\frac{1}{2}q(q + 1), q, q]_2$,

$C_p(\mathcal{D}) = [\frac{1}{2}q(q+1), q, 2q]_p$ if p is an odd prime, and $Aut(C_p(\mathcal{D})) = Aut(\mathcal{D}) = S_{q+1}$ for all p.

(iv) If g is not an involution, the design $\mathcal{D}$ is a $1-(q(q+1), 2q, 2)$ design with $q+1$ blocks and $Aut(\mathcal{D}) = 2^{\frac{1}{2}q(q+1)}{:}S_{q+1}$. Furthermore $C_2(\mathcal{D}) = [q(q+1), q, 2q]_2$, $C_p(\mathcal{D}) = [q(q+1), q+1, 2q]_p$ if p is an odd prime, and $Aut(C_p(\mathcal{D})) = Aut(\mathcal{D}) = 2^{\frac{1}{2}q(q+1)}{:}S_{q+1}$ for all p.

Proof. For the proof we refer to [68, Method 2]. □

- Now a $g \in 3A$ is an element of order 3 having cycle type $1^2 3^2$ and g fixes exactly two points. So by Theorem 4.2.12 2.iv, we obtain a 1-(56, 14, 2) design $\mathcal{D}$ with 8 blocks and $Aut(\mathcal{D}) \cong 2^{28}{:}S_8$. The binary code of $\mathcal{D}$ is a $[56, 7, 14]_2$ code with $\text{Aut}(C) \cong \text{Aut}(\mathcal{D})$. The weight enumerator of C is

$$W(x) = 1 + 8x^{14} + 28x^{24} + 56x^{30} + 35x^{32}.$$

- A $g \in 7A$ is an element of order 7 having cycle type $1^1 7^1$ and g fixes exactly one point. Thus by Theorem 4.2.12 1.i we obtain a 1-(24, 3, 1) design $\mathcal{D}$ with 8 blocks and with $\text{Aut}(\mathcal{D}) \cong {S_3}^8{:}S_8$. The code of $\mathcal{D}$ is a $[24, 8, 3]_2$ code C with $\text{Aut}(C) \cong \text{Aut}(\mathcal{D})$. The weight enumerator of C is

$$W(x) = 1 + 8x^3 + 28x^6 + 56x^9 + 70x^{12} + 56x^{15} + 28x^{18} + 8x^{21} + x^{24}.$$

Max. sub.	Index	nX	Design ($\mathcal{D}$)	Code (C)	Aut($\mathcal{D}$)	Aut(C)
S_4	7	2A	1-(21,9,3)	$[21, 4, 9]_2$	${S_3}^7{:}L_2(7)$	${S_3}^7{:}L_2(7)$
		3A	1-(56, 8, 1)	$[56, 7, 8]_2$	${S_8}^7{:}S_7$	${S_8}^7{:}S_7$
		4A	1-(42, 6, 1)	$[42, 7, 6]_2$	${S_6}^7{:}S_7$	${S_6}^7{:}S_7$
7:3	8	3A	1-(56, 14, 2)	$[56, 7, 14]_2$	$2^{28}{:}S_8$	$2^{28}{:}S_8$
		7A	1-(24, 3, 1)	$[24, 8, 3]_2$	${S_3}^8{:}S_8$	${S_3}^8{:}S_8$

Table 4.2.6: Codes and designs from $L_2(7)$ using Key-Moori Method 2.

4.2.3 Observations

Proposition 4.2.11 reflects a common property to the designs constructed from Method 3.2.5. That method gives us a 1-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, that is every point belongs to exactly λ blocks. Here we follow the recent work in a forthcoming paper by T. Le and J. Moori (see [64]). If we consider a point x and if we denote by I_x the intersection of the λ blocks containing x, then $n = |I_x|$ is the same for any point of $\mathcal{P}$. Moreover if $y \in I_x$, then $I_x = I_y$. If we define on $\mathcal{P}$ the equivalence relation $\sim$ by $y \sim x \Leftrightarrow y \in I_x$, then the number of equivalence classes corresponding to $\sim$ is $v' = v/n$. If $[x]$ is a class of $\sim$, then $\mathcal{P}' = \{[x_1], \ldots, [x_{v'}]\}$ is the point set of a 1-(v', k', λ) design $\mathcal{D}'$ with $k' = k/n$; ${S_n}^{v'} \trianglelefteq \text{Aut}(\mathcal{D})$ and $\text{Aut}(\mathcal{D})$ / ${S_n}^{v'} \simeq \text{Aut}(\mathcal{D}')$. Table 4.2.7 gives a summary of the value of the parameter $n = |I_x|$ and the design $\mathcal{D}'$ corresponding to each design $\mathcal{D}$ in Table

4.2.6, we have used MAGMA to compute n and to construct $\mathcal{D}'$ directly from $\mathcal{D}$.
In Table 4.2.8 we can also notice that if $C' = [m', p, d']_2$ is the code from $\mathcal{D}'$ then the code from $\mathcal{D}$ is $C = [m, p, d]_2$ with $m = m'n$ and $d = d'n$.

1-(v, k, λ) design $\mathcal{D}$	$\lvert I_x \rvert$	1-(v', k', λ) design $\mathcal{D}'$	Aut($\mathcal{D}'$)	Aut($\mathcal{D}$) $\simeq {S_n}^{v'}$:Aut($\mathcal{D}'$)
1-(21, 9, 3)	3	1-(7, 3, 3)	$L_2(7)$	${S_3}^7{:}L_2(7)$
1-(56, 8, 1)	8	1-(7, 1, 1)	S_7	${S_8}^7{:}S_7$
1-(42, 6, 1)	6	1-(7, 1, 1)	S_7	${S_6}^7{:}S_7$
1-(56, 14, 2)	2	1-(28, 7, 2)	S_8	$2^{28}{:}S_8$
1-(24, 3, 1)	3	1-(8, 1, 1)	S_8	${S_3}^8{:}S_8$

Table 4.2.7: Designs from the conjugacy classes of $L_2(7)$

$\mathcal{D}$	n	$\mathcal{D}'$	code(C')	code(C)	Aut(C')	Aut(C)
1-(21, 9, 3)	3	1-(7, 3, 3)	$[7, 4, 3]_2$	$[21, 4, 9]_2$	$L_2(7)$	${S_3}^7{:}L_2(7)$
1-(56, 8, 1)	8	1-(7, 1, 1)	$[7, 7, 1]_2$	$[56, 7, 8]_2$	S_7	${S_8}^7{:}S_7$
1-(42, 6, 1)	6	1-(7, 1, 1)	$[7, 7, 1]_2$	$[42, 7, 6]_2$	S_7	${S_6}^7{:}S_7$
1-(56, 14, 2)	2	1-(28, 7, 2)	$[28, 7, 7]_2$	$[56, 7, 14]_2$	S_8	$2^{28}{:}S_8$
1-(24, 3, 1)	3	1-(8, 1, 1)	$[8, 8, 1]_2$	$[24, 8, 3]_2$	S_8	${S_3}^8{:}S_8$

Table 4.2.8

4.3 Designs and codes from $L_2(q)$, $q \in \{8, 9, 11, 13, 16\}$

For i, q_i and n_i in Table 4.3.1, we applied Method 3.2.1 to the groups $G = L_2(q_i)$ with Ω the set of all cosets of a maximal subgroup of index n_i. Using MAGMA [15] we considered the primitive representation of degree n_i of $L_2(q_i)$ and we formed the orbits of the point stabilizer. In [26, 27] M.R. Darafsheh et al. used the first method of Key and Moori and constructed 1-designs from the individual non-trivial orbits (that is not from the unions) of $L_2(q_i)$, but here we considered not only the non-trivial orbits but also any of their unions and constructed symmetric 1-designs according to Theorem 3.2.2. We look at the associated binary codes and the structures of the automorphism groups of the designs and the codes. We note here that in [26, 27] none of these codes were investigated. Table 4.3.1 gives the structure of the maximal subgroup of index n_i of $L_2(q_i)$, the number of orbits of the point stabilisers corresponding to the action of $L_2(q_i)$ on the cosests of these maximal subgroups and the lengths of the orbits of the point stabilisers with the number of orbits with that length in parenthesis in case there is more than one such orbits.

i	q_i	n_i	Max. Sub.	rank	length
1	8	28	D_{18}	4	1, 9(3)
2	8	36	D_{14}	5	1, 7(3), 14
3	9	15	S_4	3	1, 8, 6
4	11	55	D_{12}	9	1, 3(2), 6(4), 12(2)
5	13	78	D_{14}	9	1, 7(5), 14(3)
6	16	68	A_5	5	1, 12, 15, 20(2)
7	16	120	D_{34}	8	1, 17(7)

Table 4.3.1: Orbits of a point-stabiliser of $L_2(q_i)$

If G is a rank-r group, then we should obtain 2^{r-2} designs with block size > 1 and the same number of codes, but some of these structures will be isomorphic. In summary we have the following proposition:

Proposition 4.3.1. *(i) There exist exactly d_i non isomorphic symmetric 1-designs with n_i points on which $L_2(q_i)$ acts primitively on points and blocks and from these d_i symmetric 1-designs we can generate exactly c_i non isomorphic binary codes. The values of d_i and c_i are given in Table 4.3.2.*

(ii) δ_{ij} of these d_i designs and γ_{ij} of these c_i codes have automorphism groups isomorphic to A_{ij} as listed in Table 4.3.3.

Remark 4.3.2. In general the number of codes c_i is smaller than the number of designs because two non-isomorphic designs may generate the same code. For instance, for $L_2(16)$ of degree 120 there exist 6 non-isomorphic designs giving the same $[120, 44, 20]_2$ binary code, more precisely 3 of these non-isomorphic designs have the same parameters 1-(120, 52, 52) and the other 3 have the parameters 1-(120, 68, 68). The block sizes give a partial information about the weight distribution of the correponding code. The code has at least 3×120 words of weight 52 and 3×120 words of weight 68. The full weight enumerator of this code is

$$\begin{aligned} W(x) &= 1 + 2040x^{20} + 29495x^{24} + 454920x^{28} + 12816045x^{32} + 308806360x^{36} \\ &+ 5956858344x^{40} + 70439478240x^{44} + 467040670200x^{48} + \mathbf{1769618341680x^{52}} \\ &+ 3923105184315x^{56} + 5119220761136x^{60} + 3923105184315x^{64} + \mathbf{1769618341680x^{68}} \\ &+ 467040670200x^{72} + 70439478240x^{76} + 5956858344x^{80} + 308806360x^{84} + 12816045x^{88} \\ &+ 454920x^{92} + 29495x^{96} + 2040x^{100} + x^{120}. \end{aligned}$$

The designs and codes of Proposition 4.3.1 are listed in Appendix B.

i	q_i	n_i	d_i	c_i
1	8	28	6	3
2	8	36	14	7
3	9	15	6	5
4	11	55	286	31
5	13	78	318	23
6	16	68	22	9
7	16	120	86	11

Table 4.3.2: Number of designs and codes

j	i	A_{ij}	δ_{ij}	γ_{ij}
1	1	S_{28}	3	2
2		$L_2(8)$	3	1
1	2	$L_2(8)$	8	2
2		S_9	4	3
3		S_{36}	2	2
1	3	S_6	2	0
2		A_8	2	2
3		S_{15}	2	3
1	4	$L_2(11)$	221	4
2		$L_2(11){:}2$	60	20
3		S_{11}	3	4
4		S_{55}	2	3
1	5	$L_2(13)$	192	6
2		$L_2(13){:}2$	124	15
3		S_{78}	2	2
1	6	$L_2(16){:}2$	9	2
2		$L_2(16){:}4$	12	4
3		S_{68}	1	2
1	7	$L_2(16)$	49	3
2		$L_2(16){:}2$	24	3
3		$L_2(16){:}4$	8	2
4		$S_4(4){:}2$	4	1
5		S_{120}	1	2

Table 4.3.3: Automorphism groups

4.3.1 Two non-isomorphic 1-(120, 52, 52) designs and two non-isomorphic $[120, 10, 52]_2$ codes

We consider $L_2(16)$ of degree 120. As we can see from Table 4.3.1, there are 7 orbits of length 17. Firstly we take three orbits of length 17 namely the second, the fifth orbit and the eighth orbit (order in MAGMA) and join them with the orbit of length 1 to obtain a block of length 52. Now we apply Method 3.2.1 to construct a 1-(120,52,52) symmetric design $\mathcal{D}_1$. The full automorphism group of $\mathcal{D}_1$ is the group $S_4(4){:}2$. From $\mathcal{D}_1$ we can generate a doubly-even $[120, 10, 52]_2$ code $\mathcal{C}_1$ with the same automorphism group as $\mathcal{D}_1$. The weight enumerator of $\mathcal{C}_1$ is

$$W(x) = 1 + 120x^{52} + 255x^{56} + 272x^{60} + 255x^{64} + 120x^{68} + x^{120}.$$

The dual of $\mathcal{C}_1$ is a $[120, 110, 4]_2$ code and $\mathcal{C}_1 \subset \mathcal{C}_1^{\perp}$.
Secondly we consider the union of the third, the fifth and the sixth orbit with the orbit of length 1. Hence, we obtain another 1-(120, 52, 52) symmetric design $\mathcal{D}_2$, non isomorphic to $\mathcal{D}_1$, and with

the full automorphism group isomorphic to $L_2(16)$:2. From $\mathcal{D}_2$ we can generate a code $\mathcal{C}_2$ with the same parameters and weight distribution as $\mathcal{C}_1$, however $\mathcal{C}_1$ and $\mathcal{C}_2$ are non isomorphic. Indeed, the full automorphism group of $\mathcal{C}_2$ is $L_2(16)$:2, which is isomorphic to a subgroup of index 240 of $\mathrm{Aut}(\mathcal{C}_1)$.

4.4 2-designs and the ternary Golay code from $L_2(11)$

With a slight modification of the main idea used in Hughes's and Carmichael-Witt's methods we can construct the unique symmetric 2-(11,5,2) design from $L_2(11)$.

Theorem 4.4.1. *The group $L_2(11)$ acts 2-transitively on a set of size 11. In that action the point stabiliser is A_5 and the two points stabiliser H is S_3. A Sylow 2-subgroup of H fixes 3 points and a Sylow 3-subgroup of H fixes 2 points.*

Proof. For the first part of the theorem see [28, Example 7.5.2]. The Sylow 2-subgroups and Sylow 3-subgroups of H are cyclic of order 2 and 3, respectively. In A_5 we can see that a Sylow 2-subgroup of H is generated by $x \in 2A$ and the Sylow 3-subgroup is generated by $y \in 3A$. The permutation character of A_5 for its action on the cosets of S_3 is given by $\chi_1 + \chi_4 + \chi_5$ and hence we can deduce that the non-identity element x of a Sylow 2-subgroup of H will fix 3 points since

$$(\chi_1 + \chi_4 + \chi_5)(x) + 1 = 1 + 0 + 1 + 1 = 3.$$

Similarly any non-identity element of the Sylow 3-subgroup of H will fix 2 points since $(\chi_1 + \chi_4 + \chi_5)(y) + 1 = 2$. □

The Sylow 3-subgroup U of H is normal in H and by applying Method 3.1.2 to $L_2(11)$ with U we will obtain a trivial 2-(11,2,1) design. A Sylow 2-subgroup of H can not be normal since $H \cong S_3$, so Method 3.1.2 does not apply.

Under the action of the Sylow 2-subgroup, the 11 points spilt into three orbits of length 1 (the points fixed by the Sylow 2-subgroup) and four orbits of size 2. By Applying Method 3.1.4 to $L_2(11)$ with a Sylow 2-subgroup of H, we can construct a 2-(11,3,3) design $\mathcal{D}$ with 55 blocks.

If we take as blocks the orbits of the union of a fixed point and two orbits of size two then we will obtain one the following designs: a 2-(11,5,10) design or a 2-(11,5,60) design or a 2-(11,5,12) design or the unique 2-(11,5,2) symmetric design mentioned in Theorem 3.1.6 (iii).

Ternary Golay code

From the 2-(11,5,2) design we can construct the $[11, 6, 5]_3$ ternary Golay code $\mathcal{C}$ with $\mathrm{Aut}(\mathcal{C}) = M_{11}$:2. Under the action of $L_2(11)$ the codewords of weight 5 split into four orbits $(5)_1$, $(5)_2$, $(5)_3$, $(5)_4$ with $L_2(11)_{w_i} = A_5$ for $i = 1$ or 2 and $L_2(11)_{w_i} = D_{12}$ for $i = 3, 4$. Table 4.4.1 gives the weight distribution of $\mathcal{C}$ and the stabiliser of a word of weight i in $L_2(11)$, M_{11} and $M_{11}{:}2$.

i	$L_2(11)_{w_i}$	Max. sub.	i	$(M_{11})_{w_i}$	Max . sub.	i	$(M_{11}{:}2)_{w_i}$	Max. sub.
$(5)_1$	A_5	Yes	$(5)_1$	S_5	No	5	S_5	No
$(5)_2$	A_5	Yes						
$(5)_3$	D_{12}	Yes	$(5)_2$	S_5	No			
$(5)_4$	D_{12}	Yes						
$(6)_1$	A_5	Yes	6	A_5	No	6	S_5	No
$(6)_2$	A_5	Yes						
$(6)_3$	S_3	No						
$(8)_1$	D_{12}	Yes	$(8)_1$	$GL_2(3)$	No	8	$GL_2(3)$	No
$(8)_2$	D_{12}	Yes						
$(8)_3$	S_3	No	$(8)_2$	$GL_2(3)$	No			
$(8)_4$	S_3	No						
$(9)_1$	D_{12}	Yes	9	$3^2{:}8$	No	9	$(3^2{:}8){:}2$	No
$(9)_2$	D_{12}	Yes						
$(11)_1$	$L_2(11)$	Yes	$(11)_1$	$L_2(11)$	Yes	11	$L_2(11)$	No
$(11)_2$	$L_2(11)$	Yes						
$(11)_3$	A_5	Yes	$(11)_2$	$L_2(11)$	Yes			
$(11)_4$	A_5	Yes						

Table 4.4.1: Stabiliser of a word w_i in $L_2(11)$, M_{11} and $M_{11}{:}2$

4.5 Projective planes and generalized Reed-Muller codes from $L_3(q)$

In this section we examine the general parameters of the designs and codes that are built by applying Carmichael-Witt's method to $G = L_3(q)$.

Theorem 4.5.1. *Let q be some power of a prime number p. Consider $L_3(q)$ as a faithful 2-transitive group on a set Ω with $|\Omega| = q^2+q+1$. Let $H < L_3(q)$ be the stabiliser of two points.*

(i) If $q = p^n$, then H is a subgroup of order $q^2(q-1)^2/(3,q-1)$ having a normal Abelian Sylow p-subgroup U of order p^{2n}.

(ii) $|Fix(U)| = q+1$.

Proof. According to [9] any 2-transitive permutation representation of $L_3(q)$ on a set of size q^2+q+1 is isomorphic to the action of G on the points of the projective space $PG(2,q)$.
(i) We have $|L_3(q)| = q^3(q^3-1)(q^2-1)/(3,q-1)$ and $|H| = |L_3(q)|/q(q+1)(q^2+q+1) = q^2(q-1)^2/(3,q-1)$. According to [67], H is a subgroup of a line stabiliser K which is a maximal subgroup of order $(q-1)^2q^3(q+1)/(3,q-1)$ of $L_3(q)$. Let's consider the line $\ell = [1{:}0{:}0]$. The stabilisers of ℓ are the 3×3 matrices of the form $\left(\begin{smallmatrix}\alpha & \mathbf{v}_2\\ 0 & M_2\end{smallmatrix}\right)$ with $\alpha \in \mathbb{F}_q$, $\mathbf{v}_2$ a vector of length 2 with

entries in $\mathbb{F}_q$ and M_2 a 2×2 matrix over $\mathbb{F}_q$ such that $\alpha \cdot \det(M_2) = 1$. The set U of matrices of the form $\begin{pmatrix} \alpha & v_2 \\ 0 & I_2 \end{pmatrix}$ is an Abelian normal subgroup of order q^2 of K and it is the Sylow p-subgroup of H.
(ii) Consequence of the fact that $U<K$ and each line of $PG(2,q)$ has $q+1$ points. □

Hence we can apply Carmichael-Witt's method to $G = L_3(q)$ and we have the following result:

Corollary 4.5.2. *From the simple group $G = L_3(q)$ we obtain a 2-$(q^2+q+1, q+1, 1)$ design $\mathcal{D}$ on which G acts 2-transitively on points and transitively on blocks.*

Proof. Direct application of Method 3.1.2 to $L_3(q)$. □

Remark 4.5.3. If B is a block in $\mathcal{B}$ then $|\mathcal{B}| = [G : G_B]$. Since a block is just a line of $PG(2,q)$, $|G_B| = (q-1)^2q^3(q+1)/(3,q-1)$, $|\mathcal{B}| = q^2+q+1$ and $\mathcal{D}$ is symmetric.

The group G is isomorphic to a subgroup of $\mathrm{Aut}(\mathcal{D})$. The question is whether $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(\mathcal{D})$ or not. In [34] we can find the list of the maximal subgroups of $G = L_3(q)$. The group G has a maximal subgroup of the form $q^2{:}GL(2,q)$ which is of index q^2+q+1, the action of G on the cosets of this maximal subgroup gives us a permutation representation of degree q^2+q+1 of G. The subgroup $q^2{:}GL(2,q)$ is the point stabiliser in that representation and $H<q^2{:}GL(2,q)$. Considering that the set of subgroups of $L_3(q)$ isomorphic to $q^2{:}GL(2,q)$ split into two conjugacy classes, if the outer automorphism of G contains an involution which fuses these two conjugacy classes then $\mathrm{Aut}(G) \not< \mathrm{Aut}(\mathcal{D})$.

Theorem 4.5.4. *Let p be a prime number and C the $\mathbb{F}_p$ code of a symmetric design 2-$(m^2+m+1, m+1, 1)$.*

(i) If $p|m$, then $2 \le \dim C \le \frac{1}{2}(m^2+m+2)$.

(ii) If $p\nmid m$ and $p|m+1$, then $\dim C = m(m+1)$.

(iii) If $p\nmid m$ and $p\nmid m+1$, then $\dim C = m^2+m+1$.

Proof. Direct application of [63, Proposition 2.6]. □

Remark 4.5.5. We see that from $L_3(q)$, using Method 3.1.2, we can obtain a $\mathbb{F}_p$-code C with $\dim(C) < m(m+1)$ only for the prime p such that $q = p^n$. More precisely we have the following results:

Theorem 4.5.6. *Let p be any prime, $q = p^n$, and $\mathcal{D}$ a $2-(q^2+q+1, q+1, 1)$ symmetric design. Then the linear code C_p over $\mathbb{F}_p$ derived from $\mathcal{D}$ is a generalized Reed-Muller code and has dimension $\binom{p+1}{2}^n + 1$. The minimum-weight words of C_p are the scalar multiples of the incidence vectors of the blocks. Further, $Hull_p(\mathcal{D})$ has minimum weight $2q$ with the minimum-weight vectors the scalar multiples of the differences of the incidence vectors of two distinct blocks of $\mathcal{D}$. The minimum weight d of $C_p^{\perp}$ satisfies $q+p \le d \le 2q$, with equality at the lower bound if $p = 2$.*

Proof. [8] and [7, Theorem 6.3.1 and Theorem 6.4.2] □

Remark 4.5.7. Table 4.5.1 gives the result that we obtained by computation using MAGMA [15] for different values of q.

