Pricing barrier options with numerical methods

C.N de Ponte

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Supervisor : Dr E.H.A Venter

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Abstract

Barrier options are becoming more popular, mainly due to the reduced cost to hold a barrier option when compared to holding a standard call/put options, but exotic options are difficult to price since the payoff functions depend on the whole path of the underlying process, rather than on its value at a specific time instant.
It is a path dependent option, which implies that the payoff depends on the path followed by the price of the underlying asset, meaning that barrier options prices are especially sensitive to volatility.
For basic exchange traded options, analytical prices, based on the Black-Scholes formula, can be computed. These prices are influenced by supply and demand. There is not always an analytical solution for an exotic option. Hence it is advantageous to have methods that efficiently provide accurate numerical solutions. This study gives a literature overview and compares implementation of some available numerical methods applied to barrier options. The three numerical methods that will be adapted and compared for the pricing of barrier options are:

- Binomial Tree Methods
- Monte-Carlo Methods
- Finite Difference Methods

Key terms

- Barrier options
- Black-Scholes
- Binomial method
- Trinomial method
- Monte Carlo simulation
- Finite difference method

Summary

Barrier options are probably the oldest of all exotic options and have been traded in the US market since 1967. The most popular standard barrier options are knock-out and knock-in options. In 1973 Merton provided the first analytical formula to price basic barrier options in continuous time.

Most real-world financial barrier options pricing have no analytical solutions, because the barrier structure is complex or discrete. There are essentially no analytical formulas for pricing discrete barrier options, and numerical pricing is difficult and slow to converge.

This dissertation offers a discussion of the theoretical background of different barrier options, an investigation of numerical techniques to determine the value of a barrier option, a description of the algorithms and shows the implementation of the algorithms in MATLAB code.

A thorough literature study was undertaken to investigate the current available pricing techniques, after which MATLAB code was implemented and experimented, with numerical data. The efficiency of the numerical methods and adaptive methods used in the valuation of financial derivatives, with special focus on barrier options was studied. Numerical methods studied include binomial methods, Monte Carlo Methods and finite difference methods. The option price was obtained using the numerical methods and was compared to the analytical solution (if it existed).

The best lattice method is the adaptation of the trinomial method using the stretch technique. The Monte Carlo method converges very slowly to obtain an accurate value, whilst the Crank-Nicolson finite difference method takes the least number of time steps to obtain an accurate value.

Declaration

I declare that, apart from the assistance acknowledged, the research presented in this dissertation is my own unaided work. It is being submitted in partial fulfilment of the requirements for the degree Master of Science in Applied Mathematics at the Potchefstroom campus of the North-West University. It has not been submitted before for any degree or examination to any other University.

Nobody, including Dr. Venter, but myself is responsible for the final version of this dissertation.

Signature.....

Date.....

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Basic Notations

K - exercise/strike price
S_t - asset price at general time
S_T - asset price at expiry date
t - general time
T - expiry date
$T - t$ - time to maturity
c_t - payoff of European call option
p_t - payoff of European put option
r - risk-free interest rate
σ - volatility
B - barrier
q - dividend yield
μ - drift
$N(x)$ - Cumulative distribution function of the standard normal distribution
M - number of asset paths sampled
$Smax$ - suitably large asset price

Contents

Chapter 1

Introduction

Options are part of derivatives instruments. Derivatives are financial instruments that derive their value from the value of other, more basic, underlying variables. These underlying variables are stochastic, i.e. random, thus taking a position in a financial instrument built on these random variables implies risk. This explains why one needs to have the correct price for a given financial product if one does not want to take inconsiderate risks.

Exotic options are not a strict defined class of options, but rather refer to options with more complicated properties than ordinary put and call options. Exotic options are usually born of the particular needs of hedgers and investors using the instruments to manage financial risk.

Barrier options are probably the oldest of all exotic options and have been traded in the US market since 1967 (Chriss 1997, 462). They are extensions of a standard stock option. Standard calls and puts have payoffs that depend on one predetermined value, the strike K, whereas barrier options have payoffs that depend on two predetermined values namely the strike K and the barrier B. The most popular standard barrier options are 'knock-out' and 'knock-in' options. A knock-in option is a contract that becomes active only when a certain price is reached. A knock-out option is a contract that starts out as ordinary call or put options, but they become null and void if the spot price ever crosses a certain predetermined knock-out barrier. A major reason for the popularity of barrier options is that, because there is a positive probability (in either case) of worthlessness, these options are cheaper than the corresponding standard stock option, and hence possibly more attractive to the speculator. They are also used by investors to gain exposure to future market scenarios more complex than can be described by standard options.

In 1973 Merton provided the first analytical formula to price basic barrier options in continuous time. It is common practice to assume that the underlying asset is continuously

monitored against the barrier. However, for many traded barrier options the barrier is only monitored at specific dates. These options are usually referred to as discrete barrier options, (Horfelt 2003).

> Most real-world financial barrier options pricing have no analytical solutions, because the barrier structure is complex or discrete, (Derman, Kani, Ergener & Bardhan 1995). There are essentially no analytical formulas for pricing discrete barrier options, and numerical pricing is difficult and slow to converge, (Broadie, Glasserman & Kou 1997). A thorough literature study was consequently undertaken to investigate the current available pricing techniques, after which MATLAB code was implemented and experiment with numerical data. The next step was to investigate the efficiency of the numerical methods and adaptive methods used in the valuation of financial derivatives, with special focus on barrier options. Numerical methods that will be studied include binomial methods, Monte Carlo Methods and finite difference methods. The option price obtained using the numerical methods will be compared to the analytical solution (if it exists). Subsequently, once the results have been obtained, their efficiency is compared.

The outline of this dissertation is as follows. Chapter 2 offers a discussion of the background and principles of standard barrier options. Barrier options are introduced in chapter 3, followed by a description of the characteristics and a derivation of the Black-Scholes for a down-and-out option. Chapter 4 presents the binomial and trinomial method in four steps. The first step includes a discussion and implementation of the methods for standard options. The second step adapts the standard lattice programs to price barrier options. Following this, the focus falls on the adaptation of the barrier lattice methods to obtain more accurate answers within less time steps. These methods include the revised binomial model, the interpolation technique and the stretch technique. The last fourth step offers a discussion of discrete barrier options and experiments with an adjusted barrier for Black-Scholes and the stretch technique. Chapter 5 discusses the Monte-Carlo method, how it is applied to barrier options and how one can use variance reductions techniques, such as antithetic variables, to obtain more accurate results. In chapter 6 the finite difference methods are introduced; explicit method, implicit method and Crank-Nicolson method, after which the last mentioned is applied to barrier options. The final chapter concludes, and the appendix provides the MATLAB codes.

Chapter 2

Introduction to Options

2.1 Fundamental concepts of options

Trading on options on an organized exchange started in the early 1980s (Hull 2000, 10). They have two primary uses: speculation and hedging. The pricing of options is a topic of some significance and interest due to their importance in financial markets. Standard option contracts are traded on option exchanges, and their prices are widely listed. However, there is also a market for more specialized option contracts such as exotic options, which are designed to tailor to more specific risk management strategies.

2.1.1 Definition of an option

Options are a class of *derivatives*, i.e., financial assets of which the value depends on another asset, called the underlying. The underlying can also be a non-financial asset, such as a commodity, or an arbitrary quantity representing a risk factor to someone, such as weather, so that setting up a market to transfer risks makes sense. Options are *contracts* with very specific rules for issuing, trading and accounting (Brandimarte 2006, 35).

Options are contracts that give their holders the right to buy or sell an underlying asset at a fixed price (called the strike, and denoted K) at a certain time in the future (the expiry date, denoted T). The holder of an option pays an agreement fee (*premium*) at the start of the option. The profit at time $t = T$ is then equal to the payoff minus the premium.
The writer of the option keeps the premium regardless of whether or not the option is ultimately exercised. It is paid by the buyer to the writer in order to enjoy the choice conferred by holding the option. The writer has no such choice, but must trade if the buyer wishes to do so.

All options of the same type (call or puts) are referred to as an option class where the call and put is defined as:
• The *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price.
• The *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price.
If you own a call option you want the asset to rise as much as possible so that you can buy the stock for a relatively small amount, then sell it and make a profit. When you own a put option you want the asset price to fall as low as possible because the lower the asset price at expiry the higher the profit.

An important relationship between the call and put is called the *put-call parity*

$$c - p = S - Ke^{-r(T-t)}. \tag{2.1}$$

It shows that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa. It is an important equation, because if (2.1) does not hold then there are arbitrage opportunities (Hull 2000, 174).

The buyer of an option does not buy the underlying instrument; he or she buys a right. If this right can be exercised only at the expiration date, then the option is *European*. If it can be exercised any time during the specified period, the option is said to be *American*. A *Bermudan* option is *in between*, given that it can be exercised at more than one of the dates during the life of the option.

In the case of a European call, the option holder has purchased the right to "buy" the underlying instrument at a certain price, strike price, at a specific date, the expiration date. In the case of European put, the option holder has again purchased the right to an action. The action in this case is to "sell" the underlying instrument at the strike price and at the expiration date.

American style options can be exercised anytime until expiration and hence may be more expensive. They may carry an early exercise premium. At the expiration date, options cease to exist (Neftci 2008, 206).

There are two sides to every option contract. On one side is the investor who has taken the *long position* (i.e., has bought the option), on the other side is the investor who has taken a *short position* (i.e., has sold or written the option) (Hull 2000, 1).

Options are referred to as *in the money*, *at the money* or *out of the money*. An in-the-money

option would give the holder a positive cash flow if it were exercised immediately. Similarly, an at-the-money option would lead to zero cash flow if it were exercised immediately, and an out-the-money option would lead to a negative cash flow if it were exercised immediately (Hull 2000, 154).

	Call option	Put option
In-the-money	$S > K$	$S < K$
At-the-money	$S = K$	$S = K$
Out-the-money	$S < K$	$S > K$

2.1.2 Payoff of the option

- *European call option*:
  If, at expiry, $S_T > K$ then the holder of the option may buy the asset for K and sell it in the market for $S(T)$, gaining an amount $S_T - K$. Alternatively, if $K \geq S(T)$ then the holder gains nothing.
  The *payoff* of the European call option is given by:

$$c = max(S_T - K, 0). \tag{2.2}$$

- *European put option*:
  At expiry, if $K > S_T$ then the holder may buy the asset as $S(T)$ in the market and exercise the option by selling it at K, gaining an amount $K - S_T$. Alternatively, if $S_T \geq K$ then the holder should do nothing.
  The *payoff* of the European put option is given by:

$$p = max(K - S_T, 0). \tag{2.3}$$

2.1.3 Factors affecting option prices

A number of mathematical models are used to value options. One of the more widely used is the Black-Scholes model. This model uses five parameters to value an option on a non-dividend-paying share.

The five parameters are:

- **the underlying share price,** S
  Changes in the underlying share price can increase or decrease the value of an option.

These price changes have opposite effects on calls and puts. For instance, as the value of the underlying share price rises, call will generally increase and the value of a put will generally decrease in price. A decrease in the underlying share price will generally have the opposite effect.

- **the strike price, K**
  The strike price determines whether or not an option has any intrinsic value. An option's premium (intrinsic value plus time value) generally increases as the option becomes further in the money, and decreases as the option becomes more deeply out of the money.

- **the time to expiry, T-t**
  Time until expiration affects the time value component of an option's premium. Generally, as expiration approaches, the levels of an option's time value, for both puts and calls, decreases or "erodes". This effect is most noticeable with at-the-money options.

- **the volatility of the underlying share, σ**
  The value of an option will increase with the volatility of the underlying share.

- **the risk-free interest rate, r**
  The effect of the risk-free interest rate has a small but measurable effect on option premiums. This effect reflects the "cost of carry" of shares in an underlying security, the interest that might be paid for margin or received from alternative investments.

In the case of a dividend-paying share, dividends can be considered a sixth factor. The price of an option is the premium paid at the outset.

2.2 Black-Scholes

The following definitions are given as referred to in (Hull 2000) in order to model option prices.

2.2.1 Random variable and stochastic processes

Random variable:

- A number of which the value is determined by the outcome of an experiment.
- The outcome is unknown.

Discrete random variable:

- Can take on only certain separated values.
- Example: the result of throwing a dice. The probability of every outcome is 1/6.

Continuous random variable:

- Can take on any real value from a range.
- Example: the price of a stock. The probability that the price is within a certain interval depends on the distribution of the random variable, for example normal distribution.

Stochastic process:

- Represents the evolution in time of a random value. It is a sequence of values of some quantity where the future values cannot be predicted with certainty.

Deterministic:

- A deterministic model/approach has predefined results and is non-random. An example of a deterministic model would be a fixed deposit, the input values are all known and do not change, thus the end result is known from the start.

2.2.2 Asset price modelling

Asset price modelling can be described as a continuous-time stochastic process where a mathematical model is used to describe random movement.

Suppose at time t the asset price is S. Consider a small time interval dt, as S changes to $S + dS$. In a risk-free environment, with each change in asset price, we associate a return defined to be the change in the price divided by the original value. The return dS/S has a contribution of μdt, where μ (drift) is the expected rate of return of a risk-free environment.

Another contribution to the return is the random change in the asset price, σdW which is a random sample from a normal distribution with mean zero where σ is a number known as the volatility and dW represents the randomness. The randomness dW can also be described as a Wiener process and has unique properties such as its variance is dt, the mean zero and dW is a random variable from a normal distribution(Dewynne, Howison & Wilmott 1995, 21) .
Thus one can then obtain the stochastic differential equation $dS/S = \sigma dW + \mu dt$. It is used to derive the Black-Scholes model to price an option.

If $\sigma = 0$, then you would have the ordinary differential equation $dS/S = \mu dt$ or $dS/t = \mu S$. And assuming that μ is constant, the ordinary differential equation can be solved exactly to give the exponential growth in the value of the asset,

$$S = S_0 e^{\mu(t)},$$

here S is the value at t.

2.2.3 Ito's lemma

In the 1940s Kiyoshi Ito developed stochastic calculus. The key result of stochastic calculus is Ito's lemma. Ito's lemma relates small changes in a function of a random variable to the small change in the random variable itself. The following concept is necessary when deriving the Black-Scholes model, and can be found with the proof in (Dewynne et al. 1995, 25) and (Bjork 2009, 51):

- **Ito's lemma**: Suppose that the value of a variable x follows the Ito process

$$dx = a(x,t)dt + b(x,t)dz \tag{2.4}$$

  where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2.
  If G is a function of x and t, then

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

  where

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

is the drift rate and

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

the variance rate (Hull 2000, 229).

Assume that the stock price S at time t, has a stochastic differential given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{2.5}$$

where μ and σ are the drift and volatility parameters, and $f(t, S)$ is a function of continuous first and second order partial derivatives. From Ito's lemma, it follows that the process followed by a function, f, of S and t is

$$df(t, S(t)) = \left[\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right]dt + \sigma S\frac{\partial f}{\partial S}dW(t). \tag{2.6}$$

2.2.4 The Black-Scholes model

One of the significant equations in financial mathematics is the Black-Scholes equation, which was developed by (Merton 1973). Its a partial differential equation that governs the value of financial derivatives, such as options.

To derive the Black-Scholes model, the following assumptions are made:

- Since a random walk becomes Brownian motion in the continuous-time limit, the assumption is that the share price follows a geometric Brownian motion or log normal model.
- Short selling is allowed. Short selling is the trading practice of borrowing a share, selling it, buying the share later and returning it to the owner.
- No transaction costs or taxes.
- All securities are infinitely divisible (it's possible to buy any fraction of a share).
- The underlying security does not pay dividends during the life of the derivative.
- No risk-less arbitrage opportunities. Arbitrage is the simultaneous purchase and sale of an asset in order to profit from a difference in the price and is done without risk involved.
- Security trading is continuous.

- It is possible to borrow and lend cash at a known constant risk-free interest rate. A risk-free interest rate is a theoretical concept and is thought to be the interest rate earned by investing in financial instruments with no risk.

The following Black-Scholes model derivation is based on the derivation in the textbook of (Dewynne et al. 1995, 42).

Given a stock price S which follows the process $dS = \mu S dt + \sigma S dW$, suppose there is an option of which the value $V(S,t)$ depends only on S and t the time. With the help of Ito's lemma and equation (2.6), one has

$$dV = \left[\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right] dt + \sigma S \frac{\partial V}{\partial S} dW. \tag{2.7}$$

One then constructs a portfolio(Π) consisting of one option and a number $-\triangle$ of the underlying asset. By choosing a portfolio of the option and the underlying asset, the Wiener process can be eliminated, for a small time period.

The value of this portfolio is

$$\Pi = V - \triangle S \tag{2.8}$$

and the change is $d\Pi = dV - \triangle dS$,

where Π follows the random walk

$$d\Pi = \left[\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \triangle S\right] dt + \sigma S \left[\frac{\partial V}{\partial S} - \triangle\right] dW. \tag{2.9}$$

Choosing

$$\Delta = \frac{\partial V}{\partial S} \tag{2.10}$$

to eliminate the random component in the random walk, one obtains a portfolio whose increment is deterministic

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt. \tag{2.11}$$

The return on an amount, Π invested in riskless assets would see a growth of $r\Pi dt$ in a time dt. If the right hand side of (2.11) was less or more then $r\Pi dt$, the arbitrager would make a riskless, no cost, instantaneous profit.

Thus

$$r\Pi dt = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt. \tag{2.12}$$

Substituting (2.8) and (2.10) into (2.12) and dividing by dt one arrives at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.13}$$

Equation (2.13) is the famous *Black-Scholes partial differential equation* governing the evolution of the price of a derivative (pricing equation), developed by (Merton 1973).

Boundary conditions on European calls and puts:

At $t = T$, the value of a call is known to be the payoff:

$$c(S, T) = max(S - K, 0). \tag{2.14}$$

When $S = 0$ one has

$$c(0, t) = 0. \tag{2.15}$$

As $S \to \infty$ the value of the option becomes that of the asset

$$c(S, t) \sim S \text{ as } S \to \infty. \tag{2.16}$$

For a put option, with value $p(S, T)$ the final condition is the payoff:

$$p(S, T) = max(K - S, 0). \tag{2.17}$$

Assuming that interest rate is constant, one finds the boundary condition at $S = 0$ to be

$$p(0, t) = Ke^{-r(T-t)}. \tag{2.18}$$

As $S \to \infty$ the option is unlikely to be exercised and so

$$p(S, t) \to 0 \text{ as } S \to \infty. \tag{2.19}$$

The Black-Scholes formulae for European Options

One of the most appealing features of the Black-Scholes model is the existence of an analytical formula for the pricing of European call and put options. For a European call option,

without the possibility of early exercise, (2.13), (2.14), (2.15) and (2.16), can be solved (Wilmott 2007, 92) exactly to give the Black-Schole values of a call option at time 0:

$$c_0(S_0, 0) = S_0 N(d_1) - Ke^{-r(T)} N(d_2). \tag{2.20}$$

For a European put option, without the possibility of early exercise, (2.13), (2.17), (2.18) and (2.19), can be solved exactly to give the Black-Schole values of a put option at time 0:

$$p_0(S_0, 0) = Ke^{-r(T)} N(-d_2) - S_0 N(-d_1), \tag{2.21}$$

where $N(.)$ is the cumulative distribution function for a standardized normal random variable given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy. \tag{2.22}$$

Further, one sees that

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)(T)}{\sigma\sqrt{T}}, \tag{2.23}$$

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)(T)}{\sigma\sqrt{T}}. \tag{2.24}$$

Or at any time t:

$$c(S, t) = SN(d_1) - Ke^{-r(T-t)} N(d_2). \tag{2.25}$$

$$p(S, t) = Ke^{-r(T-t)} N(-d_2) - SN(-d_1), \tag{2.26}$$

and

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{2.27}$$

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.28}$$

Where

c - European call option price (premium),

p - European put option price (premium),

S - Underlying asset's price at time t,

K - Predetermined option strike price,

r - Risk-free interest rate,

$N(x)$ - Cumulative distribution function of the standard normal distribution,

σ - Standard deviation.

Chapter 3

Theory of Barrier Options

Barrier options are a class of exotic options, they are considered to be one of the simplest types of path dependent options. Their payoff, and therefore value, depends on the path taken by the asset S up to expiry. The path dependence is weak because the only factor considered is whether or not the barrier B has been triggered (Wilmott 2007, 385). Barrier options were created to provide risk managers with a cheaper means to hedge their exposures without paying for price ranges that they believe unlikely to occur. They are also used by investors to gain exposure to future markets more complex than the simple bullish or bearish expectations embodied in standard options, (Kotze 1999).

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time (Hull 2000, 462). A barrier option has a payoff that depends on whether or not a specified level of the underlying is reached before expiration (Wilmott 2007, 381).

3.1 Characteristics of barrier options

The following definitions can be found in (Chriss 1997, 434).

- *Knock-out options* start out as ordinary call or put options, but they become null and void if the spot price ever crosses a certain predetermined knock-out barrier, even before the expiration date.
- *Knock-in options* start their lives inactive, in a sense null and void, and only become active on the event that the stock price crosses the knock-in barrier, then it becomes an ordinary call or put option.

The barrier option can be further portrayed by the position of the barrier relative to the initial value of underlying.

