Symmetry analysis and solutions of the generalized (2 + 1)-dimensional Calogero-Bogoyavlenskii-Schiff equation and the (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equation

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Contents

Declaration

I Motshidisi Charity Moroke student number 21443122, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Ms M. C. MOROKE

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this dissertation.

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MC Moroke, B Muatjetjeja, AR Adem, Solutions and Conservation laws for a (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation. To be submitted.

Dedication

To my mum and dad for their continuous support. To my husband for always believing in me and constantly encouraging me to grow.

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List of Acronyms

DE:	Differential equation
PDE:	Partial differential equation
NLPDE:	Nonlinear partial differential equation
NLEE:	Nonlinear evolution equation
KdV:	Korteweg-de Vries
Kdv1:	(3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation
CBS:	(2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation

Abstract

In this dissertation we study three nonlinear partial differential equations namely the Korteweg-de Vries (KdV) equation as an illustrative example, a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation and the (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation. We invoke the multiplier approach to obtain conservation laws. In addition, we use the classical symmetry method to obtain group invariant solutions for the aforementioned equations.

Introduction

Nonlinear partial differential equations (NLPDEs) have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics and geochemistry. Thus, it is important to investigate the exact solutions of NLPDEs. In general it is not easy to determine exact solutions of these equations and rarely can one write down their solutions explicitly.

Conservation laws [1–3] have been used to determine the existence, uniqueness, stability of solutions and exact solutions of DEs. Steudel [3] introduced a different approach of constructing conservation laws, that involves writing a conserved vector in a characteristic form, where the characteristics are the multipliers of the differential equation. Conservation laws play a remarkable role in the study of differential equations. The mathematical idea of conservation laws comes from the formulation of well-known physical conserved quantities such as mass, momentum and energy. Finding the conservation laws of differential equations is often the first step towards finding the exact solutions. Thus, it is essential to study conservation laws of partial differential equations (PDEs).

In the last few decades, a variety of effective methods for finding exact solutions, such as homogeneous balance method [4], the ansatz method [5, 6], variable separation approach [7], inverse scattering transform method [8], Backlund transformation [9], Darboux transformation [10] and Hirota's bilinear method [11] were successfully applied to NLPDEs.

In this dissertation we consider the Korteweg-de Vries (KdV) equation as an illustrative example, a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation and the (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation. Firstly, we consider the Korteweg-de Vries equation of the form

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

as an illustrative example.

Secondly, we study a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation [12] that is given by

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \tag{2}$$

where a and b are arbitrary constants and $u = u(t, x, y)$ denotes the wave profile with t, x and y representing time and space variables respectively. The (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff is a typical so-called breaking soliton equation describing the (2+1)-dimensional interaction of a Riemann wave propagation along the y-axis with a long wave along the x-axis.

Lastly, we consider a (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation [13]

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z) = 0, \tag{3}$$

where $u = u(t, x, y, z)$ denotes the wave profile.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the multiplier method and an illustrative example using multiplier approach to construct conservation laws of the Korteweg-de Vries equation are presented.

In Chapter two, the multiplier method will be employed to derive conservation laws of a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation. Furthermore, the classical symmetry method will be used to construct Exact Solution of (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation.

In Chapter three, the conservation laws of a (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation are obtained by employing the multiplier approach. In addition, we derive the group invariant solutions of the (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli equation.

In Chapter four, a summary of the results of the dissertation is presented.

A bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter, we present some basic methods on how to obtain conservation laws and Lie point symmetries of differential equations, which will be utilized in this dissertation. We further derive the conservation laws for the Korteweg-de Vries equation via multiplier approach.

1.1 Fundamental relation on multiplier and Lie point symmetries

In this section, we present the notation that will be used to construct conservation laws of (1) by the multiplier method [14]. We will also give some basic definitions on Lie point symmetries [15, 16].

Consider a kth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the *total* *derivative operator* with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n. \tag{1.2}$$

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{1.3}$$

The n-tuple vector $\mathbf{T} = (T^1, T^2, \ldots, T^n), \quad T^j \in \mathcal{A}, \quad j = 1, \ldots, n$, is a *conserved vector* of (1.1) if T^i satisfies

$$D_i T^i|_{(1.1)} = 0. \tag{1.4}$$

The equation (1.4) defines a *local conservation law* of system (1.1).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.5}$$

that holds identically. Here we will consider multipliers of the zeroth order, i.e., $\Lambda_\alpha = \Lambda_\alpha(t, x, u)$. The right hand side of (1.5) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.6}$$

Definition 1.1 (Point symmetry) The vector field

$$\mathbf{X} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \tag{1.7}$$

is said to be a *point symmetry* of the kth-order partial differential equation (1.1), if

$$\mathbf{X}^{[k]}(E_\alpha) = 0, \tag{1.8}$$

whenever $E_\alpha = 0$. This can also be written as

$$\mathbf{X}^{[k]} E_\alpha \big|_{E_\alpha = 0} = 0, \tag{1.9}$$

where the symbol $|_{E_\alpha = 0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.2 (Generator) Consider the vector field

$$\mathbf{X} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{1.10}$$

which has the second-order prolongation

$$\mathbf{X}^{[2]} = \mathbf{X} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}, \tag{1.11}$$

where

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{tt} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),\\
\zeta_{tx} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),\\
\zeta_{xx} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).
\end{aligned}$$

and

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots,
\end{aligned}$$

1.2 An illustrative example on finding conservation laws via multiplier approach

In this section we will employ the multiplier method to construct the first-order multiplier of the Korteweg-de Vries equation. Thereafter, we will derive the associated conservation laws of the KdV equation.

Consider the Korteweg-de Vries equation of the form

$$u_t + uu_x + u_{xxx} = 0 \tag{1.12}$$

and take the first order multiplier $\Lambda(t, x, u, u_t, u_x)$. Then from equation (1.6), we have

$$\frac{\delta}{\delta u}[\Lambda(t, x, u, u_t, u_x)(u_t + uu_x + u_{xxx})] = 0, \tag{1.13}$$

where the Euler-Langrage operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} \\ &\quad + D_xD_t\frac{\partial}{\partial u_{xt}} + \cdots \end{aligned} \tag{1.14}$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots.$$

Expanding equation (1.13) leads to

$$\left[\frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} - D_tD_x^2\frac{\partial}{\partial u_{txx}} + D_x^4\frac{\partial}{\partial u_{xxxx}}\right]$$
$$(\Lambda(u_t + uu_x + u_{xxx})) = 0,$$

$$\Lambda_u(u_t + uu_x + u_{xxx}) + u_x\Lambda - D_t(\Lambda_{u_t}(u_t + uu_x + u_{xxx}) + \Lambda)$$

$$-D_x(\Lambda_{u_x}(u_t + uu_x + u_{xxx}) + u\Lambda) - D_x^3(\Lambda) = 0.$$

Since Λ depends only on t, x, u, u_t and u_x, the coefficients of the like derivatives can be equated to zero to yield the following system of overdetermined linear partial differential equations:

$$\Lambda_{u_t} = 0, \tag{1.15}$$

$$\Lambda_{u_x} = 0, \tag{1.16}$$

$$\Lambda_t = -u\Lambda_x, \tag{1.17}$$

$$\Lambda_{uu} = 0, \tag{1.18}$$

$$\Lambda_{ux} = 0, \tag{1.19}$$

$$\Lambda_{xx} = 0. \tag{1.20}$$

Integrating equations (1.15) and (1.16) with respect to u_t and u_x, respectively gives

$$\Lambda = b(t, x, u). \tag{1.21}$$

Differentiating equation (1.21) with respect to u twice, we obtain

$$\Lambda_{uu} = b_{uu}. \tag{1.22}$$

Then equation (1.18) implies that

$$b_{uu} = 0. \tag{1.23}$$

Integrating (1.23) twice with respect to u, we obtain

$$b = uc(t, x) + d(t, x). \tag{1.24}$$

Then equation (1.21) becomes

$$\Lambda = uc(t, x) + d(t, x). \tag{1.25}$$

Again differentiating equation (1.25) twice with respect to x, we obtain

$$\Lambda_{xx} = uc_{xx} + d_{xx}. \tag{1.26}$$

Now using equation (1.20), we get

$$uc_{xx} + d_{xx} = 0. \tag{1.27}$$

Splitting the above equation with respect to u yields

$$u \quad : \quad c_{xx} = 0, \tag{1.28}$$

$$1 \quad : \quad d_{xx} = 0. \tag{1.29}$$

Integrating equation (1.28) and (1.29) respectively, with respect to x gives

$$c(t, x) = xe(t) + f(t), \tag{1.30}$$

and

$$d(t, x) = xg(t) + h(t). \tag{1.31}$$

Therefore equation (1.25) yields the following

$$\Lambda = xue(t) + uf(t) + xg(t) + h(t). \tag{1.32}$$

Again by differentiating equation (1.32) with respect to t and x, we obtain

$$\Lambda_t = xue_t + uf_t + xg_t + h_t, \tag{1.33}$$

$$\Lambda_x = ue + g. \tag{1.34}$$

Substituting equation (1.33) and (1.34) into (1.17), yields

$$xue_t + uf_t + xg_t + h_t + u^2 e + ug = 0. \tag{1.35}$$

Splitting the above equation with respect to the powers of u, we get

$$u^2 \quad : \quad e = 0, \tag{1.36}$$

$$u \quad : \quad xe_t + f_t + g = 0, \tag{1.37}$$

$$1 \quad : \quad xg_t + h_t = 0. \tag{1.38}$$

Again, splitting (1.37) and (1.38) respectively with respect to x, gives

$$x \quad : \quad e_t = 0, \tag{1.39}$$

$$1 \quad : \quad f_t + g = 0, \tag{1.40}$$

and

$$x \quad : \quad g_t = 0, \tag{1.41}$$

$$1 \quad : \quad h_t = 0. \tag{1.42}$$

Now, simplifying the above equations, yields

$$g = k_1, \tag{1.43}$$

$$f = -tk_1 + k_2, \tag{1.44}$$

$$h = k_3. \tag{1.45}$$

Substituting (1.43), (1.44) and (1.45) into (1.32), we get the following multiplier

$$\Lambda = (x - ut)k_1 + uk_2 + k_3. \tag{1.46}$$

We now recall equation (1.5), viz.,

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.47}$$

to construct the conservation laws of equation (1.12). Employing the above equation (1.47), we get

$$\begin{aligned}
&[xk_1 - tuk_1 + uk_2 + k_3](u_t + uu_x + u_{xxx}) \\
&= \left[\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{ttx}\frac{\partial}{\partial u_{tx}}\right] T^1(t, x, u, u_x, u_{xx}) \\
&+ \left[\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{txx}\frac{\partial}{\partial u_{tx}}\right] T^2(t, x, u, u_x, u_{xx}).
\end{aligned}$$

