



On a Class of Subdiagonal Algebras

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Abstract

We investigate some new classes of operator algebras which we call semi- σ -finite subdiagonal and Riesz approximable. These constitute the most general setting to date for a noncommutative Hardy space theory based on Arveson's subdiagonal algebras. We develop this theory and study the properties of these new classes.

1 Introduction

From its inception in the remarkable thesis and 1967 paper of Arveson [1], the theory of subdiagonal algebras has now developed into a very refined theory of quantum H^p spaces which has attracted very many international researchers. Primary structural facts regarding classical H^p spaces and their abstract function theoretic generalizations from the 1960's, now have beautiful quantum analogues in this setting. We mention for example the F & M Riesz theorem, Beurling invariant subspace theorem, Jensen and Szegő theorems, Gleason–Whitney theorem, inner-outer factorization, and so on. The quantum theory is moreover no mere verbatim clone of the classical theory. Rather one often finds in the quantum world that a theory which is fairly simple to formulate in the classical setting, unfolds into a very intricate kaleidoscope in the quantum world.

In a series of papers the authors extended the theory of generalized H^p spaces for function algebras from the 1960s to the setting of Arveson's *finite maximal subdiagonal*

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algebras of a von Neumann algebra M possessing a faithful normal tracial state. As stated in [9], “as an example of what some might call ‘mathematical quantization’, or noncommutative (operator algebraic) generalization of a classical theory, the program succeeds to a degree of ‘faithfulness to the original’ which seems to be quite rare”. Much of this theory is summarized in the latter survey paper. After this, many researchers turned their attention to generalizing parts of the theory to subalgebras of semi- and σ -finite von Neumann algebras. Important structural results were obtained by Ji, Ohwada, Saito, Bekjan, Xu, the authors, and others, although other results start to break down in successively more general settings. In the present paper we develop the theory in the most general setting to date. We propose new classes of algebras, that we call semi- σ -finite subdiagonal and Riesz approximable, that contain both the semi- and σ -finite case. We then work out some of the generalized Hardy space theory for these algebras. The essence of what we present is a theory of subdiagonal subalgebras conditioned to von Neumann algebras equipped with a strictly semifinite faithful normal weight. In fact an extension of the theory to general von Neumann algebras is possible, as will be shown in the forthcoming paper [37]. However we believe that the present setting may well be the most general setting in which some of the more delicate properties of noncommutative H^p -spaces—like the F & M Riesz theorem and the Gleason–Whitney theorem—hold true. As such this context deserves to be studied in its own right.

We will see that if A is a semi- σ -finite subdiagonal algebra in M then A is an increasing limit of maximal σ -finite subdiagonal algebras A_i . This allows a very convenient simplification of Haagerup’s reduction method, and its application to subdiagonal algebras. The Hardy space $H^p(A)$ is simply the completion of the increasing union of the Hardy spaces $H^p(A_i)$. The theory of maximal subdiagonal algebras in the σ -finite case has been developed by many authors: Ji, Saito, Labuschagne, Bekjan and others (see e.g. [5, 11, 29–32, 35, 49]). Thus most of the Hardy space theory of maximal semi- σ -finite subdiagonal algebras may be viewed as a ‘limiting case’ of the theory of maximal σ -finite subdiagonal algebras. This is similar to the way in which most of the theory of maximal *semifinite* subdiagonal algebras, developed by Bekjan, Ueda, Sager and others, is a ‘limiting case’ of the theory of maximal (tracially) finite subdiagonal algebras (see e.g. [2, 4, 41, 47]).

We summarize briefly the structure of our paper. In Sect. 2 we define semi- σ -finite subdiagonal algebras, and give some alternative characterizations and examples. In Sects. 3 and 4, we work out some aspects of the basic Hardy space theory for these algebras, including some Beurling invariant subspace theory. In Sect. 5.2 we introduce a much larger class of algebras that we call *Riesz approximable*, and check that the Lebesgue decomposition, F & M Riesz, and Gleason–Whitney theorems all hold for these. More generally we generalize several of our results from [8, 11] related to the latter theorems to a wider setting. Some of this is related to the recent paper [16].

We now turn to notation and some background facts. We refer the reader to [1] and our survey [9] for the basic ideas, and for the ‘tracially finite’ variant of the theory. Suppose that M is a von Neumann algebra with a faithful normal semifinite weight ω , and modular group (σ_t^ω) .

Definition 1.1 A weak* closed unital subalgebra A of M is said to be an *analytically conditioned* subalgebra if

- $\sigma_t^\omega(A) = A$ for all $t \in \mathbb{R}$,
- $\omega|_D$ is semifinite where $D = A \cap A^*$,
- the faithful normal conditional expectation $\mathbb{E} : M \rightarrow D = A \cap A^*$ satisfying $\omega \circ \mathbb{E} = \omega$ (ensured by the above condition by [44, Theorem IX.4.2]), is multiplicative on A .

A subalgebra A of M is said to be *subdiagonal with respect to ω* if in addition to being analytically conditioned, it also satisfies the requirement that $A + A^*$ is weak* dense in M . We say that a subdiagonal algebra is *maximal subdiagonal* (with respect to $\mathbb{E}$) if it is not properly contained in any larger proper subdiagonal algebra in M with respect to $\mathbb{E}$. For a subspace F of M we define F_0 to be $F \cap \text{Ker}(\mathbb{E})$, as is usual in the subdiagonal theory.

Our analysis requires some background on noncommutative L^p -spaces (see e.g. [21–23, 28]). By necessity, our discussion shall be brief. Full details of all claims made below may be found in [22]. Throughout this discussion ω will be a fixed faithful normal semifinite reference weight on a von Neumann algebra M . The theory is a lot more accessible in the case where the reference weight is tracial (that is $\omega(a^*a) = \omega(aa^*)$ for each $a \in M$). It is convention to denote the reference weight by τ in the tracial case. In the tracial case the algebra M may be enlarged to the so-called algebra of τ -measurable operators $\tilde{M}$ which is defined to be the set of all densely defined closed operators f affiliated to M which satisfy the requirement that for some $\epsilon > 0$ we have that $\tau(\chi_{(\epsilon, \infty)}(|f|)) < \infty$. This enlarged space turns out to be a complete metrizable algebra which is large enough to contain all the L^p -spaces and which admits an extension of τ to $\tilde{M}$. Given any $1 \leq p < \infty$ the space $L^p(M, \tau)$ is then simply defined to be $L^p(M, \tau) = \{f \in \tilde{M} : \tau(|f|^p) < \infty\}$, with the norm given by $\|f\|_p = \tau(|f|^p)^{1/p}$.

The construction of L^p -spaces for general von Neumann algebras is much more challenging. We shall follow Haagerup's approach to construct these spaces. The first step in this approach is to use the modular automorphism group σ_t^ω ($t \in \mathbb{R}$) induced by ω to construct the crossed product $M \rtimes_\omega \mathbb{R}$. Here the dual action of $\mathbb{R}$ on this crossed product is induced by a group of automorphisms (θ_s) on $M \rtimes_\omega \mathbb{R}$ satisfying

$$\theta_s(\pi(a)) = \pi(a) \text{ and } \theta_s(\lambda_t) = e^{-ist} \lambda_t \text{ for each } a \in M \text{ and } t, s \in \mathbb{R}. \quad (1.1)$$

With $\tilde{\omega}$ denoting the dual weight on $M \rtimes_\omega \mathbb{R}$, it is moreover possible to show that

$$\sigma_t^{\tilde{\omega}}(f) = \lambda_t f \lambda_t^* \text{ with } \sigma_t^{\tilde{\omega}}(\pi(a)) = \pi(\sigma_t^\omega(a)) \text{ for all } f \in (M \rtimes_\omega \mathbb{R}) \text{ and all } a \in M. \quad (1.2)$$

By Stone's theorem there exists a nonsingular positive operator h affiliated to $M \rtimes_\omega \mathbb{R}$ for which we have that $\lambda_t = h^{it}$ for all $t \in \mathbb{R}$. The fact that $\sigma_t^{\tilde{\omega}}$ is implemented as above ensures that $M \rtimes_\omega \mathbb{R}$ is in fact semifinite, with the prescription $\tau(\cdot) = \tilde{\omega}(h^{-1} \cdot)$ yielding an fns (that is, faithful normal semifinite) trace on $M \rtimes_\omega \mathbb{R}$ for which we have that $\tau \circ \theta_s = e^{-s} \tau$ for all $s \in \mathbb{R}$. So by construction h is just the Pedersen–Takesaki Radon–Nikodym derivative $\frac{d\tilde{\omega}}{d\tau}$ of $\tilde{\omega}$ with respect to τ [39]. The semifiniteness ensures

that $\mathfrak{M} = M \rtimes_{\omega} \mathbb{R}$ may be enlarged to the algebra $\tilde{\mathfrak{M}}$ of τ -measurable operators, with each θ_s in addition extending continuously to this enlarged algebra in the topology of convergence in measure. For each $1 \leq p < \infty$ the Haagerup L^p -space is then defined to be the space

$$L^p(M) = \{a \in \tilde{\mathfrak{M}} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}.$$

It is known that $L^{\infty}(M)$ corresponds to the canonical copy of M inside $M \rtimes_{\omega} \mathbb{R}$. The space $L^1(M)$ admits a so-called tracial functional tr , which can be used to realise the norm on $L^p(M)$ by means of the prescription $\|a\|_p = tr(|a|^p)^{1/p}$.

In the case where e is a projection in the centralizer of ω with ω semifinite on eMe , one may canonically identify $L^p(eMe)$ (constructed using the restriction of ω to eMe) with $eL^p(M)e$. What lies behind this, is the fact that the assumption that e is a fixed point of the modular group, allows one to canonically identify $e(M \rtimes_{\omega} \mathbb{R})e$ with $(eMe) \rtimes_{\omega_e} \mathbb{R}$ where $\omega_e = \omega|_{eMe}$. Details of the computation may be found in the discussion following Lemma 8 of [48] or in [23].

Each $1 \leq p < \infty$ moreover admits a norm-dense bijective embedding $i^{(p)} : \mathfrak{m}_{\omega} \rightarrow L^p(M)$ of the algebra $\mathfrak{m}_{\omega}$ into $L^p(M)$. Formally this embedding may be thought of as a prescription taking the form $a \rightarrow h^{1/2p} a h^{1/2p}$. Since here h may not be τ -measurable, there are however significant challenges regarding existence and closability which need to be overcome, regarding which we shall not elaborate here. Details of the construction of these embeddings may be found in Section 2 of [24]. In the case considered above where e is a projection in the centralizer of ω , the fact that σ_t^{ω} is implemented by the unitary group h^{it} inside the crossed product will, when combined with the fact that e is a fixed point of the modular group, ensure that e commutes with each h^{it} and hence strongly with h itself. On tracing the construction of the embeddings $i^{(p)}$ in [24], this fact in turn ensures that we will have that $i^{(p)}(eae) = ei^{(p)}(a)e$ for any $a \in \mathfrak{m}_{\omega}$.

One interesting fact which we shall have occasion to apply to conditional expectations, concerns positive maps $T : M \rightarrow M$ satisfying the condition $\omega \circ T \leq \omega$. For such maps we clearly have that $T(\mathfrak{m}_{\omega}) \subset \mathfrak{m}_{\omega}$. However as can be seen from [28], the prescription $T^{(p)}(i^{(p)}(a)) = i^{(p)}(T(a))$ ($a \in \mathfrak{m}_{\omega}$) yields a $\|\cdot\|_p$ -norm bounded operator uniquely extending to a bounded operator on $L^p(M)$.