G Aut(G)	Degree	G_{xy}	p	U	$\lvert\mathrm{Fix}(U)\rvert$	$\mathcal{D}$	$\mathcal{C}_p(\mathcal{D})$	Aut($\mathcal{D}$)	Aut($\mathcal{C}_p(\mathcal{D})$)
$L_3(3)$ $L_3(3){:}2$	13 on the cosets of $3^2{:}2S_4 \cong 3^2{:}GL_2(3)$	$(S_3)^2$	3	3^2	4	2-(13,4,1)	$[13,7,4]_3$	$L_3(3)$	$2{\times}L_3(3)$
$L_3(4)$ $L_3(4){:}(2\times S_3)$	21 on the cosets of $2^4{:}A_5 \cong 2^4{:}GL_2(4)$	$2^4{:}3$	2	2^4	5	2-(21,5,1)	$[21,10,5]_2$	$L_3(4){:}S_3$	$L_3(4){:}S_3$
$L_3(5)$ $L_3(5){:}2$	31 on the cosets of $5^2{:}GL_2(5)$	$(5{:}4)^2$	5	5^2	6	2-(31,6,1)	$[31,16,6]_5$	$L_3(5)$	$L_3(5){:}2^2$
$L_3(7)$ $L_3(7){:}S_3$	57 on the cosets of $7^2{:}2{\cdot}L_2(7){:}2$ $\cong 7^2{:}GL_2(7)$	$((7^2{:}3){:}2):2$	7	7^2	8	2-(57,8,1)	$[57,29,8]_7$	$L_3(7){:}3$	$(6{\times}L_3(9)){:}3$
$L_3(8)$ $L_3(8){:}6$	73 on the cosets of $2^6{:}(7{\times}L_2(8))$ $\cong 2^6{:}GL_2(8)$	$(2^3{:}7)^2$	2	2^6	9	2-(73,9,1)	$[73,28,9]_2$	$L_3(8){:}3$	$L_3(8){:}3$
$L_3(9)$ $L_3(9){:}2^2$	91 on the cosets of $3^4{:}GL_2(9)$	$(3^2{:}8)^2$	3	3^4	10	2-(91,10,1)	$[91,37,10]_3$	$L_3(9){:}2$	1
$L_3(11)$ $L_3(11){:}2$	133 on the cosets of $11^2{:}(5{\times}2L_2(11).2)$ $\cong 11^2{:}GL_2(11)$	$(11{:}10)^2$	11	11^2	12	2-(133,12,1)	$[133,67,12]_{11}$	$L_3(11)$	1

Table 4.5.1: Some designs and codes from $L_3(q)$

4.6 Designs and codes from $U_3(q)$, $q \in \{4, 9\}$

4.6.1 Designs and codes from 2-transitive actions

We apply Method 3.1.2 to the 2-transitive groups $U_3(q)$ for $q = 2^2$ and $q = 3^2$. Method 3.1.2 apply to these groups because of the following result:

Theorem 4.6.1. *(i) Let $q = p^2$ with p a prime number. Then $U_3(q)$ is doubly-transitive on the set X of isotropic points with $|X| = 1 + q^3$.*

(ii) The subgroup H fixing two points is cyclic of order $(q^2 - 1)$ and H has a unique normal subgroup W of order $q + 1$ which fixes $q + 1$ points.

Proof. See [74]. □

Corollary 4.6.2. *From the groups $U_3(4)$ and $U_3(9)$ we can construct 2-$(65, 5, 1)$ and 2-$(730, 10, 1)$ designs $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. The automorphism groups of $\mathcal{D}_1$ and $\mathcal{D}_2$ are $P\Gamma U_3(4)$ and $P\Gamma U_3(9)$, respectively.*

1 Could not be obtained in a reasonable time with the computational resources we had

Proof. In $U_3(4)$, the subgroup H is a subgroup of order 15 and according to Theorem 4.6.1 the unique Sylow 5-subgroup of H is the subgroup W itself and it fixes 5 points. In $U_3(9)$, the subgroup H is a subgroup of order 80 and since W is a subgroup of order 10, the Sylow 5-subgroup U of H is the unique subgroup of order 5 of W and U fixes 10 points. Now by applying Method 3.1.2 we obtain the above designs. We can get the automorphism of the corresponding designs by direct computation with MAGMA or just by noticing that the designs are hermitian unitals with automorphism groups isomorphic to $P\Gamma U_3(q)$ (cf. [74]). □

Remark 4.6.3. The unitals can be constructed by using some combinatorial properties of the set of isotropic subspaces of a 3-dimensional vector space over a finite field V endowed with a bilinear form b (see [74]).

Theorem 4.6.4. *Let $\mathcal{D}$ be a 2-$(q^3+1, q+1, 1)$ design, p be a prime number and A be the incidence matrix of $\mathcal{D}$ then*

$$rank_p(A) = \begin{cases} q(q^2-q+1), & \text{if } p|q+1 \\ q^3+1, & \text{otherwise.} \end{cases}$$

Proof. Conjectured in [2] and proved in [44] □

Remark 4.6.5. We can see that the only p-ary codes with dimension less than 730 that we can construct from $\mathcal{D}_2$ are binary and 5-ary codes. The only p-ary codes with dimension less than 65 that we can derive from $\mathcal{D}_1$ are 5-ary codes.

Corollary 4.6.6. *From $\mathcal{D}_1$ we can construct a $[65, 52, 5]_5$ linear code C_1 and from $\mathcal{D}_2$ we can construct linear codes $C_2 = [730, 657]_5$ and $C_2' = [730, 657]_2$.*

Remark 4.6.7. The minimum distance of C_2 is between 5 and 10, $C_2^{\perp} = [730, 73]_5$ and the minimum distance of $C_2^{\perp}$ is between 6 and 462. The minimum distance of C_2' is between 6 and 10, $C_2'^{\perp} = [730, 73]_2$ and the minimum distance of $C_2'^{\perp}$ is between 10 and 180 . The dual of C_1 is $C_1^{\perp} = [65, 13, 36]_5$ and $\mathrm{Aut}(C_1) = P\Gamma U_3(4) \times 4$.

4.6.2 Designs and codes from primitive representations

We applied Key-Moori Method 1 and 2 to the primitive representations of $U_3(4)$ and $U_3(9)$. The 2-transitive actions mentioned in Theorem 4.6.1 correspond to the primitive representations of degree 65 and 730 of $U_3(4)$ and $U_3(9)$, respectively and their point stabilisers have orbits of length 1, 64 and 1, 729, respectively. Hence Applying Key-Moori Method 1 on these groups with these representations will give us 1-(65, 64, 64) and 1-(730, 729, 729) symmetric designs.

Case of $U_3(4)$

Apart from the representation of degree 65 used in the previous section (Section 4.6.1), the group $U_3(4)$ has three others primitive representations corresponding to the maximal subgroups listed in Table 4.6.1.

Max. sub	Degree
2^{2+4}:15	65
$5\times A_5$	208
5^2:S_3	416
13:3	1600

Table 4.6.1: Maximal subgroups of $U_3(4)$

We applied Key-Moori Method 1 to the primitive representations of degree 208 and 416 of $U_3(4)$. In the primitive representation of degree 208 the point stabiliser is $5\times A_5$ and it has 5 orbits of length 1, 12, 60, 60 and 75 respectively. For each of these orbits we found the designs, as described in Key-Moori Method 1, and their corresponding codes.

1. For the orbit of length 12, and the two orbits of length 60 we obtain a 1-(208,12,12) design and two isomorphic 1-(208,60,60) designs respectively. They have the same full automorphism group isomorphic to $U_3(4)$:4. These designs give the same $[208, 80, 12]_2$ binary code with the full automorphism group $U_3(4)$:4.

2. For the orbit of length 75 we obtain a 1-(208,75,75) design with the full automorphism group $U_3(4)$:4 and a $[208, 144, 10]_2$ binary code.

The group $U_3(4)$ contains 195 elements of order 2 that form a unique conjugacy class 2A. For the three smallest maximal subgroups of $U_3(4)$ we applied Key-Moori Method 2 with 2A. Table 4.6.2 lists the designs and codes hence constructed.

Max . sub.	Design ($\mathcal{D}$)	Aut($\mathcal{D}$)	Code ($\mathcal{C}$)	Aut($\mathcal{C}$)
2^{2+4}:15	1-(195,3,1)	$(S_3)^{65}$:S_{65}	$[195, 65, 3]_2$	$(S_3)^{65}$:S_{65}
$5\times A_5$	1-(195,15,16)	$(S_3)^{65}$:$(U_3(4)$:4)	$[195, 65, 3]_2$	$(S_3)^{65}$:S_{65}
5^2:S_3	1-(195,15,32)	$U_3(4)$:4	$[195, 117, 6]_2$	[2]

Table 4.6.2: Designs and codes from the conjugacy class of element of order 2 of $U_3(4)$

2 Could not be obtained in a reasonable time with the computational resources we had.

Case of $U_3(9)$

Apart from the representation of degree 730 used in Section 4.6.1, $U_3(9)$ has four primitive representations of degree 5913, 59130, 70956 and 194400, respectively. These four degrees are too large to compute with MAGMA.

4.7 Some designs and codes from the symplectic groups $S_n(2)$, $n \in \{6, 8\}$

4.7.1 Designs and codes from 2-transitive actions

Let $n \in \mathbb{N}$ and V be a vector space of dimension $2n$ over $\mathbb{F}_2$, V is endowed with a symplectic form $b : V \times V \to \mathbb{F}_2$, $(u, v) \mapsto b(u, v) = ufv^T$ with $f = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = e - e^T$ where $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The quadratic form θ_0 is defined over V by $\theta_0(u) = ueu^T$, furthermore we define the sets $\Omega = \{\theta_a | a \in V\}$ where $\theta_a(u) = \theta_0(u) + b(u, a)$, $\Omega^+ = \{\theta_a | \theta_0(a) = 0\}$ and $\Omega^- = \{\theta_a | \theta_0(a) = 1\}$.

Theorem 4.7.1. *For $n \geq 2$ the group $S_{2n}(2)$ acts 2-transitively on $\Omega^\pm$ and in these actions we have:*

(i) The point stabiliser is the group $GO^\pm_{2n}(2)$ with $[S_{2n}(2){:}GO^\pm_{2n}(2)] = 2^{n-1}(2^n \pm 1)$,

(ii) the two points stabiliser is $2^{2n-2}{:}GO^\pm_{2n-2}(2)$.

Proof. (i) [28, Theorem 7.7A] and [10].
(ii) We fix a point, according to [87, Section 3.7.2] the $2^{n-1}(2^n \pm 1) - 1 = (2^n \mp 1)(2^{n-1} \pm 1)$ remaining points can be seen as the isotropic vectors of a vector space of dimension $2n$. Therefore the two points stabiliser is just the stabiliser of an isotropic vector in $GO^\pm(2n, 2)$ which is $2^{2n-2}{:}GO^\pm_{2n-2}(2)$. □

We applied Method 3.1.4 to the groups $S_{2n}(2)$ for $n = 3, 4$.

Computation for $S_6(2)$ when acting on Ω^-

The symplectic group $S_6(2)$ is 2-transitive in its action on the 28 cosets of its maximal subgroup $GO^-_6(2) \cong U_4(2){:}2$. In this action the two points stabiliser H is $2^4{:}GO^-_4(2) \cong 2^4{:}S_5$.

1. The group H has fifteen Sylow 2-subgroups of order 2^7 which are isomorphic to $D_8^2{:}2$ and each fixes exactly 2 points. So by applying Method 3.1.4 to $S_6(2)$ we obtain a trivial 2-(28,2,1) design.

2. The group H has forty Sylow 3-subgroups which are cyclic groups of order 3 and generated by elements of order 3 from the class 3A of $S_6(2)$.

Proposition 4.7.2. *From the fixed points of a Sylow 3-subgroup of H we can construct a 2-(28,10,40) design $\mathcal{D}$ with $Aut(\mathcal{D}) \cong S_6(2)$.*

Proof. Using the character table of $G = S_6(2)$, the character of the permutation representation of degree 28 of G is equal to $\chi_1 + \chi_6 = 1a + 27a$ and hence we have $|\mathrm{Fix}(U)| = (\chi_1 + \chi_6)(g) = 1 + 9$ where $g \in 3A$ and $U = \langle g \rangle$. Using MAGMA we can easily see that $[H{:}H_{\mathrm{Fix}(U)}] = 40$. Thus by applying Method 3.1.4 to G with a Sylow 3-subgroup of H we can construct a 2-(28,10,40) design $\mathcal{D}$. Computation with MAGMA shows that $|\mathrm{Aut}(\mathcal{D})| = |S_6(2)|$. Since $S_6(2)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathcal{D})$, we have $\mathrm{Aut}(\mathcal{D}) \cong S_6(2)$. $\square$

From the incidence matrix of $\mathcal{D}$ we can construct a binary code C which is a $[28, 21, 4]_2$ code with the weight enumerator

$$\begin{aligned} W(x) = {} & 1 + 315x^4 + 6048x^6 + 47817x^8 + 206976x^{10} + 472059x^{12} + 630720x^{14} + 472059x^{16} \\ & + 206976x^{18} + 47817x^{20} + 6048x^{22} + 315x^{24} + x^{28}. \end{aligned}$$

The full automorphism group of C is $S_6(2)$ and $C^\perp$ is a $[28, 7, 12]_2$ code. The weight enumerator of $C^\perp$ is $W(x) = 1 + 63x^{12} + 63x^{16} + x^{28}$.

3. The group H has ninety six Sylow 5-subgroups which are cyclic groups of order 5, they are generated by elements of order 5 from the class 5A of $S_6(2)$. In this case the application of Method 3.1.4 to $S_6(2)$ gives the following proposition:
   Proposition 4.7.3. *From the fixed points of a Sylow 5-subgroups of H we can construct a 2-(28,3,16) design $\mathcal{D}$ with $Aut(\mathcal{D}) \cong S_6(2)$.*

   Proof. Same arguments as in Proposition 4.7.2 with $g \in 5A$ and U a Sylow 5-subgroup of H. $\square$

   From the incidence matrix of $\mathcal{D}$ we can construct a $[28, 27, 2]_3$ code C.

Computation for $S_6(2)$ when acting on Ω^+

The symplectic group $S_6(2)$ is 2-transitive in its action on the 36 cosets of its maximal subgroup $GO_6^+(2) \cong S_8$. In this action the two points stabiliser H is $2^4{:}GO_4^+(2) \cong (S_4)^2{:}2$.

1. The group H has nine Sylow 2-subgroups which are isomorphic to ${D_8}^2{:}2$, using MAGMA we can prove that each of them fixes exactly 2 points. So by applying Method 3.1.4 to $S_6(2)$ we obtain a trivial 2-(36,2,1) design.

2. The group H has sixteen Sylow 3-subgroups which are isomorphic to 3^2, using MAGMA we can prove that each of them fixes 3 points. Hence by applying Method 3.1.4 to $S_6(2)$ we will produce a 2-(36,3,16) design $\mathcal{D}$ with 3360 blocks. The full automorphism of $\mathcal{D}$ is $S_6(2)$.

Construction with MAGMA shows that for $p \in \{2,3,5,7\}$ the dimension of the p-ary code from this design is at least 35.

Computation for $S_8(2)$ when acting on Ω^-

The group $S_8(2)$ is 2-transitive on the 120 cosets of its maximal subgroup $GO_8^-(2)$. The two points stabiliser H is $2^6{:}GO_6^-(2) \cong 2^6{:}U_4(2)$.

1. A Sylow 2-subgroup of H is isomorphic to $(4{\times}2){:}D_8$, using MAGMA we can prove that it fixes 2 points. So by applying Method 3.1.4 to $S_8(2)$ we will obtain a trivial 2-(120,2,1) design.
2. A Sylow 3-subgroup of H is isomorphic to $3^3{:}3$ and it fixes 3 points. Thus by applying Method 3.1.4 to $S_8(2)$ we can construct a 2-(120,3,64) design $\mathcal{D}$ with $\mathrm{Aut}(\mathcal{D}) = S_8(2)$. For $p \in \{2,3,5,7\}$ the dimension of the p-ary code from $\mathcal{D}$ is at least 119.
3. A Sylow 5-subgroup of H is a cyclic group of order 5 and computation with MAGMA shows that it fixes 10 points. In this case the application of Method 3.1.4 to $S_8(2)$ gives the following proposition:

   Proposition 4.7.4. *From the fixed points of a Sylow 5-subgroup of H we can construct a 2-(120,10,3456) design $\mathcal{D}$ with $Aut(\mathcal{D}) \cong S_8(2)$.*

   Proof. Same arguments as in Proposition 4.7.2. □

   A binary code $C = [120, 111, 4]_2$ can be constructed from the incidence matrix of $\mathcal{D}$ with $\mathrm{Aut}(C) = S_8(2)$. The dual of C is $C^\perp = [120, 9, 56]_2$. The weight enumerator of $C^\perp$ is $W(x) = 1 + 255x^{56} + 255x^{64} + x^{120}$.

4.7.2 Designs and codes from primitive representations

We consider the primitive representations of $S_6(2)$ and $S_8(2)$. For each group we construct designs as described in Key-Moori Method 1 and 2. and the binary codes derived from these designs.

Case of $S_6(2)$

The symplectic group $S_6(2)$ has eight primitive representations of degree 28, 36 ,63, 120 , 135, 315, 336 and 960, respectively . The first column of Table 4.7.1 gives the degree of these primitive representations, the second column gives the structure of the point stabilisers which are maximal subgroups of $S_6(2)$, the third one contains the number of orbits of the point stabiliser and the last column gives the length of each orbits. The representation of degree 28 and 36 are 2-transitive and we used them in the previous section to construct 2-designs. For each of the remaining primitive non 2-transitive representations we formed symmetric 1-designs as described in Key Moori Method

1. We found also the binary codes associated with these designs. The results are summarized in Table 4.7.2, the column 'Block' gives the length of the orbits constituting the blocks of the designs.

Degree	Max. sub.	Rank	Length
28	$U_4(2)$:2	2	1, 27
36	S_8	2	1, 35
63	2^5:S_6	3	1, 30, 32
120	$U_3(3)$:2	3	1, 56, 63
135	2^6:$L_3(2)$	4	1, 14, 56, 64
315	$(2^{1+4}\times 2^2)$:$(S_3)^2$	5	1, 18, 24, 128, 144
336	$S_3\times S_6$	5	1, 20, 45, 90, 180
960	$L_2(8)$:3	6	1, 56, 63, 84, 252, 504

Table 4.7.1: Maximal subgroups and their orbits

Max. sub.	Block	Design ($\mathcal{D}$)	Aut($\mathcal{D}$)	Code ($\mathcal{C}$)	Aut($\mathcal{C}$)
2^5:S_6	1,30	1-(63,31,31)	$L_6(2)$	$[63,7,31]_2$	$L_6(2)$
	32	1-(63,32,32)	$L_6(2)$	$[63,6,32]_2$	$L_6(2)$
$U_3(3)$:2	1,63	1-(120,64,64)	$O_8^+(2)$	$[120,9,56]_2$	$S_8(2)$
	56	1-(120,56,56)	$O_8^+(2)$	$[120,8,56]_2$	$O_8^+(2)$
2^6:$L_3(2)$	1,14	1-(135,15,15)	$O_8^+(2)$	$[135,43,15]_2$	$O_8^+(2)$
	1,14,56	1-(135,71,71)	$O_8^+(2)$:2	$[135,9,63]_2$	$O_8^+(2)$:2
$2^{1+4}\times 2^2$:$(S_3)^2$	24	1-(315,24,24)	$S_6(2)$	$[315,96,24]_2$	$S_6(2)$
	128	1-(315,128,128)	$S_6(2)$	$[315,14,120]_2$	$S_6(2)$
	24,128	1-(315,152,152)	$S_6(2)$	$[315,96,24]_2$	$S_6(2)$
	24,144	1-(315,168,168)	$S_6(2)$	$[315,104,24]_2$	$S_6(2)$
	128,144	1-(315,272,272)	$S_6(2)$	$[315,48,48]_2$	$S_6(2)$
$S_3\times S_6$	1,45	1-(336,46,46)	$S_6(2)$	$[336,50,46]_2$	$S_6(2)$

Table 4.7.2: Symmetric 1-designs and binary codes from $S_6(2)$

The group $S_6(2)$ has 5103 elements of order 2 that split into four conjugacy classes 2A, 2B, 2C, 2D of size 63, 315, 945 and 3780, respectively. We consider the class 2A and applied Key-Method 2 on the different maximal subgroups of $S_6(2)$. The results of the computation are gathered in Table 4.7.3 and the details of the weight distribution of each code are in Table 4.7.4 .

Remark 4.7.5. The centralizer of an element of 2A is the maximal subgroup $2^5{:}S_6$, hence the 1-design obtained from the conjugacy class 2A inside $2^5{:}S_6$ is a symmetric 1-design which is isomorphic to the symmetric 1-(63,31,31) design constructed from the union of the orbits of length 1 and 30 of $2^5{:}S_6$. This symmetric 1-design is in fact a symmetric 2-(63,31,15) design.

Max. sub.	Design ($\mathcal{D}$)	Aut($\mathcal{D}$)	Code ($\mathcal{C}$)	Aut($\mathcal{C}$)
$U_4(2){:}2$	1-(63,36,16)	$S_6(2)$	$[63,7,28]_2$	$S_6(2)$
S_8	1-(63,28,16)	$S_6(2)$	$[63,7,28]_2$	$S_6(2)$
$2^5{:}S_6$	2-(63,31,15)	$L_6(2)$	$[63,7,31]_2$	$L_6(2)$
$U_3(3){:}2$	1-(63,7,15)	$S_6(2)$	$[63,36,7]_2$	$S_6(2)$
$2^6{:}L_3(2)$	1-(63,15,75)	$S_6(2)$	$[63,21,15]_2$	$S_6(2)$
$2^{1+4}{\times}2^2{:}(S_3)^2$	1-(63,18,96)	$S_6(2)$	$[63,56,4]_2$	$L_6(2)$

Table 4.7.3: Designs and codes from the conjugacy class 2A of $S_6(2)$

Table 4.7.4: Weight distributions of the codes from Table 4.7.3

i	$[63,7,28]_2$	$[63,7,28]_2$	$[63,7,31]_2$	$[63,36,7]_2$	$[63,21,15]_2$	$[63,56,4]_2$
0	1	1	1	1	1	1
4						9765
6						1057224
7				135		
8				945		60544953
10						1996794072
11				6048		
12				26208		41694856749
13				60480		
14				216000		584173436400
15				970599	315	
16				2911797	945	5724932809365
17				7197120		
18				18392640		40448633569680
19				48394080		
20				106466976		210758816714985
21				192487680		
22				367476480		823875120414360
23				744822099	54936	
24				1241370165	91560	2447745309517725
25				1707713280		
26				2495888640		5580858785942664
27				3872806336	193536	
28	36	36		4979322432	248832	9832942289229633

Continued on next page

Table 4.7.4 – continued from previous page

i	[63,7,28]	[63,7,28]	[63,7,31]	[63,36,7]	[63,21,15]	[63,56,4]
29				5304700800		
30				6011994240		13449656041565856
31			63	7256513187	458451	
32	63	63	63	7256513187	458451	14317376396958243
33				6011994240		
34				5304700800		11867343566087520
35				4979322432	248832	
36	28	28		3872806336	193536	7647844002734159
37				2495888640		
38				1707713280		3818482327223928
39				1241370165	91560	
40				744822099	54936	1468647185710635
41				367476480		
42				192487680		431553634502760
43				106466976		
44				48394080		95799462143175
45				18392640		
46				7197120		15827726179440
47				2911797	945	
48				970599	315	1908310936455
49				216000		
50				60480		163568562192
51				26208		
52				6048		9621890019
54						369776680
55				945		
56				135		8649279
58						109368
60						651
63			1	1	1	

Case of $S_8(2)$

The symplectic group $S_8(2)$ has eleven primitive permutation representations. Table 4.7.5 list the three smallest. In Section 4.7.1, we used the 2-transitive representation of degree 120 to built 2-designs.