- If the barrier is above the initial asset value, one has an up option.
- If the barrier is below the initial asset value, one has a down option.

Summary of characteristics

(Hull 2000, 662) offers the following definitions:

- **Down-and-out** An option that terminates when the price of the underlying asset declines to a predetermined level.
- **Up-and-out** An option that terminates when the price of the underlying asset increases to a predetermined level.
- **Down-and-in** An option that comes into existence when the price of the underlying asset declines to a predetermined level.
- **Up-and-in** An option that comes into existence when the price of the underlying asset increases to a predetermined level.

Terms associated with barrier options

Barrier options can also have cash *rebates* associated with them. This is a consolation prize paid to the holder of the option when an out barrier is knocked out or when an in barrier is never knocked in. The rebate can be nothing or it could be some fraction of the premium. Rebates are usually paid immediately when an option is knocked out, however, payments can be deferred to the maturity of the option, (Kotze 1999).

A barrier can be either *continuous* or *discrete*. Once a continuously monitored barrier is reached the option is immediately knocked in or out, while in discretely monitored conditions, barriers only come into effect in discrete monitored time, for example at close of every market day, every quarter, every month, or every half year. Analytic formulas present methods to price barrier options in continuous time. Pricing discretely monitored barrier options is not as easy as pricing continuously monitored barrier options, since there is essentially no closed solution (Wilmott 2007, 372).

In-out parity is the barrier option answer to put-call parity. The principal is the same as in put-call parity (2.1). In-out parity says that the "in" option value plus the "out" option value is equal to the value of the vanilla option

$$c = c_{in} + c_{out}. \tag{3.1}$$

Barrier options can be divided into two groups: the intrinsic and non-intrinsic barrier options. The non-intrinsic options are those that do not have any intrinsic value when the barrier is crossed. The down-and-in call is a non-intrinsic option since the barrier is set at a level below the strike price when the option is issued. Thus when the option starts to exist, $t = 0$, there is no intrinsic value in it.
The intrinsic options do have an intrinsic value until the barrier is breached. The up-and-out call is an intrinsic option since the barrier is set at a level above the strike price when the option is issued. That indicates that the option loses all the intrinsic value once the barrier is touched. In the table below the intrinsic and non-intrinsic barrier options types are summarized:

Intrinsic	Non-intrinsic
up-and-out call	up-and-out put
up-and-in call	up-and-in put
down-and-out put	down-and-in call
down-and-in put	down-and-out call

3.2 Black-Scholes and barrier option pricing

(Merton 1973) proposed the first analytic formula for a down-and-out call option and later (Reiner & Rubinstein 1991) provided the formulas for all four types of barrier on both call and put options.

The price of a barrier option will depend on the standard Black-Scholes parameters, as well as on the barrier level, B.

The outline in (Higham 2004, 197) is followed to derive the value of a Barrier Option. In section 2.2.4 on page 11 showed a derivation of the Black-Schole partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.2}$$

Suppose that the function $V(S,t)$ satisfies the Black-Scholes partial differential equation. Set

$$\hat{\mathrm{V}}(S,t) = S^{1-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right). \tag{3.3}$$

Thus

$$\frac{\partial \hat{\mathrm{V}}}{\partial t} = S^{1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial t}\left(\frac{B}{S}, t\right),$$

$$\frac{\partial \hat{\mathrm{V}}}{\partial S} = \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right) - BS^{-1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial S}\left(\frac{B}{S}, t\right),$$

$$\begin{aligned} \frac{\partial^2 \hat{\mathrm{V}}}{\partial S^2} &= \left(1 - \frac{2r}{\sigma^2}\right)\left(\frac{-2r}{\sigma^2}\right) S^{-1-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right) \\ &\quad + \frac{\partial V}{\partial S}\left(\frac{B}{S}, t\right)\left(\frac{4Br}{\sigma^2}\right) S^{-2-\frac{2r}{\sigma^2}} \\ &\quad + B^2 S^{-3-\frac{2r}{\sigma^2}} \frac{\partial^2 V}{\partial S^2}\left(\frac{B}{S}, t\right). \end{aligned}$$

Substitute the above partial fraction expressions in (3.2). So,

$$\begin{aligned}
&\frac{\partial \hat{\mathrm{V}}}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 \hat{\mathrm{V}}}{\partial S^2}+rS\frac{\partial \hat{\mathrm{V}}}{\partial S}-r\hat{\mathrm{V}}\\
= \;& S^{1-\frac{2r}{\sigma^2}}\frac{\partial V}{\partial t}\left(\frac{B}{S},t\right)\\
&-\left(1-\frac{2r}{\sigma^2}\right)(r)S^{1-\frac{2r}{\sigma^2}}V\left(\frac{B}{S},t\right)\\
&+2BrS^{-\frac{2r}{\sigma^2}}\frac{\partial V}{\partial S}\left(\frac{B}{S},t\right)+\frac{1}{2}\sigma^2B^2S^{-1-\frac{2r}{\sigma^2}}\frac{\partial^2 V}{\partial S^2}\left(\frac{B}{S},t\right)\\
&+r\left(1-\frac{2r}{\sigma^2}\right)S^{1-\frac{2r}{\sigma^2}}V\left(\frac{B}{S},t\right)-rBS^{-\frac{2r}{\sigma^2}}\frac{\partial V}{\partial S}\left(\frac{B}{S},t\right)\\
&-rS^{1-\frac{2r}{\sigma^2}}V\left(\frac{B}{S},t\right)\\
= \;& S^{1-\frac{2r}{\sigma^2}}\frac{\partial V}{\partial t}\left(\frac{B}{S},t\right)\\
&+B^2S^{-1-\frac{2r}{\sigma^2}}\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\left(\frac{B}{S},t\right)\\
&+rBS^{-\frac{2r}{\sigma^2}}\frac{\partial V}{\partial S}\left(\frac{B}{S},t\right)\\
&-rVS^{1-\frac{2r}{\sigma^2}}\left(\frac{B}{S},t\right)\\
= \;& S^{1-\frac{2r}{\sigma^2}}\\
&\left[\frac{\partial V}{\partial t}\left(\frac{B}{S},t\right)+\frac{1}{2}\sigma^2\left(\frac{B}{S}\right)^2\frac{\partial^2 V}{\partial S^2}\left(\frac{B}{S},t\right)+r\frac{B}{S}\frac{\partial V}{\partial S}\left(\frac{B}{S},t\right)-rV\left(\frac{B}{S},t\right)\right].
\end{aligned}$$

The term inside the block brackets is zero since V satisfies the Black-Scholes PDE. Thus $\hat{\mathrm{V}}$ solves the Black-Scholes PDE.

Similar to the derivation above, it can be proven that

$$c_{down-in}(S,t)=\left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}}c\left(\frac{B^2}{S},T-t\right),$$

solves the Black-Schole PDE, where $c\left(\frac{B^2}{S},T-t\right)$ is calculated using equation (2.25).

One can then calculate the option value of a down-and-out call option, for $K>B$, on the

domain $0 \leq t \leq T$, $B \leq S$ by using the *in-out parity*.

$$\begin{aligned} c(S,t) &= c_{down-in}(S,t) + c_{down-out}(S,t) \\ c_{down-out}(S,t) &= c(S,t) - c_{down-in}(S,t) \\ &= c(S,t) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} c\left(\frac{B^2}{S}, T-t\right). \end{aligned}$$

An interesting observation
From equation above one can immediately observe that the down-and-out call is worth less than the European call.

From section 2.2.4, equation (2.25) one finds

$$c(S,t) = SN\left(\frac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}N\left(\frac{\log(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

where $N(.)$ is the cumulative distribution function for a standardized normal random variable given by

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy.$$

Thus

$$\begin{aligned} c_{down-out}(S,t) &= c(S,t) - c_{down-in}(S,t) \\ &= c(S,t) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} c\left(\frac{B^2}{S}, T-t\right) \\ &= SN\left(\frac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}N\left(\frac{\log(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad -B\left(\frac{S}{B}^{-2r\sigma^{-2}}\right)N\left(\frac{\log(B^2/SK) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad -\left(\frac{S}{B}^{1-2r\sigma^{-2}} Ke^{-r(T-t)}\right)N\left(\frac{\log(B^2/SK) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Unless the barrier is crossed, the Black-Scholes partial differential equation (2.13) is applicable, thus $c_{down-out}(S,t)$ must satisfy the partial differential equation on the domain $0 \leq t \leq T, B \leq S$, where

c_t - European call option price(premium),

p_t - European put option price (premium),

S_t - Underlying asset's price at time t,

K - Predetermined option strike price,

r - Risk-free interest rate,

σ - Volatility of the underlying.

When $S(t^*) \leq B$ for some t^*, then the option becomes worthless:

$$c_{down-out}(B,t) = 0, \text{ for } t^* \leq t \leq T.$$

At expiry, when $S(t) > B$ for $0 \leq t \leq T$, then one recovers the European value, so that:

$$c_{down-out}(S,t) = c(S,t), \text{ for } B \leq S,$$

where $c(S,t)$ is from equation (2.25).

The method below will be referred to as the *standard method*:
According to (Wilmott 2007, 408), the continuously monitored barrier option *values* can be calculated using the following formula mentioned below, where N(.) denotes the cumulative distribution function for a standardized normal variable, B the barrier position, q the dividend yield , S the stock price and K the strike price. (Dewynne et al. 1995, 207) derives the formulae below.

The following are formulae for the *value* of the call and put barrier options:

Consider a *up-and-in* and *up-and-out* **call**, with the barrier above the initial stock price $B > S_0$.
Up-and-out call:

1. K >B:

$$c_{up-out} = 0.$$

   If $K > B$ then $K > S_0$. According to (2.14) this means that the value of the option will be 0.

2. K < B:

$$\begin{aligned} c_{up-out} = {} & Se^{-q(T-t)}(N(d_1) - N(d_3) - b(N(d_6) - N(d_8))) - \\ & Ke^{-r(T-t)}(N(d_2) - N(d_4) - a(N(d_5) - N(d_7))). \end{aligned}$$

   If $K < B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

Up-and-in call:

1. K >B:

$$c_{up-in} = 0.$$

If $K > B$ then $K > S_0$. According to (2.14) this means that the value of the option will be 0.

2. K < B:

$$\begin{aligned} c_{up-in} = & Se^{-q(T-t)}(N(d_3) + b(N(d_6) - N(d_8))) - \\ & Ke^{-r(T-t)}(N(d_4) + a(N(d_5) - N(d_7)))). \end{aligned}$$

If $K < B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

Consider a *down-and-out* and *down-and-in* **put**, with the barrier below the initial stock price $B < S_0$.
Down-and-out put:

1. K >B:

$$\begin{aligned} p_{down-out} = & -Se^{-q(T-t)}(N(d_3) - N(d_1) - b(N(d_8) - N(d_6))) + \\ & Ke^{-r(T-t)}(N(d_4) - N(d_2) - a(N(d_7) - N(d_5))). \end{aligned}$$

If $K > B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

2. K < B:

$$p_{down-out} = 0.$$

If $K < B$ then $K < S_0$. According to (2.17) this means that the value of the option will be 0.

Down-and-in put:

1. K >B:

$$\begin{aligned} p_{down-in} = & -Se^{-q(T-t)}(1 - N(d_3) + b(N(d_8) - N(d_6))) + \\ & Ke^{-r(T-t)}(1 - N(d_4) + a(N(d_7) - N(d_5))). \end{aligned}$$

If $K > B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

2. K < B:

$$p_{down-in} = 0.$$

If $K < B$ then $K < S_0$. According to (2.17) this means that the value of the option will be 0.

Consider a *down-and-out* and *down-and-in* **call**, with the barrier below the initial stock price $B < S_0$.
Down-and-out call:

1. K > B:

$$\begin{aligned} c_{down-out} &= Se^{-q(T-t)}(N(d_1) - b(1 - N(d_8))) - \\ &\quad Ke^{-r(T-t)}(N(d_2) - a(1 - N(d_7))). \end{aligned}$$

   If $K > B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

2. K <B:

$$\begin{aligned} c_{down-out} &= Se^{-q(T-t)}(N(d_3) - b(1 - N(d_6))) - \\ &\quad Ke^{-r(T-t)}(N(d_4) - a(1 - N(d_5))). \end{aligned}$$

   If $K < B$ then $K < S_0$. According to (2.14) this means that the value of the option is calculated using the above formula.

Down-and-in call:

1. K > B:

$$c_{down-in} = Se^{-q(T-t)}b(1 - N(d_8)) - Ke^{-r(T-t)}a(1 - N(d_7))).$$

   If $K > B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

2. K <B:

$$\begin{aligned} c_{down-in} &= Se^{-q(T-t)}(N(d_1) - N(d_3) + b(1 - N(d_6))) - \\ &\quad Ke^{-r(T-t)}(N(d_2) - N(d_4) + a(1 - N(d_5))). \end{aligned}$$

   If $K < B$ then $K < S_0$. According to (2.14) this means that the value of the option is calculated using the above formula.

Consider a *up-and-out* and *up-and-in* **put**, with the barrier above the initial stock price $B > S_0$.
Up-and-out put:

1. K > B:

$$\begin{aligned} p_{up-out} &= -Se^{-q(T-t)}(1 - N(d_3) - bN(d_6)) + \\ &\quad Ke^{-r(T-t)}(1 - N(d_4) - aN(d_5)). \end{aligned}$$

   If $K > B$ then $K > S_0$. According to (2.17) this means that the value of the option is calculated using the above formula.

2. K <B:

$$\begin{aligned} p_{up-out} &= -Se^{-q(T-t)}(1-N(d_1)-bN(d_8))+ \\ &\quad Ke^{-r(T-t)}(1-N(d_2)-aN(d_7)). \end{aligned}$$

If $K < B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

Up-and-in put:

1. K > B:

$$\begin{aligned} p_{up-in} &= -Se^{-q(T-t)}(N(d_3)-N(d_1)+bN(d_6))+ \\ &\quad Ke^{-r(T-t)}(N(d_4)-N(d_2)+aN(d_5)). \end{aligned}$$

If $K > B$ then $K > S_0$. According to (2.17) this means that the value of the option is calculated using the above formula.

2. K <B:

$$p_{up-in} = -Se^{-q(T-t)}bN(d_8)+Ke^{-r(T-t)}aN(d_7).$$

If $K < B$ then one can have $K \leq S$ or $K \geq S$ and the value of the option is calculated using the above formula.

with

$$a = \left(\frac{B}{S}\right)^{-1+2(r-q)/\sigma^2}$$

$$b = \left(\frac{B}{S}\right)^{1+2(r-q)/\sigma^2}$$

$$d_1 = \frac{\log(S/K)+(r-q+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/K)+(r-q-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 = \frac{\log(S/B) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_4 = \frac{\log(S/B) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_5 = \frac{\log(S/B) - (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_6 = \frac{\log(S/B) - (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_7 = \frac{\log(SK/B^2) - (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_8 = \frac{\log(SK/B^2) - (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

where c_t - European call option price,

S_t - Underlying asset's price at time t,

K - Predetermined option strike price,

r - Risk-free interest rate,

q - Dividend yield,

σ - Standard deviation.

Example 3.2.1 *Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$.*

Once again the formula for this option is given by:

$$\begin{aligned} c_{down-out} &= S(N(d_1) - b(1 - N(d_8))) - \\ &\quad Ke^{-r(T)}(N(d_2) - a(1 - N(d_7))) \end{aligned}$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}$$

$$d_7 = \frac{\log(SK/B^2) - (r - \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}$$

$$d_8 = \frac{\log(SK/B^2) - (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}$$

with the help of the MATLAB code in section 8.1.1, $bs\_daoc(S, K, sigma, r, T, B)$, the option value is $c_{down-out} = 6,3076$.

One can now calculate the option value of a down-and-in call option by using the *in-out parity*.

$$\begin{aligned} c &= c_{in} + c_{out} \\ c_{in} &= c - c_{out} \\ &= 6,3441 - 6,3076 \\ &= 0,0365. \end{aligned}$$

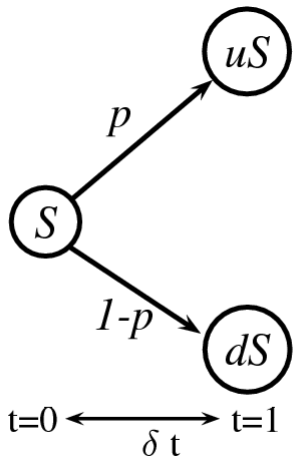
Chapter 4

Binomial and Trinomial Method

Analytical solutions do not always exist for real-world barrier options, either because the barrier structure is complex or because it is discrete, (Derman et al. 1995). There are essentially no analytical formulas for pricing discrete barrier options, and numerical pricing is difficult and slow to converge, (Broadie et al. 1997). It is consequently necessary to investigate the efficiency of a number of numerical methods and adaptive methods.

4.1 Binomial method

The figure below is an illustration of a one step binomial tree with two branches. The initial asset price is S at time $t = 0$, with a risk-free rate r. Time T is the date of maturity of the option and is a discrete measured time period. With $\delta t = \frac{T-t}{n} = T$. The only assumption needed is that arbitrage opportunities do not exist.



To generalise, consider an asset whose price is S_0 and an option on the asset whose current price is V. Assume that the option lasts for time T, during the life of the option the asset price can either move up from S_0 to a new level, S_0u, where $u > 1$ and the percentage decrease of the asset price is $u - 1$.
Or move down from S_0 to a new level, S_0d, where $d < 1$ and the percentage decrease of the asset price is $1 - d$.
Thus if the asset price moves up to S_0u, the pay-off from the option is V_u and if it moves down to S_0d its V_d.

Suppose there exists a portfolio consisting of a long position in $\triangle$ shares and a short position in the option.
One can now calculate the value of $\triangle$ that makes the portfolio risk-less: If there is an up-movement in the asset price, the value of the portfolio at T is $S_0u\triangle - V_u$. If there is a down-movement in the asset price, the value of the portfolio at T is $S_0d\triangle - V_d$.
They are equal when $S_0u\triangle - V_u = S_0d\triangle - V_d$ or

$$\triangle = \frac{V_u - V_d}{S_0u - S_0d}.$$

Thus $\triangle$ is the ratio of change in the option price to the change in the asset price as one moves between the nodes.

If we denote the risk-free interest rate by r, the present value of the portfolio is $(S_0u\triangle - V_u)e^{-rT}$. The cost of setting up the portfolio is $S_0\triangle - V$. It follows that $S_0\triangle - V) = (S_0u\triangle - V_u)e^{-rT}$ or $V = S_0(1 - ue^{-rT}) + V_ue^{-rT}$.
Substituting from equation $\triangle = \frac{V_u - V_d}{S_0u - S_0d}$ and simplifying,

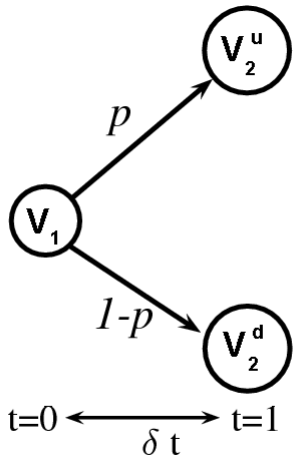
$$V = e^{-rT}(pV_u + (1 - p)V_d),$$

where

$$p = \frac{e^{rT} - d}{u - d}.$$

One could interpret p as the probability of an up movement and (1-p) as the probability of a down movement in a risk neutral word. Then $pV_u + (1 - p)V_d$ is the expected payoff from the option and $V = e^{-rT}(pV_u + (1 - p)V_d)$ represents the value of the option as its expected future payoff discounted at the risk-free rate (Hull 2000, 285). Hence the option pricing formula can be interpreted as a risk-neutral valuation.

Thus one can determine the value of an option at time t, V_1.



For one time step,

$$V_1 = e^{-r\delta t}((p)V_2^u + (1-p)V_2^d)$$

where

$$V_2^u = \max(uS - K, 0), \tag{4.1}$$
$$V_2^d = \max(dS - K, 0) \tag{4.2}$$

and K = strike price.

To generalize one can determine the value of an option at time t, V_t, as

$$V_t = e^{-r\delta t}(pV_{t+\delta t}^u + (1-p)V_{t+\delta t}^d) \tag{4.3}$$

where $V_{t+\delta t}^u$ is the value of the option at time $t + \delta t$ if the value of the asset increases, and $V_{t+\delta t}^d$ is its value if the asset price decreases.

Thus, in order to price an option, one divides the period of the contract $[0, T]$ into a certain number of subintervals, with a binomial process occurring in each time interval. One then uses the payoff equation (2.14) or (2.17) at the end of the interval $[t = T]$.

Then one uses equation (4.3) to move backwards in time through the tree. Since risk neutrality is assumed, the value at each node at time $T - \delta t$ can be calculated as the expected value at time T discounted at rate r for a time period δt. In the same way, the value at each node at time $T - 2\delta t$ can be calculated as the expected value at time $T - \delta t$ discounted for a time period δt at rate r, and so on until one reaches a price at the beginning of the interval, which is the value of the option (Hull 2000, 390).

Derivation of u, d and p:

There are different methods to determine the values of u, d and p.
These parameters must give correct values for the mean and variance of asset price changes during a time interval of length δt. By applying the risk neutral assumption (that any risk-free portfolio must grow at the risk-free rate), (Hull 2000, 389) derives specific values of u, d and p chosen to match the volatility (σ) of the asset price.