Simplifying the above equation gives

$$\begin{aligned}
&[xk_1 - tuk_1 + uk_2 + k_3](u_t + uu_x + u_{xxx}) = \\
&T^1_t + u_t T^1_u + u_{tx} T^1_{u_x} + u_{txx} T^1_{u_{xx}} + T^2_x + u_x T^2_u + u_{xx} T^2_{u_x} + u_{xxx} T^2_{u_{xx}}.
\end{aligned} \tag{1.48}$$

Splitting equation (1.48) on u_t, u_{tx}, u_{txx} and u_{xxx} yields

$$u_t \; : \; T^1_u = xk_1 - tuk_1 + uk_2 + k_3, \tag{1.49}$$

$$u_{tx} \; : \; T^1_{u_x} = 0, \tag{1.50}$$

$$u_{txx} \; : \; T^1_{u_{xx}} = 0, \tag{1.51}$$

$$u_{xxx} \; : \; T^2_{u_{xx}} = xk_1 - tuk_1 + uk_2 + k_3, \tag{1.52}$$

$$1 \; : \; T^1_t + T^2_x + u_x T^2_u + u_{xx} T^2_{u_x} = k_1 uu_x(x - tu) + uu_x(uk_2 + k_3). \tag{1.53}$$

We can now solve the above equations for T^1 and T^2. From equation (1.49), we obtain

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + A(t, x, u_x, u_{xx}), \tag{1.54}$$

where $A(t, x, u_x, u_{xx})$ is an arbitrary function of t, x, u_x and u_{xx}. Using equation (1.50) and equation (1.54), we get

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + B(t, x, u_{xx}), \tag{1.55}$$

where $B(t, x, u_{xx})$ is an arbitrary function of t, x and u_{xx}. Differentiating equation (1.55) with respect to u_{xx}, gives

$$T^1_{u_{xx}} = B_{u_{xx}}. \tag{1.56}$$

Equating the above equation to zero using equation (1.51), we obtain

$$B_{u_{xx}} = 0. \tag{1.57}$$

Integrating equation (1.57) with respect to u_{xx}, yields

$$B = C(t, x). \tag{1.58}$$

Thus equation (1.55) becomes

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + C(t, x), \tag{1.59}$$

where $C(t, x)$ is an arbitrary function of t and x only. The integration of equation (1.52) with respect to u_{xx}, yields

$$T^2(t, x, u, u_x, u_{xx}) = xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 + D(t, x, u, u_x), \tag{1.60}$$

where $D(t, x, u, u_x)$ is an arbitrary function of t, x, u and u_x. Differentiating equation (1.59) with respect to t and equation (1.60) with respect to x, u and u_x respectively and substituting the resulting derivatives into equation (1.53), we obtain

$$\begin{aligned}&-\frac{u^2k_1}{2} + C_t + u_{xx}k_1 + D_x - tu_xu_{xx}k_1 + u_xu_{xx}k_2 + u_xD_u + u_{xx}D_{u_x} = xuu_xk_1\\ &-tu^2u_xk_1 + u^2u_xk_2 + uu_xk_3.\end{aligned} \tag{1.61}$$

Splitting the above equation with respect to u_{xx}, yields

$$u_{xx} \;:\; k_1 - tu_xk_1 + u_xk_2 + D_{u_x} = 0, \tag{1.62}$$

$$1 \;:\; -\frac{u^2k_1}{2} + C_t + D_x + u_xD_u = k_1uu_x(x - tu) + uu_x(uk_2 + k_3). \tag{1.63}$$

Equation (1.62), implies that

$$D = -u_xk_1 + \frac{tu_x^2k_1}{2} - \frac{u_x^2k_2}{2} + E(t, x, u). \tag{1.64}$$

Since $D_x = E_x$ and $D_u = E_u$, then equation (1.63), implies that

$$\begin{aligned} C(t,x) &= xtuu_xk_1 - \frac{t^2u^2u_xk_1}{2} + tu^2u_xk_2 + tuu_xk_3 + \frac{tu^2k_1}{2} \\ &\quad - \int E_x dt - u_x \int E_u dt. \end{aligned} \tag{1.65}$$

Therefore equations (1.59) and (1.60) reduce to

$$\begin{aligned} T^1(t,x,u,u_x,u_{xx}) &= xuk_1 + \frac{u^2k_2}{2} + uk_3 + xtuu_xk_1 - \frac{t^2u^2u_xk_1}{2} \\ &\quad + tu^2u_xk_2 + tuu_xk_3 - \int E_x dt - u_x \int E_u dt, \qquad (1.66) \\ T^2(t,x,u,u_x,u_{xx}) &= xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 \\ &\quad + \frac{tu_x^2k_1}{2} - \frac{u_x^2k_2}{2} + E. \qquad (1.67) \end{aligned}$$

Now differentiating equation (1.66) with respect to u_x and using equation (1.50), we obtain

$$\int E_u dt = txuk_1 - \frac{t^2u^2k_1}{2} + tu^2k_2 + tuk_3. \tag{1.68}$$

Simplifying equation (1.68), yields

$$E(t,x,u) = \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F(t,x). \tag{1.69}$$

Now we have

$$\begin{aligned} E_x &= \frac{u^2k_1}{2} + F_x, \qquad (1.70) \\ \int E_u dt &= txuk_1 - \frac{t^2u^2k_1}{2} + tu^2k_2 + tuk_3, \qquad (1.71) \\ \int E_x dt &= \frac{tu^2k_1}{2} + \int F_x dt + G(x,u). \qquad (1.72) \end{aligned}$$

Back substitution of equations (1.69)-(1.72) into (1.66) and (1.67) respectively, we obtain

$$\begin{aligned} T^1(t,x,u,u_x,u_{xx}) &= xuk_1 + \frac{u^2k_2}{2} + uk_3 - \frac{tu^2k_1}{2} - \int F_x dt - G, \qquad (1.73) \\ T^2(t,x,u,u_x,u_{xx}) &= xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 + \frac{tu_x^2k_1}{2} \\ &\quad - \frac{u_x^2k_2}{2} + \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F. \qquad (1.74) \end{aligned}$$

Differentiating equation (1.73) with respect to u, gives

$$T_u^1 = xk_1 + uk_2 + k_3 - tuk_1 - G_u. \tag{1.75}$$

Using equation (1.49), equation (1.75) can be written as

$$xk_1 + uk_2 + k_3 - tuk_1 - G_u = xk_1 - tuk_1 + uk_2 + k_3. \tag{1.76}$$

Now equation (1.76), becomes

$$G_u = 0. \tag{1.77}$$

Equation (1.77) implies that

$$G = I(x). \tag{1.78}$$

Then equations (1.73) and (1.74), become

$$T^1 = xuk_1 + \frac{u^2k_2}{2} + uk_3 - \frac{tu^2k_1}{2} - \int F_x dt - I, \tag{1.79}$$

$$\begin{aligned} T^2 = {} & xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 + \frac{tu_x^2k_1}{2} \\ & - \frac{u_x^2k_2}{2} + \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F. \end{aligned} \tag{1.80}$$

Therefore the components of the conserved vectors associated with the above multiplier (1.46) are

$$\begin{aligned} T_1^t &= xu - \frac{tu^2}{2}, \\ T_1^x &= xu_{xx} - tuu_{xx} - u_x + \frac{tu_x^2}{2} + \frac{xu^2}{2} - \frac{tu^3}{3}, \\ T_2^t &= \frac{u^2}{2}, \\ T_2^x &= uu_{xx} - \frac{u_x^2}{2} + \frac{u^3}{3}, \\ T_3^t &= u, \\ T_3^x &= u_{xx} + \frac{u^2}{2}, \end{aligned}$$

respectively.

Chapter 2

Exact solutions and conservation laws of a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation

2.1 Introduction

In this chapter we study a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \tag{2.1}$$

where a and b are arbitrary constants and $u = u(t, x, y)$ denotes the brevity. The CBS equation was initially constructed by Bogoyavlenskii and Schiff in different ways [17–19]. Several authors have studied this equation for various values of the parameters a and b. See for example [20–22].

When setting $a = 4$ and $b = 2$, Eq. (2.1) becomes the (2+1)-dimesional Calogero-Bogoyavlenskii-Schiff equation, namely

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0. \tag{2.2}$$

When setting $a = -4$ and $b = -2$, Eq. (2.1) becomes the (2+1)-dimesional breaking soliton equation [23], viz.,

$$u_{xt} - 4u_x u_{xy} - 2u_{xx} u_y + u_{xxxy} = 0. \tag{2.3}$$

When setting $a = 4$ and $b = 4$, Eq. (2.1) becomes the (2+1)-dimesional Bogoyavlenskii's breaking soliton equation [24], namely

$$u_{xt} + 4u_x u_{xy} + 4u_{xx} u_y + u_{xxxy} = 0. \tag{2.4}$$

In [12] Mohammed used the modified simple equation method to construct exact solutions of a generalized (2 + 1)-dimensional nonlinear evolution equation (2.1).

The objective of this work is twofold. Firstly we seek to establish new exact solution of the (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation (2.1) using the Lie symmetry method. Thereafter, we aim to derive local conservation laws of equation (2.1) using the invariance and multiplier approach based on the well known results that the Euler-Lagrange operator annihilates the total divergence.

2.2 Symmetry analysis of (2.1)

In this section we obtain the Lie point symmetries of the Calagero-Bogoyavlenskii-Schiff equation (2.1).

