



Lie group analysis of modified Camassa-Holm and 3D Kadomtsev-Petviashvili-like equations

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Declaration

I ISAAC KEOAGILE MOHAPI, student number 24479268, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed:

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This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

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Declaration of Publications

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Dedication

To my family

Acknowledgements

First and foremost, I would like to give praises to the Almighty God for carrying me throughout this research. Moreover, I give my deepest gratitude to my research supervisor Prof. C.M. Khalique for his immense support, guidance and profound knowledge. I would also like to thank Dr. O.D. Adeyemo for fruitful discussions. Additionally I thank North-West University for providing me with necessary resources and their post graduate bursary scheme for granting me with the financial support in completing my research. Last but not least, I thank my loved ones for their endless support, encouragement and love that they have been providing me without boundaries.

Abstract

In this work, we investigate two nonlinear partial differential equations that possess real-world applications, namely the generalized simplified Camassa-Holm and (3+1)-dimensional Kadomtsev-Petviashvili like equation. Invoking Lie group analysis, we reduce the differential order of the equations or systematically find optimal variable changes, with an aim of finding analytical solutions. In addition, techniques inclusive of direct integration, simplest equation method and Kudryashov's approach are implored in this research. Furthermore, we construct conserved vectors for the underlying equations where diverse dimensional providings of selected solutions are presented. Method for securing conserved vectors of differential equations which comprise Noether's approach, multiplier technique as well as Ibragimov's conservation theorem are implored.

Keywords: Lie symmetry analysis; Nonlinear (3+1)-dimensional Kadomtsev-Petviashvili-like and generalized simplified Camassa-Holm equations; Group-invariant and exact solutions; Conservation laws.

Introduction

Nonlinear partial differential equations (NLPDEs) are complex mathematical models used to approximate and understand natural and man-made occurrences. Some famous NLPDEs include, but are not limited to, the Benjamin-Bona-Mahony equation [1] of fluid mechanics which is a refinement of the famous Korteweg-de Vries equation [2], the Kadomtsev-Petviashvili equation [3] for shallow water waves, the Logarithmic Schrödinger equation [4] for super-fluids and quantum gravity, the nonlinear Klein-Gordon equation [5] for relativistic quantum mechanics. Countless other NLPDEs exist and they have captivated researchers in diverse fields such as engineering, weather systems, fluid and plasma physics, financial markets and so on.

Inevitably, this has brought about immense innovation in solution methods for NLPDEs and perhaps the resurgence of Lie groups, a unified approach to obtaining analytical solutions of DEs. Lie groups were first introduced to the world by the renowned mathematician Sophus Lie during the late nineteenth century [?, 6–9, 23].

Numerous ad-hoc approaches for acquiring exact solutions of DEs have also been developed in the last several decades, for example, the homogeneous balance approach [10], the (G'/G) –expansion approach [11], the simplest equation approach [12], the Jacobi elliptic function expansion approach [13], the Kudryashov’s approach [14, 15], the Lie symmetry method [?, 6–9, 23].

Conserved vectors present another interesting aspect of NLPDEs. Great scholars have come up with different methods for their derivation such as Noether’s theorem [17, 18], multiplier technique [7] and Ibragimov’s method [19, 20]. Conserved vectors can, amongst other things, be used to reduce the order of NLPDEs making it possible

to solve them using the usual techniques.

Outline of the chapters

Mathematical concepts of Lie group analysis and some algorithms for computing conservation laws are presented in chapter one.

In the second chapter, we exhibit an elucidative example of this work. We consider the potential Burgers equation and compute its Lie point symmetries and then use them to construct a one parameter groups of transformation and group-invariant solutions. Moreover, we procure the conserved vectors of the equation utilizing the multiplier technique.

Next, we investigate the generalized simplified Camassa-Holm equation in chapter three. Similarity reductions alongside simplest equation techniques, Kudryashov's method and direct integration method are invoked in achieving exact solutions of the equation. In addition, conserved vectors are procured using the multiplier technique, Ibragimov's technique and Noether's theorem.

In chapter four, we explore the (3+1)-dimensional Kadomtsev-Petviashvili like equation in nonlinear science and engineering where closed-form solutions are secured through the use of symmetry reductions and the Kudryashov's method. Thereafter, the multiplier technique for conserved currents are implored in deriving the conserved vectors.

Finally, an outline of findings specified in this work is given and prospective work is alleged in chapter five.

Bibliography is provided at the end of this work.

Chapter 1

Preliminaries

In this chapter, some basic concepts of Lie's theory together with conserved vectors of differential equations (DEs) are presented, which are used throughout this work. Selected techniques for finding solutions and conserved vectors of DEs are also outlined.

1.1 Introduction

DEs have proven to be of great importance in describing physical phenomena in nature and solving them by quadrature has been a priority over the ages. During the late 18th century a Norwegian mathematician Marius Sophus Lie (1842–1899), using group theoretic method, inspired by E. Galois, discovered a revolutionary symmetry-based technique for finding results to DEs. This algorithmic technique, also known as Lie group analysis, provides a more organized and reliable method of producing the exact solutions to DEs. Many publications have been made on Lie group analysis of late. See for example refer to [6–9, 23].

1.2 Lie group analysis

1.2.1 One-parameter groups of continuous transformations

The setting is in Euclidean spaces $\mathbb{R}^n$ of $x = (x^1, \dots, x^n)$ independent variables and $\mathbb{R}^m$ of $u = (u^1, \dots, u^\lambda)$ dependent variables. Consider the transformations

$$T_\varepsilon : \bar{x}^i = \phi^i(x, u, \varepsilon), \quad \bar{u}^\lambda = \varrho^\lambda(x, u, \varepsilon) \quad (1.1)$$

accompanied by ε as a real constant which perpetually takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of ε equal to zero with differentiable functions ϕ^i and ϱ^λ .

Definition 1.1 A set H of transformations given by (1.1) is a local Lie group that satisfies the properties:

- (1) (Closure) If $T_{\varepsilon_1}, T_{\varepsilon_2} \in H$ for $\varepsilon_1, \varepsilon_2 \in \mathcal{D}' \subset \mathcal{D}$ then $T_{\varepsilon_3} = T_{\varepsilon_2} T_{\varepsilon_1} \in H$, where $\varepsilon_3 = \psi(\varepsilon_1, \varepsilon_2) \in \mathcal{D}$.
- (2) (Identity) There exists a unique $T_0 \in H$ if and only if $\varepsilon = 0$ such that $T_0 T_{\varepsilon_1} = T_{\varepsilon_1} T_0 = T_{\varepsilon_1}$.
- (3) (Inverse) There exists a unique $T_{\varepsilon_1} \in H$ for every transformation $T_\varepsilon^{-1} = T_{\varepsilon^{-1}} \in H$, where $\varepsilon \in \mathcal{D}' \subset \mathcal{D}$ and $\varepsilon^{-1} \in \mathcal{D}$ such that $T_\varepsilon T_{\varepsilon^{-1}} = T_{\varepsilon^{-1}} T_\varepsilon = T_0$.

Associativity property from (1.1) is satisfied from (1).

1.2.2 Prolongations

The differential variables [9] are defined as

$$u_i^\lambda = D_i(u^\lambda), \quad u_{ij}^\lambda = D_j D_i(u_i^\lambda), \dots \quad (1.2)$$

accompanied by the total differentiation operator with respect to x^i given as

$$D_i = \frac{\partial}{\partial x^i} + u_i^\lambda \frac{\partial}{\partial u^\lambda} + u_{ij}^\lambda \frac{\partial}{\partial u_j^\lambda} + \dots \quad (1.3)$$

The partial derivatives given below

$$u_{(1)} = \{u_i^\lambda\}, \quad u_{(2)} = \{u_{ij}^\lambda\}, \quad \dots, \quad u_{(k)} = \{u_{i_1 \dots i_k}^\lambda\}$$

are the first-order, second-order and k th-order, respectively.

Prolonged or extended groups

Consider the local Lie group H given by the transformations

$$\begin{aligned} \bar{x}^i &= \phi^i(x, u, \varepsilon), & \phi^i|_{\varepsilon=0} &= x^i, \\ \bar{u}^\lambda &= \varrho^\lambda(x, u, \varepsilon), & \varrho^\lambda|_{\varepsilon=0} &= u^\lambda. \end{aligned} \tag{1.4}$$

The construction of the Lie group H is equivalent to the computation of corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + \xi^i(x, u)\varepsilon, \quad \phi^i|_{\varepsilon=0} = x^i, \quad \bar{u}^\lambda \approx u^\lambda + \eta^\lambda(x, u)\varepsilon, \quad \varrho^\lambda|_{\varepsilon=0} = u^\lambda, \tag{1.5}$$

obtained from equation (1.1) by a Taylor series expansion of the functions $\phi^i(x, u, \varepsilon)$ and $\varrho^\lambda(x, u, \varepsilon)$ in ε about $\varepsilon = 0$. Consequently, we have

$$\xi^i(x, u) = \left. \frac{\partial \phi^i(x, u, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}, \quad \eta^\lambda(x, u) = \left. \frac{\partial \varrho^\lambda(x, u, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}. \tag{1.6}$$

The infinitesimal transformation is used to write (1.5) as

$$\bar{x}^i \approx x^i(1 + \varepsilon X), \quad \bar{u}^\lambda \approx u^\lambda(1 + \varepsilon X),$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\lambda(x, u) \frac{\partial}{\partial u^\lambda} \tag{1.7}$$

is the generator of the group H .

To find transformed derivatives from (1.1), we use the change of variable formula

$$D_i = D_i(\phi^j) \bar{D}_j \tag{1.8}$$

with $\bar{D}_j$ as total differentiation in the variables $\bar{x}^i$. So

$$\bar{u}_i^\lambda = \bar{D}_j(u^\lambda), \quad \bar{u}_{ij}^\lambda = \bar{D}_j(\bar{u}_i^\lambda) = \bar{D}_i(\bar{u}_j^\lambda), \dots$$

Applying (1.8) and (1.5), we have

$$\begin{aligned} D_i(\varrho^\lambda) &= D_i(\phi^j)\bar{D}_j(\bar{u}^\lambda) \\ &= D_i(\phi^j)\bar{u}_j^\lambda. \end{aligned} \quad (1.9)$$

Expansion of (1.9) yields

$$\left(\frac{\partial \phi^j}{\partial x^i} + u_i^\beta \frac{\partial \phi^j}{\partial u^\beta} \right) \bar{u}_j^\lambda = \frac{\partial \varrho^\lambda}{\partial x^i} + u_i^\beta \frac{\partial \varrho^\lambda}{\partial u^\beta}. \quad (1.10)$$

The variable $\bar{u}_j^\lambda$ can be written as functions of $x^i, u^\lambda, u_{(i)}$, that is

$$\bar{u}_i^\lambda = \psi_i^\lambda(x^i, u^\lambda, u_{(1)}, \varepsilon), \quad \psi_i^\lambda|_{\varepsilon=0} = u_i^\lambda. \quad (1.11)$$

The transformations in space of variables $x^i, u^\lambda, u_{(1)}$ given in (1.5) and (1.11) form the first prolongation group of $H^{[1]}$.

Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\lambda \approx u_i^\lambda + \varepsilon \zeta_i^\lambda, \quad (1.12)$$

where $\zeta_i^\lambda = \zeta_i^\lambda(x, u, u_{(1)}, \varepsilon)$.

Prolonged generators

By using the relation given in (1.9) together with (1.5), we get

$$\begin{aligned} D_i(\phi^j)(\bar{u}_j^\lambda) &= D_i(\varrho^\lambda) \\ D_i(x^j + a\xi^j)(u_j^\lambda + a\zeta_j^\lambda) &= D_i(u^\lambda + a\eta^\lambda) \\ u_i^\lambda + a\zeta_i^\lambda + au_j^\lambda D_i\xi^j &= u_i^\lambda + aD_i\eta^\lambda \\ \zeta_i^\lambda &= D_i(\eta^\lambda) - u_j^\lambda D_i(\xi^j), \end{aligned} \quad (1.13)$$

labelled as the formula of first prolongation . Equivalently, we can get higher order prolongations

$$\begin{aligned} \zeta_{ij}^\lambda &= D_j(\eta_i^\lambda) - u_{ip}^\lambda D_j(\xi^p), \\ &\vdots \end{aligned}$$

$$\zeta_{i_1, i_2, \dots, i_k}^\lambda = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{k-1}}^\lambda) - u_{i_1, i_2, \dots, i_{k-1} j}^\lambda D_{i_k}(\xi^j). \quad (1.14)$$

The prolonged generators of the prolongations $H^{[1]}, \dots, H^{[k]}$ of the group H are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\lambda \frac{\partial}{\partial u_i^\lambda}, \\ &\vdots \\ X^{[k]} &= X^{[k-1]} + \zeta_{i_1, \dots, i_k}^\lambda \frac{\partial}{\partial u_{i_1, \dots, i_k}^\lambda} \quad k \geq 1 \end{aligned} \quad (1.15)$$

with

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\lambda(x, u) \frac{\partial}{\partial u^\lambda}. \quad (1.16)$$

1.2.3 Group admitted by a partial differential equation

Consider a k^{th} -order system of partial differential equations (PDEs)

$$E_\lambda(x, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)}, \dots, u_{(k)}) = 0, \quad (1.17)$$

The PDEs system (1.17) is invariant under the generator

$$\mathbf{X} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\lambda(x, u) \frac{\partial}{\partial u^\lambda} \quad (1.18)$$

if and only if

$$\mathbf{X}^{[k]} E_\lambda(x, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)}, \dots, u_{(k)})|_{(1.17)} = 0, \quad (1.19)$$

which are known as the determining equations of all symmetries of (1.17).

Here $\mathbf{X}^{[k]}$ denotes the k th prolongation of $\mathbf{X}$ and is given by

$$\mathbf{X}^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\lambda \frac{\partial}{\partial u^\lambda} + \zeta_i^\lambda \frac{\partial}{\partial u_i^\lambda} + \dots + \zeta_{i_1 i_2 \dots i_k}^\lambda \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\lambda} \quad (1.20)$$

with

$$\zeta_i^\lambda = D_i(\eta^\lambda) - u_j^\lambda D_i(\xi^j), \quad (1.21)$$

$\vdots$

$$\zeta_{i_1 \dots i_s}^\lambda = D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\lambda) - u_{j i_1 \dots i_{s-1}}^\lambda D_{i_s}(\xi^j), \quad s > 1. \quad (1.22)$$

The generator (1.18) of the symmetry group is called infinitesimal symmetry of (1.17).

1.3 Euler formula

The Euler operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (1.23)$$

and the Lie-Bäcklund operator is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\lambda \frac{\partial}{\partial u^\lambda} + \sum_{s \geq 1} \zeta_{i_1 \dots i_s}^\lambda \frac{\partial}{\partial u_{i_1 \dots i_s}^\lambda}. \quad (1.24)$$

1.4 Exact solutions

This section provides a concise demonstration of few techniques for finding exact or analytic solutions of DEs.

1.4.1 The simplest equation approach

The simplest equation approach can be used for finding analytic solutions of nonlinear partial differential equations (NLPDEs). It was developed by Kudryashov [24, 25]. Many researchers have recently used this approach on different NLPDEs. It has been proven to be a useful and influential mathematical tool for solving NLPDEs in different disciplines of sciences applied. See, for example, the papers [26, 27, 29–31]. These are the fundamental steps of the strategy:

Consider the NLPDE of the form

$$W_1(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots) = 0. \quad (1.25)$$

Utilizing the transformation

$$u(t, x) = F(y), \quad y = k_1 t + k_2 x + k_3, \quad (1.26)$$

reduces equation (1.25) to an nonlinear ordinary differential equation (NLODE)

$$W_2[F(y), k_1 F'(y), k_2 F'(y), k_3 F'(y), k_1^2 F''(y), k_2^2 F''(y), k_3^2 F''(y), \dots] = 0. \quad (1.27)$$

The simplest equations that we use here are the Bernoulli equation

$$H'(y) = aH(y) + bH^2(y), \quad (1.28)$$

the Riccati equation

$$G'(y) = aG^2(y) + bG(y) + c \quad (1.29)$$

with a, b and c as constants [25, 30, 31]. We look for solutions of the NLODE (1.27) that are of the form

$$F(y) = \sum_{i=0}^M A_i (G(y))^i, \quad (1.30)$$

where $G(y)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \dots, A_M$ are parameters to be determined.

The solution of Bernoulli equation (1.28) that we use here is given by

$$H(y) = a \left\{ \frac{\cosh[a(y+S)] + \sinh[a(y+S)]}{1 - b \cosh[a(y+S)] - b \sinh[a(y+S)]} \right\} \quad (1.31)$$

where S is an integration constant. For the Riccati equation (1.29), the solutions to be used are

$$G(y) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (y+S) \right] \quad (1.32)$$

and

$$G(y) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta y \right) + \frac{\operatorname{sech} \left(\frac{\theta y}{2} \right)}{S \cosh \left(\frac{\theta y}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta y}{2} \right)} \quad (1.33)$$

with $\theta = \sqrt{b^2 - 4ac}$ and S is an integration constant.

1.4.2 Kudryashov's approach

This approach is utilized in finding the exact solutions of NLPDEs and it is described in [14]. It can also be seen, for example, in the papers [32, 33].

Consider the NLPDE

$$D_1(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{xxx}, \dots) = 0. \quad (1.34)$$

The algorithm involved in Kudryashov's approach is as follows:

Step 1. The substitution $u(x, t) = U(\psi)$, $\psi = sx + \omega t$, with constants s and ω , transforms (1.34) to the NLODE

$$D_2(U, \omega U', sU', \omega^2 U'', s^2 U'', \dots) = 0. \quad (1.35)$$

Step 2. Suppose that the exact solutions of (1.35) are presented as

$$U(\psi) = \sum_{n=0}^N a_n R^n(\psi), \quad (1.36)$$

with a_n ($n = 0, 1, 2, \dots, N$) as constants to be determined, such that $a_N \neq 0$, and $R(\psi)$ is the solution of the first-order NLODE

$$R(\psi) = R^2(\psi) - R(\psi). \quad (1.37)$$

The equation (1.37) has the solution

$$R(\psi) = \frac{1}{1 + e^\psi}. \quad (1.38)$$

Step 3. Next we substitute the value of $U(\psi)$ into (1.35) and then use (1.37) to generate an equation which involves the powers of R .

Step 4.

Equating various powers of R to zero yields the system of algebraic equations

$$P_n(a_N, a_{N-1}, \dots, a_0, s, \omega, \dots) = 0, \quad (n = 0, \dots, N). \quad (1.39)$$

Step 5. Finally, the solution of the system of algebraic equations produces the values of coefficients $a_0, a_1, \dots, a_{N-1}, a_N$ and relations for parameters of (1.35). Thus, we acquire the exact solutions of (1.35) in the form expressed in (1.36).

1.5 Conserved vectors

A conserved vector of (1.17) is an n -tuple $T = (T^1, T^2, \dots, T^n)$, $T^i \in \mathcal{A}$, $i = 1, 2, \dots, n$, ($\mathcal{A}$ is the space of differential functions) such that

$$D_i T^i = 0 \quad (1.40)$$

holds for all solutions of (1.17). Equation (1.40) is called a local conservation law.

1.5.1 The multiplier approach

This approach involves the variational derivative. See for example [7]. Consider $Q_\lambda(x, u, u_{(1)} \dots)$ as a multiplier that has the property

$$Q_\lambda E_\lambda = D_i T^i \quad (1.41)$$

holds identically. The multiplier Q_λ has the determining equation

$$\frac{\delta}{\delta u^\beta} (Q_\lambda E_\lambda) = 0, \quad (1.42)$$

which holds for functions $u(x^1, x^2, x^3, x^4, \dots, x^n)$. Once we get the multipliers from (1.42), we may find conserved vectors by invoking (1.41) [7].

1.5.2 Noether's theorem

If $L(x, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)} \dots, u_{(s)}) \in \mathcal{A}$, $s \leq k$ is a function, such that system

$$E_\beta(x, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)}, \dots, u_{(k)}) = 0$$

is equivalent to

$$\frac{\delta L}{\delta u^\beta} = 0, \quad (1.43)$$

then L is called a Lagrangian of the system. Equations (1.43) are therefore the corresponding Euler-Lagrange differential equations.

