Toeplitz-like operators with rational symbol having poles on the unit circle

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Thesis accepted for the degree *Doctor of Philosophy in Mathematics* at the North-West University

This degree is jointly supervised between North-West University, South Africa and Vrije Universiteit Amsterdam, The Netherlands under a cotutelle joint PhD Agreement.

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Graduation May 2020

29927064

VRIJE UNIVERSITEIT

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ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor of Philosophy aan
de Vrije Universiteit Amsterdam en North-West University,
op gezag van de rector magnificus en vise-kanselier
prof.dr. V. Subramaniam en prof.dr. N.D. Kgwadi,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de Faculteit der Bètawetenschappen
op vrijdag 12 juni 2020 om 9.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door
Jacob Jaftha
geboren te Kaapstad, Zuid-Afrika

Acknowledgements

It is with pleasure that I thank the following people who contributed to this thesis.

I am especially grateful to prof. dr. M.A. Kaashoek for suggesting the topic and for his generosity in sharing his insights, even after his retirement. In particular, his insistence on understanding the scalar case first before attempting the more general matrix case.

I am deeply grateful to my supervisors. To prof. dr. A.C.M Ran and prof. dr. S. ter Horst for the way they have guided my research. I have learnt a lot from their expert opinions and their approach to mathematics. To prof. dr. G.J. Groenewald for his meticulous attention to detail as many an oversight would have occurred if not for his keen eye.

Being away from home for a longish period in a different country can be trying and it helps to have someone to talk to. I had the pleasure to share an office with dr. F. Van Schagen, who always seem to have an interesting titbit to share. And some of my visits to Amsterdam coincided with visits of dr. D. Janse van Rensburg, who didn't mind sharing a dig or two on common South Africa - Netherlands experiences. I'm really thankful for M. van der Aa who tried to make my stays in Amsterdam close to home away from home.

I thank my colleagues in the Numeracy Centre, who have gracefully endured the vagaries required by my studies, which did not always coincide with the academic terms of our university. Doing a PhD part time created many challenges in scheduling but they all did the necessary and so I could focus on the task at hand. Vera Frith, the centre's coordinator, and all my colleagues Jumani Clarke, Michelle Henry, Pam Lloyd, Nompelo Lungisa, Muzi Manzini, Duncan Mhakure, Sheena Rughubar-Reddy and Renee Rix: a big thank you.

Visits to Amsterdam and Potchefstroom were made possible through substantive support from various institutions. My home institution, University of Cape Town, for sabbatical support and a research development grant; the North-West University, for support for the regular visits to Potchefstroom and a postgraduate bursary; the Vrije Universiteit Amsterdam, for financial support and accommodation on my visits; the South African Department of Higher Education and Training (DHET) for grants under the National Mathematics and Statistical Sciences Teaching Development collaborative project.

My family and friends who are always available for support and encouragement. Especially Nicolé, my wife and partner, who is grounded in the day to day management of the household, did not allow me to become too engrossed in my own world but to enjoy other pleasures of life. I take especially delight in the life of our children Benita, Tracy, Lance and Stuart whose questions, quips and understanding of life never fails to bring on a smile, even in the most awkward situations.

Abstract

Let H^p be the Hardy space of p-integrable functions on the unit circle $\mathbb{T}$ in the complex plane that have an analytic extension to the open unit disk $\mathbb{D}$. Suppose that ω is a rational function with poles on the unit circle. The topic of this thesis is the analysis of a Toeplitz-like operator T_ω in H^p generated by such an ω. We investigate Fredholm properties, the spectrum and the adjoint in case ω is a scalar function and explore the Fredholm properties of T_Ω in case Ω is a rational matrix function with poles on $\mathbb{T}$.

We show that, in general, the operator T_ω is a well-defined, closed, densely defined linear operator whose domain contains the polynomials. It is shown that the operator is Fredholm if and only if the symbol has no zeroes on the unit circle, and a formula for the index is given as well. A matrix representation of the operator is discussed.

A description of the spectrum of T_ω and its various parts, i.e., point, residual and continuous spectrum, is given, as well as a description of the essential spectrum. In this case, it is shown that the essential spectrum need not to be connected in $\mathbb{C}$. Various examples illustrate the results.

The adjoint operator T_ω^* is described. In the case where $p = 2$ and ω has poles only on the unit circle $\mathbb{T}$, a description is given for when T_ω^* is symmetric and when T_ω^* admits a selfadjoint extension. We compare the operator with unbounded Toeplitz operators studied earlier and show that if ω is a proper rational function, then T_ω^* coincides with an unbounded Toeplitz operator studied earlier by Sarason.

We extend the analysis of the Toeplitz-like operator to the case where it is generated by a rational matrix function having poles on $\mathbb{T}$. A Wiener-Hopf type factorization of rational matrix functions with poles and zeroes on $\mathbb{T}$ is introduced and then used to analyse the Fredholm properties of Toeplitz-like operators. A formula for the index, based on the factorization, is given. Furthermore, it is shown that the determinant of ω having no zeroes on $\mathbb{T}$ is not sufficient for T_ω being Fredholm, which is in contrast to the classical case, where the symbol has no zeroes on $\mathbb{T}$ is sufficient for the operator T_ω being Fredholm.

Keywords: Toeplitz operators, unbounded operators, Hardy Space, Fredholm operators, Adjoint operator, Wiener-Hopf factorization, rational symbol

Summary

Let H^p be the Hardy space of p-integrable functions on the unit circle $\mathbb{T}$ in the complex plane that have an analytic extension to the open unit disk $\mathbb{D}$. Let $\mathcal{P}$ denote the space of complex polynomials in z, i.e., $\mathcal{P} = \mathbb{C}[z]$, and $\mathcal{P}_n \subset \mathcal{P}$ the subspace of polynomials of degree at most n. Let Rat denote the space of rational complex functions, and Rat_0 the subspace of strictly proper rational complex functions. We will also need the subspaces $\mathrm{Rat}(\mathbb{T})$ and $\mathrm{Rat}_0(\mathbb{T})$ of Rat consisting of the rational functions in Rat with all poles on $\mathbb{T}$ and the strictly proper rational functions in Rat with all poles on $\mathbb{T}$, respectively.

Suppose that ω is a rational function with poles on the unit circle. The topic of this thesis is the analysis of a Toeplitz-like operators T_ω in H^p generated by ω: in this case we say ω is the symbol of the operator T_ω. We investigate Fredholm properties, the spectrum and the adjoint in case ω is a scalar function and explore the Fredholm properties of T_Ω in case Ω is a rational matrix with poles on $\mathbb{T}$.

In their study of the spectral properties of Toeplitz operators, in articles appearing 1950 and 1954, P. Hartman and A. Wintner showed that the Toeplitz operator generated by ω on H^2 is bounded if and only if ω is essentially bounded on $\mathbb{T}$ or equivalently $\omega \in L^\infty(\mathbb{T})$. Here they introduced the Toeplitz matrix on ℓ^2 that is equivalent to the Toeplitz-like operator on H^2 generated by a function $a \in L^2$, which is not necessarily bounded but is a closed, densely defined unbounded operator. Toeplitz operators generated by functions f in $H^p, p < 1$ were studied by H. Helson (1988) in his study on large analytic functions. D. Sarason, in 2008, looked at unbounded Toeplitz operators in H^2 generated by functions in the Smirnov class, N^+. Note that the Smirnov class includes $L^p, p < 1$ and the Toeplitz operator of Helson and Sarason are closed, densely defined operators, but their domains generally do not include all the polynomials, whereas the domain of the Toeplitz operator of Hartman and Wintner does include all polynomials.

For $\omega \in \mathrm{Rat}$, possibly having poles on $\mathbb{T}$, we define a Toeplitz-like operator $T_\omega(H^p \to H^p)$, for $1 < p < \infty$, as follows:

$$\mathrm{Dom}(T_\omega) = \{g \in H^p \,|\, \omega g = f + \rho \text{ with } f \in L^p, \rho \in \mathrm{Rat}_0(\mathbb{T})\}\,,\ T_\omega g = \mathbb{P} f.$$

where $\mathbb{P}$ is the Riesz projection of L^p onto H^p, due to M. Riesz and not the Riesz projection in spectral operator theory, due to F. Riesz . Note that in case ω has no poles on $\mathbb{T}$, then $\omega \in L^\infty$ and the Toeplitz-like operator T_ω defined above coincides with the classical Toeplitz operator T_ω on H^p. If ω, however, is a rational function with poles on $\mathbb{T}$ then $\omega \notin L^p$, $p \geq 1$ and if, furthermore, ω has poles in $\mathbb{D}$ then it is not in N^+.

Basic properties: In general, for $\omega \in \mathrm{Rat}$, the operator T_ω is a well-defined, closed, densely defined linear operator. By the Euclidean division algorithm, one easily verifies that all polynomials are contained in $\mathrm{Dom}(T_\omega)$. Moreover, it can be verified that $\mathrm{Dom}(T_\omega)$ is invariant under the forward shift operator T_z and that the following classical result holds:

$$T_{z^{-1}} T_\omega T_z f = T_\omega f, \qquad f \in \mathrm{Dom}(T_\omega).$$

For $\omega \in \mathrm{Rat}(\mathbb{T})$ we have a complete description of the domain and range of the operator as below:

Theorem A. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+ having roots only inside, on, or outside $\mathbb{T}$. Then*

$$\mathrm{Ker}(T_\omega) = \{r_0/s_+ \mid \deg(r_0) < \deg(q) - \deg(s_- s_0)\};$$
$$\mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{\deg(q)-1}; \quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}},$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1}\} \subset \mathcal{P}_{\deg(s)-1}.$$

Furthermore, $H^p = \overline{\mathrm{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ forms a direct sum decomposition of H^p, where

$$\widetilde{\mathcal{Q}} = \mathcal{P}_{k-1} \quad \text{with} \quad k = \max\{\deg(s_-) - \deg(q), 0\},$$

following the convention $\mathcal{P}_{-1} := \{0\}$.

Fredholm properties: For the case that ω has no poles on $\mathbb{T}$, when T_ω is a classical Toeplitz operator, the operator T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$, a result of R. Douglas. We show that this result remains true in case $\omega \in \mathrm{Rat}$. Note that a closed Fredholm operator in a Banach space necessarily has a closed range.

We establish the following analogue of Wiener-Hopf factorization: for $\omega \in \mathrm{Rat}$ we can write

$$\omega(z) = \omega_-(z)(z^\kappa \omega_0(z))\omega_+(z)$$

where κ is the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$, ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and ω_0 has all its poles and zeroes on $\mathbb{T}$. Based on the choice of the domain it can then be shown that

$$T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}.$$

This factorization eventually allows to reduce the questions on Fredholm properties to the case where ω has only poles on $\mathbb{T}$. It also allows us to characterize invertibility of T_ω and to give a formula for the inverse of T_ω in case it exists.

Theorem B. *Let $\omega \in \mathrm{Rat}$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. In case T_ω is Fredholm, the Fredholm index of T_ω is given by*

$$\mathrm{Index}\,(T_\omega) = \sharp \left\{ \begin{matrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{matrix} \right\} - \sharp \left\{ \begin{matrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{matrix} \right\},$$

and T_ω is either injective or surjective. In particular, T_ω is injective, invertible or surjective if and only if $\mathrm{Index}\,(T_\omega) \leq 0$, $\mathrm{Index}\,(T_\omega) = 0$ *or* $\mathrm{Index}\,(T_\omega) \geq 0$, *respectively.*

It should be noted that when we talk of poles and zeroes of ω these do not include the poles or zeroes at infinity.

Matrix representation: Since all polynomials are in the domain of T_ω we can write down the matrix representation of T_ω with respect to the standard basis of H^p. It turns out that this matrix representation has the form of a Toeplitz matrix. In addition, there is an assertion on the growth of the coefficients in the upper triangular part of the matrix.

Theorem C. *Let $\omega \in \mathrm{Rat}$ possibly with poles on $\mathbb{T}$. Then we can write the matrix representation $[T_\omega]$ of T_ω with respect to the standard basis $\{z^n\}_{n=0}^\infty$ of H^p as*

$$[T_\omega] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \vdots & & & \ddots & & \end{pmatrix}.$$

In addition $a_{-j} = O(j^{M-1})$ for $j \geq 1$ where M is the largest order of the poles of ω in $\mathbb{T}$ and $(a_j)_{j=0}^\infty \in \ell^2$.

The spectrum: Using the fact that $\lambda I_{H^p} - T_\omega = T_{\lambda-\omega}$, our extended analysis of the operator T_ω enables us to describe the spectrum of T_ω, and its various parts. Our first main result is a description of the essential spectrum of T_ω, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\lambda I_{H^p} - T_\omega$ is not Fredholm.

Theorem D. *Let $\omega \in \mathrm{Rat}$. Then the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ of T_ω is an algebraic curve in $\mathbb{C}$ which is given by*

$$\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) := \{\omega(e^{i\theta}) \mid 0 \leq \theta \leq 2\pi,\ e^{i\theta} \textit{ not a pole of } \omega\}.$$

Furthermore, the map $\lambda \mapsto$ $\mathrm{Index}\,(T_{\lambda-\omega})$ *is constant on connected components of $\mathbb{C}\backslash\omega(\mathbb{T})$ and the intersection of the point spectrum, residual spectrum and resolvent set of T_ω with $\mathbb{C}\backslash\omega(\mathbb{T})$ coincides with sets of $\lambda \in \mathbb{C}\backslash\omega(\mathbb{T})$ with* $\mathrm{Index}\,(T_{\lambda-\omega})$ *being strictly positive, strictly negative and zero, respectively.*

We show that the algebraic curve $\omega(\mathbb{T})$, and thus the essential spectrum of T_ω, need not be connected in $\mathbb{C}$. Our next main result provides a description of the spectrum of T_ω and its various parts.

Theorem E. *Let $\omega \in \mathrm{Rat}$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Define*

$$\begin{aligned} k_q &= \sharp\{\textit{roots of } q \textit{ inside } \overline{\mathbb{D}}\} = \sharp\{\textit{poles of } \lambda - \omega \textit{ inside } \overline{\mathbb{D}}\}, \\ k_\lambda^- &= \sharp\{\textit{roots of } \lambda q - s \textit{ inside } \mathbb{D}\} = \sharp\{\textit{zeroes of } \lambda - \omega \textit{ inside } \mathbb{D}\}, \\ k_\lambda^0 &= \sharp\{\textit{roots of } \lambda q - s \textit{ on } \mathbb{T}\} = \sharp\{\textit{zeroes of } \lambda - \omega \textit{ on } \mathbb{T}\}, \end{aligned}$$

where in all these sets, multiplicities of the roots, poles and zeroes are to be taken into account. Then the resolvent set $\rho(T_\omega)$, point spectrum $\sigma_{\mathrm{p}}(T_\omega)$, residual spectrum $\sigma_{\mathrm{r}}(T_\omega)$ and continuous spectrum $\sigma_{\mathrm{c}}(T_\omega)$ of T_ω are given by

$$\begin{aligned} \rho(T_\omega) &= \{\lambda \in \mathbb{C} \mid k_\lambda^0 = 0 \textit{ and } k_q = k_\lambda^-\}, \\ \sigma_{\mathrm{p}}(T_\omega) &= \{\lambda \in \mathbb{C} \mid k_q > k_\lambda^- + k_\lambda^0\}, \quad \sigma_{\mathrm{r}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_q < k_\lambda^-\}, \\ \sigma_{\mathrm{c}}(T_\omega) &= \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0 \textit{ and } k_\lambda^- \le k_q \le k_\lambda^- + k_\lambda^0\}. \end{aligned}$$

Furthermore, $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0\}$.

Examples will be given where T_ω has a bounded resolvent set, even with an empty resolvent set. This is in sharp contrast to the case where ω has no poles on the unit circle $\mathbb{T}$. For in this case the operator is bounded, the resolvent set is a nonempty unbounded set and the spectrum a compact set, and the essential spectrum is connected.

The adjoint: In case ω has no poles on $\mathbb{T}$, in fact for any $\omega \in L^\infty$, the adjoint of the Toeplitz operator T_ω on H^p can be identified with the Toeplitz operator T_{ω^*} on $H^{p'}$, with $1 < p' < \infty$ such that $1/p + 1/p' = 1$ and with ω^* defined as $\omega^*(z) = \overline{\omega(z)}$ on $\mathbb{T}$.

For the Toeplitz-like operators generated by rational functions with poles on the unit circle the situation is more complicated. However, we do obtain that T_ω^* can be identified with the restriction of the Toeplitz-like operator T_{ω^*} on $H^{p'}$ to a dense subspace of its domain. Like for the operator T_ω, in case ω is in $\mathrm{Rat}(\mathbb{T})$ we obtain a more explicit description of T_ω^*.

The degree of a polynomial $r \in \mathcal{P}$ is denoted as $\deg(r)$. Given $r \in \mathcal{P}$ with $\deg(r) = k$, we define the polynomial $r^\sharp$ by $r^\sharp(z) = z^k \overline{r(1/\overline{z})}$. The following theorem describes the basic properties of the adjoint.

Theorem F. *Let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime and $1 < p < \infty$. Factor $s = s_- s_0 s_+$ and $q = q_- q_0 q_+$ with s_-, q_- having roots only inside $\mathbb{T}$, s_0, q_0 having roots only on $\mathbb{T}$, and s_+, q_+ having roots only outside $\mathbb{T}$. Set $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$ $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$ and let $1 < p' < \infty$ with $1/p + 1/p' = 1$. Then*

$$\mathrm{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^* = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}.$$

Furthermore, we have

$$\begin{aligned} \mathrm{Ran}(T_\omega^*) &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} Q_{n_0+n_--m_0-m_-} (s_0)^\sharp H^{p'}, \\ \mathrm{Ker}(T_\omega^*) &= \left\{ \frac{(q_-)^\sharp (q_0)^\sharp r}{(s_-)^\sharp} \mid \deg(r) < n_- - m_- - m_0 \right\}. \end{aligned}$$

Here $Q_k = I_{H^{p'}} - P_{\mathcal{P}_{k-1}}$*, with* $P_{\mathcal{P}_{k-1}}$ *the standard projection in* $H^{p'}$ *onto* $\mathcal{P}_{k-1} \subset H^{p'}$ *to be interpreted as* 0 *if* $k \leq 0$*, i.e.,* $Q_k = I_{H^{p'}}$ *if* $k \leq 0$*. Thus, for* $n_0 + n_- \leq m_0 + m_-$ *we have* $\mathrm{Ran}(T_\omega^*) = T_{z^{m-n}/(q_+)^\sharp}(s_+ s_0)^\sharp H^{p'}$*. Moreover,*

$$\dim \mathrm{Ker}(T_\omega^*) = \max\left\{0, \#\{\textit{zeroes of}\ \omega\ \textit{inside}\ \mathbb{D}\} - \#\{\textit{poles of}\ \omega\ \textit{in}\ \overline{\mathbb{D}}\}\right\},$$

where the multiplicities of the zeroes and poles are taken into account. Hence, $\dim \mathrm{Ker}(T_\omega^*)$ *is the maximum of* 0 *and* $n_- - m_- - m_0$*. In particular,* T_ω^* *is injective if and only if the number of poles of* ω *inside* $\overline{\mathbb{D}}$ *is greater than or equal to the number of zeroes of* ω *inside* $\mathbb{D}$*, multiplicities taken into account.*

By comparing the results on T_ω and T_ω^* on H^2, it is obvious T_ω cannot be selfadjoint, except when ω has no poles on $\mathbb{T}$. Below we describe in terms of ω when T_ω^* is symmetric, in which case $T_\omega^* \subset T_\omega$, and whenever T_ω^* is symmetric we describe when T_{ω^*} admits a selfadjoint extension.

Theorem G. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime. Consider* T_ω *on* H^2*. Then*

$$T_\omega^*\ \textit{is symmetric} \quad \Longleftrightarrow \quad \omega(\mathbb{T}) \subset \mathbb{R}.$$

In particular, if T_ω^* *is symmetric, then* $\deg(s) \leq \deg(q) \leq 2\deg(s)$*. Furthermore, if* T_ω^* *is symmetric, then* T_ω^* *admits a selfadjoint extension if and only if the number of roots of* $s - iq$ *and* $s + iq$ *in* $\mathbb{D}$*, counting multiplicities, coincide. This happens in particular if* $\omega(\mathbb{T}) \neq \mathbb{R}$*, but cannot happen in case* $\deg(q)$ *is odd.*

We show that if $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper, then the adjoint operator T_ω^* is precisely a Toeplitz-like operator of the type studied by Sarason, i.e. Toeplitz operators generated by functions in the Smirnov class. Hence in this case our Toeplitz-like operator $T_\omega = T_\omega^{**}$ coincides with the adjoint of the Toeplitz-like operator considered by Sarason. We also show that $H(\overline{\mathbb{D}})$, the space of functions analytic on a neighborhood of $\overline{\mathbb{D}}$, is contained in $\mathrm{Dom}(T_\omega)$ and in fact is a core of T_ω.

Sarason introduces a class of closed, densely defined Toeplitz-like operators on H^2 determined by algebraic properties, which was further investigated by Rosenfeld. In particular, this class of Toeplitz-like operators contains the unbounded Toeplitz-like operator studied by Sarason and is closed under taking adjoints, and hence contains our Toeplitz-like operators with proper symbols in $\mathrm{Rat}(\mathbb{T})$. In fact, we show that T_ω is contained in the class of Toeplitz-like operators introduced by Sarason for any ω in Rat.

Matrix symbols: Let $\Omega \in \mathrm{Rat}^{m\times m}$ with possibly poles on $\mathbb{T}$ and $\det \Omega(z) \not\equiv 0$. Define $T_\Omega\,(H^p_m \to H^p_m)$ by

$$\mathrm{Dom}(T_\Omega) = \left\{\ f \in H^p_m : \quad \begin{array}{c} \Omega f = h + r \text{ where } h \in L^p_m(\mathbb{T}), \\ \text{and} \quad r \in \mathrm{Rat}_0^m(\mathbb{T}) \end{array}\ \right\}, \qquad T_\Omega f = \mathbb{P} h$$

where $\mathbb{P}$ is the Riesz projection of $L^p_m(\mathbb{T})$ onto H^p_m. We will consider the square case only for simplicity but many of the results in this chapter extend to the non-square case, i.e. $m \neq n$.

The Wiener-Hopf factorization of matrices with no poles on the unit circle allows one to determine invertibility conditions and Fredholm-properties of the block Toeplitz operator generated by rational matrix functions with entries in $L^\infty(\mathbb{T})$.

Using an adaptation of the construction of the Wiener-Hopf factorization due to N. Wiener and E. Hopf, and made accessible by K. Clancy and I. Gohberg, we prove a Wiener-Hopf type factorization for a rational matrix function with poles on the unit circle. We show, for $\Omega \in \mathrm{Rat}^{m\times m}$ with $\det\Omega \not\equiv 0$ that we can write

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

for some $k > 0$. Here Ω_- and $(\Omega_-)^{-1}$ are minus functions , Ω_+ and $(\Omega_+)^{-1}$ are plus functions, $\Omega_0 = \mathrm{Diag}_{j=1}^m(\phi_j)$ with ϕ_j a scalar rational function with poles and zeroes only on $\mathbb{T}$ and P_0 is a lower triangular matrix polynomial with $\det(P_0(z)) = z^N$ for some $N \geq 0$.

Note that, if we choose P_0 to be lower triangular then the degree of the entries on the diagonal need not be in any order (increasing or decreasing), and conversely, if we choose to have either increasing or decreasing order of the degree of the entries on the diagonal, P_0 is not necessarily lower triangular. This is in sharp contrast to the classical Wiener-Hopf factorization result where the entries on the diagonal has increasing degree. As is the case for rational matrix functions in L^∞, the Wiener-Hopf type factorization of rational matrix functions with poles on $\mathbb{T}$, $\Omega(z)$, allows us to write

$$T_\Omega = T_{\Omega_-}T_{z^{-k}}T_{\Omega_0}T_{P_0}T_{\Omega_+}$$

where $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ is the Wiener-Hopf type factorization of Ω. From this we show that the following classical result holds:

$$T_{z^{-1}I_m}T_\Omega T_{zI_m}f = T_\Omega f, \qquad f \in \mathrm{Dom}(T_\Omega)$$

where I_m is the n-dimensional indentity matrix.

Using the factorization we reduce the questions on Fredholm properties to the case where the operator is generated by a rational matrix function that is the product of a diagonal rational matrix function, $\Omega_0(z) \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$ with zeroes only on $\mathbb{T}$, and a lower triangular polynomial matrix $P_0(z)$ with determinant z^N, $N \geq 0$.

Theorem H. *Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ be the Wiener-Hopf type factorization of Ω described above.*

Then T_Ω is Fredholm if and only if T_{Ω_0} is Fredholm, and, in particular, if and only if each of the entries, ϕ_j, of Ω_0 has no zeroes on $\mathbb{T}$. In case T_Ω is Fredholm,

$$\begin{aligned}\mathrm{Index}\,T_\Omega &= mk + \mathrm{Index}\,T_{\Omega_0} + \mathrm{Index}\,T_{P_0}\\ &= mk + \sum_{j=1}^m \deg q_j - \sum_{j=1}^m k_j\end{aligned}$$

where $\phi_j = \frac{s_j}{q_j}$ and q_j has roots only on $\mathbb{T}$ and k_j are the powers of z on the diagonal of P_0.

Further analysis of the factor $z^{-k}\Omega_0(z)P_0(z)$ reveals that when T_Ω is Fredholm, we have

$$\dim \operatorname{Ker} T_\Omega = \sum_{j=1}^{m} \max(k + \deg q_j - k_j, 0), \quad \operatorname{codim} \operatorname{Ran} T_\Omega = \sum_{j=1}^{m} \max(k_j - \deg q_j - k, 0)$$

and that T_Ω is invertible exactly when $k + \deg q_j = k_j$ for each j.

Contents

Preface

This thesis is presented in article format. In slightly altered form, the content of the following journal articles are included in Chapters 2 to 4:

1. G.J. Groenewald, S. ter Horst, J. Jaftha and A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle I: Fredholm properties, *Oper. Theory Adv. Appl.*, Vol. 271 (2018), 239 – 268.
2. G.J. Groenewald, S. ter Horst, J. Jaftha, and A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle II: the spectrum, *Oper. Theory Adv. Appl.*, Vol. 272 (2019), 133 - 154.
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The first two sections of Chapter 1 is a literature review with well known results from the theory of Toeplitz operators that are relevant to the results in this thesis. The rest of the chapter is a summary of the results found in chapters 2 - 5 and a comprehensive example that illustrates the main results. Chapter 5 is in the form of an article that will be submitted for publication later.

The origin for the study of this dissertation was a manuscript on the matrix case that I submitted but that was not accepted for publication. Following the rejection, a series of discussions were initiated by Rien Kaashoek with Sanne ter Horst, Gilbert Groenewald and Andre Ran at the North-West University at Potchefstroom on the substance of the submitted manuscript, and the necessity to understand better the scalar case first. These discussions lead to the PhD proposal and the participants became the thesis supervisors. *Toeplitz-like operators with rational symbol having poles on the unit circle* was the thrust of the original manuscript and the ideas embedded therein became points of departure for the second chapter as well as the final chapter of the dissertation. The plan for the contents of chapters 3 and 4 were constructed through discussions where I took a leading role in the initial research plan.

All co-authors of the articles are thesis supervisors and so permission for inclusion of the articles in the thesis is implicit.

Chapter 1

Introduction

In this thesis we analyse a Toeplitz-like operator on $H^p (1 < p < \infty)$ generated by a rational function having a pole on the unit circle. We explore Fredholm properties of the operator, the adjoint of the operator and the spectrum where the operator is generated by a scalar function, and we end with a chapter on Fredholm properties for the case where the operator is generated by a matrix valued function. Apart from Sections 1.3 and 1.4, this chapter is a literature review with well known results from the theory of Toeplitz operators.

Let ω be a rational function with poles on the unit circle, $\mathbb{T}$. Then $\omega \notin L^p \quad (1 \leq p \leq \infty)$ and so a Toeplitz operator T_ω generated by ω will not be bounded. In particular, $\omega \notin L^2$, and so the Toeplitz operator generated by ω is not in the class of unbounded Toeplitz operators on H^2 studied by Hartman and Wintner (cf. [18]). Furthermore, if ω has poles in $\mathbb{D}$ then ω is not analytic and so ω is not in the Smirnov class. The Toeplitz operator generated by such an ω will not be in the class of unbounded Toeplitz operators on H^2 studied by H. Helson (cf. [20]) and by Sarason (cf. [26]).