We applied Key-Moori Method 1 to the primitive representation of degree 255.

1. The group $L_2(8)$ is the full automorphism group of the 1-(255,127,127) design (obtained by taking the union of the orbits of length 126 and 1) and it is also the automorphism group of the $[255, 9, 127]_2$ binary codes of this design.

Degree	Max. sub.	Rank	Length
120	$O_8^-(2){:}2$	2	1, 119
136	$O_8^+(2){:}2$	2	1, 135
255	$2^7{:}S_6(2)$	3	1, 126, 128

Table 4.7.5: Maximal subgroups and their orbits

2. From the orbit of length 128 we obtain a 1-(255,128,128) design with $L_2(8)$ as the automorphism group, it is also the automorphism group of the $[255, 8, 128]_2$ code C from the design. The weight enumerator of C is $W(x) = 1 + 128x^{255}$. The generator matrix of C is the $8 \times (2^8 - 1)$ matrix of all non zero binary vectors of length 8, so $C^{\perp}$ is the Hamming code $[255, 247, 3]_2$.

5 Designs and codes from A_n

In this chapter we will look at the designs and codes that we can obtain by applying our four methods to the alternating group A_n . In Section 5.1 we study the parameters of the symmetric 1-designs and the corresponding codes that we have produced by applying Key-Moori Method 1 on the primitive representation of degree $\binom{n}{2}$ of A_n. We will illustrate these results by giving the designs and codes that we obtain for $n \in \{6, 7, 8, 9, 10, 11\}$. For the same values of n, in Section 5.2, we will look at the 1-designs and codes that we can construct from some other primitive representations of A_n. In Section 5.3 we will study, in general, the structure of the 2-designs that can be constructed from the natural 2-transitive action of A_n on n points. We will devote the last section (Section 5.4) to study the design and code that we can construct using Carmichael-Witt's method on the primitive representation of degree 15 of A_7.

5.1 1-designs and codes from the primitive representations of degree $\binom{n}{2}$

The alternating group A_n acts as a rank-3 primitive group on the cosets of S_{n-2}. The orbits of the point stabiliser, namely $\{\alpha\}$, Δ_1, Δ_2 have lengths 1, $2(n-2)$ and $(n-2)(n-3)/2$, respectively. When Key-Moori Method 1 is applied on each Δ_1 and Δ_2 , we obtain the following results:

Proposition 5.1.1. *Using Key-Moori Method 1 with the orbits of lengths $2(n-2)$ and $(n-2)(n-3)/2$, we can construct 1-$(\binom{n}{2}, 2(n-2), 2(n-2))$ and 1-$(\binom{n}{2}, \binom{n-2}{2}, \binom{n-2}{2})$ designs, respectively. The binary code of the first design is*

1.1 a $[\binom{n}{2}, n-1, n-1]_2$ code when n is odd,

1.2 a $[\binom{n}{2}, n-2, 2(n-2)]_2$ code when n is even.

The binary code of the second design is

2.1 the full space $\mathbb{F}_2^{\binom{n}{2}}$ if $n \equiv 0 \pmod 4$,

2.2 a $[\binom{n}{2}, \binom{n}{2} - n + 1, 3]_2$ code if $n \equiv 1 \pmod 4$,

2.3 a $[\binom{n}{2}, \binom{n}{2} - 1, 2]_2$ code if $n \equiv 2 \pmod 4$,

2.4 a $[\binom{n}{2}, \binom{n}{2} - n, 4]_2$ code if $n \equiv 3 \pmod 4$.

Proof. The group S_{n-2} has $\binom{n}{2}$ conjugates in A_n and the parameters of the designs come from a direct application of Key-Moori Method 1. For the codes we refer to [57] and [31] where the authors use the results of [39] on the binary codes of a triangular graph. □

For $n \in \{6,7,8,9,10,11\}$, we applied also Key-Moori Method 1 on the union of any two orbits, namely $\{\alpha\} \cup \Delta_1$, $\{\alpha\} \cup \Delta_2$ and $\Delta_2 \cup \Delta_1$, to obtain 1-$(\binom{n}{2}, 2n-1, 2n-1)$, 1-$(\binom{n}{2}, \binom{n-2}{2}+1, \binom{n-2}{2}+1)$ and 1-$(\binom{n}{2}, \binom{n}{2}-1, \binom{n}{2}-2)$ symmetric designs, respectively. Table 5.1.1 gives a summary of the parameters of the obtained designs and the corresponding codes. The column "Block" gives the length of the orbits that form the blocks of the designs.

Table 5.1.1: Designs and codes from the representation of degree $\binom{n}{2}$

n	Block	design ($\mathcal{D}$)	Aut($\mathcal{D}$)	code ($\mathcal{C}$)	Aut ($\mathcal{C}$)
	6	1-(15,6,6)	S_6	$[15,4,2]_2$	S_{15}
	1,6	1-(15,7,7)	A_8	$[15,5,7]_2$	A_8
6	8	1-(15,8,8)	A_8	$[15,4,8]_2$	A_8
	1,8	1-(15,9,9)	S_6	$\mathbb{F}_2^{15}$	S_{15}
	6,8	1-(15,14,14)	S_{15}	$[15,4,2]_2$	S_{15}
	10	1-(21,10,10)	S_7	$[21,14,4]_2$	S_7
7	1,10	1-(21,11,11)	S_7	$[21,15,3]_2$	S_7
	10	1-(21,10,10)	S_7	$[21,6,6]_2$	S_7
	1,10	1-(21,11,11)	S_7	$[21,7,6]_2$	S_7
	12	1-(28,12,12)	S_8	$[28,6,12]_2$	S_8
	1,12	1-(28,13,13)	S_8	$\mathbb{F}_2^{28}$	S_{28}
8	15	1-(28,15,15)	S_8	$\mathbb{F}_2^{28}$	S_{28}
	1,15	1-(28,16,16)	S_8	$[28,7,12]_2$	$S_6(2)$
	12,15	1-(28,27,27)	S_{28}	$\mathbb{F}_2^{28}$	S_{28}
	14	1-(36,14,14)	S_9	$[36,9,8]_2$	S_9
	1,14	1-(36,15,15)	S_9	$[36,28,3]_2$	S_9
9	21	1-(36,21,21)	S_9	$[36,8,8]_2$	S_9
	1,21	1-(36,22,22)	S_9	$[36,28,3]_2$	S_9
	14,21	1-(36,35,35)	S_{36}	$\mathbb{F}_2^{36}$	S_{36}
	16	1-(45,16,16)	S_{10}	$[45,8,16]_2$	S_{10}
	1,16	1-(45,17,17)	S_{10}	$\mathbb{F}_2^{45}$	S_{45}
10	28	1-(45,28,28)	S_{10}	$[45,44,2]_2$	S_{45}
	1,28	1-(45,29,29)	S_{10}	$[45,9,16]_2$	S_{10}
	16,28	1-(45,44,44)	S_{45}	$[45,44,2]_2$	S_{45}
	18	1-(55,18,18)	S_{11}	$[55,10,10]_2$	S_{11}
11	1,18	1-(55,19,19)	S_{11}	$[55,45,3]_2$	S_{11}
	36	1-(55,36,36)	S_{11}	$[55,44,4]_2$	S_{11}
	1,36	1-(55,37,37)	S_{11}	$[55,11,10]_2$	S_{11}
	18,36	1-(55,54,54)	S_{55}	$[55,54,2]_2$	S_{55}

5.2 1-designs and codes from the other primitive representations

For $n \in \{7,8,9,10,11\}$, Table 5.2.1 gives the list of some maximal subgroups of A_n, the degree of its primitive representations corresponding to the actions on the cosets of these maximal subgroups, the number of orbits of the point stabilisers (rank) and the length of these orbits. For each of these representations, we found the designs as described by Key-Moori Method 1. The primitive representations of degree d corresponding to rank-2 actions will give us a 1-$(d, d-1, d-1)$ symmetric designs with automorphism groups isomorphic to S_d. For A_7 of degree 35, A_8 of degree $d \in \{35, 56\}$, A_9 of degree $d \in \{84, 120\}$, A_{10} of degree $d \in \{120, 126\}$ and A_{11} of degree 165, we formed the symmetric 1-design from any union of the orbits of the point stablisers and the corresponding binary codes. The results are summarized in Table 5.2.2.

n	Max . sub.	Degree	Rank	Length
7	A_6	7	2	1,6
	$L_2(7)$	15	2	1,14
	$(A_4 \times 3)$:2	35	3	1,4,12,18
8	A_7	8	2	1,7
	2^3:$L_3(2)$	15	2	1,7
	2^4:$(S_3)^2$	35	3	1,16,18
	$(A_5 \times 3)$:2	56	4	1,10,15,30

n	Max. sub.	Degree	Rank	Length
9	$A_6 \times 3$:2	84	4	1,18,20,45
	$L_2(8)$:3	120	3	1,56,63
10	A_9	10	2	1,9
	$A_7 \times 3$:2	120	4	1,21,35,63
	$(A_5 \times A_5)$:4	126	3	1,25,100
11	A_{10}	11	2	1,10
	$A_8 \times 3$:2	165	4	1,24,56,84

Table 5.2.1: Some maximal subgroups of A_n and their orbits

Table 5.2.2: 1-designs and codes from A_n $(7 \leq n \leq 11)$

i	n	Block	design $(\mathcal{D}_i)$	Aut$(\mathcal{D}_i)$	code $(\mathcal{C}_i)$	Aut$(\mathcal{C}_i)$
1	7	18	1-(35,18,18)	S_8	$[35,34,2]_2$	S_{35}
2		4 12	1-(35,16,16)	S_8	$[35,6,16]_2$	S_8
3		4 18 12	1-(35,34,34)	S_{35}	$\cong \mathcal{C}_1$	
4		18 12	1-(35,30,30)	S_7	$[35,20,6]_2$	S_7
5		1 12	1-(35,13,13)	S_7	$[35,15,5]_2$	S_7
6		12	1-(35,12,12)	S_7	$[35,20,4]_2$	S_7
7		1 18	1-(35,19,19)	S_8	$[35,7,15]_2$	S_8
8		1 18 12 4	1-(35,35,1)	S_{35}	$[35,1,35]_2$	S_{35}
9		1 12 4	1-(35,17,17)	S_8	$[35,35,1]_2$	S_{35}
10		4	1-(35,4,4)	S_7	$\cong \mathcal{C}_6$	
11		1 4	1-(35,5,5)	S_7	$[35,21,5]_2$	S_7
12		1 18 4	1-(35,23,23)	S_7	$[35,21,4]_2$	S_7
13		18 4	1-(35,22,22)	S_7	$[35,14,8]_2$	S_7
14		1 18 12	1-(35,31,31)	S_7	$\cong \mathcal{C}_{12}$	
15		1 18	1-(35,19,19)	S_8	$[35,7,15]_2$	S_8
17		16	1-(35,16,16)	S_8	$[35,6,16]_2$	S_8

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Table 5.2.2 – continued from previous page

i	n	Block	design ($\mathcal{D}_i$)	Aut($\mathcal{D}_i$)	code ($\mathcal{C}_i$)	Aut ($\mathcal{C}_i$)
18		18	1-(35,18,18)	S_8	$[35,34,2]_2$	S_{35}
19		16 1 18	1-(35,35,1)	S_{35}	$[35,1,35]_2$	S_{35}
20		16 18	1-(35,34,34)	S_{35}	$\cong \mathcal{C}_{17}$	
21	8	16 1	1-(35,17,17)	S_8	$[35,35,1]_2$	S_{35}
24		18 4	1-(35,22,22)	S_7	$[35,14,8]_2$	S_7
25		10 15	1-(56,25,25)	S_8	$[56,8,21]_2$	S_8
26		10 30 15	1-(56,55,55)	S_{56}	$[56,56,1]_2$	S_{56}
27		30 15	1-(56,45,45)	S_8	$[56,22,11]_2$	S_8
28		1 15	1-(56,16,16)	S_8	$[56,15,16]_2$	S_8
29		1 10 30 15	1-(56,56,1)	S_{56}	$[56,1,56]_2$	S_{56}
30		1 10 15	1-(56,26,26)	S_8	$[56,49,4]_2$	S_8
31		10 30	1-(56,40,40)	S_8	$[56,14,16]_2$	S_8
32		45	1-(84,45,45)	S_9	$[84,76,3]_2$	S_9
33	9	18 20	1-(84,38,38)	S_9	$[84,8,38]_2$	S_9
34		18 20 45	1-(84,83,83)	S_{84}	$[84,84,1]_2$	S_{84}
35		20 45	1-(84,65,65)	S_9	$[84,28,7]_2$	S_9
36		1 20	1-(84,21,21)	S_9	$[84,36,7]_2$	S_9
37		20	1-(84,20,20)	S_9	$[84,48,8]_2$	S_9
38		1 45	1-(84,46,46)	S_9	$[84,9,28]_2$	S_9
39		1 18 20 45	1-(84,84,1)	S_{84}	$[84,1,84]_2$	S_{84}
40		1 18 20	1-(84,39,39)	S_9	$\cong \mathcal{C}_{32}$	
41		18	1-(84,18,18)	S_9	$[84,56,4]_2$	S_9
42		1 18	1-(84,19,19)	S_9	$\cong \mathcal{C}_{35}$	
43		1 18 45	1-(84,64,64)	S_9	$[84,49,8]_2$	S_9
44		18 45	1-(84,63,63)	S_9	$\cong \mathcal{C}_{36}$	
45		1 20 45	1-(84,66,66)	S_9	$[84,57,4]_2$	S_9
46		1 63	1-(120,64,64)	$O_8^+(2)$:2	$[120,9,56]_2$	$S_8(2)$
47		56	1-(120,56,56)	$O_8^+(2)$:2	$[120,8,56]_2$	$O_8^+(2)$:2
48		63	1-(120,63,63)	$O_8^+(2)$:2	$[120,120,1]_2$	S_{120}
49		56 1 63	1-(120,120,1)	S_{120}	$[120,1,120]_2$	S_{120}
50		56 63	1-(120,119,119)	S_{120}	$\cong \mathcal{C}_{48}$	
51		56 1	1-(120,57,57)	$O_8^+(2)$:2	$\cong \mathcal{C}_{48}$	
52		63	1-(120,63,63)	S_{10}	$[120,120,1]_2$	S_{120}
53		35 21	1-(120,56,56)	S_{10}	$[120,10,36]_2$	S_{10}
54		35 21 63	1-(120,119,119)	S_{120}	$\cong \mathcal{C}_{52}$	
55		35 63	1-(120,98,98)	S_{10}	$[120,28,22]_2$	S_{10}
56		1 35	1-(120,36,36)	S_{10}	$[120,36,24]_2$	$S_8(2)$
57		35	1-(120,35,35)	S_{10}	$\cong \mathcal{C}_{52}$	
58		1 63	1-(120,64,64)	S_{10}	$\cong \mathcal{C}_{53}$	
59		1 35 21 63	1-(120,120,1)	S_{120}	$[120,1,120]_2$	S_{120}
60	10	1 35 21	1-(120,57,57)	S_{10}	$\cong \mathcal{C}_{52}$	
61		21	1-(120,21,21)	S_{10}	$\cong \mathcal{C}_{52}$	
62		1 21	1-(120,22,22)	S_{10}	$\cong \mathcal{C}_{55}$	

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Table 5.2.2 – continued from previous page

i	n	Block	design ($\mathcal{D}_i$)	Aut($\mathcal{D}_i$)	code ($\mathcal{C}_i$)	Aut ($\mathcal{C}_i$)
63		1 21 63	1-(120,85,85)	S_{10}	$\cong \mathcal{C}_{52}$	
64		21 63	1-(120,84,84)	S_{10}	$\cong \mathcal{C}_{56}$	
65		1 35 63	1-(120,99,99)	S_{10}	$\cong \mathcal{C}_{52}$	
66		1 100	1-(126,101,101)	S_{10}	$[126,126,1]_2$	S_{126}
67		25	1-(126,25,25)	S_{10}	$\cong \mathcal{C}_{66}$	
68		100	1-(126,100,100)	S_{10}	$[126,26,32]_2$	S_{10}
69		25 100 1	1-(126,126,1)	S_{126}	$[126,1,126]_2$	S_{126}
70		25 100	1-(126,125,125)	S_{126}	$\cong \mathcal{C}_{66}$	
71		25 1	1-(126,26,26)	S_{10}	$[126,27,26]_2$	S_{10}
72		84	1-(165,84,84)	S_{11}	$[165,164,2]_2$	S_{165}
73		56 24	1-(165,80,80)	S_{11}	$[165,10,72]_2$	S_{11}
74		56 24 84	1-(165,164,164)	S_{165}	$\cong \mathcal{C}_{72}$	
75	11	56 84	1-(165,140,140)	S_{11}	$[165,44,16]_2$	S_{11}
76		56 1	1-(165,57,57)	S_{11}	$[165,55,9]_2$	S_{11}
77		56	1-(165,56,56)	S_{11}	$[165,120,4]_2$	S_{11}
78		1 84	1-(165,85,85)	S_{11}	$[165,11,45]_2$	S_{11}
79		56 1 84 24	1-(165,165,1)	S_{165}	$[165,1,165]_2$	S_{165}
80		56 1 24	1-(165,81,81)	S_{165}	$[165,165,1]_2$	S_{11}
81		24	1-(165,24,24)	S_{11}	$\cong \mathcal{C}_{77}$	
82		24 1	1-(165,25,25)	S_{11}	$[165,45,9]_2$	S_{11}
83		24 1 84	1-(165,109,109)	S_{11}	$[165,121,4]_2$	S_{11}
84		24 84	1-(165,108,108)	S_{11}	$[165,54,16]_2$	S_{11}

5.3 2-designs

Let $n \geq 5$. The alternating group A_n is $(n-2)$-transitive in its natural action on the set of n points $\{1, 2, \ldots, n\}$, in particular we can consider A_n as a 2-transitive group on these n points. In that representation the point stabilizer is A_{n-1} and the two points stabilizer is A_{n-2}. Hence for $n \geq 7$, Method 3.1.2 cannot be applied to A_n in its natural representation since A_{n-2} is simple for $n \geq 7$ and therefore the 2 points stabilizer does not have a normal Sylow p-subgroup for a prime p dividing $n!$. For $n = 6$ Carmichael-Witt's method can be applied to A_6 since the 2 points stabilizer will be A_4 which has a unique Sylow 2-subgroup of order 4, namely the Klein group K. The group K fixes 2 points and we obtain a trivial design 2-(6,2,1). We apply Hughes's Method on A_n in its 2-transitive natural action on n points.

Notation. For an integer m and a prime p, we define inductively the group $p^{\wr m}$ by $p^{\wr 1} = \mathbb{Z}_p$ and $p^{\wr m} = p^{\wr m-1} \wr \mathbb{Z}_p$.

Theorem 5.3.1. *Let p be a prime divisor of $n!$ and let $a_0 + a_1p + \cdots + a_\alpha p^\alpha$ be the p-base expansion of n. If P is a Sylow p-subgroup of S_n, then*

$$P \cong \left(p^{\wr 1}\right)^{a_1} \times \left(p^{\wr 2}\right)^{a_2} \times \cdots \times \left(p^{\wr \alpha}\right)^{a_\alpha}.$$

Proof. [49, 4.1.22, 4.1.23 and 4.1.24]. □

Let $\mathcal{D}$ be a 2-design constructed from A_n with Method 3.1.4, we have the following result:

Theorem 5.3.2. *If U is a Sylow p-subgroup of the two points stabilizer of A_n and if the p-base expansion of $n-2$ is $a_0 + a_1p + \cdots + a_\alpha p^\alpha$, then $\mathcal{D}$ is a 2-$(n, 2+a_0, \binom{n-2}{a_0})$ design with $\binom{n}{a_0+2}$ blocks.*

Proof. For $m \leq n-2$, A_n acts m-transitively on $\{1, 2, \ldots, n\}$ in other words A_n acts transitively on the set of m-subsets of $\{1, 2, \ldots, n\}$, therefore if $|\mathrm{Fix}(U)| = k$ then $\mathcal{B} = \mathrm{Fix}(U)^{A_n}$ is just the set of k-subsets of $\{1, 2, \ldots, n\}$ and $|\mathcal{B}| = \binom{n}{k}$. To complete any couple of points of $\{1, 2, \ldots, n\}$ into a block we just have to choose $k-2$ points from the $n-2$ remaining points which gives us $\binom{n-2}{k-2}$ choices. According to Theorem 5.3.1, the size of the support of the permutations of U is at most $s = a_1p + \cdots + a_\alpha p^\alpha$ and $|\mathrm{Fix}(U)| = (n-2) - s + 2 = a_0 + 2$. □

5.4 A Steiner triple system of order 15 and a Hamming code from A_7

There are exactly 80 Steiner triple systems of order 15 (see for example [40]). A detailed study of these 80 Steiner systems and their codes have been done in [84]. Here we will give a construction of one of them using Theorem 3.1.2 and we will construct a Hamming code from its incidence matrix.

Theorem 5.4.1. *The group A_7 is represented as a faithful 2-transitive group on a set of 15 points.*

(i) The two points stabilizer is A_4 and the Klein group V_4 is a normal Sylow 2-subgroup of A_4.

(ii) $|Fix(V_4)| = 3$.

Proof. (i) The group A_7 is 2-transitive in its action on the 15 cosets of its maximal subgroup $L_2(7)$ (see [25]). In that action the point stabilizer is $L_2(7)$. If H is the two points stabilizer, then H is a subgroup of order 12 of $L_2(7)$ so that $H < S_4$. The alternating group A_4 is the only subgroup of order 12 of S_4, therefore $H \cong A_4$ and V_4 is a normal Sylow 2-subgroup of order 4 of A_4.
(ii) We can check with MAGMA [15] that in its representation of degree 15, an element of order 2 of A_7 is a cycle of type $1^3 2^6$. □

Corollary 5.4.2. *From A_7 we can construct a 2-(15,3,1) Steiner system $\mathcal{S}$ and $Aut(\mathcal{S}) \cong A_8$.*

Proof. The first part of the corollary is just an application of Method 3.1.2. Now, the 15 points of $\mathcal{S}$ can be considered as the 15 non zeros vectors of the vector space $\mathbb{F}_2^4$, so a block of $\mathcal{S}$ is just a set of three vectors which are linearly dependent. Therefore $\mathrm{Aut}(\mathcal{S}) = GL(4,2) \cong A_8$. $\square$

Remark 5.4.3. The Steiner system $\mathcal{S}$ can also be seen as $PG(3,2)$ the projective space of dimension three over $\mathbb{F}_2$. The set of subgroups of A_7 that are isomorphic to $L_2(7)$ split into two conjugacy classes and S_7 the automorphism group of A_7 has an outer automorphism which fuses these two conjugacy classes so $\mathrm{Aut}(A_7) \cong S_7 \not< \mathrm{Aut}(\mathcal{S}) \cong A_8$.