If there exists a vector $\mathbf{A} = (\mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^n)$ such that

$$X(L) + LD_i(\xi^i) = D_i(\mathbf{A}^i) \quad (1.44)$$

then the Lie-Bäcklund operator X is a Noether symmetry generator associated with the Lagrangian L of Euler-Lagrange equations.

For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, there corresponds a vector $T = (T^1, T^2, \dots, T^n)$, known as Noether conserved vector, defined by

$$T^i = B^i - \xi^i L - W^\beta \frac{\delta L}{\delta u_i^\beta} - \sum_{s \geq 1} D_{i_1 \dots i_s}(W^\beta) \frac{\delta L}{\delta u_{i_1 \dots i_s}^\beta}, \quad (1.45)$$

where $W^\beta = \eta^\beta - \xi^j u_j^\beta$, are the characteristic functions.

Noether conserved vector T can also be written as [34]

$$\begin{aligned} T^k = & \xi^i \mathcal{L} + (\eta - u_{x^j} \xi^j) \left\{ \frac{\partial \mathcal{L}}{\partial u_{x^k}} - \sum_{l=k}^n D_{x^i} \left(\frac{\partial \mathcal{L}}{\partial u_{x^j x^k}} \right) \right\} \\ & + \sum_{l=k}^n \{ \zeta_l - u_{x^l x^j} \xi^j \} \frac{\partial \mathcal{L}}{\partial u_{x^k x^l}} - B^i. \end{aligned} \quad (1.46)$$

1.5.3 Ibragimov's theorem of conserved vectors

Definition 1.2 The adjoint equations of the system given by (1.17) are

$$E_\lambda^*(x^i, u^\lambda, v^\lambda, \dots, v_{(s)}, u_{(s)}) \equiv \frac{\delta}{\delta u^\lambda} (v^\beta E_\beta) = 0, \quad (1.47)$$

where v is the new dependent variables [35].

Definition 1.3 Formal Lagrangian $\mathcal{L}$ of the system (1.17) and its adjoint equations (1.47) is

$$\mathcal{L} = v^\lambda E_\lambda(x^i, u^\lambda, u_{(1)}, \dots, u_{(s)}). \quad (1.48)$$

Theorem 1.1 Every infinitesimal symmetry X of the system given by (1.17) leads to conservation laws (1.40), where the conserved vectors are [35]

$$C^i = \xi^i \mathcal{L} + W^\lambda \left[\frac{\partial \mathcal{L}}{\partial u_i^\lambda} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\lambda} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\lambda} \right) + \dots \right]$$

$$+ D_j(W^\lambda) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\lambda} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\lambda} + \dots \right] + D_j D_k(W^\lambda) \frac{\partial \mathcal{L}}{\partial u_{ijk}^\lambda} + \dots . \quad (1.49)$$

1.6 Conclusion

In this chapter, we outlined the basic introduction to Lie group theory. We also presented three solution approaches for finding the exact solutions of partial differential equations and three approaches for deriving the conserved vectors. These various approaches mentioned in this chapter will be utilized in this research.

Chapter 2

Symmetry reductions, solutions and conserved vectors for the potential Burgers equation: an example

The Lie point symmetries of the generalized potential Burgers equation (gpBe) are presented in this chapter. Then, using Lie equations, we identify one-parameter group transformations for each of the Lie point symmetries. Thereafter, we use these Lie point symmetries to construct the group-invariant solutions of the pBe. Finally, using the multiplier method, we get the equation's conserved vectors.

2.1 Introduction

Burgers equation [36] is regularly encountered in several applied mathematics disciplines such as gas dynamics, fluid mechanics and nonlinear acoustics, with applications in hydrodynamics and other areas due to the fact that it is the simplest partial differential equation with properties of nonlinear effects and dissipative characteristics. Furthermore, in mathematics, it is the prototype of an equation that is

linearizable through a direct coordinate transformation. The following is the basic form of the Burgers equation:

$$v_t - 2vv_x - v_{xx} = 0. \quad (2.1)$$

Let

$$v = u_x,$$

replacing v in (2.1) with u_x and integrating with respect to x , we get the potential form of the Burgers equation (pBe), named as the potential Burgers equation [8, 37]

$$u_t - u_x^2 - u_{xx} = 0, \quad (2.2)$$

with $u = u(t, x)$.

In this chapter, we shall investigate the pBe

$$u_t - \alpha u_x^2 - \beta u_{xx} = 0, \quad (2.3)$$

with α and β as non-zero constants.

2.2 Solutions of the pBe

In this section we determine the Lie point symmetries of the pBe (2.3) and then proceed to find its group-invariant solutions of by using the Lie point symmetries of equation (2.3).

2.2.1 Lie symmetries

The vector field

$$G = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.4)$$

is a Lie point symmetry of pBe (2.3) if

$$G^{(2)} \Delta|_{\Delta=0} = 0, \quad (2.5)$$

with

$$\Delta \equiv u_t - \alpha u_x^2 - \beta u_{xx} = 0,$$

and the second prolongation of G is defined by

$$G^{[2]} = G + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}, \quad (2.6)$$

with ζ_1 and ζ_2 given by (1.13), and ζ_{11} , ζ_{12} and ζ_{22} given by (1.14) which can simply be expressed as

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \end{aligned}$$

and

$$\begin{aligned} \zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\ \zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \end{aligned}$$

where the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \quad (2.7)$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots. \quad (2.8)$$

Thus equation (2.5) becomes

$$\begin{aligned} &\left[\tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} \right. \\ &\left. + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} \right] \left[u_t - \alpha u_x^2 - \beta u_{xx} \right] \Big|_{(2.3)} = 0, \end{aligned}$$

which gives

$$\zeta_1 - 2\alpha u_x \zeta_2 - \beta \zeta_{22} \Big|_{(2.3)} = 0. \quad (2.9)$$

Substituting the values of ζ_1 , ζ_2 and ζ_{22} in equation (2.9), we acquire the following determining equation:

$$\eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u - 2\alpha u_x \eta_x - 2\alpha u_x^2 \eta_u + 2\alpha u_t u_x \tau_x$$

$$\begin{aligned}
& +2\alpha u_t u_x^2 \tau_u + 2\alpha u_x^2 \xi_x + 2\alpha u_x^3 \xi_u - \beta \eta_{xx} - 2\beta u_x \eta_{xu} - \beta u_x^2 \eta_{uu} - \beta u_{xx} \eta_u \\
& + 2\beta u_{tx} \tau_x + \beta u_t \tau_{xx} + 2\beta u_t u_x \tau_{xu} + 2\beta u_{tx} u_x \tau_u + \beta u_t u_x^2 \tau_{uu} + \beta u_t u_{xx} \tau_u \\
& + 2\beta u_{xx} \xi_x + \beta u_x \xi_{xx} + 2\beta u_x^2 \xi_{xu} + \beta u_x^3 \xi_{uu} + 3\beta u_x u_{xx} \xi_u \Big|_{u_{xx}=\frac{1}{\beta}(u_t-\alpha u_x^2)} = 0
\end{aligned}$$

By replacing u_{xx} by $\frac{1}{\beta}(u_t - \alpha u_x^2)$ in the above equation, we obtain

$$\begin{aligned}
& \eta_t - u_t \tau_t - u_x \xi_t + 2u_t u_x \xi_u - 2\alpha u_x \eta_x - \alpha u_x^2 \eta_u + 2\alpha u_t u_x \tau_x + \alpha u_t u_x^2 \tau_u \\
& + 2\beta u_t u_x \tau_{xu} - \alpha u_x^3 \xi_u - \beta \eta_{xx} - 2\beta u_x \eta_{xu} - \beta u_x^2 \eta_{uu} + 2\beta u_{tx} \tau_x + \beta u_t \tau_{xx} \\
& + 2\beta u_{tx} u_x \tau_u + \beta u_t u_x^2 \tau_{uu} + 2u_t \xi_x + \beta u_x \xi_{xx} + 2\beta u_x^2 \xi_{xu} + \beta u_x^3 \xi_{uu} = 0. \quad (2.10)
\end{aligned}$$

Splitting the above equation on the derivatives of u , we have

$$u_{tx} : \quad \tau_x = 0, \quad (2.11)$$

$$u_x u_{tx}^2 : \quad \tau_u = 0, \quad (2.12)$$

$$u_t u_x : \quad \xi_u = 0, \quad (2.13)$$

$$u_t : \quad 2\xi_x - \tau_t = 0, \quad (2.14)$$

$$u_x^2 : \quad \beta \eta_{uu} + \alpha \eta_u = 0, \quad (2.15)$$

$$u_x : \quad \beta \xi_{xx} - 2\beta \eta_{xu} - 2\alpha \eta_x - \xi_t = 0, \quad (2.16)$$

$$\text{rest} : \quad \eta_t - \beta \eta_{xx} = 0. \quad (2.17)$$

Solving (2.11) and (2.12) produces

$$\tau = p(t),$$

with arbitrary function p . Solving equation (2.13), we obtain

$$\xi = s(t, x),$$

with arbitrary function s depending on x and t . Substituting ξ and τ into (2.14) and integrating with respect to x yields

$$s(t, x) = \frac{1}{2} x p'(t) + c(t),$$

with arbitrary function c depending on t . Hence, the value of ξ becomes

$$\xi = \frac{1}{2} x p'(t) + c(t). \quad (2.18)$$

Equation (2.15) gives

$$\eta(t, x, u) = \vartheta(x, t) + \chi(x, t)e^{-\frac{\alpha u}{\beta}}, \quad (2.19)$$

where ϑ and χ are arbitrary functions that depend on t and x . Next, with the substitution of the values of ξ and η from (2.18) and (2.19) into (2.16) we get

$$\vartheta_x(x, t) = -\frac{1}{4\alpha}p''(t) - \frac{1}{2\alpha}c'(t).$$

By integrating the above equation with respect to x , we get

$$\vartheta(t, x) = -\frac{1}{8\alpha}x^2a''(t) - \frac{1}{2\alpha}xc'(t) + \varphi(t),$$

with φ being an arbitrary function that depends on t . The value of η now becomes

$$\eta = -\frac{1}{8\alpha}x^2a''(t) - \frac{1}{2\alpha}xc'(t) + \varphi(t) + \vartheta(t, x)e^{-\frac{\alpha}{\beta}u}. \quad (2.20)$$

Substituting (2.20) into (2.17) produces

$$\begin{aligned} & -\frac{1}{8\alpha}x^2p'''(t) - \frac{1}{2\alpha}xc''(t) + \varphi'(t) + \frac{\beta}{4\alpha}p''(t) + \chi_t(t, x)e^{-\frac{\alpha}{\beta}u} \\ & - \beta\chi_{xx}(t, x)e^{-\frac{\alpha}{\beta}u} = 0. \end{aligned} \quad (2.21)$$

Splitting equation (2.21) on $e^{-\frac{\alpha}{\beta}u}$, we get

$$e^{-\frac{\alpha}{\beta}u} : \chi_t(t, x) - \beta\chi_{xx}(t, x) = 0, \quad (2.22)$$

$$\text{rest} : \varphi'(t) - \frac{1}{8\alpha}x^2p'''(t) - \frac{1}{2\alpha}xc''(t) + \frac{\beta}{4\alpha}p''(t) = 0. \quad (2.23)$$

If we split (2.23) on the powers of x , we have

$$x^2 : p'''(t) = 0, \quad (2.24)$$

$$x^1 : c''(t) = 0, \quad (2.25)$$

$$\text{rest} : \frac{\beta}{4\alpha}p''(t) + \varphi'(t) = 0. \quad (2.26)$$

Integrating on (2.24) and (2.25) with respect to t , respectively, yields

$$p(t) = \frac{1}{2}Z_1t^2 + Z_2t + Z_3 \quad (2.27)$$

and

$$c(t) = Z_4 t + Z_5, \quad (2.28)$$

with constants Z_1, Z_2, Z_3, Z_4 and Z_5 . It follows from (2.26) and (2.27) that

$$\varphi(t) = -\frac{\beta}{4\alpha} Z_1 t + Z_6, \quad (2.29)$$

where Z_6 is a constant. Therefore, the general solution of the system (2.11) to (2.17) is

$$\begin{aligned} \tau &= \frac{1}{2} Z_1 t^2 + Z_2 t + Z_3, \\ \xi &= \frac{1}{2} Z_1 t x + \frac{1}{2} Z_2 x + Z_4 t + Z_5, \\ \eta &= -\frac{1}{8\alpha} Z_1 x^2 - \frac{1}{2\alpha} Z_4 x - \frac{\beta}{4\alpha} Z_1 t + Z_6 + \chi(t, x) e^{-\frac{\alpha}{\beta} u}, \end{aligned}$$

Finally, the Lie symmetries of the pBe (2.3) are given by

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t}, \\ G_2 &= \frac{\partial}{\partial x}, \\ G_3 &= \frac{\partial}{\partial u}, \\ G_4 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\ G_5 &= 2\alpha t \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}, \\ G_6 &= 4\alpha t^2 \frac{\partial}{\partial t} + 4\alpha t x \frac{\partial}{\partial x} - (x^2 + 2\beta t) \frac{\partial}{\partial u}, \\ G_\beta &= \chi(t, x) e^{-\frac{\alpha}{\beta} u} \frac{\partial}{\partial u}, \end{aligned}$$

which generate a Lie algebra of infinite dimension, with $\chi(t, x)$ satisfying the heat equation $\chi_t = \beta \chi_{xx}$.

2.2.2 Commutator table for the Lie point symmetries

We work on the commutation relations for the symmetry generators. By the definition of Lie bracket, we first evaluate

$$[G_1, G_6] = G_1 G_6 - G_6 G_1$$

$$\begin{aligned}
&= \partial_t \left(4\alpha t^2 \partial_t + 4\alpha t x \partial_x - (x^2 + 2\beta t) \partial_u \right) - \left(4\alpha t^2 \partial_t + 4\alpha t x \partial_x - (x^2 + 2\beta t) \partial_u \right) \partial_t \\
&= 4\alpha \left(2t \partial_t + x \partial_x \right) - 2\beta \partial_u \\
&= 4\alpha G_4 - 2\beta G_3.
\end{aligned}$$

Similarly, one can calculate all the remaining commutation relations using the above method. The table below represents all the commutation relations of the Lie symmetries of the pBe.

$[G_i, G_j]$	G_1	G_2	G_3	G_4	G_5	G_6
G_1	0	0	0	$2G_1$	$2\alpha G_2$	$4\alpha G_4 - 2\beta G_3$
G_2	0	0	0	G_2	$-G_3$	$2G_5$
G_3	0	0	0	0	0	0
G_4	$-2G_1$	$-G_2$	0	0	G_5	$2G_6$
G_5	$-2\alpha G_2$	G_3	0	$-G_5$	0	0
G_6	$2\beta G_3 - 4\alpha G_4$	$-2G_5$	0	$-2G_6$	0	0

2.2.3 One-parameter groups G_i of transformations

The Lie equations below can be used to find the corresponding one parameter group of transformations

$$\begin{aligned}
\frac{d\bar{t}}{d\epsilon} &= \xi^1(t, x, u), \quad \bar{t}|_{\epsilon=0} = t, \\
\frac{d\bar{x}}{d\epsilon} &= \xi^2(t, x, u), \quad \bar{x}|_{\epsilon=0} = x, \\
\frac{d\bar{u}}{d\epsilon} &= \eta(t, x, u), \quad \bar{u}|_{\epsilon=0} = u.
\end{aligned}$$

We now find the one-parameter group of transformations for each Lie point symmetry of the pBe. For each G_i , suppose $T_{\epsilon i}$ is the corresponding group. Let us first start by calculating the one parameter group corresponding to infinitesimal generator G_4 , namely

$$G_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{d\epsilon} = 2\bar{t}, \quad \bar{t}|_{\epsilon=0} = t, \quad \frac{d\bar{x}}{d\epsilon} = \bar{x}, \quad \bar{x}|_{\epsilon=0} = x, \quad \frac{d\bar{u}}{d\epsilon} = 0, \quad \bar{u}|_{\epsilon=0} = u.$$

Solving the above equations we obtain

$$\bar{t} = te^{2\epsilon}, \quad \bar{x} = xe^\epsilon, \quad \bar{u} = u.$$

Hence the one parameter group T_{ϵ_4} corresponding to the operator G_4 is

$$W_{\epsilon_4} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{2\epsilon_4}, xe^{\epsilon_4}, u).$$

Similarly, following the very same method as above, we obtain the following one parameter groups:

$$W_{\epsilon_1} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + \epsilon_1, x, u),$$

$$W_{\epsilon_2} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + \epsilon_2, u),$$

$$W_{\epsilon_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, u + \epsilon_3),$$

$$W_{\epsilon_5} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + 2\alpha\epsilon_5t, u - \alpha\epsilon_5^2t - \epsilon_5x),$$

$$W_{\epsilon_6} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \left(\frac{t}{1 - 4\alpha\epsilon_6t}, \frac{x}{1 - 4\alpha\epsilon_6t}, u + \frac{\beta}{2\alpha} \ln(1 - 4\alpha\epsilon_6t) - \frac{\epsilon_6x^2}{1 - 4\alpha\epsilon_6t} \right),$$

$$W_{\epsilon_7} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \left(t, x, \frac{\beta}{\alpha} \ln \left\{ \epsilon_7 \frac{\alpha}{\beta} \chi(x, t) + e^{\frac{\alpha}{\beta}u} \right\} \right).$$

Since each group W_i is a symmetry group, if $u = h(t, x)$ is a solution of the pBe, so are the functions

$$W_{\epsilon_2} : u = h(t, x + \epsilon_2),$$

$$W_{\epsilon_3} : u = h(te^{2\epsilon_4}, xe^{\epsilon_4}),$$

$$W_{\epsilon_5} : u = h(t, x + 2\alpha\epsilon_5t) + \alpha\epsilon_5^2t + \epsilon_5x,$$

$$W_{\epsilon_6} : u = h\left(\frac{t}{1 - 4\alpha\epsilon_6t}, \frac{x}{1 - 4\alpha\epsilon_6t} \right) - \frac{\beta}{2\alpha} \ln(1 - 4\alpha\epsilon_6t) + \frac{\epsilon_6x^2}{1 - 4\alpha\epsilon_6t},$$

$$W_\chi : u = \frac{\beta}{\alpha} \ln[e^{\alpha h(x, t)/\beta} - \alpha\epsilon_\chi \chi(x, t)/\beta].$$

2.2.4 Constructing group invariant solutions of (2.3)

Given a Lie symmetry

$$G = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.30)$$

the invariant solutions under the one parameter group generated by G are determined as follows. We determine two linearly independent invariants

$$H_1 = \varrho(t, x) \text{ and } H_2 = \sigma(t, x, u)$$

by solving the quasi-linear of order one partial differential equation

$$G(H) \equiv \xi^1(t, x, u) \frac{\partial H}{\partial t} + \xi^2(t, x, u) \frac{\partial H}{\partial x} + \eta(t, x, u) \frac{\partial H}{\partial u} = 0,$$

with characteristic equations also known as the associated Lagrange system

$$\frac{dt}{\xi^1(t, x, u)} = \frac{dx}{\xi^2(t, x, u)} = \frac{du}{\eta(t, x, u)}.$$

Then one of the invariants is written as a function of the other, for example

$$H_2 = k(H_1), \tag{2.31}$$

with k as a function of H_1 and solve (2.31) for u . In conclusion, the expression of u is substituted in (2.3) and an ordinary differential equation is achieved for the unidentified function k . This method decreases the number of independent variables by one.

Let us now demonstrate the above technique by taking into account the six linearly independent Lie symmetries G_1, G_2, G_3, G_4, G_5 and G_6 , by composing invariant solutions under these symmetries.