There are many areas in which Toeplitz operators play an important role such as information and control theory, physics, probability theory, and several other areas. Examples of the interplay between quite diverse topics in mathematics such as operator theory, complex analysis, harmonic analysis and the theory of Banach algebras, can be found in the study of Toeplitz and Wiener-Hopf operators, which, together with differential operators, forms an important class of non-selfadjoint operators. For example, Toeplitz operators are very useful for proving index theorems in the framework of non-commutative geometry. Many mathematicians such as A. Böttcher, A.P. Calderon, A. Devinatz, R.G. Douglas, I. Gohberg, M.A. Kaashoek, B.V. Khvedelidze, M.G. Krein, S.G. Mikhlin, S. Prößdorf, D. Sarason, B. Silbermann, I.B. Simonenko, H. Widom and others, have contributed to the study of Toeplitz -and Wiener-Hopf operators in the past half century.

Good references for bounded Toeplitz operators are the texts *Analysis of Toeplitz Operators* by A. Böttcher and B. Silbermann [3], *Basic Classes of Linear operators* chapters III and IV [15] and *Classes of Linear Operators* Vol II, Part VI and Chapter XXXII by I. Gohberg, S. Goldberg and M.A. Kaashoek [14] and for unbounded Toeplitz operators the articles *Unbounded Toeplitz operators* by D. Sarason [26] and *On the spectra of Toeplitz Matrices* by P. Hartman and A. Wintner [18], and references contained therein. Selected references for applications of Toeplitz operators are J. Andersen and J. Blaavand [1] in Topological Quantum Field Theory, Z. Zhu and M. Wakin [31] in Information Theory and Signal Processing and P. Deift, A. Its and I. Krasovky [8] in Statistical Mechanics.

1.1 Bounded Toeplitz operators

In this section we gather background results on bounded Toeplitz operators on H^p and start with the origins and some applications of Toeplitz operators.

1.1.1 Toeplitz operators: origins and applications

In his Habilitationsschrift Otto Toeplitz initiated the study of quadratic forms $\sum \psi_{jk} x_j y_k$ with coefficients $\psi_{jk} = \psi_{j-k}$ and the associated Toeplitz matrices $T_n(\psi) = \{\psi_{j-k}\}_{0 \leq j,k \leq n-1}$ with determinants $D_n(\psi) = \det(T_n(\psi))$. In [28] Toeplitz showed that the polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

has an analytic extension $\psi(z)$ in the unit disk with

$$\psi(z) = d_0 + d_1 z + d_2 z^2 + \cdots + d_n z^n + d_{n+1} z^{n+1} + \cdots$$

where $d_j = c_j, 0 \leq j \leq n$ and with nonnegative real part $(\operatorname{Re}(\psi(z)) \geq 0)$ in the unit disk if and only if the $(n+1) \times (n+1)$ Toeplitz matrix

$$\begin{pmatrix} 2c_0 & c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_{-1} & 2c_0 & c_1 & \cdots & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2c_0 & \cdots & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ c_{-n+1} & \vdots & \vdots & & & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & \cdots & 2c_0 \end{pmatrix}$$

with $c_{-k} = \bar{c}_k$ is non-negative. This provided an algebraic characterisation of a problem introduced by C. Carathéodory ([4]) and which Carathéodory characterised by using Minkowski's theory of convex bodies.

Toeplitz matrices were applied in the analysis of the Ising Model in Statistical Mechanics. W. Lenz introduced the Ising model in statistical mechanics named after a student of his, E. Ising [21]. In the two-dimensional Ising model it represents a model of ferromagnetism based on the interaction of random spins $\sigma_{i,j} = \pm 1$ at site $(i, j) \in \mathbb{Z}^2$. B. Kaufman and L. Onsager in articles [23], [24] and [25] discussed an analysis of the two-dimensional Ising model without a magnetic field present. In their analysis

$$M_0 = \lim_{n \to \infty} \langle \sigma_{1,1}, \sigma_{1,1+n} \rangle^{\frac{1}{2}} \tag{1.1.1}$$

where M_0 is the spontaneous magnetization of the two-dimensional Ising model and $\langle \sigma_{1,1}, \sigma_{1,1+n} \rangle$ is a correlation function. Kaufman and Onsager produced two approaches to solving Equation (1.1.1). Firstly, $\langle \sigma_{1,1}, \sigma_{1,1+n} \rangle$ is written as a sum of two $n \times n$ Toeplitz determinants. This requires the computation of the asymptotics of $n \times n$ Toeplitz determinants for which G. Szegö provided a solution in the celebrated Szegö Strong Limit Theorem of 1952, cf. [27].

The other approach is to write $\langle \sigma_{1,1}, \sigma_{1,1+n} \rangle$ as a single $n \times n$ Toeplitz determinant. Onsager recognised the eigenvalue equation for the Toeplitz determinant as a discrete analogue of a Milne-type integral equation which could be solved using the Wiener-Hopf technique, a technique Onsager was familiar with. See, for example, [8] for additional discussion of the application of Toeplitz operators in the analysis of the Ising model.

A more recent application of Toeplitz operators is in the area of Topological Quantum Field Theory (TQFT). In [1], J.E. Andersen and J.L. Blaavand show how Toeplitz operators defined on the holomorphic part of the space of endomorphisms on a compact manifold are used to confirm that the geometric constructions proposed by E. Witten in [30] satisfy the Atiyah-Segal-Witten TQFT axioms.

1.1.2 Bounded multiplication and Toeplitz operators on L^p

In this section, we review relevant material on bounded multiplication and Toeplitz operators on L^p generated by a function $a \in L^\infty$. Results contained in this section concerning the scalar case can be found in [3] and [15] and those results concerning the matrix case, in [14].

Let $\mathbb{T}$ be the unit circle in the complex plane. For $1 \leq p < \infty$, we denote by $L^p(\mathbb{T})$, or simply L^p, the Banach space of complex valued measurable functions f on $\mathbb{T}$ such that $|f|^p$ is integrable, i.e.,

$$f \in L^p \iff \|f\|_p := \left(\int_{\mathbb{T}} |f|^p dm \right)^{1/p} < \infty,$$

where m is the Lebesgue measure. By L^∞ we shall mean the measurable and essentially bounded functions on $\mathbb{T}$, i.e.,

$$f \in L^\infty \iff \|f\|_\infty := \operatorname*{ess\,sup}_{z \in \mathbb{T}} |f(z)| < \infty.$$

Then

$$L^\infty \subset L^r \subset L^s \subset L^1 \qquad \text{for } 1 \leq s \leq r \leq \infty.$$

For $a \in L^\infty$ and $1 < p < \infty$ define the multiplication operator M_a by

$$M_a : L^p \longrightarrow L^p, f \longmapsto af.$$

Then M_a is called the multiplication operator generated by a or equivalently with symbol a. Clearly, M_a is a bounded operator on L^p and, in fact, M_a is a bounded operator on L^p if and only if $a \in L^\infty$ with

$$\|M_a\|_p = \|a\|_\infty.$$

Let $H^p \subset L^p \, (1 \leq p \leq \infty)$ be the Hardy space of functions in L^p that have an analytic extension to the open unit disk in the complex plane. Then

$$H^\infty \subset H^r \subset H^s \subset H^1 \qquad \text{for } 1 \leq s \leq r \leq \infty.$$

For $1 < p < \infty$, let $\mathbb{P} : L^p \to H^p$ be the projection due to M. Riesz, see for example pages 149 - 153 in [16]. We shall refer to this as the Riesz projection, not be confused with the Riesz projection occurring in spectral theory, which is due to F. Riesz, see for example pages 9 - 13 in [13]. Then $\mathbb{P}$ is a bounded projection for $1 < p < \infty$. For $p = 2$ the Riesz projection coincides with the orthogonal projection of L^2 onto H^2.

For $a \in L^\infty$, the Toeplitz operator T_a on $H^p \, (1 < p < \infty)$ generated by a is defined by

$$T_a : H^p \to H^p, \qquad f \mapsto \mathbb{P}(af).$$

Note that

$$T_a = \mathbb{P} M_a|_{H^p}$$

and so T_a is bounded and we say that a is the symbol of T_a. It is well known that the operator T_a is bounded if and only if $a \in L^\infty$ (cf. Hartman and Wintner, 1950, [18]). If $p = 2$ then

$$\|T_a\|_2 = \|M_a\|_2 = \|a\|_\infty$$

and for $p \neq 2$ we have

$$\|T_a\|_p \leq \|\mathbb{P}\| \|a\|_\infty$$

where $\mathbb{P}$ is the Riesz projection on L^p.

1.1.3 Shift invariance

Define the bilateral shift M_z on $L^p \, (1 \leq p \leq \infty)$ as the multiplication operator generated by z and the unilateral shift T_z on $H^p \, (1 \leq p \leq \infty)$ as the Toeplitz operator generated by z. Thus

$$M_z : L^p \to L^p, \quad f(z) \mapsto zf(z), z \in \mathbb{T}$$

and

$$T_z : H^p \to H^p, \quad f(z) \mapsto \mathbb{P}(zf(z)), z \in \mathbb{T}.$$

The backward bilateral shift $M_{z^{-1}}$ is the multiplication operator generated by z^{-1} and the backward unilateral shift $T_{z^{-1}}$ is the Toeplitz operator generated by z^{-1}.

Note that

1. $(T_z)^n = T_{z^n}$ and $(T_{z^{-1}})^n = T_{z^{-n}}$, $n \in \mathbb{Z}_+$,

2. $T_{z^{-1}}T_z = I$ on H^p but $T_zT_{z^{-1}}$ is not the identity operator on H^p,

3. T_z^n is an isometry on H^p, has closed complemented range with codimension n,

4. $T_{z^{-1}}^n$ is surjective with dimension of its null-space n.

Let $a \in L^\infty$ then

$$M_aM_z = M_zM_a \quad \text{and } T_{z^{-1}}T_aT_z = T_a$$

and we say that the multiplication operators are shift invariant. In fact, bounded multiplication and Toeplitz operators on L^p and H^p, respectively, are the only operators with the above property, as the next result shows.

Proposition 1.1.1. *Let A be a bounded operator on $L^p \quad (1 < p < \infty)$ and B a bounded operator on $H^p \quad (1 < p < \infty)$ with $M_{z^{-1}}AM_z = A$ and $T_{z^{-1}}BT_z = B$ where M_z is the bilateral shift on L^p and T_z the unilateral shift on H^p. Then there are $a, b \in L^\infty$ with $M_a = A$ and $T_b = B$ where M_a is the multiplication operator generated by a and T_b the Toeplitz operator generated by b.*

For $a, b \in L^\infty$ the product of multiplication operators M_a, M_b generated by a and b is again a multiplication operator, $M_{ab} = M_aM_b$ but it is not always the case that T_aT_b is again a Toeplitz operator. However, we have the following result.

Proposition 1.1.2. *Let $a, b \in L^\infty$. Then T_aT_b is a Toeplitz operator if and only if $\overline{a}$ or b is analytic where $\overline{a}(z) = a(\overline{z})$. In case $\overline{a}$ and b are analytic, for any $c \in L^\infty$ we have*

$$T_{acb} = T_aT_cT_b$$

1.1.4 Fourier coefficients and the matrix representation

Define the function $e_n \, (n \in \mathbb{Z})$ by $e_n(z) = z^n \, (z \in \mathbb{T})$ which we will denote by z^n. Then $\{z^n\}_{n\in\mathbb{Z}}$ is a basis for $L^p \, (1 \le p \le \infty)$ and an orthogonal basis for L^2. The set $\{z^n\}_{n\ge 0}$ is a basis for $H^p \, (1 \le p \le \infty)$ and an orthogonal basis for H^2.

Given $f \in L^p$ we define its Fourier coefficient $f_n (n \in \mathbb{Z})$ by

$$f_n = (f(z), z^n) := \frac{1}{2\pi}\int_{\mathbb{T}} f(z)\overline{z^n}dm(z) = \frac{1}{2\pi}\int_{\mathbb{T}} f(z)z^{-n}dm(z).$$

Since $\{z^n\}_{n\in\mathbb{Z}}$ is a basis for $L^p \, (1 \le p \le \infty)$ and $\{z^n\}_{n\ge 0}$ is a basis for $H^p \, (1 \le p \le \infty)$, for $f \in L^p$ and $g \in H^p$ we can write

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^n \qquad \text{and} \qquad g(z) = \sum_{n\ge 0} g_n z^n, \quad z \in \mathbb{T},$$

where f_n and g_n are the n-th Fourier coefficients of f and g respectively.

The harmonic extension of $f \in L^1$ is the function $\widehat{f}$ defined on $\mathbb{D}$ by

$$\widehat{f}(z) = \sum_{n \in \mathbb{Z}} f_n r^{|n|} e^{in\theta} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi)$$

where $z = re^{i\theta} \in \mathbb{D}$ and f_n is the n-the Fourier coefficient of f. If $f \in H^1$ then its harmonic extension is analytic and so is referred to as its analytic extension with

$$\widehat{f}(z) = \sum_{n \geq 0} f_n z^n \quad (z \in \mathbb{D}).$$

Note that for $f \in H^p$ the non-tangential limits

$$\lim_{z \to e^{i\theta}} \widehat{f}(z) \quad (z \in \mathbb{D})$$

exist almost everywhere and coincide with $f(e^{i\theta})$. Thus we will identify the analytic extension $\widehat{f}$ of $f \in H^p$ with f.

Note that for $f \in L^p$, in terms of Fourier coefficients, we have

$$(\mathbb{P}f)_n = f_n \text{ for } n \geq 0, \qquad (\mathbb{P}f)_n = 0 \text{ for } n < 0$$

where $(\mathbb{P}f)_n$ is the n-th Fourier coefficient of $\mathbb{P}f$.

Let $a \in L^\infty$ and T_a be the Toeplitz operator on $H^p\,(1 < p < \infty)$ with symbol a. Then

$$(T_a z^j, z^k) = \frac{1}{2\pi} \int_{\mathbb{T}} a z^j z^{-k} dm(z) = \frac{1}{2\pi} \int_{\mathbb{T}} a z^{-(k-j)} dm(z) = a_{k-j}$$

the $(k-j)$-th Fourier coefficient of a. This is a defining characteristic of Toeplitz operators, as can be seen in the next result.

Theorem 1.1.3. *Let A be a bounded operator on $H^p\,(1 < p < \infty)$, and suppose $(a_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers with $(Az^j, z^k) = a_{k-j}$ for all $k, j \in \mathbb{Z}_+$. Then there is an $a \in L^\infty$ such that $A = T_a$ and a_n is the n-th Fourier coefficient of a.*

Let $a \in L^\infty$ then the set of polynomials, $\mathcal{P}$, is contained in the domain of T_a and so we can determine the action of T_a as an infinite matrix. From Theorem 1.1.3 the matrix representation is a Toeplitz matrix, i.e., the matrix $[T_a]$ of T_a is one with constant diagonals that are parallel to the main diagonal

$$[T_a] = M = [m_{ij}]_{i,j=0}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $a_j \in \mathbb{C}$ is the j-th Fourier coefficient of a. Note that we can write $m_{ij} = a_{i-j}$.

For $p = 2$, the Toeplitz operator T_a on H^2 and the Toeplitz matrix $[T_a]$ defined on ℓ^2 are unitarily equivalent through the isomorphism

$$H^2 \to \ell^2, \qquad \sum_{n \in \mathbb{Z}_+} \varphi_n z^n \mapsto \{\varphi_n\}_{n \in \mathbb{Z}_+}. \tag{1.1.2}$$

1.1.5 Rational symbols

Let $\mathcal{C}$ be the space of all continuous complex-valued functions on $\mathbb{T}$. Suppose $a \in L^\infty$ is a rational function, then a has no poles on $\mathbb{T}$ and $a \in \mathcal{C}$. Because the Toeplitz-like operators we consider only have rational symbols, we discuss the case of rational L^∞ symbols here, but many results remain true for continuous symbols. For the rest of this section we assume that $a \in L^\infty$ is a rational function.

A function is called a *plus function* if its Fourier coefficients with negative index are all zero, and called a *minus function* if its Fourier coefficients with strictly positive index are all zero. Suppose a is a rational function with no poles on $\mathbb{T}$. If a has no zeroes on $\mathbb{T}$ we can write

$$a(z) = a_-(z) z^\kappa a_+(z), \quad z \in \mathbb{T},$$

for some $\kappa \in \mathbb{Z}$ where a_- and $(a_-)^{-1}$ are minus functions, and a_+ and $(a_+)^{-1}$ are plus functions. Here κ is the difference between the number of zeroes of a in $\mathbb{D}$ and poles of a in $\mathbb{D}$, counting multiplicities, and is equal to the winding number of a. This factorization is called the Wiener-Hopf factorization of a relative to $\mathbb{T}$.

Theorem 1.1.4. *Let $a \in L^\infty$ be a rational function without zeroes on $\mathbb{T}$ and T_a the Toeplitz operator with symbol a on H^p $(1 < p < \infty)$. Suppose*

$$a(z) = a_-(z) z^\kappa a_+(z), \quad z \in \mathbb{T},$$

is the Wiener-Hopf factorization of a relative to $\mathbb{T}$. If $\kappa \geq 0$ then

$$T_a = T_{a_-} T_z^\kappa T_{a_+}$$

and if $\kappa < 0$ then

$$T_a = T_{a_-} (T_{z^{-1}})^{-\kappa} T_{a_+}.$$

1.1.6 Fredholm Theory

An operator A from Banach space X to Banach space Y is called Fredholm if it has a closed range with finite dimensional nullspace (Ker$A$) and its range (Ran$A$) has finite codimension. The index of a Fredholm operator A is given by

$$\operatorname{Index} A = \operatorname{Dim} \operatorname{Ker} A - \operatorname{codim} \operatorname{Ran} A. \tag{1.1.3}$$

Theorem 1.1.5. *Let T_a be the Toeplitz operator generated by $a \in L^\infty$ on $H^p\,(1 < p < \infty)$. Then T_a is injective or has dense range. In particular, T_a is invertible on H^p if and only if T_a is Fredholm with* $\operatorname{Index} T_a = 0$.

Let $\mathbb{T}^\circ = \mathbb{T} \backslash \{-1\}$ and suppose $a \in \mathcal{C}$ with no zeroes on $\mathbb{T}$. Then there is a real-valued function b on $\mathbb{T}^\circ$ with $a = |a|e^{i\pi b}$. The increment in b as the result of a counter-clockwise rotation on $\mathbb{T}$ is an integer dependent on a only and not on the choice of a particular b. This integer is called the winding number of a and denoted by $\operatorname{wind}(a)$.

Theorem 1.1.6. *Let T_a be the Toeplitz operator generated by $a \in L^\infty$ on $H^p\,(1 < p < \infty)$ and suppose a is a rational function. Then T_a is Fredholm if and only if $a(z) \neq 0$ for $z \in \mathbb{T}$. If T_a is Fredholm, then* $\operatorname{Index} T_a = -\operatorname{wind}(a)$.

Let T_a be the Toeplitz operator generated by $a \in L^\infty$ on $H^p(1 < p < \infty)$. From the above we have that T_a is Fredholm if and only if a has no zeroes on $\mathbb{T}$ and in this case $\operatorname{Index} T_a = -\operatorname{wind}(a)$ which is equal to the difference between the number of zeroes of a and the number of poles of a in $\mathbb{D}$, counting multiplicities. Furthermore, T_a is invertible if and only if T_a is Fredholm with index zero and so in this case a has the same number of poles as zeroes in $\mathbb{D}$, multiplicities counted.

Theorem 1.1.7. *Let $a \in L^\infty$ be a rational function without zeroes on $\mathbb{T}$ and T_a the Toeplitz operator with symbol a on $H^p\,(1 < p < \infty)$. Suppose*

$$a(z) = a_-(z) z^\kappa a_+(z), \quad z \in \mathbb{T}$$

is the Wiener-Hopf factorization of a relative to $\mathbb{T}$. Then $\operatorname{Index} T_a = -\kappa$ *and so T_a is invertible if and only if $\kappa = 0$ and in this case*

$$(T_a)^{-1} = T_{(a_+)^{-1}} T_{(a_-)^{-1}}.$$

If $\kappa > 0$ then T_a is left invertible with a left inverse given by

$$T_a^+ = T_{(a_+)^{-1}} (T_{z^{-1}})^\kappa T_{(a_-)^{-1}}.$$

If $\kappa < 0$ then T_a is right invertible with a right inverse given by

$$T_a^+ = T_{(a_+)^{-1}} (T_z)^{-\kappa} T_{(a_-)^{-1}}.$$

The essential range $R(a)$ of an L^∞ function a is defined by

$$R(a) := \{\lambda \in \mathbb{C} : m\{z \in \mathbb{T} : |a(z) - \lambda| < \varepsilon\} > 0 \quad \text{for every } \varepsilon > 0\}$$

where m is the Lebesgue measure. Observe that $R(a) = a(\mathbb{T})$ if a is continuous. For an operator A on a Banach space X the spectrum $\sigma(A)$ is defined by

$$\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda I \quad \text{is not invertible}\}$$

and the essential spectrum $\sigma_{ess}(A)$ by

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \quad \text{is not Fredholm}\}.$$

The point spectrum of A is the set

$$\sigma_p(A) = \{\lambda \in \sigma(A) : A - \lambda I \quad \text{is not injective}\},$$

the continuous spectrum of A is the set

$$\sigma_c(A) = \{\lambda \in \sigma(A) : A - \lambda I \quad \text{is injective and has dense range}\}$$

and the residual spectrum of A is

$$\sigma_r(A) = \{\lambda \in \sigma(A) : A - \lambda I \quad \text{is (injective and) but does not have dense range}\}.$$

Let T_a be the Toeplitz operator generated by $a \in L^\infty$ on $H^p\,(1 < p < \infty)$. Hartman and Wintner in [18] showed that the point spectrum of T_a on H^2 is empty in case a is rational real-valued and posed the problem of specifying the spectral properties. In case a is continuous, Gohberg in [11], and more explicitly in [12], showed that T_a is Fredholm exactly when the symbol has no zeroes on $\mathbb{T}$, and in this case the index of the operator coincides with the negative of the winding number of the symbol with respect to zero. This implies that the essential spectrum of a Toeplitz operator with continuous symbol is the image of the unit circle under a.

Hartman and Wintner in [19] followed up on their earlier question by showing that if a is real valued, then the spectrum of T_a on H^2 is contained in the interval bounded by the essential lower and upper bounds of a on $\mathbb{T}$. They also showed that the point spectrum is empty whenever a is not a constant. Halmos, after posing in [17] the question whether the spectrum of a Toeplitz operator is connected, with Brown in [2] showed that the spectrum cannot consist of only two points. Widom, in [29], established that T_a on H^2 has connected spectrum, and later extended the result for general H^p, with $1 \leq p \leq \infty$. That the essential (Fredholm) spectrum of a bounded Toeplitz operator in H^2 is connected was shown by Douglas in [9]. For the case of bounded Toeplitz operators in H^p it is posed as an open question in Böttcher and Silbermann in [3, Page 70] whether the essential (Fredholm) spectrum of a Toeplitz operator in H^p is necessarily connected.

For $\lambda \in \mathbb{C}$ and $a \in L^\infty$ the operator $T_a - \lambda I$ is the Toeplitz operator $T_{a-\lambda}$ so that questions on the spectrum of T_a can be related to questions on Toeplitz operators with an additional complex parameter. For $a = \frac{s}{q} \in L^\infty$ rational we can describe the spectrum and its subsets in terms of the winding number of a and thus in terms of the difference between the poles of a in $\mathbb{D}$ and the zeroes of $\lambda - a$ in $\mathbb{D}$. This is summarised as follows:

Theorem 1.1.8. *Let $a \in L^\infty$ be a rational function and suppose wind $(a|\lambda)$ is the winding number of a with respect to $\lambda \in \mathbb{C}$. Then*

1. *The essential spectrum of* T_a *is given by*

$$\sigma_{ess}(T_a) = a(\mathbb{T}).$$

2. *The spectrum of* T_a *is given by*

$$\sigma(T_a) = a(\mathbb{T}) \cup \{\lambda \in \mathbb{C} : \text{ wind } (a|\lambda) \neq 0\}.$$

   (a) *The point spectrum of* T_a *is the set*

   $$\sigma_p(T_a) = \{\lambda \in \mathbb{C} : \text{ wind } (a|\lambda) < 0\}.$$

   (b) *The continuous spectrum of* T_a *is the set*

   $$\sigma_c(T_a) = a(\mathbb{T})$$

   (c) *The residual spectrum of* T_a *is the set*

   $$\sigma_r(T_a) = \{\lambda \in \mathbb{C} : \text{ wind } (a|\lambda) > 0\}.$$

1.1.7 The adjoint operator and selfadjoint Toeplitz operators

The dual space of H^p $(1 < p < \infty)$ is $(H^p)^* = L^{p'}/\mathbb{P}(L^{p'})$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ where $\mathbb{P}$ is the Riesz projection in $L^{p'}$. There is an isomorphism between $(H^p)^*$ and $H^{p'}$ given by

$$G(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f\overline{g}dm, \qquad G \in (H^p)^* \mapsto g \in H^{p'}.$$

This is an isometry for $p = 2$.

With this identification, it is clear that for $a \in L^\infty$, the adjoint T_a^* of T_a in H^p can be identified with $T_{\overline{a}}$ on $H^{p'}$, the Toeplitz operator generated by $\overline{a}$.

Let a be a real valued function, i.e., $[T_a]$ Hermitian matrix, then T_a is a self-adjoint operator on H^2. Furthermore, $\sigma(T_a) = R(a) = [m, M]$ where m, M are the (essential) lower and upper bounds of a. Also, if a is not a constant, then the point spectrum $\sigma_p(T_a)$ of T_a is empty.

1.1.8 Matrix symbols

Many results similar to the scalar case holds for the matrix case as well. We will confine ourselves to results in Fredholm theory related to rational matrix symbols in this section.

By L^p_m and H^p_m we shall mean complex-valued functions, f, on $\mathbb{T}$ with m-components with each component in L^p and H^p, respectively. Let $a = (a_{ij})_{i,j=1}^n$ be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$. Then $a(z)$ is continuous on $\mathbb{T}$. The block Toeplitz operator T_a defined on $H_n^p\,(1 < p < \infty)$ is given by

$$T_a : H_n^p \to H_n^p, \qquad (f_j)_{j=1}^n \mapsto \left(\sum_{j=1}^n T_{a_{kj}} f_j \right)_{k=1}^n$$

where $T_{a_{kj}}$ is the Toeplitz operator on H^p generated by a_{kj}, the kj-th entry of a. We say that T_a is the block Toeplitz operator with defining function a or equivalently T_a is generated by a.

Theorem 1.1.9. *Let a be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$ and suppose T_a is the block Toeplitz operator generated by a. Then T_a is Fredholm if and only if*

$$\det a(z) \neq 0, \qquad z \in \mathbb{T}.$$

Theorem 1.1.10. *Let a be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$ and suppose T_a is the block Toeplitz operator generated by a. Assume that $\det a(z) \neq 0$ for all $z \in \mathbb{T}$. Then T_a is Fredholm and the index of T_a is the negative of the winding number relative to the origin of the curve parametrised by the function*

$$\rho : [-\pi, \pi] \to \mathbb{C}, \qquad \rho(t) = \det a(e^{it}).$$

Recall that a Toeplitz operator which is Fredholm with zero index is invertible (Theorem 1.1.5). There is a decisive difference between the scalar case and matrix case ($n > 1$) in that a block Toeplitz operator with index zero need not be invertible. A case in point is $a(z) = \operatorname{Diag}(z, z^{-1})$. Here the block Toeplitz operator T_a generated by a is clearly not invertible but both $\dim \operatorname{Ker}(T_a)$ and $\operatorname{codim} \operatorname{Ran}(T_a)$ are equal to 1.