The expected return from a asset is the risk-free interest rate, r, since a risk neutral environment is assumed. Thus, the expected value of the asset price at the end of a time interval of length δt is $Se^{r\delta t}$.

It then follows that

$$\begin{aligned} Se^{r\delta t} &= puS + (1-p)dS, & (4.4)\\ e^{r\delta t} &= pu + (1-p)d. & (4.5) \end{aligned}$$

The stochastic process assumed for the asset price implies that the variance of the proportional change in the asset price in a small time interval of length δt is $\sigma^2\delta t$.
Since the variance of a stochastic variable Q is defined as $E(Q^2) - [E(Q)]^2$, the variance of the asset price can be given as

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2\delta t. \quad (4.6)$$

One then substitutes equation (4.5) to obtain

$$e^{r\delta t}(u+d) - ud - e^{2r\delta t} = \sigma^2\delta t. \quad (4.7)$$

Equations (4.5) and (4.7) impose two conditions on p, u and d. An extra condition, proposed by (Cox, Ross & Rubinstein 1979) (CRR), is

$$u = \frac{1}{d}.$$

This condition ensures that the tree reconnects at each time level, and hence minimizes the amount of nodes.

These three conditions helps one to derive u, d and p:

From (4.5):

$$\begin{aligned} puS + (1-p)dS &= E[S_{t+\delta t}] \\ &= Se^{r\delta t}. \end{aligned}$$

One divides by S

$$\begin{aligned} pu + (1-p)d &= e^{r\delta t} \\ pu + d - pd &= e^{r\delta t} \\ p(u-d) &= e^{r\delta t} - d. \end{aligned}$$

Thus

$$p = \frac{e^{r\delta t} - d}{u - d}, \tag{4.8}$$

$$\text{var}(S_{t+\delta t}) = E[S_{t+\delta t}^2] - E[S_{t+\delta t}]^2.$$

Thus from the variance equation (4.6):

$$\begin{aligned} \sigma^2 \delta t &= pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 \\ &= pu^2 + (1-p)d^2 - p^2u^2 - 2p(1-p)ud - (1-p)^2d^2 \\ &= u^2(p - p^2) + [(1-p) - (1-p)^2]d^2 - 2p(1-p)ud \\ &= u^2p(1-p) + (1-p)[1-(1-p)]d^2 - 2p(1-p)ud \\ &= u^2p(1-p) + (1-p)(p)d^2 - 2p(1-p)ud \\ &= p(1-p)[u^2 - 2ud + d^2]. \end{aligned}$$

Hence

$$\sigma^2 \delta t = p(1-p)(u-d)^2. \tag{4.9}$$

Using (4.8) one obtains

$$\begin{aligned} p(1-p) &= p - p^2 \\ &= \frac{e^{r\delta t} - d}{u-d} - \frac{e^{2r\delta t} - 2de^{r\delta t} + d^2}{(u-d)^2} \\ &= \frac{e^{r\delta t}u - ud - e^{r\delta t}d + d^2 - e^{2r\delta t} + 2de^{r\delta t} - d^2}{(u-d)^2} \\ &= \frac{e^{r\delta t}(u - d + 2d) - ud - e^{2r\delta t}}{(u-d)^2} \\ &= \frac{e^{r\delta t}(u+d) - ud - e^{2r\delta t}}{(u-d)^2}. \end{aligned}$$

Hence

$$p(1-p) = \frac{e^{r\delta t}(u+d) - ud - e^{2r\delta t}}{(u-d)^2}. \tag{4.10}$$

Substitute (4.10) in (4.9) to obtain

$$\sigma^2\delta t = e^{r\delta t}(u+d) - ud - e^{2r\delta t}$$

$d = 1/u$, thus

$$\sigma^2\delta t = e^{r\delta t}\left(u + \frac{1}{u}\right) - u\frac{1}{u} - e^{2r\delta t}$$

and

$$\begin{aligned} u + \frac{1}{u} &= \frac{\sigma^2\delta t + 1 + e^{2r\delta t}}{e^{r\delta t}} \\ &= e^{-r\delta t}\sigma^2\delta t + e^{-r\delta t} + e^{r\delta t}. \end{aligned}$$

Using Taylor's theorem one can say:
As $r\sigma^2\delta t \to 0$

$$\begin{aligned} e^{-r\delta t} &\approx (1 - r\delta t) \\ e^{r\delta t} &\approx (1 + r\delta t). \end{aligned}$$

Therefore

$$\begin{aligned} u + \frac{1}{u} &= \sigma^2\delta t + 2 \\ u^2 + 1 &= \sigma^2\delta t u + 2u \\ u^2 - (\sigma^2\delta t + 2)u + 1 &= 0. \end{aligned}$$

Hence

$$\begin{aligned} u &= \frac{(\sigma^2\delta t + 2) \pm \sqrt{(\sigma^2\delta t + 2)^2 - 4}}{2} \\ &= \frac{(\sigma^2\delta t + 2) \pm \sqrt{\sigma^4\delta t^2 + 4\sigma^2\delta t + 4 - 4}}{2} \\ &= \frac{\sigma^2\delta t}{2} + 1 \pm \sigma\sqrt{\delta t}. \end{aligned}$$

Since $\sqrt{\delta t}$ is larger than δt for a small δt and σ^2 is relatively smaller than σ, one can ignore the first term $(\sigma^2\delta t)/2$.
Thus

$$\begin{aligned} u &\approx 1 \pm \sigma\sqrt{\delta t} \\ &\approx e^{+\sigma\sqrt{\delta t}} \end{aligned}$$

because $u > 1$.

Therefore

$$u = e^{\sigma\sqrt{\delta t}},$$

$$d = e^{-\sigma\sqrt{\delta t}}$$

and

$$p = \frac{e^{r\delta t} - d}{u - d}.$$

■

(Higham 2004, 153) chooses $p = 0,5$ and then obtains

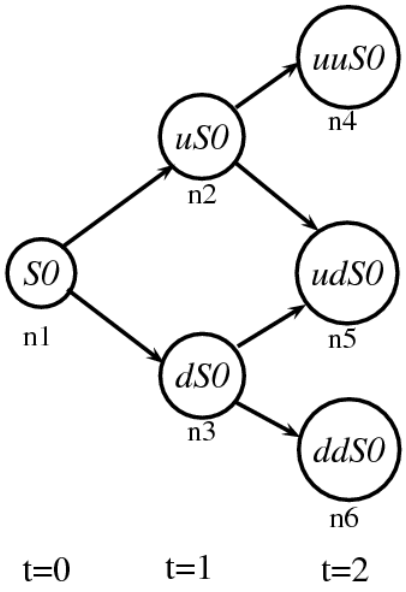
$$u = e^{\sigma\sqrt{\delta t}+(r-\frac{1}{2}\sigma^2)\delta t}$$

and

$$d = e^{-\sigma\sqrt{\delta t}+(r-\frac{1}{2}\sigma^2)\delta t}.$$

Standard binomial method algorithm

Assume there are two *time steps (N)* and that S_0 is the initial asset price.



1. First work out the value of the asset at every node.
   Thus, at $t = 0$

$$S_{n1} = S_0.$$

At $t = 1$

$$S_{n2} = u \times S_0,$$
$$S_{n3} = d \times S_0.$$

At t=2

$$S_{n4} = u^2 \times S_0,$$
$$S_{n5} = u \times d \times S_0,$$
$$S_{n6} = d^2 \times S_0,$$

where S_0 is the initial asset value, u the up movement and d the down movement.

2. Then work out the option value at each terminal node (n4, n5 and n6) by applying the payoff equation (2.14) for a call option and (2.17) for a put option.
Lets consider a call option, subsequently n4, n5 and n6 have the option values

$$c_{n4} = max(S_{n4} - K, 0),$$
$$c_{n5} = max(S_{n5} - K, 0),$$
$$c_{n6} = max(S_{n6} - K, 0)$$

respectively, where K is the strike price.

3. Next apply backward induction thus using equation (4.3); n2, n3 and n1 have the option values

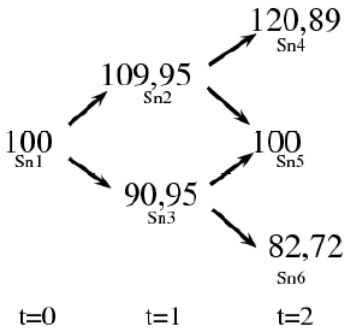
$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5}),$$
$$c_{n3} = e^{-r\delta t}(pc_{n5} + (1-p)c_{n6}),$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

respectively.

Example 4.1.1 *Consider a call option with two time steps with* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$*;* $N = 2$ *and* $\delta t = \frac{T}{N} = 0,1$.

Using CRR (section 4.1) parameters; $u = e^{\sigma\sqrt{\delta t}}$, $d = e^{-\sigma\sqrt{\delta t}}$ and $p = (e^{r\delta t} - d)/(u - d)$, and substituting the given values; $u = 1,0995$, $d = 0,9095$ and $p = 0,5292$.

Calculating the asset value at every node, one obtains:



Then calculate the option value at each terminal node by using the payoff equation $c(S,t) = max(S-K,0)$. Nodes n4, n5 and n6 option values are then

$$c_{n4} = 20,89;$$
$$c_{n5} = 0;$$
$$c_{n6} = 0$$

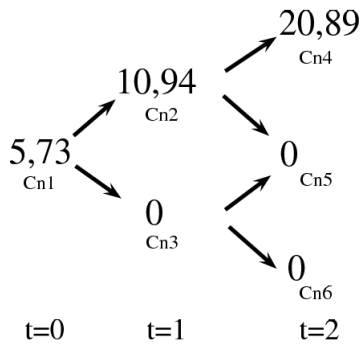
respectively.
(*For example:* $c_{n4} = max(120,89 - 100;0) = 20,89$)

Then apply backward induction to get the options values of nodes n2, n3 and finally n1. Using the equations

$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5}),$$
$$c_{n3} = e^{-r\delta t}(pc_{n5} + (1-p)c_{n6}),$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

and substitute the valid numerical data.
(*For example:* $c_{n2} = e^{-0.1\times 0.1}(0,5292 \times 20,89 + (1-0,5292)\times 0) = 10,94$)

One can then obtain the option value at each node:



The MATLAB code in section 8.1.2.1, $Call\_bm(S, K, sigma, r, T, N)$, gives the option value as $c = 5,7351$, for the same input values.

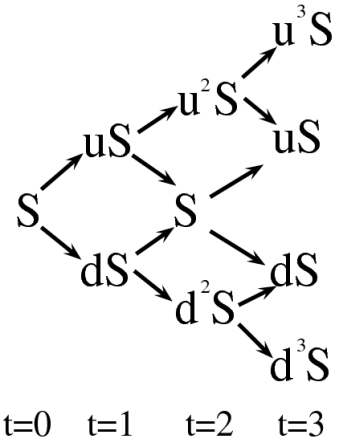
To obtain a more accurate value, one should increase N, the number of time steps.

N	**Binomial**
20	6,2776
50	6,3174
100	6,3307
150	6,3352
200	6,3374
400	6,3408
800	6,3424
1000	6,3428
2000	6,3434
4000	6,3438
Analytical answer:	6,3441

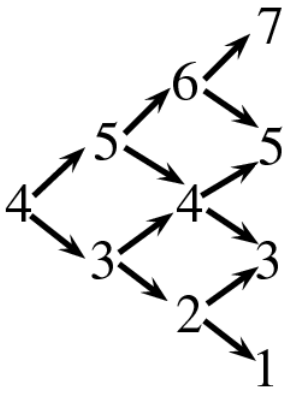
The pricing algorithm for a call option:

The following pricing algorithm as by (Brandimarte 2006, 413) says that if one sets up the lattice in the following way, the CPU time can be improved. Since $u \times d$ equals one, memory can be saved therefore the CPU time is improved, by using a vector to store the underlying asset prices, rather than a two-dimensional matrix.

With 3 time steps there are only 7 different values used for the underlying asset price.



Thus, if there are N time steps, there are $2N + 1$ different price values.



The numbers shown in the picture above are locations in the vector. Thus in element 1 the lowest value is stored resulting from a sequence of down steps only (dS). Note that you obtain the same values in different locations in the tree (i.e. at t=0 and t=2 the location indicated by 4 will have the same value).
Even-numbered entries correspond to the second-to-last time layer and odd-numbered entries correspond to the last time layer. Depending on the number of time steps, the number of branches of the lattice may be even or odd-numbered.

Brandimarte uses this pattern to store option values in his program, thus using one vector of $2N + 1$ elements elements instead of using huge matrix's that take a lot of memory (i.e. in the picture above, 7 storage places are used instead of 10 individual storage places). Note that as the number of time steps increase so will the storage places, making Brandimarte's program efficient. Below is steps explaining how the program is complied to give the final option price.

1. First precompute invariant quantities, including the discounted probabilities, in the first section of the code.

2. Then write the vector $S(i)$ of underlying asset prices, start with the smallest element, which is Sd^N. Next multiply by u, storing S_0 in element $S(N + 1)$, which is the mid-element, and then proceed both up and down.

3. When one works with call values $c(i)$, the index steps over by two, which amounts to alternating odd- and even-indexed values corresponding to consecutive time layers.

4. When time to maturity is τ, one needs to consider only the $2(N - \tau) + 1$ innermost elements of the array $c(i)$. The option price is stored in the root of the lattice, which corresponds to position $N + 1$ (Brandimarte 2006, 412).

   set the initial option value $S(1) = Sd^N$
   set the rest of the Option values:

for $i = 2$ to $2N + 1$
  compute: $S(i) = uS(i - 1)$
end
set the terminal call values:
for $i = 1$ to 2 to $2N + 1$
  compute: c(i)=maximum(S(i) - K,0)
end
Work backwards to set the rest of the call values
for tau equal 1 to N
  for i equal $(tau + 1)$ to 2 to $(2N + 1 - tau)$
    compute: $c(i) = e^{-r\delta t}(pc(i + 1) + (1 - p)(c(i - 1)))$
  end
end
Final call value = c(N+1)

where

c - European call option price,
S - Underlying asset's price,
K - Predetermined option strike price,
r - Risk-free interest rate,
σ - Standard deviation,
p - Probability of upward or downward movement,
u - upward movement factor,
d - downward movement factor,
t - time to maturity,
δt - length of time interval,
N - number of time steps.

In section 8.1.2.1, the program Call_bm is based on the above described algorithm.

4.1.1 Accuracy of option value

In real world financial problems, the number of time steps (N) are not a known factor. Thus, to find the option value accurately, it is important to have a program that can efficiently calculate N. A proficient way of doing this is by modifying the previous method, by adding a tolerance(tol).

The resulting algorithm can be summarized as follows:

set Option value old = 0
for N=1 to 5000
  compute: Option value new = binomial method option value
  if |Option value new- Option value old| < tolerance
    break *(stop)*
  end
  set Option value old= Option value new
end

Example 4.1.2 *Consider a call option with* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$ *and* $T = 0,2$*.*

With the help of the MATLAB code found in section 8.1.2.2, $newtest\_bm(S, K, sigma, r, T, tol)$, one obtains the following table:

N	Tol	c
262	0,01	6,3390
2615	0,001	6,3446
Analytical answer: 6,3441		

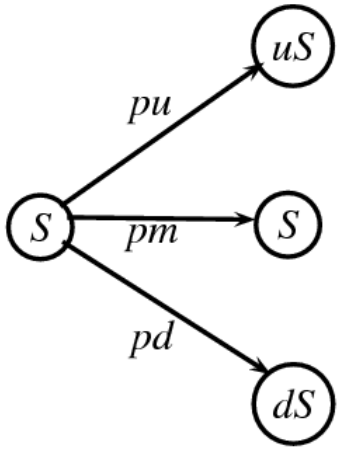
On an intel core i5 processor pc, when the $tol = 0,01$ it took $0,042$ seconds and when the $tol = 0,001$ it took $30,338$ seconds.

According to Black-Scholes the analytic solution for the call option is $6,3441$.

4.2 Trinomial method

Trinomial trees provide an effective method of numerical calculation of option prices within Black-Scholes share pricing model. Trinomial trees can be used as an alternative to binomial trees.

(Ritchken 1995) notes that the trinomial trees have a distinct advantage over binomial trees. The asset price can move in three directions from a given node, thus the number of time steps can be reduced and one can still attain the same accuracy as in the binomial tree. The trinomial tree offers more flexibility than the binomial tree and is therefore useful when pricing complex derivatives.



To discretize a geometric Brownian motion, the jump sizes and probabilities must match the mean and variance. A possible choice to determine the jump sizes, is to build a trinomial tree where the asset price at each node can go up, stay at the same level, or go down (Haug 1998, 300).

The parameters p_u, p_d and p_m that are considered in this chapter are derived in (Haug 1998, 300).

There are different methods to determine these parameters, such as in (Haug 1998, 300):

$$\begin{aligned} u &= e^{\sigma\sqrt{2\delta t}}, \\ d &= e^{-\sigma\sqrt{2\delta t}}, \end{aligned}$$

$$p_u = \left(\frac{e^{r\delta t/2} - e^{-\sigma\sqrt{\delta t/2}}}{e^{\sigma\sqrt{\delta t/2}} - e^{-\sigma\sqrt{\delta t/2}}} \right)^2,$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\delta t/2}} - e^{r\delta t/2}}{e^{\sigma\sqrt{\delta t/2}} - e^{-\sigma\sqrt{\delta t/2}}} \right)^2,$$
$$p_m = 1 - p_u - p_d,$$

and in (Hull 2000, 405)

$$\begin{aligned} u &= e^{\sigma\sqrt{3\delta t}}, \\ d &= \frac{1}{u}, \end{aligned}$$

$$p_u = \sqrt{\frac{\delta t}{12\sigma^2}} \left(r - \frac{\sigma^2}{2} \right) + \frac{1}{6},$$
$$p_d = -\sqrt{\frac{\delta t}{12\sigma^2}} \left(r - \frac{\sigma^2}{2} \right) + \frac{1}{6},$$
$$p_m = \frac{2}{3}.$$

After building the asset price tree, the value of the option can be found in the standard way by using backward induction.

Divide the period of the contract [0,T] into a certain number of subintervals, with a trinomial process occurring in each time interval. Use the payoff equation (2.14) or (2.17) at the end of the interval $[t = T]$.
Then use equation

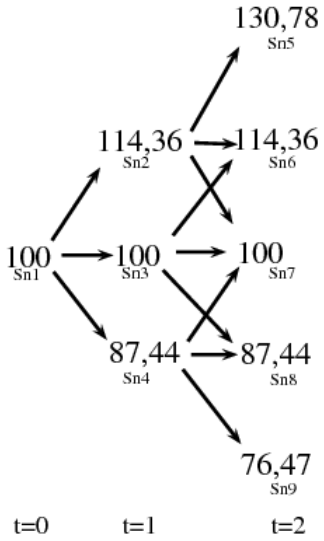
$$V_t = e^{-r\delta t}(p_u V^u_{t+\delta t} + p_m V^m_{t+\delta t} + p_d V^d_{t+\delta t})$$

to move backwards in time through the tree. Since risk neutrality is assumed, the value at each node at time $T-\delta t$ can be calculated as the expected value at time T discounted at rate r for a time period δt. In the same way, the value at each node at time $T-2\delta t$ can be calculated as the expected value at time $T-\delta t$ discounted for a time period δt at rate r, and so on until one reaches a price at the beginning of the interval, which is the value of the option.

Example 4.2.1 *Consider a call option with two time steps with* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$*;* $N = 2$ *and* $\delta t = \frac{T}{N} = 0,1$.

Using the parameters in (Haug 1998, 300) to determine (u, d, p_u, p_m and p_d), and substituting the above values, one has $u = 1,14$; $d = 0,87$; $p_u = 0,27$; $p_m = 0,5$ and $p_d = 0,23$.

Calculating the asset value at every node:



Calculate the option value at each terminal node by using the equation $c(S,t) = max(S - K, 0)$, where K = strike price.

Nodes n5, n6, n7, n8 and n9 option values are then

$$\begin{aligned} c_{n5} &= 30,78; \\ c_{n6} &= 14,36; \\ c_{n7} &= 0; \\ c_{n8} &= 0; \\ c_{n9} &= 0 \end{aligned}$$

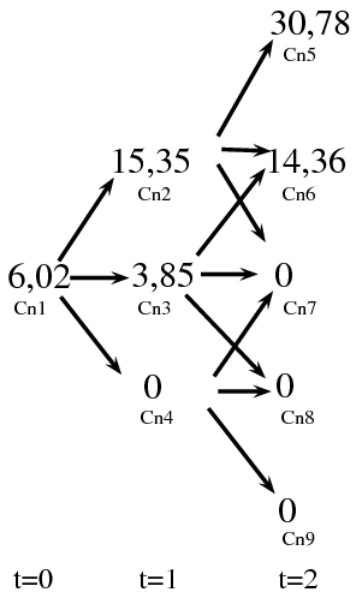
respectively.
(*For example:* $c_4 = max(130,78 - 100; 0) = 30,78$).