Consider the vector field

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u}, \tag{2.5}$$

where ξ^i,i= 1, 2, 3 and η depend on t, x, y and u, is a generator of Lie point symmetries of the (2+1)-dimensional Bogoyavlenskii-schiff equation (2.1) if and only if

$$X^{[4]}(u_{xt} + au_x u_{xy} + bu_{xx} u_y + u_{xxxy})|_{(2.1)} = 0. \tag{2.6}$$

Here $X^{[4]}$ is the fourth prolongation of the vector field X. The invariance condition (2.6) yields the determining equations, which are systems of linear partial differential equations, namely

$$
\begin{aligned}
\xi^2_y &= 0,\\
\xi^2_u &= 0,\\
\xi^1_y &= 0,\\
\xi^3_x &= 0,\\
\xi^1_x &= 0,\\
\xi^3_u &= 0,\\
\xi^1_u &= 0,\\
\eta_{yu} &= 0,\\
\eta_{uu} &= 0,\\
\xi^2_{xx} &= 0,\\
\eta_{xt} &= 0,\\
\eta_{xu} &= 0,\\
\eta_u + \xi^2_x &= 0,\\
b\eta_y - \xi^2_t &= 0,\\
a\eta_x - \xi^3_t &= 0,\\
a\eta_{yx} - \xi^2_{xt} + \eta_{tu} &= 0,\\
\xi^3_y - \xi^1_t - \eta_u + \xi^2_x &= 0,\\
\xi_x &= 0,\\
\xi_y &= 0,\\
\xi_t &= 0,\\
\eta_u &= 0.
\end{aligned}
$$

Solving the above systems of equation, prompts the following cases

Case 1. a and b arbitrary but not in the form in Cases 2-3

This case gives us seven symmetries, viz.,

$$
\begin{aligned}
X_1 &= -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u},\\
X_2 &= y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t},\\
X_3 &= \frac{\partial}{\partial y},\\
X_4 &= at\frac{\partial}{\partial y} + x\frac{\partial}{\partial u},\\
X_5 &= \frac{\partial}{\partial t},\\
X_6 &= bF(t)\frac{\partial}{\partial x} - yF'\frac{\partial}{\partial u},\\
X_7 &= G(t)\frac{\partial}{\partial u}.
\end{aligned}
$$

Case 2. $a = 2b$

Here we obtain eight symmetries, namely

$$
\begin{aligned}
X_1 &= -\frac{\partial}{\partial x}x - 2\frac{\partial}{\partial t}t + \frac{\partial}{\partial u}u,\\
X_2 &= \frac{\partial}{\partial y}y + \frac{\partial}{\partial t}t,\\
X_3 &= (btu - xy)\frac{\partial}{\partial u} - \frac{\partial}{\partial x}xtb - 2\frac{\partial}{\partial y}ytb - 2\frac{\partial}{\partial t}bt^2,\\
X_4 &= \frac{\partial}{\partial y},\\
X_5 &= 2\frac{\partial}{\partial y}tb + \frac{\partial}{\partial u}x,\\
X_6 &= \frac{\partial}{\partial t},\\
X_7 &= \frac{\partial}{\partial x}F_{10}(t)b + \frac{\partial}{\partial u}F_{10}'y,\\
X_8 &= \frac{\partial}{\partial u}F_{19}(t).
\end{aligned}
$$

Case 3. $a = b$

In this case we get seven symmetries, viz.,

$$
\begin{aligned}
X_1 &= -\frac{\partial}{\partial x}x - 2\frac{\partial}{\partial t}t + \frac{\partial}{\partial u}, \\
X_2 &= \frac{\partial}{\partial y}y + \frac{\partial}{\partial t}t, \\
X_3 &= \frac{\partial}{\partial y}, \\
X_4 &= \frac{\partial}{\partial y}ta + \frac{\partial}{\partial u}x, \\
X_5 &= \frac{\partial}{\partial t}, \\
X_6 &= \frac{\partial}{\partial x}F_{10}(t)a - \frac{\partial}{\partial u}F_{10}^{'}y, \\
X_7 &= \frac{\partial}{\partial u}F_{19}(t).
\end{aligned}
$$

2.2.1 Symmetry reduction and exact solutions of (2.1)

In this subsection we compute exact solutions and reductions of (2.1). Firstly, we will consider the symmetries obtained in Case 1. We shall first begin with X_4. The characteristics equations are

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{at} = \frac{du}{x}.$$

Solving the above characteristics equation we get the following three invariants

$$f = x, g = t, \theta = \frac{uat - xy}{at}. \tag{2.7}$$

Now considering θ as the new dependent variable and g and f as new independent variables, (2.1) transforms to a nonlinear PDE in two independent variables, viz.,

$$bf\theta_{ff} + ag\theta_{gf} + \theta_f = 0. \tag{2.8}$$

Solving equation (2.8) yields

$$\theta\left(f,g\right) = C\left(g\right) + \int E\left(gf^{-\frac{a}{b}}\right)f^{-\frac{a}{b}}\,\mathrm{d}f, \tag{2.9}$$

where C and E are arbitrary functions of g and f. Now using the invariants (2.7) together with equation (2.9), we conclude that the group-invariant solution of equation

(2.1) for arbitrary a and b is

$$u(t,x,y)=\frac{1}{at}\Bigg(atC\left(t\right)+at\int E\left(tx^{-\frac{a}{b}}\right)x^{-\frac{a}{b}}\,\mathrm{d}x+xy\Bigg). \tag{2.10}$$

Now choosing X_1, the associated Lagrange system is

$$\frac{-dt}{2t}=\frac{-dx}{x}=\frac{dy}{0}=\frac{du}{u}. \tag{2.11}$$

Solving the above Lagrange system yields the three invariants, namely

$$f=y, g=\frac{x}{\sqrt{t}}, \theta=u\sqrt{t}. \tag{2.12}$$

Now treating θ as the new dependent variable and g and f as new independent variables, (2.1) reduces to

$$2b\theta_f\theta_{gg}+2a\theta_{fg}\theta_g-g\theta_{gg}+2\theta_{fggg}-2\theta_g=0. \tag{2.13}$$

When finding the Lie symmetries of equation (2.13), we get

$$\begin{aligned}
\Upsilon_1 &= 2b\frac{\partial}{\partial g}-f\frac{\partial}{\partial\theta},\\
\Upsilon_2 &= 2f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}+\theta\frac{\partial}{\partial\theta},\\
\Upsilon_3 &= \frac{\partial}{\partial f},\\
\Upsilon_4 &= \frac{\partial}{\partial\theta}.
\end{aligned}$$

The use of the combination $\Upsilon=\Upsilon_3+p\Upsilon_4$ (p is a constant), gives us the following lagrange system

$$\frac{d\theta}{p}=\frac{df}{1}=\frac{dg}{0}. \tag{2.14}$$

Thus equation (2.14) gives the following two invariants

$$r=g, \phi=-fp+\theta. \tag{2.15}$$

Invoking the above invariants (2.15), equation (2.13) reduces to

$$(2bp-r)\frac{d^2\phi}{dr^2}-2\frac{d\phi}{dr}=0. \tag{2.16}$$

The intergration of the above equation and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1):

$$u(t,x,y) = \frac{2\,C1\,bp - C1\,y - C2}{t\,(2\,bp - y)}, \tag{2.17}$$

where C_1 and C_2 are abitrary constants of intergration.

Considering X_2, we obtain the following associated Lagrange system

$$\frac{dt}{t} = \frac{dx}{0} = \frac{dy}{y} = \frac{du}{0}. \tag{2.18}$$

When solving the above Lagrange system yields the three invariants, namely

$$f = x, g = \frac{y}{t}, \phi = u. \tag{2.19}$$

Now treating ϕ as the new dependent variable and g and f as new independent variables, (2.1) reduces to

$$a\phi_f\phi_{gf} + b\phi_{ff}\phi_g - g\phi_{gf} + \phi_{fffg} = 0. \tag{2.20}$$

When finding the Lie symmetries of equation (2.20), we get

$$\begin{aligned}
\Sigma_1 &= \frac{\partial}{\partial f},\\
\Sigma_2 &= f\frac{\partial}{\partial f} + \phi\frac{\partial}{\partial \phi},\\
\Sigma_3 &= \frac{\partial}{\partial \phi},\\
\Sigma_4 &= a\frac{\partial}{\partial g} + \frac{\partial}{\partial \phi}.
\end{aligned}$$

The use of Σ_4 gives us the following lagrange system

$$\frac{d\phi}{f} = \frac{df}{0} = \frac{dg}{a}. \tag{2.21}$$

Thus, the above system gives the following two invariants

$$s = f, \theta = \frac{a\phi - fg}{a}. \tag{2.22}$$

Invoking the above invariants, equation (2.20) reduces to

$$bs\theta^{''} + a\theta^{'} = 0. \tag{2.23}$$

The intergration of equation (2.23) and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1):

$$u(t,x,y) = \frac{C_2 x^{-\frac{a}{b}+1} ta + C_1 ta + xy}{ta}, \tag{2.24}$$

where C_1 and C_2 are intergration constants.

We now consider X_1 and the associated characteristic equations are

$$\frac{-dt}{2t} = \frac{-dx}{x} = \frac{dy}{0} = \frac{du}{u}. \tag{2.25}$$

Solving equation (2.25) , we obtain the following three invariants

$$f = y, g = \frac{x}{\sqrt{t}}, \theta = u\sqrt{t}. \tag{2.26}$$

Now considering θ as the new dependent variable and g and f as new independent variables, equation (2.1) transforms to

$$2b\theta_f \theta_{gg} + 2a\theta_{fg}\theta_g - g\theta_{gg} + 2\theta_{fggg} - 2\theta_g = 0. \tag{2.27}$$

When finding the Lie symmetries of equation (2.27), we get

$$\begin{aligned}
\Omega_1 &= 2b\frac{\partial}{\partial g} - f\frac{\partial}{\partial \theta},\\
\Omega_2 &= 2f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} + \theta\frac{\partial}{\partial \theta},\\
\Omega_3 &= \frac{\partial}{\partial f},\\
\Omega_4 &= \frac{\partial}{\partial \theta}.
\end{aligned}$$

Now choosing Ω_2 gives us the following lagrange system

$$\frac{d\theta}{\theta} = \frac{df}{2f} = \frac{dg}{-g}. \tag{2.28}$$

Solving the above Lagrange system yields the two invariants, namely

$$r = fg^2, \phi = g\theta. \tag{2.29}$$

Now treating ϕ as the new dependent variable and r as the new independent variable, equation (2.27) reduces to

$$4ar^2\phi^{''}\phi^{'} + 4br^2\phi^{''}\phi^{'} + 8r^3\phi^{''''} - 2ar\phi\phi^{''} + 2ar\phi^{'2} - 2br\phi^{'2}$$
$$+24r^2\phi^{'''} - 2r^2\phi^{''} - a\phi\phi^{'} + 2b\phi\phi^{'} + 6r\phi^{''} - r\phi^{'} = 0. \quad (2.30)$$

Consequently, the general group invariant solution of equation (2.1) is

$$u = \frac{\sqrt{t}}{x}\phi\left(\frac{yx^2}{t^2}\right), \quad (2.31)$$

where ϕ is any solution of the ordinary differential equation (2.30).

Now using X_2, we get the following Lagrange system

$$\frac{-dt}{t} = \frac{dx}{0} = \frac{dy}{y} = \frac{du}{0}. \quad (2.32)$$

The above system gives the following three invariants

$$f = x, g = \frac{y}{t}, \theta = u. \quad (2.33)$$

Invoking the above invariants, equation (2.1) reduces to

$$-a\theta_f\theta_{gf} - b\theta_{ff}\theta_g + g\theta_{gf} - \theta_{gfff} = 0. \quad (2.34)$$

When finding the Lie symmetries of equation (2.34), we get

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial f}, \\
\Gamma_2 &= f\frac{\partial}{\partial f} + \theta\frac{\partial}{\partial \theta}, \\
\Gamma_3 &= \frac{\partial}{\partial \theta}, \\
\Gamma_4 &= a\frac{\partial}{\partial g} + f\frac{\partial}{\partial \theta}.
\end{aligned}$$

When using Γ_2, the lagrange system is

$$\frac{d\theta}{\theta} = \frac{df}{f} = \frac{dg}{0}. \quad (2.35)$$

Solving the above Lagrange system yields the two invariants, namely

$$m = g, \phi = \frac{\theta}{f}. \quad (2.36)$$

If we treat ϕ as the new dependent variable and m as the new independent variable, equation (2.34) reduces to

$$a\phi\phi^{'} - m\phi^{'} + \phi^{'''} = 0. \tag{2.37}$$

Consequently, the general group invariant solution of equation (2.1) is

$$u(t, x, y) = x\phi\left(\frac{y}{t}\right), \tag{2.38}$$

where ϕ is any solution of the ordinary differential equation (2.37).