Let a be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$. We call a a plus function if a has no poles on the closed unit disk $\overline{\mathbb{D}}$ which means that each entry of a has no poles on the closed unit disk $\overline{\mathbb{D}}$. This is equivalent to the fact that the Fourier coefficients a_k with negative index are zero, i.e.

$$a_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} a(e^{it}) dt = 0, \qquad k = -1, -2, \ldots.$$

We call a a minus function if a has no poles on $|z| \geq 1$ with the point at infinity included.

As with the scalar case, it is not the case that if a and b are $n \times n$ defining functions of block Toeplitz operators that the product of block Toeplitz operators T_a and T_b would again be a block Toeplitz operator with defining function ab.

Theorem 1.1.11. *Let a_1 and a_2 be $n \times n$ rational matrix functions with no poles on $\mathbb{T}$. If a_1 is a minus function or a_2 is a plus function then*

$$T_{a_1} T_{a_2} = T_{a_1 a_2}$$

where T_{a_1}, T_{a_2} and $T_{a_1 a_2}$ are the block Toeplitz operators generated by a_1, a_2 and $a_1 a_2$ respectively.

Recall that a (scalar) function a with no zeroes or poles on $\mathbb{T}$ has a (non-canonical) Wiener-Hopf factorisation relative to $\mathbb{T}$ in the form

$$a(z) = a_-(z)z^k a_+(z), z \in \mathbb{T}$$

where a_- and a_-^{-1} are minus functions and a_+ and a_+^{-1} are plus functions. The result below provides a matrix equivalent of the Wiener-Hopf factorization.

Theorem 1.1.12. *Let a be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$ and assume that $\det a(z) \neq 0$ for $z \in \mathbb{T}$. Then there exist integers $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ and rational matrix functions a_- and a_+, that have no poles on $\mathbb{T}$, such that*

$$a(z) = a_-(z) \begin{pmatrix} z^{\kappa_1} & & & \\ & z^{\kappa_2} & & \\ & & \ddots & \\ & & & z^{\kappa_n} \end{pmatrix} a_+(z)$$

and

1. *a_+ has no poles on the closed unit disk, $|z| \leq 1$,*
2. *$\det a_+(z) \neq 0$ for $|z| \leq 1$,*
3. *a_- has no poles for $|z| \geq 1$ (point at infinity included),*
4. *$\det a_-(z) \neq 0$ for $|z| \geq 1$ (point at infinity included).*

In particular a_-^{-1} and a_+^{-1} exist, the functions a_-, a_-^{-1} are minus functions and a_+, a_+^{-1} are plus functions.

The factorization in the theorem above is called a (non-canonical) Wiener-Hopf factorization of the rational matrix a relative to $\mathbb{T}$ and the indices $\kappa_j, j = 1, 2, \ldots, n$ are uniquely determined by a. The factors a_- and a_+ are, however, not uniquely determined by a.

Theorem 1.1.13. *Let a be an $n \times n$ rational matrix function with no poles on $\mathbb{T}$ and assume that $\det a(z) \neq 0$ for $z \in \mathbb{T}$ with*

$$a(z) = a_-(z)\mathrm{Diag}\left(z^{\kappa_1}, z^{\kappa_2}, \ldots, z^{\kappa_n}\right) a_+(z), \qquad z \in \mathbb{T}$$

a Wiener-Hopf factorization of a relative to $\mathbb{T}$. Then the block Toeplitz operator T_a generated by a is Fredholm with

$$\dim \mathrm{Ker}(T_a) = \sum_{\kappa_j \leq 0} \kappa_j, \qquad \mathrm{codim}\, \mathrm{Ran}(T_a) = \sum_{\kappa_j \geq 0} \kappa_j$$

and a generalised inverse of T_a is given by the operator

$$T_a^+ = T_{a_+^{-1}} \begin{pmatrix} T_z^{-\kappa_1} & & & & & & & & & \\ & T_z^{-\kappa_2} & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & T_z^{-\kappa_r} & & & & & & \\ & & & & I & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & I & & & \\ & & & & & & & T_{z^{-1}}^{\kappa_{s+1}} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & T_{z^{-1}}^{\kappa_n} \end{pmatrix} T_{a_-^{-1}}$$

where T_z is the unilateral (forward) shift operator on H^p, $T_{z^{-1}}$ the unilateral backward shift operator, $\kappa_1, \ldots, \kappa_r$ are the negative factorization indices and $\kappa_{s+1}, \ldots, \kappa_n$ are the positive factorization indices. Note that for the middle factor in the generalised inverse we use the identification $H_k^p \cong \oplus_{j=1}^k H^p$.

From the above theorem it follows that T_a will be invertible exactly when all indices are zero, i.e. $\kappa_1 = \kappa_2 = \cdots = \kappa_n = 0$, and in this case we have

$$T_a^{-1} = T_{a_+^{-1}} T_{a_-^{-1}}.$$

1.2 Unbounded Toeplitz operators: L^2 symbols and symbols in the Smirnov class

In this section we discuss results on two types of unbounded Toeplitz operators that have been considered in the literature. Of course we cannot do justice to the breadth of topics in the literature on these classes of operators and so we will confine ourselves to results relating to the aim of this thesis, namely basic results in Fredholm characteristics, the spectrum and the adjoint.

1.2.1 Unbounded Toeplitz operators on H^2 with L^2 symbols

Let $f \in L^2$ and $g \in H^2$ then $fg \in L^1$ but not necessarily in L^2. In [18] and [19], P. Hartman and A. Wintner investigated the spectra of infinite Hermitian Toeplitz matrices on ℓ^2 where they assert that the related problem for infinite Laurent matrices are comparatively simple. Here they introduced the infinite unbounded Toeplitz matrix associated with the Toeplitz operator T_f^{Hr}, defined below, to fill the gap in the literature on the location of the spectra of such infinite Toeplitz matrices. Note that the infinite Toeplitz matrix on ℓ^2 is isometrically equivalent to the Toeplitz operator on H^2, cf. Equation 1.1.2. In this subsection we review some results contained in [18] and [19].

The Toeplitz operator $T_f^{\mathrm{Hr}}(H^2 \to H^2)$ with symbol $f \in L^2$ is defined by

$$\mathrm{Dom}(T_f^{\mathrm{Hr}}) = \{h \in H^2 \mid fh = g_1 + g_2 \in L^1, g_1 \in H^2, g_2 \in L^1, (g_2)_n = 0, n \geq 0\}, \qquad T_f^{\mathrm{Hr}} h = g_1,$$

where $(g_2)_n$ is the n-th Fourier coefficient of g_2. Note that T_f^{Hr} is not bounded, unless if $f \in L^\infty$. Its domain contains the polynomials and so T_f^{Hr} is densely defined. Let M be the subspace of H^2 whose elements have only a finite number of nonzero Fourier coefficients, i.e. M is the space of polynomials in H^2, and suppose that $T_{\overline{f}}^{\circ}$ is the restriction of the Toeplitz operator $T_{\overline{f}}^{\mathrm{Hr}}$ with symbol $\overline{f}$ restricted to M. Then

$$(T_{\overline{f}}^{\circ})^* = T_f^{\mathrm{Hr}}, \qquad (T_f^{\mathrm{Hr}})^* = \overline{T_{\overline{f}}^{\circ}}$$

where $\overline{T_{\overline{f}}^{\circ}}$ is the smallest closed extension of $T_{\overline{f}}^{\circ}$. From this it follows that T_f^{Hr} is a closed operator as it is an adjoint operator and

$$T_f^{\circ} \subset (T_{\overline{f}}^{\mathrm{Hr}})^* \subset T_f^{\mathrm{Hr}}, \qquad (T_f^{\mathrm{Hr}})^{**} = T_f^{\mathrm{Hr}}.$$

Suppose that $f \in L^2$ is real-valued. Then $(T_f^{\mathrm{Hr}})^*$ is a closed symmetric operator and so

$$T_f^{\circ} \subset (T_f^{\mathrm{Hr}})^* \subset T_f^{\mathrm{Hr}} = (T_f^{\mathrm{Hr}})^{**}.$$

The following results contain conditions that ensures that T_f^{Hr} is selfadjoint or has a selfadjoint restriction.

Lemma 1.2.1. *Let $f \in L^2$ be real-valued. Then T_f^{Hr} is self-adjoint if and only if $\overline{\mathrm{Ran}(T_g^{\circ})} = H^2$ where $g = f \pm i$.*

Proposition 1.2.2. *Let $f \in L^2$ be real-valued. If f is bounded below , i.e. $f(z) \geq c$ for some $c \in \mathbb{R}^+$, then T_f^{Hr} is self-adjoint.*

Note that there exist real-valued functions $f \in L^2$ such that T_f^{Hr} is not self-adjoint but have a self-adjoint restriction (see for example (I*) on page 879 in [18]).

1.2.2 Unbounded Toeplitz operators on H^2 with Smirnov class symbols

A function $\phi \in H^\infty$ is called an inner function if $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$. A function $g \in H^1$ is said to be outer if its analytic extension, $\widehat{g}$, can be represented in the form

$$\widehat{g}(z) = c \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\psi(e^{i\theta})d\theta\right),$$

where $c \in \mathbb{T}$, $\psi \in L^1$, $\psi \geq 0$ a.e. on $\mathbb{T}$, $\log \psi \in L^1$. The inner-outer factorization theorem says that for every function $f \in H^p \quad (1 \leq p \leq \infty)$ which is not identically zero there is an inner function $\varphi \in H^\infty$ and an outer function $g \in H^p$ such that $f = \varphi g$. This factorization is unique up to a multiplicative constant. Note that if $f \in H^1$ is outer, then

1. its analytic extension $\widehat{f}(z) \neq 0$ for $z \in \mathbb{D}$, and
2. fH^p is dense in H^p.

The Smirnov class N^+ is the class of analytic functions that are quotients of H^∞ functions where the denominator is an outer function. An alternate definition for the Smirnov class (cf. [20]) is

$$N^+ = \{f \text{ analytic on } \mathbb{D} : \log |f| \in L^1, \text{ for } g \in H^2, fg \in L^2 \Rightarrow fg \in H^2\}.$$

Every $\phi \in N^+$ can be written uniquely as $\phi = \frac{b}{a}$ where a, b are in the unit ball of H^∞, a an outer function with $a(0) > 0$ and $|a(z)|^2 + |b(z)|^2 = 1$ for $z \in \mathbb{T}$. This is called the canonical form of $\phi \in N^+$.

For $\phi = \frac{b}{a} \in N^+$ define the Toeplitz operator $T_\phi^{\mathrm{Sa}}(H^2 \to H^2)$ (cf. [20], [26]) by

$$\mathrm{Dom}(T_\phi^{\mathrm{Sa}}) = \{h \in H^2 \mid \phi h \in H^2\}, \qquad T_\phi^{\mathrm{Sa}} h = \phi h.$$

Suppose $\phi = \frac{b}{a} \in N^+$ is the canonical form of ϕ then $\mathrm{Dom}(T_\phi^{\mathrm{Sa}}) = aH^2$ and so T_ϕ^{Sa} is closed and is densely defined as a is an outer function.

Let $H(\overline{\mathbb{D}})$ be the space of functions that are analytic on a neighbourhood of $\mathbb{D}$. Then $H(\overline{\mathbb{D}}) \subset \mathrm{Dom}(T_\phi^{\mathrm{Sa}})^*$ for $\phi \in N^+$.

Let $\phi \in N^+$ and define $T_{\overline{\phi}}^{\circ}(H(\overline{\mathbb{D}}) \to H^2)$ by

$$T_{\overline{\phi}}^{\circ}(H(\overline{\mathbb{D}}) \to H^2), \qquad f \mapsto \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \overline{\phi_n} f_{m+n} \right) z^m$$

where ϕ_n and f_m are the n-th and m-th Fourier coefficients of ϕ and f respectively. Then $T_{\overline{\phi}}^{\circ}$ is closable and $\overline{T_{\overline{\phi}}^{\circ}} = (T_\phi^{\mathrm{Sa}})^*$. We denote $(T_\phi^{\mathrm{Sa}})^*$ by $T_{\overline{\phi}}^{\mathrm{Sa}}$.

Let $\phi \in N^+$ and $\psi \in H^\infty$. For $f \in \mathrm{Dom} T_{\overline{\phi}}^{\mathrm{Sa}}$ we have

$$T_{\overline{\phi}}^{\mathrm{Sa}} T_{\overline{\psi}} f = T_{\overline{\phi\psi}}^{\mathrm{Sa}} f = T_{\overline{\psi}} T_{\overline{\phi}}^{\mathrm{Sa}} f.$$

Let $\phi \in N^+$ be real-valued on $\mathbb{T}$, for example $\phi(z) = -i\frac{z-1}{z+1}$. Then

1. T_ϕ^{Sa} is symmetric, and
2. if ϕ is non-negative then T_ϕ^{Sa} has a selfadjoint extension but is not selfadjoint.

1.3 Toeplitz-like operators with rational symbol having a pole on the unit circle

In this section we discuss the main results contained in the thesis. But, first we introduce some notation. By Rat and Rat_0 we shall mean the space of rational functions and the space of strictly proper rational functions, respectively. By $\mathrm{Rat}(\mathbb{T})$ and $\mathrm{Rat}_0(\mathbb{T})$ we shall mean the spaces of rational functions with poles only on $\mathbb{T}$ and the strictly proper rational functions with poles only on $\mathbb{T}$, respectively. By $\mathcal{P}$ and $\mathcal{P}_k$ we shall mean the spaces of polynomials and polynomials of degree at most k, respectively.

By $\mathrm{Rat}^{m\times n}$ and $\mathrm{Rat}_0^{m\times n}$ we shall mean the matrix equivalent, i.e. the space of $m \times n$ rational matrix functions with each entry a rational function and the space of strictly proper rational functions (identically zero at infinity), respectively. The other symbols are extended to the matrix case in a similar manner.

1.3.1 Basic and Fredholm Properties

Results in this section appear in Chapter 2 and in Chapter 3.

Let $\omega \in \mathrm{Rat}$ with poles on $\mathbb{T}$. Then $\omega \notin L^p$ for $1 \leq p \leq \infty$, and in particular $\omega \notin L^2$. To see this, suppose ω has a pole of order n at $\alpha \in \mathbb{T}$. Then $|\omega(z)| \approx |z-\alpha|^{-n}$ as $z \to \alpha$. From this it follows that $\int_{\mathbb{T}} |\omega|^p dz$ diverges. Furthermore, if $\omega \in \mathrm{Rat}$ has poles in $\mathbb{D}$ as well as on $\mathbb{T}$ then $\omega \notin N^+$. From this it follows that Toeplitz operators generated by rational functions with a pole in $\mathbb{D}$ will be an unbounded Toeplitz operator not covered by the classes investigated in Section 1.2.

Let $\omega \in \mathrm{Rat}$ with possible poles on $\mathbb{T}$. Define the Toeplitz-like operator $T_\omega(H^p \to H^P)$ for $1 < p < \infty$ as follows:

$$\mathrm{Dom}(T_\omega) = \{g \in H^p \,|\, \omega g = f + \rho \text{ with } f \in L^p, \rho \in \mathrm{Rat}_0(\mathbb{T})\}, \; T_\omega g = \mathbb{P} f, \tag{1.3.1}$$

where $\mathbb{P}$ is the Riesz projection from L^p onto H^p.

Note that in case ω has no poles on $\mathbb{T}$ then $\omega \in L^\infty$ and the Toeplitz-like operator T_ω defined above coincides with the classical Toeplitz operator T_ω on H^p discussed in Section 1.1.2. In general, for $\omega \in \mathrm{Rat}$, the operator T_ω is a well-defined, closed and densely defined linear operator. By the Euclidean division algorithm, one easily verifies that all polynomials are contained in $\mathrm{Dom}(T_\omega)$. Moreover, it can be verified that $\mathrm{Dom}(T_\omega)$ is invariant under the unilateral (forward) shift operator T_z and that the following classical result holds:

$$T_{z^{-1}} T_\omega T_z f = T_\omega f, \qquad f \in \mathrm{Dom}(T_\omega). \tag{1.3.2}$$

For $\omega = \frac{s}{q} \in \mathrm{Rat}(\mathbb{T})$, the definition as the Toeplitz-like operator in Equation (1.3.1) above can be simplified as

$$\mathrm{Dom}(T_\omega) = \left\{g \in H^p \,|\, \omega g = h + \frac{r}{q} \text{ with } h \in H^p, r \in \mathcal{P}_{\deg(q)-1}\right\}, \; T_\omega g = h, \tag{1.3.3}$$

which looks decidedly similar to the definition of T_f^{Hr} in Section 1.2.1.

In case $\omega \in \mathrm{Rat}(\mathbb{T})$, we have a complete description of the operator.

Theorem 1.3.1. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+ having roots only inside, on, or outside $\mathbb{T}$. Then*

$$\begin{gathered} \mathrm{Ker}(T_\omega) = \{r_0/s_+ \mid \deg(r_0) < \deg(q) - \deg(s_- s_0)\}\,; \\ \mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{\deg(q)-1}; \quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}, \end{gathered} \tag{1.3.4}$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1}\} \subset \mathcal{P}_{\deg(s)-1}. \tag{1.3.5}$$

Furthermore, $H^p = \overline{\mathrm{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ forms a direct sum decomposition of H^p, where

$$\widetilde{\mathcal{Q}} = \mathcal{P}_{k-1} \quad \text{with} \quad k = \max\{\deg(s_-) - \deg(q), 0\}, \tag{1.3.6}$$

following the convention $\mathcal{P}_{-1} := \{0\}$.

If $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with q having at least one root on $\mathbb{T}$ then qH^p is dense in H^p. Note that if the polynomial q has a root on $\mathbb{T}$ then $qH^p + \mathcal{P}_{\deg q - 1} \neq H^p$ but is only dense in $H^p (1 \leq p \leq \infty)$.

Theorem 1.3.2. *Let $s \in \mathcal{P}$, $s \not\equiv 0$. Then $H^p = sH^p + \mathcal{P}_{\deg(s)-1}$ if and only if s has no roots on the unit circle $\mathbb{T}$.*

The description of the nullspace and range of T_a in case $a \in \mathrm{Rat}(\mathbb{T})$ can be used to determine the index of T_a similarly as in Theorem 1.1.5 for $a \in L^\infty$.

Theorem 1.3.3. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. If T_ω is Fredholm, the Fredholm index of T_ω is given by*

$$\mathrm{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix}.$$

and T_ω is either injective or surjective. In particular, T_ω is injective, invertible or surjective if and only if $\mathrm{Index}\,(T_\omega) \leq 0$, $\mathrm{Index}\,(T_\omega) = 0$ *or* $\mathrm{Index}\,(T_\omega) \geq 0$, *respectively.*

For $\omega \in \mathrm{Rat}$ we construct the following analogue of the Wiener-Hopf factorization:

$$\omega(z) = \omega_-(z)(z^\kappa \omega_0(z))\omega_+(z)$$

where κ is the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$, ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and ω_0 has all its poles and zeroes on $\mathbb{T}$. Based on the choice of the domain in equation (1.3.1) it can then be shown that

$$T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}. \tag{1.3.7}$$

This factorization, together with the fact that T_ω is shift invariant (Equation 1.3.2) trivially extends Proposition 1.1.2 to this case, namely

$$T_a T_\omega T_b = T_{a\omega b}$$

where $\overline{a}$ and b are rational functions that are analytic on $\mathbb{D}$ and $\omega \in \text{Rat}$ with possibly poles on $\mathbb{T}$. In addition, this factorization eventually allows the questions on Fredholmness to be reduced to the case where ω has only poles on $\mathbb{T}$. It thus allows us to characterize invertibility of T_ω and to give a formula for the inverse of T_ω in case it exists.

Theorem 1.3.4. *Let $\omega \in \text{Rat}$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. In case T_ω is Fredholm, the Fredholm index of T_ω is given by*

$$\text{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix},$$

and T_ω is either injective or surjective. In particular, T_ω is injective, invertible or surjective if and only if $\text{Index}\,(T_\omega) \leq 0$, $\text{Index}\,(T_\omega) = 0$ *or* $\text{Index}\,(T_\omega) \geq 0$, *respectively.*

The result of Theorem 1.3.4 may also be expressed in terms of the winding number as follows: $\text{Index}\,(T_\omega) = -\lim_{r\downarrow 1} \text{wind}\,(\omega | r\mathbb{T})$. In the case where ω is continuous on the unit circle and has no zeroes there, it is well-known that the index of the Fredholm operator T_ω is given by the negative of the winding number of the curve $\omega(\mathbb{T})$ with respect to zero (Theorem 1.1.6). However, if ω has poles on the unit circle, the limit $\lim_{r\downarrow 1}$ cannot be replaced by either $\lim_{r\to 1}$ or $\lim_{r\uparrow 1}$ in this formula.

Proposition 1.3.5. *Let $\omega \in \text{Rat}$ with at least one pole on $\mathbb{T}$ and let κ be the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$. Then T_ω is invertible if and only if ω has no zeroes on $\mathbb{T}$ and κ is also equal to the number of poles of ω on $\mathbb{T}$. In that case ω factorizes as*

$$\omega(z) = \omega_-(z) \frac{z^\kappa}{q_0(z)} \omega_+(z),$$

where ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and q_0 is a polynomial of degree κ with all its roots on $\mathbb{T}$, and moreover,

$$T_\omega^{-1} = T_{\omega_+^{-1}} T_{\frac{q_0}{z^\kappa}} T_{\omega_-^{-1}}.$$

Since the polynomials $\mathcal{P}$ are contained in the domain of the closed operator T_ω defined in (1.3.1), by inspecting the action of T_ω on the monomials z^n and expressing the result as a power series, it is possible to determine a matrix representation $[T_\omega]$ of the operator $T_\omega(H^p \to H^p)$ with respect to the basis $\{z^n\}_{n=0}^\infty$. This matrix representation $[T_\omega]$ has a Toeplitz structure, i.e., $[T_\omega] = [a_{m-n}]_{m,n=0}^\infty$ for some sequence $(a_n)_{n\in\mathbb{Z}}$. Here a_n has a polynomial bound, $a_n = O(n^j)$ for some $j \in \mathbb{N}$.

Theorem 1.3.6. *Let $\omega \in \mathrm{Rat}$ possibly with poles on $\mathbb{T}$. Then we can write the matrix representation $[T_\omega]$ of T_ω with respect to the standard basis $\{z^n\}_{n=0}^\infty$ of H^p as*

$$[T_\omega] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \vdots & & & \ddots & & \end{pmatrix}.$$

In addition $a_{-j} = O(j^{M-1})$ for $j \geq 1$ where M is the largest order of the poles of ω in $\mathbb{T}$ and $(a_j)_{j=0}^\infty \in \ell^2$.

1.3.2 The spectrum

Results in this section appear in Chapter 3.

Using the fact that $\lambda I_{H^p} - T_\omega = T_{\lambda-\omega}$ enables us to describe the spectrum of T_ω, and its various parts. We start with the essential spectrum.

Theorem 1.3.7. *Let $\omega \in \mathrm{Rat}$. Then the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ of T_ω is an algebraic curve in $\mathbb{C}$ which is given by*

$$\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) := \{\omega(e^{i\theta}) \mid 0 \leq \theta \leq 2\pi,\ e^{i\theta} \text{ not a pole of } \omega\}.$$

Furthermore, the map $\lambda \mapsto \mathrm{Index}\,(T_{\lambda-\omega})$ is constant on connected components of $\mathbb{C}\backslash\omega(\mathbb{T})$ and the intersection of the point spectrum, residual spectrum and resolvent set of T_ω with $\mathbb{C}\backslash\omega(\mathbb{T})$ coincides with sets of $\lambda \in \mathbb{C}\backslash\omega(\mathbb{T})$ with $\mathrm{Index}\,(T_{\lambda-\omega})$ being strictly positive, strictly negative and zero, respectively.

In the next two results the various parts of the spectrum of T_ω are described using the facts that $\lambda I_{H^p} - T_\omega = T_{\lambda-\omega}$ and that $\mathrm{Index}\,(T_\omega) = -\lim_{r\downarrow 1} \mathrm{wind}\,(\omega|r\mathbb{T})$.

Theorem 1.3.8. *Let $\omega \in \mathrm{Rat}$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Define*

$$\begin{aligned} k_q &= \sharp\{\text{roots of } q \text{ inside } \overline{\mathbb{D}}\} = \sharp\{\text{poles of } \lambda - \omega \text{ inside } \overline{\mathbb{D}}\}, \\ k_\lambda^- &= \sharp\{\text{roots of } \lambda q - s \text{ inside } \mathbb{D}\} = \sharp\{\text{zeroes of } \lambda - \omega \text{ inside } \mathbb{D}\}, \qquad (1.3.8) \\ k_\lambda^0 &= \sharp\{\text{roots of } \lambda q - s \text{ on } \mathbb{T}\} = \sharp\{\text{zeroes of } \lambda - \omega \text{ on } \mathbb{T}\}, \end{aligned}$$

where in all these sets multiplicities of the roots, poles and zeroes are to be taken into account. Then the resolvent set $\rho(T_\omega)$, point spectrum $\sigma_{\mathrm{p}}(T_\omega)$, residual spectrum $\sigma_{\mathrm{r}}(T_\omega)$ and continuous spectrum $\sigma_{\mathrm{c}}(T_\omega)$ of T_ω are given by

$$\begin{gathered}\rho(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 = 0 \text{ and } k_q = k_\lambda^-\},\\ \sigma_{\mathrm{p}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_q > k_\lambda^- + k_\lambda^0\}, \quad \sigma_{\mathrm{r}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_q < k_\lambda^-\},\\ \sigma_{\mathrm{c}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0 \text{ and } k_\lambda^- \leq k_q \leq k_\lambda^- + k_\lambda^0\}.\end{gathered} \tag{1.3.9}$$

Furthermore, $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0\}$.

For the case where $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper we can be a bit more specific.

Theorem 1.3.9. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$ be proper, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Thus* $\deg(s) \leq \deg(q)$ *and all roots of q are on $\mathbb{T}$. Let a be the leading coefficient of q and b the coefficient of s corresponding to the monomial $z^{\deg(q)}$, hence $b = 0$ if and only if ω is strictly proper. Then $\sigma_{\mathrm{r}}(T_\omega) = \emptyset$, and the point spectrum is given by*

$$\sigma_{\mathrm{p}}(T_\omega) = \omega(\mathbb{C}\backslash\overline{\mathbb{D}}) \cup \{b/a\}.$$

Here $\omega(\mathbb{C}\backslash\overline{\mathbb{D}}) = \{\omega(z) \mid z \in \mathbb{C}\backslash\overline{\mathbb{D}}\}$. In particular, if ω is strictly proper, then $0 = b/a$ is in $\sigma_{\mathrm{p}}(T_\omega)$. Finally,

$$\sigma_{\mathrm{c}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0 \text{ and all roots of } \lambda q - s \text{ are in } \overline{\mathbb{D}}\}.$$

Examples for $\omega \in \mathrm{Rat}$ with poles on $\mathbb{T}$ so that T_ω has a bounded resolvent set, even with an empty resolvent set, are given below. This is in sharp contrast to the case where ω has no poles on the unit circle $\mathbb{T}$. For in this case the operator is bounded, the resolvent set is a nonempty unbounded set and the spectrum a compact set, and the essential spectrum is connected. Furthermore, an example is given where the essential spectrum is not connected.

Let $\omega(z) = \frac{1}{(z-1)^k}$ for some integer $k > 1$. This is an example where the Toeplitz-like operator generated by ω has an empty resolvent set. In this case,

$$\sigma_{\mathrm{p}}(T_\omega) = \sigma(T_\omega) = \mathbb{C}, \qquad \sigma_{\mathrm{c}}(T_\omega) = \sigma_{\mathrm{r}}(T_\omega) = \rho(T_\omega) = \emptyset,$$

and the essential spectrum is given by

$$\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = \left\{(it - \frac{1}{2})^k | t \in \mathbb{R}\right\}$$

which is a parabola $\mathrm{Re}(z) = \frac{1}{4} - \mathrm{Im}(z)^2$ for $k = 2$. See Figure 1.1.

Let $\omega(z) = \frac{z}{z^2+1}$. This is an example where the Toeplitz-type operator generated by ω has an essential spectrum that is the union of disjoint sets. In this case,

$$\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = (-\infty, -1] \cup [1, \infty) = \sigma_{\mathrm{c}}(T_\omega), \quad \sigma_{\mathrm{p}}(T_\omega) = \mathbb{C}\backslash\omega(\mathbb{T})$$

and so $\sigma_{\mathrm{r}}(T_\omega) = \rho(T_\omega) = \emptyset$.