It is clear from the definition of these Haagerup L^p spaces that they each have a different “phase”. So to make sense of concepts like intersections of these spaces in a manner which harmonises with the classical setting, these spaces first need to be embedded into a common superspace where their orientation with respect to each other is in line with that of the classical setting. The superspace we use is the noncommutative analogue of $L^1 + L^{\infty}$ constructed using the theory of noncommutative Orlicz spaces for general von Neumann algebras as espoused in [23, 36]. The so-called fundamental function of $(L^1 + L^{\infty})(\mathbb{R})$ is given by $\varphi_{1+\infty}(t) = \min(1, t)$ (where $t \geq 0$). We then use this function to define the space $L^{1+\infty}(M)$ by the prescription

$$L^{1+\infty}(M) = \left\{ a \in \tilde{\mathfrak{M}} : \theta_s(a) = v_s^{1/2} a v_s^{1/2} \text{ for all } s \in \mathbb{R} \right\},$$

where each v_s is a bounded element of $\mathfrak{M}$ given by $v_s = \varphi_{1+\infty}(h) \varphi_{1+\infty}(h)^{-1}$. (When the fundamental function of $L^p(\mathbb{R})$, namely $t \rightarrow t^{1/p}$, is substituted for $\varphi_{1+\infty}(h)$,

we get exactly Haagerup's definition of $L^p(M)$.) When equipped with the topology inherited from $\tilde{\mathfrak{M}}$, this space turns out to be a quasi-Banach space.

It is known that each $L^p(M)$ admits a canonical embedding $\iota^{(p)}$ of $L^p(M)$ into $L^{1+\infty}(M)$ [22, Proposition 7.22]. We write $\mathcal{L}^p(M)$ for $\iota^{(p)}(L^p(M))$. The embedding of M into $L^{1+\infty}(M)$ takes the form $\iota^{(\infty)}(a) = \varphi_{1+\infty}(h)^{1/2} a \varphi_{1+\infty}(h)^{1/2}$ for all $a \in M$, whereas for each $L^p(M)$ ($1 \leq p < \infty$) the embedding takes the form $\iota^{(p)}(a) = \eta_p(h)^{1/2} a \eta_p(h)^{1/2}$ for all $a \in L^p(M)$ where

$$\eta_p(t) = \varphi_{1+\infty}(t) t^{-1/p} = \begin{cases} t^{1/q} & 0 \leq t \leq 1 \\ t^{-1/p} & t > 1 \end{cases}$$

In support of the claim that inside $L^{1+\infty}(M)$ the spaces $L^p(M)$ are appropriately oriented with respect to each other, we cite the fact that for any $1 \leq p < \infty$ we have that $\iota^{(p)}(i^{(p)} m_\omega) = \iota^{(\infty)}(m_\omega)$. When working with the representations of $L^p(M)$ inside $L^{1+\infty}(M)$, we shall write $\mathcal{L}^p(M)$ for $\iota^{(p)}(L^p(M))$ equipped with the norm inherited from $L^p(M)$. We shall define $H^p(A)$ shortly, and then we shall similarly write $\mathcal{H}^p(A)$ for $\iota^{(p)}(H^p(A))$ equipped with the norm inherited from $L^p(M)$.

2 Emergent Paradigms: Semi- σ -Finite Subdiagonal Algebras

Our framework is a von Neumann algebra M and weak* continuous inclusions of weak* closed operator algebras $D = A \cap A^* \subset A \subset M$. In this section we will consider a setting in which M admits σ -finite weak* closed $*$ -subalgebras M_i whose union is weak* dense in M , with each $A_i = A \cap M_i$ a maximal subdiagonal algebra in M_i , and with $\bigcup_i A_i$ weak* dense in A . Within this class a specific subclass now suggests itself:

Theorem 2.1 *Assume that M is a von Neumann algebra with a faithful normal semifinite weight ω , with weak* continuous inclusions of weak* closed operator algebras $D = A \cap A^* \subset A \subset M$ as above. In this setting, the following conditions are equivalent:*

- (1) *A is a subdiagonal algebra in M with respect to ω , and ω is strictly semifinite on D ;*
- (2) *A is a subdiagonal algebra in M with respect to ω , and there are ω -finite projections $e_i \in D \cap M_\omega$ with $e_i \nearrow 1$ (here M_ω is the centralizer of ω [44]);*
- (3) *there are ω -finite projections $e_i \in D \cap M_\omega$ with $e_i \nearrow 1$ such that $A_i = e_i A e_i$ is a maximal subdiagonal algebra in the σ -finite von Neumann algebra $M_i = e_i M e_i$ with respect to the weight $\omega|_{M_i}$.*

A subalgebra A satisfying these conditions is a maximal subdiagonal algebra, and we have $D \cap M_\omega = D_\omega$.

Proof (2) $\Rightarrow$ (3) We have $D_i = e_i D e_i \subset A_i \subset M_i = e_i M e_i$, and A_i is subdiagonal in M_i . Indeed $A_i + A_i^* = e_i(A + A^*)e_i$ is weak* dense in $e_i M e_i$, and we have $A_i \cap A_i^* = e_i(A \cap A^*)e_i = D_i$. The restriction ω_i of ω to $(M_i)_+$ is finite and extends

to a faithful positive normal functional on M_i , so M_i is σ -finite. Note that ω_i has a modular automorphism group which is the restriction of σ_t^ω to M_i , and this leaves A_i invariant. Also the restriction of $\mathbb{E}$ to M_i is ω_i -preserving. So by the main result in [49] (and the remark after it), A_i is maximal subdiagonal in M_i .

(3) $\Rightarrow$ (2) The restriction ω_i of ω to M_i has modular automorphism group which is the restriction of σ_t^ω to M_i . As part of the condition in (3) that A_i is maximal subdiagonal with respect to $\omega|_{M_i}$ we are assuming that there is a $\omega|_{M_i}$ -preserving conditional expectation $\mathbb{E}_i : M_i \rightarrow D_i$ which is multiplicative on A_i . Then σ_t^ω leaves A_i and D_i invariant by the result of Ji, Ohwada and Saito cited in [49] above the main theorem there. Hence σ_t^ω leaves A and D invariant. By Takesaki's theorem there exists a unique normal ω -preserving conditional expectation $\mathbb{E} : M \rightarrow D$, and by uniqueness its restriction to M_i is $\mathbb{E}_i$. It follows that $\mathbb{E}$ is multiplicative on A . (To see this note that the A_i 's are increasing.) Finally, since $\cup_i M_i$ is weak* dense in M , and $A_i + A_i^*$ is weak* dense in M_i , we have A subdiagonal in M .

(2) $\Rightarrow$ (1) This follows from Exercise VIII.2.1 of [44] applied to the Claim: $D \cap M_\omega \subset D_\omega$. To prove this claim we will use [44, Theorem VIII.2.6]. Suppose that $x \in D \cap M_\omega$ and $y \in \mathfrak{m}_\omega^D$, with $y = n_1^* n_2$ say for $n_i \in \mathfrak{n}_\omega^D = D \cap \mathfrak{n}_\omega^M$. Since by [44, Theorem VIII.2.6] we have that $n_2^* n_2 x \in \mathfrak{m}_\omega^M$ and hence that $x^* n_2^* n_2 x \in \mathfrak{m}_\omega^M$, it is clear that $n_2 x \in \mathfrak{n}_\omega^D$. Thus $yx = n_1^* n_2 x \in \mathfrak{m}_\omega^D$. Similarly $xy \in \mathfrak{m}_\omega^D$. Hence we clearly have that $x \in D_\omega$.

(1) $\Rightarrow$ (2) Since by definition σ_t^ω leaves A invariant, it clearly leaves D invariant, so that the restriction of σ_t^ω to D is the automorphism group of $\omega|_D$. So $D \cap M_\omega = D_\omega$ and using one of the well known characterizations of strictly semifinite weights (see e.g. [22, 23]) we obtain an increasing net $e_i \nearrow 1$ as in (2).

To see that A is indeed maximal suppose that $A \subset \mathcal{B} \subset M$ where $\mathcal{B}$ is a subdiagonal algebra with respect to $\mathbb{E}$. In particular $\mathcal{B} \cap \mathcal{B}^* = D$. Then

$$D_i \subset A_i \subset \mathcal{B}_i = e_i \mathcal{B} e_i \subset M_i,$$

and $\mathcal{B}_i$ is easily seen as in the last proof to be subdiagonal in the σ -finite von Neumann algebra M_i . So $A_i = \mathcal{B}_i$ by the main theorem in [49]. Thus any $x \in \mathcal{B}$ has $x = \lim_i e_i x e_i \in A$. So A is a maximal subdiagonal algebra. $\square$

An algebra A satisfying all of the above conditions will be called a *semi- σ -finite subdiagonal algebra*.

This class has essentially appeared very briefly in the literature, on the last page of [49] where it is defined in terms of a strictly semifinite weight (to see the connection with (1) one may use the remark after [49, Theorem 1.1]). A different route to the maximality of such A was stated there with hints. It is true but not at present clear that all of the algebras considered on the last page of [49] are in fact semi- σ -finite subdiagonal algebras; this needs a generalization of [32, Theorem 2.4] that will appear in forthcoming work of the second author.

Semi- σ -finite subdiagonal algebras generalize maximal subdiagonal algebras in both semifinite and in σ -finite von Neumann algebras:

Proposition 2.2 *A maximal subdiagonal algebra in a semifinite or a σ -finite von Neumann algebra is a semi- σ -finite subdiagonal algebra.*

Proof In the σ -finite case we are considering the maximal subdiagonal algebra studied in e.g. [32]. Here ω is finite, so strictly semifinite on D , and we may take $e_i = 1$ in the definition of a semi- σ -finite subdiagonal algebra. In the semifinite case we are assuming (as in the papers on subdiagonal algebra in the semifinite case referred to above) that ω is a semifinite trace on both M and D and that A is subdiagonal with respect to ω . Here σ_t^ω is the identity map. The existence of ω -finite projections $e_i \in D = D \cap M_\omega$ with $e_i \nearrow 1$ is immediate from the semifiniteness. $\square$

We end this section by mentioning further examples of semi- σ -finite subdiagonal algebras. Every von Neumann algebra M is a semi- σ -finite subdiagonal algebra (so $A = M$). This follows from the fact that every von Neumann algebra possesses a strictly semifinite faithful normal weight.

Example 2.3 If A is a maximal σ -finite (or semifinite) subdiagonal algebra in M , then $A \bar{\otimes} B(H)$ is a semi- σ -finite subdiagonal algebra. Indeed for any von Neumann algebra N we have that $A \bar{\otimes} N$ is a semi- σ -finite subdiagonal algebra (in $M \bar{\otimes} N$). More generally the latter is also true if A is a semi- σ -finite subdiagonal algebra.