Secondly, we now consider the symmetries obtained in Case 2. We start with X_1. The Lagrange equations are

$$\frac{dt}{2t} = \frac{dx}{-x} = \frac{dy}{0} = \frac{du}{u}.$$

After solving the above Lagrange system, we obtain the following three invariants, namely

$$f = y, g = \frac{x}{\sqrt{t}}, \phi = u\sqrt{t}. \tag{2.39}$$

If we consider ϕ as the new dependent variable and f and g as new independent variables, (2.1) reduces to

$$4b\phi_g\phi_{gf} + 2b\phi_{gg}\phi_f + g\phi_{gg} + 2\phi_g + 2\phi_{gggf} = 0. \tag{2.40}$$

When finding the Lie symmetries of equation (2.40), we get

$$\begin{aligned}
\Delta_1 &= 2b\frac{\partial}{\partial g} - f\frac{\partial}{\partial \phi},\\
\Delta_2 &= 2f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + \phi\frac{\partial}{\partial \phi},\\
\Delta_3 &= \frac{\partial}{\partial f},\\
\Delta_4 &= \frac{\partial}{\partial \phi}.
\end{aligned}$$

The use of the combination $\Delta = \Delta_3 + p\Delta_4$ (p is a constant), gives us the following characteristic equation

$$\frac{d\phi}{p} = \frac{df}{1} = \frac{dg}{0}. \tag{2.41}$$

Solving the above characteristic equation yields the two invariants, namely

$$r = g, \theta = -fp + \phi. \tag{2.42}$$

Now treating θ as the new dependent variable and r as the new independent variable, equation (2.40) reduces to

$$pb\theta^{''} - \frac{1}{2}r\theta^{''} - \theta^{'} = 0. \tag{2.43}$$

The intergration of the above equation and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1):

$$u(t,x,y) = \frac{2\,C1\,bp - C1\,y - C2}{t\,(2\,bp - y)}, \tag{2.44}$$

where C_1 and C_2 are arbitrary constants of integration.

If we consider X_2, we obtain the following characteristic

$$\frac{dt}{t} = \frac{dx}{0} = \frac{dy}{y} = \frac{du}{0}.$$

The solution for the above Lagrange system is

$$f = x, g = \frac{y}{t}, \phi = u. \tag{2.45}$$

Invoking the above invariants, equation (2.1) reduces to

$$g\phi_{gf} - 2b\phi_f\phi_{gf} - b\phi_{ff}\phi_g - \phi_{gfff} = 0. \tag{2.46}$$

When finding the Lie symmetries of equation (2.46), we get

$$\begin{aligned}
\daleth_1 &= \frac{\partial}{\partial\phi}, \\
\daleth_2 &= 2b\frac{\partial}{\partial g} + f\frac{\partial}{\partial\phi}, \\
\daleth_3 &= \frac{\partial}{\partial f}, \\
\daleth_4 &= f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g} - \phi\frac{\partial}{\partial\phi}.
\end{aligned}$$

The use of $\daleth_2$, gives us the following Lagrange system

$$\frac{d\phi}{f} = \frac{df}{0} = \frac{dg}{2b}. \tag{2.47}$$

Solving the Lagrange system (2.47) yields the two invariants, namely

$$m = f, \theta = \frac{1}{2}\left(\frac{2b\phi - fg}{b}\right). \tag{2.48}$$

Now treating θ as the new dependent variable and m as the new independent variable, equation (2.46) reduces to

$$\theta^{'} - \frac{1}{2}m\theta^{''} = 0. \tag{2.49}$$

The intergration of the above equation and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1)

$$u(t, x, y) = \frac{1}{2}\left(\frac{2C_1btx + 2C_2bt + x^2y}{btx}\right). \tag{2.50}$$

where C_1 and C_2 are arbitrary constants of integration.

Now choosing $\urcorner_4$, gives us the following characteristic equation

$$\frac{d\phi}{\phi} = \frac{df}{f} = \frac{dg}{2g}. \tag{2.51}$$

Solving equation (2.51), we obtain the following two invariants, namely

$$m = f\sqrt{g}, \theta = \frac{\phi}{\sqrt{g}}. \tag{2.52}$$

We now consider θ as the new dependent variable and m as the new independent variable, equation (2.46) reduces to

$$3bm\theta^{'}\theta^{''} + b\theta\theta^{''} + 4b\theta^{''} - m\theta^{''} + m\theta^{''''} - 2\theta^{'} + 4\theta^{'''} = 0. \tag{2.53}$$

Thus, we conclude that

$$u(t, x, y) = \sqrt{\frac{y}{t}}\theta\left(x\sqrt{\frac{y}{t}}\right), \tag{2.54}$$

is a general group invariant solution of equation (2.1), where θ is any solution of equation (2.53).

Now we consider X_1, the Lagrange system is

$$\frac{dt}{-2t} = \frac{dx}{-x} = \frac{dy}{0} = \frac{du}{u}. \tag{2.55}$$

Solving the Lagrange system (2.55) yields the three invariants, namely

$$f = y, g = \frac{x}{\sqrt{t}}, \phi = u\sqrt{t}. \tag{2.56}$$

Taking ϕ as the new dependent variable and f and g as new independent variables,(2.1) reduces to

$$2\phi_g - 4b\phi_g\phi_{gf} - 2b\phi_{gg}\phi_f + g\phi_{gg} - 2\phi_{gggf} = 0. \tag{2.57}$$

When finding the Lie symmetries of equation (2.57), we get

$$\begin{aligned}
\daleth_1 &= 2b\frac{\partial}{\partial g} + f\frac{\partial}{\partial \phi}, \\
\daleth_2 &= 2f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} + \phi\frac{\partial}{\partial \phi}, \\
\daleth_3 &= \frac{\partial}{\partial f}, \\
\daleth_4 &= \frac{\partial}{\partial \phi}.
\end{aligned}$$

The use of $\daleth_2$, gives us the following Lagrange system

$$\frac{d\phi}{\phi} = \frac{df}{2f} = \frac{dg}{-g}. \tag{2.58}$$

Solving equation (2.58) yields the two invariants, namely

$$r = fg^2, \theta = g\phi. \tag{2.59}$$

Invoking the above invariants, equation (2.57) reduces to

$$12br\theta^{'}\theta^{''} + 8r^2\theta^{''''} - 4b\theta\theta^{''} + 2b\theta^{'2} + 24r\theta^{'''} - 2r\theta^{''} - \theta^{'} + 6\theta^{''} = 0. \tag{2.60}$$

Therefore, we conclude that

$$u(t, x, y) = \frac{1}{x}\theta\left(\frac{yx^2}{t}\right), \tag{2.61}$$

is a general group invariant solution of equation where ϕ is any solution of (2.60)

Lastly, we now consider the symmetries obtained in Case 3. We begin with X_2. The Lagrange equations are

$$\frac{dy}{y} = \frac{dt}{t} = \frac{du}{0} = \frac{dx}{0}. \tag{2.62}$$

When solving the Lagrange equation (2.62), we obtain these three invariants

$$f = x, g = \frac{y}{t}, \psi = u. \tag{2.63}$$

Now treating ψ as the new dependent variable and f and g as new independent variables, (2.1) reduces to

$$g\psi_{fg} - a\psi_f\psi_{fg} - a\psi_{ff}\psi_g - \psi_{fffg} = 0. \tag{2.64}$$

When finding the Lie symmetries of equation (2.64), we get

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial \psi}, \\
\Xi_2 &= a\frac{\partial}{\partial g} + f\frac{\partial}{\partial \psi}, \\
\Xi_3 &= \frac{\partial}{\partial f}, \\
\Xi_4 &= f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g} - \psi\frac{\partial}{\partial \psi}.
\end{aligned}$$

If we consider Ξ_2, we get the following lagrange system

$$\frac{d\psi}{f} = \frac{df}{0} = \frac{dg}{a}. \tag{2.65}$$

Thus equation (2.65) gives the following two invariants

$$v = f, \lambda = \frac{a\psi - fg}{a}. \tag{2.66}$$

Considering λ as the new dependent variable and v as independent variables, (2.64) reduces to

$$v\lambda'' + \lambda' = 0. \tag{2.67}$$

Now intergrating and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1)

$$u(t, x, y) = \frac{atC_2 \ln x + atC_1 + xy}{at}, \tag{2.68}$$

where C_1 and C_2 are arbitrary constants of intergration.

Now choosing X_1, we get the following characteristic equations

$$\frac{dy}{0} = \frac{dt}{2t} = \frac{du}{u} = \frac{dx}{-x}. \tag{2.69}$$

Solving the characteristic equation (2.69) yields the three invariants, namely

$$f = y, g = \frac{x}{\sqrt{t}}, \psi = u\sqrt{t}. \tag{2.70}$$

Now if we treat ψ as the dependent variable and f and g as independent variables, (2.1) reduces to

$$g\psi_{gg} - 2a\psi_f\psi_{gg} - 2a\psi_{fg}\psi_g - 2\psi_{fggg} + g\psi_g = 0. \tag{2.71}$$

When finding the Lie symmetries of equation (2.71), we get

$$\begin{aligned}
G_1 &= 2a\frac{\partial}{\partial g} + f\frac{\partial}{\partial \psi}, \\
G_2 &= 2f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} + \psi\frac{\partial}{\partial \psi}, \\
G_3 &= \frac{\partial}{\partial f}, \\
G_4 &= \frac{\partial}{\partial \psi}.
\end{aligned}$$

The use of the combination $G = G_3 + dG_4$ (d is a constant), gives us the following Lagrange system

$$\frac{d\psi}{d} = \frac{df}{1} = \frac{dg}{0}. \tag{2.72}$$

The above Lagrange system yields the following two invariants:

$$v = g, \lambda = -fp + \psi. \tag{2.73}$$

If we consider λ as the dependent variable and v as the independent variable, then equation (2.71) reduces to

$$ap\lambda^{''} + \frac{1}{2}v\lambda^{''} - \lambda^{'} = 0. \tag{2.74}$$

The intergration of equation (2.74) and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1):

$$u(t, x, y) = \frac{2\,C1\,ap - C1\,y - C2}{t\,(2\,ap - y)}, \tag{2.75}$$

where C_1 and C_2 are arbitrary constants of integration.