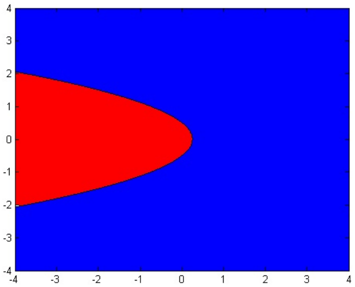

Figure 1.1: Spectrum of T_ω where $\omega(z) = \frac{1}{(z-1)^2}$. The black curve is the essential spectrum and the colors in the other regions codify the Fredholm index, where red indicates index 2 and blue indicates index 1.

1.3.3 The adjoint operator and selfadjoint extensions

Results in this section appear in Chapter 4.

Given $r \in \mathcal{P}$ with $\deg(r) = k$, say $r(z) = r_0 + zr_1 + \cdots + z^k r_k$, we define the polynomial $r^\sharp$ by

$$r^\sharp(z) = z^k\overline{r(1/\overline{z})} = \overline{r_0}z^k + \overline{r_1}z^{k-1} + \cdots + \overline{r_k}.$$

Note that on $\mathbb{T}$, ω^* is defined as $\omega^*(z) = \overline{\omega(z)}$, i.e., $\omega^*(z) = \overline{s(z)}/\overline{q(z)}$. Then $\omega^*(z) = z^{m-n}s^\sharp(z)/q^\sharp(z)$, but it is not always a representation as the ratio of two polynomials. Note in particular that $\omega^* \in \mathrm{Rat}(\mathbb{T})$ in case ω is proper, while this need not be the case if ω is not proper for then $z^{m-n}s^\sharp(z)$ is not a polynomial.

From a theorem of Cohn [6] it follows that in the transformation $r \to r^\sharp$, the nonzero roots of r (including multiplicity) transfer via the map $\alpha \mapsto 1/\overline{\alpha} = |\alpha|^{-2}\alpha$, while the degree decreases by the multiplicity of 0 as a root of r. We write $(s_+)^\sharp$ rather than $s_+^\sharp$, etc., to avoid confusion with what one may interpret as $(s^\sharp)_+$.

In case $\omega \in \mathrm{Rat}$ has no poles on $\mathbb{T}$, in fact for any $\omega \in L^\infty$ as we saw earlier in Section 1.1.7, the adjoint of the Toeplitz operator T_ω on H^p can be identified with the Toeplitz operator $T_{\overline{\omega}}$ on $H^{p'}$, with $1 < p' < \infty$ such that $1/p + 1/p' = 1$.

For the Toeplitz-like operators with a rational symbol having a pole on $\mathbb{T}$, the situation is more complicated than for Toeplitz operators with L^∞ symbols.

Theorem 1.3.10. *Let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime and $1 < p < \infty$. Factor $s = s_- s_0 s_+$ and $q = q_- q_0 q_+$ with s_-, q_- having roots only inside $\mathbb{T}$, s_0, q_0 having roots only on $\mathbb{T}$, and s_+, q_+ having roots only outside $\mathbb{T}$. Set $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$ $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$ and let $1 < p' < \infty$ with $1/p + 1/p' = 1$. Then*

$$\mathrm{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^* = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}. \tag{1.3.10}$$

Furthermore, we have

$$\begin{aligned} \mathrm{Ran}(T_\omega^*) &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} \mathcal{Q}_{n_0+n_- - m_0 - m_-}(s_0)^\sharp H^{p'}, \\ \mathrm{Ker}(T_\omega^*) &= \left\{ \frac{(q_-)^\sharp (q_0)^\sharp r}{(s_-)^\sharp} \mid \deg(r) < n_- - m_- - m_0 \right\}. \end{aligned} \tag{1.3.11}$$

Here $Q_k = I_{H^{p'}} - P_{\mathcal{P}_{k-1}}$*, with* $P_{\mathcal{P}_{k-1}}$ *the standard projection in* $H^{p'}$ *onto* $\mathcal{P}_{k-1} \subset H^{p'}$ *to be interpreted as* 0 *if* $k \leq 0$*, i.e.,* $Q_k = I_{H^{p'}}$ *if* $k \leq 0$*. Thus, for* $n_0 + n_- \leq m_0 + m_-$ *we have* $\operatorname{Ran}(T_\omega^*) = T_{z^{m-n}/(q_+)^\sharp}(s_+ s_0)^\sharp H^{p'}$*. Moreover,*

$$\dim \operatorname{Ker}(T_\omega^*) = \max\left\{0, \#\{\textit{zeroes of } \omega \textit{ inside } \mathbb{D}\} - \#\{\textit{poles of } \omega \textit{ in } \overline{\mathbb{D}}\}\right\},$$

where the multiplicities of the zeroes and poles are taken into account. Hence, $\dim \operatorname{Ker}(T_\omega^*)$ *is the maximum of* 0 *and* $n_- - m_- - m_0$*. In particular,* T_ω^* *is injective if and only if the number of poles of* ω *inside* $\overline{\mathbb{D}}$ *is greater than or equal to the number of zeroes of* ω *inside* $\mathbb{D}$*, multiplicities taken into account.*

For $\omega \in \mathrm{Rat}$, the second adjoint T_ω^{**} is well-defined and $T_\omega^{**} = T_\omega$. Now consider $\omega \in \mathrm{Rat}(\mathbb{T})$ and $p = 2$. From Theorem 1.3.10 it is obvious that $T_\omega \neq T_\omega^*$, except in the degenerate case where q is constant. Consequently, T_ω cannot be selfadjoint. Next, we describe when is T_{ω^*} symmetric and when is it selfadjoint. Note that a polynomial $r \neq 0$ is called self-inversive in case $r = \gamma r^\sharp$ for a constant $\gamma \in \mathbb{C}$.

Theorem 1.3.11. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime. Set* $n = \deg(s)$ *and* $m = \deg(q)$*. Then the following are equivalent.*

(1) T_ω^* *is symmetric;*

(2) $\omega(\mathbb{T}) \subset \mathbb{R}$*;*

(3) $\omega(z) = \widetilde{\omega}(-i\frac{z+1}{z-1})$ *with* $\widetilde{\omega}$ *a real rational function with poles only on* $\mathbb{R}$*;*

(4) the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ *of* T_ω *is contained in* $\mathbb{R}$*;*

(5) ω *is proper,* $s = z^{m-n}\widetilde{s}$ *with* $\widetilde{s}$ *self-inversive and* $q_0\overline{s_n} = \overline{q_m}s_{m-n}$ *holds, where* $s(z) = \sum_{k=0}^n s_k z^k$ *and* $q(z) = \sum_{k=0}^m q_k z^k$*.*

Moreover, if T_ω^* *is symmetric, then* $T_\omega^* \subset T_\omega$ *and if* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *then* $\deg(s) \leq \deg(q) \leq 2\deg(s)$*.*

The next result uses the fact that the Fredholm index of T_ω is constant on connected subsets of the spectrum to provide a condition for T_ω^* to have a selfadjoint extension.

Proposition 1.3.12. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$*, with* $s, q \in \mathcal{P}$ *coprime, be such that* T_ω^* *is symmetric. Then* $T_{\omega+i}$ *and* $T_{\omega-i}$ *are both Fredholm and* T_ω^* *admits a selfadjoint extension if and only if the Fredholm indices of* $T_{\omega+i}$ *and* $T_{\omega-i}$ *coincide.*

We have the following corollaries.

Corollary 1.3.13. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$*, with* $s, q \in \mathcal{P}$ *coprime, be such that* T_ω^* *is symmetric. Assume* $\omega(\mathbb{T}) \neq \mathbb{R}$*. Then* T_ω^* *admits a selfadjoint extension.*

Corollary 1.3.14. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, be such that T_ω^* has a selfadjoint extension. Then $\deg(q)$ is even.*

It is not the case, however, that if $\deg(q)$ is even that T_ω^* must admit a self-adjoint extension.

Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Set $n = \deg(s)$ and $m = \deg(q)$. Assume ω is proper, i.e., $n \leq m$. Then $\omega^*(z) = z^{m-n} s^\sharp / q^\sharp \in \mathrm{Rat}(\mathbb{T})$. Since $q^\sharp$ has zeroes only on $\mathbb{T}$ it is outer and thus $\omega^* \in N^+$. If, however, ω is not proper i.e. $m > n$ then ω^* is not in N^+ as it has a pole at 0. While in general T_ω and T_ω^{Sa} are different, the following proposition shows that in case ω is proper, T_ω coincides with $T_{\omega^*}^{\mathrm{Sa}}$, and hence $T_\omega = T_\omega^{**} = (T_{\omega^*}^{\mathrm{Sa}})^*$. Here T_ω^{Sa} is defined as in Section 1.2.2.

Proposition 1.3.15. *Let $\widetilde{\omega} = \widetilde{s}/\widetilde{q} \in \mathrm{Rat}(\mathbb{T})$ with $\widetilde{s}, \widetilde{q} \in \mathcal{P}$ co-prime. Then $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = \widetilde{q}H^2$ and $T_{\widetilde{\omega}}^{\mathrm{Sa}} = T_{\widetilde{\omega}}|_{\widetilde{q}H^2}$. In particular, if $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper, then $T_\omega^* = T_{\omega^*}^{\mathrm{Sa}}$.*

1.3.4 Matrix symbols

The results in this section appear in Chapter 5.

As indicated earlier, the spaces $\mathrm{Rat}^{m\times n}$, $\mathrm{Rat}^{m\times n}(\mathbb{T})$, $\mathrm{Rat}_0^{m\times n}$ and $\mathrm{Rat}_0^{m\times n}(\mathbb{T})$ are the $m\times n$ matrix equivalent of the rational functions, i.e., each entry is a rational function. Similarly, $\mathcal{P}^{m\times n}$ and $\mathcal{P}_k^{m\times n}$ are the matrix equivalent of the polynomials. As before in Section 1.1.8, by L_m^p and H_m^p we shall mean functions with m-components and each component is in L^p and H^p, respectively.

A pole of rational matrix function is a pole of any of its entries. The zero of a square rational matrix function, $\Omega(z)$, is a pole of its inverse, $\Omega^{-1}(z)$, see for example [10]. For a square matrix polynomial $P(z)$, this means that a zero is a point $z \in \mathbb{C}$ where $\det P(z) = 0$.

Let $\omega \in \mathrm{Rat}^{m\times m}$, i.e. ω is an $m \times m$ rational matrix function with possibly poles on $\mathbb{T}$, and $\det \omega \not\equiv 0$. We define the block Toeplitz-like operator T_ω generated by ω as

$$\mathrm{Dom}(T_\omega) = \left\{ f \in H_m^p : \begin{array}{c} \omega f = h + r \text{ where } h \in L_m^p(\mathbb{T}), \\ \text{and} \quad r \in \mathrm{Rat}_0^m(\mathbb{T}) \end{array} \right\}, \qquad T_\omega f = \mathbb{P}h$$

where $\mathbb{P}$ is the Riesz projection of L_m^p onto H_m^p.

Using similar arguments as in the scalar case, cf., Chapter 2, we determine various basic properties of the Toeplitz-like operator T_Ω. Some of these results can be derived by restricting to the entries of Ω. For example, we show that T_Ω is a well-defined, closed, densely defined linear operator on H_m^p. More specifically, $\mathcal{P}^m \subset \mathrm{Dom}(T_\Omega)$. As with Toeplitz-like operators with scalar symbols, we show that

$$S_- T_\Omega S_+ f = T_\Omega f \qquad f \in \mathrm{Dom}(T_\Omega),$$

where $S_+ = T_{zI_m}$ and $S_- = T_{z^{-1}I_m}$ on H^p_m.

Unlike the scalar case, though, getting a complete description of the domain, range and kernel of operator beyond the diagonal case appear significantly more complicated. The following lemma contains the best results we have at this stage.

Lemma 1.3.16. *Let* $\Omega \in \mathrm{Rat}^{m\times m}$ *and write* $\Omega = \Omega_1 + \Omega_2$ *where* $\Omega_1 \in L^{m\times m}_\infty$ *and* $\Omega_2 \in \mathrm{Rat}^{m\times m}_0(\mathbb{T})$*. Then* $\mathrm{Dom}(T_\Omega) = \mathrm{Dom}(T_{\Omega_2})$*. Let* $\Omega_2 = q^{-1}P$ *for some polynomial* q *with zeroes only on* $\mathbb{T}$ *and* P *a matrix polynomial. Suppose* $\Omega_2 = \left[\frac{s_{ij}}{q_{ij}}\right]$ *and let* $q_j = LCM^m_{i=1}\{q_{ij}\}$*, i.e., the least common multiple of the denominators in the* j*-th column of* Ω_2*. Then*

$$qH^p_m + \mathcal{P}^m_{\deg q-1} \subset \oplus_j \left(q_jH^p + \mathcal{P}_{\deg q_j-1}\right) \subset \mathrm{Dom}(T_\omega) \tag{1.3.12}$$

and both inclusions can be strict.

We prove the construction of a Wiener-Hopf type factorization of rational matrix functions with poles and zeroes on $\mathbb{T}$. The construction relies on the Smith decomposition of polynomial matrices and is an adaptation of the proof of Theorem 2.1 in [7].

Theorem 1.3.17. *Let* $\Omega \in \mathrm{Rat}^{m\times m}$ *with* $\det \Omega \not\equiv 0$*. Then*

$$\Omega = z^{-k}\Omega_-\Omega_0P_0\Omega_+ \tag{1.3.13}$$

for some $k > 0$*,* Ω_- *and* $(\Omega_-)^{-1}$ *are minus functions,* Ω_+ *and* $(\Omega_+)^{-1}$ *are plus functions,* $\Omega_0 = \mathrm{Diag}^m_{j=1}(\phi_j)$ *with* ϕ_j *a scalar rational function with poles and zeroes only on* $\mathbb{T}$ *and* P_0 *is a lower triangular matrix polynomial with* $\det(P_0(z)) = z^N$ *for some* $N \geq 0$*.*

We call the factorization in (1.3.13) a Wiener-Hopf type factorization for a rational matrix function, Ω, with poles and zeroes on $\mathbb{T}$. If Ω has no poles and zeroes on $\mathbb{T}$, then it coincides with the classical Wiener-Hopf factorization where Ω_0 is the identity and P_0 is a diagonal matrix with each of its entries a non-negative power of z. Note that, if we choose P_0 in (1.3.13) to be lower triangular then the degree of the entries on the diagonal need not be in any order (increasing or decreasing), and conversely, if we choose to have either increasing or decreasing order of the degree of the entries on the diagonal, P_0 is not necessarily lower triangular. This is in sharp contrast to the classical Wiener-Hopf factorization result where the entries on the diagonal has increasing degree. Using this factorization allows us to factorize the Toeplitz-like operator T_Ω as

$$T_\Omega = T_{\Omega_-}T_{z^{-k}}T_{\Omega_0}T_{P_0}T_{\Omega_+} \tag{1.3.14}$$

where $\Omega = z^{-k}\Omega_-\Omega_0P_0\Omega_+$ is the factorization of Ω in Theorem 1.3.17.

The factorization in (1.3.14) reduces the question on Fredholm properties to the case where Ω is a product of a diagonal matrix function, $\Omega_0(z) \in \mathrm{Rat}^{m\times m}_0(\mathbb{T})$ with zeroes only on $\mathbb{T}$, and a lower triangular polynomial matrix $P_0(z)$ with determinant z^N, $N \geq 0$.

Theorem 1.3.18. *Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ be the Wiener-Hopf type factorization of Ω as in Theorem 1.3.17. Then T_Ω is Fredholm if and only if T_{Ω_0} is Fredholm, which happens exactly when each of the entries ϕ_j of of Ω_0 has no zeroes on $\mathbb{T}$. In case T_Ω is Fredholm, we have*

$$\begin{aligned}\operatorname{Index} T_\Omega &= mk + \operatorname{Index} T_{\Omega_0} + \operatorname{Index} T_{P_0} \\ &= mk + \sum_{j=1}^{m} \deg q_j - \sum_{j=1}^{m} k_j,\end{aligned}$$

where $\phi_j = \frac{s_j}{q_j}$ and q_j has roots only on $\mathbb{T}$ and k_j are the powers of z on the diagonal of P_0.

In addition we can use the factorization to find the dimension of the kernel, the codimension of the range as well as when the Toeplitz-like operator is invertible. This is expressed in terms of the factor $z^{-k}\Omega_0 P_0$ in the Wiener-Hopf factorization in Theorem 1.3.17.

Proposition 1.3.19. *Let*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

be the Wiener-Hopf type factorization of Ω as in Theorem 1.3.17. Suppose that T_Ω is Fredholm then

$$\dim \operatorname{Ker} T_\Omega = \sum_{j=1}^{m} \max(k + \deg q_j - k_j, 0)$$

and

$$\operatorname{codim} \operatorname{Ran} T_\Omega = \sum_{j=1}^{m} \max(k_j - \deg q_j - k, 0)$$

where $\phi_j = \frac{1}{q_j}$ are the entries of Ω_0, q_j has roots only on $\mathbb{T}$ and k_j are the powers of z on the diagonal of P_0.

Corollary 1.3.20. *Let*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

be the Wiener-Hopf type factorization of Ω as in Theorem 1.3.17. Then T_Ω is invertible exactly when $k + \deg q_j = k_j$ for each j.

1.3.5 An example

We conclude with an example that illustrates some of the main results. In this example we discuss all the topics for the scalar case whereas other examples in the text only discuss one aspect.

Let $\omega(z) = \frac{z+\alpha}{(z-1)^2}, \alpha \neq -1$.

Fredholm properties: Since $\omega \in \mathrm{Rat}_0(\mathbb{T})$, from Theorem 1.3.1 the Toeplitz-like operator $T_\omega(H^p \to H^p), 1 < p < \infty$ with symbol ω has

$$\mathrm{Dom}(T_\omega) = (z-1)^2 H^p + \mathcal{P}_1 \quad \text{and} \quad \mathrm{Ran}(T_\omega) = (z+\alpha)H^p + \mathbb{C}.$$

From Theorem 1.3.3 we have that T_ω is Fredholm if and only if $\alpha \notin \mathbb{T}$. In this case, i.e., $|\alpha| \neq 1$, T_ω is surjective.

Furthermore, if $|\alpha| > 1$ then

$$\mathrm{Ker}(T_\omega) = \left\{ \frac{r}{z+\alpha} \mid r \in \mathcal{P}_1 \right\}, \qquad \mathrm{Index}\,(T_\omega) = 2,$$

and if $|\alpha| < 1$ then

$$\mathrm{Ker}(T_\omega) = \mathbb{C}, \qquad \mathrm{Index}\,(T_\omega) = 1.$$

Matrix representation: Write

$$\omega(z) = \frac{z+\alpha}{(z-1)^2} = \frac{1}{z-1} + \frac{1+\alpha}{(z-1)^2}.$$

For $N \geq 2$, by Lemma 2.6.1 twice, we have

$$\begin{aligned} z^N &= (z-1)\sum_{j=0}^{N-1} z^j + 2 \\ &= (z-1)^2 \sum_{j=0}^{N-2} (N-j-1)z^j + (z-1) + 2 \end{aligned} \tag{1.3.15}$$

and so

$$\begin{aligned} \omega(z)z^N &= \frac{z+\alpha}{(z-1)^2} z^N \\ &= \sum_{j=0}^{N-1} z^j + (1+\alpha)\sum_{j=0}^{N-2}(N-j-1)z^j + \frac{2}{z-1} + (1+\alpha)\frac{2+N(z-1)}{(z-1)^2} \\ &= \sum_{j=0}^{N-2} \left((N-j-1)(1+\alpha)+1\right) z^j + z^{N-1} + \frac{2(1+\alpha)+(N(1+\alpha)+2)(z-1)}{(z-1)^2}. \end{aligned}$$

Thus, for $N \geq 1$,

$$T_\omega z^N = \sum_{j=0}^{N-2} \left((N-j-1)(1+\alpha)+1\right) z^j + z^{N-1}$$

from which it follows that the matrix representation $[T_\omega]$ of T_ω is given by

$$[T_\omega] = \begin{pmatrix} 0 & 1 & 2+\alpha & 3+2\alpha & \cdots \\ 0 & 0 & 1 & 2+\alpha & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The spectrum: For the parts of the spectrum, details of the proofs can be found in Example 3.6.1 in Section 3.6. Note that

$$\sigma_{\text{ess}}(T_\omega) = \omega(\mathbb{T}) = \{(it - \tfrac{1}{2}) + (1+\alpha)(it - \tfrac{1}{2})^2 \mid t \in \mathbb{R}\}. \tag{1.3.16}$$

Define the circle

$$\mathbb{T}(-\tfrac{1}{2}, \tfrac{1}{2}) = \{z \in \mathbb{C} \mid |z + \tfrac{1}{2}| = \tfrac{1}{2}\},$$

and write $\mathbb{D}(-\frac{1}{2}, \frac{1}{2})$ for the open disc formed by the interior of $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ and $\mathbb{D}^c(-\frac{1}{2}, \frac{1}{2})$ for the open exterior of $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$.

For $\alpha \notin \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ the curve $\omega(\mathbb{T})$ is equal to the parabola in $\mathbb{C}$ given by

$$\begin{aligned} &\omega(\mathbb{T}) = \{-(\alpha+1)(x(y) + iy) \mid y \in \mathbb{R}\}, \qquad \text{where} \\ &x(y) = \frac{|\alpha+1|^4}{(|\alpha|^2 + \text{Re}(\alpha))^2} y^2 + \frac{(\text{Re}(\alpha)+1)|\alpha+1|^2 \text{Im}(\alpha)}{(|\alpha|^2 + \text{Re}(\alpha))^2} y + \frac{|\alpha|^2(1-|\alpha|^2)}{(|\alpha|^2 + \text{Re}(\alpha))^2}, \end{aligned}$$

while for $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ the curve $\omega(\mathbb{T})$ becomes the half-line given by

$$\omega(\mathbb{T}) = \left\{ -(\alpha+1)r - \frac{(\alpha+1)(1+2\overline{\alpha})}{4(1-|\alpha|^2)} \mid r \geq 0 \right\}.$$

As ω is strictly proper, we have $\sigma_{\text{r}}(T_\omega) = \emptyset$. For the remaining parts of the spectrum we consider three cases.

(i) For $\alpha \in \mathbb{D}(-\frac{1}{2}, \frac{1}{2})$ the points $-\frac{1}{2}$ and 0 are separated by the parabola $\omega(\mathbb{T})$ and the connected component of $\mathbb{C} \backslash \omega(\mathbb{T})$ that contains $-\frac{1}{2}$ is equal to $\rho(T_\omega)$, while the connected component that contains 0 is equal to $\sigma_{\text{p}}(T_\omega)$. Finally, $\sigma_{\text{ess}}(T_\omega) = \omega(\mathbb{T}) = \sigma_{\text{c}}(T_\omega)$.

(ii) For $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ we have

$$\rho(T_\omega) = \emptyset, \quad \sigma_{\text{c}}(T_\omega) = \omega(\mathbb{T}) = \sigma_{\text{ess}}(T_\omega), \quad \sigma_{\text{p}}(T_\omega) = \mathbb{C} \backslash \omega(\mathbb{T}),$$

and for each $\lambda \in \omega(\mathbb{T})$, $\lambda - \omega$ has two zeroes on $\mathbb{T}$.

(iii) For $\alpha \in \mathbb{D}^c(-\frac{1}{2}, \frac{1}{2})$ we have $\sigma_{\text{p}}(T_\omega) = \mathbb{C}$, and hence $\rho(T_\omega) = \sigma_{\text{c}}(T_\omega) = \emptyset$.

The adjoint: Since $\omega(z) = \frac{s(z)}{q(z)} = \frac{z+\alpha}{(z-1)^2} \in \mathrm{Rat}_0(\mathbb{T})$ we have

$$\omega^*(z) = \frac{z^{m-n}s^\sharp(z)}{q^\sharp(z)} = \frac{z(z+\overline{\alpha})}{(z-1)^2}$$

and so from Theorem 1.3.10

$$\mathrm{Dom}(T_\omega^*) = (z-1)^2 H^p$$

and for $g = (z-1)^2\psi \in (z-1)^2 H^p$,

$$T_\omega^* g = z(z+\overline{\alpha})\psi.$$

Since $\omega(\mathbb{T})$ is a parabola and a degenerate parabola (or half-line) when $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$, the adjoint T_ω^* of T_ω can be symmetric only for $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$. But $\omega(\mathbb{T}) \subset \mathbb{R}$ only when $\alpha \in \mathbb{R}$ and so T_ω^* is symmetric when $\alpha \in \mathbb{R} \cap \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$, i.e., $\alpha = 0, -1$. If $\alpha = -1$ then $\omega(z) = \frac{1}{z-1}$ and so T_ω^* does not have a selfadjoint extension for $\alpha = -1$. So T_ω^* has a self-adjoint extension only when $\alpha = 0$.

1.4 Outline of rest of the thesis

In Chapter 2, the basic properties, as well as Fredholm characteristics of the Toeplitz-like operator $T_\omega(H^p \to H^p)$, where ω is a rational function with poles on $\mathbb{T}$ are discussed. In addition, a matrix representation is illustrated. This chapter appeared as:

G.J. Groenewald, S. ter Horst, J. Jaftha, A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle I: Fredholm properties, *Oper. Theory AdvȦppl.*, Vol. 271 (2018), 239 - 268.

Chapter 3 continuous the analysis of a Toeplitz-like operator with a rational symbol having a pole on $\mathbb{T}$. A description of the spectrum of the Toeplitz-like operator, T_ω, is given, as well its various parts, i.e., the point spectrum, residual spectrum and the continuous spectrum. Various examples are given that illustrates the results. This chapter appeared as:

G.J. Groenewald, S. ter Horst, J. Jaftha, A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle II: The adjoint, *Oper. Theory Adv. Appl.*, Vol. 272 (2019), 133 - 154.

In Chapter 4, we continue the analysis of the Toeplitz-like operator with rational symbol having a pole on $\mathbb{T}$. The adjoint of the Toeplitz-like operator, $T_\omega(H^p \to H^p)$, is discussed. In case $p = 2$, conditions are given when $T^\star$ is symmetric and when it has a self-adjoint extension. This chapter is accepted for publication and will appear as:

G.J. Groenewald, S. ter Horst, J. Jaftha, A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle III: The spectrum, *Integral Equations and Operator Theory* (to appear).

In Chapter 5, the analysis of the Toeplitz-like operators is extended to operators with matrix valued functions as symbols. The basic properties and the Fredholm characteristics of Toeplitz-like operator $T_\Omega(H^p_m \to H^p_m)$ are discussed. A Wiener-Hopf type factorization of rational matrix functions with poles and zeroes on $\mathbb{T}$ are introduced that is used to describe the Fredholm characteristics.

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Chapter 2

Fredholm properties

Abstract. In this paper a definition is given for an unbounded Toeplitz-like operator with rational symbol which has poles on the unit circle. It is shown that the operator is Fredholm if and only if the symbol has no zeroes on the unit circle, and a formula for the index is given as well. Finally, a matrix representation of the operator is discussed. [1]

Mathematics Subject Classification (2010). Primary 47B35, 47A53; Secondary 47A68.
Keywords. Toeplitz operators, unbounded operators, Fredholm properties.

2.1 Introduction

The Toeplitz operator T_ω on $H^p = H^p(\mathbb{D})$, $1 < p < \infty$, over the unit disc $\mathbb{D}$ with rational symbol ω having no poles on the unit circle $\mathbb{T}$ is the bounded linear operator defined by

$$T_\omega : H^p \to H^p, \quad T_\omega f = \mathbb{P}\omega f \ (f \in H^p),$$

with $\mathbb{P}$ the Riesz projection of $L^p = L^p(\mathbb{T})$ onto H^p. This operator, and many of its variations, has been extensively studied in the literature, cf., [1, 3, 5, 16] and the references given there.