To see these, we use several facts from e.g. [42, Chapters 8 and 9]: recall that $M \bar{\otimes} N$ has a canonical faithful normal semifinite weight built from the two weights on M and N . Then $A \bar{\otimes} N + A^* \bar{\otimes} N$ is certainly weak* dense in $M \bar{\otimes} N$. That $(A \bar{\otimes} N) \cap (A^* \bar{\otimes} N) = D \bar{\otimes} N$ follows easily from Tomiyama's 'Fubini' slice map theorem: any $x \in (A \bar{\otimes} N) \cap (A^* \bar{\otimes} N)$ has its left slices $L_\psi(x) \in A \cap A^* = D$ for $\psi \in N_*$. If E is the weight preserving expectation onto $D = A \cap A^*$ then $E \otimes I_N$ is a weight preserving normal conditional expectation onto $D \bar{\otimes} N$, and it is multiplicative on $A \bar{\otimes} N$. The modular group for $M \bar{\otimes} N$ is $\sigma_t^M \otimes \sigma_t^N$, and this preserves $A \bar{\otimes} N$. Finally suppose that $e_t \nearrow 1$ in M and $f_s \nearrow 1$ in N are in the centralizers and are finite with respect to the respective weights. Then $e_t \otimes f_s \nearrow 1 \otimes 1$, and $(\sigma_t^M \otimes \sigma_s^N)(e_t \otimes f_s) = e_t \otimes f_s$. So $A \bar{\otimes} N$ is a semi- σ -finite subdiagonal algebra (in $M \bar{\otimes} N$).

Now suppose we are given a maximal subdiagonal subalgebra A of a σ -finite von Neumann algebra M equipped with a faithful normal state ω . It is a well know fact that upon taking the crossed product with the dyadic rationals $\mathbb{Q}_D$, A may be enlarged to a subdiagonal subalgebra $A \rtimes_\omega \mathbb{Q}_D$ of $M \rtimes_\omega \mathbb{Q}_D$ which can be approximated with subdiagonal subalgebras of finite von Neumann algebras. Details of this construction may be found in [49]. However more is true. As we shall see below if one takes the crossed product with modular group, then A in fact enlarges to a semi- σ -finite subdiagonal subalgebra $A \rtimes_\omega \mathbb{R}$ of $M \rtimes_\omega \mathbb{R}$.

Example 2.4 As before let M be a σ -finite von Neumann algebra equipped with a faithful normal state ω and let A be a weak* closed unital subalgebra which is maximal subdiagonal with respect to ω . Let G be a unimodular abelian locally compact group admitting an action α on M . We shall write λ_t for the unitary defined on $L^2(G; H) \equiv H \otimes L^2(G)$ by $\lambda_t(\xi)(s) = \xi(t^{-1}s)$, and ℓ_t for the corresponding unitary defined on $L^2(G)$ by $\ell_t(f)(s) = f(t^{-1}s)$. Clearly $\lambda_t = \mathbb{1} \otimes \ell_t$.

The group then admits a neighbourhood base $\mathcal{U}$ of the group unit $e \in G$ consisting of compact symmetric neighbourhoods for which $\{U.U : U \in \mathcal{U}\}$ is again a neighbourhood base (see [18, Proposition 2.1]). The collection $\{\frac{1}{|U|}\chi_U : U \in \mathcal{U}\}$ is a left and right approximate identity for G in the sense described in [18, Proposition 2.44]. We leave it as an exercise to verify that the assumptions made with regard to $\mathcal{U}$ ensure that then the collection $\{\zeta_U : U \in \mathcal{U}\}$ where $\zeta_U = \frac{1}{|U|^2}\chi_U * \chi_U$, is again a left and right approximate identity in the sense of [18, Proposition 2.44]. Now we have that $\{\zeta_U : U \in \mathcal{U}\} \subset C_0(G)$ [18, Proposition 2.41]. It is an exercise to show that each ζ_U is in fact supported on $U.U$.

With conjugation of functions in $L^1(G)$ defined as in Section 3.2 of [18], the unimodularity G and symmetry of the neighbourhoods $U \in \mathcal{U}$, ensure that $\chi_U^* = \chi_U$ for every $U \in \mathcal{U}$. Recall that ℓ_f ($f \in L^1(G)$) is just the convolution operator $x \rightarrow f * x$ on $L^2(G)$. The fact just noted now ensures that

$$\ell_{\zeta_U} = \frac{1}{|U|^2} \ell_{\chi_U^* * \chi_U} = \frac{1}{|U|^2} \ell_{\chi_U}^* \ell_{\chi_U} \geq 0.$$

By [18, Proposition 2.44] the net $\{\frac{1}{|U|}\ell_{\chi_U} : U \in \mathcal{U}\} \subset \text{VN}_I(G)$ is strongly convergent to $\mathbb{1}$. The matching net $\{\frac{1}{|U|}\lambda_{\chi_U} : U \in \mathcal{U}\}$ lives inside the von Neumann subalgebra of $M \rtimes_\alpha G$ generated by the λ_t 's, which may by mild abuse of notation be written as $\mathbb{1} \otimes \text{VN}_I(G)$. With $\tilde{\omega}$ denoting the dual weight on $M \rtimes_\alpha G$, it follows from what we noted above and equation (1.5) of [28] that¹

$$\frac{1}{|U|^2} \tilde{\omega}(\lambda_{\chi_U}^* \lambda_{\chi_U}) = \tilde{\omega}(\lambda_{\zeta_U}) = \zeta_U(e) < \infty$$

for each U . Thus $\{\frac{1}{|U|}\lambda_{\chi_U} : U \in \mathcal{U}\}$ not only converges strongly to $\mathbb{1}$ but is also contained $\mathfrak{n}_{\tilde{\omega}}(M \rtimes_\alpha G)$. This is enough to ensure that $(\mathbb{1} \otimes \text{VN}(G)) \cap \mathfrak{n}_{\tilde{\omega}}(M \rtimes_\alpha G)$ is weak* dense in $\mathbb{1} \otimes \text{VN}(G)$ and hence that the restriction of $\tilde{\omega}$ to $\mathbb{1} \otimes \text{VN}(G)$ is a faithful normal semifinite weight. The abelianess of G ensures that $\mathbb{1} \otimes \text{VN}(G)$ is abelian. Hence we may appeal to the theory of tracial von Neumann algebras to conclude that there must then be a net (e_γ) of projections in $(\mathbb{1} \otimes \text{VN}(G)) \cap \mathfrak{m}_{\tilde{\omega}}(M \rtimes_\alpha G)$ increasing to $\mathbb{1}$.

Now pass to the case where G is just the additive reals $\mathbb{R}$, with the crossed product being taken with the modular automorphism group of (M, ω) . The modular automorphism group for the pair $(M \rtimes_\omega \mathbb{R}, \tilde{\omega})$ is of course implemented by the λ_t 's [22, Proposition 6.61], which ensures that each λ_s (and therefore every element of $\{\lambda_t : t \in \mathbb{R}\}'' = \mathbb{1} \otimes \text{VN}(\mathbb{R})$) is a fixed point of $\sigma_t^{\tilde{\omega}}$. So in this case the net (e_γ) is in fact contained in the centralizer $(M \rtimes_\omega \mathbb{R})_{\tilde{\omega}}$. A similar argument to that used in [49, Lemma 3.1] can now be used to show that $A \rtimes_\omega \mathbb{R}$ is a subdiagonal subalgebra of $M \rtimes_\omega \mathbb{R}$ with respect to $\tilde{\omega}$. Since by construction $(e_\gamma) \subset D \rtimes_\omega \mathbb{R}$, it follows that then $A \rtimes_\omega \mathbb{R}$ is a semi- σ -finite subdiagonal subalgebra of $M \rtimes_\omega \mathbb{R}$ with respect to $\tilde{\omega}$.

¹ The authors believe that [26, Lemma 2.3] and the related results in [27] are more appropriate justification for the claims made in equation (1.5) of [28] and the discussion surrounding it.

3 Basic Structural Theory of H^p Spaces of Semi- σ -Finite Subdiagonal Subalgebras

As usual, for $1 \leq p < \infty$ we define $H^p(A)$ to be the closure in the p -norm of $i^{(p)}(A \cap \mathfrak{m}_\omega^M)$, and $H_0^p(A)$ to be the closure in the p -norm of $i^{(p)}(A_0 \cap \mathfrak{m}_\omega^M)$. The utility of the following theorem is the approximation it affords. (See the comment following the theorem for details.)

Theorem 3.1 *Suppose that we have weak* continuous inclusions of weak* closed operator algebras $D = A \cap A^* \subset A \subset M$, and M is a von Neumann algebra with a faithful normal semifinite weight ω and a normal ω -preserving conditional expectation $E : M \rightarrow D$. We also assume that ω is semifinite on D , and moreover that there is an increasing net of ω -finite projections $e_i \in D \cap M_\omega$ with $e_i \nearrow 1$. Let $1 \leq p \leq \infty$, and let M_i, A_i, D_i be the usual compressions by e_i . Then $L^p(M_i) = e_i L^p(M) e_i$ for each i , these are increasing with i , and the union of the $L^p(M_i)$ is dense in $L^p(M)$. Similarly $H^p(A_i) = [i^{(p)}(A_i)]_p$ equals $e_i H^p(A) e_i$, and these sets are increasing with i , and the p -closure of their union is $H^p(A)$. Similarly $H_0^p(A_i) = [i^{(p)}((A_i)_0)]_p = e_i H_0^p(A) e_i$, and these are increasing with the p -closure of their union being $H_0^p(A) = H^p(A) \cap \text{Ker}(E_p)$.*

In all of these increasing closures $F = \overline{\bigcup_i e_i F e_i}$ above, any $x \in F$ is a p -norm limit $x = \lim_i e_i x e_i$, if $1 \leq p \leq \infty$ (a SOT limit if $p = \infty$).

Proof We said already that $L^p(M_i) = e_i L^p(M) e_i$. Of course e_i acts on the left and right as contractive projections on $L^p(M)$, and $e_i \mathfrak{n}_\omega^M e_i \subset \mathfrak{n}_\omega^M$. Similarly, if $e_i \leq e_j$ then $L^p(M_i) \subset L^p(M_j)$. That the union of the $L^p(M_i)$ is dense in $L^p(M)$ follows from the Claim: If $e_i \nearrow 1$ in M then for each $\xi \in L^p(M)$ both (ξe_i) and $(e_i \xi)$ converge to ξ in p -norm. To see this we first note that $e_i \xi \rightarrow \xi$ weakly in $L^p(M)$. Indeed this follows by definition of the $L^p(M)$ and trace duality: if $\nu \in L^q(M)$ then $r = \xi \nu \in L^1(M)$ and

$$\text{tr}(\nu e_i \xi) = \text{tr}(e_i r) \rightarrow \text{tr}(r) = \text{tr}(\nu \xi),$$

where tr is the tracial functional on L^1 . Similarly $\xi e_i \rightarrow \xi$ weakly. Thus if $\xi e_i = 0$ (or $e_i \xi = 0$) for all i then $\xi = 0$. It then follows from [33, Lemma 2.4] that $\xi e_i \rightarrow \xi$ in p -norm. By continuity of $*$ in the p -norm (see 4.12(ii) and 7.12 in [22]) we have $e_i \xi \rightarrow \xi$ in p -norm. Then

$$\|e_i \xi e_i - \xi\|_p \leq \|e_i(\xi e_i - \xi)\|_p + \|e_i \xi - \xi\|_p \rightarrow 0.$$

We defined $H^p(A_i)$ to be $[i^{(p)}(A_i)]_p$ in $L^p(M_i)$. Claim: $[i^{(p)}(A_i)]_p = e_i [i^{(p)}(A \cap \mathfrak{m}_\omega^M)]_p e_i$. That $[i^{(p)}(A_i)]_p \subset e_i [i^{(p)}(A \cap \mathfrak{m}_\omega^M)]_p e_i$ is easy to see, and the converse is clear since $e_i(A \cap \mathfrak{m}_\omega^M) e_i \subset A_i$. Similarly the $[i^{(p)}(A_i)]_p$ are increasing, and the p -closure F of their union is $[i^{(p)}(A \cap \mathfrak{m}_\omega^M)]_p$. Indeed the $[i^{(p)}(A_i)]_p$ are clearly contained in $[i^{(p)}(A \cap \mathfrak{m}_\omega^M)]_p$ and hence also F . If $a \in A \cap \mathfrak{m}_\omega^M$ then $i^{(p)}(e_i a e_i) \in [A_i]_p$, and as we saw above $\|i^{(p)}(e_i a e_i) - i^{(p)}(a)\|_p \rightarrow 0$. So $[i^{(p)}(A \cap \mathfrak{m}_\omega^M)]_p$ is contained in F . Claim: $(A_i)_0 = e_i A_0 e_i$. Indeed clearly $e_i A_0 e_i \subset A_i \cap A_0 = (A_i)_0$.