Considering X_4, we get the following characteristic equations

$$\frac{dy}{ta} = \frac{dt}{0} = \frac{du}{x} = \frac{dx}{0}. \tag{2.76}$$

Solving (2.76) yields the three invariants, namely

$$f = x, g = t, \psi = \frac{aut - xy}{at}. \tag{2.77}$$

Invoking the above invariants, equation (2.1) reduces to

$$f\psi_{ff} - g\psi_{fg} + \psi_f = 0. \tag{2.78}$$

Now intergrating and reverting back to the original variables, we obtain the following group-invariant solution of equation (2.1):

$$u(t, x, y) = \frac{atF_2(t) + \left[\int \frac{F_1(\frac{t}{x})}{x} dx\right] at + xy}{at} \tag{2.79}$$

where F_1 and F_2 are arbitrary functions.

Remark 2.1 Likewise, one can construct more general group invariant solutions of the generalized (3+1) dimensional Calogaro-Bogoyavlenskii-Schiff equation using the other symmetries.

2.3 Conservation laws

In this section we construct the conservation laws of equation (2.1). We will consider multipliers of the second order $\Lambda(t, x, y, u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{tx}, u_{xy}, u_{ty})$. Thus equation (1.6) becomes

$$\frac{\delta}{\delta u}[\Lambda(t, x, y, u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xy}, u_{tx}, u_{ty})(u_{xt}+au_xu_{yx}+bu_{xx}u_y+u_{yxxx})] = 0, \tag{2.80}$$

where the Euler-Langrage Operator $\delta/\delta u$ is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}}$$

$$+D_y^2\frac{\partial}{\partial u_{yy}}+D_xD_t\frac{\partial}{\partial u_{xt}}+D_xD_y\frac{\partial}{\partial u_{xy}}+D_tD_y\frac{\partial}{\partial u_{ty}}+\cdots \tag{2.81}$$

and the total differential operators are defined by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+u_{ty}\frac{\partial}{\partial u_y}+\cdots,\\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+u_{xy}\frac{\partial}{\partial u_y}+\cdots.\\
D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{ty}\frac{\partial}{\partial u_t}+u_{xy}\frac{\partial}{\partial u_y}+\cdots.
\end{aligned}$$

Thus equation (2.80) gives

$$\begin{aligned}
&\Big(\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_t^2\frac{\partial}{\partial u_{tt}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}\\
&+D_xD_t\frac{\partial}{\partial u_{xt}}+D_xD_y\frac{\partial}{\partial u_{xy}}+D_tD_y\frac{\partial}{\partial u_{ty}}-D_yD_x^2\frac{\partial}{\partial u_{yxx}}+D_x^3D_y\frac{\partial}{\partial u_{yxxx}}\Big)\\
&(\Lambda(u_{xt}+au_xu_{yx}+bu_{xx}u_y+u_{yxxx}))=0.
\end{aligned}\tag{2.82}$$

The expansion of equation (2.82) and split the monomial and simplify yields

$$\Lambda_{u_{xt}}=-\Lambda_y(a-b),$$

$$\Lambda_{yy}=0,$$

$$\Lambda_{u_{xy}}=0,$$

$$\Lambda_{u_xu_x}=0,$$

$$\Lambda_x=0,$$

$$\Lambda_u=0,$$

$$\Lambda_{u_t}=0,$$

$$\Lambda_{u_y}=0,$$

$$\Lambda_{u_{tt}}=0.$$

Solving the above overdetermined systems of partial differential equations gives us the following cases for the multiplier Λ.

Case 1: a and b arbitrary but not in the form of Case 2-3

$$\Lambda_1=yF_2^{'}-u_x(a-b)F_2(t)+C_1u_x+F_3(t)$$

Case 2: $a = 2b$

$$\begin{aligned}\Lambda_2 &= \frac{1}{2}\Big[y^2F_6''(t) - 2yF_7' - 2yu_xF_6' + \left(\frac{3}{2}bu_x^2 + u_{xxx}\right)F_6 + F_7(t)u_x + F_8(t)\\ &\quad -2C_1bt^2u_t - 2C_1btyu_y - C_1btxu_x - C_1btu - 2C_3btu_y + 4C_2tu_t\\ &\quad +2C_2yu_y + C_1xy + C_2xu_x + C_2u + C_3x + C_4u_t + C_5u_y\Big]\end{aligned}$$

Case 3: $a = b$

$$\Lambda_1 = C_1u_x + F_2(t, y)$$

Using equation (1.5) on Λ_1 and solving the resulting determining equation, we obtain the following conserved vectors corresponding to C_1, F_2 and F_3 respectively

$$\begin{aligned}T_1^t &= -\frac{1}{4}\,uu_{xx} + \frac{1}{4}\,u_x^2,\\ T_1^x &= \frac{1}{3}auu_x\,u_{xy} - \frac{1}{3}\,buu_x\,u_{xy} + \frac{1}{3}\,bu_x{}^2u_y + \frac{1}{6}\,au_x^2\,u_y\\ &\quad +\frac{1}{4}\,uu_{tx} + \frac{1}{8}\,uu_{xxxy} + \frac{1}{4}\,u_t\,u_x + \frac{5}{8}\,u_x\,u_{xxy}\\ &\quad -\frac{3}{8}\,u_{xx}\,u_{xy} + \frac{1}{8}\,u_{xxx}\,u_y,\\ T_1^y &= -\frac{1}{8}\,uu_{xxxx} + \frac{1}{6}\,au_x{}^3 + \frac{1}{4}\,u_x\,u_{xxx} - \frac{1}{3}\,auu_x\,u_{xx} + \frac{1}{3}\,buu_x\,u_{xx} -\\ &\quad \frac{1}{8}\,u_{xx}{}^2;\\ T_2^t &= \frac{1}{4}\,aF_2\,(t)\,uu_{xx} - \frac{1}{4}\,bF_2uu_{xx} - \frac{1}{4}\,aF_2u_x{}^2 + \frac{1}{4}\,bF_2u_x{}^2\\ &\quad +\frac{1}{2}\,(F_2'\,(t))\,u_x\,y,\\ T_2^x &= -\frac{1}{4}\,F_2u_{xx} - \frac{1}{2}\,byF_2uu_{xy} + \frac{1}{4}\,ayF_2u_x\,u_y + \frac{1}{2}\,byF_2u_x\,u_y\\ &\quad -\frac{1}{3}\,a^2F_2uu_x\,u_{xy} - \frac{1}{3}\,b^2F_2uu_x\,u_{xy} - \frac{1}{6}\,abF_2u_x{}^2u_y\\ &\quad +\frac{1}{4}\,ayF_2uu_{xy} - \frac{1}{6}\,a^2F_2u_x{}^2u_y + \frac{1}{3}\,b^2F_2u_x{}^2u_y - \frac{3}{8}\,bF_2u_{xx}\,u_{xy}\\ &\quad -\frac{1}{8}\,aF_2u_{xxx}\,u_y + \frac{1}{8}\,bF_2u_{xxx}\,u_y - \frac{1}{4}\,bF_2'uu_x - \frac{1}{4}\,aF_2U\,u_{tx}\\ &\quad +\frac{1}{4}\,bF_2uu_{tx} - \frac{1}{8}\,aF_2uu_{xxxy} + \frac{1}{8}\,bF_2uu_{xxxy} - \frac{1}{4}\,aF_2u_t\,u_x\\ &\quad +\frac{1}{4}\,bF_2u_t\,u_x - \frac{5}{8}\,aF_2u_x\,u_{xxy} + \frac{5}{8}\,bF_2u_x\,u_{xxy} + \frac{3}{8}\,aF_2u_{xx}\,u_{xy}\\ &\quad +\frac{1}{2}\,yF_2'u_t + \frac{3}{4}\,yF_2'u_{xxy} - \frac{1}{2}\,ab(F_2'u_y + \frac{2}{3}\,F_2uu_x\,u_{xy})\end{aligned}$$

$$
\begin{aligned}
T_2^y &= \frac{1}{4}\,yF_2'u_{xxx} - \frac{2}{3}\,abF_2uu_x\,u_{xx} + \frac{1}{3}\,a^2F_2uu_x\,Uxx + \frac{1}{3}\,b^2F_2uu_x\,u_{xx} \\
&\quad -\frac{1}{4}\,ayF_2'uu_{xx} + \frac{1}{2}\,byF_2uu_{xx} - \frac{1}{6}\,a^2F_2{u_x}^3 + \frac{1}{8}\,aF_2{u_{xx}}^2 \\
&\quad -\frac{1}{8}\,bF_2{u_{xx}}^2 + \frac{1}{6}\,abF_2{u_x}^3 + \frac{1}{4}\,ayF_2{u_x}^2 + \frac{1}{8}\,aF_2uu_{xxxx} \\
&\quad -\frac{1}{8}\,bF_2uu_{xxxx} - \frac{1}{4}\,aF_2u_x\,u_{xxx} + \frac{1}{4}\,bF_2u_x\,u_{xxx}\,; \\
T_3^t &= \frac{1}{2}\,F_3u_x, \\
T_3^x &= \frac{1}{4}\,aF_3uu_{xy} - \frac{1}{2}\,bF_3uu_{xy} + \frac{1}{4}\,aF_3u_x\,u_y + \frac{1}{2}\,bF_3u_x\,u_y \\
&\quad +\frac{1}{2}\,F_3u_t + \frac{3}{4}\,F_3u_{xxy} - \frac{1}{2}\,F_3'u, \\
T_3^y &= \frac{1}{4}\,aF_3{u_x}^2 + \frac{1}{4}\,F_3u_{xxx} - \frac{1}{4}\,aF_3uu_{xx} + \frac{1}{2}\,bF_3uu_{xx}.
\end{aligned}
$$