Then apply backward induction and get the options values using the equations

$$\begin{aligned} c_{n2} &= e^{-r\delta t}(p_u c_{n5} + p_m c_{n6} + p_d c_{n7}), \\ c_{n3} &= e^{-r\delta t}(p_u c_{n6} + p_m c_{n7} + p_d c_{n8}), \\ c_{n4} &= e^{-r\delta t}(p_u c_{n7} + p_m c_{n8} + p_d c_{n9}), \\ c_{n1} &= e^{-r\delta t}(p_u c_{n2} + p_m c_{n3} + p_d c_{n4}), \end{aligned}$$

and substitute the valid numerical data.
(*For example:* $c_{n2} = e^{-0,1\times 0,1}(0,27 \times 30,78 + 0,5 \times 14,36 + 0,23 \times 0) = 15,35$).

Obtain the option value at each node:



The MATLAB code found in section 8.1.2.3, $Call\_tm(S, K, sigma, r, T, N)$, using the same input values gives the option value as $c = 6,0229$, which is similar to the hand worked out example above.

To obtain a more accurate value, one should increase N.

N	**Trinomial**
20	6,3108
50	6,3307
100	6,3374
150	6,3397
200	6,3408
400	6,3424
800	6,3433
1000	6,3434
2000	6,3438
4000	6,3439
Analytical answer:	6,3441

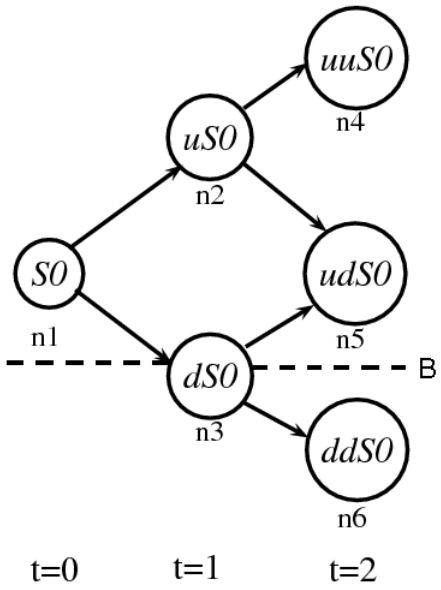
4.3 Barrier options for the binomial and trinomial method

The procedures for pricing knock-out and knock-in options through binomial/trinomial tree methods work on exactly the same principal as standard options and the pricing method is a two-step procedure. First value the options at every point above (up) or below (out) the barrier. With the knock-out options, values below the barrier are equal to the value of the rebate, unless there is no rebate, then the barrier values are equal to zero. Whereas with the knock-in options, barrier values are equal to the value of an ordinary put or call settled when the spot price breaches the barrier. Once the values are determined along the barrier, use the payoff equation at the remaining terminal $[t = T]$ nodes and then perform backward induction. The technique is similar to what is done with ordinary puts and calls, except one works backwards from every point along the barrier, not just from the terminal nodes (Chriss 1997, 452).

Barrier binomial method algorithm

The adaptation of the trinomial method is discussed below, to price a down-and-out barrier option with initial price $S0$, two time steps and a barrier.

Assume there are two *time steps (N)*.



1. First work out the value of the stock at every node.
   Thus at $t = 0$

$$S_{n1} = S_0.$$

At $t = 1$

$$S_{n2} = u \times S_0,$$
$$S_{n3} = d \times S_0.$$

At t=2

$$S_{n4} = u^2 \times S_0,$$
$$S_{n5} = u \times d \times S_0,$$
$$S_{n6} = d^2 \times S_0,$$

where S_0 is the initial stock value, u the up movement and d the down movement.

2. Second, value the options at every point below or equal to the barrier. Values below or equal to the barrier are equal to zero, thus

$$\begin{aligned} c_{n3} &= 0 \\ c_{n6} &= 0 \end{aligned}$$

3. Then work out the option value at each terminal node (n4 and n5) by applying the payoff equation (2.14) for a call option and (2.17) for a put option.
Since its a down-and-out call option, subsequently n4 and n5 have the option values

$$c_{n4} = max(S_{n4} - K, 0),$$
$$c_{n5} = max(S_{n5} - K, 0)$$

respectively, where K is the strike price.

4. Next apply backward induction thus using equation (4.3), n2 and n1 have the option values

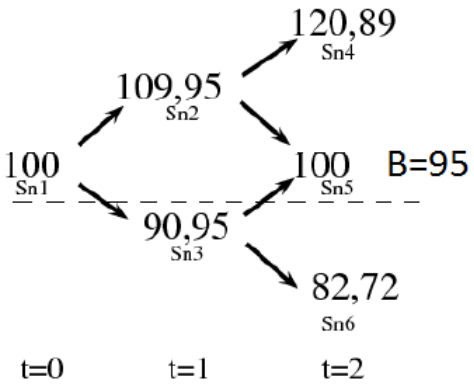
$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5}),$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

respectively.

Example 4.3.1 *Consider a down-and-out call option with two time steps, with* $S = 100$, $K = 100$, $\sigma = 0,3$; $r = 0,1$; $T = 0,2$; $N = 2$ *and* $B = 95$.

Using CRR (section 4.1) parameters; $u = e^{\sigma\sqrt{\delta t}}$, $d = e^{-\sigma\sqrt{\delta t}}$ and $p = (e^{r\delta t} - d)/(u - d)$, and substituting the above values, one has $u = 1,0995$, $d = 0,9095$ and $p = 0,5292$.

Calculating the stock value at every node:



First value the options at every point below the barrier, with down-and-out options. Values below or equal to the barrier are equal to zero, thus $c_{n3} = 0$ and $c_{n6} = 0$.

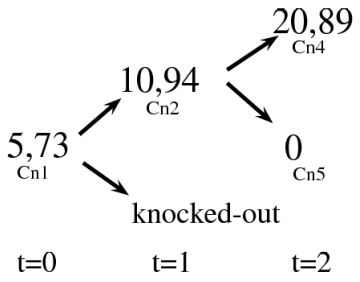
Then calculate the option price at each terminal node by using the payoff equation $c(S,t) = max(S - K, 0)$, where K = strike price. Thus $c_{n4} = 20,89$ and $c_{n5} = 0$.
(*For example:* $c_4 = max(120,89 - 100; 0) = 20,89$)

Then apply backward induction using equations

$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5})$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

and substitute the valid numerical data.
(*For example:* $c_{n2} = e^{-0,1\times 0,1}(0,5292 \times 20,89 + (1 - 0,5292) \times 0) = 10,94$)

Obtain the option value at each node:



The MATLAB code found in section 8.1.2.1, $KOCall\_bm(S, K, sigma, r, T, N, B)$, gives the option value as $c = 5,7351$, which is similar to the example above that has been worked out by hand.

The trinomial method with a barrier can be applied in a similar way as in the binomial method above. The MATLAB code, $KOCall\_tm(S, K, sigma, r, T, N, B)$, for the trinomial tree can be found in section 8.1.3.2.

Thus, the following table is obtained, with $S = 100$, $K = 100$, $\sigma = 0,3$; $r = 0,1$; $T = 0,2$; and $B = 95$:

N	**Binomial**	**Trinomial**
20	4,8036	4,4139
50	4,6703	5,0290
100	4,5166	5,1768
150	4,5707	4,6697
200	4,6718	5,0368
400	4,5172	4,7245
800	4,4428	4,6987
1000	4,5895	4,4047
2000	4,5338	4,4317
4000	4,4860	4,4050
Analytical answer: 4,3975		

Note that the convergence is non-monotonic, see section 4.3.2.

4.3.1 Direct barrier method

The following section is a naïve approach of the binomial method for barrier options, it makes use of Black-Scholes, ordinary calls/puts, in-out parity and a change in parameters. One can refer back to section 3.2.

$$\begin{aligned} c(S,T) &= c_{down-in}(S,T) + c_{down-out}(S,T) \\ c_{down-out}(S,T) &= c(S,T) - c_{down-in}(S,T) \\ &= c(S,T) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} c\left(\frac{B^2}{S},T\right). \end{aligned}$$

Example 4.3.2 *Consider a down-and-out call option*

$$c_{down-out}(S,T) = c(S,T) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} c\left(\frac{B^2}{S},T\right) \tag{4.11}$$

with $K > B$*;* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$*;* $N = 2$ *and* $B = 95$*.*

The MATLAB code, $DAOC(S,K,r,sigma,T,B,N)$, gives the option value as $c_{down-out} = 5,7351$. This code uses the binomial method for barrier options (section 4.3) for $c_{BM}[S,K,r,\sigma,T]$ and also for $c_{BM-down-in}[S,B,K,r,\sigma,T] = \left(\frac{B}{S}\right)^{\gamma-1} c_{BM}\left[\frac{B^2}{S},K,r,\sigma,T\right]$, where BM = binomial method.

Consider the above example with $B = 85$ and calculate the option value for various time steps.

N	$c_{down-out}$
50	6,2960
100	6,3113
150	6,3030
200	6,3134
400	6,3125
800	6,3116
1000	6,3111
2000	6,3109
3000	6,3098
4000	6,3086

One can find the analytical solution with $B = 85$, as $c_{down-out} = 6,3076$. Note that the rate of convergence becomes slower in the neighbourhood of the analytical value.

In addition, when $B = 95$ then $c_{down-out} = 4,3975$, the price of a down-and-out call option decreases as the barrier increases towards the initial stock price S and vice versa. In all the examples in this dissertation $S = 100$, thus the probability of the barrier being hit when the barrier target is closer to the stock price is greater, therefore the price is lower(for a down-and-out call option).

The source of the problem arises from the location of the barrier with respect to adjacent layers of nodes in the tree (lattice). The errors may be quite significant if the layers of the tree are set up so that the barrier falls between layers of the lattice, (Boyle & Lau 1994). In section 4.3.2 the error is described.

To avoid the above-mentioned error, (Boyle & Lau 1994) suggests constraining the time partitions so that the resulting lattice has layers that are as close as possible to the barrier. Section 4.3.4 explains how this is accomplished.

4.3.2 Errors with the binomial method for a barrier option

1. *Stock price quantization error*

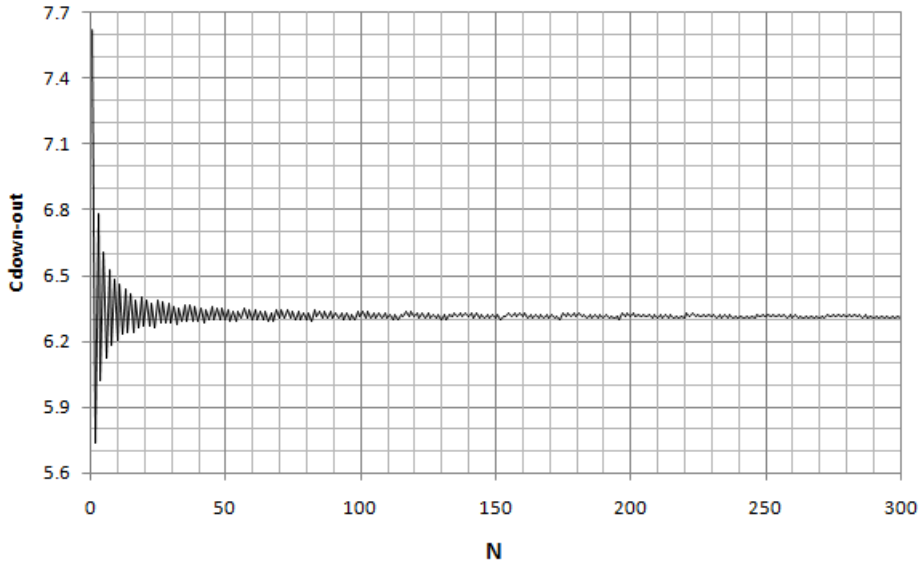
   The tree (lattice) "quantizes" the stock price and the instants in time at which it can be observed. When using a lattice, the stock is allowed to take the values of *only* those points on the lattice, thus valuing an option on a stock that moves directly. This is known as the unrealistic lack of continuity *quantization error*, (Derman et al. 1995).

2. *Option specification error*

   The inability of the lattice to accurately represent the terms of the option is the second type of error. The available stock prices are fixed once a lattice is chosen. If the exercise price or barrier level of the option does not coincide with one of the available stock prices, effectively one has to move the exercise price or barrier to the closest stock price available. Thus the options valued have contractual terms that differ from those of the actual option and this is the known as the *specification error*, (Derman et al. 1995).

3. *Non-monotonic convergence*

   Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$; $N = 1 : 300$ and $B = 85$.



Non-monotonic convergence of a down-and-out call option.

For barrier options, specification errors vanish much more slowly than quantization errors. Lattice methods can be adjusted for the specification error and one can obtain much improved results, (Derman et al. 1995).

4.3.3 A comparison of parameters for the trinomial method

Section 4.2 showed that u, p, p_u, p_m and p_d can have different parameters. This section considers whether the choice of these parameters will make an effective difference.

Consider a down-and-out call option priced using the trinomial method with $S = 100$, $K = 100$, $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$.

Using (Haug 1998, 300) parameters

$$\begin{aligned} u &= e^{\sigma\sqrt{2\delta t}}, \\ d &= e^{-\sigma\sqrt{2\delta t}}, \end{aligned}$$

$$p_u = \left(\frac{e^{r\delta t/2} - e^{-\sigma\sqrt{\delta t/2}}}{e^{\sigma\sqrt{\delta t/2}} - e^{-\sigma\sqrt{\delta t/2}}} \right)^2,$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\delta t/2}} - e^{r\delta t/2}}{e^{\sigma\sqrt{\delta t/2}} - e^{-\sigma\sqrt{\delta t/2}}} \right)^2,$$

$$p_m = 1 - p_u - p_d,$$

and (Hull 2000, 405) parameters

$$\begin{aligned} u &= e^{\sigma\sqrt{3\delta t}}, \\ d &= \frac{1}{u}, \end{aligned}$$

$$p_u = \sqrt{\frac{\delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6},$$

$$p_d = -\sqrt{\frac{\delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6},$$

$$p_m = \frac{2}{3},$$

the following results are obtained:

N	**Haug**	**Hull**
20	6,2890	6,2736
50	6,3209	6,2851
100	6,3134	6,2952
150	6,3149	6,3106
200	6,3202	6,3041
400	6,3177	6,3047
800	6,3146	6,3089
1000	6,3152	6,3158
2000	6,3119	6,3127
3000	6,3079	6,3118
4000	6,3114	6,3117
5000	6,3093	6,3102

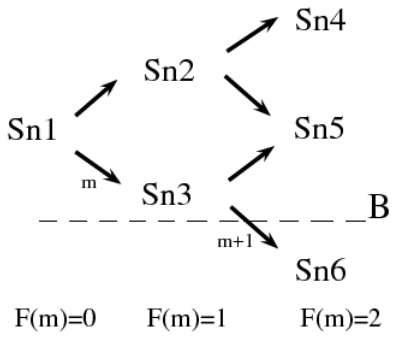
The analytical down-and-out call option price to this example is $6,3076$. Thus the parameters make little difference, a large number of time steps are required for both to obtain an accurate answer. Also note that the convergence is not monotonic.

4.3.4 Revised binomial model

In the binomial lattice, the barrier B will in general lie between two horizontal sets of nodes, this causes an inaccuracy into the calculations (called the *specification error*). Thus (Boyle & Lau 1994) sets up the following technique to try overcome the specification error.

This section considers a down-and-out call option.

For a CRR (section 4.1) tree one sets the number of time steps equal to $F(m)$, which is also the periods to expiration. The barrier B generally lies between the stock price after m down jumps and the stock price after $m+1$ down jumps so that ${S_d}^m > B > {S_d}^{m+1}$.



$F(m)$ the new N (number of time steps) is given by

$$F(m) = \frac{m^2\sigma^2 T}{\log \frac{S}{B}}$$

with $m = 1, 2, ...$

The barrier is now close to but just above a layer of horizontal nodes in the tree. This $N = F(m)$ makes the size of the price move just large enough that m of them will take the price beyond the barrier. The formula is the same whether the option is an up ($B > S$) or down ($B < S$) option. The resulting algorithm to find F_m can be summarized as follows:

for m=1 to 100
  Set $F_m = \frac{m^2\sigma^2 T}{\log \frac{S}{B}}$
  Set N = rounded off value of F_m
  compute rest of binomial or trinomial lattice barrier program
end

Example 4.3.3 *Consider a down-and-out call option and the binomial method, with* $K > B$*;* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$ *and* $B = 85$*.*

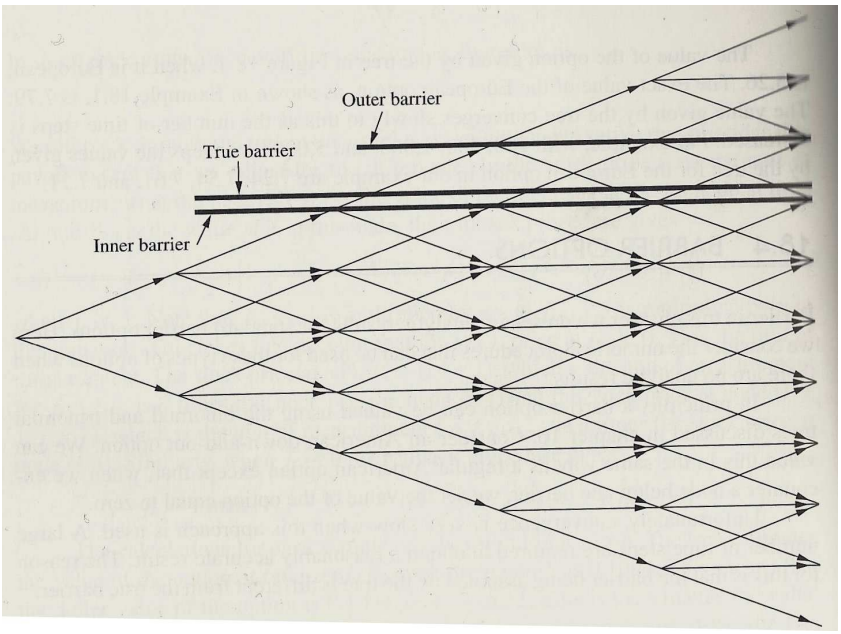
m	**F(m)**	$\mathbf{c}_{down-out}$
1	4	6,0229
2	18	6,2654
3	40	6,2890
4	72	6,3062
5	113	6,3321
6	163	6,3242
7	222	6,3143
8	290	6,3082
9	368	6,3124
10	454	6,3070
11	549	6,3155
12	654	6,3061
13	767	6,3128
14	890	6,3123

The Black-Scholes answer is $c_{down-out} = 6,3076$, and after 454 time steps $c_{down-out} = 6,3070$, where as in section 4.3.1 one observes that after 5000 time steps it is still uncertain if the answer has been passed correctly to 3 decimal places.

One should notice that the result of Hull's technique $c_{down-out} = 6,3047$ when N is randomly chosen as 400, which is only 6.305 rounded off. In this example, with increments of 1 for $m = 10$, $F(m) = N = 457$ with $c_{down-out} = 6,3070$, which is only 6.307 rounded off. Remember that the Black-Scholes answer is $c_{down-out} = 6,3076$. Thus one can conclude that this technique is possibly a little more convenient, relatively fast and easy to implement. Refining the partition size may not necessarily produce better results according to (Boyle & Lau 1994), if m increases, one does not necessary obtain better results. In this example when m increases to $m = 11$, $F(m) = 549$ the solution is $c_{down-out} = 6,3155$. When m increases to $m = 12$, $F(m) = 654$ the solution is $c_{down-out} = 6,3061$. And when m increase to $m = 13$, $F(m) = 767$ the solution is $c_{down-out} = 6,3128$.

4.3.5 Interpolation technique

As seen above, working with binomial and trinomial trees, a large number of time steps are required to obtain reasonably accurate results, hence convergence is very slow. The barrier being assumed by the tree is different from the true barrier. Define the *inner barrier* as the barrier formed by the nodes just on the inside of the true barrier and the *outer barrier* as the barrier formed by nodes just outside the true barrier (Hull 2000, 478).



Barriers assumed by trinomial trees (Hull 2000, 478).

The *interpolation algorithm*:

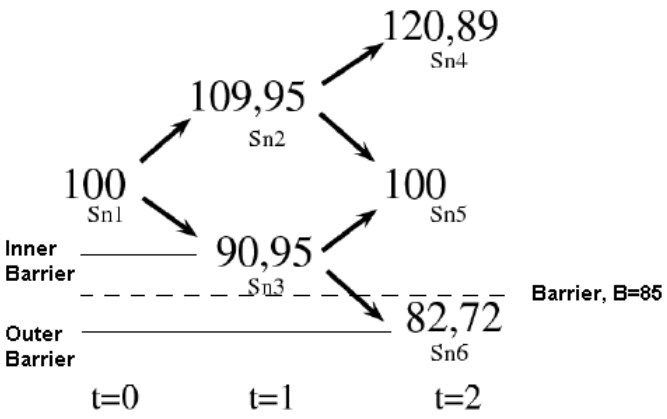
1. Calculate the value of the option on the assumption that the inner barrier is the true barrier.
2. Calculate the value of the option on the assumption that the outer barrier is the true barrier.
3. Interpolate between the two prices.

$$V = V_{\text{inner barrier}} + (\text{true barrier} - \text{inner barrier})\frac{V_{outer\ barrier} - V_{inner\ barrier}}{\text{outer barrier} - \text{inner barrier}}.$$

Example 4.3.4 *Consider a down-and-out call option and the binomial method, with* $K > B$*;* $S = 100$*;* $K = 95$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$ *and* $B = 85$*.*

Using CRR (section 4.1) parameters; $u = e^{\sigma\sqrt{\delta t}}$, $d = e^{-\sigma\sqrt{\delta t}}$ and $p = (e^{r\delta t} - d)/(u - d)$, and substituting the given values; $u = 1,0995$, $d = 0,9095$ and $p = 0,5292$.