Case 1. Considering the symmetry $G_1 = \partial/\partial t$. The characteristic equations of the symmetry G_1 as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which gives the two invariants $H_1 = x$ and $H_2 = u$. Thus the invariant solution is given by $H_2 = \lambda(H_1)$, i.e.,

$$u = \lambda(x).$$

Substituting the value of u into (2.3) yields the ordinary differential equation of order two

$$\beta \lambda''(x) + \alpha \lambda'^2(x) = 0,$$

whose solution is

$$\lambda(x) = \frac{\beta}{\alpha} \ln(\alpha x + C_1) + C_2, \quad (2.32)$$

where C_1 and C_2 are integration constants. Thus, the invariant solution of (2.3) under G_1 is

$$u(t, x) = \frac{\beta}{\alpha} \ln(\alpha x + C_1) + C_2. \quad (2.33)$$

Case 2. Next, we consider the space translation symmetry $G_2 = \partial/\partial x$. The lagrange system associated with this operator yields the invariants $H_1 = t$ and $H_2 = u$. Thus, the invariant solution is given by

$$\mathcal{U}(t) = u.$$

Substitution of $u = \mathcal{U}(t)$ into (2.3), yield

$$\mathcal{U}'(t) = 0.$$

Eventually solving the above equation, we obtain

$$\mathcal{U} = M,$$

with M as arbitrary constant of integration. Therefore, the invariant solution of (2.3) under G_1 is

$$u(t, x) = M.$$

Case 3. The symmetry operator $G_3 = \partial/\partial u$ does not have invariant solutions.

Case 4. Now considering symmetry

$$G_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x},$$

with characteristic equations

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{0}.$$

Thus, one invariant is $H_1 = u$.

$$\frac{dt}{2t} = \frac{dx}{x}$$

gives invariant $H_2 = x/\sqrt{t}$.

Therefore, the group invariant solution is given by $H_1 = F(H_2)$ with F as an arbitrary function, i.e.,

$$u = F\left(\frac{x}{\sqrt{t}}\right).$$

Then

$$u_t = -\frac{1}{2} \frac{x}{t^{\frac{3}{2}}} F'\left(\frac{x}{\sqrt{t}}\right),$$

$$u_x = \frac{1}{\sqrt{t}} F'\left(\frac{x}{\sqrt{t}}\right),$$

$$u_{xx} = \frac{1}{t} F''\left(\frac{x}{\sqrt{t}}\right).$$

Substituting u_t , u_x and u_{xx} obtained above into (2.3) we get the ordinary differential equation of order two

$$\alpha F'^2(z) + \frac{1}{2} z F'(z) + \beta F''(z) = 0,$$

with $z = x/\sqrt{t}$. By letting $Q(z) = F'(z)$, we obtain

$$2 \frac{dQ}{dz} + \frac{1}{\beta} z Q = -2 \frac{\alpha}{\beta} Q^2.$$

The above is a Bernoulli equation and its solution is

$$Q(z) = \left[\sqrt{\beta} e^{-\frac{z^2}{4\beta}} \left\{ \alpha \sqrt{\pi} \operatorname{erf}\left(\frac{y}{2\sqrt{\beta}}\right) + \sqrt{\beta} C_1 \right\} \right]^{-1},$$

here C_1 is an integration constant. Hence

$$\frac{dF}{dz} = \left[\sqrt{\beta} e^{-\frac{z^2}{4\beta}} \left\{ \alpha \sqrt{\pi} \operatorname{erf}\left(\frac{y}{2\sqrt{\beta}}\right) + \sqrt{\beta} C_1 \right\} \right]^{-1}.$$

Integrating the above equation, we get

$$F(z) = \frac{\beta}{\alpha} \ln \left\{ \left| \frac{\alpha}{\sqrt{\beta}} \sqrt{\pi} \operatorname{erf}\left(\frac{z}{2\sqrt{\beta}}\right) C_1 + C_2 \right| \right\},$$

where C_2 is an integration constant. Reverting back to the original variables, we get

$$u(t, x) = \frac{\beta}{\alpha} \ln \left\{ \left| \frac{\alpha}{\sqrt{\beta}} \sqrt{\pi} \operatorname{erf} \left(\frac{x}{2\sqrt{\beta t}} \right) C_1 + C_2 \right| \right\}. \quad (2.34)$$

Case 5. Now we construct the invariant solution under the symmetry generator G_5 defined as

$$G_5 = 2\alpha t \frac{\partial}{\partial x} - x \frac{\partial}{\partial u},$$

with characteristic equations

$$\frac{dt}{0} = \frac{dx}{2\alpha t} = \frac{du}{-x}.$$

Then one of the invariants is $H_1 = t$ and

$$\frac{dx}{2\alpha H_1} = \frac{du}{-x},$$

yields $H_2 = u + x^2/4\alpha t$. Hence the invariant solution of (2.3) under G_5 is $H_2 = \Phi(H_1)$, i.e.,

$$u = \Phi(t) - \frac{x^2}{4\alpha t}, \quad (2.35)$$

with arbitrary function Φ depending on t . Substitution of (2.35) into (2.3), gives the ordinary differential equation of order one

$$\frac{d\Phi}{dt} + \frac{\beta}{2\alpha t} = 0.$$

Solving for Φ , we get

$$\Phi(t) = -\frac{\beta}{2\alpha} \ln(t) + \mu,$$

where μ is a constant. Hence, the invariant solution of G_5 is

$$u(t, x) = -\left\{ \frac{\beta}{2\alpha} \ln(t) + \frac{x^2}{4\alpha t} \right\} + \mu. \quad (2.36)$$

Case 6. Lastly we consider the symmetry operator G_6

$$G_6 = 4\alpha t^2 \frac{\partial}{\partial t} + 4\alpha t x \frac{\partial}{\partial x} - (x^2 + 2\beta t) \frac{\partial}{\partial u},$$

with characteristic equations

$$\frac{dt}{4\alpha t^2} = \frac{dx}{4\alpha t x} = \frac{du}{-(x^2 + 2\beta t)}.$$

Therefore, one invariant is $H_1 = x/t$.

$$\frac{dt}{4\alpha t^2} = \frac{du}{-(x^2 + 2\beta t)}$$

gives

$$H_2 = u + \frac{\beta}{2\alpha} \ln t + \frac{x^2}{4\alpha t}. \quad (2.37)$$

Thus, the group invariant solution is $H_2 = \Gamma(H_1)$, i.e.,

$$u = \Gamma\left(\frac{x}{t}\right) - \frac{x^2}{4\alpha t} - \frac{\beta}{2\alpha} \ln t.$$

Now

$$u_t = -\frac{x}{t^2} \Gamma'\left(\frac{x}{t}\right) + \frac{x^2}{4\alpha t^2} - \frac{\beta}{2\alpha t},$$

$$u_x = \frac{1}{t} \Gamma'\left(\frac{x}{t}\right) - \frac{x}{2\alpha t},$$

$$u_{xx} = \frac{1}{t^2} \Gamma''\left(\frac{x}{t}\right) - \frac{1}{2\alpha t}.$$

Substituting the above values of u_t , u_x and u_{xx} into (2.3) gives us the second-order nonlinear ordinary differential equation

$$\beta \Gamma''(z) + \alpha \Gamma'^2(z) = 0,$$

where $z = x/t$. The above equation is called Liouville ordinary differential equation, and with the aid of Maple we get its solution as

$$\Gamma(z) = \frac{\beta}{\alpha} \ln \left\{ \frac{\alpha}{\beta} (C_1 z + C_2) \right\}, \quad (2.38)$$

with C_2 and C_1 as integration constants. Back substitution to the original variables, the group-invariant solution of (2.3) under G_6 is

$$u(t, x) = \frac{\beta}{\alpha} \ln \left\{ \frac{\alpha}{\beta} \left(C_1 \frac{x}{t} + C_2 \right) \right\} - \frac{x^2}{4\alpha t} - \frac{\beta}{2\alpha} \ln t. \quad (2.39)$$

2.3 Conserved vectors of the pBe

In this section we determine the conserved vectors for the pBe (2.3) by applying the method of multiplier. We remember the standard Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \dots, \quad (2.40)$$

with the total derivatives D_t and D_x given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ttx} \frac{\partial}{\partial u_{tx}} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{txx} \frac{\partial}{\partial u_{tx}} + \dots. \end{aligned} \quad (2.41)$$

The above mentioned is reached by applying the multiplier $Q(t, x, u)$ of order zero. The determining equation for the multiplier $Q(t, x, u)$ is

$$\frac{\delta}{\delta u} [Q(t, x, u)(u_t - \alpha u_x^2 - \beta u_{xx})] = 0. \quad (2.42)$$

By expanding the above equation we get

$$u_t Q_u - \alpha u_x^2 Q_u - \beta u_{xx} Q_u - D_t Q + 2\alpha D_x (Q u_x) - \beta D_x^2 Q = 0. \quad (2.43)$$

Substituting the values of D_t and D_x from equation (2.41) into (2.43), gives

$$\begin{aligned} &\alpha u_x^2 Q_u - 2\beta u_{xx} Q_u - Q_t + 2\alpha u_x Q_x + 2\alpha u_{xx} Q - \beta Q_{xx} \\ &- 2\beta u_x Q_{ux} - \beta u_x^2 Q_{uu} = 0. \end{aligned} \quad (2.44)$$

If we split equation (2.44) on derivatives of , we get the following system of PDEs:

$$u_{xx} : \alpha Q - \beta Q_u = 0, \quad (2.45)$$

$$u_x^2 : \alpha Q_u - \beta Q_{uu} = 0, \quad (2.46)$$

$$u_x : \alpha Q_x - \beta Q_{ux} = 0, \quad (2.47)$$

$$\text{rest} : \beta Q_{xx} + Q_t = 0. \quad (2.48)$$

Solving (2.45) gives

$$Q(t, x, u) = \theta(t, x) e^{\frac{\alpha u}{\beta}}, \quad (2.49)$$

with arbitrary function θ that depends on x and t . It is important to consider that (2.49) satisfy (2.47) and (2.46). Substituting the value of Q from (2.49) into (2.48), we get

$$\beta\theta_{xx}e^{\frac{\alpha u}{\beta}} + \theta_t e^{\frac{\alpha u}{\beta}} = 0.$$

Thus, becoming

$$\beta\theta_{xx} + \theta_t = 0. \quad (2.50)$$

To solve (2.50), we assume that

$$\theta(t, x) = X(x)T(t). \quad (2.51)$$

Now, replacing θ from (2.51) into (2.50) gives

$$\beta X''T + T'X = 0.$$

Hence

$$-\frac{T'}{\beta T} = \frac{X''}{X} - \mu^2,$$

with μ as arbitrary constant. Hence we have

$$T' - \beta\mu^2 T = 0 \quad (2.52)$$

and

$$X'' + \mu^2 X = 0. \quad (2.53)$$

Solving (2.52), we obtain

$$T(t) = C_1 e^{\beta\mu^2 t},$$

with C_1 being an integration constant. Solving equation (2.53) yields

$$X(x) = k_1 \sin \mu x + k_2 \cos \mu x$$

where k_1 and k_2 are arbitrary constants. Thus,

$$\theta(t, x) = C_1 e^{\beta\mu^2 t} (k_1 \sin \mu x + k_2 \cos \mu x).$$

Letting $C_1 k_1 = B_1$ and $C_1 k_2 = B_2$, the above equation becomes

$$\theta(t, x) = e^{\beta \mu^2 t} (B_1 \sin \mu x + B_2 \cos \mu x).$$

Now equation (2.49) can be written in the form

$$Q(t, x, u) = e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} (B_1 \sin \mu x + B_2 \cos \mu x),$$

which produces two multipliers

$$Q_1(t, x, u) = \left(e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \right) \sin \mu x$$

and

$$Q_2(t, x, u) = \left(e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \right) \cos \mu x$$

for (2.3).

Firstly we calculate the conserved vectors of (2.3) connected to the multiplier $Q_1(t, x, u)$.

Hence the following

$$Q_1(t, x, u) \{u_t - \alpha u_x^2 - \beta u_{xx}\} = D_t T^t + D_x T^x, \quad (2.54)$$

with the flux T^x and density T^t described as

$$T^t = T^t(t, x, u) \text{ and } T^x = T^x(t, x, u, u_x). \quad (2.55)$$

Hence (2.54) becomes

$$e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x (u_t - \alpha u_x^2 - \beta u_{xx}) = D_t T^t + D_x T^x.$$

The above equation yields

$$e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x (u_t - \alpha u_x^2 - \beta u_{xx}) = T_t^t + u_t T_u^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x. \quad (2.56)$$

Splitting (2.56) on the derivatives of u , we get

$$u_{xx} : \quad T_{u_x}^x = -\beta e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x, \quad (2.57)$$

$$u_t : \quad T_u^t = e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x, \quad (2.58)$$

$$\text{rest : } T_t^t + T_x^x + u_x T_u^x = -\alpha u_x^2 e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x. \quad (2.59)$$

Integrating (2.57) with respect to u_x gives

$$T^x = -\beta u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x + \psi(t, x, u). \quad (2.60)$$

Equation (2.58) gives

$$T^t = \frac{\beta}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x + v(t, x), \quad (2.61)$$

with arbitrary function v that depends on t and x . Substituting the aforementioned values of T^x and T^t into (2.59), we get

$$\psi_x + v_t + u_x \psi_u + \frac{\beta^2}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x - \beta \mu u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x = 0.$$

Splitting the above equation with u derivatives produces

$$u_x : \quad \psi_u = \beta \mu e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x, \quad (2.62)$$

$$\text{rest : } \quad \psi_x + v_t + \frac{\beta^2}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x = 0. \quad (2.63)$$

Integrating (2.62) with respect to u gives

$$\psi = \frac{\beta^2}{\alpha} \mu e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x + \Theta(t, x), \quad (2.64)$$

with arbitrary function Θ that depends on t and x . Replacing this value of ψ into (2.63) gives

$$v_t + \Theta_x = 0.$$

We set $v = \Theta = 0$, because the two play a part in the trivial part of the conserved vector, and therefore, (2.63) is satisfied. Hence flux T^x and the density T^t above changes to be

$$\begin{aligned} T_1^t &= \frac{\beta}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x, \\ T_1^x &= \frac{\beta^2}{\alpha} \mu e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x - \beta u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x, \end{aligned}$$

being the conserved vectors of (2.3) of the multiplier $Q_1(t, x, u)$.

Lastly the conserved vectors of (2.3) for the multiplier $Q_2(t, x, u)$ were calculated. Thus, the following

$$Q_2(t, x, u)\{u_t - \alpha u_x^2 - \beta u_{xx}\} = D_t T^t + D_x T^x, \quad (2.65)$$

which implies

$$e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x (u_t - \alpha u_x^2 - \beta u_{xx}) = D_t T^t + D_x T^x.$$

The above equation results in

$$e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x (u_t - \alpha u_x^2 - \beta u_{xx}) = T_t^t + u_t T_u^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x. \quad (2.66)$$

Splitting (2.66) on derivatives of u , we obtain

$$u_{xx} : \quad T_{u_x}^x = -\beta e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x, \quad (2.67)$$

$$u_t : \quad T_u^t = e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x, \quad (2.68)$$

$$\text{rest} : \quad T_t^t + T_x^x + u_x T_u^x = -\alpha u_x^2 e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x. \quad (2.69)$$

Integrating (2.67) with respect to u_x , we obtain

$$T^x = -\beta u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x + d(t, x, u), \quad (2.70)$$

with arbitrary function d depending on t , x and u . (2.68) yields

$$T^t = \frac{\beta}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x + e(t, x), \quad (2.71)$$

with arbitrary function e having x and t as independent variables. Substituting the aforementioned values of T^x and T^t in equation (2.69), we get

$$e_t + d_x + u_x d_u + \frac{\beta^2 \mu^2}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x + \beta \mu u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x = 0.$$

If we split the above equation on u derivatives we obtain

$$u_x : \quad d_u + \beta \mu e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x = 0, \quad (2.72)$$

$$\text{rest} : \quad e_t + d_x + \frac{\beta^2 \mu^2}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x = 0. \quad (2.73)$$

Integrating (2.72) with respect to u gives

$$d = -\frac{\beta^2 \mu}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x + f(t, x),$$

with arbitrary function f having x and t as independent variables. Replacing this value of d into (2.73) gives

$$f_x + e_t = 0.$$

We put $f = e = 0$, as the two add to the trivial part of the conserved vectors. Thus, flux T^x and density T^t are

$$\begin{aligned} T_2^t &= \frac{\beta}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x, \\ T_2^x &= -\beta u_x e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \cos \mu x - \frac{\beta^2 \mu}{\alpha} e^{\frac{\beta^2 \mu^2 t + \alpha u}{\beta}} \sin \mu x, \end{aligned}$$

which are the conserved vectors of (2.3) associated with $Q_2(t, x, u)$ multiplier.

2.4 Concluding remarks

We investigated Lie group analysis of the pBe (2.3) in this chapter. Firstly, we constructed Lie symmetries. Then, we used the obtained Lie symmetries to perform a symmetry reductions. Moreover, we constructed the invariant solution for the equation (2.3). Furthermore, we used the approach of multiplier to calculate the conserved vectors for the pBe (2.3).

Chapter 3

Solutions and conservation laws for the gsmCH equation

3.1 Introduction

The equation [38]

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} \quad (3.1)$$

represents an important family of NLPDEs for different values of b . These equations have been studied in one form or another over the years. These forms include, amongst others, the famous Camassa-Holm (CH) equation [39]

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (3.2)$$

which is obtained when $b = 2$. Equation (3.2) was derived by Camassa and Holm in [39] and it describes one-directional movement of shallow water waves in a medium of even depth. Camassa and Holm found travelling wave solutions, N-soliton solutions as well as the Bi-Hamiltonian structure of equation (3.2). The reader should note that a general form of the Camassa-Holm equation also referred to in [39], is given by

$$u_t + 2\alpha u_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (3.3)$$

where α is a constant denoting critical wave velocity in shallow water. When $b = 3$, equation (3.1) becomes the Degasperis-Processi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (3.4)$$

which was derived in [40] by applying the asymptotic integrability technique. The same equation is seen in [41], where the author proved that the smooth solutions of equation (3.4) have a limitless velocity of propagation which loses the property of having compact support. Modified forms of equations (3.2) and (3.4) have been studied as well, for instance, in [38] Wazwaz changed the nonlinear convection term in equations (3.2) and (3.4) from uu_x to u^2u_x , thereby increasing the strength of nonlinearity, see [42], this led to the modified CH equation [38]

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx} \quad (3.5)$$

and the modified DP equation

$$u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx}. \quad (3.6)$$

The author in [38] investigated the effect of introducing the u^2u_x term by using the sine-cosine and tanh algorithms and obtained bell-shaped solitons which are in contrast to the expected peaked solitons associated with the family of equations (3.1). The two equations (3.5) and (3.6) are derived from the modified DP-CH family [38], given by

$$u_t - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + uu_{xxx}. \quad (3.7)$$

Evidently, by letting $b = 2$ in equation (3.7) we obtain the modified CH equation (3.5). Tian and Song [43] investigated a modified form of equation (3.3), known as the generalised CH equation

$$u_t + 2\alpha u_x - u_{xxt} + \beta u^n u_x = 2u_xu_{xx} + uu_{xxx}, \quad (3.8)$$

where $\alpha, \beta \neq 0$, and n is the strength of nonlinearity of the convection term. They obtained travelling wave and peakon solutions by making use of direct integration.

Based on the work of Tian and Song, Wazwaz in [42] considered equation (3.8) without the two extra terms on the right hand side, that is,

$$u_t + 2\alpha u_x - u_{xxt} + \beta u^n u_x = 0. \quad (3.9)$$

According to Boyd [44], the two terms on the right hand side of equation (3.8) are small for amplitude waves in shallow media. By making use of the sine-cosine method in [42], bell-shaped solitons, compactons and periodic solutions were obtained. For the case $n = 2$

$$u_t + 2\alpha u_x - u_{xxt} + \beta u^2 u_x = 0 \quad (3.10)$$

was studied in several papers, for example in [45], explicit solitary wave solutions of equation (3.10) were found using the exponential function method. Equation (3.10) was studied again in [46] where the researchers introduced a fractional complex transform to the exponential function method to find and analyse its solitary wave solutions. In this work, we aim to investigate the generalized simplified modified Camassa-Holm (gsmCH) equation (3.9).

We aim to obtain more solutions of the gsmCH equation (3.9) by utilising Lie symmetry method coupled with direct integration technique, Kudryashov's method, as well as the simplest equation method. Furthermore, we derive the conservation laws of equation (3.9) by using the multiplier approach, and theorems by Ibragimov and Noether.