Conversely, $(A_i)_0 \subset A_i \cap A_0 \subset e_i A_0 e_i$. A similar argument to the above applied to $(A_i)_0 = e_i A_0 e_i$ shows that we have $H_0^p(A_i) = [i^{(p)}((A_i)_0)]_p = e_i H_0^p(A) e_i$, and that these are increasing with the p -closure of their union being $[i^{(p)}(A_0 \cap m_\omega^M)]_p$. $\square$

The hypotheses of the theorem hold if A is a semi- σ -finite subdiagonal subalgebra (note that the last proof did not use subdiagonality of A or that A is invariant under the modular group). Throughout the rest of this part we will assume that A is a semi- σ -finite subdiagonal subalgebra of a von Neumann algebra M with respect to a given faithful normal semifinite weight ω , in the sense described in Theorem 2.1.

The key point is that if A is a semi- σ -finite subdiagonal algebra in M then the A_i are maximal σ -finite subdiagonal algebra in M_i , with the last theorem then providing a very convenient simplification of Haagerup's reduction method and its application to general subdiagonal algebras. In particular $L^p(M)$ is simply the completion of the increasing union vector space $\cup_i L^p(M_i)$ in the p -norm, and $H^p(A)$ is simply the completion of the increasing union of subdiagonal Hardy spaces $H^p(A_i)$. As we said in the introduction, the σ -finite Hardy space theory has been developed by many authors. Using this technology, most of the theory of semi- σ -finite subdiagonal algebras will follow as a 'limiting case' of the theory of maximal σ -finite subdiagonal algebras. We give several examples of this approach:

Corollary 3.2 (L^2 -density) *For a semi- σ -finite subdiagonal algebra the closure of $H^2(A) + H^2(A)^*$ is $L^2(M)$.*

Proof Clearly in the setting of the previous result $L^2(M)$ contains the unions of the $[i^{(p)}(A_i + A_i^*)]_2$. So for L^2 -density in $L^2(M)$ in such a setting we just in addition need L^2 -density in $L_2(M_i)$, i.e. that $i^{(p)}(A_i + A_i^*)$ is 2-dense in $L_2(M_i)$. This is well known for maximal subdiagonal algebras in a σ -finite von Neumann algebra such as our M_i . $\square$

For a semi- σ -finite subdiagonal algebra we saw that we have $L^p(D) \subset L^p(M)$, and by duality it follows easily that there exists an expectation $E_p : L^p(M) \rightarrow L^p(D)$ that is modular over m_ω^D . Generally conditional expectations $L^p(M) \rightarrow L^p(D)$ have been studied in [20, 28, 34], and the existence of E_p in general is shown e.g. in [23]. We sketch a quick proof in our setting: The proof of [34, Lemma 2.2] shows that the expectation $E_p^i : L^p(M_i) \rightarrow L^p(D_i)$ is the dual of the inclusion $L^p(D_i) \subset L^p(M_i)$. Thus if $i : L^p(D) \rightarrow L^p(M)$ is the inclusion then by Banach space duality and the fact that $L^p(M_i) = e_i L^p(M) e_i$ it is easy to see that $i|_{L^p(M_i)}^*$ maps into the copy of $L^p(D_i)$ and agrees with E_p^i . Thus $i^* : L^p(M) \rightarrow L^p(D)$ is an expectation, and by density it will have the desired properties of the expectation (e.g. most of the assertions of [34, Propositions 2.2 and 2.3]). Also, $E_p = \mathbb{E}^{(p)}$ on $i^{(p)}(m_\omega)$.

Corollary 3.3 *For a semi- σ -finite subdiagonal algebra we have $L^p(D) \subset H^p(A)$, and the expectation $E_p : L^p(M) \rightarrow L^p(D)$ maps $H^p(A)$ onto $L^p(D)$.*

Proof We of course have $i^{(p)}(D_i) \subset [i^{(p)}(A_i)]_p \subset L^p(M_i)$, and so $L^p(D) \subset H^p(A)$. The last assertion follows by topology since $E_p(H^p(A_i)) \subset L^p(D_i) \subset L^p(D)$ with E_p restricting to the identity on $i^{(p)}(D_i)$. $\square$

In the language of e.g. [22] the trace functional on $L^1(M_i)$ is $\text{tr}(e_i a e_i)$, where tr is the trace on $L^1(M)$. Note that $M_i \subset \mathfrak{m}_\omega$. We see that $\text{tr} = \text{tr}_i$ on M_i . In particular, $\text{tr}(x) = \lim_i \text{tr}(e_i x e_i) = \lim_i \text{tr}_i(e_i x e_i)$.

We also have a trace L^p duality. If $x \in L^p$, $y \in L^q$ then $xy \in L^1$ and $e_i x e_i \rightarrow x$ in p -norm, $e_i y e_i \rightarrow y$ in q -norm, $e_i x e_i y e_i \rightarrow xy$ in 1-norm. So

$$\langle x, y \rangle = \text{tr}(xy) = \lim_i \text{tr}_i(e_i x y e_i) = \lim_i \text{tr}_i(e_i x e_i y e_i).$$

As an easy consequence of the trace duality of noncommutative L^p -spaces we obtain the following:

Proposition 3.4 *For a semi- σ -finite subdiagonal algebra A we have that*

- (1) $H^p(A) = H_0^q(A)^\perp$ in $L^p(M)$ and $H_0^p(A) = H^q(A)^\perp$ for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$;
- (2) $\mathcal{H}^q(A) \cap \mathcal{H}^p(A) = \mathcal{H}^q(A) \cap \mathcal{L}^p(A)$ and $\mathcal{H}_0^q(A) \cap \mathcal{H}^p(A) = \mathcal{H}_0^q(A) \cap \mathcal{L}^p(A)$ for all $p, q \geq 1$.

Proof Let $\xi \in L^p(M)$ with $\xi \in H_0^q(A)^\perp$. So using the duality above,

$$\langle \xi, e_i \eta e_i \rangle = \langle e_i \xi e_i, e_i \eta e_i \rangle = 0, \quad \eta \in H_0^q(A_i).$$

By the σ -finite case [30, Corollary 3.4] we have $e_i \xi e_i \in H^q(A_i)$ so that $\xi = \lim_i e_i \xi e_i \in H^q(A)$. The second assertion of (1) is similar.

The proof of the second set of assertions is as in [2, Proposition 3.2 (ii)]. $\square$

Corollary 3.5 *If A is a semi- σ -finite subdiagonal algebra then $H^p(A)H^r(A) \subset H^s(A)$ if $1/s = 1/p + 1/r$ and $p, r, s \geq 1$.*

Proof Let $x \in H^p(A)$, $y \in H^r(A)$. Then $xy \in L^s(M)$ by e.g. [22, Proposition 7.24]. Suppose that $p, r < \infty$. Let $x_i = e_i x e_i \rightarrow x$ in p -norm, and $y_i = e_i y e_i \rightarrow y$ in r -norm. Since $e_i x e_i \in H^p(A_i) \subset L^p(M_i)$, $e_i y e_i \in H^r(A_i) \subset L^r(M_i)$, and A_i is maximal subdiagonal in σ -finite M_i , we have $x_i y_i \in H^s(A)$ with norm there $\leq \|x_i\| \|y_i\| \leq K$. Moreover

$$\begin{aligned} \|x_i y_i - xy\|_s &\leq \|x_i(y_i - y)\|_s + \|(x_i - x)y\|_s \\ &\leq \|x_i\|_p \|y_i - y\|_r + \|x_i - x\|_p \|y\|_s \rightarrow 0. \end{aligned}$$

(That $x_i y_i \in H^s(A)$ is clear from such arguments and the density of A_i in $H^p(A_i)$ for any $p \geq 1$.) Thus xy is in $H^s(A)$ since the latter is closed in $L^p(M)$. Suppose that $p = \infty$ and $r < \infty$, with $y_i = e_i y e_i \rightarrow y$ in r -norm. Then $e_j x y_i \rightarrow e_j x y$ in r -norm for each j . So $e_j x y \in H^r(A)$ from which it is easy to see that $xy \in H^r(A)$. $\square$

In general the converse of the last result, Riesz factorization, is not even known for maximal subdiagonal algebras in the σ -finite case. However if the A_i do have Riesz factorization then so will A by a weakly converging subnet argument, at least for finite $p, r > 1$.

Corollary 3.6 (The Hilbert transform) *Let A be a semi- σ -finite subdiagonal algebra. For $1 < p < \infty$ there exists a bounded linear map $H : L^p(M) \rightarrow L^p(M)$ such that $x + iHx \in H^p(A)$ for all $x \in L^p(M)$. This is the unique bounded extension of the map $i^{(p)}(a + d + b^*) \mapsto i i^{(p)}(b^* - a)$ for $a, b \in A_0 \cap m_\omega^M, d \in D \cap m_\omega^D$.*

Proof Indeed by [30] we have unique norm bounded $H_i : L^p(M_i) \rightarrow L^p(M_i)$ such that $x + iH_i x \in H^p(A_i)$ for all $x \in L^p(M_i)$, with H_i extending the map

$$i^{(p)}(a + d + b^*) \mapsto i i^{(p)}(b^* - a) \text{ for } a, b \in (A_i)_0, d \in D_i.$$

By the uniqueness the H_i are ‘compatible’, hence they simultaneously extend to a bounded $H : L^p(M) \rightarrow L^p(M)$. For $x \in L^p(M)$ we have $x + iHx = \lim_i e_i x e_i + iH_i(e_i x e_i) \in H^p(A)$. Note that for $a, b \in A_0 \cap m_\omega^M, d \in D \cap m_\omega^D$

$$\begin{aligned} H(i^{(p)}(a + d + b^*)) &= \lim_i H(i^{(p)}(e_i a e_i + e_i d e_i + e_i b^* e_i)) \\ &= \lim_i i i^{(p)}(e_i b^* e_i - e_i a e_i) = i(b^* - a). \end{aligned}$$

So by density H is the unique bounded extension of the map $a + d + b^* \mapsto i(b^* - a)$ for $a, b \in A_0 \cap m_\omega^M, d \in D \cap m_\omega^D$. $\square$

Corollary 3.7 *Let A be a semi- σ -finite subdiagonal algebra. For $1 < p < \infty$, $L^p(M) = H_0^p \oplus L^p(D) \oplus (H_0^p)^*$.*

This is just as in Corollary 3.3 in [30] if $p > 1$. Note that e.g. $H_0^p \cap (H_0^p)^* = (0)$ for $p \geq 1$, since if $x \in H_0^p \cap (H_0^p)^*$ then $e_i x e_i \in H_0^p(A_i) \cap (H_0^p(A_i))^* = (0)$, so that $x = 0$.