Calculating the stock value at every node, one obtains:



First calculate the value of the option on the assumption that the *inner barrier* is the true barrier:

Value the options at every point below or equal to the barrier. The values below or equal to the barrier are equal to zero, thus $c_{n3} = 0$ and $c_{n6} = 0$

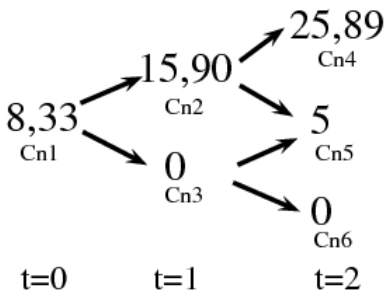
Then calculate the option price at each terminal node by using the payoff equation $c(S,t) = max(S - K, 0)$, where K =strike price. Thus $c_{n4} = 25,89$ and $c_{n5} = 5$.
(For example: $c_{n4} = max(120,89 - 95; 0) = 25,89$

Then apply backward induction using equations

$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5});$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

and substitute the valid numerical data.
(*For example:* $c_{n2} = e^{-0,1\times 0,1}(0,5292 \times 25,89 + (1 - 0,5292) \times 5) = 15,90$)

The option value obtained at each node:



Next, calculate the value of the option on the assumption that the *outer barrier* is the true barrier:

Now, value the options at every point below or equal to the barrier. The values below or equal to the barrier are equal to zero, thus $c_{n6} = 0$

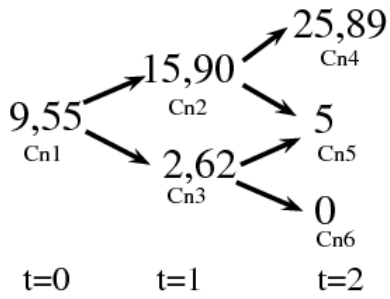
Then calculate the option price at each terminal node by using the payoff equation $c(S,t) = max(S-K,0)$, where K =strike price. Thus $c_{n4} = 25,89$ and $c_{n5} = 5$.
(For example: $c_{n4} = max(120,89 - 95;0) = 25,89$

Then apply backward induction using equations

$$c_{n2} = e^{-r\delta t}(pc_{n4} + (1-p)c_{n5});$$
$$c_{n3} = e^{-r\delta t}(pc_{n5} + (1-p)c_{n6});$$
$$c_{n1} = e^{-r\delta t}(pc_{n2} + (1-p)c_{n3})$$

and substitute the valid numerical data.
(*For example:* $c_{n2} = e^{-0,1\times 0,1}(0,5292 \times 25,89 + (1-0,5292)\times 5) = 15,90$)

The option value obtained at each node:



Then, interpolate between the two prices:

$$\begin{aligned} C_{\text{barrier}} &= C_{\text{inner barrier}} + (\text{true barrier} - \text{inner barrier})\frac{C_{outer\ barrier} - C_{inner\ barrier}}{\text{outer barrier} - \text{inner barrier}} \\ &= 8,33 + (85 - 90,95)\frac{9,55 - 8,33}{82,72 - 90,95} \\ &= 9,21. \end{aligned}$$

As one can observe from this simple example the end result is rather bad since the example is for a big time step ($\delta t = 1$) and $N = 2$. Note however that this example intention is to describe how the interpolation method works. The end result improves as N increase and the time step decreases.

Example 4.3.5 *Consider a down-and-out call option and the trinomial interpolation method, with $K > B$; $S = 100$; $K = 95$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$.*

The Black-Scholes answer is $c_{down-and-out} = 6,3076$. With the help of the MATLAB code KOCall_tm_int, found under section 8.1.4, one obtains the values in the following table.

N	$c_{down-and-out}$
20	6,2678
50	6,2955
100	6,2989
150	6,3019
200	6,3043
400	6,3059
800	6,3066
1000	6,3069
2000	6,3072
3000	6,3074
4000	6,3074
5000	6,3074

4.3.6 Stretch technique

This technique is due to the trinomial lattice implementation by (Kamrad & Ritchken 1991).

Since (Kamrad & Ritchken 1991) applied their technique to the logarithm of the underlying asset, the first step in this section is to explain why it can be done.

Simulation

There are two natural ways to simulate a continuous time price process:

1. Consider the process $\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$.
   Then take an interval length δt and set $S(t_0) = S_0$ (a given initial price at $t = t_0$).
   The corresponding simulation equation is
   $$S(t_{k+1}) - S(t_k) = \mu S(t_k)\delta t + \sigma S(t_k)\varepsilon(t_k)\sqrt{\delta t},$$
   where the $\varepsilon(t_k)$ values are uncorrelated normal random variables of mean 0 and standard deviation 1.
   This leads to the multiplicative model
   $$S(t_{k+1}) = [1 + \mu\delta t + \sigma\varepsilon(t_k)\sqrt{\delta t}]S(t_k).$$
   The random coefficient is not log normal but normal, thus the simulation method does not produce log normal price distributions.
2. An alternative approach is to use the log form $d\ln S(t) = \mu dt + \sigma dt$.
   In discrete form this is
   $$\ln S(t_{k+1}) - \ln S(t_k) = \mu\delta t + \sigma\varepsilon(t_k)\sqrt{\delta t}.$$
   This leads to the multiplicative model
   $$S(t_{k+1}) = e^{\mu\delta t + \sigma\varepsilon(t_k)\sqrt{\delta t}}S(t_k),$$
   with a log normal random coefficient.

The two methods are not exactly equivalent but their differences tend to cancel in the long run.
Therefore, either method is about as good as the other in practice, (Luenberger 1998). The simulation methods are described in section 5.1.

Assume the log of the underlying asset follows a geometric Wiener process, with drift $\mu = r - \sigma^2/2$, where r is the riskless rate, and σ is the instantaneous volatility. Then, with $S(t)$ representing the price at date t, one obtains

$$ln\{S(t + \Delta t)\} = ln\{S(t)\} + \xi(t),$$

with $S(0)$ given, where $\xi(t)$ is a normal random variable with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$. For a parameter $\lambda \geq 1$, define p_u, p_d and p_m as:

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}, \tag{4.12}$$

$$p_d = \frac{1}{2\lambda^2} - \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}, \tag{4.13}$$

$$p_m = 1 - \frac{1}{\lambda^2}, \tag{4.14}$$

λ is referred to as the stretch parameter. It can be viewed as a parameter that indicates the spacing between the underlying movements from one time period to another. If $\lambda = 1$ then $p_m = 0$ and the lattice is then a binomial lattice.

For a down-and-out call option one can choose the stretch parameter λ such that the barrier is hit exactly. η_0 is the largest integer smaller then η, where η is given by:

$$\eta = \frac{ln(S_0/B)}{\sigma\sqrt{\Delta t}}.$$

If η is an integer, then one shall maintain λ, thus $1 \leq \lambda < 2$. In most case η is not an integer, then

$$\eta = \eta_0\lambda.$$

The resulting algorithm to find λ can be summarized as follows:

Set $\eta = \dfrac{ln(S/B)}{\sigma\sqrt{\Delta t}}$
if $\eta > 2$
    $\lambda = \eta/$ rounded off (η)
else
    $\lambda = \eta$
end

Thus

1. Determine λ.
2. Precompute invariant quantities u, d, p_u, p_m and p_d.
3. Calculate stock prices.
4. Calculate option price.

Example 4.3.6 *Consider a down-and-out call option with $K > B$; $S = 10$; $K = 9$; $\sigma = 0,25$; $r = 0,05$; $T = 1$ and $B = 5$.*

The Black-Scholes answer is $c_{down-and-out} = 1,8141$. With the help of the MATLAB code stretch_tm, found under section 8.1.5, one obtains the values in the following table.

N	$c_{down-and-out}$	λ
20	1,8156	1,0333
50	1,8145	1,0319
100	1,8142	1,0269
150	1,8140	1,0290
200	1,8141	1,0054
400	1,8143	1,0082
800	1,8141	1,0054
1000	1,8141	1,0078

4.3.7 A numerical comparison of a down-and-out call option using various techniques

Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$. The value of the down-and-out call option is calculated with all the techniques mentioned in this chapter to give a brief comparison.

N	**Binomial**	**Trinomial**	**Interpolation**	λ	**Stretch**
20	6,2680	6,2890	6,2678	1,0835	6,3320
50	6,2960	6,3209	6,2955	1,0707	6,3079
100	6,3113	6,3134	6,2989	1,0095	6,3083
150	6,3030	6,3149	6,3019	1,0597	6,3078
200	6,3134	6,3202	6,3043	1,0077	6,3210
400	6,3125	6,3177	6,3059	1,0095	6,3176
800	6,3116	6,3146	6,3066	1,0077	6,3077
1000	6,3111	6,3152	6,3069	1,0081	6,3077
2000	6,3109	6,3119	6,3072	1,0032	6,3076
3000	6,3098	6,3079	6,3074	1,0053	6,3115
4000	6,3086	6,3114	6,3074	1,0081	6,3076
5000	6,3084	6,3093	6,3074	1,0077	6,3076
Analytical answer: 6,3076					

One can conclude that the stretch technique has given the most accurate answers for the chosen time steps.

At 50 time steps the other techniques are not even close to being correct to 3 decimal places, where the stretch techniques has an answer of $6,3079$.

At 2000 time steps the stretch technique obtains the analytical answer correct to 4 decimal places.

The interpolation technique seems to have a monotonic convergence while the convergence of the other techniques is not monotonic. The stretch technique is a good example of the convergence not being monotonic, at 150 time steps the answer is correct to 3 decimal places and then at 200 and even 400 it is not close. The numerical answer oscillates around the correct value.

Now, with the barrier closer to the strike price, $B = 95$, the following table is obtained:

N	**Binomial**	**Trinomial**	**Interpolation**	λ	**Stretch**
20	4,8036	4,4139	4,2567	1,7098	4,3812
50	4,6703	5,0290	4,3640	1,3517	4,3977
100	4,5166	5,1768	4,3569	1,2744	4,3982
150	4,5707	4,6697	4,3712	1,1706	4,3985
200	4,6718	5,0368	4,3832	1,0814	4,8756
400	4,5172	4,7245	4,3862	1,0923	4,3980
800	4,4428	4,6987	4,3921	1,0814	4,3978
1000	4,5895	4,4047	4,3928	1,0075	4,6095
2000	4,5338	4,4317	4,3967	1,0058	4,5491
3000	4,4050	4,5895	4,3965	1,0470	4,3976
4000	4,4860	4,4050	4,3971	1,0075	4,5059
5000	4,4901	4,4633	4,3971	1,0013	4,3976
Analytical answer: 4,3975					

Note that these methods struggle to obtain the analytical answer when the barrier is very close to the strike price, K, which is also mentioned in various literature such as in (Derman et al. 1995). Another method that was not investigate but that can be studied further is the adaptive mesh model described in (Ahn, Figlewski & Gao 1999), this method tries to overcome the problem of the barrier being too close to the strike price.

One can conclude, in general, that the better two lattice techniques for barriers are the stretch technique and the interpolation technique. The stretch technique reaches the analytical answer with the least number of time steps though it converges non-monotonic and the interpolation technique converges rather well.

4.4 Discrete barrier options

A barrier can be either *continuous* or *discrete*. Once a continuously monitored barrier is reached the option is immediately knocked in or out, while in discretely monitored conditions, barriers only come into effect in discrete monitored time, for example at close of every market day, every quarter, every month, or every half year (Wilmott 2007, 372).

Barrier monitoring is often assumed to be continuous, but in practice it is often discrete, which may lead to significant pricing errors, (Forsyth, Vetzal & Zvan 2000). In the continuous time, the payoffs depend on the price of the underlying asset throughout the life of the option. While in discrete time, the payoffs are determined by underlying prices at a finite set of time, (Broadie, Glasserman & Kou 1999).

Discretely monitored barrier options are popular and important, although pricing them is not as easy as their continuous counterparts. There is in essence no direct analytical solution. Many traded options are based on discrete price, though nearly all closed-form expressions available for pricing path-dependent options are based on continuous-time paths, (Broadie et al. 1999). Thus, in this chapter the question becomes how best to use a continuous formula to approximate the price of a discrete barrier option.

4.4.1 The barrier adjustment technique for Black-Scholes

This method developed by (Broadie et al. 1997) shows that discrete barrier options can be priced with remarkable accuracy using continuous barrier formulas by applying a simple continuity correction to the barrier (B). The method utilizes the formulae for the prices of continuous barrier options, but shifts the barrier to correct for discrete monitoring by adapting the barrier to:

$$B_{new} = Be^{\pm\alpha\sigma\sqrt{\frac{T}{m}}}, \tag{4.15}$$

where

$$
\begin{aligned}
\alpha &= \alpha_{up} \text{ or } \alpha_{down}, \\
\alpha_{up} &= 0,5863 \quad \text{(for a up barrier)}, \\
\alpha_{down} &= -0,5826 \quad \text{(for a down barrier)}, \\
B &= \text{Barrier}, \\
\sigma &= \text{volatility}, \\
T &= \text{time period}, \\
m &= \text{number of times the underlying asset price is monitored over } [0,T].
\end{aligned}
$$

The shift is determined exclusively by the asset volatility, a constant α and the monitoring frequency, (Broadie et al. 1997). Equation (4.15), the adjustment to the barrier, pushes the barrier away from the initial stock price to compensate for the excessive frequency of the barrier being breached when observations are continuous. For each time interval δt on the interval $[0,T]$, check whether the Barrier has been hit.

Example 4.4.1 *Consider a discrete down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ $B = 85$ and $m = 50$.*

The program in section 8.1.1 is used with the adjusted barrier $B_{new} = Be^{\pm\alpha\sigma\sqrt{\frac{T}{m}}}$.

Once again the Black-Scholes formula for this option is given by:

$$
\begin{aligned}
c_{down-out} &= S(N(d_1) - b(1 - N(d_8))) - \\
&\quad Ke^{-r(T)}(N(d_2) - a(1 - N(d_7)))
\end{aligned}
$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}},$$

$$d_7 = \frac{\log(SK/B_{new}^2) - (r - \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}},$$

$$d_8 = \frac{\log(SK/B_{new}^2) - (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}.$$

The option value is $c_{down-out} = 6,32213$.

The result is similar to the result obtained in (Broadie et al. 1997). Note that the discrete answer is higher than the continuous answer obtained in section 3.2. When a continuously monitored barrier is reached the option is immediately knocked in or out, while in discretely monitored conditions, barriers only come into effect in discrete monitored time.

4.4.2 The barrier adjustment technique for the trinomial method

Numerical methods are necessary for accurate evaluation of discrete option prices. They are based on discretization of time, but typically a much finer one than specified in the terms of an option. One has to adapt the time discretization for a discrete option, the process involves two time increments, (Broadie et al. 1999):

- the time intervals between the monitory times,
- the time steps of the numerical method.

Hence when adapting the trinomial method for a discrete barrier option, N (the number of time steps in the tree diagram), should be divisible by m (the number of monitoring times), (Broadie et al. 1999).

This section discusses a correction term that improves the approximation of discrete-time barrier options. The correction shifts the barrier to price a discrete option using the trinomial method for continuous monitored barriers.

(Broadie et al. 1999) adapted the techniques of (Boyle & Lau 1994), (Kamrad & Ritchken 1991) and (Ritchken 1995) that were discussed in chapter 4.

In the case of a discrete barrier option, one can use a trinomial method that puts a row of nodes at the level of the shifted barrier. Thus, making use of the *stretch technique* in section 4.3.6 and article, (Broadie et al. 1999).

Then we adapt the pricing algorithm in (Broadie et al. 1999) as follows:

1. Determine λ

   Set $\eta = \dfrac{ln(S/B)}{\sigma\sqrt{\Delta t}}$
   if $\eta > 2$

$\lambda = \eta$/ rounded off (η)
else
$\lambda = \eta$
end

2. Precompute invariant quantities u, d, p_u, p_m and p_d

$$\begin{aligned} u &= e^{\lambda\sigma\sqrt{\frac{T}{N}}} \\ d &= e^{-\lambda\sigma\sqrt{\frac{T}{N}}} \\ p_u &= \frac{1}{2\lambda^2} + \frac{(r - \frac{\sigma^2}{2})\sqrt{\frac{T}{N}}}{2\lambda\sigma} \\ p_m &= 1 - \frac{1}{\lambda^2} \\ p_d &= 1 - p_u - p_m \end{aligned}$$

3. Calculate stock prices on the nodes of the tree.

4. Set $B_{new} = Be^{\pm 0{,}5\lambda\sqrt{\frac{T}{m}}}$, where B is the initial barrier. For an up option use

$$B_{new} = Be^{0{,}5\lambda\sqrt{\frac{T}{m}}}$$

and for a down option use

$$B_{new} = Be^{-0{,}5\lambda\sqrt{\frac{T}{m}}}.$$

5. Calculate option price with the discrete shift barrier (B_{new}), similar to the technique described in section 4.3.6.

Example 4.4.2 *Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ $B = 85$ and $m = 50$.*

The program in section 8.1.5 is used with the adjusted barrier $B_{new} = Be^{\pm 0{,}5\lambda\sqrt{\frac{T}{m}}}$.

N	λ	$c_{down-out}$
100	1,0095	6,3250
200	1,0077	6,3295
300	1,0491	6,3271
400	1,0095	6,3249
500	1,0032	6,3234
1000	1,0081	6,3241
1500	1,0199	6,3218
2000	1,0032	6,3233
2500	1,0095	6,3221
3000	1,0053	6,3210
T*rue value*: 6,3221		

The *"true" value* is determined from the modified trinomial procedure described in (Broadie et al. 1999). In order to compute the *true value* to within $0,001$ on average, this modified trinomial procedure needs to run the routine using 80000 time steps, according to (Broadie et al. 1997).

In (Broadie et al. 1999) the algorithm is adapted to find another set of values for λ and N, to improve the results. However, the algorithm is not described in detail and could not be tested.

When one experiments with the example in table 1 of (Broadie et al. 1999), using $S = K = 100$, $B = 95$, $\sigma = 0,6$, $r = 0,1$, $m = 3$ and $T = 0,2$ and our algorithm, one obtains for $N = 504$; $\lambda = 1,0728$ and $c_{down-out} = 9,461$.
The result compares sufficiently with their algorithm since for $N = 504$ their answer is $c_{down-out} = 9,493$.

Chapter 5

Monte Carlo Method

5.1 Monte Carlo simulation

Option value calculations can be regarded as calculating expected values. The Monte Carlo method calculates the expected values of random variables.
The basic principle of pricing options via Monte Carlo simulation can be described as; "the value of an option is the present value of the expected payoff under a risk neutral random walk" (Wilmott 2007, 338).

The method can be extended to calculate option values where payoffs depend on several market variables. Monte Carlo simulation is one of the most powerful and most easily implemented methods for the calculation of option values. It provides a method to compute option values in cases where no analytical formulas are available. However, it is computationally expensive and cannot handle American style options with ease, (Hull 2000, 406).

Generally the method first simulates sample paths of the underlying asset over the relevant time interval. Then it simulates these according to the risk-neutral measure. Subsequently the method evaluates the discounted cash flows of a security on each sample path, it averages the discounted cash flows over the sample paths.

Generating option paths:

Assuming that the underlying stock S follows a geometric brownian motion, one finds the stochastic differential equation (SDE) for the stock return to be (Hull 2000, 407)

$$dS = \mu S dt + \sigma S dW, \tag{5.1}$$

where μ is the expected return, σ is the volatility and dW the Weiner process. To simulate

one path for S one has to divide the lifespan of the option into M intervals of length δt (process of discretization), and approximate (5.1) by

$$\delta t \to 0, \qquad S(t+\delta t) - S(t) = \mu S(t)\delta t + \sigma S(t)\varepsilon\sqrt{\delta t}.$$

Where
$S(t)$: Stock price at time t,
$S(t+\delta t)$: Stock price at one time interval,
ε : A random value from a sample from the normal distribution with mean zero, and standard deviation one,
δt : Interval length.

Refer back to section 2.2.3 on page 9 and apply formula

$$df = \left[\mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right] dt + \sigma S_t \frac{\partial f}{\partial S} dW_t,$$

to the function

$$f(S,t) = \ln S.$$

Thus in this case

$$\begin{aligned} \frac{\partial f}{\partial S}(S_t,t) &= \frac{1}{S_t}, \\ \frac{\partial f}{\partial t}(S_t,t) &= 0, \\ \frac{\partial^2 f}{\partial S^2}(S_t,t) &= -\frac{1}{S_t^2}, \end{aligned}$$

and the result is

$$\begin{aligned} d(\ln S_t) &= \left[\mu S_t \frac{1}{S_t} - \frac{1}{2}\sigma^2 S_t^2 \frac{1}{S_t^2}\right] dt + \sigma S_t \frac{1}{S_t} dW_t \qquad (5.2) \\ &= \left[\mu - \frac{1}{2}\sigma^2\right] dt + \sigma dW_t. \qquad (5.3) \end{aligned}$$

Thus $d(\log S_t)$ is normally distributed with mean $\left[\mu - \frac{1}{2}\sigma^2\right] dt$ and variance $\sigma^2 dt$, (Roman 2004).