Theorem 5.4.4. *Given a Steiner triple system $\mathcal{D}$ of order $v > 3$,*

$$rank_p(\mathcal{D}) = \begin{cases} v \text{ for every prime } p \neq 2,3, \\ v - (d_P + 1), \text{ if } p = 2, \\ v - (d_A + 1), \text{ if } p = 3, \end{cases}$$

where d_P and d_A are the projective and affine dimension of the system

Proof. It is the main result in [29]. $\square$

Remark 5.4.5. For our 2-(15,3,1) Steiner system $\mathcal{S}$, we have $d_P = 3$ and $d_A = 0$.

Corollary 5.4.6. *If C_p is the linear code over $\mathbb{F}_p$ induced from $\mathcal{S}$ then*

(i) $\dim(C_p) = 15$ *for* $p \neq 2,3$,

(ii) $\dim(C_2) = 11$,

(iii) $\dim(C_3) = 14$.

Proof. It's an application of Theorem 5.4.4 to $\mathcal{S}$. $\square$

Corollary 5.4.7. *From $\mathcal{S}$ we can construct the Hamming code $C_2 = [15,11,3]_2$ with $Aut(C_2) = A_8$ and $C_2^\perp = [15,4,8]_2$.*

Proof. Using MAGMA we can get the weight enumerator of C_2 as follows

$$W(x) = 1 + 35x^3 + 105x^4 + 168x^5 + 280x^6 + 280x^7 + 435x^8 + 280x^9 + 168x^{10} + 105x^{11} + 35x^{12} + x^{15}$$

and thus we can see that the minimum weight of C_2 is 3. The dual of C_2 is a code of dimension 4 with the weight enumerator $W(x) = 1 + 15x^8$. The generator matrix of $C_2^\perp$ which is the parity check matrix of C_2 is constituted by all the nonzero binary columns of length 4, thus C_2 is a binary Hamming code. $\square$

6 Designs and codes from sporadic simple groups

In this chapter, we investigate the designs and codes obtained from the small Mathieu groups and the Higman-Sims group. Designs and codes obtained by using Key-Moori Method 1 on sporadic simple groups such as J_1, J_2, McL, Ru and HS, have been studied in [55,70–73]. In Section 6.1, we look at the code and designs that we can obtain by applying Key-Moori Method 1 and 2 on the small Mathieu groups M_{11} and M_{12}. In Section 6.2 we apply Key-Moori Method 1 and Hughes's Method on the primitive permutation representations of degree 100 and 176 of the Higman-Sims group, respectively.

6.1 Small Mathieu groups

The five Mathieu groups are the first generation of sporadic simple groups, they were first discovered by Émille Mathieu in the 1860s and 1870s [65,66]. The large Mathieu groups M_{24}, M_{23}, M_{22} and the small Mathieu groups M_{11}, M_{12} can be obtained by successive transitive extensions starting from the 3-transitive groups $L_3(4)$ and $M_{10} \cong A_6{}^{\cdot}2$, respectively [77, Theorem 9.52, 9.53, 9.55, 9.56, 9.57]. The largest Mathieu group M_{24} is 5-transitive on the 24 cosets of its maximal subgroup, the fourth Mathieu group, M_{23}. In this action, the stabiliser of five points is 2^4:3 and its unique Sylow 2-subgroup, 2^4, fixes exactly 8 points. Hence, by applying Carmichael-Witt's method we will obtain the unique Witt's 5-(24,8,1) design. The blocks of Witt's design are called *octad* and any two of them intersect in either 0, 2 or 4 points. If two octads are meeting in 2 points then their symmetric difference is a set of 12 points called *dodecad*. The stabiliser of a dodecad in M_{24} is the fourth Mathieu group M_{12}. If Δ is a dodecad and Ω an octad that meet Δ in at least 5 points then Δ meets Ω in exactly 6 points called *hexad* [28, Lemma 6.8C.]. There exist exactly 132 octads intersecting a dodecad in 6 points. Indeed any 5 points of the dodecads determine an octad so a priori we have $\binom{12}{5}$ octads intersecting it. Since an octad intersects the dodecad on 6 points and the 5-subsets from these 6 points determine the same octad, we finally have the $\binom{12}{5}/\binom{6}{5} = 132$ octads. Now From the property of the Witt's design any 5-subset of the hexads will belong to exactly one of these 132 octads. Thus from the 12 points of a dodecad we can construct a 5-(12,6,1) design and M_{12} is an automorphism group of this design. We have $|M_{12}| = 132 \times |S_6|$ and hence M_{12} is transitive on the blocks. The set-wise stabiliser of a block, S_6, is 5-transitive on the points of that block. It follows that M_{12} is 5-flag-transitive and hence 5-transitive on the 12 points of the dodecad. The point stabiliser in this action is the fifth Mathieu group M_{11}.

6.1.1 Computation for M_{12}

Designs and codes from the primitive representation of degree 66

Proposition 6.1.1. *From M_{12} of degree 66, we can construct 1-(66,20,20), 1-(66,21,21), 1-(66,45,45) and 1-(66,46,46) symmetric designs with full automorphism groups isomorphic to S_{12}. From the 1-(66,20,20) and 1-(66,46,46) designs we can generate $[66,10,20]_2$ and $[66,11,20]_2$ codes with automorphism groups isomorphic to S_{12}, respectively. The 1-(66,21,21) and 1-(66,45,45) designs will produce the full space $(\mathbb{F}_2)^{66}$.*

Proof. The Mathieu group M_{12} is a maximal subgroup of A_{12}. The action of M_{12} on 66 points is just the restriction of the action of A_{12} on the 66 duads from 12 points and the result follows from Proposition 5.1.1. □

The Mathieu group M_{12} acts primitively on the 12 cosets of M_{11}. We take the conjugacy classes of elements of order 2 and 3 of M_{12} inside M_{11} and form as indicated in Key-Moori Method 2 non-symmetric 1-designs. For the following we denote M_{11} by M.

1-(495,11,4) design and $[495,11,165]_2$ code

The Mathieu group M_{12} has two conjugacy classes of element of order 2, namely 2A and 2B, of size 396 and 495, respectively. We have $|M \cap 2B| = 11$ and $|M \cap 2A| = 0$. Using the character of degree 12 of M_{12}, we have $\chi_M = \chi_1 + \chi_2 = 1a + 11a$ and for $g \in 2B$, $\chi_M(g) = 1 + 3 = 4$. Hence we construct a non-symmetric 1-(495,11,4) design $\mathcal{D}$ with 12 blocks. Computation with MAGMA shows that the full automorphism group of $\mathcal{D}$ is $\mathrm{Aut}(\mathcal{D}) \cong S_{12}$. From the design $\mathcal{D}$ we can generate a binary code $[495,11,165]_2$ with the automorphism group S_{12} and the weight enumerator

$$W(x) = 1 + 12x^{165} + 528x^{240} + 792x^{245} + 495x^{256} + 220x^{261}.$$

1-(1760,440,3) design and $[1760,12,440]_2$ code

The Mathieu group M_{12} has two conjugacy classes of element of order 3, namely 3A and 3B of size 1760 and 2640 respectively, $|M \cap 3A| = 440$ and $|M \cap 3B| = 0$. For $g \in 3A$, $\chi_M(g) = 1 + 2 = 3$. Thus we obtain a 1-(1760,440,3) design $\mathcal{D}$. Computation with MAGMA shows that $\mathrm{Aut}(\mathcal{D})$ is $(S_8)^{220}{:}S_{12}$. From $\mathcal{D}$ we can generate a $[1760,12,440]_2$ code with the full automorphism group isomorphic to $\mathrm{Aut}(\mathcal{D})$ and weight enumerator

$$W(x) = 1 + 12x^{440} + 66x^{720} + 495x^{832} + 792x^{840} + 220x^{872} + 924x^{880}$$
$$+ 220x^{888} + 792^{920} + 495x^{928} + 66x^{1040} + 12x^{1320} + x^{1760}.$$

6.1.2 Computation for M_{11}

Designs and codes from the primitive representation of degree 55

Proposition 6.1.2. *From M_{11} of degree 55, we can construct a 1-(55,18,18), 1-(55,19,19), 1-(55,36,36) and a 1-(55,37,37) symmetric design with the full automorphism group isomorphic to S_{11}. From these designs we can obtain $[55, 10, 10]_2$, $[55, 45, 3]_2$, $[55, 44, 4]_2$ and $[55, 11, 10]_2$ codes respectively, with full automorphism groups isomorphic to S_{11}.*

Proof. Same argument as in Proposition 6.1.1. □

Designs and codes from the primitive representation of degree 66

The Mathieu group M_{11} acts as a rank-4 primitive group on the cosets of S_5. The orbits of the point stabiliser have lengths, 1, 15, 20 and 30. Applying Key-Moori Method 1 on all possible unions of these 4 orbits and ignoring the trivial design 1-(66,66,1) and the 1-(66,65,65) design, we produce 14 non isomorphic designs from which we can generate 7 non isomorphic binary codes. Table 6.1.1 summarizes the results we obtain The column 'Block' gives the length of the orbits constituting the blocks of the designs.

Table 6.1.1: Designs with 66 points and codes of length 66 from M_{11}

i	Block	design ($\mathcal{D}_i$)	Aut($\mathcal{D}_i$)	code ($\mathcal{C}_i$)	Aut ($\mathcal{C}_i$)
1	30	1-(66,30,30)	M_{11}	$[66,54,4]_2$	S_{12}
2	20 15	1-(66,35,35)	M_{11}	$[66,22,11]_2$	M_{11}
3	20 30 15	1-(66,65,65)	S_{66}	$[66,66,1]_2$	S_{66}
4	20 30	1-(66,50,50)	M_{11}	$[66,44,8]_2$	M_{11}
5	1 20	1-(66,21,21)	S_{12}	$\cong \mathcal{C}_3$	
6	20	1-(66,20,20)	S_{12}	$[66,10,20]_2$	S_{12}
7	1 30	1-(66,31,31)	M_{11}	$\cong \mathcal{C}_2$	
8	1 20 30 15	1-(66,66,1)	S_{66}	$[66,1,66]_2$	S_{66}
9	1 20 15	1-(66,36,36)	M_{11}	$[66,55,4]_2$	S_{12}
10	15	1-(66,15,15)	M_{11}	$\cong \mathcal{C}_2$	
11	1 15	1-(66,16,16)	M_{11}	$[66,45,8]_2$	M_{11}
12	1 30 15	1-(66,46,46)	S_{12}	$[66,11,20]_2$	S_{12}
13	30 15	1-(66,45,45)	S_{12}	$\cong \mathcal{C}_3$	
14	1 20 30	1-(66,51,51)	M_{11}	$\cong \mathcal{C}_2$	

Designs and codes from conjugacy classes of elements

The group M_{11} has 10 conjugacy classes of elements, some of them are listed in Table 6.1.2. We applied Key-Moori Method 2 to the maximal subgroups $A_6{}^{\cdot}2$ and $L_2(11)$ and some of the conjugacy classes of M_{11} to construct 1-designs. We looked also at the corresponding binary codes.

nX	$\lvert nX\rvert$	$C_G(g)$	Maximal Centralizer
2A	165	$2^{\cdot}S_4$	Yes
3A	440	$3\times S_3$	No
4A	990	8	No
5A	1584	5	No
6A	1320	6	No
8A	990	8	No
11A	720	6	No

Table 6.1.2: Some conjugacy classes of M_{11}

$A_6{}^{\cdot}2$:

2A : $\mathcal{D}$ = 1-(165,45,3); $\mathcal{C} = [165, 11, 45]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = S_{11}$;
$W_{\mathcal{C}}(x) = 1 + 11x^{45} + 55x^{72} + 330x^{77} + 627x^{80} + 627x^{85} + 330x^{88} + 55x^{93} + 11x^{120} + x^{165}$.

3A : $\mathcal{D}$ =1-(440,80,2): $\mathcal{C} = [440, 10, 80]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_8)^{55}{:}S_{11}$;
$W_{\mathcal{C}}(x) = 1 + 11x^{80} + 55x^{144} + 165x^{192} + 330x^{224}, +462x^{240}$.

5A : $\mathcal{D}$ = 1-(1584,144,1); $\mathcal{C} = [1584, 11, 144]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_{144})^{11}{:}S_{11}$;
$W_{\mathcal{C}}(x) = 1 + 11x^{144} + 55x^{258} + 165x^{432} + 330x^{576} + 462x^{720} + 462x^{864} + 330x^{1008} + 165x^{1152} + 55x^{1296} + 11x^{1440} + x^{1584}$.

8A : $\mathcal{D}$ = 1-(990,90,1): $\mathcal{C} = [990, 11, 90]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_{90})^{11}{:}S_{11}$;
$W_{\mathcal{C}}(x) = 1 + 11x^{90} + 55x^{180} + 165x^{270} + 330x^{360} + 462x^{450} + 462x^{540} + 330x^{630} + 165x^{720} + 55x^{810} + 11x^{900} + x^{990}$.

$L_2(11)$:

2A : $\mathcal{D}$ = 1-(165,55,4): $\mathcal{C} = [165, 11, 55]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = M_{11}$;
$W_{\mathcal{C}}(x) = 1 + 12x^{55} + 55x^{72} + 660x^{79} + 627x^{80} + 220x^{87} + 330x^{88} + 132x^{95} + 11x^{120}$.

3A : $\mathcal{D}$ =1-(440,110,3) ; $\mathcal{C} = [440, 12, 110]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = 2^{220}{:}S_{12}$; $W_{\mathcal{C}}(x) = 1 + 12x^{110} + 66x^{180} + 495x^{208} + 792x^{210} + 220x^{218} + 924x^{220} + 220x^{222} + 792x^{230} + 495x^{232} + 66x^{260} + 12x^{330} + x^{440}$.

5A : $\mathcal{D}$ = 1-(1584,264,2); $\mathcal{C} = [1584, 11, 264]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_{24})^{66}{:}S_{12}$;
$W_{\mathcal{C}}(x) = 1 + 12x^{264} + 66x^{480} + 220x^{648} + 495x^{768} + 792x^{840} + 462x^{864}$.

6A : $\mathcal{D}$ =1-(1320,110,1); $\mathcal{C} = [1320, 12, 110]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_{110})^{12}{:}S_{12}$;
$W_{\mathcal{C}}(x) = 1 + 12x^{110} + 66x^{220} + 220x^{330} + 495x^{440} + 792x^{550} + 924x^{660} + 792x^{770} + 495x^{880} + 220x^{990} + 66x^{1100} + 12x^{1210} + x^{1320}$.

11A : $\mathcal{D}$ = 1-(720,60,1); $\mathcal{C} = [720, 12, 60]_2$; $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) = (S_{60})^{12}{:}S_{12}$;
$W_{\mathcal{C}}(x) = 1 + 12x^{60} + 66x^{120} + 220x^{180} + 495x^{240} + 792x^{300} + 924x^{360} + 792x^{420} + 495x^{480} + 220x^{540} + 66x^{600} + 12x^{660} + x^{720}$.

6.2 Higman-Sims group

In [42], using the 77 blocks and the 22 points of a S(3,6,22) Steiner system with an additional point as vertices, Donald Higman and Charles Sims constructed a regular graph with 100 vertices. They proved that the automorphism group of this graph is HS:2, where HS a new simple group of order $44,352,000$. Later, in [43], Higman rediscovered HS as the full automorphism group of a 2-(176,50,14) design acting 2-transitively on the 176 points .

Designs and codes from the primitive representation of degree 176

As we have seen earlier, HS acts 2-transitively on 176 points. In that action, the point stabilizer is the group $U_3(5){:}2$ and the two points stabilizer H is $A_6{^\cdot}2^2$. The group H has Sylow 2-subgroups, Sylow 3-Subgroups and Sylow 5-subgroups. A Sylow 3-subgroup or a Sylow 5-subgroup of H cannot obviously be normal since they are subgroups of A_6. We can easily show that H has forty five Sylow 2-subgroups and hence a Sylow 2-subgroup of H is not normal.

We applied Method 3.1.4 to HS. A Sylow 2-subgroup and 3-subgroup of H are isomorphic to $(2{\times}D_8)$ and 3^2, respectively. Each of them fixes exactly 2 points, so applying Method 3.1.4 to HS with a Sylow 2-subgroup or a Sylow 3-subgroup will produce a trivial 2-(176,2,1) design. A Sylow 5-subgroup of H is a cyclic group of order 5 which fixes 6 points and we have the following result.

Proposition 6.2.1. *From HS we can construct a 2-(176,6,36) design $\mathcal{D}$ with $Aut(\mathcal{D}) = HS$.*

Proof. We apply Method 3.1.4 to HS with a Sylow 5-subgroup of H to construct the design $\mathcal{D}$. Computation with MAGMA shows that $|\mathrm{Aut}(\mathcal{D})| = |\mathrm{HS}|$. Since HS $<$ $\mathrm{Aut}(\mathcal{D})$, we have HS $\cong \mathrm{Aut}(\mathcal{D})$. □

Remark 6.2.2. The code C associated to the design $\mathcal{D}$ is a $[176, 155, 6]_2$ code with $\mathrm{Aut}(C)$ =HS. The dual $C^{\perp}$ is a $[176, 21, 56]_2$ code. In [18], $C^{\perp}$ is presented as a subspace of a linear code $[176, 22, 50]_2$ which is constructed from a monomial representation of HS.

Now, a word of weight i is denoted by w_i and the set of words of weight i is denoted by W_i. By acting HS on W_i, the orbits form the blocks of a 2-$(176, i, k_i)$ design $\mathcal{D}_{w_i}$ where $k_i = |(w_i)^{\mathrm{HS}}|{\times}i/176$. The number of blocks of $\mathcal{D}_{w_i}$ is $[\mathrm{HS} : (\mathrm{HS})_{w_i}]$. Table 6.2.1 gives the parameters of the design $\mathcal{D}_{w_i}$, and the structure of HS_{w_i} for the words of minimum weight of C. Table 6.2.2 gives the same results for the weight distribution of $C^{\perp}$.

i	$\mathcal{D}_{w_i}$	No. of blocks	$(\mathrm{HS})_{w_i}$	Maximality
6_1	2-(176,6,36)	36960	$(5{:}4)\times A_5$	Yes
6_2	2-(176,6,90)	92400	$(SL_2(5){:}2){:}2$	No

Table 6.2.1: 2-design $\mathcal{D}_{w_i}$ from C and the stabiliser in HS of a word w_i of C

Table 6.2.2: 2-design $\mathcal{D}_{w_i}$ from $C^{\perp}$ and the stabiliser in HS of a word w_i of $C^{\perp}$

i	$\mathcal{D}_{w_i}$	No. of blocks	$(\mathrm{HS})_{w_i}$	Maximality
56	2-(176,56,110)	1100	$L_3(4){:}2_1$	Yes
64	2-(176,64,540)	4125	$2^3.(2^3.L_2(7))$	No
72	2-(176,72,2556)	15400	$2\times A_6\cdot 2^2$	Yes
$(80)_1$	2-(176,80,790)	3850	$2^4.S_6$	Yes
$(80)_2$	2-(176,80,47400)	231000	$(((2\times D_8){:}2){:}3){:}2$	No
$(80)_3$	2-(176,80,75840)	369600	S_5	No
$(88)_1$	2-(176,88,38280)	154000	$(A_4\times A_4){:}2$	No
$(88)_2$	2-(176,88,172260)	693000	$((\mathbb{Z}_4\times\mathbb{Z}_4){:}2){:}2$	No
$(96)_1$	2-(176,96,1140)	3850	$2^4.S_6$	Yes
$(96)_2$	2-(176,96,68400)	231000	$(((2\times D_8){:}2){:}3){:}2$	No
$(96)_3$	2-(176,96,109440)	369600	S_5	No
104	2-(176,104,5356)	15400	$(2\times A_6)\cdot 2^2$	Yes
112	2-(176,112,1665)	4125	$2^3.(2^3.L_2(7))$	No
120	2-(176,120,510)	1100	$L_3(4){:}2_1$	Yes

Remark 6.2.3. Under the action of HS the words of weight 6 of C split into 2 orbits W_{6_1} and W_{6_2} of length 36960 and 92400, respectively. The code words in W_{6_1} form a 2-(176,6,36) design $\mathcal{D}_{w_{6_1}}$ and the generators of the code C belong to W_{6_1}. The code words in W_{6_2} form a 2-(176,6,90) design $\mathcal{D}_{w_{6_2}}$ with corresponding $[176,154,6]_2$ code C'. The dual of C' is a $[176,22,50]_2$ code and from its words of weight 50 we can construct the unique symmetric 2-(176,50,14) design admitting HS as the full automorphism group mentioned in Theorem 3.1.6.

The full space $\mathbb{F}_2^{176}$ can be seen as sum of HS-modules as follows

$$\mathbb{F}_2^{176} = \mathbb{F}_2 \jmath \oplus C_{20} \oplus \mathbb{F}_2 \jmath \oplus C_{132} \oplus \mathbb{F}_2 \jmath \oplus C_{20} \oplus \mathbb{F}_2 \jmath,$$

we have $C = \mathbb{F}_2 J \oplus C_{132} \oplus \mathbb{F}_2 J \oplus C_{20} \oplus \mathbb{F}_2 J$, $C^{\perp} = \mathbb{F}_2 J \oplus C_{20}$ and $\mathrm{Soc}(\mathbb{F}_2^{176}) = \mathbb{F}_2 J$.

Designs and codes from the primitive representation of degree 100

The HS group acts as a rank-3 primitive group on the cosets of the third Mathieu group M_{22}. The orbits of the point stabiliser have length 1, 22, 77, respectively. We consider the orbit of length 22 and constructed according to Key-Moori Method 1, a symmetric 1-(100,22,22) design $\mathcal{D}$ with $\mathrm{Aut}(\mathcal{D}) \cong$ HS:2. From the incidence matrix of $\mathcal{D}$ we can generate a $[100, 22, 22]_2$ binary code $\mathcal{C}$ with $\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\mathcal{D}) \cong$ HS:2 and weight enumerator

$$\begin{aligned}
W(x) &= 1 + 100x^{22} + 1100x^{30} + 3850x^{32} + 4125x^{36} + 38500x^{38} \\
&+ 92400x^{40} + 193600x^{42} + 347600x^{44} + 485100x^{36} + 600600x^{48} \\
&+ 660352x^{50} + 600600x^{52} + 485100x^{54} + 347600x^{56} + 193600x^{58} \\
&+ 92400x^{60} + 38500x^{62} + 4125x^{64} + 3850x^{68} + 1100x^{70} + 100x^{78} + x^{100}.
\end{aligned}$$

Now, by applying Key-Moori Method 1 on the orbit of length 77, we obtain a 1-(100,77,77) symmetric design that yields a $[100, 23, 23]_3$ ternary code. Both the design and its code has the group HS:2 as the full automorphism group.