3.2 Lie point symmetries of (3.9)

The vector field

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}$$

is a Lie point symmetry of equation (3.9) if and only if

$$X^{(3)}\Delta|_{\Delta=0} = 0, \quad (3.11)$$

where

$$\Delta \equiv u_t + 2\alpha u_x + \beta u^n u_x - u_{txx},$$

and $X^{[3]}$ is the third prolongation [7] of X defined as

$$X^{[3]} = X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{txx} \frac{\partial}{\partial u_{txx}}. \quad (3.12)$$

Here ζ_t , ζ_x , ζ_{tx} and ζ_{txx} are determined by

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ \zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\ \zeta_{txx} &= D_x(\zeta_{tx}) - u_{ttx} D_x(\tau) - u_{txx} D_x(\xi), \end{aligned} \quad (3.13)$$

where the total derivatives D_t and D_x are defined as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots. \end{aligned}$$

The expansion of the ζ 's (3.13) gives

$$\begin{aligned} \zeta_t &= \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u, \\ \zeta_x &= \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u, \\ \zeta_{txx} &= \eta_{txx} + u_{xx} \eta_{tu} + 2u_{tx} u_x \eta_{uu} + u_{txx} \eta_u + u_t \eta_{xxu} + u_{xx} u_t \eta_{uu} + 2u_t u_x \eta_{xuu} \\ &\quad + u_t u_x^2 \eta_{uuu} + 2u_{tx} \eta_{xu} + u_x^2 \eta_{tuu} + 2u_x \eta_{txu} - u_t u_x^3 \xi_{uuu} - u_x^3 \xi_{tuu} - u_{tx} \xi_{xx} \\ &\quad - 3u_{tx} u_x^2 \xi_{uu} - 2u_{xx} u_t \xi_{xu} - 2u_{txx} \xi_x - 2u_t u_x^2 \xi_{xuu} - 2u_x^2 \xi_{txu} - 4u_{tx} u_x \xi_{xu} \\ &\quad - 3u_{txx} u_x \xi_u - 3u_{xx} u_t u_x \xi_{uu} - u_t u_x \xi_{xxu} - 3u_{xx} u_x \xi_{tu} - u_{xxx} \xi_t - 2u_{xx} \xi_{tx} \\ &\quad - u_{xxx} u_t \xi_u - 3u_{xx} u_{tx} \xi_u - u_x \xi_{txx} - 4u_t u_{tx} u_x \tau_{uu} - u_t u_x^2 \tau_{tuu} - u_{tt} u_x^2 \tau_{uu} \\ &\quad - u_t \tau_{txx} - u_t^2 u_x^2 \tau_{uuu} - 2u_{ttx} u_x \tau_u - 2u_{tt} u_x \tau_{xu} - 2u_t^2 u_x \tau_{xuu} - 2u_t \tau_{txu} u_x \\ &\quad - 2u_{tx} \tau_{tu} u_x - 2u_{tx}^2 \tau_u - 2u_t u_{txx} \tau_u - u_{xx} u_{tt} \tau_u - u_{xx} u_t^2 \tau_{uu} - 2u_{ttx} \tau_x \\ &\quad - 4u_t u_{tx} \tau_{xu} - u_{tt} \tau_{xx} - u_t^2 \tau_{xxu} - u_{txx} \tau_t - u_{xx} u_t \tau_{tu} - 2u_{tx} \tau_{tx}. \end{aligned}$$

Expanding the determining equation (3.11), we obtain

$$(\zeta_t + 2\alpha \zeta_x + \beta u^n \zeta_x + n\beta u^{n-1} u_x \eta - \zeta_{txx}) \big|_{(3.9)} = 0,$$

which upon substituting the values of ζ_t, ζ_x and ζ_{txx} gives

$$\begin{aligned}
& \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u + 2\alpha (\eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x \\
& - u_x^2 \xi_u) + \beta u^n (\eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u) + n\beta u^{n-1} u_x \eta + \eta_{txx} - u_{xx} \eta_{tu} \\
& - 2u_{tx} u_x \eta_{uu} - u_{txx} \eta_u - u_t \eta_{xxu} - u_{xx} u_t \eta_{uu} - 2u_t u_x \eta_{xuu} - u_t u_x^2 \eta_{uuu} - 2u_{tx} \eta_{xu} - u_x^2 \eta_{tuu} \\
& - 2u_x \eta_{txu} + u_t u_x^3 \xi_{uuu} + u_x^3 \xi_{tuu} + u_{tx} \xi_{xx} + 3u_{tx} u_x^2 \xi_{uu} + 2u_{xx} u_t \xi_{xu} + 2u_{txx} \xi_x + 2u_t u_x^2 \xi_{xuu} \\
& + 2u_x^2 \xi_{txu} + 4u_{tx} u_x \xi_{xu} + 3u_{txx} u_x \xi_u + 3u_{xx} u_t u_x \xi_{uu} + u_t u_x \xi_{xxu} + 3u_{xx} u_x \xi_{tu} + u_{xxx} \xi_t + 2u_{xx} \xi_{tx} \\
& + u_{xxx} u_t \xi_u + 3u_{xx} u_{tx} \xi_u + u_x \xi_{txx} + 4u_t u_{tx} u_x \tau_{uu} + u_t u_x^2 \tau_{tuu} + u_{tt} u_x^2 \tau_{uu} + u_t \tau_{txx} + u_t^2 u_x^2 \tau_{uuu} \\
& + 2u_{tt} u_x \tau_u + 2u_{tt} u_x \tau_{xu} + 2u_t^2 u_x \tau_{xuu} + 2u_t \tau_{txu} u_x + 2u_{tx} \tau_{tu} u_x + 2u_{tx}^2 \tau_u + 2u_t u_{txx} \tau_u + u_{xx} u_{tt} \tau_u \\
& + u_{xx} u_t^2 \tau_{uu} + 2u_{tt} \tau_x + 4u_t u_{tx} \tau_{xu} + u_{tt} \tau_{xx} + u_t^2 \tau_{xxu} + u_{txx} \tau_t + u_{xx} u_t \tau_{tu} + 2u_{tx} \tau_{tx} \Big|_{(3.9)} = 0.
\end{aligned}$$

Replacing u_{txx} by $u_t + 2\alpha u_x + \beta u^n u_x$ in the above equation, we obtain

$$\begin{aligned}
& \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u + 2\alpha \eta_x + 2\alpha u_x \eta_u - 2\alpha u_t \tau_x - 2\alpha u_t u_x \tau_u \\
& - 2\alpha u_x \xi_x - 2\alpha u_x^2 \xi_u + \beta u^n \eta_x + \beta u^n u_x \eta_u - \beta u^n u_t \tau_x - \beta u^n u_t u_x \tau_u - \beta u^n u_x \xi_x \\
& - \beta u^n u_x^2 \xi_u + n\beta u^{n-1} u_x \eta + \eta_{txx} - u_{xx} \eta_{tu} - 2u_{tx} u_x \eta_{uu} - u_t \eta_u - 2\alpha u_x \eta_u - \beta u^n u_x \eta_u \\
& - u_t \eta_{xxu} - u_{xx} u_t \eta_{uu} - 2u_t u_x \eta_{xuu} - u_t u_x^2 \eta_{uuu} - 2u_{tx} \eta_{xu} - u_x^2 \eta_{tuu} - 2u_x \eta_{txu} + u_t u_x^3 \xi_{uuu} \\
& + u_x^3 \xi_{tuu} + u_{tx} \xi_{xx} + 3u_{tx} u_x^2 \xi_{uu} + 2u_t \xi_x + 2u_{xx} u_t \xi_{xu} + 4\alpha u_x \xi_x + 2\beta u^n u_x \xi_x + 2u_t u_x^2 \xi_{xuu} \\
& + 2u_x^2 \xi_{txu} + 4u_{tx} u_x \xi_{xu} + 3u_t u_x \xi_u + 6\alpha u_x^2 \xi_u + 3\beta u^n u_x^2 \xi_u + 3u_{xx} u_t u_x \xi_{uu} + u_t u_x \xi_{xxu} \\
& + 3u_{xx} u_x \xi_{tu} + u_{xxx} \xi_t + 2u_{xx} \xi_{tx} + u_{xxx} u_t \xi_u + 3u_{xx} u_{tx} \xi_u + u_x \xi_{txx} + 4u_t u_{tx} u_x \tau_{uu} + u_t u_x^2 \tau_{tuu} \\
& + u_{tt} u_x^2 \tau_{uu} + u_t \tau_{txx} + u_t^2 u_x^2 \tau_{uuu} + 2u_{tt} u_x \tau_u + 2u_{tt} u_x \tau_{xu} + 2u_t^2 u_x \tau_{xuu} + 2u_t \tau_{txu} u_x + 2u_{tx} \tau_{tu} u_x \\
& + 2u_{tx}^2 \tau_u + 2u_t^2 \tau_u + 4\alpha u_x u_t \tau_u + 2\beta u^n u_x u_t \tau_u + u_{xx} u_{tt} \tau_u + u_{xx} u_t^2 \tau_{uu} + 2u_{tt} \tau_x + 4u_t u_{tx} \tau_{xu} \\
& + u_{tt} \tau_{xx} + u_t^2 \tau_{xxu} + u_t \tau_t + 2\alpha u_x \tau_t + \beta u^n u_x \tau_t + u_{xx} u_t \tau_{tu} + 2u_{tx} \tau_{tx} = 0.
\end{aligned}$$

Splitting the above equation on derivatives of u gives the following over determined system of linear PDEs:

$$\tau_x = 0, \quad \tau_u = 0, \quad (3.14)$$

$$\xi_u = 0, \quad \xi_t = 0, \quad (3.15)$$

$$\eta_{uu} = 0, \quad (3.16)$$

$$\eta_{tu} = 0, \quad (3.17)$$

$$\xi_{xxx} - 2\xi_x = 0, \quad (3.18)$$

$$\eta_{xxu} - 2\xi_x = 0, \quad (3.19)$$

$$\beta n u^{n-1} \eta - \xi_t + \beta u^n \xi_x + 2\alpha \xi_x + 2\alpha \tau_t + \beta u^n \tau_t = 0, \quad (3.20)$$

$$\eta_t + 2\alpha \eta_x + \beta u^n \eta_x + \eta_{txx} = 0. \quad (3.21)$$

From equations (3.14), we get

$$\tau(t) = A(t),$$

with $A(t)$ as an arbitrary function of t . Equation (3.15) implies that

$$\xi = B(x),$$

with B as arbitrary function of x . Integrating equation (3.16) twice with respect to u yields

$$\eta = D(t, x)u + C(t, x),$$

where D and C are arbitrary functions of t and x . Substituting the above value of η into (3.17) we get $D_t = 0$, thus $D = D(x)$. Hence

$$\eta = D(x)u + C(t, x).$$

Substituting the above value of η into equation (3.21), we get

$$C_t + 2\alpha D' u + 2\alpha C_x + \beta u^{n+1} D' + \beta u^n C_x + C_{txx} = 0.$$

Splitting the above equation on powers of u we get

$$u^{n+1} : D'(x) = 0, \quad (3.22)$$

$$u^n : C_x = 0, \quad (3.23)$$

$$u^0 : C_t + C_{txx} = 0. \quad (3.24)$$

From (3.22) we have $D = k_1$, where k_1 is a constant of integration. Equation (3.23) implies that $C = C(t)$. Substituting this value of C into equation (3.24) we get that

$C'(t) = 0$. This can easily be solved to obtain $C(t) = k_2$, with k_2 being an arbitrary constant. Therefore

$$\eta = k_1 u + k_2. \quad (3.25)$$

Substituting ξ into equation (3.19) we have $B'(x) = 0$, and this gives $B = k_3$, where k_3 is an arbitrary constant. Hence

$$\xi = k_3. \quad (3.26)$$

Now substituting the values of η , ξ and τ into equation (3.20), we obtain

$$\beta n u^n k_1 + \beta n u^{n-1} k_2 + 2\alpha A' + \beta u^n A' = 0.$$

Splitting the above equation on powers of u we have

$$u^{n-1} : k_2 = 0, \quad (3.27)$$

$$u^n : n k_1 + A' = 0, \quad (3.28)$$

$$u^0 : A' = 0. \quad (3.29)$$

Hence from (3.28), we obtain $k_1 = 0$ and solving equation (3.29) we get $A = k_4$, where k_4 is an arbitrary constant. It can be easily seen that the obtained value of ξ satisfies equation (3.18). Therefore the infinitesimals of equation (3.9) are

$$\tau = k_4, \quad \xi = k_3, \quad \eta = 0.$$

Consequently, the Lie point symmetries of the gsmCH equation (3.9) are given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

We note that these symmetries represent translations in t and x respectively.

3.3 Symmetry reductions and solutions of (3.9)

In this subsection we use the symmetries obtained in the previous subsection to perform symmetry reductions and obtain exact solutions of the gsmCH equation (3.9).

Case 1. We start with the time translation symmetry $X_1 = \partial/\partial t$.

This symmetry leads to the two invariants

$$J_1 = x, \quad J_2 = u.$$

Thus the group-invariant solution is given by $u = f(x)$, where f is an arbitrary function of x . Substituting this value of u into equation (3.9) we obtain the nonlinear ordinary differential equation (NLODE)

$$2\alpha f' + \beta(f)^n f' = 0.$$

Solving the above NLODE we get

$$u(t, x) = K,$$

where K is an arbitrary constant of integration.

Case 2. We consider the space translation symmetry $X_2 = \partial/\partial x$.

This symmetry has the invariants

$$J_1 = t, \quad J_2 = u.$$

Thus its group-invariant solution is given by $u = \phi(t)$. Substituting this value into equation (3.9) yields $\phi'(t) = 0$, whose solution is $\phi(t) = C_1$, where C_1 is an arbitrary constant of integration. Hence the group-invariant solution corresponding to this operator is given by

$$u(t, x) = C_1.$$

Case 3. $X = X_2 + cX_1$.

Solving the characteristic equations of $X = X_2 + cX_1$ gives the invariants

$$u(t, x) = U(\xi), \quad \xi = x - ct. \tag{3.30}$$

Inserting the invariant solution (3.30) into equation (3.9), gives the NLODE

$$-cU' + 2\alpha U' + \beta U^n U' - cU''' = 0. \tag{3.31}$$

Employing the transformation

$$U(\xi) = F^{1/n}(\xi)$$

transforms equation (3.31) to the NLODE

$$\begin{aligned} (2\alpha n^2 - cn^2)F^2F' + n^2\beta F^3F' + (c - 3nc + 2n^2c)F'^3 + (3nc - 3n^2c)FF'F'' \\ + n^2cF^2F''' = 0. \end{aligned} \quad (3.32)$$

3.4 Solutions of (3.9) using the Kudryashov's method

In this subsection we apply Kudryashov's method in order to solve equation (3.32).

This is done by first assuming that the solution of (3.32) takes the form

$$F(\xi) = \sum_{k=0}^N A_k R^k(\xi), \quad (3.33)$$

where N is a positive integer whose value is obtained by employing the balancing procedure, and $R(\xi)$ satisfies the Riccati equation

$$R'(\xi) = R^2(\xi) - R(\xi), \quad (3.34)$$

whose solution

$$R(\xi) = \frac{1}{1 + e^\xi}. \quad (3.35)$$

The balancing procedure of (3.32) gives $N = 2$, thus equation (3.33) becomes

$$F(\xi) = A_0 + A_1 R(\xi) + A_2 R^2(\xi). \quad (3.36)$$

Substituting equation (3.36) into equation (3.32) and using (3.34) gives

$$\begin{aligned} n^2cA_1^3R(\xi)^3 + 2n^2cA_2^3R(\xi)^6 + 2n^2\alpha A_1^3R(\xi)^4 - 2n^2\alpha A_1^3R(\xi)^3 + 4n^2\alpha A_2^3R(\xi)^7 \\ - 4n^2\alpha A_2^3R(\xi)^6 + n^2\beta A_1^4R(\xi)^5 - n^2\beta A_1^4R(\xi)^4 + 2n^2\beta A_2^4R(\xi)^9 + 18cA_1^2A_2R(\xi)^5 \\ + 6cA_1^2A_2R(\xi)^7 - 18cA_1^2A_2R(\xi)^6 + 12cA_1A_2^2R(\xi)^8 - 36cA_1A_2^2R(\xi)^7 - 2n^2\beta A_2^4R(\xi)^8 \\ - 6cA_1^2A_2R(\xi)^4 + 36cA_1A_2^2R(\xi)^6 - 12cA_1A_2^2R(\xi)^5 + 3ncA_1^3R(\xi)^4 + 12ncA_2^3R(\xi)^7 \end{aligned}$$

$$\begin{aligned}
& + 6n^2cA_0A_1A_2R(\xi)^3 + 3ncA_1^3R(\xi)^6 - 6ncA_1^3R(\xi)^5 + 12ncA_2^3R(\xi)^9 - 24ncA_2^3R(\xi)^8 \\
& + 2n^2cA_1^3R(\xi)^6 - 3n^2cA_1^3R(\xi)^5 + 4n^2cA_2^3R(\xi)^9 - 6n^2cA_2^3R(\xi)^8 + 18n^2cA_0A_1A_2R(\xi)^4 \\
& + 12n^2\alpha A_0A_1A_2R(\xi)^4 - 12n^2\alpha A_0A_1A_2R(\xi)^3 + 9n^2\beta A_0^2A_1A_2R(\xi)^4 - cA_1^3R(\xi)^3 \\
& + 24cA_2^3R(\xi)^7 - 9n^2\beta A_0^2A_1A_2R(\xi)^3 + 12n^2\beta A_0A_1^2A_2R(\xi)^5 - 12n^2\beta A_0A_1^2A_2R(\xi)^4 \\
& + 3cA_1^3R(\xi)^4 + 15n^2\beta A_0A_1R(\xi)^6A_2^2 - 15n^2\beta A_0A_1R(\xi)^5A_2^2 + 66ncA_0A_1R(\xi)^4A_2 \\
& - 6n^2cA_0^2A_2R(\xi)^2 - 18ncA_0A_1A_2R(\xi)^3 + 30ncA_0A_1R(\xi)^6A_2 - 78ncA_0A_1R(\xi)^5A_2 \\
& - 54n^2cA_0A_1R(\xi)^5A_2 + 30n^2cA_0A_1A_2R(\xi)^6 - 8cA_2^3R(\xi)^6 + cA_1^3R(\xi)^6 - 3cA_1^3R(\xi)^5 \\
& + 8cA_2^3R(\xi)^9 - 24cA_2^3R(\xi)^8 - 66ncA_1R(\xi)^7A_2^2 + 84ncA_0A_2^2R(\xi)^5 - 24ncA_0A_2^2R(\xi)^4 \\
& + 24n^2cA_0^2A_2R(\xi)^5 - 54n^2cA_0^2A_2R(\xi)^4 + 12ncA_0A_1^2R(\xi)^3 - 12n^2cA_0^2A_1R(\xi)^3 \\
& + 12n^2cA_0A_2^2R(\xi)^7 - 12n^2cA_0A_2^2R(\xi)^6 + 12n^2cA_1^2R(\xi)^7A_2 - 21n^2cA_1^2R(\xi)^6A_2 \\
& + 12n^2cA_1R(\xi)^8A_2^2 - 18n^2cA_1R(\xi)^7A_2^2 + 3n^2cA_0A_1^2R(\xi)^2 - 12n^2cA_0A_2^2R(\xi)^5 \\
& + 12n^2cA_0A_2^2R(\xi)^4 + 6n^2cA_1^2R(\xi)^5A_2 + 3n^2cA_1^2R(\xi)^4A_2 + 6n^2cA_1R(\xi)^5A_2^2 \\
& + 2n^2\alpha A_0^2A_1R(\xi)^2 - 2n^2\alpha A_0^2A_1R(\xi) + 4n^2\alpha A_0^2A_2R(\xi)^3 - 4n^2\alpha A_0^2A_2R(\xi)^2 \\
& + 4n^2\alpha A_0A_1^2R(\xi)^3 - 4n^2\alpha A_0A_1^2R(\xi)^2 + 8n^2\alpha A_0A_2^2R(\xi)^5 - 8n^2\alpha A_0A_2^2R(\xi)^4 \\
& + 8n^2\alpha A_1^2R(\xi)^5A_2 - 8n^2\alpha A_1^2R(\xi)^4A_2 + 10n^2\alpha A_1R(\xi)^6A_2^2 - 10n^2\alpha A_1R(\xi)^5A_2^2 \\
& - 3n^2\beta A_0A_1^3R(\xi)^3 + 6n^2\beta A_0A_2^3R(\xi)^7 - 6n^2\beta A_0A_2^3R(\xi)^6 + 5n^2\beta A_1^3R(\xi)^6A_2 \\
& - 5n^2\beta A_1^3R(\xi)^5A_2 + 9n^2\beta A_1^2R(\xi)^7A_2^2 - 9n^2\beta A_1^2R(\xi)^6A_2^2 + 7n^2\beta A_1R(\xi)^8A_2^3 \\
& - 7n^2\beta A_1R(\xi)^7A_2^3 + n^2\beta A_0^3A_1R(\xi)^2 - n^2\beta A_0^3A_1R(\xi) + 2n^2\beta A_0^3A_2R(\xi)^3 \\
& - 2n^2\beta A_0^3A_2R(\xi)^2 + 3n^2\beta A_0^2A_1^2R(\xi)^3 - 3n^2\beta A_0^2A_1^2R(\xi)^2 + 6n^2\beta A_0^2A_2^2R(\xi)^5 \\
& - 6n^2\beta A_0^2A_2^2R(\xi)^4 + 3n^2\beta A_0A_1^3R(\xi)^4 + 6ncA_0A_1^2R(\xi)^5 - 15ncA_0A_1^2R(\xi)^4 \\
& - 96ncA_0A_2^2R(\xi)^6 + 18ncA_1^2R(\xi)^7A_2 - 39ncA_1^2R(\xi)^6A_2 + 30ncA_1R(\xi)^8A_2^2 \\
& + 24ncA_1^2A_2R(\xi)^5 - 9n^2cA_0A_1^2R(\xi)^4 - 6ncA_1A_2^2R(\xi)^5 + 6n^2cA_0^2A_1R(\xi)^4 \\
& - 3ncA_0A_1^2R(\xi)^2 - 3ncA_1^2A_2R(\xi)^4 + 42ncA_1A_2^2R(\xi)^6 + 6n^2cA_0A_1^2R(\xi)^5 \\
& + 6n^2cA_0^2A_1R(\xi)^2 + 36n^2cA_0^2A_2R(\xi)^3 + 36ncA_0A_2^2R(\xi)^7 = 0. \tag{3.37}
\end{aligned}$$