For $p = 1$ we have L^1 -density: the 1-closure of $H^1(A) + H^1(A)^*$ is $L^1(M)$ since it contains the unions of the $[i^{(1)}(A_i + A_i^*)]_1 = L_1(M_i)$. In the last line we used L^1 -density in the σ -finite case. (A proof of the latter: if $x \in M_i$ annihilates $i^{(1)}(A_i + A_i^*)$, then by [24, Proposition 2.10] $i^{(1)}(x)$ annihilates $A_i + A_i^*$, and hence also M_i by weak* density. So $i^{(1)}(x) = 0$ whence $x = 0$.) We have $H^1(A) = H_0^1(A) \oplus L^1(D)$. Indeed suppose that $x \in [A_0]_1 \cap [D]_1$ and $x_i = e_i x e_i \rightarrow x$ in the 1 norm. Our earlier analysis then shows that $(x_i) \subset [(A_i)_0]_1 \cap [D_i]_1$. Then $0 = E(x_i) = x_i = E_p(x_i)$, whence $x = 0$. So $[A_0]_1 \cap [D]_1 = (0)$.

4 A Beurling Theory for Semi- σ -Finite Subdiagonal Subalgebras

In the present context we define a (right) A -invariant subspace of $L^p(M)$, to be a closed subspace K of $L^p(M)$ such that $KA \subset K$. For consistency, we will not consider left invariant subspaces at all, leaving the reader to verify that entirely symmetric results pertain in the left invariant case. Invariant subspaces may be classified in accordance with their structure. In this regard we say that an invariant subspace K is *simply invariant* if in addition the closure of KA_0 is properly contained in K .

Given a right A -invariant subspace K of $L^2(M)$, we define the *right wandering subspace* of K to be the space $W = K \ominus [KA_0]_2$, and then say that K is *type 1* if W generates K as an A -module (that is, $K = [WA]_2$), and *type 2* if $W = (0)$.

The development of especially the L^p -version of the theory of closed right A -invariant subspaces, makes deep use of the concept of a ‘column L^p -sum’ as introduced in [33]. Given $1 \leq p < \infty$ and a collection $\{X_i : i \in I\}$ of closed subspaces of $L^p(M)$, the *external* column L^p -sum $\oplus_i^{col} X_i$ is defined to be the closure of the restricted algebraic sum in the norm $\|(x_i)\|_p \stackrel{def}{=} \text{tr}((\sum_i x_i^* x_i)^{\frac{p}{2}})^{\frac{1}{p}}$. That this is a norm for $1 \leq p < \infty$ is verified in [31]. If X is a subspace of $L^p(M)$, and if $\{X_i : i \in I\}$ is a collection of subspaces of X , which together densely span X , with the added property that $X_i^* X_j = \{0\}$ if $i \neq j$, then we say that X is the *internal* column L^p -sum $\oplus_i^{col} X_i$. We shall not need the concept of an external column sum. So wherever column sum is mentioned below, it shall refer to an internal column sum. Note that if J is a finite subset of I , and if $x_i \in X_i \subset L^p$ for all $i \in J$, then we have that

$$\text{tr} \left(\left| \sum_{i \in J} x_i \right|^p \right)^{1/p} = \text{tr} \left(\left(\left| \sum_{i \in J} x_i \right|^2 \right)^{\frac{p}{2}} \right)^{1/p} = \text{tr} \left(\left(\sum_{i \in J} x_i^* x_i \right)^{\frac{p}{2}} \right)^{1/p}.$$

This shows that X is then isometrically isomorphic to the external column L^p -sum $\oplus_i^{col} X_i$. Since the projections onto the summands are clearly contractive, it follows by routine arguments (or by [31, Lemma 2.4]) that if $(x_i) \in \oplus_i^{col} X_i$, then the net $(\sum_{j \in J} x_j)$, indexed by the finite subsets J of I , converges in norm to (x_i) .

The first cycle of results we present are extensions of corresponding results in §2 of [35]. The first result in this regard is basically a restatement of [35, Theorem 2.4]. The exact same proof offered in [35] goes through in the general setting and hence we forgo the proof.

Theorem 4.1 *Let A be an analytically conditioned algebra in M .*

- (1) *Suppose that X is a subspace of $L^2(M)$ of the form $X = Z \oplus^{col} [YA]_2$ where Z, Y are closed subspaces of X , with Z a type 2 invariant subspace, and*

$$\{y^* x : y, x \in Y\} = Y^* Y \subset L^1(D).$$

Then X is simply right A -invariant if and only if $Y \neq \{0\}$.

- (2) *If X is as in (1), then $[YD]_2 = X \ominus [XA_0]_2$ (and $X = [XA_0]_2 \oplus [YD]_2$).*
- (3) *If X is as described in (1), then that description also holds if Y is replaced by $[YD]_2$. Thus (after making this replacement) we may assume that Y is a D -submodule of X .*
- (4) *The subspaces $[YD]_2$ and Z in the decomposition in (1) are uniquely determined by X . So is Y if we take it to be a D -submodule (see (3)).*
- (5) *If A is maximal subdiagonal, then any right A -invariant subspace X of $L^2(M)$ is of the form described in (1), with Y the right wandering subspace of X .*

Building on Theorem 4.1, we are now able to present the following rather elegant decomposition of the right wandering subspace. This extends [35, Proposition 2.5]. The proof of the general case is quite a bit more tricky than that of the σ -finite case, and hence full details need to be provided.

Theorem 4.2 *Let A be an analytically conditioned algebra in M . Suppose that X is as in Theorem 4.1, and that W is the right wandering subspace of X . Then W may be decomposed as an orthogonal direct sum $\bigoplus_i^{\text{col}} u_i L^2(D)$, where u_i are partial isometries in M for which $u_i \left(\frac{d\bar{\omega}}{d\tau_L} \right)^{1/2} a \in W$ for each $a \in A \cap \mathfrak{n}_\omega$, with $u_i^* u_i \in D$, and $u_j^* u_i = 0$ if $i \neq j$. If W has a cyclic vector for the D -action, then we need only one partial isometry in the above.*

Proof By the theory of representations of a von Neumann algebra (see e.g. the discussion at the start of Section 3 in [33]), any normal Hilbert D -module is an L^2 direct sum of cyclic Hilbert D -modules, and if K is a normal cyclic Hilbert D -module, then K is spatially isomorphic to $eL^2(D)$, for an orthogonal projection $e \in D$.

Suppose that the latter isomorphism is implemented by a unitary D -module map ψ . Let (f_λ) be a net in $\mathfrak{n}(D)_\omega^* \cap \mathfrak{n}(D)_\omega$ converging strongly to $\mathbb{1}$ (in a more general situation this is guaranteed by Lemma 9 in [45]). If in addition $K \subset W$, we will then have that $g_\lambda = \psi(e[f_\lambda h^{1/2}]) \in W$ for each λ , where $h = \frac{d\bar{\omega}}{d\tau_L}$. Then

$$\text{tr}(d^* g_\lambda^* g_\lambda d) = \|\psi(e[f_\lambda h^{1/2}]d)\|_2^2 = \text{tr}(d^* (h^{1/2} f_\lambda^*) e[f_\lambda h^{1/2}]d),$$

for each $d \in D$, and so $g_\lambda^* g_\lambda = (h^{1/2} f_\lambda^*) e[f_\lambda h^{1/2}] = |e[f_\lambda h^{1/2}]|^2$. Hence there exists a partial isometry u_λ majorised by e such that $g_\lambda = u_\lambda e[f_\lambda h^{1/2}] = u_\lambda [f_\lambda h^{1/2}]$. By the modular action of ψ we will then have that $\psi(e[f_\lambda h^{1/2}]d) = g_\lambda d = u_\lambda [f_\lambda h^{1/2}]d$ for any $d \in D$. Since $L^2(D)$ is the closure of $\{(h^{1/2}d) : d \in \mathfrak{n}(D)_\omega\}$ (see [22, Proposition 7.40 & Theorem 7.45]), and since $\psi(e[f_\lambda h^{1/2}]d) = u_\lambda [f_\lambda h^{1/2}]d = (u_\lambda f_\lambda)(h^{1/2}d)$ for each $d \in \mathfrak{n}(D)_\omega^*$, it follows $\psi(e f_\lambda b) = u_\lambda f_\lambda b$ for all $b \in L^2(D)$.

When working with D , we may of course assume that D is in standard form (see [25]), in which case the Haagerup–Terp standard form enables us to further identify $L^2(D)$ with the underlying Hilbert space of D . But then the σ -strong* convergence of (f_λ) to $\mathbb{1}$ ensures that $e f_\lambda b$ will for any $b \in L^2(D)$ converge in L^2 -norm to eb . Since $\|eb - e f_\lambda b\|_2 = \|\psi(eb - e f_\lambda b)\|_2 = \|\psi(eb) - u_\lambda f_\lambda b\|_2$, this in turn ensures that $(u_\lambda f_\lambda b)$ converges to $\psi(eb)$ in L^2 -norm. Given that the net $(u_\lambda f_\lambda)$ is in the unit ball of M , it must admit a subnet $(u_\gamma f_\gamma)$ which converges to some element u_e of the unit ball of M . For any $b \in L^2(D)$ the net $(u_\gamma f_\gamma b)$ will then converge to $u_e b$ in the L^2 -weak topology. But $(u_\gamma f_\gamma b)$ is also a subnet of $(u_\lambda f_\lambda b)$ which converges to $\psi(eb)$, and will therefore itself still be L^2 -norm convergent to $\psi(eb)$. It is therefore clear that $\psi(eb) = u_e b$ for each $b \in L^2(D)$ and hence that $(u_\lambda f_\lambda b)$ is for each $b \in L^2(D)$, L^2 -norm convergent to $u_e b$. For any $b \in L^2(D)$, we now also have that

$$\begin{aligned} \text{tr}(d^* b^* e b d) &= \|e b d\|_2^2 \\ &= \lim_\lambda \|e f_\lambda b d\|_2^2 \\ &= \lim_\lambda \|\psi(e f_\lambda b d)\|_2^2 \\ &= \lim_\lambda \|\psi(e f_\lambda b) d\|_2^2 \\ &= \lim_\lambda \|u_\lambda f_\lambda b d\|_2^2 \end{aligned}$$

$$\begin{aligned}
&= \|u_e b d\|_2^2 \\
&= \operatorname{tr}(d^* b^* u_e^* u_e b d).
\end{aligned}$$

This equality firstly ensures that $b^* e b = b^* u_e^* u_e b$ for all $b \in L^2(D)$, which then in turn ensures that $u_e^* u_e = e$. It follows that u_e is a partial isometry with initial projection e , and that $\psi(e L^2(D)) = u_e L^2(D)$.

Given u_i and u_j with $i \neq j$, we have that $u_i L^2(D), u_j L^2(D) \subset W$. Hence $L^2(D) u_j^* u_i L^2(D) \subset L^1(D)$. Since we have that $\operatorname{tr}([d_1^* h^{1/2}] u_j^* u_i (h^{1/2} d_0))$ equals

$$\operatorname{tr}(\psi(e_j(h^{1/2} d_1))^* \psi(e_i(h^{1/2} d_0))) = \operatorname{tr}([d_1^* h^{1/2}] e_j e_i (h^{1/2} d_0)) = 0,$$

for any $d_0, d_1 \in \mathfrak{n}(D)_\omega$, the density of $\{(h^{1/2} d) : d \in \mathfrak{n}(D)_\omega\}$ in $L^2(D)$ now ensures that $u_j^* u_i = 0$. In the case where $i = j$ we of course have that $u_i^* u_i = e_i \in D$. Putting these facts together, we see that W is of the desired form. $\square$

The first corollary of the above theorem corresponds to [35, Corollary 2.5]. Here too the proof of the general case requires more delicacy than that of the σ -finite case, and hence we state the proof in full.