Since the underlying option's returns follow a log normal distribution, applying Ito's lemma as above, leads to (5.3), so that

$$\ln S(t+\delta t) - \ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\varepsilon\sqrt{\delta t}, \qquad (5.4)$$

or

$$S(t+\delta t) = S(t)e^{\left(\mu-\frac{1}{2}\sigma^2\right)\delta t+\sigma\varepsilon\sqrt{\delta t}}. \tag{5.5}$$

The more time steps that are used in the model and the smaller the time steps size are in the model, the closer the model comes to a real asset price movement.

The pricing algorithm (Wilmott 2007, 924) :

1. Assume that σ and r (risk-free interest rate), are known. Simulate the risk-neutral random walk starting at today's value of the stock over the required time horizon. This gives one realization of the underlying price path.
2. For this realization calculate the option payoff.
3. Perform many more such realizations over the time horizon.
4. Calculate the average payoff over all realizations.
5. Take the present value of this average, this is the option value.

for $i = 1$ to M
  compute an N(0,1) sample εi
  set $S_i = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\varepsilon i}$
  set $V_i = e^{-rT}\Lambda(S_i)$
end
set $a_M = \frac{1}{M}\sum_{i=1}^{M} V_i$
set $b_M^2 = \left(\frac{1}{(M-1)}\right)\sum_{i=1}^{M}(V_i - a_M)^2$

Where $\Lambda(x)$ denotes the payoff function, such that $\Lambda(x) = max(x-E,0)$ for a call and $\Lambda(x) = max(x-E,0)$ for a put (Higham 2004, 115).

The output provides an approximate option price a_M and an approximate 95% confidence interval using the formula

$$\left[a_M - \frac{1,96b_M}{\sqrt{M}}, a_M + \frac{1,96b_M}{\sqrt{M}}\right],$$

a_M is the price of the option (Higham 2004).

5.1.1 Advantages of Monte Carlo

1. The mathematics needed to perform Monte Carlo simulations are simple.
2. The model can be adjusted without too much trouble.
3. To get a better accuracy, one just has to run more simulations.
4. The technique readily carries over to some European exotic and path-dependent contracts.

5.1.2 Disadvantages of Monte Carlo

1. A limitation that typically makes it impossible to get very high accuracy from a Monte Carlo approximation, is that the size of the confidence interval shrinks like the inverse square root of the number of samples. To reduce the 'error' by a factor of 10 requires a hundredfold increase in the sample size. Hence quite a large M is necessary to simulate the different paths to obtain accurate prices (Higham 2004, 143).
2. The Monte Carlo method in its basic form is only used to estimate the option price at one point in S. With American options, to know when it is optimal to exercise the option, one must calculate the option price for all values of S and t up to expiry in order to check that at no time there is arbitrage opportunities. Thus the Monte Carlo application to American options is far from straightforward (Higham 2004, 196).
3. The square root of the variance, of the random variable under consideration is the size of the confidence interval, which is directly proportional to the standard deviation. It is highly desirable in practice to use the technique *variance reduction*, which occurs in section 5.3. This technique transforms the problem $\mathrm{E}(X)$ to the problem of $\mathrm{E}(Y)$ where Y is another random sample that has the same mean as X, but a smaller variance (Higham 2004, 143).

5.2 Barrier options

The Monte Carlo method extends easily to handle path dependency options such as Barrier Options.

The following algorithm values a down-and-out call option. Here, M is the number of asset paths that are sampled.

for $i = 1$ to M
 for $j = 0$ to $N - 1$
  compute an N(0,1) sample εj
  set $S_{j+1} = S_j e^{(r-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\varepsilon j}$
 end
  set $S_i^{max} = max_{0\leq j\leq N} S_j$
  if $S_i^{max} > B$
   set $V_i = e^{-rT} max(S_N - K, 0)$,
   else set $V_i = 0$
end
set $a_M = \frac{1}{M}\sum_{i=1}^{M} V_i$
set $b_M^2 = \left(\frac{1}{(M-1)}\right)\sum_{i=1}^{M}(V_i - a_M)^2$

The result gives an approximate option price a_M and an approximate 95% confidence interval using the formula,

$$\left[a_M - \frac{1,96 b_M}{\sqrt{M}}, a_M + \frac{1,96 b_M}{\sqrt{M}}\right]$$

(Higham 2004, 194).

Example 5.2.1 *Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$, where CI is the confidence interval.*

The Black-Scholes answer is $c_{down-and-out} = 6,3076$. With the help of the MATLAB code monte(S,K,sigma,r,T,Dt,N,M) found in section 8.1.6, one obtains the values in the following table.

M	$c_{down-and-out}$	CI	Time (seconds)
$\Delta t = 10^{-2}$			
10^2	6,3467	4,6773; 8,0162	0,006
10^3	6,3363	5,7897; 6,8832	0,008
10^4	6,2668	6,0908; 6,4428	0,065
10^5	6,3658	6,3093; 6,4223	0,677
$\Delta t = 10^{-3}$			
10^2	5,9269	4,2526; 7,6011	0,003
10^3	6,0213	5,4812; 6,5614	0,016
10^4	6,3526	6,1756; 6,5296	0,150
10^5	6,3790	6,3224; 6,4356	1.414
$\Delta t = 10^{-4}$			
10^2	4,4912	3,1162; 5,8863	0,012
10^3	6,3876	5,8176; 6,9580	0,101
10^4	6,3469	6,1690; 6,5247	0,925
10^5	6,4070	6,3502; 6,4639	9,153
$\Delta t = 10^{-5}$			
10^2	7,7065	5,7045; 9,7084	0,085
10^3	6,5526	5,9814; 7,1238	0,607
10^4	6,4942	6,3128; 6,6757	5,854
10^5	6,3300	6,2735; 6,3864	59,329
Analytical answer: $6,3076$			

5.3 Variance reduction

The Monte Carlo method gives a simple and flexible technique for option valuation. Yet, it can be expensive in terms of computation time. A very large value of M (number of random variables) is usually necessary to estimate the option value with reasonable accuracy. This section illustrates how one can speed up the convergence.

The Monte Carlo method uses the sample mean

$$a_m := \frac{1}{M} \sum_{i=1}^{M} X_i \tag{5.6}$$

to approximate the expected value of random variable X. The width of the corresponding confidence interval is inversely proportional to $\sqrt{M}$. Thus to improve the approximation by taking more samples becomes an expensive affair. The confidence interval width scales with $\sqrt{\text{var}(X_i)}$. Hence replacing X_i in (5.6) with another sequence of random variables that have the same mean as the X_i but with smaller variance, motivates the idea of *variance reduction*.

5.3.1 Antithetic variates

The antithetic technique involves the calculation of two estimates for an option value using the one set of random numbers.

This is done by calculating the first value V_1 using our normal random numbers (ε_j) to generate one realization of the asset price path, an option payoff and its present value.
Now to determine the second value V_2 take the same set of random numbers but change their signs, thus replace ε_j with $-\varepsilon_j$, where ε is the random sample value. Then again generate a realization, and calculate the option payoff and its present value.
The estimate for the option value is the average of these two values. This operation is performed many times to get an accurate estimate for the option value (Wilmott 2007, 935).
Thus the antithetic variable technique requires twice as many evaluations of the exponential and square root functions.

This technique works because of the symmetry in the normal distribution, when one value is above the true value, the other tends to be below and vice versa (Hull 2000, 411).

The corresponding antithetic variates technique algorithm values a down-and-out call option (Higham 2004, 224):

for $i = 1$ to M
 for $j = 0$ to $N - 1$
  compute an N(0,1) sample εj
  set $S_{j+1} = S_j e^{(r-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\xi j}$
  set $\bar{S}_{j+1} = \bar{S}_j e^{(r-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}\xi j}$
 end
  set $S_i^{max} = max_{0\leq j\leq N} S_j$
  set $\bar{S}_i^{max} = max_{0\leq j\leq N} \bar{S}_j$
  if $S_i^{max} > B$
   set $V_i = e^{-rT} max(S_N - K, 0)$,
   else set $V_i = 0$
  if $\bar{S}_i^{max} > B$
   set $\bar{V}_i = e^{-rT} max(\bar{S}_N - K, 0)$,
   else set $\bar{V}_i = 0$
  set $\hat{V}_i = \frac{1}{2}(V_i + \bar{V}_i)$
end
set $a_M = \frac{1}{M}\sum_{i=1}^{M} \hat{V}_i$
set $b_M^2 = \left(\frac{1}{(M-1)}\right)\sum_{i=1}^{M}(\hat{V}_i - a_M)^2$

Example 5.3.1 *Consider a down-and-out call option with $K > B$; $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$, where CI is the confidence interval.*

The Black-Scholes answer is $c_{down-and-out} = 6,3076$. With the help of the MATLAB code monte_anti(S,K,sigma,r,T,Dt,M,B) found in section 8.1.7, one obtains the values in the following table.

M	$c_{down-and-out}$	CI	Time (seconds)
$\Delta t = 10^{-2}$			
10^2	6,8449	5,8947; 7,7951	0,016
10^3	6,4445	6,1619; 6,7272	0,033
10^4	6,3448	6,2545; 6,4351	0,162
10^5	6,3385	6,3098; 6,3671	1,306
$\Delta t = 10^{-3}$			
10^2	6,2356	5,3459; 7,1252	0,009
10^3	6,2455	5,9701; 6,5208	0,049
10^4	6,2696	6,1801; 6,3591	0,294
10^5	6,3453	6,3166; 6,3741	2,698
$\Delta t = 10^{-4}$			
10^2	5,2406	4,5647; 5,9166	0,041
10^3	6,1168	5,8361; 6,3975	0,222
10^4	6,2447	6,1545; 6,3349	1,682
10^5	6,3483	6,3195; 6,3771	16,438
$\Delta t = 10^{-5}$			
10^2	6,8957	5,9602; 7,8311	0,168
10^3	6,2186	5,9422; 6,4948	1,146
10^4	6,3099	6,2185; 6,4013	10,311
10^5	6,3228	6,2942; 6,3515	97,908
Analytical answer: $6,3076$			

5.3.2 Comparison

$\Delta t = 10^{-5}$			
M	$c_{down-and-out}$	CI	Time (seconds)
Monte Carlo Simulation			
10^2	7,7065	5,7045; 9,7084	0,085
10^3	6,5526	5,9814; 7,1238	0,607
10^4	6,4942	6,3128; 6,6757	5,854
10^5	6,3300	6,2735; 6,3864	59,329
Antithetic Variable Technique			
10^2	6,8957	5,9602; 7,8311	0,168
10^3	6,2186	5,9422; 6,4948	1,146
10^4	6,3099	6,2185; 6,4013	10,311
10^5	6,3228	6,2942; 6,3515	97,908
Analytical answer: $6,3076$			

At $M = 10^4$ the standard Monte Carlo simulation confidence interval does not even include

the correct answer, whereas the antithetic variable technique has an answer of 6,3099.

One can conclude that the antithetic variable technique has given more accurate answers in comparison to the standard Monte Carlo simulation. Observe that the technique shrinks the confidence intervals. However, twice as many evaluations of the exponential and square root functions are required (Higham 2004, 218).

The control variate technique is another variance reduction method that one can study further.

It is important to note that the overall accuracy of the Monte Carlo process depends not only on the error in the Monte Carlo approximation to the mean, but also on the error arising from the time discretization. (Higham 2004, 224).

Example 5.3.2 *Consider a down-and-out call option with* $K > B$*;* $S = 10$*;* $K = 9$*;* $\sigma = 0,25$*;* $r = 0,05$*;* $T = 1$ *and* $B = 5$*.*

M	$\delta t = 10^{-3}$	$\delta t = 10^{-4}$
Monte Carlo Simulation		
10^4	1,8241	1,8527
10^5	1.8279	1.8137
Antithetic Variable Technique		
10^4	1,8068	1,8085
10^5	1.8133	1.8088
Analytical answer: $1,8141$		

Chapter 6

Finite Difference Methods

Finite difference methods are designed for finding numerical solutions of differential equations. They value a derivative by solving the differential equation that the derivative satisfies.

The finite difference method involves converting the Black-Scholes differential equation into a set of algebraic equations which can then be solved numerically using an appropriate technique. Consider the Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{6.1}$$

Where

f = the option value,
S = the value of the underlying asset at time t,
r = the risk-free interest rate,
σ = constant volatility.

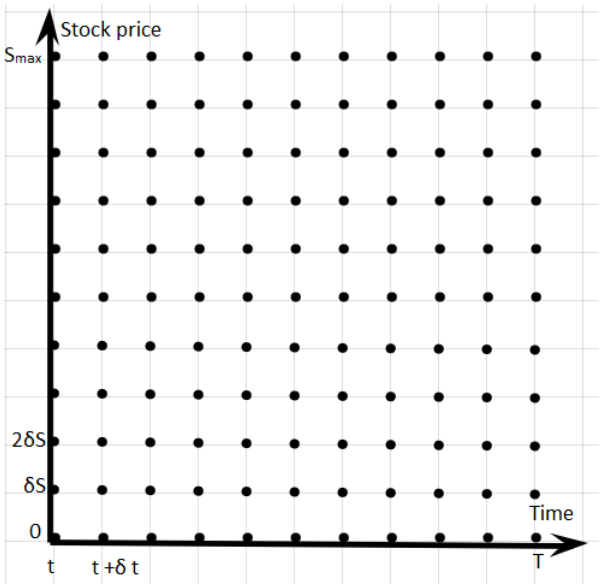
The finite difference method can calculate, with increased computational effort, option prices in cases where there are several state variables.

To solve the partial difference equation (PDE) by finite difference methods one must set up a discrete grid, with respect to time and asset prices.
Note that for the discretization process and the algorithm, the notation $f(t, S)$ is used to indicate the option value at time t, if the asset price is $S(t)$, to fit the explanation in Hull (Hull 2000).

Let T be option maturity, divide this into N equally spaced intervals of length $\delta t = T/N$. Let S_{max} be a suitably large asset price, that one assumes cannot be reached by $S(t)$ within

the time horizon considered.
Define $\delta S = S_{max}/M$ and consider a total of $M+1$ equally spaced stock prices.
The grid consists of points (t, S) such that

$$S = 0, \delta S, 2\delta S, \ldots, M\delta S \equiv S_{max},$$
$$t = 0, \delta t, 2\delta t, \ldots, N\delta t \equiv T.$$



The grid notation used is $f_{i,j} = f(i\delta t, j\delta S)$, where $f_{i,j}$ denotes the value of the option at the (i, j) point. The (i, j) point on the grid is the point that corresponds to time $i\delta t$ and stock price $j\delta S$. The grid above is defined by a total of (M+1)(N+1) stock and time points (Hull 2000).

Next, construct the linear set of difference equations, firstly by replacing the partial derivatives of equation (6.1) with approximations based on Taylor expansion.
First approximate $\frac{\partial f(t,S)}{\partial S}$ at point (i, j). The respective expansions of $f(t, S+\delta S)$ and $f(t, S-\delta S)$ in Taylor series are

$$f(t, S+\delta S) = f(t, S) + \frac{\partial f}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\delta S^2 + \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\delta S^3 + O(\delta S^4), \tag{6.2}$$

$$f(t, S-\delta S) = f(t, S) - \frac{\partial f}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\delta S^2 - \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\delta S^3 + O(\delta S^4). \tag{6.3}$$

Using equation (6.2) and ignoring higher order terms, the *forward difference operator* at point (i, j) is given by

$$\frac{\partial f}{\partial S}(t, S) = \frac{f(t, S+\delta S) - f(t, S)}{\delta S} + O(\delta S) \tag{6.4}$$

$$\approx \frac{f(i, j+1) - f(i, j)}{\delta S} \tag{6.5}$$

and similarly using equation (6.3), the corresponding *backward difference operator* is given by

$$\frac{\partial f}{\partial S}(t,S) = \frac{f(t,S)-f(t,S-\delta S)}{\delta S}+O(\delta S^2) \tag{6.6}$$

$$\approx \frac{f(i,j)-f(i,j-1)}{\delta S}. \tag{6.7}$$

Secondly discretize the equations. Subtracting equation (6.3) from (6.2) and taking the first order partial derivatives results in the central difference operator at point (i,j) given by

$$\frac{\partial f}{\partial S}(t,S) = \frac{f(t,S+\delta S)-f(t,S-\delta S)}{2\delta S}+O(\delta S^3) \tag{6.8}$$

$$\approx \frac{f(i,j+1)-f(i,j-1)}{2\delta S}. \tag{6.9}$$

Then adding equation (6.2) and (6.3), take the second order partial derivative and approximate $\frac{\partial^2 f}{\partial S^2}$ at the point (i,j) to obtain

$$\frac{\partial^2 f}{\partial S^2}(t,S) = \frac{f(t,S+\partial S)-2f(t,S)+f(t,S-\delta S)}{\delta S^2}+O(\delta S^3) \tag{6.10}$$

$$\approx \frac{f(i,j+1)-2f(i,j)+f(i,j-1)}{\delta S^2}. \tag{6.11}$$

This approximation is preferred because its symmetry preserves the reflectional symmetry of the second order partial derivative.
Approximate $f(t+\partial t,S)$ at point (i,j) using Taylor series to obtain

$$f(t+\partial t,S) = f(t,S)+\frac{\partial f}{\partial t}\delta t+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\delta t^2+\frac{1}{6}\frac{\partial^3 f}{\partial t^3}\delta t^3+O(\delta t^4). \tag{6.12}$$

Thus, the forward difference operator for time is then given by

$$\frac{\partial f}{\partial t}(t,S) = \frac{f(t+\delta t,S)-f(t,S)}{\delta t}+O(\delta t) \tag{6.13}$$

$$\approx \frac{f(i+1,j)-f(i,j)}{\delta t}. \tag{6.14}$$

To identify the 3 different approaches within the finite difference method,

1. Implicit method
2. Explicit method
3. Crank-Nicolson method

replace the first and second derivatives in the Black-Scholes PDE equation (6.1), with equations (6.9), (6.11) and (6.14). The results follow in section 6.1, 6.2 and 6.3.

Boundary conditions

As was the case with the binomial tree model, use option payoff values to determine option prices on the boundaries of the grid (Brandimarte 2006, 477).
The terminal condition at expiration for a call is

$$f(T, S) = \max(S - K, 0),$$

and for a put is

$$f(T, S) = \max(K - S, 0),$$

for all values of S, with K the strike price.

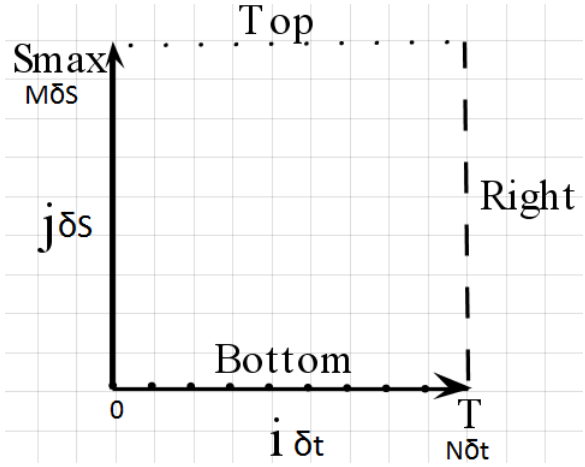
For an European call, when $S(t)$ is very large then

$$f(t, S_{max}) = S_{max} - Ke^{-r(T-t)}.$$

When $S(t) = 0$, then

$$f(t, 0) = 0.$$

Thus:



where

Bottom Boundary: $f_{i,0} = 0, \qquad i = 0, 1, \ldots, N$
Right Boundary: $f_{N\delta t,j} = \max(j\delta S - K, 0), \qquad j = 0, 1, \ldots, M$
Top Boundary: $f_{i,M\delta S} = M\delta S - Ke^{-r(N-i)\delta t}, \qquad i = 0, 1, \ldots, N$

Then similarly for an European put, when $S(t)$ is very large then

$$f(t, S_{max}) = 0.$$

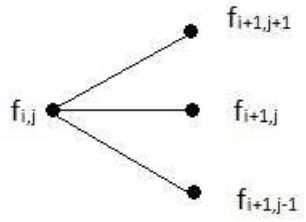
When $S(t) = 0$, then

$$f(t, 0) = Ke^{-r(T-t)}.$$

Thus:

Bottom Boundary: $f_{i,0} = Ke^{-r(N-i)\delta t}, \qquad i = 0, 1, \ldots, N$
Right Boundary: $f_{N\delta t,j} = \max(K - j\delta S, 0), \qquad j = 0, 1, \ldots, M$
Top Boundary: $f_{i,M\delta S} = 0, \qquad i = 0, 1, \ldots, N$

6.1 Explicit method



Since the value of the option at the maturity time is known, it is now possible to give an expression that gives one the previous value $f_{i,j}$ explicitly in terms of the given values $f_{i+1,j+1}$, $f_{i+1,j}$ and $f_{i+1,j-1}$ (Hull 2000, 419).