In this chapter the case where ω is allowed to have poles on the unit circle is considered. Let Rat denote the space of rational complex functions, and Rat_0 the subspace of strictly proper rational complex functions. We will also need the subspaces $\text{Rat}(\mathbb{T})$ and $\text{Rat}_0(\mathbb{T})$ of Rat consisting of the rational functions in Rat with all poles on $\mathbb{T}$ and the strictly proper rational functions in Rat with all poles on $\mathbb{T}$, respectively. For $\omega \in \text{Rat}$, possibly having poles on $\mathbb{T}$, we define a Toeplitz-like operator $T_\omega(H^p \to H^p)$, for $1 < p < \infty$, as follows:

$$\text{Dom}(T_\omega) = \{g \in H^p \,|\, \omega g = f + \rho \text{ with } f \in L^p, \rho \in \text{Rat}_0(\mathbb{T})\}, \ T_\omega g = \mathbb{P}f. \tag{2.1.1}$$

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[1] Up to some minor modifications, this chapter has been published as
G.J. Groenewald, S. ter Horst, J. Jaftha and A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle I: Fredholm properties, *Oper. Theory Adv. Appl.*, Vol. 271 (2018), 239 – 268.

Note that in case ω has no poles on $\mathbb{T}$, then $\omega \in L^\infty$ and the Toeplitz-like operator T_ω defined above coincides with the classical Toeplitz operator T_ω on H^p. In general, for $\omega \in \mathrm{Rat}$, the operator T_ω is a well-defined, closed, densely defined linear operator. By the Euclidean division algorithm, one easily verifies that all polynomials are contained in $\mathrm{Dom}(T_\omega)$. Moreover, it can be verified that $\mathrm{Dom}(T_\omega)$ is invariant under the forward shift operator T_z and that the following classical result holds:

$$T_{z^{-1}} T_\omega T_z f = T_\omega f, \qquad f \in \mathrm{Dom}(T_\omega).$$

These basic properties are derived in Section 2.2.

This definition is somewhat different from earlier definitions of unbounded Toeplitz-like operators, as discussed in more detail in a separate part, later in this introduction. The fact that all polynomials are contained in $\mathrm{Dom}(T_\omega)$, which is not the case in several of the definitions in earlier publications, enables us to determine a matrix representation with respect to the standard basis of H^p and derive results on the convergence behaviour of the matrix entries; see Theorem 2.1.3 below.

In this chapter we are specifically interested in the Fredholm properties of T_ω. For the case that ω has no poles on $\mathbb{T}$, when T_ω is a classical Toeplitz operator, the operator T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$, a result of R. Douglas; cf., Theorem 2.65 in [1] and Theorem 10 in [17]. This result remains true in case $\omega \in \mathrm{Rat}$. We use the standard definitions of Fredholmness and Fredholm index for an unbounded operator, as given in [4], Section IV.2: a closed linear operator which has a finite dimensional kernel and for which the range has a finite dimensional complement is called a Fredholm operator, and the index is defined by the difference of the dimension of the kernel and the dimension of the complement of the range. Note that a closed Fredholm operator in a Banach space necessarily has a closed range ([4], Corollary IV.1.13).The main results on unbounded Fredholm operators can be found in [4], Chapters IV and V.

Theorem 2.1.1. *Let $\omega \in \mathrm{Rat}$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. Moreover, in that case the index of T_ω is given by*

$$\mathrm{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix}.$$

It should be noted that when we talk of poles and zeroes of ω these do not include the poles or zeroes at infinity.

The result of Theorem 2.1.1 may also be expressed in terms of the winding number as follows: $\mathrm{Index}\,(T_\omega) = -\lim_{r\downarrow 1} \mathrm{wind}\,(\omega|r\mathbb{T})$. In the case where ω is continuous on the unit circle and has no zeroes there, it is well-known that the index of the Fredholm operator T_ω is given by the negative of the winding number of the curve $\omega(\mathbb{T})$ with respect to zero (see, e.g., [1], or [6], Theorem XVI.2.4). However, if ω has poles on the unit circle, the limit $\lim_{r\downarrow 1}$ cannot be replaced by either $\lim_{r\to 1}$ or $\lim_{r\uparrow 1}$ in this formula.

The proof of Theorem 2.1.1 is given in Section 2.5. It relies heavily on the following analogue of Wiener-Hopf factorization given in Lemma 2.5.1: for $\omega \in \mathrm{Rat}$ we can write

$\omega(z) = \omega_-(z)(z^\kappa\omega_0(z))\omega_+(z)$ where κ is the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$, ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and ω_0 has all its poles and zeroes on $\mathbb{T}$. Based on the choice of the domain as in (2.1.1) it can then be shown that $T_\omega = T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}$. This factorization eventually allows to reduce the proof of Theorem 2.1.1 to the case where ω has only poles on $\mathbb{T}$. It also allows us to characterize invertibility of T_ω and to give a formula for the inverse of T_ω in case it exists.

If ω has only poles on $\mathbb{T}$, i.e., $\omega \in \mathrm{Rat}(\mathbb{T})$, then we have a more complete description of T_ω in case it is a Fredholm operator. Here and in the remainder of the chapter, we let $\mathcal{P}$ denote the space of complex polynomials in z, i.e., $\mathcal{P} = \mathbb{C}[z]$, and $\mathcal{P}_n \subset \mathcal{P}$ the subspace of polynomials of degree at most n.

Theorem 2.1.2. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. Assume s has no roots on $\mathbb{T}$ and factor s as $s = s_-s_+$ with s_- and s_+ having roots only inside and outside $\mathbb{T}$, respectively. Then*

$$\begin{aligned} &\mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{\deg(q)-1}, \quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}, \\ &\mathrm{Ker}(T_\omega) = \left\{ \frac{r_0}{s_+} \mid \deg(r_0) < \deg(q) - \deg(s_-) \right\}. \end{aligned} \tag{2.1.2}$$

Here $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}_{\deg(s)-1}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1}\}.$$

Moreover, a complement of $\mathrm{Ran}(T_\omega)$ in H^p is given by $\mathcal{P}_{\deg(s_-)-\deg(q)-1}$ (to be interpreted as $\{0\}$ in case $\deg(s_-) \leq \deg(q)$). In particular, T_ω is either injective or surjective, and both injective and surjective if and only if $\deg(s_-) = \deg(q)$, and the Fredholm index of T_ω is given by

$$\mathrm{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \text{poles of } \omega \text{ multi.} \\ \text{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \text{zeroes of } \omega \text{ in } \mathbb{D} \text{ multi.} \\ \text{taken into account} \end{Bmatrix}.$$

The proof of Theorem 2.1.2 is given in Section 2.4. In case ω has zeroes on $\mathbb{T}$, so that T_ω is not Fredholm, part of the claims of Theorem 2.1.2 remain valid, after slight reformulation. For instance, the formula for $\mathrm{Ker}(T_\omega)$ holds provided that the roots of s on $\mathbb{T}$ are included in s_+ (see Lemma 2.4.1) and of the identities for $\mathrm{Dom}(T_\omega)$ and $\mathrm{Ran}(T_\omega)$ only one-sided inclusions are proved in case zeroes on $\mathbb{T}$ are present (see Proposition 2.4.5 for further detail).

Since all polynomials are in the domain of T_ω we can write down the matrix representation of T_ω with respect to the standard basis of H^p. It turns out that this matrix representation has the form of a Toeplitz matrix. In addition, there is an assertion on the growth of the coefficients in the upper triangular part of the matrix.

Theorem 2.1.3. *Let $\omega \in \mathrm{Rat}$ possibly with poles on $\mathbb{T}$. Then we can write the matrix representation $[T_\omega]$ of T_ω with respect to the standard basis $\{z^n\}_{n=0}^\infty$ of H^p as*

$$[T_\omega] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \vdots & & & \ddots & & \end{pmatrix}.$$

In addition $a_{-j} = O(j^{M-1})$ for $j \geq 1$ where M is the largest order of the poles of ω in $\mathbb{T}$ and $(a_j)_{j=0}^\infty \in \ell^2$.

In subsequent chapters we will discuss further properties of the class of Toeplitz operators given by (2.1.1). In particular, in [7] the spectral properties of such operators are discussed. In further subsequent chapters a formula for the adjoint will be given, and several properties of the adjoint will be presented, and the matrix case will be discussed.

Connections to earlier work on unbounded Toeplitz operators Several authors have considered unbounded Toeplitz operators before. In the following we shall distinguish between several definitions by using superscripts.

For $\omega : \mathbb{T} \to \mathbb{C}$ the Toeplitz operator is defined usually by $T_\omega f = \mathbb{P}\omega f$ with domain given by $\mathrm{Dom}(T_\omega) = \{f \in H^p \mid \omega f \in L^p\}$, see e.g., [9]. Note that for ω rational with a pole on $\mathbb{T}$ this is a smaller set than in our definition (2.1.1). To distinguish between the two operators, we denote the classical operator by T_ω^{cl}. Hartman and Wintner have shown in [9] that the Toeplitz operator T_ω^{cl} is bounded if and only if its symbol is in L^∞, as was established earlier by Otto Toeplitz in the case of symmetric operators. Hartman, in [8], investigated unbounded Toeplitz operators on ℓ^2 (equivalently on H^2) with L^2-symbols. The operator in [8] is given by

$$\mathrm{Dom}(T_\omega^{\mathrm{Hr}}) = \{f \in H^2 \mid \omega f = g_1 + g_2 \in L^1, g_1 \in H^2, g_2 \in \overline{zH^1}\}, \quad T_\omega^{\mathrm{Hr}} f = g_1.$$

Observe the similarity with the definition (2.1.1). These operators are not bounded, unless $\omega \in L^\infty$. Note that the class of symbols discussed in the current chapter does not fall into this category, as a rational function with a pole on $\mathbb{T}$ is not in L^2. The Toeplitz operator T_ω^{Hr} with L^2-symbol is necessarily densely defined as its domain would contain the polynomials. The operator T_ω^{Hr} is an adjoint operator and so it is closed. Necessary and sufficient conditions for invertibility have been established for the case where ω is real-valued on $\mathbb{T}$ in terms of $\omega \pm i$. Of course, T_ω^{Hr} is symmetric in this case.

In [14] Rovnyak considered a Toeplitz operator in H^2 with real-valued L^2 symbol W such that $\log(W) \in L^1$. The operator is symmetric and densely defined via a construction of a resolvent involving a Reproducing Kernel Hilbert Space. This leads to a self-adjoint operator and clearly, the construction is very different from the approach taken in the current chapter.

Janas, in [11], considered Toeplitz operators in the Bargmann-Segal space B of Gaussian square integrable entire functions in $\mathbb{C}^n$. The Bargmann-Segal space is also referred to as the

Fock space or the Fisher space in the literature. The symbol of the operator is a measurable function. A Toeplitz-like operator, T_ω^{J}, is introduced as

$$\mathrm{Dom}(T_\omega^{\mathrm{J}}) = \{f \in B \mid \omega f = h + r,\ h \in B, \int r\overline{p}\,\mathrm{d}\mu = 0,\ \text{for all } p \in \mathcal{P}\},\ T_\omega^{\mathrm{J}} f = h.$$

Again, observe the similarity with the definition (2.1.1). Consider also the operator $T_\omega^{\mathrm{cl,B}}$ on the domain $\{f \in B \mid \omega f \in L_2(\mu)\}$ with $T_\omega^{\mathrm{cl,B}} f = \mathbb{P}\omega f$. It is shown in [11] that

1. $T_\omega^{\mathrm{cl,B}} \subset T_\omega^{\mathrm{J}}$, i.e. T_ω^{J} is an extension of the Toeplitz operator $T_\omega^{\mathrm{cl,B}}$,
2. T_ω^{J} is closed,
3. $T_\omega^{\mathrm{cl,B}}$ is closable whenever $\mathrm{Dom}(T_\omega^{\mathrm{cl,B}})$ is dense in B,
4. if $\mathcal{P} \subset \mathrm{Dom}(T_\omega^{\mathrm{cl,B}})$ and ω is an entire function then $T_\omega^{\mathrm{cl,B}} = T_\omega^{\mathrm{J}}$.

Let N^+ be the Smirnov class of holomorphic functions in $\mathbb{D}$ that consists of quotients of functions in H^∞ with the denominator an outer function. Note that a nonzero function $\omega \in N^+$ can always be written uniquely as $\omega = \frac{b}{a}$ where a and b are in the unit ball of H^∞, a an outer function, $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$, see [15, Proposition 3.1]. This is called the canonical representation of $\omega \in N^+$. For $\omega \in N^+$ the Toeplitz operator T_ω^{He} on H^2 is defined by Helson in [10] and Sarason in [15] as the multiplication operator with domain

$$\mathrm{Dom}(T_\omega^{\mathrm{He}}) = \{f \in H^2 \mid \omega f \in H^2\}$$

and so this is a closed operator. Note that although a rational function with poles only on the unit circle is in the Smirnov class, the definition of the domain in (2.1.1) is different from the one used in [15]. In fact, for $\omega \in \mathrm{Rat}(\mathbb{T})$, the operator (2.1.1) is an extension of the operator T_ω^{He}, i.e., $T_\omega^{\mathrm{He}} \subset T_\omega$. In [15] it is shown that if $\mathrm{Dom}(T_\omega^{\mathrm{He}})$ is dense in H^2 then $\omega \in N^+$. Also, if ω has canonical representation $\omega = \frac{b}{a}$ then $\mathrm{Dom}(T_\omega^{\mathrm{He}}) = aH^2$; compare with (2.1.2) to see the difference. By extending our domain as in (2.1.1), our Toeplitz-like operator T_ω is densely defined for any $\omega \in \mathrm{Rat}$, i.e., poles inside $\mathbb{D}$ are allowed.

Helson in [10] studied T_ω^{He} in H^2 where $\omega \in N^+$ with ω real-valued on $\mathbb{T}$. In this case T_ω^{He} is symmetric, and Helson showed among other things that T_ω^{He} has finite deficiency indices if and only if ω is a rational function.

Overview The chapter consists of six sections, including the current introduction. In Section 2.2 we prove several basic results concerning the Toeplitz-like operator T_ω. In the following section, Section 2.3, we look at division with remainder by a polynomial in H^p. The results in this section form the basis of many of the proofs in subsequent sections, and may be of independent interest. Section 2.4 is devoted to the case where ω is in $\mathrm{Rat}(\mathbb{T})$. Here we prove Theorem 2.1.2. In Section 2.5 we prove the Fredholm result for general $\omega \in \mathrm{Rat}$, Theorem 2.1.1, and in Section 2.6 we prove Theorem 2.1.3 on the matrix representation of T_ω. Finally, in Section 2.7 we discuss three examples that illustrate the main results of the chapter.

Notation We shall use the following notation, most of which is standard: $\mathcal{P}$ is the space of polynomials (of any degree) in one variable; $\mathcal{P}_n$ is the subspace of polynomials of degree at most n. Throughout, K^p denotes the standard complement of H^p in L^p; $\mathcal{W}_+$ denotes the analytic Wiener algebra on $\mathbb{D}$, that is, power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with absolutely summable Taylor coefficients, hence analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. In particular, $\mathcal{P} \subset \mathcal{W}_+ \subset L^p$ for each p.

2.2 Basic properties of T_ω

In this section we derive some basic properties of the Toeplitz-like operator T_ω as defined in (2.1.1). The main result is the following proposition.

Proposition 2.2.1. *Let* $\omega \in \mathrm{Rat}$*, possibly having poles on* $\mathbb{T}$*. Then* T_ω *is a well-defined closed linear operator on* H^p *with a dense domain which is invariant under the forward shift operator* T_z*. More specifically, the subspace* $\mathcal{P}$ *of polynomials is contained in* $\mathrm{Dom}(T_\omega)$*. Moreover,* $T_{z^{-1}}T_\omega T_z f = T_\omega f$ *for* $f \in \mathrm{Dom}(T_\omega)$*.*

The proof of the well-definedness relies on the following well-known result.

Lemma 2.2.2. *Let* $\psi \in \mathrm{Rat}$ *have a pole on* $\mathbb{T}$*. Then* $\psi \notin L^p$*. In particular, the intersection of* $\mathrm{Rat}_0(\mathbb{T})$ *and* L^p *consists of the zero function only.*

Indeed, if $\psi \in \mathrm{Rat}$ has a pole at $\alpha \in \mathbb{T}$ of order n, then $|\psi(z)| \sim |z-\alpha|^{-n}$ as $z \to \alpha$, and therefore the integral $\int_{\mathbb{T}} |\psi(z)|^p \,\mathrm{d}z$ diverges.

Proof of well-definedness claim of Proposition 2.2.1. Let $g \in \mathrm{Dom}(T_\omega)$ and assume $f_1, f_2 \in L^p$ and $\rho_1, \rho_2 \in \mathrm{Rat}_0(\mathbb{T})$ such that $f_1 + \rho_1 = \omega g = f_2 + \rho_2$. Then $f_1 - f_2 = \rho_2 - \rho_1 \in L^p \cap \mathrm{Rat}_0(\mathbb{T})$. By Lemma 2.2.2 we have $f_1 - f_2 = \rho_2 - \rho_1 = 0$, i.e., $f_1 = f_2$ and $\rho_1 = \rho_2$. Hence f and ρ in the definition of $\mathrm{Dom}(T_\omega)$ are uniquely determined. From this and the definition of T_ω it is clear that T_ω is a well-defined linear operator. □

In order to show that T_ω is a closed operator, we need the following alternative formula for $\mathrm{Dom}(T_\omega)$ for the case where $\omega \in \mathrm{Rat}(\mathbb{T})$.

Lemma 2.2.3. *Let* $\omega \in \mathrm{Rat}(\mathbb{T})$*, say* $\omega = s/q$ *with* $s, q \in \mathcal{P}$ *co-prime. Then*

$$\mathrm{Dom}(T_\omega) = \{g \in H^p \colon \omega g = h + r/q,\ h \in H^p,\ r \in \mathcal{P}_{\deg(q)-1}\},\ T_\omega g = h. \tag{2.2.1}$$

Moreover, $\mathrm{Dom}(T_\omega)$ *is invariant under the forward shift operator* T_z *and*

$$T_{z^{-1}}T_\omega T_z f = T_\omega f \qquad f \in \mathrm{Dom}(T_\omega).$$

Proof. Assume $g \in H^p$ with $\omega g = h + r/q$, where $h \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$. Since $H^p \subset L^p$ and $r/q \in \mathrm{Rat}_0(\mathbb{T})$, clearly $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = \mathbb{P}h = h$. Thus it remains to prove the reverse implication.

Assume $g \in \mathrm{Dom}(T_\omega)$, say $\omega g = f + \rho$, where $f \in L^p$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$. Since $\rho \in \mathrm{Rat}_0(\mathbb{T})$, we can write $q\rho$ as $q\rho = r_0 + \rho_0$ with $r_0 \in \mathcal{P}_{\deg(q)-1}$ and $\rho_0 \in \mathrm{Rat}_0(\mathbb{T})$. Then

$$sg = q\omega g = qf + q\rho = qf + r_0 + \rho_0, \quad \text{i.e.,} \quad \rho_0 = sg - qf - r_0 \in L^p.$$

By Lemma 2.2.2 we find that $\rho_0 \equiv 0$. Thus $sg = qf + r_0$. Next write $f = h + k$ with $h \in H^p$ and $k \in K^p$. Then qk has the form $qk = r_1 + k_1$ with $r_1 \in \mathcal{P}_{\deg(q)-1}$ and $k_1 \in K^p$. Thus

$$sg = qh + qk + r_0 = qh + k_1 + r_1 + r_0, \quad \text{i.e.,} \quad k_1 = sg - qh - r_1 - r_0 \in H^p.$$

Since also $k_1 \in K^p$, this shows that $k_1 \equiv 0$, and we find that $sg = qh + r$ with $r = r_0 + r_1 \in \mathcal{P}_{\deg(q)-1}$. Dividing by q gives $\omega g = h + r/q$ with $h \in H^p$ as claimed.

Finally, we prove that $\mathrm{Dom}(T_\omega)$ is invariant under T_z. Let $f \in \mathrm{Dom}(T_\omega)$, say $sf = qh + r$ with $h \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$. Then $szf = qzh + zr$. Now write $zr = cq + r_0$ with $c \in \mathbb{C}$ and $r_0 \in \mathcal{P}_{\deg(q)-1}$. Then $szf = q(zh + c) + r_0$ is in $qH^p + \mathcal{P}_{\deg(q)-1}$. Thus $zf \in \mathrm{Dom}(T_\omega)$, and $T_\omega T_z f = zh + c$. Hence $T_{z^{-1}} T_\omega T_z f = h = T_\omega f$ as claimed. □

Lemma 2.2.4. *Let* $\omega \in \mathrm{Rat}$. *Then* $\omega = \omega_0 + \omega_1$ *with* $\omega_0 \in \mathrm{Rat}_0(\mathbb{T})$ *and* $\omega_1 \in \mathrm{Rat}$ *with no poles on* $\mathbb{T}$. *Moreover,* ω_0 *and* ω_1 *are uniquely determined by* ω *and the poles of* ω_0 *and* ω_1 *correspond to the poles of* ω *on and off* $\mathbb{T}$*, respectively.*

Proof. The existence of the decomposition follows from the partial fraction decomposition of ω into the sum of a polynomial and elementary fractions of the form $c/(z - z_k)^n$.

To obtain the uniqueness, split ω_1 into the sum of a strictly proper rational function ν_1 and a polynomial p_1. Assume also $\omega = \omega_0' + \nu_1' + p_1'$ with ω_0' in $\mathrm{Rat}_0(\mathbb{T})$, $\nu_1' \in \mathrm{Rat}_0$ with no poles on $\mathbb{T}$ and p_1' a polynomial. Then $(\omega_0 - \omega_0') + (\omega_1 - \omega_1') = p_1' - p_1$ is in $\mathrm{Rat}_0 \cap \mathcal{P}$, and hence is zero. So $p_1 = p_1'$. Then $\omega_0 - \omega_0' = \omega_1' - \omega_1$ is in Rat_0 and has no poles on $\mathbb{C}$, and hence it is the zero function. □

Proof of closedness claim of Proposition 2.2.1. By Lemma 2.2.4, $\omega \in \mathrm{Rat}$ can be written as $\omega = \omega_0 + \omega_1$ with $\omega_0 \in \mathrm{Rat}_0(\mathbb{T})$ and $\omega_1 \in \mathrm{Rat}$ with no poles on $\mathbb{T}$, hence $\omega_1 \in L^\infty$. Then $T_\omega = T_{\omega_0} + T_{\omega_1}$ and T_{ω_1} is bounded on H^p. It follows that T_ω is closed if and only if T_{ω_0} is closed. Hence, without loss of generality we may assume $\omega \in \mathrm{Rat}_0(\mathbb{T})$, which we will do in the remainder of the proof.

Say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime, q having roots only on $\mathbb{T}$ and $\deg(s) < \deg(q)$. Let $g_1, g_2, \ldots$ be a sequence in $\mathrm{Dom}(T_\omega)$ such that in H^p we have

$$g_n \to g \in H^p \quad \text{and} \quad T_\omega g_n \to h \in H^p \quad \text{as } n \to \infty. \tag{2.2.2}$$

We have to prove that $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = h$. Applying Lemma 2.2.3 above, we know that $\omega g_n = h_n + r_n/q$ with $h_n \in H^p$ and $r_n \in \mathcal{P}_{\deg(q)-1}$. Moreover $h_n = T_\omega g_n \to h$. Using (2.2.2) it follows that

$$r_n = sg_n - qh_n \to sg - qh =: r \quad \text{as } n \to \infty \text{, with convergence in } H^p.$$

Since $\deg(r_n) < \deg(q)$ for each n, it follows that $r = \lim_{n\to\infty} r_n$ is also a polynomial with $\deg(r) < \deg(q)$. Thus $r/q \in \mathrm{Rat}_0(\mathbb{T})$, and $r = sg - qh$ implies that $\omega g = h + r/q$. Thus $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = h$. We conclude that T_ω is closed. □

Proof of Proposition 2.2.1. In the preceding two parts of the proof we showed all claims except that $\mathrm{Dom}(T_\omega)$ contains $\mathcal{P}$ and is invariant under T_z. Again write ω as $\omega = \omega_0 + \omega_1$ with $\omega_0 \in \mathrm{Rat}_0(\mathbb{T})$ and $\omega_1 \in \mathrm{Rat}$ with no poles on $\mathbb{T}$. Let $r \in \mathcal{P}$. Then $\omega r = \omega_0 r + \omega_1 r$. We have $\omega_1 r \in \mathrm{Rat}$ with no poles on $\mathbb{T}$, hence $\omega_1 r \in L^p$. By Euclidean division, $\omega_0 r = \psi + r_0$ with $\psi \in \mathrm{Rat}_0(\mathbb{T})$ (having the same denominator as ω_0) and $r_0 \in \mathcal{P} \subset L^p$. Hence $\omega r \in L^p + \mathrm{Rat}_0(\mathbb{T})$, so that $r \in \mathrm{Dom}(T_\omega)$. This shows $\mathcal{P} \subset \mathrm{Dom}(T_\omega)$. Finally, we have $\mathrm{Dom}(T_\omega) = \mathrm{Dom}(T_{\omega_0})$ and it follows by the last claim of Lemma 2.2.3 that $\mathrm{Dom}(T_{\omega_0})$ is invariant under T_z. □

2.3 Intermezzo: Division with remainder by a polynomial in H^p

Let $s \in \mathcal{P}$, $s \not\equiv 0$. The Euclidean division algorithm says that for any $v \in \mathcal{P}$ there exist unique $u, r \in \mathcal{P}$ with $v = us + r$ and $\deg(r) < \deg(s)$. If $\deg(v) \geq \deg(s)$, then $\deg(v) = \deg(s) + \deg(u)$. We can reformulate this as:

$$\mathcal{P} = s\mathcal{P} \dot{+} \mathcal{P}_{\deg(s)-1} \quad \text{and} \quad \mathcal{P}_n = s\mathcal{P}_{n-\deg(s)} \dot{+} \mathcal{P}_{\deg(s)-1}, \quad n \geq \deg(s),$$

with $\dot{+}$ indicating direct sum. What happens when $\mathcal{P}$ is replaced with a class of analytic functions, say by H^p, $p \geq 0$? That is, for $s \in \mathcal{P}$, $s \not\equiv 0$, when do we have

$$H^p = sH^p + \mathcal{P}_{\deg(s)-1}? \tag{2.3.1}$$

Since $\mathcal{P} \subset H^p$, we know that $\mathcal{P} = s\mathcal{P} \dot{+} \mathcal{P}_{\deg(s)-1} \subset sH^p + \mathcal{P}_{\deg(s)-1}$. Hence $sH^p + \mathcal{P}_{\deg(s)-1}$ contains a dense (non-closed) subspace of H^p. Thus question (2.3.1) is equivalent to asking whether $sH^p + \mathcal{P}_{\deg(s)-1}$ is closed. The following theorem provides a full answer to the above question.

Theorem 2.3.1. *Let $s \in \mathcal{P}$, $s \not\equiv 0$. Then $H^p = sH^p + \mathcal{P}_{\deg(s)-1}$ if and only if s has no roots on the unit circle $\mathbb{T}$.*

Another question is, even if s has no roots on $\mathbb{T}$, whether $sH^p + \mathcal{P}_{\deg(s)-1}$ is a direct sum. This does not have to be the case. In fact, if s has only roots outside $\mathbb{T}$, then $1/s \in H^\infty$ and $sH^p = H^p$, so that $sH^p + \mathcal{P}_{\deg(s)-1}$ is not a direct sum, unless if s is constant. Clearly, a similar phenomenon occurs if only part of the roots of s are outside $\mathbb{T}$. In case all roots of s are inside $\mathbb{T}$, then the sum is a direct sum.

Proposition 2.3.2. *Let $s \in \mathcal{P}$, $s \not\equiv 0$ and having no roots on $\mathbb{T}$. Write $s = s_- s_+$ with $s_-, s_+ \in \mathcal{P}$ having roots inside and outside $\mathbb{T}$, respectively. Then $H^p = sH^p + \mathcal{P}_{\deg(s_-)-1}$ is a direct sum decomposition of H^p. In particular, $sH^p + \mathcal{P}_{\deg(s)-1}$ is a direct sum if and only if s has all its roots inside $\mathbb{T}$.*

We also consider the question whether there are functions in H^p that are not in $sH^p + \mathcal{P}_{\deg(s)-1}$ and that can be divided by another polynomial q. This turns out to be the case precisely when s has a root on $\mathbb{T}$ which is not a root of q.