Decomposing equation (3.37) into various powers of $R(\xi)$, we obtain

$$n^2\beta A_0^3A_1 + 2n^2\alpha A_0^2A_1 = 0,$$

$$\begin{aligned}
& \beta n^2 A_2^4 + 2cn^2 A_2^3 + 6cnA_2^3 + 4cA_2^3 = 0, \\
& 7\beta n^2 A_1 A_2^3 - 2\beta n^2 A_2^4 + 12cn^2 A_1 A_2^2 - 6cn^2 A_2^3 + 30cnA_1 A_2^2 - 24cnA_2^3 + 12cA_1 A_2^2 \\
& - 24cA_2^3 = 0, \\
& n^2 \beta A_0^3 A_1 - 2\beta n^2 A_0^3 A_2 - 3\beta n^2 A_0^2 A_1^2 + 2n^2 \alpha A_0^2 A_1 - 4\alpha n^2 A_0^2 A_2 - 4\alpha n^2 A_0 A_1^2 \\
& + 6cn^2 A_0^2 A_1 - 6cn^2 A_0^2 A_2 + 3cn^2 A_0 A_1^2 - 3cnA_0 A_1^2 = 0, \\
& 6\beta n^2 A_0 A_2^3 + 9\beta n^2 A_1^2 A_2^2 - 7\beta n^2 A_1 A_2^3 + 4\alpha n^2 A_2^3 + 12cn^2 A_0 A_2^2 + 12cn^2 A_1^2 A_2 \\
& - 18cn^2 A_1 A_2^2 + 36cnA_0 A_2^2 + 18cnA_1^2 A_2 - 66cnA_1 A_2^2 + 12cnA_2^3 + 6cA_1^2 A_2 \\
& - 36cA_1 A_2^2 + 24cA_2^3 = 0, \\
& 2\beta n^2 A_0^3 A_2 + 3\beta n^2 A_0^2 A_1^2 - 9\beta n^2 A_0^2 A_1 A_2 - 3\beta n^2 A_0 A_1^3 + 4\alpha n^2 A_0^2 A_2 + 4\alpha n^2 A_0 A_1^2 \\
& - 12\alpha n^2 A_0 A_1 A_2 - 2\alpha n^2 A_1^3 - 12cn^2 A_0^2 A_1 + 36cn^2 A_0^2 A_2 + 6cn^2 A_0 A_1 A_2 + cn^2 A_1^3 \\
& + 12cnA_0 A_1^2 - 18cnA_0 A_1 A_2 - cA_1^3 = 0, \\
& 15\beta n^2 A_0 A_1 A_2^2 - 6\beta n^2 A_0 A_2^3 + 5\beta n^2 A_1^3 A_2 - 9\beta n^2 A_1^2 A_2^2 + 10\alpha n^2 A_1 A_2^2 - 4\alpha n^2 A_2^3 - 8cA_2^3 \\
& + 30cn^2 A_0 A_1 A_2 - 12cn^2 A_0 A_2^2 + 2cn^2 A_1^3 - 21cn^2 A_1^2 A_2 + 2cn^2 A_2^3 + 30cnA_0 A_1 A_2 \\
& - 96cnA_0 A_2^2 + 3cnA_1^3 - 39cnA_1^2 A_2 + 42cnA_1 A_2^2 + cA_1^3 - 18cA_1^2 A_2 + 36cA_1 A_2^2 = 0, \\
& 9\beta n^2 A_0^2 A_1 A_2 - 6\beta n^2 A_0^2 A_2^2 + 3\beta n^2 A_0 A_1^3 - 12\beta n^2 A_0 A_1^2 A_2 - \beta n^2 A_1^4 + 12\alpha n^2 A_0 A_1 A_2 \\
& - 8\alpha n^2 A_0 A_2^2 + 2\alpha n^2 A_1^3 - 8\alpha n^2 A_1^2 A_2 + 6cn^2 A_0^2 A_1 - 54cn^2 A_0^2 A_2 - 9cn^2 A_0 A_1^2 \\
& + 18cn^2 A_0 A_1 A_2 + 12cn^2 A_0 A_2^2 + 3cn^2 A_1^2 A_2 - 15cnA_0 A_1^2 + 66cnA_0 A_1 A_2 - 24cnA_0 A_2^2 \\
& + 3cnA_1^3 - 3cnA_1^2 A_2 + 3cA_1^3 - 6cA_1^2 A_2 = 0, \\
& 6\beta n^2 A_0^2 A_2^2 + 12\beta n^2 A_0 A_1^2 A_2 - 15\beta n^2 A_0 A_1 A_2^2 + \beta n^2 A_1^4 - 5\beta n^2 A_1^3 A_2 + 8\alpha n^2 A_0 A_2^2 \\
& + 8\alpha n^2 A_1^2 A_2 - 10\alpha n^2 A_1 A_2^2 + 24cn^2 A_0^2 A_2 + 6cn^2 A_0 A_1^2 - 54cn^2 A_0 A_1 A_2 - 12cn^2 A_0 A_2^2 \\
& - 3cn^2 A_1^3 + 6cn^2 A_1^2 A_2 + 6cn^2 A_1 A_2^2 + 6cnA_0 A_1^2 - 78cnA_0 A_1 A_2 + 84cnA_0 A_2^2 - 6cnA_1^3 \\
& + 24cnA_1^2 A_2 - 6cnA_1 A_2^2 - 3cA_1^3 + 18cA_1^2 A_2 - 12cA_1 A_2^2 = 0.
\end{aligned}$$

Solving the above system using Maple gives

$$A_0 = 0, A_1 = \frac{2c(n^2 + 3n + 2)}{\beta n^2}, A_2 = -\frac{2c(n^2 + 3n + 2)}{\beta n^2}, c = \frac{2\alpha n^2}{n^2 - 1}. \quad (3.38)$$

Therefore the solution of the equation (3.9) is

$$u(t, x) = \left\{ \frac{2c(n+1)(n+2)}{\beta n^2} \left(\frac{e^{(x-ct)}}{1 - e^{(x-ct)}} \right) \right\}^{1/n}. \quad (3.39)$$

Solitary wave solution

Integrating the NLODE (3.31) twice with respect to ξ and taking constants of integration to be zero gives

$$\frac{1}{2}cU'^2 - \frac{\beta}{(n+1)(n+2)}U^{n+2} - \frac{2\alpha - c}{2}U^2 = 0. \quad (3.40)$$

Next we let $U(\xi) = V(\xi)^{1/n}$ to transform the NLODE (3.40) to

$$V'^2 - \frac{n^2(2\alpha - c)}{c}V^2 - \frac{2n^2\beta}{c(n+1)(n+2)}V^3 = 0 \quad (3.41)$$

whose solution is

$$V(\xi) = \frac{(n+1)(n+2)(c-2\alpha)}{2\beta} \left\{ \sec \left(\sqrt{\frac{n^2(2\alpha - c)}{4c}} (\xi + C) \right) \right\}^2, \quad (3.42)$$

where C constant of integration. Thus going back to our original variables the solution of equation (3.9) is

$$u(t, x) = \left[\frac{(n+1)(n+2)(c-2\alpha)}{2\beta} \left\{ \sec \left(\sqrt{\frac{n^2(2\alpha - c)}{4c}} (x - ct + C) \right) \right\}^2 \right]^{1/n}. \quad (3.43)$$

3.5 Exact solutions of (3.9) using the simplest equation method

In this subsection, we use the simplest equation method [47–49] to solve the third-order NLODE (3.31) for $n = 1$ and $n = 2$. The simplest equations that we use are the Bernoulli equation

$$H'(z) = aH(\xi) + bH^2(\xi) \quad (3.44)$$

and the Riccati equation

$$G'(\xi) = aG^2(\xi) + bG(\xi) + c, \quad (3.45)$$

where a , b and c are constants [31]. We look for solutions of the NLODE (3.31) that are of the form

$$W(\xi) = \sum_{i=0}^M A_i (G(\xi))^i, \quad (3.46)$$

where $G(\xi)$ satisfies the Bernoulli equation or Riccati equation, M is a positive integer that can be determined by a balancing procedure and $A_0, \dots, A_M$ are parameters to be determined.

The solution of Bernoulli equation (3.44) we use here is given by

$$H(\xi) = a \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\},$$

where C is a constant of integration. For the Riccati equation (3.45), the solutions to be used are

$$G(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (\xi + C) \right], \quad (3.47)$$

and

$$G(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)}, \quad (3.48)$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

Solutions of (3.9) using the Bernoulli equation as the simplest equation

Case 1. $n = 1$

In this case, the balancing procedure yields $M = 2$ and solutions of (3.31) are of the form

$$W(\xi) = A_0 + A_1 G + A_2 G^2. \quad (3.49)$$

We insert the above value of $W(\xi)$ into equation (3.31). Then, using the Bernoulli equation (3.44) and equating the coefficients of G^i to zero, we obtain an algebraic system of five equations in terms of A_0, A_1, A_2 , namely

$$12cm^3A_2 + \beta mA_2^2 = 0,$$

$$cl^3A_1 + \beta lA_0A_1 + 2\alpha lA_1 - clA_1 = 0,$$

$$54clm^2A_2 + 6cm^3A_1 + 2\beta lA_2^2 + 3\beta mA_1A_2 = 0,$$

$$38cl^2mA_2 + 12clm^2A_1 + 3\beta lA_1A_2 + 2\beta mA_0A_2 + \beta mA_1^2 + 4\alpha mA_2 - 2cmA_2 = 0,$$

$$8cl^3A_2 + 7cl^2mA_1 + 2\beta lA_0A_2 + \beta lA_1^2 + \beta mA_0A_1 + 4\alpha lA_2 + 2\alpha mA_1 - 2clA_2 - cmA_1 = 0.$$

With the aid of Maple, we solve the above system and obtain

$$A_0 = A_0, \quad A_1 = \frac{12lcmA_0}{cl^2 + 2\alpha - c}, \quad A_2 = \frac{12cm^2A_0}{cl^2 + 2\alpha - c}.$$

Therefore the solution of (3.9) is given by

$$\begin{aligned} u(t, x, y) = & A_0 + aA_1 \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\} \\ & + A_2 a^2 \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\}^2, \end{aligned} \quad (3.50)$$

where C is an arbitrary constant of integration.

Case 2. $n = 2$

In this case, the balancing procedure yields $M = 1$ and solutions of (3.31) are of the form

$$W(\xi) = A_0 + A_1 G. \quad (3.51)$$

We insert the above value of $W(\xi)$ into equation (3.31). Then, using the Bernoulli equation (3.44) and equating the coefficients of powers of G to zero, we obtain an algebraic system equations in terms of A_0, A_1 and solving the system using maple we obtain

$$A_0 = \pm \frac{1}{2} \frac{\sqrt{6\beta} cl}{\beta}, \quad A_1 = \pm \frac{\sqrt{6\beta} cm}{\beta}.$$

Therefore, the solution of equation (3.9) is given by

$$u(t, x, y) = A_0 + aA_1 \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\},$$

where C is an arbitrary constant of integration.

Solutions of (3.9) using the Riccati equation as the simplest equation

Case 1. $n = 1$

For this case, the balancing procedure gives $M = 2$, equation (3.46) becomes

$$W(\xi) = A_0 + A_1 G + A_2 G^2. \quad (3.52)$$

The insertion of the value of $W(\xi)$ into equation (3.31) and making use of the Riccati equation (3.45) yields the following algebraic system of equations in terms of A_0, A_1, A_2

$$\begin{aligned}
12cl^3A_2 + \beta lA_2^2 &= 0, \\
6cl^3A_1 + 54cl^2mA_2 + 3\beta lA_1A_2 + 2\beta mA_2^2 &= 0, \\
2cln^2A_1 + cm^2nA_1 + 6cmn^2A_2 + \beta nA_0A_1 + 2\alpha nA_1 - cnA_1 &= 0, \\
12cl^2mA_1 + 40cl^2nA_2 + 38clm^2A_2 + 2\beta lA_0A_2 + \beta lA_1^2 + 3\beta mA_1A_2 \\
+ 2\beta nA_2^2 + 4\alpha lA_2 - 2clA_2 &= 0, \\
8clmnA_1 + 16cln^2A_2 + cm^3A_1 + 14cm^2nA_2 + \beta mA_0A_1 + 2\beta nA_0A_2 \\
+ \beta nA_1^2 + 2\alpha mA_1 + 4\alpha nA_2 - cmA_1 - 2cnA_2 &= 0.
\end{aligned}$$

The solution of the above system using Maple gives

$$A_0 = A_0, \quad A_1 = \frac{12mclA_0}{8cln + cm^2 + 2\alpha - c}, \quad A_2 = \frac{12cl^2A_0}{8cln + cm^2 + 2\alpha - c}.$$

Consequently, the solutions of the gsmCH equation (3.9) are

$$\begin{aligned}
u(t, x, y) = & A_0 + A_1 \left[-\frac{b}{2a} - \frac{\theta}{2a} \tanh \left\{ \frac{1}{2} \theta (\xi + C) \right\} \right] \\
& + A_2 \left[-\frac{b}{2a} - \frac{\theta}{2a} \tanh \left\{ \frac{1}{2} \theta (\xi + C) \right\} \right]^2,
\end{aligned}$$

and

$$\begin{aligned}
u(t, x, y) = & A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)} \right\} \\
& + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)} \right\}^2,
\end{aligned}$$

where C is an arbitrary constant of integration.

Case 2: $n = 2$

For this case, the balancing procedure gives $M = 1$, equation (3.46) becomes

$$W(\xi) = A_0 + A_1 G. \quad (3.53)$$

The insertion of the value of $W(\xi)$ into equation (3.31) and making use of the Riccati equation (3.45) yields the algebraic system of equations in terms of A_0, A_1 whereby with the aid of maple solving these equations we obtain

$$A_0 = \pm \frac{1}{2} \frac{\sqrt{6\beta cm}}{\beta}, \quad A_1 = \pm \frac{\sqrt{6\beta cl}}{\beta}.$$

Consequently, the solutions of equation (3.9) are

$$u(t, x, y) = A_0 + A_1 \left[-\frac{b}{2a} - \frac{\theta}{2a} \tanh \left\{ \frac{1}{2} \theta (\xi + C) \right\} \right], \quad (3.54)$$

and

$$u(t, x, y) = A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)} \right\}, \quad (3.55)$$

where C is an arbitrary constant of integration.

Solution of (3.9) using direct integration

In this subsection, we derive a solution of an ODE equation (3.31) by using direct integration when $n = 1$. Integrating equation (3.31) twice gives

$$U'^2 = \frac{2C_1}{c}U + \frac{(2\alpha - c)}{c}U^2 + \frac{2\beta}{3c}U^3 + \frac{2C_2}{c}, \quad (3.56)$$

where C_1, C_2 are arbitrary constants. To gain the solution of equation (3.56), we assume that r_1, r_2 and r_3 are roots of cubic polynomial

$$U^3 + \frac{3(2\alpha - c)}{2\beta}U^2 + \frac{3c_1}{\beta}U + \frac{3C_2}{\beta} = 0,$$

with $r_1 > r_2 > r_3$. Equation (3.56) can be written as

$$U'^2 = \frac{2\beta}{3c}(U - r_1)(U - r_2)(U - r_3),$$

whose solution [24, 25] is

$$U(\xi) = r_2 + (r_1 - r_2) \operatorname{cn}^2 \left\{ \sqrt{\frac{\beta(r_1 - r_3)}{6c}} (\xi - \xi_0), R^2 \right\}, \quad R^2 = \frac{r_1 - r_2}{r_1 - r_3},$$

where ξ_0 is a constant and cn is the Jacobi cosine function. Consequently, the solution for gsmCH equation (3.9) is

$$u(t, x) = r_2 + (r_1 - r_2)\text{cn}^2 \left\{ \sqrt{\frac{\beta(r_1 - r_3)}{6c}}(\xi - \xi_0), R^2 \right\}.$$

3.6 Conservation laws of (3.9)

In this section we derive conservation laws of gsmCH equation (3.9). We use three different methods of finding conservation laws, namely, the multiplier method, Ibragimov's method and the classical Noether's theorem.

3.6.1 Conservation laws of (3.9) using multiplier method

In this subsection we employ the multiplier method to construct conservation laws for the gsmCH equation (3.9). We look for the zeroth-order multipliers $\Lambda = \Lambda(t, x, u)$ that are obtained by solving the determining equation

$$\frac{\delta}{\delta u} \left\{ \Lambda(t, x, u) (u_t + 2\alpha u_x + \beta u^n u_x - u_{txx}) \right\} = 0,$$

which gives

$$\begin{aligned} & u_t \Lambda_u + 2\alpha u_x \Lambda_u + \beta u^n u_x \Lambda_u - u_{txx} \Lambda_u - u_x^2 \Lambda_u - u u_{xx} + \beta n u^{n-1} u_x \Lambda \\ & - D_t \Lambda - 2\alpha D_x \Lambda - \beta D_x (u^n \Lambda) + D_t D_x^2 \Lambda = 0. \end{aligned}$$

The above equation expands to

$$\begin{aligned} & -\Lambda_t - 2\alpha \Lambda_x - \beta u^n \Lambda_x + \Lambda_{txx} + 2u_x \Lambda_{txu} + u_x^2 \Lambda_{tuu} + u_{xx} \Lambda_{tu} + u_t \Lambda_{xxu} \\ & + 2u_t u_x \Lambda_{xuu} + 2u_{tx} \Lambda_{xu} + u_t u_x^2 \Lambda_{uuu} + 2u_x u_{tx} \Lambda_{uu} + u_t u_{xx} \Lambda_{xuu} = 0. \end{aligned}$$

Simplifying and splitting the above equation on the derivatives of u , we obtain

$$u_{tx} : \Lambda_{xu} = 0, \tag{3.57}$$

$$u_{xx} : \Lambda_{tu} = 0, \tag{3.58}$$

$$u_t u_{xx} : \Lambda_{uu} = 0, \tag{3.59}$$

$$\text{rest} : \Lambda_{txx} - \Lambda_t - 2\alpha\Lambda_x - \beta u^n \Lambda_x = 0. \quad (3.60)$$

Integrating equation (3.59) twice with respect u yields

$$\Lambda = A(t, x)u + B(t, x),$$

where $A(t, x)$ and $B(t, x)$ are arbitrary function whose arguments are t and x . Substituting the above value of Λ into equation (3.57), we have that $A_x = 0$, which leads to $A = A(t)$, where A is an arbitrary function of t . Hence

$$\Lambda = A(t)u + B(t, x).$$

Substituting this new value of Λ into equation (3.58) we get that $A'(t) = 0$, which implies that $A(t) = C_1$, thus

$$\Lambda = C_1 u + B(t, x).$$

where C_1 is a constant of integration. Equation (3.60) implies that

$$B_{txx} - B_t - 2\alpha B_x - \beta u^n B_x = 0.$$

Splitting the above equation over powers of u yields

$$u^n : B_x = 0, \quad (3.61)$$

$$\text{rest} : B_{txx} - B_t - 2\alpha B_x = 0. \quad (3.62)$$

Equation (3.61) yields $B = D(t)$. Substituting the value of B into equation (3.62), we get $D'(t) = 0$, which is easily solved to obtain

$$D(t) = C_2,$$

where C_2 is a constant. Therefore $\Lambda = C_1 u + C_2$ is the multiplier for equation (3.9).