7 Hadamard designs and codes from Key-Moori Method 1

In Example 2.4.8 we have seen that from a Hadamard matrix of order $4t$ we can construct a 2-$(4t+3, 2t+1, t)$ design called Hadamard design and reciprocally from the incidence matrix of a design with such parameters we can construct a Hadamard matrix. There are several methods of constructing a Hadamard matrix: we can obtain a Hadamard matrix from the tensor product of two Hadamard matrices [79, Chapter 8, Theorem 2.4], in [75] Paley developed a method using Legendre symbols in $GF(p^k)$, in [86] Williamson developed a new method and ameliorated the results obtained by Paley. Here we develop the construction of some Hadamard designs using Key-Moori Method 1.

7.1 Hadamard designs from symplectic groups

Applying Key-Moori Method 1 to the symplectic groups $S_{2n}(2)$ (for $n \geq 3$) gives a 1-$(2^{2n}-1, 2^{2n-1}, 2^{2n-1})$ symmetric design which is a 2-$(2^{2n}-1, 2^{2n-1}-1, 2^{2n-1})$ symmetric design [56]. The incidence matrix of this design can be used to construct a Hadamard matrix of order 2^{2n} which belongs to a special family of Hadamard matrices called Sylvester H-matrices [82].

7.2 Hadamard designs from Frobenius groups

Definition 7.2.1. *A Frobenius group G is a non-regular transitive group such that the identity is the only element of G which fixes more than one point. The set $N = \{e\} \cup \{g \in G\colon |Fix(g)| = 0\}$ is called the Frobenius kernel of G.*

Theorem 7.2.2. *A group G is a Frobenius group if and only if it has a proper subgroup H such that $H \cap H^g = \{e\}$ for all $g \in G\backslash H$.*

Proof. See [38]. □

Theorem 7.2.3. *Let G be a Frobenius group acting on a set X of size n.*

(i) The subgroup G_x has only one orbit of length 1, it acts regularly on each of its orbit on $X\backslash\{x\}$ and the size of an orbit is equal to $|G_x|$.

(ii) The Frobenius kernel N is a normal subgroup of G and $G \cong N{:}G_x$.

Proof. (i) The singleton $\{x\}$ is clearly one of the orbits of G_x on X. According to [77, Theorem 9.1] G_x acts transitively on each of its orbit on $X\backslash\{x\}$, since G_x is a subset of the Frobenius group G then apart from the identity element an element of G_x can not have any more fixed point on these orbits. Hence G_x acts regularly on each of its orbit on $X\backslash\{x\}$. The second part of the statement comes from [77, Theorem 9.8 (iii)].
(ii) See [48, Theorem 7.2] and [77, p. 254-255]. □

Theorem 7.2.4. *If G is a non-Abelian finite group of order pq, where $p < q$ are prime and $p \mid q-1$ then we can consider G as a primitive Frobenius group of degree q isomorphic to $\mathbb{Z}_q{:}\mathbb{Z}_p$.*

Proof. It is known that if $p \mid q-1$ then up to isomorphism $\mathbb{Z}_q{:}\mathbb{Z}_p$ is the only non Abelian group of order pq. The group G acts on the n_p conjugates of $\mathbb{Z}_p$ and it is clear that this action is transitive. For $g \in G\backslash\mathbb{Z}_p$ we have $\mathbb{Z}_p \cap {\mathbb{Z}_p}^g = \{e\}$ otherwise ${\mathbb{Z}_p}^g = \mathbb{Z}_p$ for all $g \in G$, that is $\mathbb{Z}_p$ would be a normal subgroup of G and G would be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$ which is absurd. Thus $N_G(\mathbb{Z}_p) = \mathbb{Z}_p$ and $n_p = [G : \mathbb{Z}_p] = q$. In that action the point stabilizer is $\mathbb{Z}_p$ which is a maximal subgroup of G (since $\mathbb{Z}_p$ is a subgroup of prime index), hence the action of G is primitive. □

Theorem 7.2.5. *For an integer $\lambda > 0$, if there exist two prime numbers v and k such that $v = 4\lambda + 3$ and $k = 2\lambda + 1$ then from the Frobenius group $\mathbb{Z}_v{:}\mathbb{Z}_k$ we can construct a symmetric design 1-(v, k, k) which is a 2-(v, k, λ) design.*

Proof. We consider the action of $G = \mathbb{Z}_v{:}\mathbb{Z}_k$ on the v conjugates of $\mathbb{Z}_k$. By Theorem 7.2.4 we can consider G as finite primitive permutation group acting on a set X of size v and the stabilizer G_x of a point $x \in X$ is $\mathbb{Z}_k$. According to Theorem 7.2.3 (i), under the action of G_x the set $X\backslash\{x\}$ splits into two orbits of length $|G_x| = k$ since $|X\backslash\{x\}| = 4\lambda + 2 = 2k$. Now if we consider $\Delta \neq \{x\}$ to be one of the orbits of size k and $\mathcal{B} = \{\Delta^g : g \in G\}$ then by applying Key-Moori Method 1 we obtain a design $\mathcal{D} = (X, \mathcal{B})$ which is a 1-(v, k, k) symmetric design.
We want to prove that for any two points of X we can always find some blocks containing them together. Let $x_1, x_2, \ldots, x_v$ be the v points of X. We denote by $\mathcal{B}_i$ the set of blocks containing x_i and we have $|\mathcal{B}_i| = k$. We can write G as $\{e\} \cup \{g \in G : |\mathrm{Fix}(g)| = 0\} \cup \{g \in G : |\mathrm{Fix}(g)| = 1\}$ and we have $\mathbb{Z}_v = \{e\} \cup \{g \in G : |\mathrm{Fix}(g)| = 0\}$ and $\mathbb{Z}_k = \{e\} \cup \{g \in G : |\mathrm{Fix}(g)| = 1\}$. Suppose that we can find two points x_i and x_j such that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$. Then we have a set $\mathcal{B}_{ij}$ of $v - 2k = 1$ block that does not contain neither x_i nor x_j. Now, if $g \in G$ sends x_i to x_j, then g sends a block of $\mathcal{B}_i$ to a block of $\mathcal{B}_j$ and fixes the block of $\mathcal{B}_{ij}$, that is $g \in \mathbb{Z}_k$ and it will be the product of k 2-cycle which is impossible. Hence for any two points of X we can always find $\lambda_{ij} > 0$ blocks containing them together.
Now we want to prove that λ_{ij} does not depend on x_i and x_j. There are k blocks containing x_i and among them there are $v - (2k - \lambda_{ij})$ that do not contain x_j. Therefore we have $\lambda_{ij} = k - (v - 2k + \lambda_{ij})$ which gives us

$$2\lambda_{ij} = k - v + 2k = 3k - v = 6\lambda + 3 - 4\lambda - 3 = 2\lambda,$$

hence $\lambda_{ij} = \lambda$ and our 1-(v, k, k) symmetric design is a 2-(v, k, λ) design. □

Remark 7.2.6. If p is a prime and n an integer such that $n|p-1$, then there is a Frobenius $\mathbb{F}_{n,p} = \langle a, b | a^p = b^n = 1, b^{-1}ab = a^u \rangle$ where u is an element of order n of $\mathbb{Z}_p{}^*$ (cf. [76]). Theorem 7.2.5 holds if instead of the simple group $\mathbb{Z}_k$ we consider a cyclic group $\mathbb{Z}_n$, of odd order $n = 2\lambda + 1$ such that $\mathbb{Z}_p{:}\mathbb{Z}_n \cong \mathbb{F}_{n,p}$. Table 7.2.1 gives us some Hadamard design from the application of Theorem 7.2.5 on different Frobenius groups $G \cong \mathbb{Z}_p{:}\mathbb{Z}_n$ where p is a prime number that can be written as $4\lambda + 3$ and n an odd integer equal to $2\lambda + 1$.

Remark 7.2.7. The Frobenius group $\mathbb{Z}_7{:}\mathbb{Z}_2$ can also be considered as a primitive group of degree 7. In that representation the point stabilizer $\mathbb{Z}_2$ has one orbit of length one and three orbits of length 2. If we apply Key-Moori Method 1 to the union of the orbit of length 1 and one orbit of length 2, we will obtain a 1-$(7, 3, 3)$ design. But in this case the design is not a 2-design.

For the dimension of the codes obtained from these 2-designs we have the following result:

Theorem 7.2.8. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-(v, k, λ) design of order, $k - \lambda$, and let p be a prime dividing n. Then*

$$rank_p(\mathcal{D}) \leq \frac{|\mathcal{B}| + 1}{2};$$

moreover, if p does not divide λ and p^2 does not divide n, then

$$\mathcal{C}_p(\mathcal{D})^{\perp} \subseteq \mathcal{C}_p(\mathcal{D})$$

and $rank_p(\mathcal{D}) \geq v/2$.

Proof. We refer to [61]. □

λ	G	**Design**$(\mathcal{D})$	**Aut**$(\mathcal{D})$	**Code**$(\mathcal{C})$	**Aut**$(\mathcal{C})$
1	7:3	2-(7,3,1)	$L_2(7)$	$[7, 4, 3]_2$	$L_2(7)$
2	11:5	2-(11,5,2)	$L_2(11)$	$[11, 6, 5]_3$	M_{11}:2
4	19:9	2-(19,9,4)	19:9	$[19, 10, 7]_5$	(19:9):4
5	23:11	2-(23,11,5)	23:11	$[23, 12, 7]_2$	M_{23}
5	23:11	2-(23,11,5)	23:11	$[23, 12, 8]_3$	(23:11):2
6	27:13	2-(27,13,6)	27:13	$[27, 14, 8]_7$	(27:13):2
7	31:15	2-(31,15,7)	31:15	$[31, 16, 7]_2$	31:15
10	43:21	2-(43,21,10)	43:21	$[43, 22, 16]_{11}$	(43:21):10

Table 7.2.1: Hadamard designs and codes from $\mathbb{Z}_p{:}\mathbb{Z}_n$

Appendix

A MAGMA codes

This appendix provides MAGMA implementations of the different method described in Chapter 3. The groups used in these programmes were mainly supplied by the following databases in MAGMA:

1. The database of all primitive groups of degree less than 2500,
2. `ATLAS`, a database that contains information on the representations of the simple groups listed at `http://brauer.maths.qmul.ac.uk/Atlas/v3/`,
3. `simgps`, a database that contains very useful information such as the list of the maximal subgroups for all simple groups of order less than a million.

Remark A.0.9. The lines of the programmes are numbered. The phrases starting with the symbol ▷ are comments and are not part of the programmes.

Programme 1 MAGMA implementation of Hughes and Carmichael-Witt's method

Require: G a faithful 2-transitive simple group

```
X := GSet(G);
H := Stabilizer(G,X,1);
H := Stabilizer(G,X,2);                         ▷ H is the 2 point stabilizer
prime := PrimeDivisors(#H);
for p in prime do                               ▷ for each Sylow p-subgroup U of H
    U := Sylow(H,p);
    orbits := Orbits(U);
    F := {};                                    ▷ find Fix(U)
    for j in [1..#orbits] do
        if #orbits[j] eq 1 then
            F := F join orbits[j];
        end if;
    end for;
    blox := Orbit(G,F);;                        ▷ blocks of the 2-design
    design := Design<2, #X|blox>;
end for;
```

Programme 2

```
union := function(lists, index) ;    ▷ a recursive function that takes as argument lists a list of subsets and index a list of indices of lists. It returns the union over the indices in index of the subsets in lists
if #index eq 1 then
else
    if #index eq 2 then
        return lists[index[1]] join lists[index[2]];
    else
        nextindex := [index[i]: i in [2..#index]];
        return lists[index[1]] join $$(lists,nextindex);    ▷ recursive call of union(lists,index)
    end if;
end if;
```

Programme 3 MAGMA implementation of Key-Moori Method 1

Require: G a primitive simple group

```
X := GSet(G);
H := Stabilizer(G,X,1);    ▷ H is the point stabilizer
v := Index(G, H);
orbits := Orbits(H);    ▷ orbits of the point stabilizer
setorbitunion := SetToSequence(Subsets(Set([1..#orbits])));
for orbitindex in setorbitunion do
    orbitunion:= SetToSequence(orbitindex);
    if #orbitunion eq 1 then    ▷ ignore the orbit of length 1
        continue;
    else    ▷ the blocks are the G-orbits of the union of any orbits of the point stabilizer
        block := union(delta,orbitunion);
        blocks := Orbit(G,block);
        design := Design⟨1, v|blocks⟩;
    end if;
end for;
```

Programme 4 MAGMA implementation of Key-Moori Method 2

Require: G a primitive simple group, Max a maximal subgroup of G

```
Cc := ConjugacyClasses(G);
for cc in Cc do
    nX := Class(G,cc[3]);
    if #(nX meet Set(Max)) eq 0 then
        continue;
    end if;
    blox := Orbit(G,(nX meet Set(Max))) ;
    design := Design⟨1, SetToIndexedSet(nX)|blox⟩;
end for;
```

B Designs and codes from $L_2(q)$ for $q \in \{8, 9, 11, 13, 16, 17\}$

B.1 Designs and codes from Key-Moori Method 1

$q = 8$

Max	Degree	rank	Length
$2^3{:}7$	9	2	1, 9
D_{18}	28	4	1, 9(3)
D_{14}	36	5	1, 7(3), 14

Table B.1.1: Maximal subgroups and their orbits

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
1	$2^3{:}7$	8	1-(9,8,8)	S_9	[9,8,2]	S_9
2		8 1	1-(9,9,1)	S_9	[9,1,9]	S_9
3	D_{18}	9	1-(28,9,9)	$L_2(8)$	[28,28,1]	S_{28}
4		9(2)	1-(28,18,18)	$L_2(8)$	[28,8,10]	$L_2(8)$
5		9(3)	1-(28,27,27)	S_{28}	$\mathcal{C}_3$	
6		1 9	1-(28,10,10)	$L_2(8)$	$\mathcal{C}_4$	
7		9(3) 1	1-(28,28,1)	S_{28}	[28,1,28]	S_{28}
8		1 9(2)	1-(28,19,19)	$L_2(8)$	$\mathcal{C}_3$	
9	D_{14}	1 7	1-(36,8,8)	$L_2(8)$	[36,16,8]	$L_2(8)$
10		7	1-(36,7,7)	$L_2(8)$	[36,28,3]	S_9
11		1 14 7(2)	1-(36,29,29)	$L_2(8)$	$\mathcal{C}_{10}$	
12		14 7	1-(36,21,21)	$L_2(8)$	[36,36,1]	S_{36}
13		1 7(3)	1-(36,22,22)	S_9	[36,9,8]	S_9
14		1 7(2)	1-(36,15,15)	$L_2(8)$	$\mathcal{C}_{12}$	
15		1 14	1-(36,15,15)	S_9	$\mathcal{C}_{10}$	
16		14 7(2)	1-(36,28,28)	$L_2(8)$	$\mathcal{C}_9$	
17		14	1-(36,14,14)	S_9	[36,8,8]	S_9
18		7(2)	1-(36,14,14)	$L_2(8)$	[36,8,14]	$L_2(8)$
19		7(3)	1-(36,21,21)	S_9	$\mathcal{C}_{10}$	
20		1 14 7(3)	1-(36,36,1)	S_{36}	[36,1,36]	S_{36}
21		1 14 7	1-(36,22,22)	$L_2(8)$	$\mathcal{C}_{18}$	
22		14 7(3)	1-(36,35,35)	S_{36}	$\mathcal{C}_{12}$	

Table B.1.2: Codes and designs from $L_2(8)$.

$q = 9$

Max	Degree	rank	Length
A_5	6	2	1, 5
$3^2{:}4$	10	2	1, 9
S_4	15	3	1, 6, 8

Table B.1.3: Maximal subgroups an their orbits

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
1	A_5	5	1-(6,5,5)	S_6	[6,6,1]	S_6
2		1 5	1-(6,6,1)	S_6	[6,1,6]	S_6
3	3^2:4	9	1-(10,9,9)	S_{10}	[10,10,1]	S_{10}
4		1 9	1-(10,10,1)	S_{10}	[10,1,10]	S_{10}
5	S_4	8 1	1-(15,9,9)	S_6	[15,15,1]	S_{15}
6		6	1-(15,6,6)	S_6	[15,14,2]	S_{15}
7		8	1-(15,8,8)	A_8	[15,4,8]	A_8
8		8 1 6	1-(15,15,1)	S_{15}	[15,1,15]	S_{15}
9		8 6	1-(15,14,14)	S_{15}	$\mathcal{C}_6$	
10		1 6	1-(15,7,7)	A_8	[15,5,7]	A_8

Table B.1.4: Codes and designs from $L_2(9)$.

$q = 11$

Max	Degree	rank	Length
11:5	12	2	1,11
A_5	11	2	1,10
D_{12}	55	9	1, 3(2), 6(4), 12(2)

Table B.1.5: Maximal subgroups and their orbits

Table B.1.6: Codes and designs from $L_2(11)$.

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
1	A_5	10	1-(11,10,10)	S_{11}	[11,10,2]	S_{11}
2		1 10	1-(11,11,1)	S_{11}	[11,1,11]	S_{11}
5	11:5	11	1-(12,11,11)	S_{12}	[12,12,1]	S_{12}
6		1 11	1-(12,12,1)	S_{12}	[12,1,12]	S_{12}
7	D_{12}	1 3 12 6(4)	1-(55,40,40)	$L_2(11)$	[55,44,4]	S_{11}
8		1 3(2) 6(3)	1-(55,25,25)	$L_2(11)$	[55,11,10]	S_{11}
9		3 12 6	1-(55,21,21)	$L_2(11)$	[55,35,5]	$L_2(11)$:2
10		12 6(2)	1-(55,24,24)	$L_2(11)$	[55,34,4]	$L_2(11)$:2
11		3 12(2) 6(3)	1-(55,45,45)	$L_2(11)$	[55,55,1]	S_{55}
12		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	[55,54,2]	S_{55}
13		1 3(2) 12 6(3)	1-(55,37,37)	$L_2(11)$	[55,35,6]	$L_2(11)$
14		1 3(2) 12(2) 6(3)	1-(55,49,49)	$L_2(11)$	$\mathcal{C}_{13}$	
15		3 12 6	1-(55,21,21)	$L_2(11)$	[55,45,3]	S_{11}
16		12(2) 6(2)	1-(55,36,36)	$L_2(11)$	[55,34,6]	$L_2(11)$
17		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_7$	
18		1 3 6(2)	1-(55,16,16)	$L_2(11)$	[55,20,10]	$L_2(11)$
19		3 6	1-(55,9,9)	$L_2(11)$	[55,21,9]	$L_2(11)$:2
20		1 3 6(2)	1-(55,16,16)	$L_2(11)$	[55,30,6]	$L_2(11)$:2
21		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
22		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$	[55,44,4]	$L_2(11)$:2
23		1 12(2) 6(3)	1-(55,43,43)	$L_2(11)$	$\mathcal{C}_{11}$	
24		1 12 6(3)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{11}$	
25		3(2) 12(2) 6(2)	1-(55,42,42)	$L_2(11)$	$\mathcal{C}_7$	
26		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
27		3 12(2) 6	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{11}$	
28		3 12 6(3)	1-(55,33,33)	$L_2(11)$	[55,45,4]	$L_2(11)$:2
29		1 12 6(3)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{15}$	
30		1 3 12(2) 6(4)	1-(55,52,52)	$L_2(11)$	$\mathcal{C}_7$	
31		3(2) 6(2)	1-(55,18,18)	$L_2(11)$:2	[55,20,10]	$L_2(11)$:2
32		3(2) 12	1-(55,18,18)	$L_2(11)$:2	$\mathcal{C}_{12}$	

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Table B.1.6 – continued from previous page

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
33		1 6	1-(55,7,7)	$L_2(11)$	[55,31,6]	$L_2(11)$:2
34		3 6(4)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
35		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	[55,34,8]	$L_2(11)$:2
36		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
37		1 3 6	1-(55,10,10)	$L_2(11)$	$\mathcal{C}_{12}$	
38		1 12	1-(55,13,13)	$L_2(11)$:2	$\mathcal{C}_{11}$	
39		3(2) 12(2) 6	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{18}$	
40		3(2) 12 6(3)	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{12}$	
41		1 12(2) 6(2)	1-(55,37,37)	$L_2(11)$:2	[55,11,18]	$L_2(11)$:2
42		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
43		1 3 12(2) 6	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{31}$	
44		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	[55,21,9]	$L_2(11)$
45		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_7$	
46		1 6(2)	1-(55,13,13)	$L_2(11)$	$\mathcal{C}_{15}$	
47		3(2) 12 6(3)	1-(55,36,36)	S_{11}	$\mathcal{C}_7$	
48		1 12 6(2)	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{28}$	
49		12 6(3)	1-(55,30,30)	$L_2(11)$	[55,44,4]	$L_2(11)$:2
50		3(2) 6(3)	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_7$	
51		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$	[55,35,4]	$L_2(11)$:2
52		1 3(2) 6(2)	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{13}$	
53		6(3)	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{49}$	
54		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
55		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$:2	[55,45,3]	$L_2(11)$:2
56		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{33}$	
57		12 6	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{16}$	
58		12(2) 6(3)	1-(55,42,42)	$L_2(11)$	[55,20,10]	$L_2(11)$:2
59		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{22}$	
60		1 3(2) 12(2)	1-(55,31,31)	$L_2(11)$:2	[55,11,15]	$L_2(11)$:2
61		3 12 6(4)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{11}$	
62		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{15}$	
63		1 3 6(3)	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{12}$	
64		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_7$	
65		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{33}$	
66		1 3 12(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
67		1 12 6	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{11}$	
68		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{35}$	
69		1 6(3)	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{44}$	
70		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{15}$	
71		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_7$	
72		3 6(3)	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{19}$	
;73		1 12 6	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{44}$	
74		1 3(2) 12 6	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{55}$	
75		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
76		3 12(2) 6(3)	1-(55,45,45)	$L_2(11)$	$\mathcal{C}_{28}$	
77		1 3(2) 12(2) 6	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{55}$	
78		6(2)	1-(55,12,12)	$L_2(11)$	$\mathcal{C}_{10}$	
79		6(2)	1-(55,12,12)	$L_2(11)$	$\mathcal{C}_{16}$	
80		1 3 6(2)	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_{20}$	
81		3 12(2) 6(3)	1-(55,45,45)	$L_2(11)$	$\mathcal{C}_{15}$	
82		12 6(2)	1-(55,24,24)	$L_2(11)$:2	$\mathcal{C}_{49}$	
83		1 3 6(2)	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_7$	
84		6(4)	1-(55,24,24)	$L_2(11)$:2	[55,10,20]	$L_2(11)$:2
85		1 3(2) 12(2) 6(3)	1-(55,49,49)	$L_2(11)$	$\mathcal{C}_{13}$	

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Table B.1.6 – continued from previous page