Corollary 4.3 *Let A be a semi- σ -finite subdiagonal subalgebra. Suppose that X is as in Theorem 4.1, and that W is the right wandering subspace of X . If $X \subset \mathcal{H}^2(A)$ then $Z \perp L^2(D)$. If additionally A is maximal subdiagonal, then the partial isometries u_i described in the preceding Proposition, all belong to A .*

Proof If indeed $X \subset \mathcal{H}^2(A)$, it is a fairly trivial observation to make that $Z = [Z A_0]_2 \subset [X A_0]_2 \subset [\mathcal{H}^2(A) A_0]_2 = \mathcal{H}_0^2(A)$. Since $\mathcal{H}^2(A) = \mathcal{H}_0^2(A) \oplus L^2(D)$, the first claim follows.

Now suppose that A is maximal subdiagonal. To see the second claim recall that in the proof of Theorem 4.2, we showed that $u_i L^2(D) \subset W$ for each i .

Let $\{e_\alpha\}$ be the net of projections in D increasing to $\mathbb{1}$ in Theorem 3.1. Given any $a_0 \in e_\alpha A_0 e_\alpha \subset A_0$, we will for any $b \in e_\alpha L^2(D) e_\alpha = L^2(e_\alpha D e_\alpha)$ therefore have that $a_0 e_\alpha u_i e_\alpha b \in a_0 W \subset A_0 H^2(A) \subset H_0^2(A)$. But $\mathbb{E}_2$ annihilates $H_0^2(A)$, and hence we must have that $0 = \mathbb{E}_2(a_0 e_\alpha u_i e_\alpha b) = \mathbb{E}(a_0 e_\alpha u_i e_\alpha) b$ for all $b \in L^2(e_\alpha D e_\alpha)$. This can of course only be if $\mathbb{E}(a_0 e_\alpha u_i e_\alpha) = 0$. Since $a_0 \in e_\alpha A_0 e_\alpha$ was arbitrary, we may now apply the sharpened Arveson maximality criterion in [31, Theorem 2.2] to see that $e_\alpha u_i e_\alpha \in e_\alpha A e_\alpha \subset A$. The fact that $\{e_\alpha\}$ is increasing now clearly ensures that we in fact have that $e_\alpha u_i e_\beta \in A$ for any α and β . Therefore $u_i = \lim_\alpha \lim_\beta e_\alpha u_i e_\beta \in A$ as claimed. $\square$

The next three results correspond to [35, Corollary 2.6, Proposition 2.7 & Theorem 2.8]. The proofs in [35] carry over to the general case, and hence we content ourselves with merely stating these results

Corollary 4.4 *If X is an invariant subspace of the form described in Theorem 4.1, then X is type I if and only if $X = \bigoplus_i^{\text{col}} u_i \mathcal{H}^2(A)$, for u_i as in Theorem 4.2.*

Proof If X is type 1, then $X = [WA]_2$ where W is the right wandering space, and so the one assertion follows from Theorem 4.2. If $X = \oplus_i^{col} u_i \mathcal{H}^2(A)$, for u_i as above, then $[XA_0]_2 = \oplus_i^{col} u_i \mathcal{H}^2(A_0)$, and from this it is easy to argue that $W = \oplus_i^{col} u_i L^2(D)$. Thus $X = [WA]_2 = \oplus_i^{col} u_i \mathcal{H}^2(A)$. $\square$

Proposition 4.5 *Let X be a closed A -invariant subspace of $L^2(M)$, where A is an analytically conditioned subalgebra of M .*

- (1) *If $X = Z \oplus [YA]_2$ as in Theorem 4.1, then Z is type 2, and $[YA]_2$ is type 1.*
- (2) *If A is a maximal subdiagonal algebra, and if $X = K_2 \oplus^{col} K_1$ where K_1 and K_2 are types 1 and 2 respectively, then K_1 and K_2 are respectively the unique spaces Z and $[YA]_2$ in Theorem 4.1.*
- (3) *If A and X are as in (2), and if X is type 1 (resp. type 2), then the space Z of Theorem 4.1 for X is (0) (resp. $Z = X$).*
- (4) *If $X = K_2 \oplus^{col} K_1$ where K_1 and K_2 are types 1 and 2 respectively, then the right wandering subspace for X equals the right wandering subspace for K_1 .*

On collating the information contained in the preceding four results, we obtain the following structure theorem for invariant subspaces of L^2 .

Theorem 4.6 *If A is a maximal subdiagonal subalgebra of M , and if K is a closed right A -invariant subspace of $L^2(M)$, then:*

- (1) *K may be written uniquely as an (internal) L^2 -column sum $K_2 \oplus^{col} K_1$ of a type 1 and a type 2 invariant subspace of $L^2(M)$, respectively.*
- (2) *If $K \neq (0)$ then K is type 1 if and only if $K = \oplus_i^{col} u_i H^2$, for u_i partial isometries with mutually orthogonal ranges and $|u_i| \in D$.*
- (3) *The right wandering subspace W of K is an $L^2(D)$ -module in the sense of Junge and Sherman, and in particular $W^*W \subset L^1(D)$.*

The theory of invariant subspaces described above carries over to yield analogous statements in the setting of L^p spaces ($1 \leq p \leq \infty$) just as the authors showed in the M ‘tracially finite’ case in [10]. However the proof of these facts requires some significant analysis which is far beyond the scope of this paper, and will be postponed to a future paper of the second author and Xu [37]. We content ourselves with merely stating the results. Details will be provided in the forthcoming paper of Labuschagne and Xu [37]. The following result extends [10, Corollary 4.3].

Theorem 4.7 *For any $1 \leq p, q \leq \infty$ there is a lattice isomorphism from the weak*-closed right A -invariant subspaces of L^p to those of L^q . We in particular have the following:*

- (1) *Given $2 \leq p < \infty$ and a right A -invariant closed subspace K of $L^p(M)$, the prescription $K \rightarrow \overline{\mathcal{S}(K)_p}^{w*}$ where $\mathcal{S}(K)_p$ is a right $A \cap \mathfrak{n}_\omega$ -invariant subspace of $\mathfrak{n}_\omega$ for which $K = [\langle \mathcal{S}(K)_p h^{1/p} \rangle]_p$, realises a lattice isomorphism from the right A -invariant subspaces of $L^p(M)$ to those of $L^\infty(M)$.*
- (2) *Given $1 \leq p < 2$ and a right A -invariant closed subspace K of $L^p(M)$, the prescription $K \rightarrow [\langle h^{1/2} \mathcal{S}(K)_p \rangle]_2$ (where $r > 0$ is chosen so that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$)*

and where $\mathcal{S}(K)_p$ is a right $A \cap \mathfrak{n}_\omega$ -invariant subspace of $\mathfrak{m}_\omega$ for which $K = [\langle h^{1/2} \mathcal{S}(K)_p h^{1/r} \rangle_p]$ realises a lattice isomorphism from the right A -invariant subspaces of $L^p(M)$ to those of $L^2(M)$.

As in the case of L^2 , we say that a closed right A -invariant subspace K of L^p is a type 2 invariant subspace if $K = [K A_0]_p$. With this concept in place, we are now able to present the following analogue of Theorem 4.6. This result extends each of [41, Theorem 4.6] and [5, Theorems 3.6 & 3.8]. The proof of the second part closely follows that of [10, Theorem 4.5].

Theorem 4.8 *Let A be a maximal subdiagonal subalgebra of M , and suppose that K is a closed A -invariant subspace of $L^p(M)$, for $1 \leq p \leq \infty$. (For $p = \infty$ we assume that K is σ -weakly closed.)*

- (1) *The space K may then be written as an L^p -column sum of the form $Z \oplus^{col} (\oplus_i^{col} u_i H^p)$ where Z is a type 2 closed right A -invariant subspace of L^p , and where the u_i 's are partial isometries in $M \cap K$ with $u_j^* u_i = 0$ if $i \neq j$ and $u_i^* u_i \in D$. Moreover, for each i , $u_i^* Z = (0)$, left multiplication by the $u_i u_i^*$ are contractive projections from K onto the summands $u_i H^p(A)$, and left multiplication by $1 - \sum_i u_i u_i^*$ is a contractive projection from K onto Z .*
- (2) *Let K be in the form $Z \oplus^{col} (\oplus_i^{col} u_i H^p)$ described above. Then there exists a contractive projection from K onto $\oplus_i^{col} u_i L^p(D)$ and along $[K A_0]_p$. The quotient $K/[K A_0]_p$ is therefore isometrically D -isomorphic to the subspace $\oplus_i^{col} u_i L^p(D)$. (Here $[\cdot]_\infty$ is as usual the σ -weak closure.)*

For $p \neq 2$ we say that a right A -invariant subspace of $L^p(M)$ is type 1 if it corresponds to a type 1 invariant subspace in $L^2(M)$ under the canonical lattice isomorphism constructed in Theorem 4.7. Then the first assertion of the preceding theorem may be interpreted as the statement that any closed right A -invariant subspace of L^p may be written as a column sum of a type 1 and type 2 invariant subspace. The following analogue of Beurling's characterization of weak* closed ideals of $H^\infty(\mathbb{D})$ now also readily follows from Theorem 4.8. This extends [10, Corollary 4.8] where this fact was noted for the case of finite von Neumann algebras.

Corollary 4.9 *If A is maximal subdiagonal, then the type 1 σ -weakly closed right ideals of A are precisely those right ideals of the form $\oplus_i^{col} u_i A$, for partial isometries $u_i \in A$ with mutually orthogonal ranges and $|u_i| \in D$.*

5 Lebesgue Decomposition, F & M Riesz, and Gleason–Whitney Theorems

5.1 F & M Riesz Spaces

Let M be a von Neumann algebra. Generalizing p. 8230 in [11], a subalgebra (resp. subspace) A of a von Neumann algebra M will be said to be an *F & M Riesz algebra* (resp. *space*) if whenever $\varphi \in M^*$ annihilates A (that is, $\varphi \in A^\perp$) then the normal and

singular parts, φ_n and φ_s , also annihilate A . Following [46], we showed in [11, Section 5] that this F & M Riesz property was a consequence of a *Lebesgue decomposition* for A^* . This idea was generalized further to linear spaces in [16]. See e.g. Theorem 2.4 in [16], although there are some subtle differences between their setup and ours. See also [17] for some background to this paper.

Write A_s^* and A_n^* for the set of restrictions to A of singular and normal functionals on M . One has:

Lemma 5.1 *A subspace A of a von Neumann algebra M is an F & M Riesz space if and only if $A_n^* \cap A_s^* = (0)$, and indeed if and only if A has a unique Lebesgue decomposition relative to M : that is, any $\varphi \in A^*$ may be written uniquely as $\varphi = \varphi_n + \varphi_s$ with $\varphi_n \in A_n^*$ and $\varphi_s \in A_s^*$. Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.*

Proof Indeed if $A_n^* \cap A_s^* = (0)$ suppose that $\varphi \in M^*$ annihilates A . Then $(\varphi_n)_{|A} = -(\varphi_s)_{|A}$ on A , so these equal 0. Conversely, if A is an F & M Riesz space and a functional in $A_n^* \cap A_s^*$ is the restriction of both φ_n and φ_s , then $\varphi_n - \varphi_s$ annihilates A . So φ_n and φ_s also annihilate A . Thus being an F & M Riesz algebra is equivalent to the assertion $A^* \cong A_n^* \oplus A_s^*$. This is because any Hahn-Banach extension ψ of a functional $\varphi \in A^*$ has a normal plus singular decomposition $\varphi_n + \varphi_s$. Indeed as in [46, Proposition 1(2)] we have

$$\|\varphi\| = \|\psi\| = \|\varphi_n\| + \|\varphi_s\| \geq \|(\varphi_n)_A\| + \|(\varphi_s)_A\| \geq \|(\varphi_n)_A + (\varphi_s)_A\| = \|\varphi\|.$$

So $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$. Conversely, the uniqueness of the Lebesgue decomposition implies that $A_n^* \cap A_s^* = (0)$. $\square$

It is well known that the F & M Riesz property above implies the Gleason–Whitney property. In an operator algebraic setting this first appeared in [8]. The proof there (the last lines of [8, Theorem 4.1] or p. 102 in [9]) generalizes vastly—see e.g. Remark 1 at the end of [46], [11, Section 5], or as noted recently in [16]. We state it in slightly greater generality.