We obtain the conservation law for the above multiplier by considering

$$\Lambda(t, x, u) \{ (u_t + 2\alpha u_x + \beta u^n u_x - u_{txx}) \} = D_t T^t + D_x T^x, \quad (3.63)$$

where density T^t and flux T^x are defined as

$$T^t = T^t(t, x, u) \text{ and } T^x = T^x(t, x, u, u_x), \quad (3.64)$$

and so equation (3.63) becomes

$$u_t + 2\alpha u_x + \beta u^n u_x - u_{txx} = D_t T^t + D_x T^x.$$

Case 1. For $\Lambda_1 = u$, equation (3.63) becomes

$$u(u_t + 2\alpha u_x + \beta u^n u_x - u_{txx}) = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x + u_{txx} T^x u_{tx}.$$

Now splitting the above equation on the third derivatives of u , we obtain

$$u_{txx} : T^x u_{tx} = -u, \quad (3.65)$$

$$\begin{aligned} \text{rest} : u(u_t + 2\alpha u_x + \beta u^n u_x) &= T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x \\ &+ u_{xx} T_{u_x}^x. \end{aligned} \quad (3.66)$$

Integration of equation (3.65) with respect to u_{tx} , we obtain the flux

$$T^x = -u u_{tx} + P(t, x, u, u_x),$$

where P is an arbitrary function of its arguments. Substituting the above value of T^x into equation (3.66) we have

$$u(u_t + 2\alpha u_x + \beta u^n u_x) = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + P_x - u_x u_{tx} + u_x P_u + u_{xx} P_{u_x}.$$

Simplifying and splitting the above equation on the second derivatives of u , we have

$$u_{xx} : P_{u_x} = 0, \quad (3.67)$$

$$u_{tx} : -u_x + T_{u_x}^t = 0, \quad (3.68)$$

$$\text{rest} : u(u_t + 2\alpha u_x + \beta u^n u_x) = T_t^t + u_t T_u^t + P_x + u_x P_u. \quad (3.69)$$

Equation (3.67) implies that $P = P(t, x, u)$. From equation (3.68) we get

$$T^t = \frac{u_x^2}{2} + C(t, x, u),$$

where C is an arbitrary function of its arguments. Substituting the above value of T^t into equation (3.69) gives

$$u(u_t + 2\alpha u_x + \beta u^n u_x) = C_t + u_t C_u + P_x + u_x P_u.$$

Splitting the above equation on derivatives of u we get

$$u_t : C_u = u, \quad (3.70)$$

$$u_x : P_u = \beta u^{n+1} + 2\alpha u, \quad (3.71)$$

$$\text{rest} : C_t + P_x = 0. \quad (3.72)$$

Equation (3.71) gives

$$P = \beta \frac{u^{n+2}}{n+2} + \alpha u^2 + D(t, x),$$

where D is an arbitrary function of t and x . Integrating equation (3.70) yields

$$C = \frac{u^2}{2} + E(t, x),$$

where E is an arbitrary function of t and x . Substituting the values of P and C into equation (3.72) gives $E_t + D_x = 0$. We take note that E and D contribute to the trivial part of the conservation law, thus we take them to be zero. Hence the conservation law corresponding to first multiplier is given by

$$\begin{aligned} T^t &= \frac{1}{2}u_x^2 + \frac{1}{2}u^2, \\ T^x &= -uu_{tx} + \frac{\beta}{n+2}u^{n+2} + \alpha u^2. \end{aligned}$$

Case 2. For the second multiplier $\Lambda_2 = 1$, we expand equation (3.63) and get

$$u_t + 2\alpha u_x + \beta u^n u_x - u_{txx} = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx} + T_{u_{tx}}^x u_{txx}.$$

Splitting the above equation on third derivatives of u yields

$$u_{txx} : T_{u_{tx}}^x = -1, \quad (3.73)$$

$$\text{Rest} : u_t + 2\alpha u_x + \beta u^n u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx}. \quad (3.74)$$

Equation (3.73) implies that

$$T^x = -u_{tx} + A(t, x, u, u_x),$$

where A is an arbitrary function of its arguments. Substituting this value of T^x into equation (3.74) we get

$$u_t + 2\alpha u_x + \beta u^n u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx}.$$

The above equation is split on second derivatives of u , to give

$$u_{tx} : T_{u_x}^t = 0, \quad (3.75)$$

$$u_{xx} : A_{u_x} = 0, \quad (3.76)$$

$$\text{rest} : u_t + 2\alpha u_x + \beta u^n u_x = T_t^t + T_u^t u_t + A_x + A_u u_x. \quad (3.77)$$

Equations (3.76) implies that

$$A = A(t, x, u).$$

From equation (3.75) we get

$$T^t = B(t, x, u),$$

where B is an arbitrary function of t , x and u . Substituting the value of T^t into (3.77) gives

$$u_t + 2\alpha u_x + \beta u^n u_x = B_t + B_u u_t + A_x + A_u u_x. \quad (3.78)$$

Splitting equation (3.78) on the derivatives of u yields

$$u_t : B_u = 1, \quad (3.79)$$

$$u_x : A_u = 2\alpha + \beta u^n, \quad (3.80)$$

$$\text{rest} : A_x + B_t = 0. \quad (3.81)$$

Integrating equation (3.79) we obtain

$$B = u + C(t, x),$$

where C is an arbitrary function of t and x . Equation (3.80) gives

$$A = 2\alpha u + \beta \frac{u^{n+1}}{n+1} + D(t, x),$$

where D is an arbitrary function of t and x . Substituting the values A and B into equation (3.81) we get $D_x + C_t = 0$. Since C and D contribute to the trivial part of the conservation law, we therefore take $C = D = 0$. Thus the conservation law corresponding to this multiplier is given by

$$T^t = u,$$

$$T^x = 2\alpha u + \frac{\beta}{n+1}u^{n+1} - u_{tx}.$$

Remark. It is noted here that since we have the multiplier $\Lambda = 1$, this implies that the gsmCH equation (3.9) is in conserved form.

3.6.2 Conservation laws using Ibragimov's method

In this section we compute conservation laws for the gsmCH equation (3.9) using the theorem due to Ibragimov as given in [35]. We begin by determining the adjoint equation using the formula

$$F^* \equiv \frac{\delta}{\delta u} \{v(u_t + 2\alpha u_x + \beta u^n u_x - u_{txx})\} = 0. \quad (3.82)$$

In this case the Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_t D_x^2 \frac{\partial}{\partial u_{txx}}, \quad (3.83)$$

where the total differential operators D_t and D_x are

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xt} \frac{\partial}{\partial u_t} + v_{xt} \frac{\partial}{\partial v_t} + \cdots. \end{aligned}$$

Equation (3.82) yields the adjoint equation

$$F^* \equiv -v_t - \beta v_x u^n - 2\alpha v_x + v_{txx} = 0. \quad (3.84)$$

We now consider the equation (3.9) and its adjoint (3.84). The Lagrangian of this system is

$$\mathcal{L} = vu_t + 2\alpha vu_x + \beta vu^n u_x - vu_{txx}. \quad (3.85)$$

To compute the conservation laws of equation (3.9), we utilise the formula [35]

$$C^i = \xi^i \mathcal{L} + W^\alpha \left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right) + D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}, \quad (3.86)$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$. Recall that equation (3.9) has the two translation symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad (3.87)$$

$$X_2 = \frac{\partial}{\partial x}. \quad (3.88)$$

Let us first consider the infinitesimal generator (3.87). It can be easily shown that extending the generator X_1 to the new variable v produces the same translation symmetry $X_1 = \partial/\partial t$. In order to determine the value of λ we apply the equation

$$X(F) = \lambda(F), \quad (3.89)$$

which yields $\lambda = 0$. Again,

$$D_t(\tau) = 0. \quad (3.90)$$

Consequently, $\eta^* = 0$ and the operator admitted by the adjoint equation (3.84) is

$$Y = \frac{\partial}{\partial t}. \quad (3.91)$$

We now use equation (3.91) to compute the Lie characteristic functions

$$W^1 = -u_t, \quad W^2 = -v_t.$$

The conserved vector for the equation (3.9) and its adjoint (3.84) corresponding to the infinitesimal generator (3.87) can thus be obtained by using (3.86). The following conserved vector is obtained

$$\begin{aligned} C^t &= 2\alpha v u_x + \beta u^n v u_x - \frac{1}{3} v_x u_{tx} - \frac{2}{3} v u_{txx}, \\ C^x &= -2\alpha v u_t - \beta u^n v u_t - \frac{1}{3} v_t u_{tx} + \frac{2}{3} v_{tx} u_t - \frac{1}{3} v_x u_{tt} + \frac{2}{3} v u_{ttx}. \end{aligned}$$

Similarly, we compute the conserved vector corresponding to the infinitesimal generator (3.88). In this case $W^1 = -u_x$ and $W^2 = -v_x$ and the conserved vector thus rendered by applying (3.86) are

$$\begin{aligned} C^t &= -v u_x - \frac{1}{3} v_x u_{xx} + \frac{1}{3} v_{xx} u_x + \frac{1}{3} v u_{xxx}, \\ C^x &= v u_t - \frac{1}{3} v_t u_{xx} - \frac{1}{3} v_x u_{tx} + \frac{2}{3} v_{tx} u_x - \frac{1}{3} v u_{ttx}. \end{aligned}$$

3.6.3 Conservation laws using Noether's approach

In this subsection we utilize Noether's approach to derive conservation laws for the gsmCH equation (3.9). This equation is of third order and as such does not have a Lagrangian. To make it variational we increase its order by letting $u = v_x$. Thus we have

$$v_{tx} + 2\alpha v_{xx} + \beta v_x^n v_{xx} - v_{txx} = 0. \quad (3.92)$$

It can be verified that equation (3.92) has a second-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2}v_t v_x - \alpha v_x^2 - \frac{1}{2}v_{tx} v_{xx} - \frac{\beta}{(n+1)(n+2)}v_x^{n+2}, \quad n \neq -1, -2,$$

as $\delta\mathcal{L}/\delta v = 0$ on the equation. Here the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t D_x \frac{\partial}{\partial v_{tx}} + D_x^2 \frac{\partial}{\partial v_{xx}} + \dots$$

The determining equation for Noether point symmetries is given by

$$X^{[2]}(\mathcal{L}) + \mathcal{L}[D_t(\tau) + D_x(\xi)] = D_t(B^1) + D_x(B^2), \quad (3.93)$$

where $B^1 = B^1(t, x, v)$ and $B^2 = B^2(t, x, v)$ are gauge terms and $X^{[2]}$ is the second prolongation of the infinitesimal generator

$$X = \tau(t, x, v) \frac{\partial}{\partial t} + \xi(t, x, v) \frac{\partial}{\partial x} + \eta(t, x, v) \frac{\partial}{\partial v}.$$

Hence expanding equation (3.93) with the Lagrangian $\mathcal{L}$ gives

$$\begin{aligned} & -\frac{1}{2}v_x \{ \eta_t + v_t \eta_v - v_t \tau_t - v_t^2 \tau_v - v_x \xi_t - v_t v_x \xi_v \} - \frac{1}{2}v_t \{ \eta_x + v_x \eta_v - v_t \tau_x - v_t v_x \tau_v \\ & - v_x \xi_x - v_x^2 \xi_v \} - 2\alpha v_x \{ \eta_x + v_x \eta_v - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} - \frac{\beta}{n+1} \{ \eta_x \\ & + v_x \eta_v - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} - \frac{1}{2}v_{xx} \{ \eta_{tx} + v_x \eta_{tv} + v_t \eta_{xv} + v_{xt} \eta_v \\ & + v_t v_x \eta_{vv} - v_{xt} \tau_{tv} - v_{xt} \xi_{xv} - v_t^2 \tau_{xv} - 2v_t v_{xt} \tau_v - v_x v_{tt} \tau_v - v_t^2 v_x \tau_{vv} - v_x \xi_{xt} \\ & - v_{xx} \xi_t - v_x^2 \xi_{tv} - 2v_x v_{xt} \xi_v - v_t v_{xx} \xi_v - v_t v_x^2 \xi_{vv} \} - \frac{1}{2}v_{tx} \{ \eta_{xx} + 2v_x \eta_{xv} + v_{xx} \eta_v \\ & + v_x^2 \eta_{vv} - 2v_{xx} \xi_x - v_x \xi_{xx} - 2v_x^2 \xi_{xv} - 3v_x v_{xx} \xi_v - v_x^3 \xi_{vv} - 2v_{xt} \tau_x - v_t \tau_{xx} \\ & - 2v_t v_x \tau_{xv} - v_t v_{xx} \tau_v - 2v_x v_{xt} \tau_v - v_t v_x^2 \tau_{vv} \} - B_t^1 - v_t B_v^1 - B_x^2 - v_x B_v^2 = 0. \end{aligned}$$

Splitting the above equation on derivatives of v yields

$$\tau_t = 0, \quad (3.94)$$

$$\tau_x = 0, \quad (3.95)$$

$$\tau_v = 0, \quad (3.96)$$

$$\xi_x = 0, \quad (3.97)$$

$$\xi_t = 0, \quad (3.98)$$

$$\xi_v = 0, \quad (3.99)$$

$$\eta_x = 0, \quad (3.100)$$

$$\eta_v = 0, \quad (3.101)$$

$$B_v^1 = 0, \quad (3.102)$$

$$B_x^1 + B_t^2 = 0. \quad (3.103)$$

Solving these PDEs gives Noether point symmetries and gauge functions. We list these below

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\ X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0, \\ X_f &= f(t) \frac{\partial}{\partial v}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2} v f'(t). \end{aligned}$$

We now construct the conservation laws corresponding to each Noether point symmetry using [34]

$$\begin{aligned} T^t &= \mathcal{L}\tau + (\eta - u_t\tau - u_x\xi) \frac{\partial \mathcal{L}}{\partial u_t} + (\zeta_2 - u_{tx}\tau - u_{xx}\xi) \frac{\partial \mathcal{L}}{u_{tx}} - B^1, \\ T^x &= \mathcal{L}\xi + (\eta - u_t\tau - u_x\xi) \left\{ \frac{\partial \mathcal{L}}{\partial u_x} - D_t \left(\frac{\partial \mathcal{L}}{\partial u_{tx}} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right\} \\ &\quad + (\zeta_2 - u_{tx}\tau - u_{xx}\xi) \frac{\partial \mathcal{L}}{u_{xx}} - B^2. \end{aligned}$$

Case 1. The first Noether symmetry $X_1 = \partial/\partial t$ has the conserved vector

$$T_1^t = -\frac{1}{(n+1)(n+2)} \beta v_x^{n+2} - \alpha v_x^2 - \frac{1}{4} v_{xxx} v_t - \frac{1}{4} v_{xx} v_{tx},$$

$$T_1^x = \frac{1}{(n+1)}\beta v_t v_x^{n+1} + \frac{1}{2}v_t^2 + \frac{1}{2}v_{tx}^2 + 2\alpha v_x v_t + \frac{1}{4}v_{xx}v_{tt} - \frac{3}{4}v_t v_{txx}.$$

Case 2. The second operator $X_2 = \partial/\partial x$ yields the consevation law

$$T_2^t = \frac{1}{2}v_x^2 + \frac{1}{4}v_{xx}^2 - \frac{1}{4}v_x v_{xxx},$$

$$T_2^x = \frac{1}{(n+1)}\beta v_x^{n+2} - \frac{1}{(n+1)(n+2)}\beta v_x^{n+2} + \alpha v_x^2 + \frac{1}{4}v_{xx}v_{tx} - \frac{3}{4}v_x v_{txx}.$$

Case 3. Finally for the operator $X_f = f(t)\partial/\partial v$, we get the conserved vector

$$T_f^t = -\frac{1}{2}f(t)v_x + \frac{1}{4}f(t)v_{xxx},$$

$$T_f^x = -\frac{1}{2}f(t)v_t + \frac{3}{4}f(t)v_{txx} - 2\alpha f(t)v_x - \frac{1}{(n+1)}\beta f(t)v_x^{n+1} - \frac{1}{4}f'(t)v_{xx}.$$

Reverting back to our original variable u , we have

$$T_1^t = -\frac{1}{(n+1)(n+2)}\beta u^{n+2} - \alpha u^2 - \frac{1}{4}u_{xx} \int u_t dx - \frac{1}{4}u_x u_t,$$

$$T_1^x = \frac{1}{(n+1)}\beta \left(\int u_t dx \right) u^{n+1} + \frac{1}{2} \left(\int u_t dx \right)^2 + \frac{1}{2}u_t^2 + 2\alpha u \int u_t dx$$

$$+ \frac{1}{4}u_x \int u_{tt} dx - \frac{3}{4} \int u_t dx u_{tx},$$

$$T_2^t = \frac{1}{2}u^2 + \frac{1}{4}u_x^2 - \frac{1}{4}u u_{xx},$$

$$T_2^x = \frac{1}{(n+1)}\beta u^{n+2} - \frac{1}{(n+1)(n+2)}\beta u^{n+2} + \alpha u^2 + \frac{1}{4}u_x u_t - \frac{3}{4}u u_{tx},$$

$$T_f^t = -\frac{1}{2}f(t)u + \frac{1}{4}f(t)u_{xx},$$

$$T_f^x = -\frac{1}{2}f(t) \int u_t dx + \frac{3}{4}f(t)u_{tx} - 2\alpha f(t)u - \frac{1}{(n+1)}\beta f(t)u^{n+1} - \frac{1}{4}f'(t)u_x.$$

3.7 Concluding remarks

In this chapter, we investigated the gsmCH equation (3.9) using Lie symmetry analysis. Firstly, Lie point symmetries were derived and used to perform symmetry reductions and obtain exact solutions of the gsmCH equation. As a result of symmetry reductions, equation (3.9) was transformed to NLODE (3.31). Consequently,

Kudryashov's method, the direct integration and the simplest equation method were used to find exact solutions which were in the form of exponential, rational, hyperbolic and elliptic functions. Lastly, the conservation laws were derived using multiplier approach, Ibragimov's method and Noether's theorem. The conservation laws that represent energy and momentum amongst others were produced with the aid of Ibragimov's method and Noether's theorem. We also realized that Noether's theorem produced local and non local conservation laws .

Chapter 4

Solutions and conservation laws of a (3+1)-dimensional KP-like equation

4.1 Introduction

In [50], Kadomtsev and Petviashvili relaxed the requirement that the waves be strictly one-dimensional and obtained the fully integrable Kadomtsev-Petviashvili (KP) equation in the form

$$(u_t + 6uu_x + u_{xxx})_x + au_{yy} = 0, \quad (4.1)$$

which describes the evolution of quasi-one-dimensional shallow-water waves when surface tension and viscosity effects are negligible. Here $u = u(t, x, y)$ is a scalar function, x and y are respectively the longitudinal and transverse spatial coordinates. The subscripts t, x, y denote partial derivatives and $a = \pm 1$. In equation (4.1) for $a = 1$ the result is called Kadomtsev-Petviashvili II (KPII) equation [51]. This equation models water waves with small surface tension. In addition when $a = -1$, we have Kadomtsev-Petviashvili I (KPI) equation. This equation can be used to model waves in thin films with high surface tension. Equation (4.1) which results

from the reduction of a system with quadratic nonlinearity that admits weakly dispersive waves in a paraxial wave approximation, is one of the most often used models in nonlinear wave theory. When just the leading order terms are kept and an asymptotic equilibrium between weak dispersion, quadratic nonlinearity, and diffraction is assumed, the equation naturally arises as a distinctive limit in the asymptotic description of such systems. The two spatial variables' distinct functions are what cause them to appear in the equation in an asymmetrical manner.

