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Chebyshev collocation method for the solution of a system of second-order boundary value problems

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Abstract. In this paper, we present a new numerical approach based on Chebyshev polynomials of the first kind for solving a system of second-order boundary value problems associated with obstacle, unilateral and contact problems. The applicability of the method is demonstrated on two numerical examples. The results obtained show the effectiveness, reliability and superiority of the new method over other existing methods in the literature. In addition, the method is simple and easy to implement.

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1. Introduction

Second-order boundary-value problems are often encountered in many areas of science and engineering such as in modeling of deflection of cantilever beams, deformation of beams and plate deflection theory, heat transfer in a solid, Troesch's problem relating to the confinement of a plasma column by radiation pressure, and fluid flow over a solid [9, 10, 11, 15, 16]. However, our focus in this study is a class of second-order boundary value problems which arise in connection with obstacle, unilateral and contact problems which have been studied by many authors (see for example, [1-4, 7, 12, 13] and the references therein).

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Chebyshev polynomials belong to the family of orthogonal polynomials and are of great importance in many areas of Mathematics. Chebyshev polynomials have many interesting and useful properties, and are used in many areas of numerical analysis such as approximation theory, economization of power series with minimal loss of accuracy and least-squares approximation. These orthogonal polynomials have been successfully applied to solve various problems [5, 6, 14, 19].

We consider, in this study, a system of second-order boundary value problem of the type

$$u''(x) = \begin{cases} f(x), & a \leq x \leq c \\ f(x) + g(x)u(x) + r, & c \leq x \leq d \\ f(x), & d \leq x \leq b \end{cases} \tag{1}$$

together with the boundary conditions:

$$u(a) = \alpha_1, \qquad u(b) = \alpha_2, \tag{2}$$

and the continuity conditions of u and u' at c and d. In this system, f and g are continuous functions on $[a, b]$ and $[c, d]$ respectively. Also, the parameter α_1, α_2 and r are real finite constants.

Meanwhile, a number of authors have investigated (1), especially a special case of it where $f(x) = 0$, $g(x) = 1$ and $r(x) = -1$ and proposed several methods, both approximate and numerical, for solving the equation. Some of these methods are: quadratic and cubic spline methods [1-2], parametric spline [7], collocation method [12], finite difference method [13], Haar wavelet [16], nonpolynomial spline method [17, 18] and Rayleigh-Ritz method [20].

The objective of this paper is to develop a simple and a more efficient numerical method based on the Chebyshev polynomials of the first kind for solving (1) with the associated boundary conditions (2). This paper is organised as follows: Section 2 is concerned with the formulation of the new method. In section 3, the derivation of a system of differential equations of the form (1) presented. Numerical examples are given in Section 4 and the results obtained are compared with the exact solutions and the results by some other methods in the literature.

2. Chebyshev collocation method

2.1. Definitions of Chebyshev polynomials

Definition 2.1. The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial of degree n in x defined by

$$T_n(x) = \cos\left(n\cos^{-1}\left(\frac{2x-q-p}{q-p}\right)\right), n \geq 0,\, x \in [p,q]. \tag{3}$$

The Chebyshev polynomial $T_n(x)$ satisfies the recurrence relation

$$T_n(x) = 2\left(\frac{2x-q-p}{q-p}\right)T_{n-1}(x) - T_{n-2}(x), n \geq 2, \tag{4}$$

with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = \frac{2x-q-p}{q-p}. \tag{5}$$

Definition 2.2. The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin\left((n+1)\cos^{-1}\left(\frac{2x-q-p}{q-p}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{2x-q-p}{q-p}\right)\right)}, \quad x \in [p,q]. \tag{6}$$

$U_n(x)$ is defined by recursively by

$$U_n(x) = 2\left(\frac{2x-q-p}{q-p}\right)U_{n-1}(x) - U_{n-2}(x), n \geq 2, \tag{7}$$

with the initial conditions

$$U_0(x) = 1, \quad U_1(x) = \frac{2x-q-p}{q-p}. \tag{8}$$

2.2. Relationship between Chebyshev polynomials of the first and second kinds

The following theorems show relationships between the Chebyshev polynomials of the first and second kinds in variable x on the interval $[p, q]$.

Theorem 2.1. *Let $T_n(x)$ and $U_n(x)$ denote the Chebyshev polynomials of degree n in x of the first and second kinds respectively, then*

$$\frac{2}{(q-p)} \int U_{n-1}(x)dx = \frac{1}{n} T_n(x) + C, \quad \forall\ n \geq 1, \tag{9}$$

where C is a constant.