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
86		1 3(2) 12(2) 6	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{55}$	
87		12 6(3)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{49}$	
88		1 3	1-(55,4,4)	$L_2(11)$	$\mathcal{C}_{12}$	
89		1 3 12(2) 6	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{12}$	
90		12(2) 6	1-(55,30,30)	$L_2(11)$	[55,10,10]	S_{11}
91		1 3(2) 6(2)	1-(55,19,19)	$L_2(11)$:2	$\mathcal{C}_{55}$	
92		3 12	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{15}$	
93		1 3(2) 6(4)	1-(55,31,31)	$L_2(11)$:2	[55,25,10]	$L_2(11)$:2
94		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{11}$	
95		1 3 12(2) 6(3)	1-(55,46,46)	$L_2(11)$	$\mathcal{C}_{35}$	
96		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{15}$	
97		12(2) 6	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{16}$	
98		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{7}$	
99		1 3 12(2) 6(3)	1-(55,46,46)	$L_2(11)$	$\mathcal{C}_{31}$	
100		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{15}$	
101		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
102		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{22}$	
103		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{33}$	
104		3(2) 12 6	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{12}$	
105		1 12 6(2)	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{44}$	
106		3(2) 6	1-(55,12,12)	$L_2(11)$	$\mathcal{C}_{12}$	
107		3 12	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{44}$	
108		1 12 6(4)	1-(55,37,37)	$L_2(11)$:2	$\mathcal{C}_{11}$	
109		3(2) 12 6(3)	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{12}$	
110		1 12(2) 6(2)	1-(55,37,37)	$L_2(11)$:2	$\mathcal{C}_{19}$	
111		12	1-(55,12,12)	$L_2(11)$:2	[55,24,10]	$L_2(11)$:2
112		1 3(2) 12 6(3)	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{13}$	
113		3(2) 12 6(4)	1-(55,42,42)	$L_2(11)$:2	$\mathcal{C}_{12}$	
114		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{41}$	
115		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{20}$	
116		1 3(2) 12 6(3)	1-(55,37,37)	S_{11}	$\mathcal{C}_{8}$	
117		12 6(2)	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{90}$	
118		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{7}$	
119		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{28}$	
120		3(2) 6(2)	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{7}$	
121		1 6(3)	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{11}$	
122		3 12(2) 6	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{15}$	
123		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{12}$	
124		3 12(2) 6	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{28}$	
125		3(2) 6(2)	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{22}$	
126		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{12}$	
127		1 12(2) 6	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{11}$	
128		1 3 6(2)	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_{12}$	
129		3(2) 12(2) 6(2)	1-(55,42,42)	$L_2(11)$:2	$\mathcal{C}_{22}$	
130		3 6(3)	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{28}$	
131		1 3 6(4)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{7}$	
132		3 6	1-(55,9,9)	$L_2(11)$	$\mathcal{C}_{11}$	
133		1 12(2) 6(3)	1-(55,43,43)	$L_2(11)$	$\mathcal{C}_{15}$	
134		3(2) 12(2)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{12}$	
135		1 6(2)	1-(55,13,13)	$L_2(11)$:2	$\mathcal{C}_{9}$	
136		3(2) 6	1-(55,12,12)	$L_2(11)$	$\mathcal{C}_{7}$	
137		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{15}$	
138		3 12(2) 6(4)	1-(55,51,51)	$L_2(11)$	$\mathcal{C}_{11}$	

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Table B.1.6 – continued from previous page

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
139		3(2) 6(3)	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{12}$	
140		3(2) 12(2) 6(3)	1-(55,48,48)	$L_2(11)$	$\mathcal{C}_7$	
141		1 12(2) 6(2)	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{15}$	
142		1 3 6	1-(55,10,10)	$L_2(11)$	$\mathcal{C}_7$	
143		1 3 6	1-(55,10,10)	$L_2(11)$	$\mathcal{C}_{22}$	
144		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_8$	
145		1 12(2) 6(2)	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{28}$	
146		1 12 6(2)	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{19}$	
147		12 6(3)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{58}$	
148		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{22}$	
149		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
150		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_7$	
151		12 6(3)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{49}$	
152		1 3(2) 12	1-(55,19,19)	$L_2(11)$:2	$\mathcal{C}_{51}$	
153		3(2) 12 6	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_7$	
154		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{44}$	
155		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{33}$	
156		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{20}$	
157		1 3 12	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_{20}$	
158		1 3 12 6(4)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{18}$	
159		3(2) 6(4)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{20}$	
160		12 6(2)	1-(55,24,24)	$L_2(11)$:2	$\mathcal{C}_{49}$	
161		1 12 6(3)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{15}$	
162		1 3(2) 12 6	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{55}$	
163		3 12(2) 6	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_9$	
164		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{28}$	
165		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{15}$	
166		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{18}$	
167		1 12(2) 6	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{15}$	
168		12 6(2)	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{84}$	
169		3 6(3)	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{44}$	
170		1 3(2) 6	1-(55,13,13)	$L_2(11)$	$\mathcal{C}_{55}$	
171		12(2) 6(2)	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{10}$	
172		12(2) 6(2)	1-(55,36,36)	$L_2(11)$:2	$\mathcal{C}_{49}$	
173		12(2) 6(4)	1-(55,48,48)	$L_2(11)$:2	$\mathcal{C}_{10}$	
174		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{11}$	
175		1 3 6(3)	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{35}$	
176		6(3)	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{49}$	
177		6	1-(55,6,6)	$L_2(11)$	$\mathcal{C}_{16}$	
178		1 3(2) 12(2) 6(2)	1-(55,43,43)	$L_2(11)$:2	[55,21,10]	$L_2(11)$:2
179		3 12(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
180		1 3 6(3)	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{22}$	
181		1 3(2) 12(2) 6(2)	1-(55,43,43)	$L_2(11)$	$\mathcal{C}_{51}$	
182		1 3(2) 12(2) 6(2)	1-(55,43,43)	$L_2(11)$	$\mathcal{C}_{13}$	
183		6	1-(55,6,6)	$L_2(11)$	$\mathcal{C}_{16}$	
184		1 3 6(3)	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_7$	
185		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_7$	
186		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
187		1 6(2)	1-(55,13,13)	$L_2(11)$	$\mathcal{C}_{28}$	
188		3(2) 12 6	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{20}$	
189		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{60}$	
190		1 3 12(2) 6(3)	1-(55,46,46)	$L_2(11)$	$\mathcal{C}_{18}$	
191		3 12 6(4)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{33}$	

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Table B.1.6 – continued from previous page

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
192		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{31}$	
193		12 6	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{16}$	
194		1 12 6(2)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{28}$	
195		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$:2	$\mathcal{C}_{55}$	
196		1 12 6	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{11}$	
197		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{35}$	
198		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{18}$	
199		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{11}$	
200		6(2)	1-(55,12,12)	$L_2(11)$:2	$\mathcal{C}_{58}$	
201		1 3(2) 6	1-(55,13,13)	$L_2(11)$	$\mathcal{C}_{178}$	
202		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{31}$	
203		1 12 6	1-(55,19,19)	S_{11}	$\mathcal{C}_{15}$	
204		1 3 6(2)	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_{18}$	
205		3 6	1-(55,9,9)	$L_2(11)$	$\mathcal{C}_{44}$	
206		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{15}$	
207		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{28}$	
208		3(2) 12(2) 6(2)	1-(55,42,42)	$L_2(11)$	$\mathcal{C}_{22}$	
209		12 6(2)	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{16}$	
210		3(2)	1-(55,6,6)	$L_2(11)$:2	$\mathcal{C}_{20}$	
211		1 3(2) 12 6	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{55}$	
212		3 12(2) 6(3)	1-(55,45,45)	$L_2(11)$	$\mathcal{C}_{28}$	
213		3(2) 12(2) 6(3)	1-(55,48,48)	$L_2(11)$	$\mathcal{C}_{20}$	
214		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{12}$	
215		1 12(2)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{33}$	
216		1 3(2) 12 6(4)	1-(55,43,43)	$L_2(11)$:2	$\mathcal{C}_{93}$	
217		1 3 12(2) 6	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{18}$	
218		12 6	1-(55,18,18)	$L_2(11)$	$\mathcal{C}_{16}$	
219		3 12(2) 6(2)	1-(55,39,39)	$L_2(11)$	$\mathcal{C}_{44}$	
220		1 12 6(2)	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{15}$	
221		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{22}$	
222		1 3(2) 6(2)	1-(55,19,19)	$L_2(11)$	$\mathcal{C}_{51}$	
223		1 12 6(4)	1-(55,37,37)	$L_2(11)$:2	$\mathcal{C}_{11}$	
224		1 3 6	1-(55,10,10)	$L_2(11)$	$\mathcal{C}_{22}$	
225		1 3(2) 12(2) 6(2)	1-(55,43,43)	$L_2(11)$:2	$\mathcal{C}_{178}$	
226		3(2) 12(2) 6(2)	1-(55,42,42)	$L_2(11)$:2	$\mathcal{C}_{35}$	
227		1 3(2) 12 6(3)	1-(55,37,37)	$L_2(11)$	$\mathcal{C}_{13}$	
228		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{7}$	
229		3 6(3)	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{11}$	
230		12	1-(55,12,12)	$L_2(11)$:2	$\mathcal{C}_{111}$	
231		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{28}$	
232		1 6	1-(55,7,7)	$L_2(11)$	$\mathcal{C}_{15}$	
233		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{7}$	
234		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{7}$	
235		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$	$\mathcal{C}_{7}$	
236		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{22}$	
237		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{11}$	
238		1 12 6(2)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{9}$	
239		6(2)	1-(55,12,12)	$L_2(11)$:2	$\mathcal{C}_{58}$	
240		3(2) 12 6	1-(55,24,24)	$L_2(11)$	$\mathcal{C}_{7}$	
241		3	1-(55,3,3)	$L_2(11)$	$\mathcal{C}_{15}$	
242		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
243		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{22}$	
244		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{13}$	

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Table B.1.6 – continued from previous page

i	Max	Block	Design $\mathcal{D}_i$	Aut($\mathcal{D}_i$)	Code $\mathcal{C}_i$	Aut($\mathcal{C}_i$)
245		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{12}$	
246		1 12(2) 6(4)	1-(55,49,49)	$L_2(11)$:2	$\mathcal{C}_{33}$	
247		1 3(2) 12	1-(55,19,19)	$L_2(11)$:2	$\mathcal{C}_{51}$	
248		3 6	1-(55,9,9)	$L_2(11)$	$\mathcal{C}_{9}$	
249		3(2) 12(2) 6(4)	1-(55,54,54)	S_{55}	$\mathcal{C}_{12}$	
250		1 3 12	1-(55,16,16)	$L_2(11)$	$\mathcal{C}_{12}$	
251		12(2)	1-(55,24,24)	$L_2(11)$:2	$\mathcal{C}_{111}$	
252		1 3 12(2) 6(2)	1-(55,40,40)	$L_2(11)$	$\mathcal{C}_{7}$	
253		1 3(2) 6(3)	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{13}$	
254		12 6(4)	1-(55,36,36)	$L_2(11)$:2	$\mathcal{C}_{10}$	
255		1 3(2) 12 6(4)	1-(55,43,43)	$L_2(11)$:2	$\mathcal{C}_{93}$	
256		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{11}$	
257		1 3 12 6(3)	1-(55,34,34)	$L_2(11)$	[55,10,18]	$L_2(11)$:2
258		1 3 12 6	1-(55,22,22)	$L_2(11)$	$\mathcal{C}_{20}$	
259		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$:2	$\mathcal{C}_{55}$	
260		12 6	1-(55,18,18)	S_{11}	$\mathcal{C}_{90}$	
261		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
262		3 12 6(2)	1-(55,27,27)	$L_2(11)$	$\mathcal{C}_{15}$	
263		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{11}$	
264		12(2) 6(3)	1-(55,42,42)	$L_2(11)$	$\mathcal{C}_{49}$	
265		1 3 12(2) 6(3)	1-(55,46,46)	$L_2(11)$	$\mathcal{C}_{12}$	
266		1 12 6(2)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{28}$	
267		1 3(2)	1-(55,7,7)	$L_2(11)$:2	$\mathcal{C}_{51}$	
268		12(2) 6(2)	1-(55,36,36)	$L_2(11)$:2	$\mathcal{C}_{49}$	
269		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
270		1 12 6(3)	1-(55,31,31)	$L_2(11)$	$\mathcal{C}_{33}$	
271		1 3 12 6(2)	1-(55,28,28)	$L_2(11)$	$\mathcal{C}_{12}$	
272		3 12 6	1-(55,21,21)	$L_2(11)$	$\mathcal{C}_{33}$	
273		3(2) 12	1-(55,18,18)	$L_2(11)$:2	$\mathcal{C}_{12}$	
274		3(2) 12 6(2)	1-(55,30,30)	$L_2(11)$:2	$\mathcal{C}_{22}$	
275		3 12 6(3)	1-(55,33,33)	$L_2(11)$	$\mathcal{C}_{19}$	
276		1 12	1-(55,13,13)	$L_2(11)$:2	$\mathcal{C}_{11}$	
277		1 12 6(2)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{9}$	
278		3(2) 12 6(3)	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{18}$	
279		3 6(2)	1-(55,15,15)	$L_2(11)$	$\mathcal{C}_{11}$	
280		3(2) 12(2) 6	1-(55,36,36)	$L_2(11)$	$\mathcal{C}_{12}$	
281		1 3(2) 6(2)	1-(55,19,19)	$L_2(11)$:2	$\mathcal{C}_{55}$	
282		1 6(2)	1-(55,13,13)	$L_2(11)$:2	$\mathcal{C}_{28}$	
283		1 3 12(2) 6	1-(55,34,34)	$L_2(11)$	$\mathcal{C}_{22}$	
284		3(2) 12 6(4)	1-(55,42,42)	$L_2(11)$:2	$\mathcal{C}_{12}$	
285		3(2) 6(2)	1-(55,18,18)	$L_2(11)$:2	$\mathcal{C}_{257}$	
286		1 3(2) 12 6	1-(55,25,25)	$L_2(11)$	$\mathcal{C}_{178}$	
287		12 6(4)	1-(55,36,36)	$L_2(11)$:2	$\mathcal{C}_{10}$	
288		12 6(2)	1-(55,24,24)	$L_2(11)$:2	$\mathcal{C}_{49}$	
289		1 3(2) 12(2) 6(4)	1-(55,55,1)	S_{55}	[55,1,55]	S_{55}
290		1 3(2) 12 6(2)	1-(55,31,31)	$L_2(11)$:2	$\mathcal{C}_{55}$	
291		1 6(4)	1-(55,25,25)	$L_2(11)$:2	$\mathcal{C}_{11}$	
292		12 6(2)	1-(55,24,24)	$L_2(11)$:2	$\mathcal{C}_{49}$	

$q = 13$

Max	Degree	rank	Length
13:6	14	2	1, 13
D_{14}	78	9	1, 7(5), 14(3)
D_{12}	91	11	1, 4(3), 6, 12(6)
A_4	91	12	1, 3(2), 6(4), 12(5)

Table B.1.7: Orbits of the maximal subgroups

Here as examples we only deal with designs and codes from the second maximal subgroup D_{14}. For the other maximal subgroups computations has been done but are not listed here.

Table B.1.8: codes and designs from $L_2(13)$.

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
1		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	[78,78,1]	S_{78}
2		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_1$	
3		14 7(2)	1-(78,28,28)	$L_2(13)$	[78,51,6]	$L_2(13)$
4	D_{14}	14 7(2)	1-(78,28,28)	$L_2(13)$	[78,51,6]	$L_2(13)$:2
5		14(3) 7(3)	1-(78,63,63)	$L_2(13)$	[78,64,4]	$L_2(13)$
6		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	[78,65,4]	$L_2(13)$:2
7		1 14(2) 7(5)	1-(78,64,64)	$L_2(13)$:2	[78,37,12]	$L_2(13)$:2
8		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_1$	
9		14(3) 7	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_1$	
10		1 14 7(3)	1-(78,36,36)	$L_2(13)$	[78,14,24]	$L_2(13)$
11		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
12		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_3$	
13		7(2)	1-(78,14,14)	$L_2(13)$:2	[78,28,14]	$L_2(13)$:2
14		1 7(3)	1-(78,22,22)	$L_2(13)$:2	$\mathcal{C}_6$	
15		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
16		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
17		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
18		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_5$	
19		1 14 7(3)	1-(78,36,36)	$L_2(13)$	[78,50,6]	$L_2(13)$:2
20		14(2) 7(4)	1-(78,56,56)	$L_2(13)$:2	$\mathcal{C}_{19}$	
21		1 14 7(3)	1-(78,36,36)	$L_2(13)$	[78,50,8]	$L_2(13)$
22		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	[78,64,4]	$L_2(13)$:2
23		1 14(3) 7(4)	1-(78,71,71)	$L_2(13)$	$\mathcal{C}_1$	
24		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
25		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_4$	
26		1 7	1-(78,8,8)	$L_2(13)$:2	$\mathcal{C}_4$	
27		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{21}$	
28		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
29		14 7(3)	1-(78,35,35)	$L_2(13)$:2	[78,64,4]	$L_2(13)$:2
30		1 7(2)	1-(78,15,15)	$L_2(13)$	$\mathcal{C}_{29}$	
31		1 14	1-(78,15,15)	$L_2(13)$:2	$\mathcal{C}_1$	
32		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
33		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_6$	
34		1 14(3) 7	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_3$	
35		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_3$	
36		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_{29}$	
37		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{29}$	
38		1 7(2)	1-(78,15,15)	$L_2(13)$	$\mathcal{C}_5$	
39		14 7(5)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_1$	
40		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{22}$	
41		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	[78,15,22]	$L_2(13)$

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Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
42		1 14 7(4)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_1$	
43		1 14 7(3)	1-(78,36,36)	$L_2(13)$	[78,29,14]	$L_2(13)$:2
44		7(3)	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_5$	
45		14 7(3)	1-(78,35,35)	$L_2(13)$:2	[78,50,6]	$L_2(13)$:2
46		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{22}$	
47		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_{29}$	
48		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
49		14(3) 7(2)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_{22}$	
50		14 7	1-(78,21,21)	$L_2(13)$:2	$\mathcal{C}_1$	
51		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{45}$	
52		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
53		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_{43}$	
54		7(3)	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_{29}$	
55		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{29}$	
56		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_{29}$	
57		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_3$	
58		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{22}$	
59		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{21}$	
60		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
61		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{29}$	
62		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_{29}$	
63		7(4)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_{21}$	
64		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_6$	
65		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_3$	
66		14(3) 7(3)	1-(78,63,63)	$L_2(13)$	$\mathcal{C}_5$	
67		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{10}$	
68		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{45}$	
69		14(3) 7(3)	1-(78,63,63)	$L_2(13)$	$\mathcal{C}_{29}$	
70		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_5$	
71		14 7	1-(78,21,21)	$L_2(13)$:2	$\mathcal{C}_1$	
72		7(2)	1-(78,14,14)	$L_2(13)$	$\mathcal{C}_{22}$	
73		1 7(3)	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_6$	
74		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_6$	
75		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_3$	
76		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_{29}$	
77		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
78		1 14(3) 7(4)	1-(78,71,71)	$L_2(13)$:2	[78,42,7]	$L_2(13)$:2
79		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{45}$	
80		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_6$	
81		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	[78,15,24]	$L_2(13)$:2
82		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{13}$	
83		1 7	1-(78,8,8)	$L_2(13)$	$\mathcal{C}_{22}$	
84		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
85		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_1$	
86		1 7(4)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_1$	
87		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
88		1 14 7(5)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_7$	
89		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
90		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$	$\mathcal{C}_{21}$	
91		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{19}$	
92		7(3)	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_5$	
93		1 7(4)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_1$	
94		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_{78}$	

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Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
95		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_1$	
96		14(3)	1-(78,42,42)	$L_2(13)$:2	[78,36,12]	$L_2(13)$:2
97		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$:2	$\mathcal{C}_{43}$	
98		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_3$	
99		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
100		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_1$	
101		14(3) 7(2)	1-(78,56,56)	$L_2(13)$:2	$\mathcal{C}_{22}$	
102		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_4$	
103		1 14(2) 7	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_4$	
104		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
105		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
106		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
107		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{19}$	
108		14(2) 7(4)	1-(78,56,56)	$L_2(13)$:2	$\mathcal{C}_{19}$	
109		14	1-(78,14,14)	$L_2(13)$:2	$\mathcal{C}_{96}$	
110		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_1$	
111		14(2) 7(5)	1-(78,63,63)	$L_2(13)$:2	$\mathcal{C}_1$	
112		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	[78,28,18]	$L_2(13)$
113		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
114		1 14 7(5)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_7$	
115		1 14(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_1$	
116		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_{78}$	
117		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
118		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
119		7(4)	1-(78,28,28)	$L_2(13)$:2	[78,14,24]	$L_2(13)$:2
120		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_3$	
121		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_5$	
122		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
123		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_6$	
124		1 7(3)	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{19}$	
125		1 14(2) 7	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_4$	
126		1 7(3)	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{21}$	
127		7(2)	1-(78,14,14)	$L_2(13)$:2	$\mathcal{C}_{13}$	
128		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_4$	
129		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{41}$	
130		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$	$\mathcal{C}_{21}$	
131		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{29}$	
132		14(3) 7(4)	1-(78,70,70)	$L_2(13)$	$\mathcal{C}_{21}$	
133		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_6$	
134		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_3$	
135		1 7(2)	1-(78,15,15)	$L_2(13)$:2	$\mathcal{C}_{29}$	
136		14(2) 7(5)	1-(78,63,63)	$L_2(13)$:2	$\mathcal{C}_1$	
137		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_6$	
138		1 14(3) 7	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{22}$	
139		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
140		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
141		1 14 7(2)	1-(78,29,29)	$L_2(13)$	[78,28,12]	$L_2(13)$:2
142		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{22}$	
143		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
144		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_{112}$	
145		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
146		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_{29}$	
147		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{41}$	

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Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
148		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_1$	
149		14 7(5)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_1$	
150		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_1$	
151		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
152		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_3$	
153		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
154		1 14(3)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{78}$	
155		1 7(3)	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{21}$	
156		7(4)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_6$	
157		14(3) 7	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_1$	
158		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$:2	[78,14,21]	$L_2(13)$:2
159		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{22}$	
160		1 14 7(4)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_1$	
161		14 7(3)	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_{45}$	
162		14(3) 7(2)	1-(78,56,56)	$L_2(13)$:2	$\mathcal{C}_{22}$	
163		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{21}$	
164		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_4$	
165		7	1-(78,7,7)	$L_2(13)$	$\mathcal{C}_1$	
166		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_1$	
167		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
168		14(3) 7(2)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_3$	
169		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$	$\mathcal{C}_{19}$	
170		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_1$	
171		7	1-(78,7,7)	$L_2(13)$	$\mathcal{C}_1$	
172		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{22}$	
173		14 7(3)	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_{45}$	
174		1 7(2)	1-(78,15,15)	$L_2(13)$	$\mathcal{C}_5$	
175		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
176		1 14 7(4)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_1$	
177		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_6$	
178		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{45}$	
179		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_{112}$	
180		14 7	1-(78,21,21)	$L_2(13)$:2	$\mathcal{C}_1$	
181		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
182		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{13}$	
183		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{21}$	
184		1 14(3) 7(4)	1-(78,71,71)	$L_2(13)$	$\mathcal{C}_1$	
185		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{29}$	
186		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_{141}$	
187		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
188		14 7(4)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_6$	
189		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_3$	
190		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{29}$	
191		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_{29}$	
192		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{19}$	
193		14(3) 7(3)	1-(78,63,63)	$L_2(13)$	$\mathcal{C}_5$	
194		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_{22}$	
195		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_{112}$	
196		7(2)	1-(78,14,14)	$L_2(13)$	$\mathcal{C}_3$	
197		7(2)	1-(78,14,14)	$L_2(13)$	$\mathcal{C}_4$	
198		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_6$	
199		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
200		14(3) 7(4)	1-(78,70,70)	$L_2(13)$	$\mathcal{C}_6$	