Many generalized KP equations have been investigated in the literature and these have become a point of interest. For example, Ma et al. [52] researched a generalized (3+1)-dimensional KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0 \quad (4.2)$$

and produced Wronskian and Grammian formulations. Through utilisation of the improved $\tan(\theta(\xi)/2)$ -expansion method, the authors in [53] obtained exact solutions of the equation (4.2) which gave them hyperbolic, trigonometric, exponential and rational functions. Furthermore, the authors in [54] utilised the Hirota bilinear method and they obtained one-soliton, two-soliton and N-soliton solutions. Wazwaz in [55] found multiple-soliton solutions and multiple singular soliton solutions. In [56] a one-parameter class of Pfaffian extensions was made for equation (4.2) by the Pfaffianization procedure. Wronski-type Pfaffian and Gramm-type Pfaffian solutions of the resulting Pfaffianized bilinear systems of the generalized KP equation were constructed. Under a dimensional reduction, the results yielded a class of Pfaffianized bilinear systems for the KP equation, which contained the standard Pfaffianized bilinear KP system. Ma and Zhu [57] utilised the multiple exp-function algorithm as a generalization of Hirota's perturbation scheme, to construct multiple wave solutions. The resulting solutions involved generic phase shifts and wave frequencies containing many existing choices. Another form of KP like equation which reads [58]

$$a(t)u_{xxxy} + \rho a(t)(u_x u_y)_x + (u_x + u_y - u_z)_t + b(t)(u_{xx} + u_{zz}) = 0, \quad (4.3)$$

where $a(t)$ and $b(t)$ are all arbitrary functions of t , and ρ , a real non-zero constant is

called the generalized (3+1)-dimensional variable-coefficient B-type equation. This equation describes the propagation of nonlinear waves in fluid dynamics. In [58], the authors used the bell polynomials to derive the bilinear form and Bäcklund transformation of the equation. In addition, the Hirota method was used to construct the one-and two-soliton solutions for equation (4.3).

The authors in [59] started to study the equation (4.3) by using Hirota bilinear method and their findings involved the lump and lump-type solutions of the generalized (3+1)-dimensional variable-coefficient B-type KP equation.

Moreover, the authors in [60] studied a new (3+1)-dimensional generalized KP problem which read as

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tz} - u_{zz} = 0 \quad (4.4)$$

and they used the simplified Hirota's direct method to derive multiple soliton solutions for the new model where the coefficients of the spatial variables are left free. In addition, the authors showed that the phase shifts depend on all these coefficients. They also proved that the new model fails the Painlevé integrability test although it gives multiple-soliton solutions.

Furthermore, the researchers in [61] studied equation (4.4) and used the linear superposition technique to obtain the resonant multi-soliton solutions for the new (3+1)-dimensional generalized KP equation.

Another KP hierarchy, known as the B-type KP (BKP) equation which is called (3+1)-dimensional BKP equation reads [62]

$$u_{tz} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0. \quad (4.5)$$

The authors in [62] studied equation (4.5) where a linear superposition principle of exponential traveling waves was analyzed for Hirota bilinear equations, with an aim to construct a specific sub-class of N -soliton solutions formed by linear combinations of exponential traveling waves. In [63], the author used the simplified form of the Hirota's method to investigate three variants of equation (4.5). Multiple-front wave

solutions were formally derived. The dispersion relation and the phase shift of each equation was established. The constraints, that arised between the coefficients of the spatial variables, that will guarantee the existence of multiple solutions, were analyzed.

In addition, the researchers in [64] used multiple exp-function method to explore the shape changing and cleaving nature of anti-kink solutions of equation (4.5).

Moreover, in [65] an efficient method for developing fresh KP-like equations was suggested. The authors in [65] used a transformation $u = 2(\ln f)_x$ to transform equation (4.5) into the Hirota bilinear form

$$(D_z D_t - D_x^3 D_y + 3D_x^2)f \cdot f = 0. \quad (4.6)$$

Furthermore, for equation (4.6) to provide much more information in physical situations, two terms were added and it was reconsidered as

$$(aD_x D_t + bD_y D_t + cD_z D_t - dD_x^3 D_y + eD_x^2)f \cdot f = 0, \quad (4.7)$$

where a, b, c, d and e are arbitrary parameters. Using the transformation $u = 2(\ln f)_x$ equation (4.7) was reversed to

$$au_{tx} + bu_{ty} + cu_{tz} - du_{xxx}y - 3u_x u_{xy} - 3u_{xx}u_y + eu_{xx} = 0. \quad (4.8)$$

In addition, equation (4.8) was studied in [66] where the authors obtained hyperbolic tangent and generic function solutions by using arbitrary linear combinations of Lie point symmetries.

In this chapter, we study the general exact solutions of equation (4.8) by using Lie group analysis together with various techniques which includes direct integration and Kudryashov's methods. Such solutions are in the form of elliptic integrals and Jacobian elliptic functions. Furthermore, we will derive the conservation laws of equation (4.8) by using the multiplier approach.

4.2 Lie symmetries of equation (4.8)

We begin by extracting the Lie symmetries associated with the NLEE (4.8). A generic infinitesimal generator of our equation takes the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \quad (4.9)$$

where $\xi^1, \xi^2, \xi^3, \xi^4$ and η are functions of (t, x, y, z) . Generator (4.9) must, however, conform to the following condition:

$$X^{[4]}(au_{tx} + bu_{ty} + cu_{tz} - du_{xxx} - 3u_x u_{xy} - 3u_{xx} u_y + eu_{xx})|_{(4.8)} = 0. \quad (4.10)$$

Here $X^{[4]}$ is a third extension of (4.9) given by [9]

$$\begin{aligned} X^{[3]} = & X + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{tz} \frac{\partial}{\partial u_{tz}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} \\ & + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} \end{aligned} \quad (4.11)$$

with $\zeta_x, \zeta_y, \zeta_{tx}, \zeta_{ty}, \zeta_{tz}, \zeta_{xx}, \zeta_{xy}$ and ζ_{xxx} defined by

$$\begin{aligned} \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4), \\ \zeta_y &= D_y(\eta) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3) - u_z D_y(\xi^4), \\ \zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - u_{ty} D_x(\xi^3) - u_{tz} D_x(\xi^4), \\ \zeta_{ty} &= D_y(\zeta_t) - u_{tt} D_y(\xi^1) - u_{tx} D_y(\xi^2) - u_{ty} D_y(\xi^3) - u_{tz} D_y(\xi^4), \\ \zeta_{tz} &= D_z(\zeta_t) - u_{tt} D_z(\xi^1) - u_{tx} D_z(\xi^2) - u_{ty} D_z(\xi^3) - u_{tz} D_z(\xi^4), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4), \\ \zeta_{xy} &= D_y(\zeta_x) - u_{xt} D_y(\xi^1) - u_{xx} D_y(\xi^2) - u_{xy} D_y(\xi^3) - u_{xz} D_y(\xi^4), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2) - u_{xxy} D_z(\xi^3) - u_{xxz} D_x(\xi^4), \\ \zeta_{xxx} &= D_y(\zeta_{xx}) - u_{xxt} D_y(\xi^1) - u_{xxx} D_y(\xi^2) - u_{xxy} D_y(\xi^3) - u_{xxz} D_y(\xi^4), \end{aligned}$$

where D_t , D_x , D_y and D_z are the total derivatives, which are given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} + \dots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} + \dots, \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_z} + \dots, \\
D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{zz} \frac{\partial}{\partial u_z} + u_{zt} \frac{\partial}{\partial u_t} + u_{zy} \frac{\partial}{\partial u_y} + u_{zx} \frac{\partial}{\partial u_x} + \dots.
\end{aligned} \tag{4.12}$$

Solving equation (4.10) leads to the following nine-dimensional Lie algebras:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}, \\
X_5 &= ct \frac{\partial}{\partial t} - az \frac{\partial}{\partial x} - bz \frac{\partial}{\partial y} - cz \frac{\partial}{\partial z}, \\
X_6 &= 3ct \frac{\partial}{\partial t} - 3(bz - cy) \frac{\partial}{\partial y} + cey \frac{\partial}{\partial u}, \\
X_7 &= 2ct \frac{\partial}{\partial t} - (az - cx) \frac{\partial}{\partial x} + (bz - cy) \frac{\partial}{\partial y} - cu \frac{\partial}{\partial u}, \\
X_{F_1} &= F_1(t) \frac{\partial}{\partial u}, \\
X_{F_2} &= F_2(z) \frac{\partial}{\partial u},
\end{aligned} \tag{4.13}$$

where F_1 and F_2 are arbitrary functions of t and z respectively.

4.3 Exact solutions of equation (4.8)

In this section we obtain different types of exact solutions of the NLEE (4.8).

4.3.1 Invariant solutions under the symmetries $X_1, \dots, X_4$

We will now explore various types of solutions emanating from the translation symmetries.

Hyperbolic solution of equation (4.8)

Using the time and space translations $X_1, \dots, X_4$ we have the group invariant

$$U(p) = u(t, x, y, z), \quad p = \alpha x + \beta y + \nu z - \gamma t, \tag{4.14}$$

where α, β, ν and γ are arbitrary constants. Using (4.14), equation (4.8) is reduced to the fourth-order ODE

$$(e\alpha^2 - a\alpha\gamma - b\beta\gamma - c\nu\gamma)U'' - 6\beta\alpha^2U''U' - d\beta\alpha^3U'''' = 0. \quad (4.15)$$

Integrating (4.15) once with respect to p gives

$$C_0 + (e\alpha^2 - a\alpha\gamma - b\beta\gamma - c\nu\gamma)U' - 3\beta\alpha^2U'^2 - d\beta\alpha^3U''' = 0, \quad (4.16)$$

where C_0 is an integration constant. Multiplying equation (4.16) by U'' and integrating once with respect to p leads to the second-order ODE

$$C_1 + C_0U' + AU'^2 - BU'^3 - EU''^2 = 0, \quad (4.17)$$

where $A = (e\alpha^2 - a\alpha\gamma - b\beta\gamma - c\nu\gamma)/2$, $B = \beta\alpha^2$, and $E = 1/2d\beta\alpha^3$.

Solution of (4.8) using direct integration

Now letting $Q(p) = U'(p)$, the resultant ODE (4.17) becomes

$$C_1 + C_0Q + AQ^2 - BQ^3 - EQ'^2 = 0. \quad (4.18)$$

Then,

$$Q'^2 = \frac{C_1}{E} + \frac{C_0Q}{E} + \frac{AQ^2}{E} - \frac{BQ^3}{E},$$

Suppose that w_1, w_2 and w_3 are real roots ($w_1 > w_2 > w_3$) of a cubic equation

$$Q^3 - \frac{AQ^2}{B} - \frac{C_0Q}{B} - \frac{C_1}{B} = 0. \quad (4.19)$$

and satisfies the conditions

$$w_1w_2w_3 = \frac{C_1}{B}, \quad w_1w_2 + w_1w_3 + w_2w_3 = -\frac{C_0}{B}, \quad w_1 + w_2 + w_3 = \frac{A}{B}.$$

Then equation (4.19) is written as

$$Q'^2 = -\frac{B}{E}(Q - w_1)(Q - w_2)(Q - w_3),$$

and has the solution

$$Q(p) = w_2 + (w_1 - w_2)\text{cn}^2 \left\{ \sqrt{\frac{B(w_1 - w_3)}{4E}} (p - p_0) \middle| X^2 \right\}, X^2 = \frac{w_1 - w_2}{w_1 - w_3}, \quad (4.20)$$

where p_0 is a constant and cn is the Jacobi cosine function. Thus by returning to the original variables we obtain the solution of (4.8) as

$$u(t, x, y, z) = \mathcal{S}_0[\text{EllipticE} \{ \text{sn}(\mathcal{S}_1(p - p_0), X^2), X^2 \}] + \left\{ w_2 - (w_1 - w_2) \frac{1 - X^4}{X^4} \right\} (p - p_0) + k_1 \quad (4.21)$$

with

$$\mathcal{S}_0 = \sqrt{\frac{4E(w_1 - w_2)^2}{(w_1 - w_3)BX^8}}, \quad \mathcal{S}_1 = \sqrt{\frac{B(w_1 - w_2)}{4E}},$$

where k_1 is a constant $p = \alpha x + \beta y + \nu z - \gamma t$.

Solution of (4.8) when C_0 and $C_1 = 0$

Now letting $U'(p) = V(p)$ and letting C_0 and $C_1 = 0$, the resultant ODE gives

$$V(p) = -\frac{A}{B} \left[\left\{ \tanh \left(\frac{\sqrt{EA}(C_3 - p)}{2E} \right) \right\}^2 - 1 \right], \quad (4.22)$$

where C_3 is an integration constant. Recall that $U(p) = \int V(p)dp$, consequently, the analytic solutions of ODE (4.15) are hyperbolic

$$\begin{aligned} u(t, x, y, z) = & -\frac{2EA}{B\sqrt{EA}} \tanh \left\{ \frac{\sqrt{EA}(C_3 - p)}{2E} \right\} - \frac{EA}{B\sqrt{EA}} \ln \left\{ \tanh \left(\frac{\sqrt{EA}(C_3 - p)}{2E} \right) - 1 \right\} \\ & + \frac{EA}{B\sqrt{EA}} \ln \left(\tanh \left\{ \frac{\sqrt{EA}(C_3 - p)}{2E} \right\} + 1 \right) + \frac{A}{B}p + \mathcal{S}_0, \end{aligned} \quad (4.23)$$

where independent variable $p = \alpha x + \beta y + \nu z - \gamma t$ and $\mathcal{S}_0$ as an integration constant .

Thus by extension, the solution of (4.23) is indeed a solution of the (3+1)-D KP-like equation (4.8).

4.3.2 Exact solutions of (4.8) using the Kudryashov method

In this section we invoke the Kudryashov method [14] to determine exact solutions of (4.8).

We begin by assuming that the solutions to the fourth-order ODE (4.15) can be written in the form

$$U(p) = \sum_{i=0}^M A_i Y^i(p), \quad (4.24)$$

where $Y(p)$ satisfies the Riccati equation

$$Y'(p) = Y^2(p) - Y(p), \quad (4.25)$$

whose solution is

$$Y(p) = \frac{1}{1 + e^p}. \quad (4.26)$$

The value of M in (4.24) can be determined by using the balancing procedure [14] and A_i $i = 0, 1, \dots, M$ are constants which we will determine. We balance the highest order derivative with the nonlinear term, that is, $U''''(p)$ and $U(p)'U''(p)$ respectively. This gives $M = 1$. Thus the solution (4.24) can be written as

$$U(z) = Y_0 + A_1 Y(z). \quad (4.27)$$

Substituting (4.27) into (4.15) and invoking (4.25) we obtain the following equation in $Q(z)$

$$\begin{aligned} & -2aa_1\alpha\gamma Y^3(p) + 3aa_1\alpha\gamma Y^2(p) - aa_1\alpha\gamma Y(p) - 2ba_1\beta\gamma Y^3(p) \\ & + 3ba_1\beta\gamma Y^2(p) - ba_1\beta\gamma Y(p) - 2ca_1\nu\gamma Y^3(p) + 3ca_1\nu\gamma Y^2(p) \\ & - ca_1\nu\gamma Y(p) - 24da_1\beta\alpha^3 Y^5(p) + 60da_1\beta\alpha^3 Y^4(p) - 50da_1\beta\alpha^3 Y^3(p) \\ & + 15da_1\beta\alpha^3 Y^2(p) - da_1\beta\alpha^3 Y(p) - 12a_1^2\alpha^2\beta Y^5(p) + 30a_1^2\alpha^2\beta Y^4(p) \\ & - 24a_1^2\alpha^2\beta Y^3(p) + 6a_1^2\alpha^2\beta Y^2(p) + 2ea_1\alpha^2 Y^3(p) - 3ea_1\alpha^2 Y^2(p) \\ & + ea_1\alpha^2 Y(p) = 0 \end{aligned} \quad (4.28)$$

Equating the coefficients of like powers of $Y(p)$ in equation (4.28) we obtain the following five algebraic equations in terms of a_0 and a_1

$$Y^5(p) : 2da_1\beta\alpha^3 + a_1^2\alpha^2\beta = 0,$$

$$Y^4(p) : da_1\beta\alpha^3 + a_1^2\alpha^2\beta = 0,$$

$$Y^3(p) : 25da_1\beta\alpha^3 + 12a_1^2\alpha^2\beta + aa_1\alpha\gamma - ea_1\alpha^2 + ba_1\beta\gamma + ca_1\nu\gamma = 0,$$

$$\begin{aligned}
Y^2(p) : 5 da_1 \beta \alpha^3 + 2 a_1^2 \alpha^2 \beta + a a_1 \alpha \gamma - e a_1 \alpha^2 + b a_1 \beta \gamma + c a_1 \nu \gamma &= 0, \\
Y^1(p) : 25 \gamma^3 \lambda a_1 + 16 \gamma^2 \lambda a_1^2 + 8 \gamma \lambda^2 a_1^2 - 4 \gamma \nu a_1 &= 0.
\end{aligned} \tag{4.29}$$

The solution of these equations is

$$\begin{aligned}
a_0 &= a_0, \\
a_1 &= \frac{2(a\alpha\gamma - \alpha^2e + b\beta\gamma + c\gamma\nu)}{\alpha^2\beta}, \\
d &= \frac{\alpha^2e - a\alpha\gamma - b\beta\gamma - c\gamma\nu}{\alpha^3\beta}.
\end{aligned} \tag{4.30}$$

Thus ,the solution of (4.8) reads

$$u(t, x, y, z) = a_0 + \frac{2 a \alpha \gamma - 2 \alpha^2 e + 2 b \beta \gamma + 2 c \gamma \nu}{\alpha^2 \beta (1 + e^{\alpha x + \beta y - \gamma t + \nu z})}. \tag{4.31}$$

4.3.3 Invariant solutions under symmetry X_5

The characteristic equations of

$$X_5 = ct \frac{\partial}{\partial t} - az \frac{\partial}{\partial x} - bz \frac{\partial}{\partial y} - cz \frac{\partial}{\partial z}$$

yield the invariant solution $G(p, q, k) = u(t, x, y, z)$, where

$$p = tz, \quad q = cx - az, \quad k = bx - ay.$$

Substituting the value of u into (4.8) produces

$$\begin{aligned}
&cpG_{pp} + cG_p + ac^3dG_{qqkq} + 3abc^2dG_{qkkq} + 3ab^2cdG_{kkkq} + ab^3dG_{kkkk} + 3ac^2G_kG_{qq} \\
&+ 9abcG_kG_{kq} + 6ab^2G_kG_{kk} + 3ac^2G_qG_{kq} + 3abcG_qG_{kk} + c^2eG_{qq} + 2bceG_{kq} \\
&+ b^2eG_{kk} = 0.
\end{aligned} \tag{4.32}$$

Equation (4.32) have five symmetries, namely;

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial q}, \quad \Gamma_2 = \frac{\partial}{\partial k}, \quad \Gamma_3 = \frac{\partial}{\partial G}, \quad \Gamma_4 = \ln p \frac{\partial}{\partial G}, \\
\Gamma_5 &= 3acp \frac{\partial}{\partial p} + (3ack - 3abq) \frac{\partial}{\partial k} + (beq - cek) \frac{\partial}{\partial G}.
\end{aligned}$$