Theorem 2.2. *Let $T_n(x)$ and $U_n(x)$ denote the Chebyshev polynomials of degree n in x of the first and second kinds respectively, then*

$$U_{n-1}(x) = 2T_{n-1}(x) + U_{n-3}(x) \quad \forall\ n \geq 3, \tag{10}$$

Theorem 2.3. *Let $T_r(x)$ and $U_{n-1}(x)$ denote the Chebyshev polynomials of degrees r and $n-1$ in x on $[p, q]$ of the first kind and second kind respectively, then*

$$U_{n-1}(x) = 2 \sum_{\substack{r=0 \\ (n-r) odd}}^{n-1} T_r(x) \quad \forall\ n \geq 1. \tag{11}$$

Note. A summation with prime denotes a sum with first term halved.

The proofs of the above theorems can be established following Ogunlaran and Oladejo [14].

2.3. Derivatives of Chebyshev polynomials $T_n(x)$

In this section, we express the derivatives of Chebyshev polynomials $T_n(x)$ of degree n in x on the interval $[p, q]$ as a sum of lower degree Chebyshev polynomials by using the relationship between Chebyshev polynomials of the first and second kinds given in Section 2.2.

Differentiating both sides of equation (9) with respect to x and substituting (11) in the result, we obtain

$$\frac{d}{dx}T_n(x) = \frac{4n}{(q-p)} \sum_{\substack{r=0 \\ (n-r) odd}}^{n-1} T_r(x). \tag{12}$$

Similarly, for the second derivative we obtain

$$\frac{d^2}{dx^2}T_n(x) = \frac{4}{(q-p)^2} \sum_{\substack{r=0 \\ (n-r) even}}^{n-1} n(n^2-r^2)T_r(x). \tag{13}$$

2.4. Properties of Chebyshev polynomials $T_n(x)$

Chebyshev polynomial $T_n(x)$ and its derivative in $[p,q]$ have some useful and interesting properties at the two endpoints of the interval. For instance, it can be deduced from theroem 2.4 that for $j=0$ and $j=n$, we obtain respectively as follows:

$$T_n(p) = (-1)^n, \tag{14}$$

and

$$T_n(q) = 1. \tag{15}$$

Theorem 2.4. Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ assumes its $(n+1)$ extrema in $[p,q]$ at $x_j = \frac{1}{2}\left(1+\cos\left(\frac{j\pi}{n}\right)\right)$ with $T_n(x_j) = (-1)^j$, for $j = 0, 1, \cdots, n$.

Proof. The proof of this theorem can be established by following Ogunlaran and Oladejo [14].

In a similar manner, we obtain from (12) as follows:

$$\frac{d}{dx}T_n(x)\Big|_{x=p} = (-1)^{n+1}\frac{2}{(q-p)}n^2, \tag{16}$$

and

$$\frac{d}{dx}T_n(x)\Big|_{x=q} = \frac{2}{(q-p)}n^2. \tag{17}$$

2.5. Implementation of the method

In order to solve (1) with the boundary conditions (2), we define our approximate solution to the solution $u(x)$ of (1)-(2) as the finite sum

$$u_n(x) = \sum_{k=0}^{n} c_k T_k(x), \tag{18}$$

where c_k are constants to be determined.

Differentiating (18) twice with respect to x, we obtain

$$u_n''(x) = \frac{4}{(q-p)^2} \sum_{k=2}^{n} \sum_{\substack{r=0 \\ (k-r) odd}}^{k=2} k(k^2 - r^2) c_k T_r(x). \tag{19}$$

We now substitute (18) and (19) into equation (1) and set $x = x_j$ in each subinterval, where the collocation points are defined as

$$x_j = p + jh, \quad h = \frac{q-p}{n}, j = 1, 2, \cdots, n-1. \tag{20}$$

Alternatively, unevenly distributed nodes may be used as collocation points. In this case, the internal extrema

$$x_j = \frac{1}{2}\left[p + q + (q-p)\cos\left(\frac{j\pi}{n}\right)\right], \ j = 1, 2, \cdots, n-1 \tag{21}$$

of the n^{th} order Chebyshev polynomial $T_n(x)$ are chosen as the collocation points.

The boundary conditions (2) yield

$$\sum_{k=0}^{n} (-1)^k c_k = \alpha_1 \qquad \text{and} \quad \sum_{k=0}^{n} c_k = \alpha_2. \tag{22}$$

Therefore, the $(n-1)$ equations together with the two boundary conditions (22) give a system of $(n+1)$ equations which can be solved to determine the Chebyshev coefficients c_k in the approximate solution (18).