Corollary 5.2 *F & M Riesz subspaces have the Gleason–Whitney property (GW1) from [8]: every Hahn–Banach extension to M of a relatively weak* continuous functional ω on an F & M Riesz subspace A of M is normal.*

Proof This follows by a modification of the argument in the last lines of our proof of [8, Theorem 4.1]. Let φ be a Hahn–Banach extension to M of a normal functional ω on A , and write $\varphi = \varphi_n + \varphi_s$. Let ξ be the restriction to A of $\varphi - \varphi_n$. This is relatively weak* continuous. So by a known variant of the Hahn–Banach theorem it extends to a weak* continuous functional ψ on M . Since $\varphi_s = \psi$ on A we must have $\xi = 0$, thus $\omega = \varphi_n$ on A . But then $\|\varphi_n\| + \|\varphi_s\| = \|\varphi\| = \|\omega\| \leq \|\varphi_n\|$, so that $\varphi_s = 0$, and φ is normal. $\square$

The following consequences are generalizations of results from [11, Section 5] and others of our earlier papers (for example for the relation with (ii) in the next result see the remark before [12, Proposition 3.4]).

Lemma 5.3 *Suppose that A is a unital subspace of a von Neumann algebra M . Consider the conditions:*

- (i) (GW2) *There is at most one normal Hahn–Banach extension to M of any functional on A .*
- (ii) *There is at most one normal state extension to M of any state on A .*
- (iii) *$A + A^*$ is weak* dense in M .*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and they all are equivalent if in addition A is a subalgebra of M .

Proof Clearly (i) $\Rightarrow$ (ii), with (ii) $\Rightarrow$ (iii) following from the last paragraph of the proof of [11, Lemma 5.8]. If A is a subalgebra then (iii) $\Rightarrow$ (i) by the other part of the proof of [11, Lemma 5.8] (we note that in that proof $\|\varphi\| = 1$). $\square$

Corollary 5.4 *Suppose that A is an F & M Riesz subspace of a von Neumann algebra M such that $A + A^*$ is weak* dense in M . If $\varphi \in M^*$ annihilates $A + A^*$ then φ is singular. Any normal functional on M is the unique Hahn–Banach extension of its restriction to $A + A^*$, and in particular is normed by $A + A^*$.*

Proof If A is an F & M Riesz subspace then so is A^* using the fact that ψ is normal (resp. singular) iff ψ^* is normal (resp. singular). The first assertion follows from this, as in [8, Corollary 3.5]. This implies the last assertion as in [8, Theorem 4.2]. $\square$

Corollary 5.5 *If A is an F & M Riesz subspace of a von Neumann algebra M such that $A + A^*$ is weak* dense in M , then $\text{Ball}(A + A^*)$ is weak* dense in $\text{Ball}(M)$.*

Actually a stronger ‘Kaplansky density’ statement holds for F & M Riesz algebras, as in the remark after [11, Corollary 5.7] referring to Theorem 3.2 there. The last lines of the proof of the latter theorem if A is unital use Lemma 3.1 there, whose generality includes the present setting.

The full ‘Gleason–Whitney property’ is a combination of both GW1 and GW2:

Corollary 5.6 [11, Corollary 5.9] *Suppose that A is a unital F & M Riesz space in a von Neumann algebra M . Then A has (GW2) if and only if every relatively weak* continuous functional on A has a unique Hahn–Banach extension to M , and if and only if every relatively weak* continuous functional on A has a unique normal Hahn–Banach extension to M . (Recall if A is also an algebra then (GW2) is equivalent to $A + A^*$ being weak* dense in M .)*

There is an obvious variant of the last result with the word ‘functional’ replaced by ‘state’.

Corollary 5.7 (Cf. [8, Section 3] and [12, Proposition 3.4]) *Suppose that A is a unital F & M Riesz space in a von Neumann algebra M . There is a completely contractive (or equivalently, completely positive) normal extension to M of a relatively weak* continuous completely contractive unital $\Phi : A \rightarrow B(H)$. Indeed any contractive unital extension to M of a contractive unital relatively weak* continuous $\Phi : A \rightarrow B(H)$ is necessarily normal. If $A + A^*$ is weak* dense in M then such extensions are unique.*

Proof The idea from [12, Proposition 3.4]: Suppose that $\Psi : M \rightarrow B(H)$ is a contractive unital extension of relatively weak* continuous contractive unital $\Phi : A \rightarrow B(H)$. The norm of $\langle \Phi(\cdot)\zeta, \zeta \rangle$ is 1 if $\|\zeta\| = 1$. Hence $\langle \Psi(\cdot)\zeta, \zeta \rangle$ is a Hahn-Banach extension. So it is weak* continuous by GW1 (Corollary 5.2). By polarization, Ψ is weak* continuous. If $A + A^*$ is weak* dense in M then since Ψ is completely positive and normal it is the necessarily unique weak* continuous extension of Φ . $\square$

Corollary 5.8 *Suppose that A is a unital F & M Riesz space in a von Neumann algebra M . If $A + A^*$ is weak* dense in M then every *-representation of M which is relatively weak* continuous on A is a boundary representation of A in the sense of 4.1.11 in [14], and is normal. (This is Arveson's notion of boundary representation with the irreducibility requirement dropped; more recently referred to as the 'unique extension property'.)*

Proof Immediate from the last corollary. $\square$

Corollary 5.9 (Cf. [12, Section 3] and [13]) *Suppose that A is a unital F & M Riesz space in a von Neumann subalgebra $M \subset B(H)$. Suppose that D is a von Neumann subalgebra of M which is contained in A with $DAD \subset A$ (so A is a D -bimodule). If $\Phi : A \rightarrow D$ is a relatively weak* continuous contractive projection onto D , then Φ is completely contractive, and extends to a complete contraction $\Psi : M \rightarrow B(H)$, and this extension is necessarily normal. If $A + A^*$ is weak* dense in M then Ψ is a normal conditional expectation onto D , and is unique.*

Proof A contractive projection from A onto D is a D -module map and hence is completely contractive. To see these use [15, Lemma 3.2], and the calculation two paragraphs above Lemma 1.1 in [12]). Let $\Psi : M \rightarrow B(H)$ be the completely contractive (normal) extension in Corollary 5.7. The rest is as in the proof of that result, and we have $\Psi(M) \subset D$ if $A + A^*$ is weak* dense in M . $\square$

Remark The above observations complement the work done in the later sections of [13] where we studied conditions under which weak* continuous contractive homomorphisms $\Phi : A \rightarrow D$ extend to normal conditional expectations $M \rightarrow D$ (see for example [13, Theorem 6.3] in the case that M has a faithful normal tracial state.)

We remark that several results in this section may be generalized beyond the case that M is a von Neumann algebra. Indeed some only use the fact that M has a Lebesgue decomposition, and in some places also that M is selfadjoint).

5.2 Riesz Approximable Algebras

In this section we give many applications that not only apply to the semi- σ -finite subdiagonal algebras of Sects. 2–4, but also in a more general setting that we call Riesz approximable algebras. Moreover they apply in settings like the Hardy space of the half-plane where the weight on M is not semifinite on D , and to other much more general settings. Indeed many of the results here do not reference the specific weight on M , their statements essentially only involving the Banach space structure of A and M .

Suppose that M has weak* closed *-subalgebras M_i whose union is weak* dense in M , and that A is a subalgebra (resp. subspace) of M such that $A_i = A \cap M_i$ is a F & M Riesz algebra (resp. space) in M_i . We will also assume that the union of the A_i is relatively weak* dense in A . We also require that the restriction of any positive singular functional on M to M_i is singular on M_i (the word ‘compatible’ is used for this property in [16]). Note that hereditary subalgebras have the latter property, as is easily seen using [43, Theorem III.3.8].

We call an algebra (resp. space) A with the properties in the last paragraph a *Riesz approximable algebra* (resp. *space*) in M . One may then consider Riesz approximable algebras that are also subdiagonal, for example a weak* closed unital Riesz approximable algebra in M for which the M_i are σ -finite and the subalgebras A_i above are maximal subdiagonal in M_i (which implies that A_i is a F & M Riesz algebra by [11, Section 5]). These will include the semi- σ -finite maximal subdiagonal algebras in previous sections.

Lemma 5.10 *Every semi- σ -finite subdiagonal algebra is a Riesz approximable algebra.*

Proof Indeed as before, since $e_i \nearrow 1$ strongly, $\cup_i M_i$ and $\cup_i A_i$ are weak* dense in M and A respectively. The A_i are F & M Riesz algebras by [11, Theorem 5.3]. Also, $M_i = e_i M e_i$ is a hereditary subalgebra of M , so as mentioned above the restriction of any positive singular functional on M to M_i is singular on M_i . $\square$

Remark One of the important ingredients of the last proof is that subdiagonal algebras in a σ -finite von Neumann algebra are F & M Riesz algebras, or equivalently have a Lebesgue decomposition. This is proved in [11, Theorem 5.3] by following Ueda’s strategy, which follows Ando’s strategy, of deducing the Lebesgue decomposition from an Amar–Lederer peak set type result. We take the opportunity to make two related comments. First, the proof of Proposition 2.3 in [11] needs correction (the result itself is true). If B is unital then in the proof that (iii) implies (i) if $a_n \in B_{sa}$ one may appeal to [38, Proposition 3.11.9] to see that $1 - q$ is open. If $b_n \searrow q$ then let $A = C^*(\{1, b_n\})$, a separable C^* -algebra with $q \in A^{**}$. In A^{**} the projection q (resp. $1 - q$) is compact so peak (resp. is open, so is a support projection) with respect to A by results in the separable case from the first two papers of the first author and Read cited there. In the last paragraph of the proof one may again appeal to the same Proposition of the first author and Read (Proposition 6.4 (1) in the 2013 paper) to see that a projection in B^{**} which is peak with respect to B^1 , is peak with respect to B .

Amar and Lederer proved for H^∞ of the disk that every ‘Lebesgue-null’ closed sets E in the maximal ideal space X of $M = L^\infty$, is contained in a null peak set in X for H^∞ . Ueda proved a noncommutative version of this for a restricted class of ‘closed sets’ (rather, closed projections), namely the supports of singular functionals on M . In [11] we showed that this result of Ueda is still valid in a σ -finite von Neumann algebra M . Recently the first author and Clouâtre have obtained the full noncommutative Amar and Lederer theorem (i.e. for all closed null projections, as opposed to supports of singular functionals) for subdiagonal algebras in such a von Neumann algebra M .

The Lebesgue decomposition, F & M Riesz, and Gleason–Whitney theorems all work for Riesz approximable algebras.