Following the same procedure as above, we get the following conserved vectors for Λ_2

$$
\begin{aligned}
T_1^t &= -\frac{4}{3}\,b^2t^2uu_x\,u_{xy} - \frac{2}{3}\,b^2t^2uu_{xx}\,u_y - \frac{1}{2}\,bt^2uu_{tx} + \frac{1}{4}\,btxuu_{xx} \\
&\quad -bt^2u_{xxxy} + \frac{1}{2}\,btyuu_{xy} - \frac{1}{2}\,bt^2u_t\,u_x - \frac{1}{4}\,btx{u_x}^2 \\
&\quad -\frac{1}{2}\,btyu_x\,u_y + \frac{1}{4}\,btuu_x + \frac{1}{2}\,xyu_x - \frac{1}{2}\,u_y, \\
T_1^x &= b^2tuu_x\,u_y + bxyu_x\,u_y - btyu_{xxy}\,u_y - \frac{1}{3}\,b^2txuu_x\,u_{xy} \\
&\quad +\frac{2}{3}\,b^2tyuu_x, u_{yy} + \frac{2}{3}\,b^2tyuu_{xy}\,u_y - \frac{1}{2}\,bt^2{u_t}^2 + \frac{1}{2}\,xyu_t \\
&\quad +\frac{3}{4}\,xyu_{xxy} + \frac{1}{2}u_x + \frac{1}{4}\,bu^2 - \frac{1}{4}\,xu_{xx} - \frac{1}{2}\,yu_{xy} + \frac{2}{3}\,b^2t^2uu_{tx}\,u_y \\
&\quad +\frac{2}{3}\,b^2t^2uu_{ty}\,u_x - \frac{4}{3}\,b^2t^2u_t\,u_x\,u_y - \frac{2}{3}\,b^2tx{u_x}^2u_y \\
&\quad -\frac{4}{3}\,b^2tyu_x\,{u_y}^2 - \frac{1}{4}\,btxuu_{tx} + \frac{1}{2}\,btyuu_{ty} - \frac{1}{8}\,btxuu_{xxxy} \\
&\quad +\frac{3}{4}\,btyuu_{xxyy} - \frac{1}{4}\,btxu_t\,u_x - \frac{1}{2}\,btyu_t\,u_y - \frac{5}{8}\,btxu_x\,u_{xxy} \\
&\quad -\frac{1}{2}\,btyu_x\,u_{xyy} + \frac{3}{8}\,btxu_{xx}\,u_{xy} + \frac{1}{4}\,btyu_{xx}\,u_{yy} - \frac{1}{8}\,btxu_{xxx}\,u_y \\
&\quad +btuu_t + \frac{1}{2}\,bt^2uu_{tt} + \frac{3}{4}\,bt^2uu_{txxy} - \frac{3}{4}\,bt^2u_t\,u_{xxy} + \frac{1}{2}\,bt^2u_{tx}\,u_{xy} \\
&\quad -\frac{1}{4}\,bt^2u_{txx}\,u_y - \frac{1}{2}\,bt^2u_{txy}\,u_x + \frac{1}{4}\,bt^2u_{ty}\,u_{xx} + \frac{1}{2}\,bty{u_{xy}}^2 \\
&\quad -\frac{1}{4}\,bxuu_x + \frac{3}{2}\,btuu_{xxy} - \frac{1}{2}\,btu_x\,u_{xy} \\
T_1^y &= -\frac{1}{4}\,yu_{xx} + \frac{2}{3}\,b^2t^2uu_{tx}\,u_x - \frac{2}{3}\,b^2ty{u_x}^2u_y + \frac{1}{8}\,btxuu_{xxxx}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3}{4}\, btyuu_{xxxy} - \frac{1}{4}\, btxu_x\, u_{xxx} - \frac{1}{4}\, btyu_x\, u_{xxy} + \frac{1}{4}\, btyu_{xx}\, u_{xy} \\
& -\frac{1}{4}\, btyu_{xxx}\, u_y - \frac{1}{3}\, b^2txu_x{}^3 + \frac{1}{3}\, b^2tuu_x{}^2 + \frac{1}{4}\, bt^2uu_{txxx} \\
& -\frac{1}{4}\, bt^2u_t\, u_{xxx} + \frac{1}{4}\, bt^2u_{tx}\, u_{xx} - \frac{1}{4}\, bt^2u_{txx}\, u_x + \frac{1}{2}\, bxyu_x{}^2 \\
& +\frac{1}{8}\, btxu_{xx}{}^2 - \frac{1}{2}\, byuu_x + \frac{3}{8}\, btuu_{xxx} - \frac{1}{8}\, btu_x\, u_{xx} \\
& -\frac{2}{3}\, b^2t^2u_t\, u_x{}^2 + \frac{1}{4}\, xyu_{xxx} - btyuu_{tx} + \frac{1}{3}\, b^2txuu_x\, u_{xx} \\
& -\frac{2}{3}\, b^2tyuu_x\, u_{xy} - \frac{2}{3}\, b^2tyuu_{xx}\, u_y\, ; \\
T_2^t \;=\;& \frac{8}{3}\, btuu_x\, u_{xy} + \frac{4}{3}\, btuu_{xx}\, u_y + tuu_{tx} - \frac{1}{4}\, xuu_{xx} + 2\, tuu_{xxxy} \\
& -\frac{1}{2}\, yuu_{xy} + tu_t\, u_x + \frac{1}{4}\, xu_x{}^2 + \frac{1}{2}\, yu_x\, u_y - \frac{1}{4}\, uu_x, \\
T_2^x \;=\;& -buu_x\, u_y - tuu_{tt} - tu_{tx}\, u_{xy} + tu_{txy}\, u_x + yu_{xxy}\, u_y + \frac{2}{3}\, bxu_x{}^2u_y \\
& +\frac{4}{3}\, byu_x\, u_y{}^2 + \frac{1}{4}\, xuu_{tx} - \frac{3}{2}\, tuu_{txxy} - \frac{1}{2}\, yuu_{ty} + \frac{1}{8}\, xuu_{xxxy} \\
& -\frac{3}{4}\, yuu_{xxyy} + \frac{1}{2}\, yu_t\, u_y + \frac{1}{2}\, tu_{txx}\, u_y - \frac{1}{2}\, tu_{ty}\, u_{xx} + \frac{5}{8}\, xu_x\, u_{xxy} \\
& +\frac{1}{2}\, yu_x\, u_{xyy} - \frac{3}{8}\, xu_{xx}\, u_{xy} - \frac{1}{4}\, yu_{xx}\, u_{yy} + \frac{1}{8}\, xu_{xxx}\, u_y \\
& +\frac{3}{2}\, tu_t\, u_{xxy} + \frac{1}{4}\, xu_t\, u_x - \frac{4}{3}\, btuu_{tx}\, u_y - \frac{4}{3}\, btuu_{ty}\, u_x \\
& +\frac{1}{3}\, bxuu_x\, u_{xy} - \frac{2}{3}\, byuu_x\, u_{yy} - \frac{2}{3}\, byuu_{xy}\, u_y + \frac{8}{3}\, btu_t\, u_x\, u_y \\
& +tu_t{}^2 - uu_t - \frac{1}{2}\, yu_{xy}{}^2 - \frac{3}{2}\, uu_{xxy} + \frac{1}{2}\, u_x\, u_{xy}, \\
T_2^y \;=\;& -\frac{4}{3}\, btuu_{tx}\, u_x - \frac{1}{3}\, bxuu_x\, u_{xx} + \frac{2}{3}\, byuu_x\, u_{xy} + \frac{2}{3}\, byuu_{xx}\, u_y \\
& +\frac{4}{3}\, btu_t\, u_x{}^2 + \frac{1}{3}\, bxu_x{}^3 + \frac{2}{3}\, byu_x{}^2u_y - \frac{1}{3}\, buu_x{}^2 + yuu_{tx} \\
& -\frac{1}{2}\, tuu_{txxx} - \frac{1}{8}\, xuu_{xxxx} + \frac{3}{4}\, yuu_{xxxy} + \frac{1}{2}\, tu_t\, u_{xxx} - \frac{1}{2}\, tu_{tx}\, u_{xx} \\
& +\frac{1}{2}\, tu_{txx}\, u_x + \frac{1}{4}\, xu_x\, u_{xxx} + \frac{1}{4}\, yu_x\, u_{xxy} - \frac{1}{8}\, xu_{xx}{}^2 - \frac{1}{4}\, yu_{xx}\, u_{xy} \\
& +\frac{1}{4}\, yu_{xxx}\, u_y - \frac{3}{8}\, uu_{xxx} + \frac{1}{8}\, u_x\, u_{xx}; \\
T_3^t \;=\;& \frac{1}{2}\, btuu_{xy} - \frac{1}{2}\, btu_x\, u_y + \frac{1}{2}\, xu_x - \frac{1}{2}, \\
T_3^x \;=\;& \frac{2}{3}\, b^2tuu_x\, u_{yy} + \frac{2}{3}\, b^2tuu_{xy}\, u_y - \frac{4}{3}\, b^2tu_x\, u_y{}^2 + \frac{1}{2}\, btuu_{ty} \\
& +\frac{3}{4}\, btuu_{xxyy} - \frac{1}{2}\, btu_t\, u_y - \frac{1}{2}\, btu_x\, u_{xyy} + bxu_x\, u_y + \frac{1}{4}\, btu_{xx}\, u_{yy}
\end{aligned}
$$