3. Application

Consider the second-order obstacle boundary value problem

$$\begin{array}{ll} -u''(x) \geq f(x) & \text{on } \ \Omega = [0,\pi] \\ u(x) \geq \psi(x) & \text{on } \ \Omega = [0,\pi] \\ \left[u''(x) + f(x)\right]\left[u(x) - \psi(x)\right] = 0 & \text{on } \ \Omega = [0,\pi] \\ u(0) = u(\pi) = 0, & \end{array} \tag{23}$$

where $f(x)$ is a given force acting on the string and ψ is the elastic obstacle.

According to [3, 4, 8, 12], problem (23) is equivalent to the variational inequality problem

$$a(u, v-u) \geq (f, v-u), \text{ for all } v \in K, \tag{24}$$

where K is the closed convex set $K = \left\{v : v \in H_0^1(\Omega) : v \geq \psi \text{ on } \Omega\right\}$.

The existence of a unique of (23) has been established through equation (24), see for example [3, 18]. Also, by using the approach of Lewy and Stampacchia [9], the inequality (24) becomes

$$u'' - \mu(u-\psi)(u-\psi) = 0, \quad 0 < x < \pi, \tag{25}$$

$$u(0) = u(\pi) = 0,$$

where $\mu(t)$ is the discontinuous function defined by

$$\mu(x) = \begin{cases} 1, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases} \tag{26}$$

is known as the penalty function and ψ is the obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leq x < \frac{\pi}{4}, \\ 1, & \text{for } \frac{\pi}{4} \leq x < \frac{3\pi}{4}, \\ -1, & \text{for } \frac{3\pi}{4} \leq x < \pi. \end{cases} \tag{27}$$

Equation (27) describes the equilibrium configuration of an obstacle string pulled at the ends and lying over elastic step of constant height 1 and unit rigidity. We can now determine the solution of the equation in the interval $[0,\pi]$ since the obstacle function ψ is known. The following system of differential equations is therefore obtained from equations (25)- (27), which is a form (1) with $g(x)=1$ and $r=-1$:

$$u''(x)=\begin{cases} f(x), & 0\le x\le \frac{\pi}{4} \text{ and } \frac{3\pi}{4}\le x\le \pi \\ u+f(x)-1, & \frac{\pi}{4}\le x\le \frac{3\pi}{4}, \end{cases} \tag{28}$$

with the boundary conditions

$$u(0)=u(\pi)=0, \tag{29}$$

and the condition of continuity of u and u' at $x=\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

4. Numerical example

In this section, we apply the new method described in Section 2 on two examples of the form (1) over the interval $[0,\pi]$. The approximate solutions obtained using evenly-spaced and unevenly-spaced collocation points for $n=4,6,8$ and 10 in (18) are evaluated at $x=x_i=\frac{i\pi}{20}$, $i=1,2,\cdots,19$; and the observed Maximum Absolute Errors (MAEs) and Maximum Relative Errors (MREs) are obtained.

Example 4.1. Consider the system of differential equations [1, 2, 7, 13, 17, 18]:

$$u''(x)=\begin{cases} 0, & 0\le x\le \frac{\pi}{4} \text{ and } \frac{3\pi}{4}\le x\le \pi, \\ u-1, & \frac{\pi}{4}\le x\le \frac{3\pi}{4}, \end{cases} \tag{30}$$

with the boundary conditions

$$u(0)=u(\pi)=0, \tag{31}$$

and the condition of continuity of u and u' at $x=\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

The analytic solution to Example 4.1 is given by

$$u(x)=\begin{cases} \frac{4x}{\gamma_1}, & 0\le x\le \frac{\pi}{4}, \\ 1-\frac{4\cosh\left(\frac{\pi}{2}-x\right)}{\gamma_2}, & \frac{\pi}{4}\le x\le \frac{3\pi}{4}, \\ \frac{4(\pi-x)}{\gamma_1}, & \frac{3\pi}{4}\le x\le \pi, \end{cases} \tag{32}$$

where $\gamma_1 = \pi + 4\coth\left(\frac{\pi}{4}\right)$ and $\gamma_2 = \pi\sinh\left(\frac{\pi}{4}\right) + 4\cosh\left(\frac{\pi}{4}\right)$.

The observed MAEs and MREs for Example 4.1 are summarized in Tables 1 and 2 for evenly-spaced and unevenly-spaced collocation points. In addition, the numerical results obtained by various existing methods are presented in Table 3. It is clearly observed from Tables $1-3$ that the new method gives better results compared to the existing methods.