Equations (6.9) and (6.11) become

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\delta S}, \\ \frac{\partial^2 f}{\partial S^2} &= \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{\delta S^2}. \end{aligned}$$

When one substitutes the above equations and equation (6.14) into equation (6.1) and note that $S = j\delta S$, one obtains

$$\frac{f_{i+1,j} - f_{i,j}}{\delta t} + rj\delta S\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\delta S} + \frac{1}{2}\sigma^2 j^2 \delta S^2 \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{\delta S^2} = rf_{i,j}, \tag{6.15}$$

for $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, M-1$.

After rearranging terms, one obtains

$$\hat{\alpha}_j f_{i+1,j-1} + \hat{\beta}_j f_{i+1,j} + \hat{\gamma}_j f_{i+1,j+1} = f_{i,j}. \tag{6.16}$$

Where

$$\begin{aligned}\hat{\alpha}_j &= \frac{1}{1+r\delta t}\left(-\frac{1}{2}rj\delta t+\frac{1}{2}\sigma^2 j^2\delta t\right),\\ \hat{\beta}_j &= \frac{1}{1+r\delta t}\left(1-\sigma^2 j^2\delta t\right),\\ \hat{\gamma}_j &= \frac{1}{1+r\delta t}\left(\frac{1}{2}rj\delta t+\frac{1}{2}\sigma^2 j^2\delta t\right).\end{aligned}$$

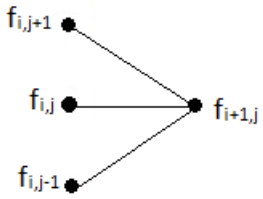
Equation 6.16 for $i = 0, 1, 2, \ldots, N-1$ and $j = 0, 1, 2, \ldots, M-1$ gives rise to a tridiagonal system. This results to the tridiagonal system given by

$$\begin{pmatrix} \hat{\alpha}_0 & \hat{\beta}_0 & & & \\ \hat{\alpha}_1 & \hat{\beta}_1 & \hat{\gamma}_1 & & \\ & \hat{\alpha}_2 & \hat{\beta}_2 & \hat{\gamma}_2 & \\ & & \cdots & \cdots & \\ & & \hat{\alpha}_{M-2} & \hat{\beta}_{M-2} & \hat{\gamma}_{M-2} \\ & & & \hat{\alpha}_{M-1} & \hat{\beta}_{M-1} \end{pmatrix} \begin{pmatrix} f_{i+1,0} \\ f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{pmatrix} = \begin{pmatrix} f_{i,0} \\ f_{i,1} \\ f_{i,2} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{pmatrix}.$$

The vector of asset prices $f_{i+1,j}$ is known at time T from our initial condition. The value of the option obtained at time zero is obtained by backward iterations.

It is easy to implement and program this method. However, the method is unstable if the ratio $\delta t/(\delta S)^2 \leq 1/2$ is not met, and the method is slow to converge (Wilmott 2007, 918) (Higham 2004, 294).

6.2 Implicit method



When one substitutes equations (6.9), (6.11) and (6.14) into equation (6.1) and note that $S = j\delta S$ (Hull 2000, 417), one obtains

$$\frac{f_{i+1,j}-f_{i,j}}{\delta t}+rj\delta S\frac{f_{i,j+1}-f_{i,j-1}}{2\delta S}+\frac{1}{2}\sigma^2 j^2\delta S^2\frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{\delta S^2}=rf_{i+1,j}, \qquad (6.17)$$

for $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, M-1$.

After rearranging terms, one obtains

$$\alpha_j f_{i,j-1}+\beta_j f_{i,j}+\gamma_j f_{i,j+1}=f_{i+1,j}. \qquad (6.18)$$

Where

$$\begin{aligned}\alpha_j &= \frac{1}{2}rj\delta t - \frac{1}{2}\sigma^2 j^2 \delta t,\\ \beta_j &= 1 + \sigma^2 j^2 \delta t + r\delta t,\\ \gamma_j &= -\frac{1}{2}rj\delta t - \frac{1}{2}\sigma^2 j^2 \delta t.\end{aligned}$$

Equation (6.18) gives rise to a tridiagonal system written as $Ax = b$, which results in the tridiagonal system given by

$$\begin{pmatrix} \beta_0 & \gamma_0 & & & \\ \alpha_1 & \beta_1 & \gamma_1 & & \\ & \alpha_2 & \beta_2 & \gamma_2 & \\ & & \cdots & \cdots & \\ & & \alpha_{M-2} & \beta_{M-2} & \gamma_{M-2} \\ & & & \alpha_{M-1} & \beta_{M-1} \end{pmatrix} \begin{pmatrix} f_{i,0} \\ f_{i,1} \\ f_{i,2} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{pmatrix} = \begin{pmatrix} f_{i+1,0} \\ f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{pmatrix}.$$

To compute $f_{i,j}$, $j = 0, 1, \ldots, M-1$ the above system for a given $f_{i+1,j}$, $j = 0, 1, \ldots, M$ is solved. Start at time $t = i + 1 = N\delta t$, using the boundary conditions, and solve the system of linear equations to obtain the price at time $t = i$.
Using the newly obtained values at time $t = i$, one can repeat the process to calculate the solution at time $t = i - 1$ until the asset price at time $t = 0$ is reached.

The system of linear equations can be solved using either the LU decomposition method (such as Thomas algorithm, section 6.3) or iterative methods (Gauss-Seidel, (Cheney & Kincaid 2004)).
The implicit method is stable, the method for any solution results in $\delta t/(\delta S)^2 > 0$, (Higham 2004).

6.3 Crank-Nicolson method

The Crank-Nicolson finite difference method is the average of the implicit and explicit methods and is introduced to improve accuracy and is unconditionally stable, hence it gives results for any ratio $\delta t/(\delta S)^2 > 0$, (Higham 2004).

We adapt the method in (Brandimarte 2006) to the grid in (Hull 2000) to derive the following equations.

The implicit scheme is given by equation (6.15) and the explicit scheme by equation (6.17).

Take the average of the two equations;

$$\frac{1}{2}\left(\frac{f_{i+1,j}-f_{i,j}}{\delta t}+rj\delta S\frac{f_{i+1,j+1}-f_{i+1,j-1}}{2\delta S}+\frac{1}{2}\sigma^2j^2\delta S^2\frac{f_{i+1,j+1}-2f_{i+1,j}+f_{i+1,j-1}}{\delta S^2}=rf_{i,j}\right),$$

and

$$\frac{1}{2}\left(\frac{f_{i+1,j}-f_{i,j}}{\delta t}+rj\delta S\frac{f_{i,j+1}-f_{i,j-1}}{2\delta S}+\frac{1}{2}\sigma^2j^2\delta S^2\frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{\delta S^2}=rf_{i+1,j}\right).$$

Then one obtains,

$$\begin{aligned}\frac{f_{i+1,j}-f_{i,j}}{\delta t}+\frac{rj\delta S}{2}\left(\frac{f_{i,j+1}-f_{i,j-1}}{2\delta S}\right)&\\ +\frac{rj\delta S}{2}\left(\frac{f_{i+1,j+1}-f_{i+1,j-1}}{2\delta S}\right)&\\ +\frac{\sigma^2j^2(\delta S)^2}{4}\left(\frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{(\delta S)^2}\right)&\\ +\frac{\sigma^2j^2(\delta S)^2}{4}\left(\frac{f_{i+1,j+1}-2f_{i+1,j}+f_{i+1,j-1}}{(\delta S)^2}\right)&\\ =\frac{r}{2}f_{i+1,j}+\frac{r}{2}f_{i,j}.&\end{aligned}$$

After rearranging the terms and multiplying by δt one obtains

$$\begin{aligned}\frac{f_{i,j-1}\delta t}{4}\left(rj-\sigma^2j^2\right)+f_{i,j}\left(1+\frac{\delta t}{2}\left(r+\sigma^2j^2\right)\right)+\frac{f_{i,j+1}\delta t}{4}\left(-rj-\sigma^2j^2\right)&=\\ \frac{f_{i+1,j-1}\delta t}{4}\left(-rj+\sigma^2j^2\right)+f_{i+1,j}\left(1-\frac{\delta t}{2}\left(r+\sigma^2j^2\right)\right)+\frac{f_{i+1,j+1}\delta t}{4}\left(rj+\sigma^2j^2\right).&\end{aligned}$$

Let

$$\begin{aligned}\tilde{\alpha}_j&=\frac{\delta t}{4}\left(\sigma^2j^2-rj\right),\\ \tilde{\beta}_j&=-\frac{\delta t}{2}\left(\sigma^2j^2+r\right),\\ \tilde{\gamma}_j&=\frac{\delta t}{4}\left(\sigma^2j^2+rj\right),\end{aligned}$$

thus

$$-\tilde{\alpha_j}f_{i,j-1}+(1-\tilde{\beta}_j)f_{i,j}-\tilde{\gamma_j}f_{i,j+1}=\tilde{\alpha_j}f_{i+1,j-1}+(1+\tilde{\beta}_j)f_{i+1,j}+\tilde{\gamma_j}f_{i+1,j+1}. \qquad (6.19)$$

One can express the system of equations in (6.19) as

$$Cf_i=Df_{i+1}.$$

This results in the tridiagonal system given by

$$\begin{pmatrix} 1-\tilde{\beta}_1 & -\tilde{\gamma}_1 & & & & \\ -\tilde{\alpha}_2 & 1-\tilde{\beta}_2 & -\tilde{\gamma}_2 & & & \\ & -\tilde{\alpha}_3 & 1-\tilde{\beta}_3 & -\tilde{\gamma}_3 & & \\ & & \cdots & \cdots & & \\ & & -\tilde{\alpha}_{M-2} & 1-\tilde{\beta}_{M-2} & -\tilde{\gamma}_{M-2} & \\ & & & -\tilde{\alpha}_{M-1} & 1-\tilde{\beta}_{M-1} \end{pmatrix} \begin{pmatrix} f_{i,1} \\ f_{i,2} \\ f_{i,3} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{pmatrix}$$

=

$$\begin{pmatrix} 1+\tilde{\beta}_1 & \tilde{\gamma}_1 & & & & \\ \tilde{\alpha}_2 & 1+\tilde{\beta}_2 & \tilde{\gamma}_2 & & & \\ & \tilde{\alpha}_3 & 1+\tilde{\beta}_3 & \tilde{\gamma}_3 & & \\ & & \cdots & \cdots & & \\ & & \tilde{\alpha}_{M-2} & 1+\tilde{\beta}_{M-2} & \tilde{\gamma}_{M-2} & \\ & & & \tilde{\alpha}_{M-1} & 1+\tilde{\beta}_{M-1} \end{pmatrix} \begin{pmatrix} f_{i+1,1} \\ f_{i+1,2} \\ f_{i+1,3} \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{pmatrix}.$$

These tridiagonal matrices can become very large. Very large matrices occupy huge accounts of memory, and processing them can take much computer time. These matrices are solved by making use of iterative methods or the Thomas algorithm.

The MATLAB code in section 8.1.8 is based on the Crank-Nicolson method described above. The system of equations is solved using the Thomas algorithm. If the asset value (S^*) is not equal to a value on the grid, one uses linear interpolation, to find the value of an option for the asset value (S^*).

The Thomas algorithm, (De Klerk 2009):
Assume the matrix equation is given as

$$Ax = b,$$

where

$$\mathbf{A} = \begin{pmatrix} d_1 & c_1 & & & & \\ a_2 & d_2 & c_2 & & & \\ & a_3 & d_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & d_{n-1} & c_{n-1} \\ & & & & a_n & d_n \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix},$$

and

$$\mathbf{b} = \begin{pmatrix} b_1 - a_1\alpha \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n - c_n\beta \end{pmatrix}.$$

Suppose now the matrix A may be written as the product of two matrices,

$$A = LU,$$

where L is a left-triangular matrix and U is a right-triangular matrix. Since A is tridiagonal, it follows that L and U only have two diagonals and the rest is zero's.

In general one can state that

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ f_2 & 1 & & & \\ & f_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & f_n & 1 \end{pmatrix}$$

and

$$\mathbf{U} = \begin{pmatrix} g_1 & c_1 & & & \\ & g_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & g_{n-1} & c_{n-1} \\ & & & & g_n \end{pmatrix},$$

with $f_2, \ldots, f_n$ and $g_1, \ldots, g_n$ unknown.

One obtains

$$\mathbf{LU} = \begin{pmatrix} g_1 & c_1 & & & \\ f_2 g_2 & f_2 c_1 + g_2 & c_2 & & \\ & f_3 g_2 & f_3 c_2 + g_3 & c_3 & \\ & & \cdots & \cdots & \\ & & & f_n g_{n-1} & f_n c_{n-1} + g_n \end{pmatrix}.$$

Since $A = LU$;

$$\begin{aligned} g_1 &= d_1, \\ f_i g_{i-1} &= a_i, \\ f_i c_{i-1} + g_i &= d_i, \end{aligned}$$

where $i = 2, \ldots, n$.

6.4 Barrier options

The Crank-Nicolson method is also more accurate than the explicit and implicit methods with an accuracy of up to $O(\delta t^2, \delta S^2)$, (Kerman 2002), and also converges faster.

Therefore the Crank-Nicolson method is implemented to find barrier option values. The only domain needed to consider for barrier options (Brandimarte 2006, 485), down and out call, is

$$B \leq S \leq S_{max},$$

with boundary conditions

$$f(t, S_{max}) = 0, \qquad f(t, B) = 0.$$

The resulting algorithm can be summarized as follows:

1. Set up the grid.
2. Set up the boundary conditions.
3. Set up the coefficients (see section 6.3).

$$\begin{aligned}\tilde{\alpha}_j &= \frac{\delta t}{4}\left(\sigma^2 j^2 - rj\right), \\ \tilde{\beta}_j &= -\frac{\delta t}{2}\left(\sigma^2 j^2 + r\right), \\ \tilde{\gamma}_j &= \frac{\delta t}{4}\left(\sigma^2 j^2 + rj\right).\end{aligned}$$

4. Solve the sequence of linear systems (using LU decomposition).

5. Return price. If the asset price S^* is not equal to a value on the grid, use linear interpolation, to find the value of an option for the asset value S^*.

Example 6.4.1 *Consider a down-and-out call option and the Crank-Nicolson method, with* $K > B$*;* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$*;* $B = 85$ *and* $S_{max} = 200$*.*

N=M	$\mathbf{c}_{down-out}$	**M=2N**	$\mathbf{c}_{down-out}$
10	6,5418	20	6,3931
20	6,3921	40	6,3131
50	6,3225	100	6,3038
100	6,3037	200	6,3069
200	6,3069	400	6,3075
300	6,3074	600	**6,3076**
400	6,3075	800	6,3076
500	**6,3076**	1000	6,3076

The Black-Scholes answer is $c_{down-out} = 6,3076$. When $M = 2N$, the finite difference method converges faster than when N and M are the same. This occurs because for each time step when $M = 2N$ there are more possible asset price values. The MATLAB code DOCallCN can be found under section 8.1.8.

Example 6.4.2 *Consider a down-and-out call option and the Crank-Nicolson method, with* $K > B$*;* $S = 100$*;* $K = 100$*;* $\sigma = 0,3$*;* $r = 0,1$*;* $T = 0,2$*;* $B = 95$ *and* $S_{max} = 200$*.*

N=M	$\mathbf{c}_{down-out}$	**M=2N**	$\mathbf{c}_{down-out}$
10	4,2919	20	4,3457
20	4,3441	40	4,3857
50	4,3917	100	4,3958
100	4,3958	200	4,3971
200	4,3971	400	4,3974
300	4,3973	600	4,3974
400	4,3974	800	4,3974
500	4,3974	1000	**4,3975**

The Black-Scholes answer is $c_{down-out} = 4,3975$. Once again, when $M = 2N$, the finite difference method converges faster than when N and M are the same.

Note that in this case, when the barrier is close to the strike price, that the Crank-Nicolson finite difference method gives more accurate results in less time steps than the lattice methods in section 4.3.7.

The MATLAB code DOCallCN can be found under section 8.1.8.

Example 6.4.3 *Consider a down-and-out call option and the Crank-Nicolson method, with* $K > B$*;* $S = 10$*;* $K = 9$*;* $\sigma = 0,25$*;* $r = 0,05$*;* $T = 1$*;* $B = 5$ *and* $S_{max} = 40$*.*

The Black-Scholes answer is $c_{down-out} = 1,8141$.

N=M	$\mathbf{c}_{down-out}$	**M=2N**	$\mathbf{c}_{down-out}$
10	1,9060	20	1,8380
20	1,8379	40	1,8248
50	1,8164	100	1,8157
100	1,8157	200	1,8143
200	1,8143	400	**1,8141**
300	**1,8141**	600	1,8141
400	1,8141	800	1,8141
500	1,8141	1000	1,8141

Chapter 7

Conclusion

Pricing barrier options with numerical methods can be difficult and certain methods can be slow to converge. This necessitates a thorough literature study to investigate the current available pricing techniques, after which MATLAB code was implemented in order to experiment with numerical data.

The standard binomial and trinomial methods converge extremely slow and with a non-monotonic convergence, as one can see in section 4.3.7. Thus, adaptive barrier methods were researched and implemented, such as the revised binomial model, the stretch technique and the interpolation technique. The interpolation technique differed in that the convergence seemed to be monotonic while the other adaptive barrier methods had non-monotonic convergence. The stretch technique gives the most accurate answers for a given time step. The stretch technique was consequently adapted to find the price of discrete barrier options, and obtained good results.

The Monte Carlo methods are easy to code though it can be slow, since tens of thousands of simulations are needed to get an accurate answer (Wilmott 2007, 387). The conclusion is that a variance reduction technique; the antithetic variable technique, has given more accurate answers in comparison to the standard Monte Carlo simulation. The technique shrinks the confidence intervals, that contains the price. However, twice as many evaluations of the exponential and square root functions are required (Higham 2004, 218), hence more computational effort is needed for a reasonable accurate answer.

The finite difference method, where the prices of barrier options on a grid (different underlying share prices and time intervals) can be calculated, was also studied. It is easy to implement and program the explicit finite difference method. However, the method is unstable if the ratio $\delta t/(\delta S)^2 \leq 1/2$ is not met, and the method is slow to converge (Wilmott 2007, 918) (Higham 2004, 294). The implicit method is stable, and results can

be obtained for any $\delta t/(\delta S)^2 > 0$, (Higham 2004). The Crank-Nicolson finite difference method is the average of the implicit and explicit methods. It improves the accuracy and is unconditionally stable, hence it gives results for any ratio $\delta t/(\delta S)^2 > 0$, (Higham 2004). The Crank-Nicolson method has a higher accuracy than the implicit method and therefore it converges faster. It is also more accurate than the explicit and implicit methods with an accuracy of up to $O(\delta t^2, \delta S^2)$, (Kerman 2002). Therefore, we can implement the Crank-Nicolson finite difference method to the barrier option.

The next table shows a down-and-out call option with $S = 100$; $K = 100$; $\sigma = 0,3$; $r = 0,1$; $T = 0,2$ and $B = 85$.
Where

S = stock price
K = exercise price
r = interest rate in percentage
σ = volatility in percentage
T = time to maturity (in a year)
B = barrier level
N = number of time steps
M = number of asset paths sampled
$Smax$ = suitably large asset price
δt = Interval length

N	**Binomial**	**Trinomial**	**Interpolation**	λ	**Stretch**	**Crank-Nicolson**
20	6,2680	6,2890	6,2678	1,0835	6,3320	6,3931
50	6,2960	6,3209	6,2955	1,0707	6,3079	6,3038
100	6,3113	6,3134	6,2989	1,0095	6,3083	6,3069
200	6,3134	6,3202	6,3043	1,0077	6,3210	6,3075
400	6,3125	6,3177	6,3059	1,0095	6,3176	6,3076
800	6,3116	6,3146	6,3066	1,0077	6,3077	
1000	6,3111	6,3152	6,3069	1,0081	6,3077	
2000	6,3109	6,3119	6,3072	1,0032	6,3076	
3000	6,3098	6,3079	6,3074	1,0053		
4000	6,3086	6,3114	6,3074	1,0081		
5000	6,3084	6,3093	6,3074	1,0077		
Analytical answer: 6,3076						

$\Delta t = 10^{-5}$			
M	$c_{down-and-out}$	CI	Time (seconds)
Monte Carlo Simulation			
10^2	7,7065	5,7045; 9,7084	0,085
10^3	6,5526	5,9814; 7,1238	0,607
10^4	6,4942	6,3128; 6,6757	5,854
10^5	6,3300	6,2735; 6,3864	59,329
Antithetic Variable Technique			
10^2	6,8957	5,9602; 7,8311	0,168
10^3	6,2186	5,9422; 6,4948	1,146
10^4	6,3099	6,2185; 6,4013	10,311
10^5	6,3228	6,2942; 6,3515	97,908
Analytical answer: $6,3076$			

As revealed by the experimental data, Monte Carlo simulation and even the variance reduction technique, take too long. The Crank-Nicolson finite difference method for the down and out call option gives an accurate value of the option in fewer time steps when compared to the other numerical methods.