Theorem 2.3.3. *Let $s, q \in \mathcal{P}$, $s, q \not\equiv 0$. Then there exists a $f \in qH^p$ which is not in $sH^p + \mathcal{P}_{\deg(s)-1}$ if and only if s has a root on $\mathbb{T}$ which is not a root of q.*

In order to prove the above results we first prove a few lemmas.

Lemma 2.3.4. *Let $s \in \mathcal{P}$ and $\alpha \in \mathbb{C}$ a root of s. Then $sH^p + \mathcal{P}_{\deg(s)-1} \subset (z-\alpha)H^p + \mathbb{C}$.*

Proof. Since $s(\alpha) = 0$, we have $s(z) = (z-\alpha)s_0(z)$ for some $s_0 \in \mathcal{P}$, $\deg(s_0) = \deg(s) - 1$. Let $f = sg + r \in sH^p + \mathcal{P}_{\deg(s)-1}$. Then $r(z) = (z-\alpha)r_0(z) + c$ for a $r_0 \in \mathcal{P}$ and $c \in \mathbb{C}$. This yields

$$f(z) = s(z)g(z) + r(z) = (z-\alpha)(s_0(z)g(z) + r_0(z)) + c \in (z-\alpha)H^p + \mathbb{C}. \qquad \square$$

Lemma 2.3.5. *Let $\alpha \in \mathbb{T}$. Then there exists a $f \in \mathcal{W}_+$ such that $f \notin (z-\alpha)H^p + \mathbb{C}$.*

Proof. By rotational symmetry we may assume without loss of generality that $\alpha = 1$. Let $h_n \downarrow 0$ such that $h(z) = \sum_{n=0}^{\infty} h_n z^n$ is analytic on $\mathbb{D}$ but $h \notin H^p$. Define $f_0, f_1, \ldots$ recursively by

$$f_0 = -h_0, \quad f_{n+1} = (h_n - h_{n+1}), \ n \geq 0.$$

Then $f(z) = (z-1)h(z)$ and $\sum_{k=0}^{N} |f_k| = 2h_0 - h_N \to 2h_0$. Hence the Taylor coefficients of $f(z) = \sum_{k=0}^{\infty} f_k z^k$ are absolutely summable and thus $f \in \mathcal{W}_+$.

Now assume $f \in (z-1)H^p + \mathbb{C}$, say $f = (z-1)g + c$ for $g \in H^p$ and $c \in \mathbb{C}$. Then $h = g + c/(z-1)$. Since the Taylor coefficients of $c/(z-1)$ have to go to zero, we obtain $c = 0$ and $h = g$, which contradicts the assumption $h \notin H^p$. $\square$

Proof of Theorem 2.3.1. Assume s has no roots on $\mathbb{T}$. Since $s \in \mathcal{P} \subset H^\infty$, we know from Theorem 8 of [17] that the range of the multiplication operator of s on H^p is closed (i.e., sH^p closed in H^p) if and only if $|s|$ is bounded away from zero on $\mathbb{T}$. Since s is a polynomial, the latter is equivalent to s having no roots on $\mathbb{T}$. Hence sH^p is closed. Since $\mathcal{P}_{\deg(s)-1}$ is a finite dimensional subspace of H^p, and thus closed, we obtain that $sH^p + \mathcal{P}_{\deg(s)-1}$ is closed [2, Chapter 3, Proposition 4.3]. Also, $sH^p + \mathcal{P}_{\deg(s)-1}$ contains the dense subspace $\mathcal{P}$ of H^p, therefore $sH^p + \mathcal{P}_{\deg(s)-1} = H^p$.

Conversely, assume s has a root $\alpha \in \mathbb{T}$. Then by Lemmas 2.3.4 and 2.3.5 we know $sH^p + \mathcal{P}_{\deg(s)-1} \subset (z-\alpha)H^p + \mathbb{C} \neq H^p$. $\square$

Proof of Proposition 2.3.2. Assume $s \in \mathcal{P}$ has no roots on $\mathbb{T}$. Write $s = s_- s_+$ with $s_-, s_+ \in \mathcal{P}$, s_- having only roots inside $\mathbb{T}$ and s_+ having only roots outside $\mathbb{T}$. Assume s has roots outside $\mathbb{T}$, i.e., $\deg(s_+) > 0$. Then $1/s_+$ is in H^∞ and $s_+H^p = H^p$ and hence $sH^p = s_-H^p$. Using Theorem 2.3.1, this implies that

$$H^p = s_-H^p + \mathcal{P}_{\deg(s_-)-1} = sH^p + \mathcal{P}_{\deg(s_-)-1}.$$

Next we show that $sH^p + \mathcal{P}_{\deg(s_-)-1}$ is a direct sum. Let $f = sh_1 + r_1 = sh_2 + r_2 \in sH^p + \mathcal{P}_{\deg(s_-)-1}$ with $h_1, h_2 \in H^p$, $r_1, r_2 \in \mathcal{P}_{\deg(s_-)-1}$. Then $r_1 - r_2 = s(h_2 - h_1)$. Clearly, each root α of s_- with multiplicity n, is also a root of s with multiplicity n. Evaluate both sides of $r_1 - r_2 = s(h_2 - h_1)$ at α, possible since $\alpha \in \mathbb{D}$, as well as the identities obtained by taking derivatives on both sides up to order $n-1$, this yields $\frac{\mathrm{d}^m}{\mathrm{d}z^m}(r_1 - r_2)(\alpha) = 0$ for $m = 0 \dots n-1$. Since $\deg(r_1 - r_2) < \deg(s_-)$, this can only occur when $r_1 - r_2 \equiv 0$, i.e., $r_1 = r_2$. We thus arrive at $s(h_2 - h_1) \equiv 0$. Since s has no roots on $\mathbb{T}$, we have $1/s \in L^\infty$ so that $h_2 - h_1 = s^{-1}s(h_2 - h_1) \equiv 0$ as a function in L^p. Hence $h_1 = h_2$ in L^p, but then also $h_1 = h_2$ in H^p. Hence we have shown $sH^p + \mathcal{P}_{\deg(s_-)-1}$ is a direct sum.

In case s has all its roots inside $\mathbb{T}$, we have $s = s_-$ and thus $\mathcal{P}_{\deg(s)-1} = \mathcal{P}_{\deg(s_-)-1}$ so that $sH^p + \mathcal{P}_{\deg(s)-1}$ is a direct sum. Conversely, if s has a root outside $\mathbb{T}$, we have $\deg(s_-) < \deg(s)$ and the identity $sH^p + \mathcal{P}_{\deg(s)-1} = sH^p + \mathcal{P}_{\deg(s_-)-1}$ shows that any $r \in \mathcal{P}_{\deg(s)-1}$ with $\deg(r) \geq \deg(s_-)$ can be written as $r = 0 + r \in sH^p + \mathcal{P}_{\deg(s)-1}$ and as $r = sh + r' \in sH^p + \mathcal{P}_{\deg(s)-1}$ with $\deg(r') < \deg(s_-)$ and $h \in H^p$, $h \not\equiv 0$. Hence $sH^p + \mathcal{P}_{\deg(s)-1}$ is not a direct sum. $\square$

Proof of Theorem 2.3.3. Assume all roots of s on $\mathbb{T}$ are also roots of q. Let $f = q\widetilde{f} \in qH^p$. Factor $s = s_+s_0s_-$ as before. Then $q = s_0\hat{q}$ for some $\hat{q} \in \mathcal{P}$. From Theorem 2.3.1 we know that $s_-s_+H^p + \mathcal{P}_{\deg(s_-s_+)-1} = H^p$. Hence $\hat{q}\widetilde{f} = s_-s_+\widehat{f} + r$ with $\widehat{f} \in H^p$ and $r \in \mathcal{P}$ with $\deg(r) < \deg(s_-s_+)$. Thus

$$fq\widetilde{f} = s_0\hat{q}\widetilde{f} = s\widehat{f} + s_0r \in sH^p + \mathcal{P}_{\deg(s)-1},$$

where we used $\deg(s_0r) = \deg(s_0) + \deg(r) < \deg(s_0) + \deg(s_-s_+) = \deg(s)$. Hence $qH^p \subset sH^p + \mathcal{P}_{\deg(s)-1}$.

Conversely, assume $\alpha \in \mathbb{T}$ such that $s(\alpha) = 0$ and $q(\alpha) \neq 0$. By Lemma 2.3.5 there exists a $\widetilde{f} \in \mathcal{W}_+ \subset H^p$ which is not in $(z-\alpha)H^p + \mathbb{C}$, and hence not in $sH^p + \mathcal{P}_{\deg(s)-1}$, by Lemma 2.3.4. Now set $f = q\widetilde{f} \in qH^p$. We have $q(z) = (z-\alpha)q_1(z) + c_1$ for a $q_1 \in \mathcal{P}$ and $c_1 = q(\alpha) \neq 0$. Assume $f \in (z-\alpha)H^p + \mathbb{C}$, say $f(z) = (z-\alpha)g(z) + c$ for a $g \in H^p$ and $c \in \mathbb{C}$. Then

$$((z-\alpha)q_1(z) + c_1)\widetilde{f}(z) = q(z)\widetilde{f}(z) = f(z) = (z-\alpha)g(z) + c.$$

Hence $\widetilde{f}(z) = (z-\alpha)(g(z) - q_1(z)\widetilde{f}(z))/c_1 + c/c_1$, $z \in \mathbb{D}$, which shows $\widetilde{f} \in (z-\alpha)H^p + \mathbb{C}$, in contradiction with our assumption. Hence $f \notin (z-\alpha)H^p + \mathbb{C}$. This implies, once more by Lemma 2.3.4, that there exists a $f \in qH^p$ which is not in $sH^p + \mathcal{P}_{\deg(s)-1}$. $\square$

The following lemma will be useful in the sequel.

Lemma 2.3.6. *Let $q, s_+ \in \mathcal{P}$, $q, s_+ \not\equiv 0$ be co-prime with s_+ having roots only outside $\mathbb{T}$. Then $s_+^{-1}(qH^p + \mathcal{P}_{\deg(q)-1}) = qH^p + \mathcal{P}_{\deg(q)-1}$.*

Proof. Set $\mathcal{R} := s_+^{-1}(qH^p + \mathcal{P}_{\deg(q)-1})$. Since s_+ has only roots outside $\mathbb{T}$, we have $s_+^{-1} \in H^\infty$ and $s_+^{-1}H^p = H^p$. Thus

$$\mathcal{R} = s_+^{-1}(qH^p + \mathcal{P}_{\deg(q)-1}) = qH^p + s_+^{-1}\mathcal{P}_{\deg(q)-1}.$$

This implies $qH^p \subset \mathcal{R}$. Next we show $\mathcal{P}_{\deg(q)-1} \subset \mathcal{R}$. Let $r \in \mathcal{P}_{\deg(q)-1}$. Since $\mathcal{P} \subset qH^p + \mathcal{P}_{\deg(q)-1}$, we have $rs_+ \in qH^p + \mathcal{P}_{\deg(q)-1}$ and thus $r = s_+^{-1}(rs_+) \in \mathcal{R}$. Hence $qH^p + \mathcal{P}_{\deg(q)-1} \subset \mathcal{R}$.

It remains to prove $\mathcal{R} \subset qH^p + \mathcal{P}_{\deg(q)-1}$. Let $g = s_+^{-1}(qh + r) \in \mathcal{R}$ with $h \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$. Since q and s_+ have no common roots, there exist polynomials $a, b \in \mathcal{P}$ with $qa + s_+b \equiv 1$ and $\deg(a) < \deg(s_+)$, $\deg(b) < \deg(q)$. Since $rb \in \mathcal{P} \subset qH^p + \mathcal{P}_{\deg(q)-1}$, we have

$$s_+^{-1}r = s_+^{-1}r(qa + s_+b) = qs_+^{-1}ra + rb \in qH^p + \mathcal{P}_{\deg(q)-1}.$$

Also $qs_+^{-1}h \in qH^p$, so we have $g \in qH^p + \mathcal{P}_{\deg(q)-1}$. This shows that $\mathcal{R} \subset qH^p + \mathcal{P}_{\deg(q)-1}$ and completes the proof. $\square$

Remark 2.3.7. For what other Banach spaces X of analytic functions on $\mathbb{D}$ do the above results hold? Note that the following properties of $X = H^p$ are used:

1. $\mathcal{P} \subset \mathcal{W}_+ \subset X$, and $\mathcal{P}$ is dense in X;
2. $\mathcal{W}_+ X \subset X$;
3. If $g = \sum_{n=0}^{\infty} g_n z^n \in X$ then $g_n \to 0$;
4. If $g \in X$ and $\alpha \in \mathbb{T}$, then $g(z/\alpha) \in X$ as well;
5. If $s \in \mathcal{P}$ has no roots on $\mathbb{T}$, then sX is closed in X.

To see item 3 for $X = H^p$: note that by Hölder's inequality $H^p \subset H^1$, and for $p = 1$ this follows from the Riemann-Lebesgue Lemma ([12, Theorem I.2.8]), actually a sharper statement can be made in that case by a theorem of Hardy, see [12, Theorem III 3.16], which states that if $f \in H^1$ then $\sum |f_n| n^{-1} < \infty$.

Other than $X = H^p$, $1 < p < \infty$, the spaces of analytic functions A^p on $\mathbb{D}$ with Taylor coefficients p-summable, c.f., [13] and reference ([1-5]) given there, also have these properties. For a function $f \in A^p$ the norm $\|f\|_{A^p}$ is defined as the l^p-norm of the sequence $(\widehat{f})_k$ of Taylor coefficients of f. Properties (1), (3) and (4) above are straightforward, property (2) is the fact that a function in the Wiener algebra is an l^p multiplier (see e.g., [13]). It remains to prove property (5).

Let s be a polynomial with no roots on $\mathbb{T}$, and let f_n be a sequence of functions in A^p such that sf_n converges to g in A^p. We have to show the existence of an $f \in A^p$ such that $g = sf$. Note that f_n and g are analytic functions, and convergence of sf_n to g in A^p means that $\|\widehat{sf_n} - \widehat{g}\|_{l^p} \to 0$. Consider the Toeplitz operator $T_s : l^p \to l^p$. Then $\widehat{sf_n} = T_s\widehat{f_n}$. So $\|T_s\widehat{f_n} - \widehat{g}\|_{l^p} \to 0$. Since s has no roots on $\mathbb{T}$ the Toeplitz operator T_s is Fredholm and has closed range, and since s is a polynomial T_s is injective. Thus there is a unique $\widehat{f} \in l^p$ such that $T_s\widehat{f} = \widehat{g}$. Now define (at least formally) the function $f(z) = \sum_{n=0}^{\infty} (\widehat{f})_k z^k$. Then $\widehat{sf} = \widehat{g}$, so at least formally $s(z)f(z) = g(z)$. It remains to show that f is analytic on $\mathbb{D}$. To see this, consider $z = r$ with $0 < r < 1$. Then by Hölder's inequality

$$\sum_{k=0}^{\infty} |(\widehat{f})_k| r^k \leq \|\widehat{f}\|_{l^p} \left(\sum_{k=0}^{\infty} r^{kq} \right)^{1/q} = \|\widehat{f}\|_{l^p} \left(\frac{1}{1 - r^q} \right)^{1/q},$$

showing that the series $f(z) = \sum_{n=0}^{\infty}(\widehat{f})_k z^k$ is absolutely convergent on $\mathbb{D}$. Since $f(z) = \frac{g(z)}{s(z)}$ is the quotient of an analytic function and a polynomial it can only have finitely many poles on $\mathbb{D}$, and since the series for $f(z)$ converges for every $z \in \mathbb{D}$ it follows that f is analytic in $\mathbb{D}$.

2.4 Fredholm properties of T_ω for $\omega \in \mathrm{Rat}(\mathbb{T})$

In this section we prove Theorem 2.1.2. We start with the formula for $\mathrm{Ker}(T_\omega)$.

Lemma 2.4.1. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Write $s = s_- s_0 s_+$ with the roots of s_-, s_0, s_+ inside, on, or outside $\mathbb{T}$, respectively. Then*

$$\mathrm{Ker}(T_\omega) = \left\{ \frac{\hat{r}}{s_+} \mid \deg(\hat{r}) < \deg(q) - (\deg(s_-) + \deg(s_0)) \right\}. \tag{2.4.1}$$

Proof. If $g = \hat{r}/s_+$ where $\deg(\hat{r}) < \deg(q) - (\deg(s_-) + \deg(s_0))$, then $sg = s_- s_0 \hat{r}$ which is a polynomial with $\deg(s_- s_0 \hat{r}) < \deg(q)$. Thus $\omega g = s_- s_0 \hat{r}/q \in \mathrm{Rat}_0(\mathbb{T})$ which implies that $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = 0$. Hence $g \in \mathrm{Ker}(T_\omega)$. This proves the inclusion $\supset$ in the identity (2.4.1).

Conversely suppose $g \in \mathrm{Ker} T_\omega$. Then $T_\omega g = 0$, i.e., by Lemma 2.2.3 we have $\omega g = \hat{r}/q$ or equivalently $sg = \hat{r}$ for some $\hat{r} \in \mathcal{P}_{\deg(q)-1}$. Hence $s_- s_0 (s_+ g) = sg = \hat{r}$. Thus $g = \widetilde{r}/s_+$ with $\widetilde{r} := s_+ g \in H^p$. Note that $s_- s_0 \widetilde{r} = \hat{r}$, so that $\widetilde{r} = \hat{r}/(s_- s_0)$. Since $\widetilde{r} \in H^p$ and $s_- s_0$ only has roots in $\overline{\mathbb{D}}$, the identity $\widetilde{r} = \hat{r}/(s_- s_0)$ can only hold in case $s_- s_0$ divides $\hat{r}$, i.e., $\hat{r} = s_- s_0 r_1$ for some $r_1 \in \mathcal{P}$. Then $\widetilde{r} = r_1 \in \mathcal{P}$ and we have

$$\deg(\widetilde{r}) = \deg(s_+ g) = \deg(\hat{r}) - \deg(s_- s_0) < \deg(q) - (\deg(s_-) + \deg(s_0)).$$

Hence g is included in the right hand side of (2.4.1), and we have also proved the inclusion $\subset$. Thus (2.4.1) holds. □

We immediately obtain the following corollaries.

Corollary 2.4.2. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$. Then*

$$\dim(\mathrm{Ker}(T_\omega)) = \\ = \max\left\{0, \sharp \begin{Bmatrix} \textit{poles of } \omega, \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \overline{\mathbb{D}}, \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} \right\}.$$

In particular, T_ω is injective if and only if the number of zeroes of ω inside $\overline{\mathbb{D}}$ is greater than or equal to the number of poles of ω (all on $\mathbb{T}$), in both cases with multiplicity taken into account.

Corollary 2.4.3. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$ with all zeroes inside $\overline{\mathbb{D}}$. Then*

$$\mathrm{Ker}(T_\omega) = \{r \mid \deg(r) < \deg(q) - \deg(s)\} \subset \mathcal{P}.$$

Corollary 2.4.4. *Let $\omega \in \mathrm{Rat}_0(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then $\mathcal{P}_{\deg(q)-\deg(s)-1} \subset \mathrm{Ker}(T_\omega)$ and thus T_ω is not injective.*

Next we prove the inclusions for $\mathrm{Dom}(T_\omega)$ and $\mathrm{Ran}(T_\omega)$ in (2.1.2).

Proposition 2.4.5. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then*

$$\begin{aligned} qH^p + \mathcal{P}_{\deg(q)-1} &\subset \mathrm{Dom}(T_\omega); \\ T_\omega(qH^p + \mathcal{P}_{\deg(q)-1}) &= sH^p + \widetilde{\mathcal{P}} \subset \mathrm{Ran}(T_\omega), \end{aligned} \tag{2.4.2}$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1}\} \subset \mathcal{P}_{\deg(s)-1}. \tag{2.4.3}$$

Proof. We start with the first inclusion of (2.4.2). Let $g \in qH^p + \mathcal{P}_{\deg(q)-1}$, i.e., $g = qh + r_1$ where $h \in H^p$ and $\deg(r_1) < \deg(q)$. Write $sr_1 = rq + r_2$ with $\deg(r_2) < \deg(q)$. Then

$$\begin{aligned} \deg(r) + \deg(q) = \deg(rq) &= \deg(rq + r_2) = \deg(sr_1) \\ &= \deg(s) + \deg(r_1) < \deg(s) + \deg(q). \end{aligned}$$

Hence $\deg(r) < \deg(s)$, and we have

$$\omega g = sh + \frac{sr_1}{q} = (sh + r) + \frac{r_2}{q} \in H^p + \mathrm{Rat}_0(\mathbb{T}).$$

Hence $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = sh + r \subset sH^p + \mathcal{P}_{\deg(s)-1}$. This proves the first inclusion in (2.4.2).

Further, observe that $sr_1 = rq + r_2$ implies $rq = sr_1 - r_2$ and we have $\deg(r_1) < \deg(q)$ and $\deg(r_2) < \deg(q)$, so that $r \in \widetilde{\mathcal{P}}$. This gives the inclusion $T_\omega(qH^p + \mathcal{P}_{\deg(q)-1}) \subset sH^p + \widetilde{\mathcal{P}}$.

To complete the proof of (2.4.2) it remains to prove the reverse inclusion. Let $f \in sH^p + \widetilde{\mathcal{P}}$, say $f = sh + r$ with $h \in H^p$, $r \in \widetilde{\mathcal{P}}$. Hence $qr = r_1 s + r_2$ with $r_1, r_2 \in \mathcal{P}_{\deg(q)-1}$. We seek $g \in qH^p + \mathcal{P}_{\deg(q)-1}$ and $\widetilde{r} \in \mathcal{P}_{\deg(q)-1}$ such that $\omega g = f + \widetilde{r}/q$, or equivalently

$$sg = qf + \widetilde{r} = qsh + qr + \widetilde{r} = sqh + sr_1 + r_2 + \widetilde{r} = s(qh + r_1) + r_2 + \widetilde{r}.$$

Since $\deg(r_2) < \deg(q)$, this is clearly satisfied for $g = qh + r_1$ and $\widetilde{r} = -r_2$. In particular, $T_\omega(qh + r_1) = sh + r$. Hence (2.4.2) holds. □

In the following lemma we determine a complement of $\widetilde{\mathcal{P}}$ in $\mathcal{P}_{\deg(s)-1}$.

Lemma 2.4.6. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Define $\widetilde{\mathcal{P}}$ by (2.4.3) and set $\widetilde{\mathcal{Q}} = \mathcal{P}_{\deg(s)-\deg(q)-1}$ if $\deg(s) > \deg(q)$ and $\widetilde{\mathcal{Q}} = \{0\}$ otherwise. Then $\mathcal{P}_{\deg(s)-1} = \widetilde{\mathcal{P}} \dot{+} \widetilde{\mathcal{Q}}$, with $\dot{+}$ indicating a direct sum. In particular, $\mathcal{P}_{\deg(s)-1} = \widetilde{\mathcal{P}}$ if and only if $\deg(s) \leq \deg(q)$.*

Proof. For $\deg(s) \leq \deg(q)$ we have $\widetilde{\mathcal{Q}} = \{0\}$. Hence it is trivial that $\widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}}$ is a direct sum. Also, in this case $\mathcal{P}_{\deg(s)-1} \subset \mathcal{P}_{\deg(q)-1}$ and consequently $sH^p + \mathcal{P}_{\deg(q)-1} = sH^p + \mathcal{P}_{\deg(s)-1}$, and this subspace of H^p contains all polynomials. In particular, for any $r \in \mathcal{P}_{\deg(s)-1}$ we have $qr \in s\mathcal{P}_{\deg(q)-1} + \mathcal{P}_{\deg(q)-1}$, which shows $r \in \widetilde{\mathcal{P}}$. Hence $\widetilde{\mathcal{P}} = \mathcal{P}_{\deg(s)-1}$.

Next, assume $\deg(s) > \deg(q)$. Let $r \in \widetilde{\mathcal{Q}}$, i.e., $\deg(r) < \deg(s) - \deg(q)$. In that case $\deg(rq) < \deg(s)$ so that if we write rq as $rq = r_1 s + r_2$ then $r_1 \equiv 0$ and $r_2 = rq$ with $\deg(rq) \geq \deg(q)$. Thus rq is not in $s\mathcal{P}_{\deg(q)-1} + \mathcal{P}_{\deg(q)-1}$ and, consequently, r is not in $\widetilde{\mathcal{P}}$. Hence $\widetilde{\mathcal{P}} \cap \widetilde{\mathcal{Q}} = \{0\}$. It remains to show that $\widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}} = \mathcal{P}_{\deg(s)-1}$. Let $r \in \mathcal{P}_{\deg(s)-1}$. Then we can write rq as $rq = r_1 s + r_2$ with $\deg(r_1) < \deg(q)$ and $\deg(r_2) < \deg(s)$. Next write r_2 as $r_2 = \widetilde{r}_1 q + \widetilde{r}_2$ with $\deg(\widetilde{r}_2) < \deg(q)$. Since $\deg(r_2) < \deg(s)$, we have $\deg(\widetilde{r}_1) < \deg(s) - \deg(q)$. Thus $\widetilde{r}_1 \in \widetilde{\mathcal{Q}}$. Moreover, we have

$$rq = r_1 s + r_2 = r_1 s + \widetilde{r}_1 q + \widetilde{r}_2 = (r_1 s + \widetilde{r}_2) + \widetilde{r}_1 q, \text{ hence } (r - \widetilde{r}_1)q = r_1 s + \widetilde{r}_2.$$

Thus $r - \widetilde{r}_1 \in \widetilde{\mathcal{P}}$, and we can write $r = (r - \widetilde{r}_1) + \widetilde{r}_1 \in \widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}}$. □

We now show that if s has no roots on $\mathbb{T}$, then the reverse inclusions in (2.4.2) also hold.

Theorem 2.4.7. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then T_ω has closed range if and only if s has no roots on $\mathbb{T}$, or equivalently, $sH^p + \widetilde{\mathcal{P}}$ is closed in H^p. In case s has no roots on $\mathbb{T}$, we have*

$$\mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{\deg(q)-1} \quad \textit{and} \quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}. \tag{2.4.4}$$

Proof. The proof is divided into three parts.

Part 1 In the first part we show that s has no roots on $\mathbb{T}$ if and only if $sH^p + \widetilde{\mathcal{P}}$ is closed in H^p. Note that for $\deg(s) \leq \deg(q)$ we have $\widetilde{\mathcal{P}} = \mathcal{P}_{\deg(s)-1}$, and the claim coincides with Theorem 2.3.1. For $\deg(s) > \deg(q)$, define $\widetilde{\mathcal{Q}}$ as in Lemma 2.4.6, viewed as a subspace of H^p. Since $\widetilde{\mathcal{Q}}$ is finite dimensional, $\widetilde{\mathcal{Q}}$ is closed in H^p. Hence, if $sH^p + \widetilde{\mathcal{P}}$ is closed, then so is $sH^p + \widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}} = H^p + \mathcal{P}_{\deg(s)-1}$. By Theorem 2.3.1, the latter is equivalent to s having no roots on $\mathbb{T}$. Conversely, if s has no roots on $\mathbb{T}$, then sH^p is closed, by Theorem 8 of [17] (see also the proof of Theorem 2.3.1). Now using that $\widetilde{\mathcal{P}}$ is finite dimensional, and thus closed in H^p, it follows that $sH^p + \widetilde{\mathcal{P}}$ is closed.

Part 2 Now we show that $sH^p + \widetilde{\mathcal{P}}$ being closed implies (2.4.4). In particular, this shows that s having no roots on $\mathbb{T}$ implies that T_ω has closed range. Note that it suffices to show $\mathrm{Dom}(T_\omega) \subset qH^p + \mathcal{P}_{\deg(q)-1}$, since the equalities in (2.4.4) then follow directly from (2.4.2). Assume $sH^p + \widetilde{\mathcal{P}}$ is closed. Then also $sH^p + \mathcal{P}_{\deg(s)-1}$ is closed, as observed in the first part of the proof, and hence $sH^p + \mathcal{P}_{\deg(s)-1} = H^p$. This also implies s has no roots on $\mathbb{T}$.