Example 4.2. Consider the system of differential equations

$$u''(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \le x \le \pi, \\ 1 + x - u, & \frac{\pi}{4} \le x \le \frac{3\pi}{4}, \end{cases} \tag{33}$$

with the boundary conditions (31).

The analytic solution to Example 4.2 is given by

$$u(x) = \begin{cases} \frac{1}{6}x^6 - \frac{1}{96}\pi^2 x, & 0 \le x \le \frac{\pi}{4}, \\ \frac{1}{4\gamma_3}\left(\gamma_1\sin(x) - \gamma_2\cos(x)\right) + 1 + x, & \frac{\pi}{4} \le x \le \frac{3\pi}{4}, \\ \frac{1}{6}x^3 - \frac{37}{96}\pi^2 x + \frac{7}{32}\pi^3, & \frac{3\pi}{4} \le x \le \pi, \end{cases} \tag{34}$$

where $\gamma_1 = 3\pi\cos\left(\frac{\pi}{4}\right) - 4\cos\left(\frac{3\pi}{4}\right) - \pi\cos\left(\frac{3\pi}{4}\right) + 4\cos\left(\frac{\pi}{4}\right)$,
$\gamma_2 = 3\pi\sin\left(\frac{\pi}{4}\right) + 4\sin\left(\frac{\pi}{4}\right) - 4\sin\left(\frac{3\pi}{4}\right) - \pi\sin\left(\frac{3\pi}{4}\right)$,
$\gamma_3 = \cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$.

The observed MAEs and MREs in the numerical solution of Example 4.2 are given in Tables 4 and 5. The results show that the method is efficient and the accuracy of the solution improves as more terms are retained in the approximate solution with a better results given by unevenly-spaced collocation points.

Table 1: Numerical results for Example 1 using evenly-spaced nodes

n	4	6	8	10
MAE	5.7317×10^{-5}	3.3588×10^{-7}	1.2109×10^{-9}	2.9657×10^{-12}
MRE	9.9412×10^{-5}	5.5559×10^{-7}	2.0046×10^{-9}	5.0075×10^{-12}

Table 2: Numerical results for Example 1 using unevenly-spaced nodes

n	4	6	8	10
MAE	3.3428×10^{-5}	6.0210×10^{-8}	8.4305×10^{-11}	7.4218×10^{-14}
MRE	4.3200×10^{-5}	3.6773×10^{-8}	3.6279×10^{-11}	2.3417×10^{-14}

Table 3: Maximum Absolute Errors (MAEs) for Example 1 by other methods

h	$\frac{\pi}{20}$	$\frac{\pi}{40}$	$\frac{\pi}{80}$
Al-said [1]	2.2×10^{-3}	5.87×10^{-4}	1.51×10^{-4}
Al-said [2]	1.94×10^{-3}	4.99×10^{-4}	1.27×10^{-4}
Khan and Aziz [8]	6.43×10^{-4}	1.83×10^{-4}	4.87×10^{-5}
Noor and Tirmzi [15]	2.50×10^{-2}	1.29×10^{-2}	6.58×10^{-3}
Noor and Tirmzi [15]	2.32×10^{-2}	1.21×10^{-2}	6.17×10^{-3}
Siraj-ul-Islam et al. [19]	2.390×10^{-4}	6.231×10^{-5}	1.622×10^{-5}
Siraj-ul-Islam et al. [20]	6.43×10^{-4}	1.83×10^{-4}	4.87×10^{-5}

Table 4: Numerical results for Example 2 using evenly-spaced nodes

n	4	6	8	10
MAE	1.1742×10^{-3}	8.7197×10^{-6}	3.7652×10^{-8}	1.0729×10^{-10}
MRE	6.8230×10^{-4}	3.7642×10^{-6}	1.3518×10^{-8}	3.3629×10^{-11}

Table 5: Numerical results for Example 2 using unevenly-spaced nodes

n	4	6	8	10
MAE	8.05574×10^{-4}	1.3667×10^{-6}	3.4887×10^{-9}	3.8544×10^{-12}
MRE	3.3780×10^{-4}	1.5785×10^{-7}	2.0552×10^{-10}	1.7212×10^{-13}

5. Conclusion

We have introduced a new numerical method for solving a certain class of system of second-order boundary value problems based on Chebyshev polynomial of the first kind. The method presents the approximate solution in form of a series and so the approximate solution and its derivatives can easily be computed at any point in the range of integration. In addition, the accuracy of the solutions improve as more terms are retained in the approximate solutions with better results obtained by unevenly-spaced nodes. Finally, the method, though simple, displays a high-level of accuracy unparalleled by the existing methods.

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