This dissertation offers a discussion of the theoretical background of different barrier options, an investigation of numerical techniques to determine the value of a barrier option, a description of the algorithms and shows the implementation of the algorithms in MATLAB code. The results are compared with the analytical solutions and results obtained in the literature. Comments are offered on the number of subintervals needed for a valid price, using the different numerical methods. The best lattice method is the adaptation of the trinomial method using the stretch technique. The Monte Carlo method converges very slowly to obtain an accurate value, whilst the Crank-Nicolson finite difference method takes the least number of time steps to obtain an accurate value.

Further research could focus on the finite difference methods and adaptation of the finite difference methods to price more exotic barrier options and discrete barrier options.

Chapter 8

Appendix

8.1 Matlab code

8.1.1 Black-Scholes for barrier options

The function bs_daoc produces the analytical value of an up-and-out call option (see also section 4.4.1), (Higham 2004):

```
function f=bs_daoc(S,K,sigma,r,T,B)

% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%
% Output argument:  f = down and out call price

randn('state',100)
% Precompute invariant quantities:
power1 = -1+(2*r)/(sigma^2);
power2 = 1+(2*r)/(sigma^2);
a = (B/S)^power1;
b = (B/S)^power2;
d1 = (log(S/K)+(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));
d2 = d1 - sigma*sqrt(T);
d3 = (log(S/B)+(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));
```

```
d4 = (log(S/B)+(r-0.5*sigma^2)*(T))/(sigma*sqrt(T));
d5 = (log(S/B)-(r-0.5*sigma^2)*(T))/(sigma*sqrt(T));
d6 = (log(S/B)-(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));
d7 = (log(S*K/(B^2))-(r-0.5*sigma^2)*(T))/(sigma*sqrt(T));
d8 = (log(S*K/(B^2))-(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));

% Normal distributions of d values:
Nd1 = 0.5*(1+erf(d1/sqrt(2)));
Nd2 = 0.5*(1+erf(d2/sqrt(2)));
Nd3 = 0.5*(1+erf(d3/sqrt(2)));
Nd4 = 0.5*(1+erf(d4/sqrt(2)));
Nd5 = 0.5*(1+erf(d5/sqrt(2)));
Nd6 = 0.5*(1+erf(d6/sqrt(2)));
Nd7 = 0.5*(1+erf(d7/sqrt(2)));
Nd8 = 0.5*(1+erf(d8/sqrt(2)));

% For down-and-out call:
if K > B
    bs_daoc = S*(Nd1-b*(1-Nd8))-K*exp(-r*T)*(Nd2-a*(1-Nd7))
else
    bs_daoc = S*(Nd3-b*(1-Nd6))-K*exp(-r*T)*(Nd4-a*(1-Nd5))
end
f=bs_daoc;
```

8.1.2 Standard options

8.1.2.1 Binomial method

The function Call_bm makes use of CRR's (section 4.1) parameters for u, d and p, and produces the value of a European call option using a binomial lattice (see also section 4.1), (Brandimarte 2006):

```
function price = Call_bm(S,K,sigma,r,T,N)

% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  N = number of time steps
%
% Output argument:  f = call price
```

```
% Precompute invariant quantities:
deltaT = T/N;
u = exp(sigma * sqrt (deltaT));
d = 1/u;
p = (exp(r*deltaT) - d)/(u-d);
discount = exp(-r*deltaT);
p_u = discount*p;
p_d = discount*(1-p);
% Set up S values:
SVals = zeros(2*N+1,1);
SVals(1) = S*d^N;
for i = 2:2*N+1
    SVals(i) = u*SVals(i-1);
end
% Set up terminal CALL values:
CVals = zeros (2*N+1,1);
for i = 1:2:2*N+1
    CVals(i) = max(SVals(i)-K,0);
end
% Work backwards:
for tau = 1 : N
    for i = (tau+1):2: (2*N+1-tau)
        CVals(i) = p_u*CVals(i+1) + p_d*CVals(i-1);
    end
end
f = CVals(N+1);
```

The function Call_bmh makes use of (Higham 2004) parameters for u, d and p, and produces the value of a European call option using a binomial lattice, the program is almost identical to the program above but with:

```
u=exp(sigma*sqrt(dt)+(r-0.5*sigma^2)*dt);
d=exp(-sigma*sqrt(dt)+(r-0.5*sigma^2)*dt);
p=0.5;
```

8.1.2.2 Accuracy of option value

The function newtest_bm returns the value of a European call option and the number of time steps using a binomial lattice (see also section 4.1.1):

```
function value=newtest_bm(S,K,sigma,r,T,tol)
```

```
% Input arguments: S = asset price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatitly in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  tol = tolerance
%
% Output arguments:  f = call price
%                     N = number of time steps
%
% Precompute invariant quantities:

f0=0;
for N = 1:5000
    deltaT = T/N;
    u = exp(sigma * sqrt (deltaT));
    d = 1/u;
    p = (exp(r*deltaT) - d)/(u-d);
    discount = exp(-r*deltaT);
    p_u = discount*p;
    p_d = discount*(1-p);

    % Set up S values:
    SVals = zeros(2*N+1,1);
    SVals(1) = S*d^N;
    for i = 2:2*N+1
        SVals(i) = u*SVals(i-1);
    end

    % Set up terminal CALL values:
    CVals = zeros (2*N+1,1);
    for i = 1:2:2*N+1
        CVals(i) = max(SVals(i)-K,0);
    end

    % Work backwards:
    for tau = 1 : N
        for i = (tau+1):2: (2*N+1-tau)
            CVals(i) = p_u*CVals(i+1) + p_d*CVals(i-1);
        end
```

```
    end
    f = CVals(N+1);

    %Apply tolerance difference
    if abs(f-f0) < tol
        break
    end
    f0 = f;
end
value=f;
display(value);
display(f0);
display(N);
end
```

8.1.2.3 Trinomial method

The function Call_tm returns the value of a European call option using a trinomial lattice (see also section 4.2), (Brandimarte 2006):

```
function f=Call_tm(S,K,sigma,r,T,N)

% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  N = number of time steps
%
% Output argument:  f = call price

% Precompute invariant quantities:
dt =T/N;
dx = sigma*sqrt(2*dt);
discount = exp(-r*dt) ;
pu = (((exp((r*dt)/2)-exp(-sigma*sqrt(dt/2))) ...
    /(exp(sigma*sqrt(dt/2))- exp(-sigma*sqrt(dt/2))))^2);
pd = (((exp(sigma*sqrt(dt/2))-exp((r*dt)/2)) ...
    /(exp(sigma*sqrt(dt/2))-exp(-sigma*sqrt(dt/2))))^2);
pm = 1 - pu -pd;
p_u = discount*pu;
p_d = discount*pd;
```

```
p_m = discount*pm;

% Set up S values:
Stree = zeros(2*N+1,1);
Stree(1) = S*exp(-N*dx);
exp_dx = exp(dx);
for j = 2:2*N+1
    Stree(j) = exp_dx*Stree(j-1);
end

% Set up terminal CALL values:
OptValues = zeros(2*N+1,2);
x = mod(N,2)+1;
for j = 1:2*N+1
    OptValues(j,x) = max(Stree(j)-K,0);
end

% Work backwards:
for x = N-1:-1:0;
    know = mod(x,2)+1;
    knext = mod(x+1,2)+1;
    for j = N-x+1:N+x+1
        OptValues(j,know) = p_d*OptValues(j-1,knext)+p_m*...
            OptValues(j,knext)+ p_u*OptValues(j+1,knext);
    end
end
f = OptValues(N+1,1);
```

8.1.3 Barrier options

8.1.3.1 Binomial method

The function KOCall_bm makes use of CRR's (section 4.1) parameters for u, d and p, and returns the value of a up-and-out European call option using a binomial lattice (see also section 4.3), (Brandimarte 2006):

```
function f=KOCall_bm(S,K,sigma,r,T,N,B)

% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
```

```
%                   T = time to maturity (in a year)
%                   B = barrier level
%                   N = number of time steps
%
% Output argument:  f = down and out call price

% Precompute invariant quantities:
    deltaT = T/N;
    u = exp(sigma * sqrt (deltaT)) ;
    d = 1/u;
    p = (exp(r*deltaT) - d)/(u-d) ;
    discount = exp(-r*deltaT);
    p_u = discount*p;
    p_d = discount*(1-p);

% set up S values
    SVals = zeros(2*N+1,1) ;
    SVals(1) = S*d^N;
    for i = 2:2*N+1
        SVals(i) = u*SVals(i-1);
    end

% set up terminal CALL values
    CVals = zeros (2*N+1,1);
    for i = 1:2:2*N+1
        if SVals(i) <= B
            CVals(i) = 0;
        else
            CVals(i) = max(SVals(i)-K,0) ;
        end
    end

% work backwards
    for tau = 1 : N
        for i = (tau+1):2: (2*N+1-tau)
            if SVals(i) <= B
                CVals(i) = 0;
            else
                CVals(i) = p_u*CVals(i+1) + p_d*CVals(i-1) ;
            end
        end
```

```
    end
f = CVals(N+1);
disp(N)
disp(f)
```

Notes:
The function KOCall_bmh makes use of (Higham 2004) parameters for u, d and p, and produces the value of a European call option using a binomial lattice, the program is almost identical to the program above, but with:

```
u=exp(sigma*sqrt(dt)+(r-0.5*sigma^2)*dt);
d=exp(-sigma*sqrt(dt)+(r-0.5*sigma^2)*dt);
p=0.5;
```

8.1.3.2 Trinomial method

The function KOCall_tm returns the value of a down-and-out European call option using a trinomial lattice (see also section 4.3), (Brandimarte 2006):

```
function f = KOCall_tm(S,K,sigma,r,T,n,B)

% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%
% Output argument:  f = down and out call price

% Precompute invariant quantities:
dt = T/N;
dx = sigma*sqrt(2*dt);
discount = exp(-r*dt) ;
pu = (((exp((r*dt)/2)-exp(-sigma*sqrt(dt/2))) ...
    /(exp(sigma*sqrt(dt/2))- exp(-sigma*sqrt(dt/2))))^2);
pd = (((exp(sigma*sqrt(dt/2))-exp((r*dt)/2)) ...
    /(exp(sigma*sqrt(dt/2))-exp(-sigma*sqrt(dt/2))))^2);
pm = 1 - pu -pd;
p_u = discount*pu;
```

```
p_d = discount*pd;
p_m = discount*pm;

% Set up S values:
Stree = zeros(2*N+1,1);
Stree(1) = S*exp(-N*dx);
exp_dx = exp(dx);
for j = 2:2*N+1
    Stree(j) = exp_dx*Stree(j-1);
end

% Set up terminal CALL values:
OptValues = zeros(2*N+1,2);
x = mod(N,2)+1;
for j = 1:2*N+1
    if Stree(j,x) <= B
        OptValues(j,x) = 0;
    else
        OptValues(j,x) = max(Stree(j,x)-K,0);
    end
end

% Work backwards:
for x = N-1:-1:0;
    know = mod(x,2)+1;
    knext = mod(x+1,2)+1;
    for j = N-x+1:N+x+1
        if Stree(j) <= B
            OptValues(j,know) = 0;
        else
            OptValues(j,know) = p_d*OptValues(j-1,knext)+p_m*...
            OptValues(j,knext)+ p_u*OptValues(j+1,knext);
        end
    end
end
f = OptValues(N+1,1);
```

Notes:
This program does not give answers for an odd number of "N" time steps. Unlike binomial lattices, one must store at least two consecutive time layers of the lattice, since there is no alternation between odd- and even- indexed entries in the array. Thus, one must make use

of a two-column array, (Brandimarte 2006).

8.1.4 Interpolation technique

The function KOCall_tm_int returns the value of a down-and-out European call option using the interpolation technique (see also section 4.3.5):

```
function f=KOCall_tm_int(S,K,sigma,r,T,N,B)
% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%
% Output argument:  f = down and out call price

% Precompute invariant quantities:
dt=T/N;
dx=sigma*sqrt(2*dt);
discount = exp(-r*dt);
pu = (((exp((r*dt)/2)-exp(-sigma*sqrt(dt/2))) ...
    /(exp(sigma*sqrt(dt/2))- exp(-sigma*sqrt(dt/2))))^2);
pd = (((exp(sigma*sqrt(dt/2))-exp((r*dt)/2)) ...
    /(exp(sigma*sqrt(dt/2))-exp(-sigma*sqrt(dt/2))))^2);
pm= 1 - pu -pd;
p_u= discount*pu;
p_d=discount*pd;
p_m=discount*pm;
exp_dx=exp(dx);

%Calculate the stock price
Stree=zeros(2*N+1,1);
Stree(1)=S*exp(-N*dx);
for j=2:2*N+1
    Stree(j)=exp_dx*Stree(j-1);
end


%Working outer barrier and inner barrier
```

```
t=mod(N,2)+1;
for j=1:2*N+1
    if Stree(j,t) >= B
               out= Stree(j,t);
               break
    end

end
for j=1:2*N+1
    if Stree(j,t) >= out
               in= Stree(j-1,t);
               break
    end

end

%KOCall_tm(S,K,sigma,r,T,N,outb)
%Calculate the option price with outter barrier
OptValues2=zeros(2*N+1,2);
for j=1:2*N+1
    if Stree(j,t) <=out
        OptValues2(j,t)=0;
    else
        OptValues2(j,t)=max(Stree(j,t)-K,0);
    end
end
for t=N-1:-1:0;
    knows= mod(t,2)+1;
    knexts=mod(t+1,2)+1;
    for j=N-t+1:N+t+1
        if Stree(j)<=out
            OptValues2(j,knows) = 0;
        else
            OptValues2(j,knows)=p_d*OptValues2(j-1,knexts)+p_m*...
            OptValues2(j,knexts)+ p_u*OptValues2(j+1,knexts);
        end
    end
end
foutb=OptValues2(N+1,1);

%KOCall_tm(S,K,sigma,r,t,n,inb)
```

```
%Calculate the option price with inner barrier
%Calculate the stock price
Stree2=zeros(2*N+1,1);
Stree2(1)=S*exp(-N*dx);
for j=2:2*N+1
    Stree2(j)=exp_dx*Stree2(j-1);
end
t=mod(N,2)+1;
OptValues=zeros(2*N+1,2);
for j=1:2*N+1
    if Stree2(j,t) <=in
        OptValues(j,t)=0;
    else
        OptValues(j,t)=max(Stree2(j,t)-K,0);
    end
end
for t=N-1:-1:0;
    knows= mod(t,2)+1;
    knexts=mod(t+1,2)+1;
    for j=N-t+1:N+t+1
        if Stree2(j)<=in
            OptValues(j,knows) = 0;
        else
            OptValues(j,knows)=p_d*OptValues(j-1,knexts)+p_m*...
            OptValues(j,knexts)+ p_u*OptValues(j+1,knexts);
        end
    end
end
finb=OptValues(N+1,1);

%Interpolate
f=finb +(B-in)*((foutb - finb)/(out - in));
display(f)
display(in)
display(out)
```

8.1.5 Stretch technique

The function stretch_tm returns the value of a down-and-out European call option using the stretch technique (see also section 4.3.6):

```
function f = stretch_tm(S,K,r,sigma,T,N,B)
% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%
% Output argument:  f = down and out call price

dt= T/N;
%Work out lamda (nn)
n=(log(S/B))/(sigma*sqrt(dt));
    if n> 2
        nn= n/floor(n);
    else
        nn=n;
    end
display(nn)
% Precompute invariant quantities:
dx=nn*sigma*sqrt(dt);
discount = exp(-r*dt);
u=r-((sigma^2)/2);
pu = (1/(2*nn^2))+((u*sqrt(dt))/(2*nn*sigma));
pd = (1/(2*nn^2))-((u*sqrt(dt))/(2*nn*sigma));
pm= 1 - 1/(nn^2);
p_u= discount*pu;
p_d=discount*pd;
p_m=discount*pm;
%Work out stock price
Stree=zeros(2*N+1,1);
Stree(1)=S*exp(-N*dx);
exp_dx=exp(dx);
for j=2:2*N+1
    Stree(j)=exp_dx*Stree(j-1);
end


%Work out option price
OptValues=zeros(2*N+1,2);
```

```
t=mod(N,2)+1;
for j=1:2*N+1
    if Stree(j,t) <=B
        OptValues(j,t)=0;
    else
        OptValues(j,t)=max(Stree(j,t)-K,0);
    end
end
for t=N-1:-1:0;
    know= mod(t,2)+1;
    knext=mod(t+1,2)+1;
    for j=N-t+1:N+t+1
        if Stree(j)<=B
            OptValues(j,know) = 0;
        else
            OptValues(j,know)=p_d*OptValues(j-1,knext)+p_m*...
            OptValues(j,knext)+ p_u*OptValues(j+1,knext);
        end
    end
end
f=OptValues(N+1,1);
```

8.1.6 Monte Carlo method

The function monte returns the value of a down-and-out European call option using a Monte Carlo simulation (Higham 2004, 200) (see also section 5.2):

```
function conf=monte(S,K,sigma,r,T,Dt,N,M)
% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%    Dt = Interval length
%    M = number of asset paths sampled
%
% Output argument: conf = confidence interval

%Monte Carlo for a knock out call option
```

```
%Smax<B for an up-and-out call option
%Smax>B for an up-and-out call option

%state uses Matlabs Marsaglias ziggurat algorithm so generate numbers and creating
%an 100x100 matrix

randn('state',100)
Dt=N/T;
f=zeros(M,1);
for i= 1:M
    Sfinal = S*cumprod(exp((r-0.5*sigma^2)*Dt+sigma*sqrt(Dt)*randn(N,1)));
    Smax=max(Sfinal);
    if Smax>B
     f(i) = exp(-r*T)*max(Sfinal(end)-K,0);
    end

end
xM = mean(f);
yM = std(f);
conf = [xM - 1.96*yM/sqrt(M), xM + 1.96*yM/sqrt(M)];
disp(xM)
disp(conf)
```

8.1.7 Antithetic variable technique

The function monte_anti returns the value of a down-and-out European call option using the antithetic variable technique (Higham 2004, 226) (see also section 5.3):

```
function conf=monte_anti(S,K,sigma,r,T,Dt,M,B)
%Monte Carlo for a out call option
% < up and > down
% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%    Dt = Interval length
%    M = number of asset paths sampled
%
```

```
% Output argument: conf = confidence interval

randn('state',100)

N=T/Dt;
f=zeros(M,1);
fanti=zeros(M,1);
for i= 1:M
    samples = randn(N,1);

    % standard Monte Carlo
    Sfinal = S*cumprod(exp((r-0.5*sigma^2)*Dt+sigma*sqrt(Dt)*samples));
    Smax=max(Sfinal);
    if Smax>B
     f(i) = exp(-r*T)*max(Sfinal(end)-K,0);
    end

     % antithetic path
    Sfinal2 = S*cumprod(exp((r-0.5*sigma^2)*Dt-sigma*sqrt(Dt)*samples));
    Smax2=max(Sfinal2);
    f2=0;
    if Smax2>B
     f2 = exp(-r*T)*max(Sfinal2(end)-K,0);
    end
    fanti(i)=0.5*(f(i)+f2);
end
xM = mean(f);
yM = std(f);
conf = [xM - 1.96*yM/sqrt(M), xM + 1.96*yM/sqrt(M)];
xManti = mean(fanti);
yManti = std(fanti);
confanti = [xManti - 1.96*yManti/sqrt(M), xManti + 1.96*yManti/sqrt(M)];
disp(xManti)
disp(confanti)
```

8.1.8 Crank-Nicolson method

The function DOCallCN (Brandimarte 2006, 487) returns the value of a down-and-out European call option using the Crank-Nicolson method (see also section 6.4):

```
function f = DOCallCN(S,K,r,T,sigma,B,Smax,N,M)
% Input arguments: S = stock price at time t
%                  K = exercise price
%                  r = interest rate in percentage
%                  sigma = volatility in percentage
%                  T = time to maturity (in a year)
%                  B = barrier level
%                  N = number of time steps
%    M = number of asset paths sampled
%    Smax = suitably large asset price
%
% Output argument: f = down and out call price

%set up grid
dt = T/N;
dS = (Smax - B)/M;
matval = zeros(M+1,N+1);
vetS = linspace(B,Smax,M+1)';
vetj = vetS/dS;

%set up boundary conditions
matval(:,N+1) = max(vetS-K,0);
matval(1,:) = 0;
matval(M+1,:) = Smax;

%set up coefficients
alpha = 0.25 * dt * (sigma^2*(vetj.^2) -r*vetj);
beta = - dt * 0.5 * (sigma^2*(vetj.^2) + r);
gamma = 0.25 * dt * (sigma^2*(vetj.^2) +r*vetj);
C = -diag(alpha(3:M) ,-1) + diag(1-beta(2:M)) - diag(gamma(2:M-1) ,1) ;
[L,U] = lu(C);
D = diag(alpha(3:M) ,-1) + diag(1+beta(2:M)) + diag(gamma(2:M-1) ,1) ;

%solve the sequence of linear systems
for i = N : -1: 1
    matval(2:M,i) = U\(L\(D*matval(2:M,i+1)));
end
%return price, possibly by linear interpolation outside the grid
f = interp1(vetS,matval(:,1),S);
```

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