Characteristic equations from combination of translation symmetries, $\Gamma = \Gamma_1 + \gamma\Gamma_2$, where γ is a constant yields the invariant solutions: $U(g) = G(p, q, k)$ where $g = k - \gamma q$. Now substituting the invariant into equation (4.32), yields ODE

$$(3abc^2d\gamma^2 + ab^3d - 3ab^2cd\gamma - ac^3d\gamma^3)U'''' + (6ac^2\gamma^2 - 12abc\gamma + 6ab^2)U''U' + (c^2e\gamma^2 + b^2e - 2bce\gamma)U'' = 0, \quad (4.33)$$

which can be written as

$$\mathcal{E}U''U' + \mathcal{F}U'' + \mathcal{Z}U'''' = 0, \quad (4.34)$$

where

$$\begin{aligned} \mathcal{Z} &= 3abc^2d\gamma^2 + ab^3d - 3ab^2cd\gamma - ac^3d\gamma^3 \\ \mathcal{E} &= 6ac^2\gamma^2 - 12abc\gamma + 6ab^2 \\ \mathcal{F} &= c^2e\gamma^2 + b^2e - 2bce\gamma \end{aligned}$$

4.3.4 Solution of (4.8) by direct integration.

In this section, we seek to find the general solution of a (3+1)-dimensional KP-like equation (4.8), we utilise NLODE (4.34). Integrating (4.34) with respect to g yield

$$\frac{\mathcal{E}}{2}U'^2 + \mathcal{F}U' + \mathcal{Z}U''' + P_0 = 0, \quad (4.35)$$

where P_0 is an arbitrary constant of integration. Let $\rho(g) = U'(g)$, then equation (4.35) becomes

$$\frac{\mathcal{E}}{2}\rho^2 + \mathcal{F}\rho + \mathcal{Z}\rho'' + P_0 = 0. \quad (4.36)$$

To integrate (4.36), we first multiply by $\rho'(g)$ to get

$$\frac{\mathcal{E}}{2}\rho^2\rho' + \mathcal{F}\rho\rho' + \mathcal{Z}\rho''\rho' + P_0\rho' = 0. \quad (4.37)$$

Integrating (4.37) with respect to g yield

$$\frac{\mathcal{Z}}{2}\rho'^2 + \frac{\mathcal{E}}{6}\rho^3 + \frac{\mathcal{F}}{2}\rho^2 + P_0\rho + P_1 = 0, \quad (4.38)$$

where P_1 is an arbitrary constant. Then,

$$\rho'^2 + \frac{\mathcal{E}}{3\mathcal{Z}}\rho^3 + \frac{\mathcal{F}}{\mathcal{Z}}\rho^2 + \frac{2}{\mathcal{Z}}P_0\rho + \frac{2}{\mathcal{Z}}P_1 = 0. \quad (4.39)$$

Suppose that r_1, r_2 and r_3 are real roots ($r_1 > r_2 > r_3$) of a cubic equation

$$\rho^3 + \frac{3\mathcal{F}}{\mathcal{E}}\rho^2 + \frac{6}{\mathcal{E}}P_0\rho + \frac{6}{\mathcal{E}}P_1 = 0. \quad (4.40)$$

and satisfies the conditions

$$r_1 r_2 r_3 = -\frac{6}{\mathcal{E}}P_1, \quad r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{6}{\mathcal{E}}P_0, \quad r_1 + r_2 + r_3 = -\frac{3\mathcal{F}}{\mathcal{E}}.$$

Then equation (4.40) is written as

$$\rho'^2 = -\frac{\mathcal{E}}{3\mathcal{Z}}(\rho - r_1)(\rho - r_2)(\rho - r_3),$$

and has the solution

$$\rho(r) = r_2 + (r_1 - r_2)\text{cn}^2 \left\{ \sqrt{\frac{\mathcal{E}(r_1 - r_3)}{12\mathcal{Z}}} (g - g_0) | K^2 \right\}, \quad K^2 = \frac{r_1 - r_2}{r_1 - r_3}, \quad (4.41)$$

where g_0 is a constant and cn is the Jacobi cosine function. Thus by returning to the original variables we obtain the solution of (4.8) as

$$u(t, x, y, z) = \mathcal{F}_0[\text{EllipticE} \{ \text{sn}(\mathcal{F}_1(g - g_0), K^2), K^2 \}] + \left\{ r_2 - (r_1 - r_2) \frac{1 - K^4}{K^4} \right\} (g - g_0) + k_1, \quad (4.42)$$

where

$$\mathcal{F}_0 = \sqrt{\frac{12\mathcal{Z}(r_1 - r_2)^2}{(r_1 - r_3)\mathcal{E}K^8}}, \quad \mathcal{F}_1 = \sqrt{\frac{\mathcal{E}(r_1 - r_2)}{12\mathcal{Z}}}$$

with k_1 as a constant and $g = bx - ay - \gamma(cx - az)$.

4.3.5 Invariant solutions under symmetry X_6

From the symmetry

$$X_6 = 3ct \frac{\partial}{\partial t} - 3(bz - cy) \frac{\partial}{\partial y} + cey \frac{\partial}{\partial u},$$

we get the invariants

$$p = x, \quad q = z, \quad k = (cy - bz)/t, \quad F(p, q, k) = u - \frac{bz \ln(bz - cy)}{3c} + \frac{e(bz - cy)}{3c}$$

and hence the group invariant solution is

$$u(t, x, y, z) = \frac{bz \ln(bz - cy)}{3c} - \frac{e(bz - cy)}{3c} + F(p, q, k). \quad (4.43)$$

Substituting (4.43) into equation (4.8) results in the PDE

$$ak^2 F_{kp} + ck^2 F_{kq} + cdk F_{ppkp} + bq F_{pp} + 3ck F_{pp} F_k + 3ck F_p F_{kp} = 0. \quad (4.44)$$

Equation (4.44) has the following operators as its Lie symmetries:

$$\begin{aligned} U_1 &= \frac{\partial}{\partial p} + F^1(q) \frac{\partial}{\partial F}, \\ U_2 &= 3c \frac{\partial}{\partial q} - \{b \ln k - F^2(q)\} \frac{\partial}{\partial F}, \\ U_3 &= (2aq - cp) \frac{\partial}{\partial p} - cq \frac{\partial}{\partial q} + 4ck \frac{\partial}{\partial k} + \{cF + cF^3(q)\} \frac{\partial}{\partial F}, \end{aligned} \quad (4.45)$$

where F^1 , F^2 and F^3 are arbitrary functions of their arguments. Now, taking F^1 , F^2 and F^3 equal to 0, we obtain the simplified Lie symmetries for equation (4.44) as:

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial p}, \\ Y_2 &= 3c \frac{\partial}{\partial q} - b \ln k \frac{\partial}{\partial F}, \\ Y_3 &= (2aq - cp) \frac{\partial}{\partial p} - cq \frac{\partial}{\partial q} + 4ck \frac{\partial}{\partial k} + cF \frac{\partial}{\partial F}. \end{aligned} \quad (4.46)$$

We use operator Y_2 to perform reductions on (4.44) which has the invariant

$$W(i, j) = F(p, q, k) - \frac{bq \ln k}{3c}, \quad (4.47)$$

where $i = p$ and $j = k$. Substituting (4.47) into (4.44) yields PDE

$$3ajW_{ij} - b + 3cdW_{iiij} + 9cW_{ii}W_j + 9cW_iW_{ij} = 0. \quad (4.48)$$

The above equation has the following operator as its Lie symmetry

$$\begin{aligned} W_1 &= \frac{\partial}{\partial i}, \\ W_2 &= 3c \frac{\partial}{\partial j} - ai \frac{\partial}{\partial W}, \\ W_3 &= \frac{\partial}{\partial W}. \end{aligned} \quad (4.49)$$

We use operator W_2 to perform reductions on (4.48) which has the invariant

$$N(s) = W(i, j) - \frac{aij}{3c}, \quad (4.50)$$

where $s = i$. Substituting (4.50) into (4.48) yields an ODE

$$b + 3asN'' + 3aN' = 0, \quad (4.51)$$

whose solution is

$$N(s) = C_1 \ln(s) - \frac{bs}{3a} + C_2, \quad (4.52)$$

where C_1 and C_2 are arbitrary constants of integration. Hence, the group-invariant solution of (4.8) under X_6 is

$$u(t, x, y, z) = \frac{bz \ln(bz - cy)}{3c} - \frac{e(bz - cy)}{3c} + C_1 \ln(x) - \frac{bx}{3a} + C_2 - \frac{bz}{3c} \ln\left(\frac{cy - bz}{t}\right).$$

4.3.6 Invariant solutions under symmetry X_7

The use of symmetry

$$X_7 = 2ct \frac{\partial}{\partial t} - (az - cx) \frac{\partial}{\partial x} + (bz - cy) \frac{\partial}{\partial y} - cu \frac{\partial}{\partial u},$$

gives the invariant solution as

$$U(p, q, k) = u(t, x, y, z) t^{\frac{1}{2}}, \quad (4.53)$$

where $p = z$, $q = (cx - az)/\sqrt{t}c$ and $k = \sqrt{t}(cy - bz)/c$. Substituting (4.53) into (4.8) gives the reduced PDE

$$2eU_{qq} - cU_p - 2dU_{qqkq} - 6U_{qq}U_k - 6U_qU_{kq} - cqU_{pq} + ckU_{kp} = 0. \quad (4.54)$$

Equation (4.54) has the following operators as its Lie symmetries:

$$\begin{aligned} Q_1 &= \frac{\partial}{\partial p}, \quad Q_2 = \frac{\partial}{\partial U}, \\ Q_3 &= 9p \frac{\partial}{\partial p} + 3q \frac{\partial}{\partial q} + (ek - 3U) \frac{\partial}{\partial U}, \\ Q_4 &= -3q \frac{\partial}{\partial q} + 9k \frac{\partial}{\partial k} + (2ek + 3U) \frac{\partial}{\partial U}. \end{aligned} \quad (4.55)$$

Case 1. We use operator Q_3 to perform reductions on (4.54) which has the invariant

$$H(i, j) = p^{\frac{1}{3}} U(p, q, k) - \frac{1}{3} p^{\frac{1}{3}} ek, \quad (4.56)$$

where $i = \frac{q}{p^{\frac{1}{3}}}$ and $j = k$. Substituting (4.56) into (4.54) yields

$$cH + 3ciH_i - 6dH_{iii} - 18H_{ii}H_j - 18H_iH_{ij} + ci^2H_{ii} - cjH_j - cijH_{ij} = 0. \quad (4.57)$$

The above equation has the following operator as its Lie symmetry

$$G = -i \frac{\partial}{\partial i} + 3j \frac{\partial}{\partial j} + H \frac{\partial}{\partial H}. \quad (4.58)$$

We use operator G to perform reductions on (4.57) which has the invariant

$$F(r) = iH(i, j), \quad (4.59)$$

where $r = ji^3$. Substituting (4.59) into (4.57) yields an ODE

$$\begin{aligned} 18FF'' - 54r^2dF'''' - 216rdF''' + (2rc - 120d)F'' + 2cF' - 108rF'F'' \\ - 72FF' = 0. \end{aligned} \quad (4.60)$$

Case 2. We consider symmetry Q_4 to find the solution of equation (4.8)

$$Q_4 = -3q \frac{\partial}{\partial q} + 9k \frac{\partial}{\partial k} + (2ek + 3U) \frac{\partial}{\partial U}. \quad (4.61)$$

We use operator Q_4 to perform reduction on (4.54) which has the invariant

$$H(i, j) = qU(p, q, k) - \frac{1}{3} ekq, \quad (4.62)$$

where $i = p$ and $j = kq^3$. Substituting the invariant (4.62) in the equation (4.54) yields the PDE

$$18H_{jj}H - 54dj^2H_{jjj} - 108H_{jj}jH_j - 216dj - 120dH_{jj} - 36H_j^2 - 2cH_{ij} = 0. \quad (4.63)$$

Equation (4.63) has the following operators as its Lie symmetries

$$T_1 = \frac{\partial}{\partial i}, \quad T_2 = i \frac{\partial}{\partial i} + j \frac{\partial}{\partial j}. \quad (4.64)$$

We use operator T_2 to perform reduction on (4.63) which has the invariant

$$W(r) = H(i, j), \quad (4.65)$$

where $r = \frac{j}{i}$. Substituting this invariant into (4.63) we get ODE

$$\begin{aligned} 2cW' - 54dr^2W'''' + 2crW'' + 216rdW''' - 108W''W'r - 120dW'' - 36W'^2 \\ + 18W''W = 0. \end{aligned} \quad (4.66)$$

Remarks

In this section we determined a nine-dimensional Lie algebra, which contained, amongst others, two-infinite dimensional Lie Algebras. From the other seven vector fields we obtained group invariant solutions which gave us hyperbolic tangent, Jacobian elliptic sine, logarithmic functions and an elliptic integral of the second kind. The presence of elliptic and Jacobian functions suggests that these solutions are general and other functions can be deduced from them. As alluded to earlier, Jacobi elliptic functions can transform into standard functions depending on the behaviour of the parameter S^2 .

4.4 Conservation laws of the (3+1)-D KP-like equation (4.8)

We now turn our attention to computing conservation laws of the (3+1)-D KP-like equation (4.8) using the multiplier approach. To find multipliers of equation (4.8) we need to solve the equation

$$\frac{\delta}{\delta u} (\Lambda E) = 0, \quad (4.67)$$

where

$$E \equiv au_{tx} + bu_{ty} + cu_{tz} - du_{xxx} - 3u_xu_{xy} - 3u_{xx}u_y + eu_{xx} = 0,$$

parameter Λ takes the dependencies $(t, x, y, z, u, u_x, u_y, u_z)$ since in this work we aim to calculate first-order multipliers (4.8) and the Euler-Lagrange operator $\delta/\delta u$ in this instance is given by

$$\begin{aligned}\frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_t D_y \frac{\partial}{\partial u_{ty}} + D_t D_z \frac{\partial}{\partial u_{tz}} \\ & + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^3 D_y \frac{\partial}{\partial u_{xxx y}}.\end{aligned}$$

The total derivatives D_t , D_x , D_y and D_z are as given in (4.12). Multiplier determining equation (4.67) produces the nine linear PDEs

$$\begin{aligned}\Lambda_x &= 0, \Lambda_{t,y} = 0, \Lambda_x = 0, \Lambda_u = 0, \Lambda_{t,u_x} = 0, \Lambda_{y,u_x} = 0, \Lambda_{z,u_z} = 0, \Lambda_{u_x,u_x} = 0, \\ b\Lambda_{t,y} + c\Lambda_{t,z} &= 0.\end{aligned}$$

Solving this system gives the three multipliers

$$\Lambda_1 = u_x, \Lambda_2 = F(z, y), \Lambda_3 = F(t, bz - cy). \quad (4.68)$$

By evaluating the divergence identity

$$D_t T + D_x X + D_y Y + D_z Z = E\Lambda \quad (4.69)$$

with density T and spatial fluxes X, Y and Z we compute the following conservation laws:

$$\begin{aligned}T_1 &= \frac{1}{4}au_x^2 - \frac{1}{4}auu_{xx} - \frac{1}{4}buu_{xy} - \frac{1}{4}cuu_{xz} + \frac{1}{4}cu_xu_z + \frac{1}{4}bu_xu_y, \\ X_1 &= -\frac{3}{2}u_x^2u_y + \frac{1}{2}eu_x^2 - \frac{1}{8}du_{xxx}u_y + \frac{3}{8}du_{xx}u_{xy} - \frac{5}{8}du_xu_{xxy} + \frac{1}{2}buu_{ty} \\ &\quad + \frac{1}{2}cuu_{tz} + \frac{1}{4}au_tu_x + \frac{1}{4}au_{tx} - \frac{1}{8}duu_{xxx y}, \\ Y_1 &= \frac{1}{8}duu_{xxx} - \frac{1}{4}buu_{tx} - \frac{1}{2}u_x^3 - \frac{1}{4}du_xu_{xxx} + \frac{1}{8}u_{xx}^2 + \frac{1}{4}bu_tu_x, \\ Z_1 &= \frac{1}{4}cu_tu_x - \frac{1}{4}cuu_{tx};\end{aligned}$$

$$\begin{aligned}T_2 &= \frac{1}{2}au_xF(z, y) - \frac{1}{2}cuF_z(z, y) - \frac{1}{2}buF_y(z, y) + \frac{1}{2}cu_zF(z, y) + \frac{1}{2}bu_yF(z, y), \\ X_2 &= \frac{3}{4}uu_xF_y(z, y) + \frac{3}{4}uu_{xy}F(z, y) - \frac{9}{4}u_xu_yF(z, y) + \frac{1}{4}du_{xx}F_y(z, y)\end{aligned}$$

$$\begin{aligned}
& + eu_x F(z, y) + \frac{1}{2} au_t F(z, y) - \frac{3}{4} du_{xy} F(z, y), \\
Y_2 &= \frac{1}{2} bu_t F(z, y) - \frac{3}{4} u_x^2 F(z, y) - \frac{3}{4} uu_{xx} F(z, y) - \frac{1}{4} du_{xxx} F(z, y) \\
Z_2 &= \frac{1}{2} cu_t F(z, y);
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{1}{2} (au_x + bu_y + cu_z) F(t, \omega), \\
X_3 &= \frac{3}{4} uu_{xy} F(t, bz - cy) - \frac{3}{4} cuu_x F_\omega(t, \omega) - \frac{1}{2} au F_t(t, \omega) \\
& - \frac{9}{4} u_x u_y F(t, \omega) - \frac{1}{4} cdu_{xx} F_\omega(t, \omega) + \frac{1}{2} au_t F(t, \omega) \\
& - \frac{3}{4} du_{xy} F(t, \omega) + eu_x F(t, \omega), \\
Y_3 &= \frac{1}{2} bu_t F(t, \omega) - \frac{1}{2} bu F_t(t, \omega) - \frac{3}{4} u_x^2 F(t, \omega) \\
& - \frac{3}{4} uu_{xx} F(t, \omega) - \frac{1}{4} du_{xxx} F(t, \omega), \\
Z_3 &= \frac{1}{2} cu_t F(t, \omega) - \frac{1}{2} cu F_t(t, \omega),
\end{aligned}$$

where $\omega = bz - cy$.

4.4.1 Conclusion

In this chapter we studied a (3+1)-D KP-like equation (4.8). We determined the Lie symmetries of this equation and symmetry reductions and reduced it to a ordinary differential equation (ODE) (4.17). Thereafter, the ODE (4.17) was solved by direct integration and its solution was expressed in terms of the incomplete elliptic integral. Secondly we considered solution of ODE (4.17) when C_0 and $C_1 = 0$ which yielded hyperbolic tangent and natural logarithmic functions. These are considered to be amongst the most general functions in Mathematics since they decompose to trigonometric or hyperbolic functions depending on certain parameter values. Furthermore, Kudrayshov's method was used to obtain the solution of the ODE (4.17). Lastly, the multiplier method using the first order was used to calculate the conservation laws of equation (4.8). Three multipliers were produced via the multiplier approach, and three local conservation laws were obtained for the equation (4.8).

Chapter 5

Concluding remarks

Amongst the plethora of real-world physical phenomena encountered such as in fluid mechanics, material science, elasticity, thermodynamics, biology, gas dynamics to mentioned a few are best described by nonlinear partial differential equations, thus studying such systems is crucial with a view to finding their exact solutions and conservation laws. Thus, in this dissertation we have studied two nonlinear partial differential equations namely generalised simplified Camassa-Holm(CH) equation and (3+1)-dimensional KP-like equation.

In chapter one, we presented some mathematical concepts of Lie group analysis and some algorithms for computing conservation laws.

In Chapter two, we presented an illustrative example of this work. We considered the potential Burgers equation and computed it's Lie point symmetries, used them to construct a one parameter groups of transformation and group-invariant solutions. Moreover, we derived the conservation laws of the equation using the multiplier approach.

In the third chapter we studied the generalized simplified modified Camassa Holm (gsmCH) equation. Reductions alongside Simplest Equation techniques, Kudryashov's method and direct integration method were invoked in achieving exact solutions of the equation. In addition, conservation laws were also derived using the multiplier

method, the theorem due to Ibragimov and Noether's theorem.

In Chapter four, the (3+1)-dimensional KP-like equation was investigated in which we had to find the symmetry reductions, group-invariant solutions and different approaches like kudryashov method, simplest equation just to mention a few were used to find solutions. Thereafter, the multiplier technique for conservation laws were implored in the construction of the conservation laws.

In future work, we plan to study variety of higher-order partial differential equations which have arbitrary functions.

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