Write $s = s_- s_+$ with $s_-, s_+ \in \mathcal{P}$, with s_- and s_+ having roots inside and outside $\mathbb{T}$ only, respectively. Let $g \in \mathrm{Dom}(T_\omega)$. Then $sg = qh + r$ for $h \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$. Note that $sH^p = s_- H^p$, since $s_+ H^p = H^p$. By Theorem 2.3.1 we have $H^p = sH^p + \mathcal{P}_{\deg(s_-)-1}$. Since

$h \in H^p = sH^p + \mathcal{P}_{\deg(s_-)-1}$, we can write $h = sh' + r'$ with $h' \in H^p$ and $r' \in \mathcal{P}_{\deg(s_-)-1}$. Note that $\deg(qr' + r) < \deg(s_-q)$. We can thus write $qr' + r = r_1 s_- + r_2$ with $\deg(r_1) < \deg(q)$ and $\deg(r_2) < \deg(s_-)$. Then

$$sg = qh + r = qsh' + qr' + r = qsh' + r_1 s_- + r_2 = s(qh' + r_1 s_+^{-1}) + r_2.$$

Hence $r_2 = s(g - qh' - r_1 s_+^{-1})$. Since $\deg(r_2) < \deg(s_-)$, we can evaluate both sides (as well as the derivatives on both sides) at the roots of s_-, to arrive at $r_2 \equiv 0$. Hence $s(g - qh' - r_1 s_+^{-1}) \equiv 0$. Dividing by s, we find $g = qh' + r_1 s_+^{-1}$. Since q and s_+ are co-prime and $r_1 \in \mathcal{P}_{\deg(q)-1}$, by Lemma 2.3.6 we have $r_1 s_+^{-1} \in qH^p + \mathcal{P}_{\deg(q)-1}$. Thus $g = qh' + r_1 s_+^{-1} \in qH^p + \mathcal{P}_{\deg(q)-1}$.

Part 3 In the last part we show that if s has roots on $\mathbb{T}$, then T_ω does not have closed range. Hence assume s has roots on $\mathbb{T}$. Also assume $\mathrm{Ran}(T_\omega)$ is closed. Since $sH^p + \widetilde{\mathcal{P}} \subset \mathrm{Ran}(T_\omega)$ and $\mathrm{Ran}(T_\omega)$ is closed, also $\overline{sH^p + \widetilde{\mathcal{P}}} \subset \mathrm{Ran}(T_\omega)$. Since $\widetilde{\mathcal{Q}}$ is finite dimensional, and hence closed, $\overline{sH^p + \widetilde{\mathcal{P}}} + \widetilde{\mathcal{Q}}$ is closed and we have

$$\overline{sH^p + \widetilde{\mathcal{P}}} + \widetilde{\mathcal{Q}} = \overline{sH^p + \widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}}} = \overline{sH^p + \mathcal{P}_{\deg(s)-1}} = H^p.$$

Therefore, we have

$$H^p = \overline{sH^p + \widetilde{\mathcal{P}}} + \widetilde{\mathcal{Q}} \subset \mathrm{Ran}(T_\omega) + \widetilde{\mathcal{Q}} \subset H^p.$$

It follows that $\mathrm{Ran}(T_\omega) + \widetilde{\mathcal{Q}} = H^p$.

Let $h \in H^p$ such that $qh \notin sH^p + \mathcal{P}_{\deg(s)-1}$, which exists by Theorem 2.3.3. Write $h = h' + r'$ with $h' \in \mathrm{Ran}(T_\omega)$ and $r' \in \widetilde{\mathcal{Q}}$. Since $h' \in \mathrm{Ran}(T_\omega)$, there exist $g \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$ such that

$$sg = qh' + r = q(h - r') + r = qh - qr' + r.$$

Write r as $r = sr_1 + r_2$ with $r_1, r_2 \in \mathcal{P}$, $\deg(r_2) < \deg(s)$. Note that $r' \in \widetilde{\mathcal{Q}}$, so that $\deg(qr') < \deg(s)$. Thus

$$qh = sg + qr' - r = sg + qr' - sr_1 - r_2 = s(g - r_1) + (qr' - r_2) \in sH^p + \mathcal{P}_{\deg(s)-1},$$

in contradiction with $qh \notin sH^p + \mathcal{P}_{\deg(s)-1}$. Hence $\mathrm{Ran}(T_\omega)$ is not closed. □

When s has no roots on $\mathbb{T}$ we have $\mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}$ and thus, by Lemma 2.4.6, $\mathrm{Ran}(T_\omega) + \widetilde{\mathcal{Q}} = H^p$. However, this need not be a direct sum in case s has roots outside $\mathbb{T}$. In the next lemma we obtain a different formula for $\mathrm{Ran}(T_\omega)$, for which we can determine a complement in H^p.

Lemma 2.4.8. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Assume s has no roots on $\mathbb{T}$. Write $s = s_- s_+$ with the roots of s_- and s_+ inside and outside $\mathbb{T}$, respectively. Define*

$$\widetilde{\mathcal{P}}_- = \{ r \in \mathcal{P} \mid rq = \widehat{r}_1 s_- + \widehat{r}_2 \text{ for } \widehat{r}_1, \widehat{r}_2 \in \mathcal{P}_{\deg(q)-1} \}$$

and define $\widetilde{\mathcal{Q}}_- = \mathcal{P}_{\deg(s_-)-\deg(q)-1}$ *if* $\deg(s_-) > \deg(q)$ *and* $\widetilde{\mathcal{Q}}_- = \{0\}$ *if* $\deg(s_-) \leq \deg(q)$. *Then*

$$\mathrm{Ran}(T_\omega) = s_- H^p \dot{+} \widetilde{\mathcal{P}}_- \quad \textit{and} \quad \mathrm{Ran}(T_\omega) \dot{+} \widetilde{\mathcal{Q}}_- = H^p.$$

In particular, $\operatorname{codim}\mathrm{Ran}(T_\omega) = \max\{0, \deg(s_-) - \deg(q)\}$.

Proof. It suffices to prove that $\mathrm{Ran}(T_\omega) = s_- H^p + \widetilde{\mathcal{P}}_-$, that is, $sH^p + \widetilde{\mathcal{P}} = s_- H^p + \widetilde{\mathcal{P}}_-$, by Theorem 2.4.7. Indeed, the direct sum claims follow since $H^p = s_- H^p \dot{+} \mathcal{P}_{\deg(s_-)}$ is a direct sum decomposition of H^p, by Proposition 2.3.2, and $\mathcal{P}_{\deg(s_-)} = \widetilde{\mathcal{P}}_- \dot{+} \widetilde{\mathcal{Q}}_-$ is a direct sum decomposition of $\mathcal{P}_{\deg(s_-)}$, by applying Lemma 2.4.6 with s replaced by s_-.

We first show that $sH^p + \widetilde{\mathcal{P}} \subset s_- H^p + \widetilde{\mathcal{P}}_-$. Let $f = sh + r$ with $h \in H^p$ and $r \in \widetilde{\mathcal{P}}$, say $rq = sr_1 + r_2$. Then $rq = s_-(s_+ r_1) + r_2$. Now write $s_+ r_1 = q\widetilde{r}_1 + \widetilde{r}_2$ with $\deg(\widetilde{r}_2) < \deg(q)$. Since $\widetilde{r}_2$ and r_2 have degree less than $\deg(q)$ and

$$q(r - s_-\widetilde{r}_1) = s_-(s_+ r_1) + r_2 - q s_- \widetilde{r}_1 = s_- \widetilde{r}_2 + r_2,$$

it follows that $r - s_-\widetilde{r}_1 \in \widetilde{\mathcal{P}}_-$. Therefore

$$f = sh + r = s_-(s_+ h + \widetilde{r}_1) + (r - s_-\widetilde{r}_1) \in s_- H^p + \widetilde{\mathcal{P}}_-.$$

Thus $sH^p + \widetilde{\mathcal{P}} \subset s_- H^p + \widetilde{\mathcal{P}}_-$.

To complete the proof we prove the reverse implication. Let $f = s_- h + r \in s_- H^p + \widetilde{\mathcal{P}}_-$ with $h \in H^p$ and $r \in \widetilde{\mathcal{P}}_-$, say $rq = s_-\widehat{r}_1 + \widehat{r}_2$ with $\widehat{r}_1, \widehat{r}_2 \in \mathcal{P}_{\deg(q)-1}$. Set

$$g = s_+^{-1}(qh + \widehat{r}_1) \in s_+^{-1}(qH^p + \mathcal{P}_{\deg(q)-1}) = qH^p + \mathcal{P}_{\deg(q)-1},$$

with the last identity following from Lemma 2.3.6. Then $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g \in sH^p + \widetilde{\mathcal{P}}$. We show that $T_\omega g = f$ resulting in $f \in sH^p + \widetilde{\mathcal{P}}$, as desired. We have

$$\begin{aligned} sg &= s_-(qh + \widehat{r}_1) = s_- qh + s_-\widehat{r}_1 = s_- qh + rq - \widehat{r}_2 \\ &= q(s_- h + r) - \widehat{r}_2 \in qH^p + \mathcal{P}_{\deg(q)-1}. \end{aligned}$$

This proves $T_\omega g = s_- h + r = f$, which completes our proof. □

Before proving Theorem 2.1.2 we first give a few direct corollaries.

Corollary 2.4.9. *Let* $\omega \in \mathrm{Rat}(\mathbb{T})$ *have no zeroes on* $\mathbb{T}$. *Then*

$$\begin{aligned} &\operatorname{codim}\mathrm{Ran}(T_\omega) = \\ &\quad = \max\left\{0, \sharp\begin{Bmatrix}\textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account}\end{Bmatrix} - \sharp\begin{Bmatrix}\textit{poles of } \omega \textit{ multi.} \\ \textit{taken into account}\end{Bmatrix}\right\}. \end{aligned}$$

In particular, T_ω *is surjective if and only if the number of zeroes of* ω *inside* $\mathbb{D}$ *is less than or equal to the number of poles of* ω *(all on* $\mathbb{T}$*), in both cases with multiplicity taken into account.*

Corollary 2.4.10. *Let $\omega \in \text{Rat}(\mathbb{T})$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. In that case the Fredholm index of T_ω is given by*

$$\text{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix}.$$

Corollary 2.4.11. *Let $\omega \in \text{Rat}(\mathbb{T})$ have no zeroes on $\mathbb{T}$. Then T_ω is either injective or surjective, and T_ω is both injective and surjective if and only if the number of poles of ω coincides with the number of zeroes inside $\mathbb{D}$.*

Proof of Theorem 2.1.2. Theorem 2.1.2 follows by combining the various results from the present section. The claim that T_ω is Fredholm if and only if ω (or equivalently s) has no zeroes on $\mathbb{T}$ along with the formula for the Fredholm index was given in Corollary 2.4.10, as a consequence of Theorem 2.4.7 and Lemma 2.4.8. The formula for $\text{Ker}(T_\omega)$ in (2.1.2) follows from Lemma 2.4.1, noting that $s_0 \equiv 1$, and the formulas for $\text{Dom}(T_\omega)$ and $\text{Ran}(T_\omega)$ follow from Theorem 2.4.7. The formula for a complement of $\text{Ran}(T_\omega)$ is obtained in Lemma 2.4.8, and, finally, the claims regarding injectivity and surjectivity of T_ω are listed in Corollary 2.4.11. □

2.5 Fredholm properties of T_ω: General case

In this section we prove Theorem 2.1.1 in the general case, i.e., for $\omega \in \text{Rat}$. In order to do this we need some preliminary results, which are closely connected to non-canonical Wiener-Hopf factorization.

Lemma 2.5.1. *Let $\omega \in \text{Rat}$ and denote by $\kappa = l^+ - l^-$ the difference between the number l^+ of zeroes of ω in $\mathbb{D}$ and the number l^- of poles of ω in $\mathbb{D}$. Then we can write*

$$\omega(z) = \omega_-(z)(z^\kappa \omega_0(z))\omega_+(z)$$

where ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and ω_0 has all its poles and zeroes on $\mathbb{T}$, i.e. $\omega_0 \in \text{Rat}(\mathbb{T})$. The functions $\omega_-, \omega_0, \omega_+$ are unique up to a multiplicative constant. In this case we have

$$T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}.$$

Proof. Suppose that $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime, and let $s = s_- s_0 s_+$ where s_- is monic and has all its roots in $\mathbb{D}$, s_0 has all its roots on $\mathbb{T}$ and s_+ has all its roots outside $\overline{\mathbb{D}}$. Let $q = q_- q_0 q_+$ be similarly defined, i.e. q_- monic with all its roots in $\mathbb{D}$, q_0 has all its roots on $\mathbb{T}$ and q_+ has all its roots outside $\overline{\mathbb{D}}$. Let $t_j, j = 1 \ldots l^+$ be the roots of s_- and $\tau_j, j = 1, \ldots l^-$ be the roots of q_-, possibly with repetitions. Then we can write

$$\frac{s_-}{q_-} = \frac{\Pi_{j=1}^{l^+}(z - t_j)}{\Pi_{j=1}^{l^-}(z - \tau_j)} = z^\kappa \frac{\Pi_{j=1}^{l^+}(1 - t_j z^{-1})}{\Pi_{j=1}^{l^-}(1 - \tau_j z^{-1})}$$

where $\kappa = l^+ - l^-$. Put $\omega_0 = \frac{s_0}{q_0}$,

$$\omega_- = \frac{1}{z^\kappa}\frac{s_-}{q_-} = \frac{1}{z^\kappa}\frac{\Pi_{j=1}^{l^+}(z-t_j)}{\Pi_{j=1}^{l^-}(z-\tau_j)}$$

and $\omega_+ = \frac{s_+}{q_+}$. Then ω_- has no zeroes or poles outside $\overline{\mathbb{D}}$ including infinity, as $\lim_{z\to\infty}\omega_-(z) = 1$, ω_+ has no poles and zeroes inside $\overline{\mathbb{D}}$, $\omega_0 \in \mathrm{Rat}(\mathbb{T})$ and we have the desired factorization $\omega = \omega_-(z^\kappa\omega_0)\omega_+$.

The uniqueness may be seen as follows: clearly κ is uniquely determined by ω. Suppose $\omega'_-\omega'_0\omega'_+ = \omega_-\omega_0\omega_+$. Then $(\omega'_-)^{-1}\omega_-\omega_0 = \omega'_0\omega'_+(\omega_+)^{-1}$, and it follows that this is a function in $\mathrm{Rat}(\mathbb{T})$. It is then easily seen that there are constants c_0, c_-, c_+ such that $\omega'_0 = c_0\omega_0$, $\omega'_- = c_-\omega_-$ and $\omega'_+ = c_+\omega_+$, with $c_-c_0c_+ = 1$.

Note that $f \in \mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+})$ if and only if

$$T_{\omega_+}f = \omega_+f \in \mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}) = \mathrm{Dom}(T_{z^\kappa\omega_0}).$$

Now let $f \in \mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+})$. So there are $h \in L^p$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$ with $z^\kappa\omega_0(\omega_+f) = h + \rho$ and $T_{z^\kappa\omega_0}\omega_+f = \mathbb{P}h$. Furthermore,

$$\omega f = \omega_-(z^\kappa\omega_0\omega_+f) = \omega_-(h+\rho) = \omega_-h + \omega_-\rho.$$

Now $\omega_-\rho$ is a rational function which has poles only in the closed unit disc. Moreover, as $\lim_{z\to\infty}\omega_-(z) = 1$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$ we have that $\omega_-\rho$ is strictly proper. Hence, by Lemma 2.2.4, we can write $\omega_-\rho$ uniquely as $\omega_-\rho = g + \rho'$ with g a rational function with poles only inside $\mathbb{D}$ and $\rho' \in \mathrm{Rat}_0(\mathbb{T})$. Then also g is a strictly proper rational function, as both $\omega_-\rho$ and ρ' are strictly proper. We conclude that

$$\omega f = (\omega_-h + g) + \rho'$$

and since $\omega_-h + g \in L^p$ we have $f \in \mathrm{Dom}(T_\omega)$ and $T_\omega f = \mathbb{P}(\omega_-h + g)$. Now since g is a rational function which is strictly proper and it has all its poles in $\mathbb{D}$, g has a realization $g(z) = c(z-A)^{-1}b$, with A a stable matrix. Then $g(z) = \sum_{j=0}^\infty \frac{cA^jb}{z^{j+1}}$, and hence $\mathbb{P}g = 0$. Thus we see that $f \in \mathrm{Dom}(T_\omega)$ and $T_\omega f = \mathbb{P}(\omega_-h)$.

On the other hand

$$T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}f = T_{\omega_-}T_{z^\kappa\omega_0}\omega_+f = T_{\omega_-}(\mathbb{P}h) = \mathbb{P}(\omega_-\mathbb{P}h).$$

Write $h = h_- + h_+$, where $h_+ = \mathbb{P}h$. Then $\mathbb{P}(\omega_-h_-) = 0$ since both ω_- and h_- are anti-analytic. Thus $\omega_-h = \omega_-h_- + \omega_-h_+$ and $\mathbb{P}(\omega_-h) = \mathbb{P}(\omega_-h_+)$. This implies that

$$T_\omega = T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+},$$

provided $\mathrm{Dom}(T_\omega)$ is equal to $\mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+})$.

We already proved that $\mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}) \subset \mathrm{Dom}(T_\omega)$. To prove the reverse inclusion, suppose that $f \in \mathrm{Dom}(T_\omega)$. Then there are $g \in L^p$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$ with $\omega f = g + \rho$. Since

$\omega_-^{-1}\rho \in \mathrm{Rat}$ has poles only in $\mathbb{D}$, by Lemma 2.2.4 we can write $\omega_-^{-1}\rho$ uniquely as $\omega_-^{-1}\rho = g' + \rho'$ with g' strictly proper and with poles only in $\mathbb{D}$ and $\rho' \in \mathrm{Rat}_0(\mathbb{T})$. Also, $\omega_-^{-1}g \in L^p$ because ω_-^{-1} has no poles on $\mathbb{T}$. But then $z^\kappa\omega_0\omega_+ f = \omega_-^{-1}g + \omega_-^{-1}\rho = (\omega_-^{-1}g + g') + \rho'$ is in $L^p + \mathrm{Rat}_0(\mathbb{T})$. Hence $\omega_+ f \in \mathrm{Dom}(T_{z^\kappa\omega_0})$, which implies $f \in \mathrm{Dom}(T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+})$. $\square$

Remark 2.5.2. Compare this with Theorem 16.2.3 of [6] and Proposition 2.14 of [1] from which it follows that if $\overline{a}, b \in H^\infty$, $c \in L^\infty$ then $T_{abc} = T_a T_c T_b$.

Observe that T_{ω_-} and T_{ω_+} are bounded and have a bounded inverse, in fact $T_{\omega_-}^{-1} = T_{\omega_-^{-1}}$ and $T_{\omega_+}^{-1} = T_{\omega_+^{-1}}$. Hence the Fredholm properties of T_ω are the same as the Fredholm properties of $T_{z^\kappa\omega_0}$. If $\kappa \geq 0$, then these properties are described by the results of the previous section. It remains to study the case where $\kappa < 0$. For this case we have the following lemma.

Lemma 2.5.3. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$ and $\kappa < 0$. Then $T_{z^\kappa\omega} = T_{z^\kappa}T_\omega$. Moreover, $T_{z^\kappa}T_\omega$ is Fredholm if and only if $T_\omega T_{z^\kappa}$ is Fredholm.*

Proof. Let $\omega = \frac{s}{q}$, where s and q are coprime and q has all its roots on $\mathbb{T}$. First we show that $T_{z^\kappa\omega} = T_{z^\kappa}T_\omega$.

Let $f \in \mathrm{Dom}(T_\omega)$, then $\omega f = h + \phi$, with $h \in L^p$ and $\phi \in \mathrm{Rat}_0(\mathbb{T})$. Then $z^\kappa\omega f = z^\kappa h + z^\kappa\phi$. Clearly $z^\kappa h \in L^p$. Write $z^\kappa\phi = g' + \phi'$, where g' is rational, strictly proper and has a pole only at zero (recall, $\kappa < 0$), and ϕ' is in $\mathrm{Rat}_0(\mathbb{T})$; see Lemma 2.2.4. Then $z^\kappa\omega f = (z^\kappa h + g') + \phi'$, and hence $f \in \mathrm{Dom}(T_{z^\kappa\omega})$. This shows $\mathrm{Dom}(T_{z^\kappa}T_\omega) = \mathrm{Dom}(T_\omega) \subset \mathrm{Dom}(T_{z^\kappa\omega})$.

Conversely, if $f \in \mathrm{Dom}(T_{z^\kappa\omega})$ then there is a $g \in L^p$ and a $\rho \in \mathrm{Rat}_0(\mathbb{T})$ such that $z^\kappa\omega f = g + \rho$ and $T_{z^\kappa\omega}f = \mathbb{P}g$. Then $\omega f = z^{-\kappa}g + z^{-\kappa}\rho$. Since $\kappa < 0$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$, we have $z^{-\kappa} \in \mathcal{P}$ and, by Euclidean division, we can write $z^{-\kappa}\rho = r + \psi$ with $r \in \mathcal{P}_{-\kappa-1}$ and $\psi \in \mathrm{Rat}_0(\mathbb{T})$. Clearly $z^{-\kappa}g \in L^p$. Thus $\omega f = (z^{-\kappa}g + r) + \psi$ is in $L^p + \mathrm{Rat}_0(\mathbb{T})$. Hence $f \in \mathrm{Dom}(T_\omega) = \mathrm{Dom}(T_{z^\kappa}T_\omega)$ and $T_\omega = \mathbb{P}z^{-\kappa}g + r$. In particular, this implies $\mathrm{Dom}(T_{z^\kappa}T_\omega) = \mathrm{Dom}(T_{z^\kappa\omega})$.

To complete the proof of the first claim, it remains to show that

$$\mathbb{P}g = T_{z^\kappa\omega}f = T_{z^\kappa}T_\omega f = T_{z^\kappa}(\mathbb{P}z^{-\kappa}g + r) = \mathbb{P}z^\kappa(\mathbb{P}z^{-\kappa}g + r).$$

Since $\deg(r) < -\kappa$, we have $\mathbb{P}z^\kappa r = 0$. Thus we have to show that $\mathbb{P}g = \mathbb{P}z^\kappa\mathbb{P}z^{-\kappa}g$. Write $g(z) = \sum_{j=-\infty}^{\infty} z^j g_j$. Then $\mathbb{P}z^{-\kappa}g = \sum_{j=\kappa}^{\infty} g_j z^{j-\kappa}$. Since $\kappa < 0$, we have

$$\mathbb{P}z^\kappa\mathbb{P}z^{-\kappa}g = \mathbb{P}\left(\sum_{j=\kappa}^{\infty} g_j z^j\right) = \sum_{j=0}^{\infty} g_j z^j = \mathbb{P}g,$$

which finalizes the proof of the claim that $T_{z^\kappa\omega} = T_{z^\kappa}T_\omega$.

To prove the second part of the statement, we show that the difference $T_{z^\kappa}T_\omega - T_\omega T_{z^\kappa}$ is a bounded finite rank operator, from which the result follows. More specifically, we show that $T_{z^\kappa}T_\omega - T_\omega T_{z^\kappa}$ is zero on $z^{-\kappa}H^p$. Note that $z^{-\kappa}\mathrm{Dom}(T_\omega)$ is dense in $z^{-\kappa}H^p$, since $\mathrm{Dom}(T_\omega)$ is dense in H^p. Thus it suffices to show that $T_{z^\kappa}T_\omega f = T_\omega T_{z^\kappa} f$ for all $f = z^{-\kappa}g \in z^{-\kappa}\mathrm{Dom}(T_\omega)$; note that by the last claim of Lemma 2.2.3 we have $z^{-\kappa}\mathrm{Dom}(T_\omega) \subset \mathrm{Dom}(T_\omega)$ since $\kappa < 0$.

Thus, let $f = z^{-\kappa}g \in z^{-\kappa}\mathrm{Dom}(T_\omega)$, say $sg = qh + r$ with $h \in H^p$ and $\deg(r) < \deg(q)$. Note that $T_{z^\kappa} f = g$, so that $T_\omega T_{z^\kappa} f = T_\omega g = h$. On the other hand, $sf = qz^{-\kappa}h + z^{-\kappa}r = q(z^{-\kappa}h + r_2) + r_1$, where $r_2, r_1 \in \mathcal{P}$, $\deg(r_2) < -\kappa$, $\deg(r_1) < \deg(q)$ are such that $z^{-\kappa}r = qr_2 + r_1$. Then $T_\omega f = z^{-\kappa}h + r_2$, which shows $T_{z^{-\kappa}}T_\omega f = \mathbb{P}(h + z^\kappa r_2) = h$, since $\deg(r_2) < -\kappa$. Thus $T_{z^\kappa}T_\omega f = T_\omega T_{z^\kappa} f$, as claimed, and the proof is complete. □

Proof of the Fredholm claim of Theorem 2.1.1. Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ where s and q are coprime. Put $\omega = \omega_- z^\kappa \omega_0 \omega_+$ as in Lemma 2.5.1 above, where ω_+ has all its poles and zeroes outside $\overline{\mathbb{D}}$, $\omega_0 \in \mathrm{Rat}(\mathbb{T})$ and ω_- has all its poles and zeroes in $\mathbb{D}$. Then $T_\omega = T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}$. Clearly T_{ω_-} and T_{ω_+} are boundedly invertible, so T_ω is Fredholm if and only if $T_{z^\kappa\omega_0}$ is Fredholm.

If $\kappa \geq 0$ it follows from Corollary 2.4.10 that $T_{z^\kappa\omega_0}$ is Fredholm if and only if ω_0 has no zeroes on $\mathbb{T}$. This proves the Fredholm claim of Theorem 2.1.1 for the case where $\kappa \geq 0$.

Now let $\kappa < 0$. Then $T_{z^\kappa\omega_0} = T_{z^\kappa}T_{\omega_0}$, by Lemma 2.5.3. Suppose first that ω_0 has no zeroes on $\mathbb{T}$, so that T_{ω_0} is Fredholm by Corollary 2.4.10. As T_{z^κ} is Fredholm as well, $T_{z^\kappa}T_{\omega_0}$ is Fredholm. Conversely, assume $T_{z^\kappa\omega_0}$ is Fredholm. Then $T_{z^\kappa}T_{\omega_0} = T_{z^\kappa\omega_0}$ is Fredholm, and thus $T_{\omega_0}T_{z^\kappa}$ is Fredholm, again using Lemma 2.5.3. Now $T_{z^\kappa}T_{z^{-\kappa}} = I$, and $T_{z^{-\kappa}}$ is Fredholm. Hence (see [4], Theorem IV.2.7) $T_{\omega_0}T_{z^\kappa}T_{z^{-\kappa}} = T_{\omega_0}$ is Fredholm. By Corollary 2.4.10 again, this implies that ω_0 has no zeroes on $\mathbb{T}$, and hence also ω has no zeroes on $\mathbb{T}$. □

For $\omega \in L^\infty$ we have the following result by L. A. Coburn (see [1] Theorem 2.38): *If $\omega \in L^\infty$ and ω does not vanish identically then either the kernel of T_ω in H^p is trivial or T_ω has dense range in H^p.*

For $\omega \in \mathrm{Rat}$ with poles in $\mathbb{T}$ the theorem of Coburn does not hold in full generality but we do have the following, which also proves the second part of Theorem 2.1.1, i.e., the statement on the index.

Theorem 2.5.4. *Let $\omega \in \mathrm{Rat}$ with possibly poles on $\mathbb{T}$. If T_ω is Fredholm then*

$$\mathrm{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix}.$$

and T_ω is invertible if and only if $\mathrm{Index}\,(T_\omega) = 0$, *i.e.*

$$\sharp \begin{Bmatrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix} = \sharp \begin{Bmatrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{Bmatrix}.$$

Furthermore, if T_ω is Fredholm then T_ω is either injective or surjective.