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Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
201		1 14(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_1$	
202		1 14(2) 7(5)	1-(78,64,64)	$L_2(13)$:2	$\mathcal{C}_7$	
203		7(3)	1-(78,21,21)	$L_2(13)$:2	$\mathcal{C}_{141}$	
204		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{21}$	
205		1 14 7	1-(78,22,22)	$L_2(13)$	$\mathcal{C}_{22}$	
206		7(5)	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_{78}$	
207		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{29}$	
208		14(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{96}$	
209		14(3) 7(2)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_3$	
210		1 14(2) 7	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_4$	
211		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{45}$	
212		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_{78}$	
213		1 7(4)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_1$	
214		14(2) 7(4)	1-(78,56,56)	$L_2(13)$:2	$\mathcal{C}_{19}$	
215		1 7(2)	1-(78,15,15)	$L_2(13)$:2	$\mathcal{C}_{45}$	
216		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
217		14(2) 7	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_1$	
218		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
219		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
220		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_{10}$	
221		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_1$	
222		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$:2	$\mathcal{C}_{141}$	
223		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_{29}$	
224		14	1-(78,14,14)	$L_2(13)$:2	$\mathcal{C}_{96}$	
225		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
226		1 7	1-(78,8,8)	$L_2(13)$	$\mathcal{C}_3$	
227		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
228		14(2) 7	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_1$	
229		1 14 7(5)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_7$	
230		14 7(3)	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_{29}$	
231		1 14(3) 7(2)	1-(78,57,57)	$L_2(13)$	$\mathcal{C}_{29}$	
232		14 7(4)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{19}$	
233		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
234		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{45}$	
235		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
236		1 14 7	1-(78,22,22)	$L_2(13)$:2	$\mathcal{C}_4$	
237		14(3) 7(3)	1-(78,63,63)	$L_2(13)$:2	$\mathcal{C}_{45}$	
238		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
239		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_3$	
240		7(2)	1-(78,14,14)	$L_2(13)$	$\mathcal{C}_3$	
241		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_{21}$	
242		14(3) 7(2)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_4$	
243		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{19}$	
244		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_{22}$	
245		1 14(3) 7	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_{81}$	
246		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_{29}$	
247		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
248		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_6$	
249		14 7(4)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{19}$	
250		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_{29}$	
251		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_5$	
252		1 7(3)	1-(78,22,22)	$L_2(13)$:2	$\mathcal{C}_6$	
253		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	

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Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
254		14(3) 7(5)	1-(78,77,77)	S_{78}	$\mathcal{C}_1$	
255		14(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{96}$	
256		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_{112}$	
257		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{22}$	
258		1 14	1-(78,15,15)	$L_2(13)$:2	$\mathcal{C}_1$	
259		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_{22}$	
260		1 7(5)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_7$	
261		14(2) 7	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_1$	
262		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_6$	
263		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$:2	$\mathcal{C}_1$	
264		1 14(2) 7(5)	1-(78,64,64)	$L_2(13)$:2	$\mathcal{C}_7$	
265		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{22}$	
266		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$	$\mathcal{C}_{21}$	
267		1 14 7(2)	1-(78,29,29)	$L_2(13)$	$\mathcal{C}_{112}$	
268		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_1$	
269		14(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{96}$	
270		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_6$	
271		14(2) 7(2)	1-(78,42,42)	$L_2(13)$	$\mathcal{C}_3$	
272		14 7(3)	1-(78,35,35)	$L_2(13)$	$\mathcal{C}_5$	
273		14 7	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_1$	
274		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_{22}$	
275		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
276		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{22}$	
277		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$	$\mathcal{C}_6$	
278		7(3)	1-(78,21,21)	$L_2(13)$	$\mathcal{C}_5$	
279		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_6$	
280		14(3) 7	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_1$	
281		1 14 7(4)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_{78}$	
282		1 14 7(3)	1-(78,36,36)	$L_2(13)$:2	$\mathcal{C}_6$	
283		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{119}$	
284		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
285		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$	$\mathcal{C}_5$	
286		14 7(4)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{19}$	
287		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
288		1 14 7	1-(78,22,22)	$L_2(13)$:2	$\mathcal{C}_4$	
289		14(2) 7(4)	1-(78,56,56)	$L_2(13)$	$\mathcal{C}_{21}$	
290		1 14	1-(78,15,15)	$L_2(13)$:2	$\mathcal{C}_1$	
291		14 7(3)	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_{29}$	
292		1 14(2) 7	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_3$	
293		14 7(5)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_1$	
294		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{29}$	
295		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$:2	$\mathcal{C}_1$	
296		7(3)	1-(78,21,21)	$L_2(13)$:2	$\mathcal{C}_{158}$	
297		14(2) 7(3)	1-(78,49,49)	$L_2(13)$	$\mathcal{C}_5$	
298		14 7(2)	1-(78,28,28)	$L_2(13)$:2	$\mathcal{C}_{22}$	
299		1 7(2)	1-(78,15,15)	$L_2(13)$	$\mathcal{C}_5$	
300		7	1-(78,7,7)	$L_2(13)$:2	$\mathcal{C}_{78}$	
301		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{45}$	
302		14 7(2)	1-(78,28,28)	$L_2(13)$	$\mathcal{C}_3$	
303		1 14(3) 7(3)	1-(78,64,64)	$L_2(13)$:2	$\mathcal{C}_{43}$	
304		14(2) 7(5)	1-(78,63,63)	$L_2(13)$:2	$\mathcal{C}_1$	
305		1 14 7(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_{29}$	
306		1 14 7(3)	1-(78,36,36)	$L_2(13)$	$\mathcal{C}_{21}$	

Continued on next page

Table B.1.8 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
307		1 14(2)	1-(78,29,29)	$L_2(13)$:2	$\mathcal{C}_1$	
308		14(3) 7(3)	1-(78,63,63)	$L_2(13)$:2	$\mathcal{C}_{29}$	
309		14(2) 7	1-(78,35,35)	$L_2(13)$:2	$\mathcal{C}_1$	
310		1 14(2) 7(2)	1-(78,43,43)	$L_2(13)$:2	$\mathcal{C}_{45}$	
311		1 14(2) 7(4)	1-(78,57,57)	$L_2(13)$:2	$\mathcal{C}_1$	
312		14(2) 7(3)	1-(78,49,49)	$L_2(13)$:2	$\mathcal{C}_{45}$	
313		1 14(3) 7(5)	1-(78,78,1)	S_{78}	[78,1,78]	S_{78}
314		14(2) 7(2)	1-(78,42,42)	$L_2(13)$:2	$\mathcal{C}_{22}$	
315		14	1-(78,14,14)	$L_2(13)$:2	$\mathcal{C}_{96}$	
316		1 14(2) 7(3)	1-(78,50,50)	$L_2(13)$:2	$\mathcal{C}_6$	
317		1 14 7	1-(78,22,22)	$L_2(13)$:2	$\mathcal{C}_4$	
318		14(3) 7(4)	1-(78,70,70)	$L_2(13)$:2	$\mathcal{C}_{19}$	

$q = 16$

Max	Degree	rank	Length
2^4:15	17	2	1,16
A_5	68	5	1, 12, 15, 20(2)
D_{34}	120	8	1, 17(7)
D_{30}	136	9	1, 4(2), 8(6), 16(6)

Table B.1.9: Maximal subgroups and their orbits

As illustration we list the designs and codes from the maximal subgroups A_5 and D_{34}.

Table B.1.10: Codes and designs from $L_2(16)$.

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
1	A_5	1 20	1-(68,21,21)	$L_2(16)$:2	[68,52,5]	$L_2(16)$:2
2		15	1-(68,15,15)	$L_2(16)$:4	$\mathcal{C}_1$	
3		1 15	1-(68,16,16)	$L_2(16)$:4	[68,17,16]	$L_2(16)$:4
4		1 20(2) 12	1-(68,53,53)	$L_2(16)$:4	$\mathcal{C}_1$	
5		20(2)	1-(68,40,40)	$L_2(16)$:4	[68,8,32]	$L_2(16)$:4
6		1 12 20 15	1-(68,48,48)	$L_2(16)$:2	[68,25,12]	$L_2(16)$:2
7		1 12 20	1-(68,33,33)	$L_2(16)$:2	[68,68,1]	S_{68}
8		20	1-(68,20,20)	$L_2(16)$:2	[68,24,12]	$L_2(16)$:2
9		12 20(2)	1-(68,52,52)	$L_2(16)$:4	[68,16,22]	$L_2(16)$:4
10		1 20(2) 15	1-(68,56,56)	$L_2(16)$:4	$\mathcal{C}_6$	
11		20 15	1-(68,35,35)	$L_2(16)$:2	$\mathcal{C}_7$	
12		20 12 15	1-(68,47,47)	$L_2(16)$:2	$\mathcal{C}_1$	
13		1 12 15	1-(68,28,28)	$L_2(16)$:4	[68,9,28]	$L_2(16)$:4
14		20(2) 15	1-(68,55,55)	$L_2(16)$:4	$\mathcal{C}_1$	
15		12	1-(68,12,12)	$L_2(16)$:4	$\mathcal{C}_8$	
16		12 15	1-(68,27,27)	$L_2(16)$:4	$\mathcal{C}_7$	
17		1 20(2) 12 15	1-(68,68,1)	S_{68}	[68,1,68]	S_{68}
18		1 12	1-(68,13,13)	$L_2(16)$:4	$\mathcal{C}_1$	
19		20 12	1-(68,32,32)	$L_2(16)$:2	$\mathcal{C}_5$	
20		1 20 15	1-(68,36,36)	$L_2(16)$:2	$\mathcal{C}_{13}$	
21		1 20(2)	1-(68,41,41)	$L_2(16)$:4	$\mathcal{C}_7$	
22		12 20(2) 15	1-(68,67,67)	$L_2(16)$:2	$\mathcal{C}_7$	

Continued on next page

Table B.1.10 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
23		17 1	1-(120,18,18)	$L_2(16)$	[120,44,18]	$L_2(16)$
24		17(3)	1-(120,51,51)	$L_2(16)$	[120,120,1]	S_{120}
25		17 1	1-(120,18,18)	$L_2(16)$:2	[120,44,18]	$L_2(16)$:2
26		17(5)	1-(120,85,85)	$L_2(16)$	C_{24}	
27		1 17(5)	1-(120,86,86)	$L_2(16)$	C_{24}	
28		1 17(5)	1-(120,86,86)	$L_2(16)$	C_{24}	
29		17(5)	1-(120,85,85)	$L_2(16)$	C_{24}	
30		1 17(2)	1-(120,35,35)	$L_2(16)$	C_{24}	
31		17(3)	1-(120,51,51)	$L_2(16)$	C_{24}	
32		1 17(2)	1-(120,35,35)	$L_2(16)$	C_{24}	
33		17(3)	1-(120,51,51)	$L_2(16)$	C_{24}	
34		1 17(3)	1-(120,52,52)	$L_2(16)$	[120,44.24]	$L_2(16)$
35		1 17(3)	1-(120,52,52)	$L_2(16)$	[120,44.20]	$L_2(16)$:2
36		17(4)	1-(120,68,68)	$L_2(16)$	C_{34}	
37		17(4)	1-(120,68,68)	$L_2(16)$	C_{34}	
38		1 17(2)	1-(120,35,35)	$L_2(16)$	C_{24}	
39		17(4)	1-(120,68,68)	$L_2(16)$	[120,10,52]	$L_2(16)$
40		1 17(3)	1-(120,52,52)	$L_2(16)$	C_{34}	
41		17(3)	1-(120,51,51)	$L_2(16)$:2	C_{24}	
42		1 17(5)	1-(120,86,86)	$L_2(16)$:2	[120,36,18]	$L_2(16)$:4
43		1 17(3)	1-(120,52,52)	$L_2(16)$	C_{34}	
44		17(3)	1-(120,51,51)	$L_2(16)$	C_{24}	
45		1 17(3)	1-(120,52,52)	$L_2(16)$	C_{34}	
46		17(4)	1-(120,68,68)	$L_2(16)$:2	C_{35}	
47		1 17(5)	1-(120,86,86)	$L_2(16)$	C_{24}	
48		17(4)	1-(120,68,68)	$L_2(16)$	C_{35}	
49		17(3)	1-(120,51,51)	$L_2(16)$	C_{24}	
50		17(2)	1-(120,34,34)	$L_2(16)$	C_{24}	
51		1 17(3)	1-(120,52,52)	$L_2(16)$	C_{39}	
52		17(4) 1	1-(120,69,69)	$L_2(16)$	C_{24}	
53		1 17(3)	1-(120,52,52)	$L_2(16)$	[120,36.24]	$L_2(16)$:4
54		17(4) 1	1-(120,69,69)	$L_2(16)$	C_{24}	
55		1 17(2)	1-(120,35,35)	$L_2(16)$:2	C_{24}	
56		17(2)	1-(120,34,34)	$L_2(16)$	C_{25}	
57		1 17(3)	1-(120,52,52)	$L_2(16)$:2	[120,10,52]	$L_2(16)$:2
58		17(4)	1-(120,68,68)	$L_2(16)$	C_{34}	
59		1 17(2)	1-(120,35,35)	$L_2(16)$:4	C_{24}	
60		17(5)	1-(120,85,85)	$L_2(16)$:4	C_{24}	
61		17(7)	1-(120,119,119)	$L_2(16)$:2	C_{24}	
62		17(3)	1-(120,51,51)	$L_2(16)$	C_{24}	
63		17(5)	1-(120,85,85)	$L_2(16)$	C_{24}	
64		17(4)	1-(120,68,68)	$L_2(16)$	C_{53}	
65		17(4) 1	1-(120,69,69)	$L_2(16)$	C_{24}	
66		1 17(5)	1-(120,86,86)	$L_2(16)$	C_{25}	
67		17(2)	1-(120,34,34)	$L_2(16)$	C_{24}	
68		17(4) 1	1-(120,69,69)	$L_2(16)$	C_{24}	
69		17(4) 1	1-(120,69,69)	$L_2(16)$	C_{24}	
70		17(6)	1-(120,102,102)	$L_2(16)$	C_{24}	
71		1 17(2)	1-(120,35,35)	$L_2(16)$	C_{24}	
72		17(5)	1-(120,85,85)	$L_2(16)$:2	C_{24}	
73		17(4) 1	1-(120,69,69)	$L_2(16)$:2	C_{24}	
74		1 17(5)	1-(120,86,86)	$L_2(16)$:2	C_{25}	
75		17(6)	1-(120,102,102)	$L_2(16)$:4	C_{42}	

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Table B.1.10 – continued from previous page

i	Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
76		17(2)	1-(120,34,34)	$L_2(16)$	$\mathcal{C}_{24}$	
77		17(4) 1	1-(120,69,69)	$L_2(16)$	$\mathcal{C}_{24}$	
78		17(4) 1	1-(120,69,69)	$L_2(16)$:2	$\mathcal{C}_{24}$	
79		17(4)	1-(120,68,68)	$L_2(16)$	$\mathcal{C}_{34}$	
80		17(3)	1-(120,51,51)	$L_2(16)$	$\mathcal{C}_{24}$	
81		17	1-(120,17,17)	$L_2(16)$	$\mathcal{C}_{24}$	
82		1 17(2)	1-(120,35,35)	$L_2(16)$:2	$\mathcal{C}_{24}$	
83		17(5)	1-(120,85,85)	$L_2(16)$:2	$\mathcal{C}_{24}$	
84		17	1-(120,17,17)	$L_2(16)$:2	$\mathcal{C}_{24}$	
85		17	1-(120,17,17)	$L_2(16)$:4	$\mathcal{C}_{24}$	
86		17(4) 1	1-(120,69,69)	$L_2(16)$	$\mathcal{C}_{24}$	
87		17(4)	1-(120,68,68)	$L_2(16)$:2	$\mathcal{C}_{57}$	
88		1 17(3)	1-(120,52,52)	$L_2(16)$:2	$\mathcal{C}_{35}$	
89		1 17(6)	1-(120,103,103)	$L_2(16)$:2	$\mathcal{C}_{24}$	
90		17(3)	1-(120,51,51)	$L_2(16)$	$\mathcal{C}_{24}$	
91		17(5)	1-(120,85,85)	$L_2(16)$	$\mathcal{C}_{24}$	
92		17(4) 1	1-(120,69,69)	$L_2(16)$:2	$\mathcal{C}_{24}$	
93		17(3)	1-(120,51,51)	$L_2(16)$:2	$\mathcal{C}_{24}$	
94		1 17(5)	1-(120,86,86)	$L_2(16)$:4	$\mathcal{C}_{42}$	
95		17(6)	1-(120,102,102)	$L_2(16)$:2	$\mathcal{C}_{25}$	
96		1 17(6)	1-(120,103,103)	$L_2(16)$	$\mathcal{C}_{24}$	
97		17(2)	1-(120,34,34)	$L_2(16)$:2	$\mathcal{C}_{25}$	
98		1 17(3)	1-(120,52,52)	$S_4(4)$:2	[120,10,52]	$S_4(4)$:2
99		1 17(6)	1-(120,103,103)	$L_2(16)$:4	$\mathcal{C}_{24}$	
100		17(4) 1	1-(120,69,69)	$S_4(4)$:2	$\mathcal{C}_{24}$	
101		17(2)	1-(120,34,34)	$L_2(16)$:2	$\mathcal{C}_{42}$	
102		1 17(3)	1-(120,52,52)	$L_2(16)$:2	$\mathcal{C}_{35}$	
103		17(7) 1	1-(120,120,1)	S_{120}	[120,1,120]	S_{120}
104		17(2)	1-(120,34,34)	$L_2(16)$:4	$\mathcal{C}_{42}$	
105		17 1	1-(120,18,18)	$L_2(16)$:4	$\mathcal{C}_{42}$	
106		17(4)	1-(120,68,68)	$L_2(16)$:2	$\mathcal{C}_{35}$	
107		17(3)	1-(120,51,51)	$S_4(4)$:2	$\mathcal{C}_{24}$	
108		17(4)	1-(120,68,68)	$S_4(4)$:2	$\mathcal{C}_{98}$	

$q = 17$

Max	Degree	rank	Length
17:8	18	2	1, 17
S_4	102	8	1, 3, 8, 12, 24(3)
D_{18}	136	12	1, 9(7), 18(4)
D_{16}	153	15	1, 4(2), 8(6), 16(6)

Table B.1.11: Maximal subgroups and their orbits

Table B.1.12: codes and designs from $L_2(17)$.

Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
S_4	3	1-(102,3,3)	$L_2(17)$	[102,84,3]	$L_2(17)$
	1 3	1-(102,4,4)	$L_2(17)$	[102,93,4]	$L_2(17)$
	6	1-(102,6,6)	$L_2(17)$	[102,92,4]	$L_2(17)$
	1 6	1-(102,7,7)	$L_2(17)$	[102,66,7]	$L_2(17)$
	8	1-(102,8,8)	$L_2(17)$	[102,56,8]	$L_2(17)$

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Table B.1.12 – continued from previous page

Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
	1 3 6	1-(102,10,10)	$L_2(17)$	[102,75,8]	$L_2(17)$
	8 3	1-(102,11,11)	$L_2(17)$	[102,84,3]	$L_2(17)$
	12	1-(102,12,12)	$L_2(17)$	[102,74,8]	$L_2(17)$
	8 6	1-(102,14,14)	$L_2(17)$	[102,92,4]	$L_2(17)$
	8 1 6	1-(102,15,15)	$L_2(17)$	[102,58,8]	$L_2(17)$
	3 12	1-(102,15,15)	$L_2(17)$	[102,58,8]	$L_2(17)$
	8 1 3 6	1-(102,18,18)	$L_2(17)$	[102,27,18]	$L_2(17)$
	12 6	1-(102,18,18)	$L_2(17)$	[102,92,4]	$L_2(17)$
	8 12	1-(102,20,20)	$L_2(17)$	[102,74,8]	$L_2(17)$
	8 1 12	1-(102,21,21)	$L_2(17)$	[102,54,10]	$L_2(17)$
	8 3 12	1-(102,23,23)	$L_2(17)$	[102,66,7]	$L_2(17)$
	24	1-(102,24,24)	$L_2(17)$	[102,48,8]	$L_2(17)$
	24	1-(102,24,24)	$L_2(17)$	[102,56,8]	$L_2(17)$
	24 1	1-(102,25,25)	$L_2(17)$	[102,54,10]	$L_2(17)$
	8 12 6	1-(102,26,26)	$L_2(17)$	[102,92,4]	$L_2(17)$
	8 1 12 6	1-(102,27,27)	$L_2(17)$	[102,84,3]	$L_2(17)$
	24 3	1-(102,27,27)	$L_2(17)$	[102,84,3]	$L_2(17)$
	8 1 3 12 6	1-(102,30,30)	$L_2(17)$	[102,49,8]	$L_2(17)$
	24 6	1-(102,30,30)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24 6	1-(102,30,30)	$L_2(17)$	[102,44,14]	$L_2(17)$
	8 24	1-(102,32,32)	$L_2(17)$	[102,56,8]	$L_2(17)$
	8 24	1-(102,32,32)	$L_2(17)$	[102,48,8]	$L_2(17)$
	24 12	1-(102,36,36)	$L_2(17)$	[102,74,8]	$L_2(17)$
	24 3 12	1-(102,39,39)	$L_2(17)$	[102,58,8]	$L_2(17)$
	24 12 6	1-(102,42,42)	$L_2(17)$	[102,92,4]	$L_2(17)$
	8 24 12	1-(102,44,44)	$L_2(17)$	[102,26,24]	$L_2(17)$
	8 24 3 12	1-(102,47,47)	$L_2(17)$	[102,66,7]	$L_2(17)$
	24(2)	1-(102,48,48)	$L_2(17)$	[102,56,8]	$L_2(17)$
	24(2)	1-(102,48,48)	$L_2(17)$	[102,48,8]	$L_2(17)$
	8 24 12 6	1-(102,50,50)	$L_2(17)$	[102,92,4]	$L_2(17)$
	8 1 12 6 24	1-(102,51,51)	$L_2(17)$	[102,84,3]	$L_2(17)$
	24(2) 3	1-(102,51,51)	$L_2(17)$	[102,36,17]	$L_2(17)$
	24(2) 3	1-(102,51,51)	$L_2(17)$	[102,84,3]	$L_2(17)$
	1 3 6 8 12 24	1-(102,54,54)	$L_2(17)$	[102,49,8]	$L_2(17)$
	24(2) 6	1-(102,54,54)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24(2) 8	1-(102,56,56)	$L_2(17)$	[102,48,8]	$L_2(17)$
	24(2) 8	1-(102,56,56)	$L_2(17)$	[102,8,48]	$L_2(17)$
	24(2) 12	1-(102,60,60)	$L_2(17)$	[102,74,8]	$L_2(17)$
	24(2) 3 12	1-(102,63,63)	$L_2(17)$	[102,66,7]	$L_2(17)$
	24(2) 12 6	1-(102,66,66)	$L_2(17)$	[102,44,14]	$L_2(17)$
	24(2) 8 12	1-(102,68,68)	$L_2(17)$	[102,74,8]	$L_2(17)$
	24(2) 8 3 12	1-(102,71,71)	$L_2(17)$	[102,58,8]	$L_2(17)$
	24(3)	1-(102,72,72)	$L_2(17)$	[102,48,8]	$L_2(17)$
	24(2) 8 12 6	1-(102,74,74)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24(2) 8 12 6 1	1-(102,75,75)	$L_2(17)$	[102,84,3]	$L_2(17)$
	24(3) 3	1-(102,75,75)	$L_2(17)$	[102,84,3]	$L_2(17)$
	1 3 6 8 12 24(2)	1-(102,78,78)	$L_2(17)$	[102,57,8]	$L_2(17)$
	24(3) 6	1-(102,78,78)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24(3) 8	1-(102,80,80)	$L_2(17)$	[102,56,8]	$L_2(17)$
	24(3) 12	1-(102,84,84)	$L_2(17)$	[102,26,24]	$L_2(17)$
	24(3) 3 12	1-(102,87,87)	$L_2(17)$	[102,58,8]	$L_2(17)$
	24(3) 12 6	1-(102,90,90)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24(3) 8 12	1-(102,92,92)	$L_2(17)$	[102,74,8]	$L_2(17)$

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Table B.1.12 – continued from previous page

Max	Block	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
	24(3) 8 3 12	1-(102,95,95)	$L_2(17)$	[102,66,7]	$L_2(17)$
	24(3) 8 12 6	1-(102,98,98)	$L_2(17)$	[102,92,4]	$L_2(17)$
	24(3) 8 12 6 1	1-(102,99,99)	$L_2(17)$	[102,84,3]	$L_2(17)$

B.2 Designs and codes from Key-Moori Method 2

$q = 8$

nX	$\|nX\|$	$C_G(g)$	Max Centralizer
2A	63	2^3	False
3A	56	9	False
7ABC	72	7	False
9ABC	56	9	False

Table B.2.1: Conjugacy classes of $L_2(8)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
D_{18}	$2A$	1-(63,9,4)	$L_2(8)$:3	$[63, 19, 9]_2$	$L_2(8)$:3
	$3A$	1-(56,2,1)	$2^{28}{:}S_{28}$	$[56, 28, 2]_2$	$2^{28}{:}S_{28}$
	$9A$	1-(56,2,1)	$2^{28}{:}S_{28}$	$[56, 28, 2]_2$	$2^{28}{:}S_{28}$
$2^3{:}7$	$2A$	1-(63,7,1)	$(S_7)^9{:}S_9$	$[63, 9, 7]_2$	$(S_7)^9{:}S_9$
	$7A$	1-(72,16,2)	$2^{36}{:}S_9$	$[72, 8, 16]_2$	$2^{36}{:}S_9$
D_{14}	$2A$	1-(63,7,4)	$L_2(8)$:3	$[63, 27, 7]_2$	$L_2(8)$:3
	$7A$	1-(72,2,1)	$2^{36}{:}S_{36}$	$[72, 36, 2]_2$	$2^{36}{:}S_{36}$

Table B.2.2: codes and designs from $L_2(8)$ using Key-Moori Method 2.