Theorem 5.11 *A Riesz approximable algebra (resp. space) A is an F & M Riesz algebra (resp. space) and has a unique Lebesgue decomposition. Thus it has all the properties of F & M Riesz spaces from the last section, such as the Gleason–Whitney theorems and corollaries there. In particular, semi- σ -finite subdiagonal algebras have all the properties mentioned in statements of the results in the last section.*

Proof Suppose that $\varphi \in A_n^* \cap A_s^*$, in the notation above Lemma 5.1. If ψ is singular on M then it is singular on M_i . We see that $\varphi|_{A_i}$ is in $(A_i)_s^*$ and $(A_i)_n^*$, so is zero by hypothesis. Since φ is the restriction of a normal functional and $\cup_i A_i$ is relatively weak* dense in A we have that $\varphi = 0$. $\square$

We may also appeal to several other known consequences of a Lebesgue decomposition. For example:

Corollary 5.12 *The predual A_* of a weak* closed Riesz approximable algebra A is an L -summand in A^* . Also, A_* has property (V^*) and is weakly sequentially complete. Hence if A_* is separable then it is a unique predual of A .*

Proof The first assertion is obvious from the theorem, and the second and third follow from results of Pfitzner and Pelczynski as in [46, Corollary 2]. The last assertion is then the main result in [40], also due to Pfitzner. $\square$

Remark The condition that A_* be separable to be a unique predual is possibly not necessary (see also Proposition 3.2 in [47]). The proof in [46] (and reprised in [11] in the σ -finite setting) for uniqueness of predual uses Property (X) of Godefroy and Talagrand, which is a sequential criterion (see e.g. [19]).

We conclude by mentioning some other Hardy space properties of our algebras. Clearly semi- σ -finite subdiagonal algebras, and more generally Riesz approximable algebras, will have the UNSEP (the unique normal state extension property). We know that they will not in general be Ueda algebras in the sense of [11, Section 5] (assuming that measurable cardinals exist), since it is shown in [11, Theorem 6.1] that then semifinite commutative von Neumann algebras need not be Ueda algebras. However semi- σ -finite subdiagonal algebras are of course an ‘increasing limit’ of Ueda algebras. Similarly, we do not know if they have ‘factorization’. Indeed in the σ -finite case authors have only able to obtain a one-sided partial factorization [29, 31] (see also [3]). Similarly it is probably not true that in general $C_e^*(A) = M$ as we have in the tracially finite case; nor probably does A even generate M as a C^* -algebra in the general case. Indeed that $C_e^*(A) = M$ in the tracially finite case follows because then we have the just mentioned (two-sided) factorization [1]. It follows more generally if A is logmodular or ‘convexly approximating in modulus’ (see [6, Proposition 4.3] or [14, Proposition 4.3.10]). If A does not generate M as a C^* -algebra then we also will not have the uniqueness of extension of completely contractive homomorphisms property in [9, Theorem 8.3], a property that does hold in the tracially finite case covered in that reference.

It is to be expected that some of the properties proved in our paper will actually turn out to be equivalent, and characterize semi- σ -finite subdiagonal algebras within

the analytically conditioned algebras, similarly to the characterizations of subdiagonal algebras in [7, 9] and some of the other papers cited above.

Author Contributions Both authors wrote the main manuscript text.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Arveson, W.B.: Analyticity in operator algebras. *Am. J. Math.* **89**, 578–642 (1967)
2. Bekjan, T.N.: Noncommutative Hardy space associated with semi-finite subdiagonal algebras. *J. Math. Anal. Appl.* **429**, 1347–1369 (2015)
3. Bekjan, T.N.: Szegő type factorization of Haagerup noncommutative Hardy spaces. *Acta Mathematica Scientia* **37B**(5), 1221–1229 (2017)
4. Bekjan, T.N., Oshanova, A.: Semifinite tracial subalgebras. *Ann. Funct. Anal.* **8**, 473–478 (2017)
5. Bekjan, T.N., Raikhan, M.: A Beurling–Blecher–Labuschagne type theorem for Haagerup noncommutative L^p spaces. *Banach J. Math. Anal.* **15**, paper no. 39, 24 pp (2021)
6. Blecher, D.P., Labuschagne, L.E.: Logmodularity and isometries of operator algebras. *Trans. Am. Math. Soc.* **355**, 1621–1646 (2003)
7. Blecher, D.P., Labuschagne, L.E.: Characterizations of noncommutative H^∞ . *Integral Equ. Oper. Theory* **56**, 301–321 (2006)
8. Blecher, D.P., Labuschagne, L.E.: Noncommutative function theory and unique extensions. *Studia Math.* **178**, 177–195 (2007)
9. Blecher, D.P., Labuschagne, L.E.: Von Neumann algebraic H^p theory. *Function Spaces: Fifth Conference on Function Spaces, Contemp. Math.*, vol. 435. Amer. Math. Soc. (2007)
10. Blecher, D.P., Labuschagne, L.E.: A Beurling theorem for noncommutative L^p . *J. Oper. Theory* **59**, 29–51 (2008)
11. Blecher, D.P., Labuschagne, L.E.: Ueda’s peak set theorem for general von Neumann algebras. *Trans. Am. Math. Soc.* **370**, 8215–8236 (2018)
12. Blecher, D.P., Labuschagne, L.E.: On vector-valued characters for noncommutative function algebras. *Complex Anal. Oper. Theory* **14**, Paper No. 31 (2020)
13. Blecher, D.P., Labuschagne, L.E.: Von Neumann algebra conditional expectations with applications to generalized representing measures for noncommutative function algebras. *Adv. Math.* **396** (2022). <https://doi.org/10.1016/j.aim.2021.108104>
14. Blecher, D.P., Le Merdy, C.: *Operator Algebras and their Modules—An Operator Space Approach*. Oxford Univ. Press, Oxford (2004)
15. Blecher, D.P., Magajna, B.: Duality and operator algebras: automatic weak* continuity and applications. *J. Funct. Anal.* **224**, 386–407 (2005)
16. Clouâtre, R., Hartz, M.: Lebesgue decompositions and the Gleason–Whitney property for operator algebras. Preprint (2022), to appear *Indiana University Mathematics Journal*, [arXiv:2211.04366](https://arxiv.org/abs/2211.04366)
17. Clouâtre, R., Timko, E.: Non-commutative measure theory: Henkin and analytic functionals on C^* -algebras. Preprint (2022), to appear *Math. Ann.*, [arXiv:2105.11295](https://arxiv.org/abs/2105.11295)
18. Folland, G.B.: *A Course in Abstract Harmonic Analysis*. Textbooks in Mathematics, 2nd edn. CRC Press, Boca Raton (2016)
19. Godefroy, G.: Existence and uniqueness of isometric preduals: a survey. In: *Banach Space Theory* (Iowa City, IA, 1987). *Contemp. Math.*, vol. 85, pp. 131–193. Amer. Math. Soc., Providence (1989)
20. Goldstein, S.: Conditional expectation and stochastic integrals in noncommutative L^p spaces. *Math. Proc. Camb. Philos. Soc.* **110**, 365–383 (1991)
21. Goldstein, S., Labuschagne, L.E.: Composition operators on Haagerup L^p -spaces. *IDAQP* **12**, 439–468 (2009)
22. Goldstein, S., Labuschagne, L.E.: Notes on noncommutative L^p and Orlicz spaces. Łódź University Press, Łódź (2020). ISBN 978-83-8220-385-1, e-ISBN 978-83-8220-386-8

23. Goldstein, S., Labuschagne, L.E.: Noncommutative Measures and L^p and Orlicz Spaces, with Applications to Quantum Physics. Oxford Univ. Press, Oxford (To appear)
24. Goldstein, S., Lindsay, J.M.: Markov semigroups KMS-symmetric for a weight. *Math. Annalen* **313**, 39–67 (1999)
25. Haagerup, U.: The standard form of von Neumann algebras. *Math. Scand.* **37**, 271–283 (1975)
26. Haagerup, U.: On the dual weights for crossed products of von Neumann algebras: I. Removing separability conditions. *Math. Scand.* **43**, 99–118 (1978)
27. Haagerup, U.: On the dual weights for crossed products of von Neumann algebras: II. Application of operator-valued weights. *Math. Scand.* **43**, 119–140 (1978)
28. Haagerup, U., Junge, M., Xu, Q.: A reduction method for noncommutative L_p -spaces and applications. *Trans. Am. Math. Soc.* **362**, 2125–2165 (2010)
29. Ji, G.-X.: A noncommutative version of H^p and characterizations of subdiagonal subalgebras. *Integr. Equ. Oper. Theory* **72**, 131–149 (2012)
30. Ji, G.-X.: Analytic Toeplitz algebras and the Hilbert transform associated with a subdiagonal algebra. *Sci. China Math* **57**, 579–588 (2014)
31. Ji, G.-X., Saito, K.-S.: Factorization in subdiagonal algebras. *J. Funct. Anal.* **159**, 191–201 (1998)
32. Ji, G.-X., Ohwada, T., Saito, K.-S.: Certain structure of subdiagonal algebras. *J. Oper. Theory* **39**, 309–317 (1998)
33. Junge, M., Sherman, D.: Noncommutative L^p modules. *J. Oper. Theory* **53**, 3–34 (2005)
34. Junge, M., Xu, Q.: Noncommutative Burkholder–Rosenthal inequalities. *Ann. Prob.* **31**, 948–995 (2003)
35. Labuschagne, L.E.: Invariant subspaces for H^2 -spaces of σ -finite algebras. *Bull. LMS* **49**, 33–44 (2017)
36. Labuschagne, L.E.: A crossed product approach to Orlicz spaces. *Proc. Lond. Math. Soc.* **107**, 965–1003 (2013)
37. Labuschagne, L.E., Xu, Q.: An extension of Haagerup’s reduction theorem with applications to subdiagonal subalgebras of general von Neumann algebras (in preparation)
38. Pedersen, G.K.: C^* -Algebras and their Automorphism Groups. Academic Press, London (1979)
39. Pedersen, G.K., Takesaki, M.: The Radon–Nikodym theorem for von Neumann algebras. *Acta Math.* **130**, 53–87 (1973)
40. Pfitzner, H.: Separable L -embedded Banach spaces are unique preduals. *Bull. Lond. Math. Soc.* **39**, 1039–1044 (2007)
41. Sager, L.: A Beurling–Blecher–Labuschagne theorem for noncommutative Hardy spaces associated with semifinite von Neumann algebras. *Integral Equ. Oper. Theory* **86**, 377–407 (2016)
42. Stratila, S.: *Modular Theory in Operator Algebras*. Cambridge-IISc Series, 2nd edn. Cambridge University Press, Delhi (2020)
43. Takesaki, M.: *Theory of Operator Algebras I*. Springer, New York (1979)
44. Takesaki, M.: *Theory of Operator Algebras II*. *Encyclopaedia of Mathematical Sciences*, vol. 125. Springer-Verlag, Berlin (2003)
45. Terp, M.: Interpolation spaces between a von Neumann algebra and its predual. *J. Oper. Theory* **8**(2), 327–360 (1982)
46. Ueda, Y.: On peak phenomena for non-commutative H^∞ . *Math. Ann.* **343**, 421–429 (2009)
47. Ueda, Y.: On the predual of non-commutative H^∞ . *Bull. Lond. Math. Soc.* **43**, 886–896 (2011)
48. Watanabe, K.: Dual of non-commutative L^p -spaces with $0 < p < 1$. *Math. Proc. Camb. Philos. Soc.* **103**, 503–509 (1988)
49. Xu, Q.: On the maximality of subdiagonal algebras. *J. Oper. Theory* **54**, 137–146 (2005)

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