$$
\begin{aligned}
& -btu_{xxy}\,u_y + \frac{1}{2}\,btu_{xy}{}^2 + \frac{1}{2}\,xu_t + \frac{3}{4}\,xu_{xxy} - \frac{1}{2}u_x,\\
T_3^y &= -\frac{2}{3}\,b^2tuu_x\,u_{xy} - \frac{2}{3}\,b^2tuu_{xx}\,u_y - \frac{2}{3}\,b^2tu_x{}^2u_y - btuu_{tx}\\
& -\frac{3}{4}\,btuu_{xxxy} + \frac{1}{2}\,bxu_x{}^2 - \frac{1}{4}\,btu_x\,u_{xxy} + \frac{1}{4}\,btu_{xx}\,u_{xy}\\
& -\frac{1}{4}\,btu_{xxx}\,u_y - \frac{1}{2}\,buu_x + \frac{1}{4}\,xu_{xxx} - u_{xx}/4;\\
T_4^t &= \frac{2}{3}\,buu_x\,u_{xy} + \frac{1}{3}\,buu_{xx}\,u_y + \frac{1}{4}\,uu_{tx} + \frac{1}{2}\,uu_{xxxy} + \frac{1}{4}\,u_t\,u_x,\\
T_4^x &= -\frac{1}{3}\,buu_{tx}\,u_y - \frac{1}{3}\,buu_{ty}\,u_x + \frac{2}{3}\,bu_t\,u_x\,u_y - \frac{1}{4}\,uu_{tt} - \frac{3}{8}\,uu_{txxy}\\
& +\frac{1}{4}\,u_t{}^2 + \frac{3}{8}\,u_t\,u_{xxy} - \frac{1}{4}\,u_{tx}\,u_{xy} + \frac{1}{8}\,u_{txx}\,u_y + \frac{1}{4}\,u_{txy}\,u_x - \frac{1}{8}\,u_{ty}\,u_{xx},\\
T_4^y &= -\frac{1}{3}\,buu_{tx}\,u_x + \frac{1}{3}\,bu_t\,u_x{}^2 - \frac{1}{8}\,uu_{txxx} + \frac{1}{8}\,u_t\,u_{xxx} - \frac{1}{8}\,u_{tx}\,u_{xx}\\
& +\frac{1}{8}\,u_{txx}\,u_x;\\
T_5^t &= -\frac{1}{4}\,uu_{xy} + \frac{1}{4}\,u_x\,u_y,\\
T_5^x &= -\frac{1}{3}\,buu_x\,u_{yy} - \frac{1}{3}\,buu_{xy}\,u_y + \frac{2}{3}\,bu_x\,u_y{}^2 - \frac{1}{4}\,uu_{ty} - \frac{3}{8}\,uu_{xxyy}\\
& +\frac{1}{4}\,u_t\,u_y + \frac{1}{4}\,u_x\,u_{xyy} - \frac{1}{8}\,u_{xx}\,u_{yy} + \frac{1}{2}\,u_{xxy}\,u_y - \frac{1}{4}\,u_{xy}{}^2,\\
T_5^y &= \frac{1}{3}\,buu_x\,u_{xy} + \frac{1}{3}\,buu_{xx}\,u_y + \frac{1}{3}\,bu_x{}^2u_y + \frac{1}{2}\,uu_{tx} + \frac{3}{8}\,uu_{xxxy}\\
& +\frac{1}{8}\,u_x\,u_{xxy} - \frac{1}{8}\,u_{xx}\,u_{xy} + \frac{1}{8}\,u_{xxx}\,u_y;\\
T_6^t &= \frac{1}{4b}\left\{-2\,b^2F_6\,(t)\,uu_x\,u_{xx} + b^2F_6u_x{}^3 + byF_6'uu_{xx} - byF_6'u_x{}^2\right\}\\
& +\frac{1}{4b}\left\{-bF_6'uu_{xxxx} + bF_6u_x\,u_{xxx} + y^2F_6''u_x\right\},\\
T_6^x &= -\frac{1}{24b}\left\{-6\,y^2F_6''u_t + 6\,yF_6''u - xx + 6\,y^2uF_6''' - 9\,y^2F_6''u_{xxy}\right\}\\
& -\frac{1}{24b}\left\{16\,b^2yF_6'u_x{}^2u_y + 3\,byF_6'u_{xxx}\,u_y + 6\,byF_6'uu_{tx}\right\}\\
& -\frac{1}{24b}\left\{15\,byF_6'u_x\,u_{xxy} + 6\,byF_6'u_t\,u_x - 9\,byF_6'u_{xx}\,u_{xy}\right\}\\
& -\frac{1}{24b}\left\{3\,byF_6'uu_{xxxy} - 18\,b^3uF_6u_x{}^2u_{xy} - 12\,b^2uF_6u_{tx}\,u_x\right\}\\
& -\frac{1}{24b}\left\{-14\,b^2uF_6u_x\,u_{xxxy} - 4\,b^2uF_6u_{xxy}\,u_{xx}\right\}\\
& -\frac{1}{24b}\left\{-14\,b^2uF_6u_{xxx}\,u_{xy} + 14\,b^2F_6u_x\,u_{xx}\,u_{xy}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{24b}\Big\{-14\,b^2F_6u_x\,u_{xxx}\,u_y\,-18\,b^3F_6{u_x}^3u_y\,-6\,b^2F_6u_t\,{u_x}^2\Big\}\\
& -\frac{1}{24b}\Big\{-5\,b^2F_6{u_x}^2u_{xxy}\,-14\,b^2F_6{u_{xx}}^2u_y\,-6\,buF_6u_{txxx}\Big\}\\
& -\frac{1}{24b}\Big\{6\,bF_6u_x\,u_{xxxxy}\,-3\,buF_6u_{xxxxxy}\,+12\,bF_6u_x\,u_{txx}\Big\}\\
& -\frac{1}{24b}\Big\{-9\,bF_6u_{xxx}\,u_{xxy}\,-6\,bF_6u_t\,u_{xxx}\,-12\,bF_6u_{tx}\,u_{xx}\Big\}\\
& -\frac{1}{24b}\Big\{6\,bF_6u_{xxxx}\,u_{xy}\,-9\,bF_6u_{xx}\,u_{xxxy}\,-3\,bF_6u_{xxxxx}\,u_y\Big\}\\
& -\frac{1}{24b}\Big\{-2\,b^2F_6'u{u_x}^2-3\,bF_6'uu_{xxx}\,+3\,bF_6'u_x\,u_{xx}\Big\}\\
& -\frac{1}{24b}\Big\{-12\,by^2F_6''u_x\,u_y\,+6\,ubF_6''yu_x+8\,b^2yF_6'uu_x\,u_{xy}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
T_6^y &= -\frac{1}{24b}\Big\{8\,b^2F_6uu_x\,u_{xx}\,-8\,b^2F_6{u_x}^3+12\,byF_6'{u_x}^2\Big\}\\
&\quad -\frac{1}{24b}\Big\{3\,bF_6uu_{xxxx}\,-6\,bF_6u_x\,u_{xxx}\,+3\,bF_6{u_{xx}}^2+6\,yF_6u_{xxx}\Big\};\\
T_7^t &= \frac{1}{2}\,F_7\,(t)\,u_x,\\
T_7^x &= bF_7u_x\,u_y\,-\frac{1}{2}\,uF_7+\frac{1}{2}\,F_7u_t+\frac{3}{4}\,F_7u_{xxy},\\
T_7^y &= \frac{1}{4}\,bF_7\left(2\,{u_x}^2+u_{xxx}\right).
\end{aligned}
$$

Again, following the same procedure as above, we get the following conserved vector associated with Λ_3

$$
\begin{aligned}
T_1^t &= -\frac{1}{4}\,uu_{xx}+\frac{1}{4}\,{u_x}^2,\\
T_1^x &= \frac{1}{2}\,a{u_x}^2u_y\,+\frac{1}{8}\,uu_{xxxy}+\frac{5}{8}\,u_x\,u_{xxy}-\frac{3}{8}\,u_{xx}\,u_{xy}+\frac{1}{8}\,u_{xxx}\,u_y\\
&\quad +\frac{1}{4}\,uu_{tx}+\frac{1}{4}\,u_x\,u_t,\\
T_1^y &= -\frac{1}{8}\,uu_{xxxx}+\frac{1}{6}\,a{u_x}^3+\frac{1}{4}\,u_{xxx}\,u_x-\frac{1}{8}\,{u_{xx}}^2;\\
T_2^t &= \frac{1}{2}\,F_2\,(t,y)\,u_x,\\
T_2^x &= -\frac{1}{4}\,yF_2u_{xx}+\frac{3}{4}\,F_2u_{xxy}+\frac{1}{2}\,F_2u_t-\frac{1}{4}\,ayuF_2u_x\,-\frac{1}{4}\,auF_2u_{xy}\\
&\quad +\frac{3}{4}\,aF_2u_x\,u_y\,-\frac{1}{2}\,uF_2t,\\
T_2^y &= \frac{1}{4}\,aF_2{u_x}^2+\frac{1}{4}\,auF_2u_{xx}\,+\frac{1}{4}\,F_2u_{xxx}.
\end{aligned}
$$

2.4 Concluding remarks

In this chapter we performed a complete multiplier classification of a generalized (2+1)-dimesional Calogero-Bogoyavlenskii-Schiff(CBS) equation. Thereafter we constructed the associated conservation laws of a generalized (2+1)-dimesional Calogero-Bogoyavlenskii-Schiff(CBS) equation.

Chapter 3

Conservation laws and exact solutions of a (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation

In this chapter we consider (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation [13] given by

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z) = 0. \tag{3.1}$$

The (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation arises in various physical phenomena such as fluids and plasma dynamics. Equation (3.1) degenerates into the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation

$$u_{yt} + u_{xxxy} - 3u_x u_{xy} - 3u_{xx}u_y = 0, \tag{3.2}$$

which occurs in the study of incompressible fluids [25, 26]. The soliton solutions of (3.1) have been obtained using the exp-function method [13] and the bilinear method through the logarithm transformation [27]. Several authors have also used various methods to obtain solutions of (3.1). See for example [13,28,29] and reference therein.

Although a great deal of research work has been devoted to finding different methods to solve nonlinear evolution equations, there is no unique method.

Thus, in this chapter we seek exact solutions of (3.1) using the classical Lie symmetry method. Furthermore, conservation laws for (3.1) will be constructed by employing the multiplier approach [30].

3.1 Lie point symmetries of (3.1)

The vector field operator

$$\begin{aligned}\mathbf{X} &= \xi^1(t,x,y,z,u)\frac{\partial}{\partial t}+\xi^2(t,x,y,z,u)\frac{\partial}{\partial x}+\xi^3(t,x,y,z,u)\frac{\partial}{\partial y}+\xi^4(t,x,y,z,u)\frac{\partial}{\partial z}\\ &\quad +\eta(t,x,y,z,u)\frac{\partial}{\partial u} \qquad (3.3)\end{aligned}$$

is a Lie point symmetry of (3.1) if

$$\mathbf{X}^{[4]}\left\{ u_{yt}+u_{zt}+u_{xxxy}+u_{xxxz}-3u_x(u_{xy}+u_{xz})-3u_{xx}(u_y+u_z)=0\right\}\Big|_{(3.1)}=0,$$

where $\mathbf{X}^{[4]}$ is the fourth extension of (3.3). Expanding the above equation and splitting the derivatives of u and simplifying, leads to an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^1 = 3F_{14}+6x\left(\frac{\partial}{\partial t}F_6\right),\\
&\xi^2 = F_9,\\
&\xi^3 = F_7,\\
&\xi^4 = 18F_6,\\
&\eta = F_{20}-x^2\left(\frac{\partial^2}{\partial t^2}F_6\right)-x\left(\frac{\partial}{\partial t}F_{14}\right)-6u\left(\frac{\partial}{\partial t}F_6\right),\\
&\frac{\partial}{\partial z}F_6(y,z,t)+\frac{\partial}{\partial y}F_6=0,\\
&\frac{\partial}{\partial y}F_{20}(y,z,t)+\frac{\partial}{\partial z}F_{20}=0,\\
&\frac{\partial}{\partial y}F_{14}(y,z,t)+\frac{\partial}{\partial z}F_{14}=0,\\
&\frac{\partial}{\partial y}F_7(y,z)+\frac{\partial}{\partial z}F_7-\frac{\partial}{\partial y}F_9(y,z)-\frac{\partial}{\partial z}F_9=0.
\end{aligned}$$

Solving the above system with the aid of Maple and taking the arbitrary functions to be linear with respect to their arguments yield the following symmetries of equation (3.1):

$$\begin{aligned}
\mathbf{X}_1 &= (3z-3y)\frac{\partial}{\partial x},\\
\mathbf{X}_2 &= 3t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u},\\
\mathbf{X}_3 &= \frac{\partial}{\partial x},\\
\mathbf{X}_4 &= (z-y)\frac{\partial}{\partial y},\\
\mathbf{X}_5 &= \frac{\partial}{\partial y},\\
\mathbf{X}_6 &= (z-y)\frac{\partial}{\partial z},\\
\mathbf{X}_7 &= \frac{\partial}{\partial z},\\
\mathbf{X}_8 &= (18z-18y)\frac{\partial}{\partial t},\\
\mathbf{X}_9 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},\\
\mathbf{X}_{10} &= \frac{\partial}{\partial t},\\
\mathbf{X}_{11} &= (z-y)\frac{\partial}{\partial u},\\
\mathbf{X}_{12} &= t\frac{\partial}{\partial u},\\
\mathbf{X}_{13} &= \frac{\partial}{\partial u}.
\end{aligned}$$