$q = 9$

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
2A	45	D_8	False
3AB	40	9	False
4A	90	4	False
5AB	72	5	False

Table B.2.3: Conjugacy classes of $L_2(9)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
A_5	$2A$	1-(45,15,2)	$(S_3)^{15}{:}S_6$	$[45, 5, 15]_2$	$(S_3)^{15}{:}S_6$
	$3A$	1-(40,20,3)	$2^{20}{:}S_6$	$[40, 6, 16]_2$	$2^{20}{:}S_6$
	$5A$	1-(72,12,1)	$(S_{12})^6{:}S_6$	$[72, 6, 12]_2$	$(S_{12})^6{:}S_6$
$3^2{:}4$	$2A$	1-(45,9,2)	S_{10}	$[45, 9, 9]_2$	S_{10}
	$3A$	1-(40,4,1)	$(S_4)^{10}{:}S_{10}$	$[40, 10, 4]_2$	$(S_4)^{10}{:}S_{10}$
	$4A$	1-(90,18,2)	$2^{45}{:}S_{10}$	$[90, 9, 18]_2$	$2^{45}{:}S_{10}$
S_4	$2A$	1-(45,9,3)	$(S_3)^{15}{:}S_6$	$[45, 10, 9]_2$	$6^{15}{:}S_6$
	$3A$	1-(40,8,3)	$2^{20}{:}S_6$	$[40, 10, 8]_2$	$2^{20}{:}S_6$
	$4A$	1-(90,6,1)	$(S_6)^{15}{:}S_6$	$[90, 15, 6]_2$	$(S_6)^{15}{:}S_6$

Table B.2.4: codes and designs from $L_2(9)$ using Key-Moori Method 2.

$q = 11$

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
2A	55	D_{12}	True
3A	110	6	False
5AB	132	5	False
6A	110	6	False
11AB	60	11	False

Table B.2.5: Conjugacy classes of $L_2(11)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
11:5	$5A$	1-(132,22,2)	$2^{66}{:}S_{12}$	$[132, 11, 22]_2$	$2^{66}{:}S_{12}$
	$11A$	1-(60,5,1)	$(S_5)^{12}{:}S_{12}$	$[60, 12, 5]_2$	$(S_5)^{12}{:}S_{12}$
A_5	$2A$	1-(55,15,3)	$L_2(11)$	$[55, 11, 15]_2$	$L_2(11){:}2$
	$3A$	1-(110,20,2)	$2^{55}{:}S_{11}$	$[110, 10, 20]_2$	$2^{55}{:}S_{11}$
	$5A$	1-(132,12,1)	$(S_{12})^{11}{:}S_{11}$	$[132, 11, 12]_2$	$(S_{12})^{11}{:}S_{11}$
D_{12}	$2A$	1-(55,7,7)	$L_2(11){:}2$	$[55, 35, 4]_2$	$L_2(11){:}2$
	$3A$	1-(110,2,1)	$2^{55}{:}S_{55}$	$[110, 55, 2]_2$	$2^{55}{:}S_{55}$

Table B.2.6: codes and designs from $L_2(11)$ using Key-Moori Method 2.

$q = 13$

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
2A	63	2^3	False
3A	56	9	False
7ABC	72	7	False
9ABC	56	9	False

Table B.2.7: Conjugacy classes of $L_2(13)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
13:6	2A	1-(91,13,2)	S_{14}	[91,13,13]	S_{14}
	3A	1-(182,26,2)	$2^{91}{:}S_{14}$	[182,13,26]	$2^{91}{:}S_{14}$
	6A	1-(182,26,2)	$2^{91}{:}S_{14}$	[182,13,26]	$2^{91}{:}S_{14}$
	13A	1-(84,6,1)	$(S_6)^{14}{:}S_{14}$	[84,14,6]	$(S_6)^{14}{:}S_{14}$
A_4	2A	1-(91,3,3)	$L_2(13){:}2$	[91,77,3]	$L_2(13){:}2$
	3A	1-(182,8,4)	$2^{91}{:}(L_2(13){:}2)$	[182,62,8]	$2^{91}{:}(L_2(13){:}2)$
D_{12}	2A	1-(91,3,3)	$L_2(13){:}2$	[91,77,3]	$L_2(13){:}2$
	3A	1-(182,8,4)	$2^{91}{:}(L_2(13){:}2)$	[182,62,8]	$2^{91}{:}(L_2(13){:}2)$
	2B	1-(91,7,7)	$L_2(13){:}2$	[91,63,6]	$L_2(13){:}2$
	3B	1-(182,2,1)	$2^{91}{:}S_{91}$	[182,91,2]	$2^{91}{:}S_{91}$
	6A	1-(182,2,1)	$2^{91}{:}S_{91}$	[182,91,2]	$2^{91}{:}S_{91}$

Table B.2.8: codes and designs from $L_2(13)$ using Key-Moori Method 2.

$q = 16$

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
2A	255	2^4	False
3A	272	15	False
5AB	272	15	False
15ABCD	272	15	False
17ABCDEFGH	240	17	False

Table B.2.9: Conjugacy classes of $L_2(16)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
2^4:15	2A	1-(255,15,1)	$(S_{15})^{17}S_{17}$	[255,17,15]	$(S_{15})^{17}S_{17}$
	3A	1-(272,32,2)	2^{136}:S_{17}	[272,16,32]	2^{136}:S_{17}
	5A	1-(272,32,2)	2^{136}:S_{17}	[272,16,32]	2^{136}:S_{17}
	15A	1-(272,32,2)	2^{136}:S_{17}	[272,16,32]	2^{136}:S_{17}
A_5	2A	1-(255,15,4)	$(S_3)^{85}$:$(L_2(16)$:4)	[255,65,17]	$(S_3)^{85}$:$(L_2(16)$:4)
	3A	1-(272,20,5)	$(S_3)^{85}$:$(L_2(16)$:4)	[272,44,20]	$(S_3)^{85}$:$(L_2(16)$:4)
	5A	1-(272,12,3)	$(S_3)^{85}$:$(L_2(16)$:4)	[272,60,12]	$(S_3)^{85}$:$(L_2(16)$:4)
D_{34}	2A	1-(255,17,8)	$L_2(16)$:4	[255,65,17]	$L_2(16)$:4
	17A	1-(240,2,1)	2^{120}:S_{120}	[240,120,2]	2^{120}:S_{120}
D_{30}	2A	1-(255,15,8)	$L_2(16)$:4	[255,81,15]	$L_2(16)$:4
	3A	1-(272,2,1)	2^{136}:S_{136}	[272,136,2]	2^{136}:S_{136}
	5A	1-(272,2,1)	2^{136}:S_{136}	[272,136,2]	2^{136}:S_{136}
	15A	1-(272,2,1)	2^{136}:S_{136}	[272,136,2]	2^{136}:S_{136}

Table B.2.10: Codes and designs from $L_2(16)$ using Key-Moori Method 2.

$q = 17$

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
2A	153	D_{16}	False
3A	272	9	False
4AB	306	2^3	False
8AB	306	2^3	False
9ABC	272	9	False
17AB	144	17	False

Table B.2.11: Conjugacy classes of $L_2(17)$

Max	nX	Design $\mathcal{D}$	Aut($\mathcal{D}$)	Code $\mathcal{C}$	Aut($\mathcal{C}$)
17:8	2A	1-(153,17,2)	S_{18}	$[153,17,17]_2$	S_{18}
	4A	1-(306,34,2)	2^{153}:S_{18}	$[306,17,34]_2$	2^{153}:S_{18}
	8A	1-(306,34,2)	2^{153}:S_{18}	$[306,17,34]_2$	2^{153}:S_{18}
	17A	1-(144,8,1)	$(S_8)^{18}$:S_{18}	$[144,18,8]_2$	$(S_8)^{18}$:S_{18}
S_4	2A	1-(153,9,6)	$L_2(17)$	$[153,83,8]_2$	$L_2(17)$
	3A	1-(272,8,3)	2^{136}:L_217	$[272,84,8]_2$	2^{136}:$L_2(17)$
	4A	1-(306,6,2)	2^{153}:L_217	$[306,101,6]_2$	2^{153}:$L_2(17)$
D_{16}	2A	1-(153,9,9)	$L_2(17)$:2	$[153,117,8]_2$	$L_2(17)$:2
	4A	1-(306,2,1)	2^{153}:S_{153}	$[306,153,2]_2$	2^{171}:S_{171}
	8A	1-(306,2,1)	2^{153}:S_{153}	$[306,153,2]_2$	2^{171}:S_{171}
D_{18}	2A	1-(153,9,8)	$L_2(17)$:2	$[153,81,9]_2$	$L_2(17)$:2
	3A	1-(272,2,1)	2^{136}:S_{136}	$[272,136,2]_2$	2^{136}:S_{136}
	9A	1-(272,2,1)	2^{136}:S_{136}	$[272,136,2]_2$	2^{136}:S_{136}

Table B.2.12: codes and designs from $L_2(17)$ using Key-Moori Method 2.

References

[1] N. H. Abel, *Mémoire sur les équations algébriques où on démontre l'impossibilité de la résolution de l'équation générale du cinquième dégré*, Facsimile, The Librarian, Faculty of Science, University of Oslo, Oslo, 1957.

[2] B. R. Andriamanalimanana, *Ovals, unitals and codes*, ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.)–Lehigh University.

[3] M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 1986.

[4] M. Aschbacher, R. Lyons, S. D. Smith, and R. Solomon, *The classification of finite simple groups*, Mathematical Surveys and Monographs, vol. 172, American Mathematical Society, Providence, RI, 2011.

[5] M. Aschbacher and S. D. Smith, *The classification of quasithin groups. I*, Mathematical Surveys and Monographs, vol. 111, American Mathematical Society, Providence, RI, 2004.

[6] ———, *The classification of quasithin groups. II*, Mathematical Surveys and Monographs, vol. 112, American Mathematical Society, Providence, RI, 2004.

[7] E. F. Assmus Jr. and J. D. Key, *Designs and their codes*, Cambridge Tracts in Mathematics, vol. 103, Cambridge University Press, Cambridge, 1992.

[8] ———, *Designs and codes: an update*, Des. Codes Cryptogr. **9** (1996), no. 1, 7–27. Second Upper Michigan Combinatorics Workshop on Designs, Codes and Geometries (Houghton, MI, 1994).

[9] E. Bannai, *Doubly transitive permutation representations of the finite projective special linear groups* $\mathrm{PSL}(n, q)$, Osaka J. Math. **8** (1971), 437–445.

[10] ———, *On some subgroups of the group* $\mathrm{Sp}(2n, 2)$, Proc. Japan Acad. **47** (1971), 769–773.

[11] R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugenics **9** (1939), 353–399.

[12] ———, *Mathematical theory of the symmetrical factorial design*, Sankhyā **8** (1947), 107–166.

[13] ———, *On some connections between the design of experiments and information theory*, Bull. Inst. Internat. Statist. **38** (1961), 257–271.

[14] R. C. Bose and S. S. Shrikhande, *A note on a result in the theory of code construction*, Information and Control **2** (1959), 183–194.

[15] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993).

[16] J. F. Box, *R. A. Fisher and the design of experiments, 1922-1926*, The American Statistician **34** (1980), no. 1, 1–7.

[17] W. Burnside, *Theory of groups of finite order*, Dover Publications Inc., New York, 1955. 2d ed.

[18] A. R. Calderbank and D. B. Wales, *A global code invariant under the Higman-Sims group*, J. Algebra **75** (1982), no. 1, 233–260.

[19] P. J. Cameron, *Finite permutation groups and finite simple groups*, Bull. London Math. Soc **13** (1981), 1–22.

[20] R. D. Carmichael, *Tactical configurations of rank two*, Amer. J. Math. **53** (1931), no. 1, 217–240.

[21] A. L. Cauchy, *Mémoire sur le nombre des valeurs qu'une fonction peut acquérir lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme*, Journal de l'École Polytechnique **x** (1815), no. XVII^{e} Cahier, 64–90.

[22] ———, *Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment,* Journal de l'École Polytechnique **x** (1815), no. XVII^{e} Cahier, 91–169.

[23] A. Cayley, LXV. *on the theory of groups, as depending on the symbolic equation* $\theta^n = 1$.*-Part* II, Philosophical Magazine Series 4 **7** (1854), no. 47, 408–409.

[24] C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of combinatorial designs*, Second, Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2007.

[25] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[26] M. R. Darafsheh, A. R. Ashrafi, and M. Khademi, *Some designs related to group actions*, Ars Combin. **86** (2008), 65–75.

[27] M. R. Darafsheh, A. Iranmanesh, and R. Kahkeshani, *Some designs and codes invariant under the groups* S_9 *and* A_8, Des. Codes Cryptogr. **51** (2009), no. 2, 211–223.

[28] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996.

[29] J. Doyen, X. Hubaut, and M. Vandensavel, *Ranks of incidence matrices of Steiner triple systems*, Math. Z. **163** (1978), no. 3, 251–259.

[30] L. Finkelstein and A. Rudvalis, *The maximal subgroups of Janko's simple group of order 50, 232, 960*, Journal of Algebra **30** (1974), no. 1-3, 122 –143.

[31] W. Fish, R. Fray, and E. Mwambene, *Binary codes from the complements of the triangular graphs*, Quaest. Math. **33** (2010), no. 4, 399–408.

[32] M. S. Ganief, *2-generations of the sporadic simple groups*, Ph.D. Thesis, 1997.

[33] C. F. Gauss, *Disquisitiones arithmeticae*, Translated into English by Arthur A. Clarke, S. J, Yale University Press, New Haven, Conn., 1966.

[34] N. Gill, PSL$(3, q)$ *and line-transitive linear spaces*, Beiträge Algebra Geom. **48** (2007), no. 2, 591–620.

[35] M. J. E. Golay, *Notes on digital coding*, Proc. IRE **37** (1949), 687.

[36] D. Gorenstein, *The classification of finite simple groups. Vol. 1*, The University Series in Mathematics, Plenum Press, New York, 1983. Groups of noncharacteristic 2 type.

[37] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1994.

[38] L. C. Grove, *Groups and characters*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1997. A Wiley-Interscience Publication.

[39] W. H. Haemers, R. Peeters, and J. M. van Rijckevorsel, *Binary codes of strongly regular graphs*, Des. Codes Cryptogr. **17** (1999), no. 1-3, 187–209.

[40] M. Hall Jr. and J. D. Swift, *Determination of Steiner triple systems of order* 15, Math. Tables Aids Comput. **9** (1955), 146–152.

[41] R. W. Hamming, *Error detecting and error correcting codes*, Bell System Tech. J. **29** (1950), 147–160.

[42] D. G. Higman and C. C. Sims, *A simple group of order* 44, 352, 000, Math. Z. **105** (1968), 110–113.

[43] G. Higman, *On the simple group of D. G. Higman and C. C. Sims*, Illinois J. Math. **13** (1969), 74–80.

[44] G. Hiss, *Hermitian function fields, classical unitals, and representations of 3-dimensional unitary groups*, Indag. Math. (N.S.) **15** (2004), no. 2, 223–243.

[45] O. Hölder, *Die einfachen gruppen im ersten und zweiten hundert der ordnungszahlen*, Math. Ann. **40** (1892), no. 1, 55–88.

[46] D. R. Hughes, *On t-designs and groups*, Amer. J. Math. **87** (1965), 761–778.

[47] H. E Huntley, *The divine proportion : a study in mathematical beauty*, New York : Dover Publications, 1970 (English).

[48] I. M. Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.

[49] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[50] W. M. Kantor, *Classification of 2-transitive symmetric designs*, Graphs Combin. **1** (1985), no. 2, 165–166.

[51] W. M. Kantor, M. E. O'Nan, and G. M. Seitz, *2-transitive groups in which the stabilizer of two points is cyclic*, J. Algebra **21** (1972), 17–50.

[52] J. D. Key, *Permutation decoding for codes from designs, finite geometries and graphs*, Information security, coding theory and related combinatorics, 2011, pp. 172–201.

[53] J. D. Key and J. Moori, *Codes, designs and graphs from the Janko groups J_1 and J_2*, J. Combin. Math. Combin. Comput. **40** (2002), 143–159.

[54] ______, *Correction to: "Codes, designs and graphs from the Janko groups J_1 and J_2" [J. Combin. Math. Combin. Comput.* **40** *(2002), 143–159]*, J. Combin. Math. Combin. Comput. **64** (2008), 153.

[55] ______, *Some irreducible codes invariant under the Janko group, J_1 or J_2*, J. Combin. Math. Combin. Comput. **81** (2012), 165–189.

[56] J. D. Key, J. Moori, and B. G. Rodrigues, *On some designs and codes from primitive representations of some finite simple groups*, J. Combin. Math. Combin. Comput. **45** (2003), 3–19.

[57] ______, *Permutation decoding for the binary codes from triangular graphs*, European J. Combin. **25** (2004), no. 1, 113–123.

[58] T. P. Kirkman, *On a problem in combinations.*, Cambridge and Dublin Math. J. **2** (1847), 191–204.

[59] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990.

[60] F. Klein, *A comparative review of recent researches in geometry*, Bull. Amer. Math. Soc. **2** (1893), no. 10, 215–249.

[61] M. Klemm, *Über den p-rang von inzidenzmatrizen*, J. Combin. Theory Ser. A **43** (1986), no. 1, 138–139.

[62] J. L. Lagrange, *Traité de la résolution des équations numériques de tous les degrés*, Paris, Bachelier, 1826.

[63] E. S. Lander, *Symmetric designs: an algebraic approach*, London Mathematical Society Lecture Note Series, vol. 74, Cambridge University Press, Cambridge, 1983.

[64] T. Le and J. Moori, *On the automorphisms of designs constructed from finite simple groups*, 2013. Submitted.

[65] É. Mathieu, *Mémoire sur l'étude des fonctions de plusieurs quantités, sur la maniére de les former et sur les substitutions qui les laissent invariables*, Journal de mathématiques pures et appliquées **6** (1861), 241–323.

[66] ______, *Sur la fonction cinq fois transitive de 24 quantités*, Journal de mathématiques pures et appliquées **18** (1873), 25–47.

[67] H. H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. **12** (1911), no. 2, 207–242.

[68] J. Moori, *Finite groups, designs and codes*, Information security, coding theory and related combinatorics, 2011, pp. 202–230.

[69] J. Moori, *AIMS course on further group theory / representation theory*, AIMS-South Africa, 2011 (English).

[70] J. Moori and B. G. Rodrigues, *A self-orthogonal doubly-even code invariant under M^cL*, Ars Combin. **91** (2009), 321–332.

[71] ———, *On some designs and codes invariant under the Higman-Sims group*, Util. Math. **86** (2011), 225–239.

[72] ———, *Some designs and binary codes preserved by the simple group Ru of Rudvalis*, J. Algebra **372** (2012), 702–710.

[73] ———, *Some ternary codes invariant under the group* $\mathrm{M^cL}$:2, Ars Combin. **104** (2012), 449–462.

[74] M. E. O'Nan, *Automorphisms of unitary block designs*, J. Algebra **20** (1972), 495–511.

[75] R. E. A. C Paley, *On orthogonal matrices*, Journal of Mathematics and Physics **12** (1933), 311–320.

[76] P. Perumal, *On the theory of the Frobenius groups*, Master's Thesis, 2012.

[77] J. J. Rotman, *An introduction to the theory of groups*, Springer, 1994.

[78] P. Ruffini, *Riflessioni intorno alla soluzione delle equazioni algebraiche generali*, Modena, 1813.

[79] H. J. Ryser, *Combinatorial mathematics*, Carus mathematical monographs, Math. Ass. of America, 1963.

[80] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379–423, 623–656.

[81] D. R. Stinson, *Combinatorial designs*, Springer-Verlag, New York, 2004. Constructions and analysis, With a foreword by Charles J. Colbourn.

[82] J. J. Sylvester, *Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers*, Philosophical Magazine **34** (1867), 461–475.

[83] L. M. G. M. Tolhuizen and W. J. van Gils, *A large automorphism group decreases the number of computations in the construction of an optimal encoder/decoder pair for a linear block code*, IEEE Trans. Inform. Theory **34** (1988), no. 2, 333–338.

[84] V. D. Tonchev and R. S. Weishaar, *Steiner triple systems of order* 15 *and their codes*, J. Statist. Plann. Inference **58** (1997), no. 1, 207–216.

[85] N. S. Whitley, *Fischer matrices and character tables of group extension*, Master's Thesis, 1994.

[86] J. Williamson, *Hadamard's determinant theorem and the sum of four squares*, Duke Math. J. **11** (1944), 65–81.

[87] R. A. Wilson, *The finite simple groups*, Graduate Texts in Mathematics, vol. 251, Springer-Verlag London Ltd., London, 2009.

[88] E. Witt, *Die 5-fach transitiven gruppen von Mathieu*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **12** (1937), 256–264 (German).

[89] ———, *Über Steinersche systeme*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **12** (1937), 265–275 (German).