3.1.1 Symmetry reductions of (3.1)

We first consider $\mathbf{X_1} + \mathbf{X_4}$. The characteristic equations are

$$\frac{dt}{0} = \frac{dz}{0} = \frac{dx}{3z-3y} = \frac{dy}{z-y}.$$

Solving the above characteristic equation we get the following three invariants

$$f = z, g = t, h = y - \frac{1}{3}x, \theta = u. \tag{3.4}$$

Now considering θ as the new dependent variable and f, g and h as new independent variables, (3.1) transforms to a nonlinear PDE in three independent variables, viz.,

$$\theta_{hg}+\frac{2}{3}\theta_{hh}\theta_h+\frac{1}{3}\theta_h\theta_{hf}-\frac{1}{3}\theta_{hh}\theta_f+\theta_{gf}-\frac{1}{27}\theta_{hhhh}-\frac{1}{27}\theta_{hhhf}=0. \tag{3.5}$$

Solving equation (3.5) yields

$$\theta\,(f,g,h)=C_6\tanh(C_3g-C_4f+C_4h+C_1)+C_5, \tag{3.6}$$

where $C_1,\cdots C_5$ are arbitrary constants. Now using the invariants (3.4) together with equation (3.6), we conclude that the group-invariant solution of equation (3.1) is

$$u(t,x,y,z)=C_6\tanh[C_3t-C_4z+C_4(y-\frac{1}{3}x)+C_1]+C_5. \tag{3.7}$$

We now consider $\mathbf{X_1}+\mathbf{X_6}$. The characteristic equations are

$$\frac{dt}{0}=\frac{dy}{0}=\frac{dx}{3z-3y}=\frac{dz}{z-y}.$$

Solving the above characteristic equation, we get the following four invariants

$$f=y,g=t,h=z-\frac{1}{3}x,\theta=u. \tag{3.8}$$

Treating θ as the new dependent variable and f, g and h as new independent variables, (3.1) reduces to a nonlinear PDE in three independent variables, viz.,

$$\theta_{gf}-\frac{1}{3}\theta_h\theta_{hf}-\frac{2}{3}\theta_{hh}\theta_h-\frac{1}{3}\theta_{hh}\theta_f+\theta_{hg}-\frac{1}{27}\theta_{hhhf}-\frac{1}{27}\theta_{hhhh}=0. \tag{3.9}$$

Solving equation (3.9), yields

$$\theta\,(f,g,h)=C_6\tanh(C_3g-C_4f+C_4h+C_1)+C_5, \tag{3.10}$$

where $C_1,\cdots C_6$ are arbitrary constants. Substituting equation (3.8) into equation (3.10), we conclude that the group invariant solution of (3.1) is

$$u(t,x,y,z)=C_6\tanh[C_3t-C_4y+C_4(z-\frac{1}{3}x)+C_1]+C_5. \tag{3.11}$$

If we choose $\mathbf{X_1}+\mathbf{X_8}$. The Lagrange equations are

$$\frac{dt}{18z-18y}=\frac{dy}{0}=\frac{dx}{3z-3y}=\frac{dz}{0}.$$

Solving the above characteristic equation, we get the following four invariants

$$f = y, g = t, h = z - \frac{1}{3}x, \Psi = u. \tag{3.12}$$

Taking Ψ as the new dependent variable and f, g and h as the new independent variables, (3.1) transforms to a nonlinear PDE in three independent variables, viz.,

$$\Psi_{hhhg} - \frac{1}{6}\Psi_{hf} - 3\Psi_h\Psi_{hf} - 3\Psi_h\Psi_{hg} - 3\Psi_{hh}\Psi_f - 3\Psi_{hh}\Psi_g - \frac{1}{6}\Psi_{hg} + \Psi_{hhhf} = 0$$

and the integration thereof yields

$$\Psi\,(f, g, h) = C_6 \tanh(C_2 f - C_2 g + C_4 h + C_1) + C_5, \tag{3.13}$$

where $C_1, \cdots C_6$ are arbitrary constants. Now using the invariants (3.12) together with equation (3.13), we get the solution of (3.1) as

$$u(t, x, y, z) = C_6 \tanh[C_2 y - C_2 z + C_4(x - \frac{1}{6}t) + C_1] + C_5. \tag{3.14}$$

We now consider $\mathbf{X_2}$. The characteristic equations are

$$\frac{dt}{0} = \frac{dy}{0} = \frac{dx}{3t} = \frac{du}{x}. \tag{3.15}$$

After solving the characteristic equation (3.15), we get the following four invariants

$$f = y, g = z, h = t, \Psi = \frac{6tu + x^2}{6t}. \tag{3.16}$$

Now considering Ψ as the new dependent variable and f, g and h as the new independent variables, (3.1) transforms to a nonlinear PDE in three independent variables, viz.,

$$h\Psi_{hf} + h\Psi_{hg} + \Psi_f + \Psi_g = 0. \tag{3.17}$$

Solving equation (3.17) yields

$$\theta\,(f, g, h) = \frac{F_1(g - f, h)h + F_2(f, g)}{h}, \tag{3.18}$$

where F_1 and F_2 are arbitrary functions of g, f and h. Now using the invariants (3.16) together with equation (3.18), we get

$$u(t, x, y, z) = \frac{1}{6}\left(\frac{6F_1(z - y, t)t - x^2 + 6F_2(y, z)}{t}\right) \tag{3.19}$$

as the group invariant solution of (3.1).

Lastly we now consider $\mathbf{X_9}$. The characteristic equations are

$$\frac{dt}{3t} = \frac{dy}{0} = \frac{dx}{x} = \frac{du}{u}.$$

Solving the characteristic equations, we get the following four invariants

$$f = y, g = z, h = xt^{-\frac{1}{3}}, \Psi = ut^{\frac{1}{3}}. \tag{3.20}$$

Now considering Ψ as the new dependent variable and f, g and h as the new independent variables, (3.1) transforms to a nonlinear PDE in three independent variables, viz.,

$$\begin{aligned} h\Psi_{hf} + h\Psi_{hg} + 9\Psi_{hh}\Psi_f + 9\Psi_{hh}\Psi_g + 9\Psi_h\Psi_{hf} + 9\Psi_h\Psi_{hg} \\ -3\Psi_{hhhf} - 3\Psi_{hhhg} + \Psi_f + \Psi_g = 0. \end{aligned} \tag{3.21}$$

Thus, we conclude that

$$u(t, x, y, z) = t^{-\frac{1}{3}}\Psi(y, z, xt^{-\frac{1}{3}}) \tag{3.22}$$

is a general group invariant solution of equations (3.1) where Ψ is any solution of (3.21)

Remark 3.1 Following the aforementioned procedure, one can obtain more group invariants solutions of the generalized (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation using the other symmetries.

3.2 Conservation laws

This section aims to compute the conservation law of equation (3.1) using the multiplier method. Here we will consider multiplier of the zeroth order,
i.e., $\Lambda = \Lambda(t, x, y, z, u)$. The right hand side of (1.6) is a divergence expression. The determining equation for the multiplier Λ is

$$\frac{\delta}{\delta u^\alpha}(\Lambda(u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z))) = 0. \tag{3.23}$$

Expanding equation (3.23) with the help of Maple computer algebra system yields

$$\begin{aligned}\Lambda_{ty}+\Lambda_{tz} &= 0,\\ \Lambda_{x} &= 0,\\ \Lambda_{u} &= 0.\end{aligned}$$

Solving the above systems of equation gives

$$\Lambda = F_1\left(z,y\right)+F_2\left(t,z-y\right).$$

Thus, we conclude that the conservation laws associated with the above multiplier Λ for F_1 are:

$$\begin{aligned}T_1^t &= -\frac{1}{2}\,uF_z\left(z,y\right)-\frac{1}{2}\,uF_y+\frac{1}{2}\,uy\,F+\frac{1}{2}\,uz\,F,\\ T_1^x &= \frac{3}{4}\,uu_x\,F_z-\frac{9}{4}\,u_x\,u_z\,F+\frac{3}{4}\,uFu_{xz}+\frac{3}{4}\,uu_x\,F_y-\frac{9}{4}\,u_x\,u_y\,F\\ &\quad+\frac{3}{4}\,uFu_{xy}-\frac{1}{4}\,u_{xx}\,F_z-\frac{1}{4}\,u_{xx}\,F_y+\frac{3}{4}\,u_{xxz}\,F+\frac{3}{4}\,u_{xxy}\,F,\\ T_1^y &= -\frac{3}{4}\,{u_x}^2F-\frac{3}{4}\,uu_{xx}\,F+\frac{1}{4}\,u_{xxx}\,F+\frac{1}{2}\,u_t\,F,\\ T_1^z &= -\frac{3}{4}\,{u_x}^2F-\frac{3}{4}\,uu_{xx}\,F+\frac{1}{4}\,u_{xxx}\,F+\frac{1}{2}\,u_t\,F.\end{aligned}$$

Likewise, the above multiplier Λ gives the following conservation laws for F_2:

$$\begin{aligned}T_2^t &= \frac{1}{2}\,u_y\,F\left(t,z-y\right)+\frac{1}{2}\,u_z\,F,\\ T_2^x &= -\frac{9}{4}\,u_x\,u_z\,F+\frac{3}{4}\,uFu_{xz}-\frac{9}{4}\,u_x\,u_y\,F+\frac{3}{4}\,uFu_{xy}\\ &\quad+\frac{3}{4}\,u_{xxz}\,F+\frac{3}{4}\,u_{xxy}\,F,\\ T_2^y &= -\frac{1}{2}\,uF_t-\frac{3}{4}{u_x}^2F-\frac{3}{4}\,uu_{xx}\,F+\frac{1}{4}\,u_{xxx}\,F+\frac{1}{2}\,u_t\,F,\\ T_2^z &= -\frac{1}{2}\,uF_t-\frac{3}{4}\,{u_x}^2F-\frac{3}{4}\,uu_{xx}\,F+\frac{1}{4}\,u_{xxx}\,F+\frac{1}{2}\,u_t\,F.\end{aligned}$$

Here we notice that due to the presence of the arbitrary functions in the conservation laws, one can generate an infinite number of conservation laws for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation.

3.3 Concluding Remarks

In this chapter, we computed the conservation laws of the generalized (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. With the aid of Maple, we have assured the correctness of the obtained solutions by substituting them back into the original equation (3.1).

Chapter 4

Conclusions and Discussion

Finding exact solutions and conservation laws for nonlinear partial differential equations is of great importance for the explanation of some practical physical problems. The aim of this work was to obtain exact solutions and derive conservation laws of some nonlinear partial differential equations by using symmetry analysis.

Chapter one provided relevant background material, such as, definitions and theorems of important concepts that were used to carry out the calculations in this work.

In chapter two a complete classification was performed on a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation. Thereafter we derived the group invariant solutions of a generalized (2+1)-Dimesional Calogaro Bogoyavlenskii-Schiff equation. Moreover, conservation laws of the underlying equation where constructed by applying the multiplier method.

In chapter three the Lie symmetry method was used to find the exact solutions of the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. We then employed the multiplier approach to derive conservation laws of the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. Lastly, in chapter four we summarised all the work that was done in this dissertation.

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