Proof. If T_ω is Fredholm then ω has no zeroes on $\mathbb{T}$. Let $\omega = \omega_- z^\kappa \omega_0 \omega_+$ be the factorization of ω as in Lemma 2.5.1. Then $T_\omega = T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}$. As T_{ω_-} and T_{ω_+} are bounded and invertible, it follows that $\mathrm{Index}\,(T_\omega) = \mathrm{Index}\,(T_{z^\kappa\omega_0})$. If $\kappa \geq 0$ then by Corollary 2.4.10

$$\mathrm{Index}\,(T_{z^\kappa\omega_0}) = \sharp \begin{Bmatrix} \text{poles of } \omega_0 \text{ multi.} \\ \text{taken into account} \end{Bmatrix} - \kappa.$$

Since $\kappa = l^+ - l^-$ is the difference between the number of zeroes of ω in the open unit disc and the number of poles of ω in the open unit disc we see that

$$\text{Index}\,(T_\omega) = \sharp \begin{Bmatrix} \text{poles of } \omega \text{ in } \overline{\mathbb{D}} \text{ multi.} \\ \text{taken into account} \end{Bmatrix} - \sharp \begin{Bmatrix} \text{zeroes of } \omega \text{ in } \mathbb{D} \text{ multi.} \\ \text{taken into account} \end{Bmatrix}$$

as stated.

If $\kappa < 0$ then, as observed in the proof of the previous lemma, $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0} = T_{\omega_0} T_{z^\kappa} + \Psi$ for some bounded Ψ of finite rank. By [4], Theorem V.2.1 we have $\text{Index}\,(T_{z^\kappa \omega_0}) = \text{Index}\,(T_{\omega_0} T_{z^\kappa})$, and by [4], Theorem IV.2.7 this is equal to $\text{Index}\,(T_{\omega_0}) + \text{Index}\,(T_{z^\kappa}) = \text{Index}\,(T_{\omega_0}) - \kappa$. From here on the proof is the same as in the case $\kappa \geq 0$.

Clearly, in case T_ω is Fredholm, T_ω is injective if and only if $T_{z^\kappa \omega_0}$ is injective, and similarly T_ω is surjective if and only if $T_{z^\kappa \omega_0}$ is surjective. In case $\kappa \geq 0$ then $z^\kappa \omega_0 \in \text{Rat}(\mathbb{T})$ and from Corollary 2.4.11 it follows that T_ω is either injective or surjective. On the other hand let $\kappa < 0$. For $h \in H^p$, $f = (z^\kappa \omega_0)^{-1} h \in \text{Dom}(T_{z^\kappa \omega_0})$ (recall that ω_0 has no zeroes on $\mathbb{T}$ as T_ω is Fredholm) with $T_{z^\kappa \omega_0} f = h$ showing that $T_{z^\kappa \omega_0}$ is surjective. In addition, $T_{z^\kappa \omega_0}$ is not injective as $\{z^j : j < M\} \subset \text{Ker}(T_{z^\kappa \omega_0})$ where $M = \kappa + \sharp\{\text{poles of } \omega_0\}$. □

We conclude this section with a characterization of invertibility of T_ω and a formula for the inverse when it exists. Here invertibility means that T_ω is bijective, so that the inverse is bounded. The classical result for continuous symbols is that T_ω is invertible if and only if ω has no zeroes on $\mathbb{T}$ and T_ω is Fredholm of index zero; the inverse is then provided using the factors in the Wiener-Hopf factorization [6, Theorem XVI.2.2].

Proposition 2.5.5. *Let $\omega \in \text{Rat}$ with at least one pole on $\mathbb{T}$ and let κ be the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$. Then T_ω is invertible if and only if ω has no zeroes on $\mathbb{T}$ and κ is also equal to the number of poles of ω on $\mathbb{T}$. In that case ω factorizes as*

$$\omega(z) = \omega_-(z) \frac{z^\kappa}{q_0(z)} \omega_+(z),$$

where ω_- has no poles or zeroes outside $\mathbb{D}$, ω_+ has no poles or zeroes inside $\overline{\mathbb{D}}$ and q_0 is a polynomial of degree κ with all its roots on $\mathbb{T}$, and moreover,

$$T_\omega^{-1} = T_{\omega_+^{-1}} T_{\frac{q_0}{z^\kappa}} T_{\omega_-^{-1}}.$$

Proof. Let $\omega = \omega_-(z^\kappa \omega_0)\omega_+$ be the factorization of Lemma 2.5.1. Then T_{ω_-} and T_{ω_+} are invertible with inverses $T_{\omega_-^{-1}}$ and $T_{\omega_+^{-1}}$, and thus it is seen that the inverse of T_ω exists if and only if the inverse of $T_{z^\kappa \omega_0}$ exists. Since an invertible operator is certainly Fredholm with index zero, it follows that ω_0 has no zeroes on $\mathbb{T}$, and so $\omega_0 = 1/q_0$ for some polynomial q_0 with roots only on the unit circle. The Fredholm index being zero implies that $\kappa = \deg(q_0)$. Conversely, if ω_0 is of this form, then $T_{z^\kappa \omega_0}$ is one-to-one by Corollary 2.4.2 and onto by Corollary 2.4.9.

It remains to show the formula for the inverse, and here too it suffices to show that

$$T^{-1}_{\frac{z^\kappa}{q_0}} = T_{\frac{q_0}{z^\kappa}}.$$

Note that $T_{\frac{q_0}{z^\kappa}}$ is a bounded operator. To show that this is the inverse of $T_{\frac{z^\kappa}{q_0}}$, let $f \in H^p$ and write $q_0(z)f(z) = z^\kappa h(z) + r(z)$ where $h \in H^p$ and r is a polynomial with $\deg(r) < \kappa$. Then $T_{\frac{q_0}{z^\kappa}} f = h$, and on the other hand, from $q_0(z)f(z) = z^\kappa h(z) + r(z)$ we have that $h \in \mathrm{Dom}(T_{\frac{z^\kappa}{q_0}})$ with $T_{\frac{z^\kappa}{q_0}} h = f$. □

2.6 Matrix representation

For $n \in \mathbb{Z}$, let e_n be the function $e_n(z) = z^n, z \in \mathbb{T}$. Then $\{e_n\}_{n=0}^\infty$ is the standard basis for H^p. Where convenient, we shall denote e_n simply by z^n.

Now let $\omega \in \mathrm{Rat}$ with possibly poles on $\mathbb{T}$. Since the polynomials $\mathcal{P}$ are contained in the domain of the closed operator T_ω defined in (2.1.1), by inspecting the action of T_ω on the monomials z^n and expressing the result as a power series, it is possible to determine a matrix representation $[T_\omega]$ of the operator $T_\omega(H^p \to H^p)$ with respect to the basis $\{e_n\}_{n=0}^\infty$.

In this section we shall prove Theorem 2.1.3, which states that this matrix representation $[T_\omega]$ has a Toeplitz structure, i.e., $[T_\omega] = [a_{m-n}]_{m,n=0}^\infty$ for some sequence $(a_n)_{n\in\mathbb{Z}}$. Here a_n has a polynomial bound, $a_n = O(n^j)$ for some $j \in \mathbb{N}$.

We first prove the following lemma, which is an explicit formulation of the Euclidean algorithm for dividing $z^N - 1$ by $(z-1)^m$, where $m < N$.

Lemma 2.6.1. *For any natural number N and any $m < N$*

$$z^N - 1 = (z-1)^m \sum_{i=0}^{N-m} \binom{i+m-1}{m-1} z^{N-m-i} + \sum_{j=0}^{m-1} \binom{N}{j} (z-1)^j. \tag{2.6.1}$$

Proof. The proof is by induction on m. The case $m = 1$ is just the well-known formula

$$z^N - 1 = (z-1)(z^{N-1} + z^{N-2} + \cdots + z + 1).$$

For $m > 1$, to show that (2.6.1) holds, we have to prove the following:

$$\sum_{i=0}^{N-m} \binom{i+m-1}{m-1} z^{N-m-i} = (z-1) \sum_{i=0}^{N-m-1} \binom{i+m}{m} z^{N-m-i-1} + \binom{N}{m}. \tag{2.6.2}$$

To see this, we shall make use of the so-called hockey-stick formula, which implies that $\sum_{i=0}^{N-m} \binom{i+m-1}{m-1} = \binom{N}{m}$. Thus, the right hand side of (2.6.2) is equal to

$$(z-1) \sum_{i=0}^{N-m-1} \binom{i+m}{m} z^{N-m-i-1} + \sum_{i=0}^{N-m} \binom{i+m-1}{m-1}. \tag{2.6.3}$$

Note that the remainder of $\sum_{i=0}^{N-m} \binom{i+m-1}{k-1} z^{N-m-i}$ upon dividing by $z-1$ is equal to $\sum_{i=0}^{N-m} \binom{i+m-1}{k-1}$, so the remainder term in (2.6.2) is correct.

To finish the proof it remains to compare the coefficients of z^k on the left and right hand sides of (2.6.2) with $k \geq 1$. A straightforward rewriting of the right hand side shows that the equality (2.6.2) follows from the basic property of binomial coefficients. $\square$

Example 2.6.2. Let $\omega(z) = (z-1)^{-m}$, i.e., $s \equiv 1$ and $q(z) = (z-1)^m$. From Proposition 2.2.1 we know $qH^p + \mathcal{P}_{m-1} \subset \mathrm{Dom}(T_\omega)$ which contains all the polynomials. Put

$$a_{-i} = \binom{i+m-1}{m-1}, i = 0, 1, 2, \ldots$$

and

$$b_{-j} = \begin{cases} 0 & j < m \\ a_{m-j} & j \geq m \end{cases}.$$

From Lemma 2.6.1 above, for $N > m$ we can write

$$\begin{aligned} z^N &= (z-1)^m \sum_{j=0}^{N-m} \binom{i+m-1}{m-1} z^{N-m-i} + \sum_{j=0}^{m-1} \binom{N}{j} (z-1)^j + 1 \\ &= q(z) \sum_{i=0}^{N-m} a_{-i} z^{N-m-i} + \sum_{j=0}^{m-1} \binom{N}{j} (z-1)^j + 1 \\ &= q(z) \sum_{j=0}^{N-m} a_{-(N-m-j)} z^j + \sum_{j=0}^{m-1} \binom{N}{j} (z-1)^j + 1. \end{aligned}$$

Put

$$r(z) = \sum_{j=0}^{m-1} \binom{N}{j} (z-1)^j + 1.$$

Then r is a polynomial with $\deg(r) < m = \deg q$ and since $s \equiv 1$,

$$s(z) z^N = q(z) \sum_{j=0}^{N-m} a_{-(N-m-j)} z^j + r$$

and so from Lemma 2.2.3, for $N > m$ we have

$$T_\omega z^N = \sum_{j=0}^{N-m} a_{-(N-m-j)} z^j = \sum_{j=0}^{N-m} b_{-(N-j)} z^j = \sum_{j=0}^{N} b_{-(N-j)} z^j$$

since $b_{-j} = 0$ for $j < m$. From Lemma 2.4.1 we have $\mathrm{Ker}(T_\omega) = \mathrm{span}\,\{z^j, j < m\}$ and so the matrix representation of T_ω will be an upper triangular Toeplitz matrix with the first m-columns zero.

Proposition 2.6.3. *Let* $\omega \in \mathrm{Rat}_0(\mathbb{T})$*, say* $\omega = s/q$ *with* $s, q \in \mathcal{P}$ *co-prime,* q *having all its roots in* $\mathbb{T}$ *and* $\deg(s) = n < m = \deg(q)$*. Then, for* $N \geq m - n$*,*

$$T_\omega z^N = \sum_{j=1}^{N-m+n+1} a_{-j} z^{N-m+n+1-j}$$

where $a_{-j} = O(j^{M-1})$ *with* M *the maximum of the orders of the poles of* ω *on* $\mathbb{T}$*. Thus the matrix representation* $[T_\omega]$ *of* T_ω *with respect to the standard basis* $\{z^n\}_{n=0}^\infty$ *of* H^p *is given by*

$$[T_\omega] = \begin{pmatrix} \underbrace{0 \cdots 0}_{m-n} & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ 0 \cdots 0 & 0 & a_0 & a_{-1} & a_{-2} & \cdots \\ 0 \cdots 0 & 0 & 0 & a_0 & a_{-1} & \cdots \\ \vdots & & & \ddots & & \end{pmatrix}$$

Proof. Since $\omega \in \mathrm{Rat}_0(\mathbb{T})$, by Corollary 2.4.4 we have $\mathcal{P}_{m-n-1} \subset \mathrm{Ker}(T_\omega)$.

Suppose $q(z) = \prod_{j=1}^t (z - \alpha_j)^{m_j}$ with $\alpha_j \in \mathbb{T}$ the poles of ω with multiplicities m_j. By partial fractions decomposition we can write $\omega = \sum_{j=1}^t \omega_j$, where

$$\omega_j(z) = \frac{s_j(z)}{(z - \alpha_j)^{m_j}}, \quad \text{for } s_j \in \mathcal{P}_{m_j - 1},\ j = 1 \cdots t.$$

Note that $[T_\omega] = \sum_{j=1}^t [T_{\omega_j}]$ and if $s_j(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_{m_j-1} z^{m_j-1}$ then $[T_{\omega_j}] = \sum_{i=0}^{m_j-1} c_i [T_{z^i/(z-\alpha_j)^{m_j}}]$. From this it follows that it suffices to prove the result for $\omega(z) = z^n/(z-\alpha)^m$. To this end, assume $\omega = s/q$ where $s(z) = z^n$ and $q(z) = (z-\alpha)^m$ for some $\alpha \in \mathbb{T}$. By Lemma 2.6.1 we have

$$z^N = (z-1)^m \sum_{j=0}^{N-m} \binom{j+m-1}{m-1} z^{N-m-j} + \sum_{j=1}^{m-1} \binom{N}{j} (z-1)^j + 1.$$

By replacing z with $\frac{z}{\alpha}$ we can write

$$\left(\frac{z}{\alpha}\right)^N = \left(\frac{z}{\alpha} - 1\right)^m \sum_{j=0}^{N-m} \binom{j+m-1}{m-1} \left(\frac{z}{\alpha}\right)^{N-m-j} + \sum_{j=1}^{m-1} \binom{N}{j} \left(\frac{z}{\alpha} - 1\right)^j + 1.$$

Multiplying with α^N results in

$$z^N = (z-\alpha)^m \sum_{j=0}^{N-m} \binom{j+m-1}{m-1} \alpha^{j-1} z^{N-m-j} + \sum_{j=1}^{m-1} \binom{N}{j} \alpha^{N-j} (z-\alpha)^j + \alpha^N.$$

So, for $N > m - n$,

$$\begin{aligned} s(z)z^N = z^{N+n} = \quad & (z-\alpha)^m \sum_{j=0}^{N+n-m} \binom{j+m-1}{m-1} \alpha^{j-1} z^{N+n-m-j} \\ + \quad & \sum_{j=1}^{m-1} \binom{N+n}{j} \alpha^{N+n-j} (z-\alpha)^j + \alpha^{N+n}. \end{aligned}$$

Put

$$a_{-i} = \binom{i+m-1}{m-1}, i = 0, 1, 2, \dots, \quad b_{-j} = \begin{cases} 0 & \text{if } j < m-n \\ a_{-(j-(m-n))} & \text{if } j \geq m-n \end{cases}$$

and

$$r(z) = \sum_{j=1}^{m-1} \binom{N+n}{j} \alpha^{N+n-j} (z-\alpha)^j + \alpha^{N+n}.$$

Then $\deg(r) < m = \deg(q)$ and

$$\begin{aligned} s(z)z^N &= q(z) \sum_{j=0}^{N+n-m} \binom{j+m-1}{m-1} \alpha^{j-1} z^{N+n-m-j} + r(z) \\ &= q(z) \sum_{j=0}^{N+n-m} a_{-j} z^{N+n-m-j} + r(z) \\ &= q(z) \sum_{j=0}^{N+n-m} a_{-((N+n)-m-j)} z^j + r(z) \\ &= q(z) \sum_{j=0}^{N+n-m} b_{-(N-j)} z^j + r(z) \end{aligned}$$

from which it follows that

$$T_\omega z^N = \sum_{j=0}^{N+n-m} b_{-(N-j)} z^j = \sum_{j=0}^{N} b_{-(N-j)} z^j$$

as $b_{-j} = 0$ for $j < m - n$. Thus the matrix representation of T_ω is given by

$$\left(\begin{array}{cccccc} \underbrace{\begin{matrix} 0 \cdots 0 \\ 0 \cdots 0 \\ 0 \cdots 0 \\ \vdots \end{matrix}}_{m-n} & \begin{matrix} a_0 \\ 0 \\ 0 \\ \end{matrix} & \begin{matrix} a_{-1} \\ a_0 \\ 0 \\ \end{matrix} & \begin{matrix} a_{-2} \\ a_{-1} \\ a_0 \\ \ddots \end{matrix} & \begin{matrix} a_{-3} \\ a_{-2} \\ a_{-1} \\ \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \end{matrix} \end{array} \right).$$

Since

$$a_{-j} = \binom{j+m-1}{m-1} = \frac{(j+m-1)(j+m-2)\cdots m}{(m-1)!} \leq \frac{(j+m-1)^{m-1}}{(m-1)!}$$

we have $a_{-j} = O(j^{m-1})$. $\square$

Proof of Theorem 2.1.3. Let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime and q having a root on $\mathbb{T}$. By Lemma 2.2.4, ω can be written uniquely as $\omega = \omega_0 + \omega_1$ with $\omega_0 \in \mathrm{Rat}_0(\mathbb{T})$ and $\omega_1 \in \mathrm{Rat}$ with no poles on $\mathbb{T}$. In particular, $\omega_1 \in L^\infty(\mathbb{T})$. Then $[T_\omega] = [T_{\omega_0}] + [T_{\omega_1}]$ and $[T_{\omega_0}]$ is as in Proposition 2.6.3. Moreover, $[T_{\omega_1}]$ has the form as in Theorem 2.1.3 and the Fourier coefficients of ω_1 are square summable. This completes the proof. $\square$

2.7 Examples

In the final section we discuss three examples.

Example 2.7.1. Let $\omega(z) = (z-1)^{-1} \in \mathrm{Rat}_0(\mathbb{T})$. Then $\omega = s/q$ with $s \equiv 1$ and $q(z) = z-1$. By Theorem 2.1.2 we have

$$\mathrm{Dom}(T_\omega) = (z-1)H^p + \mathbb{C}, \qquad \mathrm{Ker}(T_\omega) = \mathcal{P}_0 = \mathbb{C}, \qquad \mathrm{Ran}(T_\omega) = H^p$$

and so T_ω is Fredholm. These facts can also be shown explicitly. By Proposition 2.4.5 it suffices to establish that $\mathrm{Dom}(T_\omega) \subset (z-1)H^p$ and so consequently $H^p \subset \mathrm{Ran}(T_\omega)$. To this end let $g \in \mathrm{Dom}(T_\omega)$. Then there are $h \in H^p$ and $c \in \mathbb{C}$ with $\omega g = (z-1)^{-1}g = h + \frac{c}{z-1}$. Then $g = (z-1)h + c$ showing than $g \in (z-1)H^p + \mathbb{C}$. For $h \in H^p$ put $g = (z-1)h$, then $g \in H^p$ with $\omega g = (z-1)^{-1}(z-1)h = h$, showing that $g \in \mathrm{Dom}(T_\omega)$ and $T_\omega g = h$.

That $\mathrm{Ker}(T_\omega) = \mathbb{C}$ is also easily verified directly, as for $c \in \mathbb{C}$, $\omega c = 0 + \frac{c}{z-1}$. Thus $T_\omega c = 0$ and so $\mathbb{C} \subset \mathrm{Ker}(T_\omega)$. The converse follows from Lemma 2.2.3 as for $g \in \mathrm{Ker}(T_\omega)$, $g = c$ for some $c \in \mathbb{C}$.

For the matrix representation, note that

$$z^n - 1 = (1 + z + z^2 + \cdots + z^{n-1})(z-1)$$

or equivalently

$$(z-1)^{-1}z^n = 1 + z + z^2 + \cdots + z^{n-1} + (z-1)^{-1}$$

and so

$$T_\omega z^n = 1 + z + z^2 + \cdots + z^{n-1}.$$

From this it follows that the matrix representation $[T_\omega]$ with respect to the standard basis $\{z^n\}_{n=0}^\infty$ of H^p is given by

$$[T_\omega] = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}.$$

Let $[T_2]$ be given by

$$[T_2] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}.$$

This is the matrix representation of $T_2 = S + P_1 - I$ where $S = T_z$ is the forward shift operator, P_1 the projection onto the first component and I the identity operator on H^p. Then T_2 is a generalised inverse of T_ω and a right-sided inverse of T_ω.

Example 2.7.2. Let $\omega(z) = \dfrac{z-\alpha}{z-1} \in \mathrm{Rat}(\mathbb{T})$ for $\alpha \in \mathbb{C}$, $\alpha \neq 1$. From Lemma 2.2.3

$$\mathrm{Dom}(T_\omega) = \left\{ g \in H^p : \omega g = h + \frac{c}{z-1}, h \in H^p, c \in \mathbb{C} \right\}.$$

For $\alpha \notin \mathbb{T}$ by Theorem 2.4.7,

$$\mathrm{Dom}(T_\omega) = (z-1)H^p + \mathbb{C}, \qquad \mathrm{Ran}(T_\omega) = (z-\alpha)H^p + \widetilde{P} = (z-\alpha)H^p + \mathbb{C},$$

since $\widetilde{P} = \{c \in \mathbb{C} : c(z-1) = c_1(z-\alpha) + c_2, \quad c_1, c_2 \in \mathbb{C}\} = \mathbb{C}$ by Lemma 2.4.6. This can be shown explicitly, and also holds for $\alpha \in \mathbb{T}$. By Proposition 2.4.5 it suffices to show $\mathrm{Ran}(T_\omega) \subset (z-\alpha)H^p + \mathbb{C}$. Note that for $h \in H^p$ we have $h \in \mathrm{Ran}(T_\omega)$ if and only if there exist $g \in H^p$ and $c \in \mathbb{C}$ such that

$$\frac{z-\alpha}{z-1} g = h + \frac{c}{z-1}, \quad \text{i.e.,} \quad \frac{z-1}{z-\alpha} h + \frac{c}{z-\alpha} = g \in H^p.$$

Now use that $\frac{(z-1)}{z-\alpha} = 1 + \frac{\alpha-1}{z-\alpha}$ and $h \in H^p$, to arrive at

$$\begin{aligned} h \in \mathrm{Ran}(T_\omega) \quad &\Longleftrightarrow \quad \frac{\alpha-1}{z-\alpha} h + \frac{c}{z-\alpha} \in H^p, \quad \text{for some } c \in \mathbb{C} \\ &\Longleftrightarrow \quad h + \frac{c}{\alpha-1} \in \frac{z-\alpha}{\alpha-1} H^p = (z-\alpha)H^p, \quad \text{for some } c \in \mathbb{C} \\ &\Longleftrightarrow \quad h \in (z-\alpha)H^p + \mathbb{C}. \end{aligned}$$

So $\mathrm{Ran}(T_\omega) = (z-\alpha)H^p + \mathbb{C}$.

For the matrix representation with respect to the basis $\{z^n\}_{n=0}^\infty$, note that $\dfrac{z-\alpha}{z-1} = 1 + \dfrac{1-\alpha}{z-1} = 1 + c(z-1)^{-1}$ where $c = 1-\alpha$. Then the matrix representation with respect to the standard basis $\{z^n\}_{n=0}^\infty$ of H^p is given by

$$[T_\omega] = \begin{pmatrix} 1 & c & c & c & c & \cdots \\ 0 & 1 & c & c & c & \cdots \\ 0 & 0 & 1 & c & c & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}.$$

From Theorem 2.1.1, T_ω is Fredholm if and only if $\alpha \notin \mathbb{T}$, which also follows from the fact that $\mathrm{Ran}(T_\omega) = (z-\alpha)H^p + \mathbb{C}$. From Theorem 2.5.4 T_ω is invertible for $\alpha \in \mathbb{D}$ and by Lemma 2.4.1,

$$\mathrm{Ker}(T_\omega) = \{c/(z-\alpha) \colon c \in \mathbb{C}\}$$

in case $\alpha \notin \mathbb{D}$.

For $\alpha \in \mathbb{D}$, let T^{+} be the operator on H^p with the matrix representation with respect to the standard basis $\{z^n\}_{n=0}^{\infty}$ of H^p be given by

$$[T^{+}] = \begin{pmatrix} 1 & -c & -c\alpha & -c\alpha^2 & -c\alpha^3 & \cdots \\ 0 & 1 & -c & -c\alpha & -c\alpha^2 & \cdots \\ 0 & 0 & 1 & -c & -c\alpha & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}$$

Then T^{+} is the bounded inverse for T_ω.

For $\alpha \notin \mathbb{D}$, T_ω is surjective and the operator $T^{\sharp}$ with the matrix representation with respect to the standard basis of H^p given by

$$[T^{\sharp}] = \alpha^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ c\alpha^{-1} & 1 & 0 & 0 & 0 & \cdots \\ c(\alpha)^{-2} & c\alpha^{-1} & 1 & 0 & 0 & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}$$

is a right-sided inverse for T_ω.

Example 2.7.3. Let $\omega(z) = \frac{z+1}{(z-1)^2} \in \mathrm{Rat}_0(\mathbb{T})$. From Lemma 2.2.3

$$\mathrm{Dom}(T_\omega) = \left\{ g \in H^p : \omega g = h + \frac{r}{(z-1)^2}, h \in H^p, \deg(r) \leq 1 \right\}.$$

From Proposition 2.4.5 we see that $(z-1)^2 H^p + \mathcal{P}_1 \subset \mathrm{Dom}(T_\omega)$ and $(z+1)H^p + \mathbb{C} \subset \mathrm{Ran}(T_\omega)$. In this case ω has a zero on $\mathbb{T}$, so T_ω is not Fredholm. Nonetheless, we will show that $(z+1)H^p + \mathbb{C} = \mathrm{Ran}(T_\omega)$ and $(z-1)^2 H^p + \mathcal{P}_1 = \mathrm{Dom}(T_\omega)$.

To this end, let $h \in \mathrm{Ran}(T_\omega)$. Then there exists an $f \in H^p$ and a polynomial r with $\deg(r) \leq 1$ with

$$f = \frac{(z-1)^2}{z+1} h + \frac{r}{z+1} \in H^p.$$

Note that $\frac{(z-1)^2}{z+1} = z - 3 + \frac{4}{z+1}$ and $\frac{r}{z+1} = c_1 + \frac{c_2}{z+1}$ for $c_1, c_2 \in \mathbb{C}$. So we have

$$\begin{aligned} h \in \mathrm{Ran}(T_\omega) &\iff (z-3)h + \frac{4}{z+1}h + c_1 + \frac{c_2}{z+1} \in H^p, \text{ for some } c_1, c_2 \in \mathbb{C} \\ &\iff \frac{4}{z+1}h + \frac{c_2}{z+1} \in H^p, \text{ for some } c_2 \in \mathbb{C} \\ &\iff h + \frac{c_2}{4} \in \frac{z+1}{4} H^p = (z+1)H^p, \text{ for some } c_2 \in \mathbb{C} \\ &\iff h \in (z+1)H^p + \mathbb{C}. \end{aligned}$$

So $\mathrm{Ran}(T_\omega) = (z+1)H^p + \mathbb{C}$. From Theorem 2.1.1 it follows that T_ω is not Fredholm, which can also be seen directly from the fact that $\mathrm{Ran}(T_\omega) = (z+1)H^p + \mathbb{C}$ which is not closed.

To show $\mathrm{Dom}(T_\omega) = (z-1)^2H^p + \mathbb{P}_1$, let $f \in \mathrm{Dom}(T_\omega)$ then there are $h \in H^p$ and $az+b = r \in \mathcal{P}_1$ with

$$\omega f = h + \frac{az+b}{(z-1)^2}, \quad \text{or equivalently,} \quad f = \frac{(z-1)^2}{z+1}h + \frac{az+b}{z+1}.$$

Since $h \in \mathrm{Ran}(T_\omega) = (z+1)H^p + \mathbb{C}$ there are $g \in H^p$ and $c \in \mathbb{C}$ with $h = (z+1)g + c$. As $\frac{(z-1)^2}{z+1} = z - 3 + \frac{4}{z+1}$ and $\frac{az+b}{z+1} = a + \frac{b-a}{z+1}$, we find that

$$\begin{aligned} f &= \frac{(z-1)^2}{z+1}\left((z+1)g + c\right) + \frac{az+b}{z+1} \\ &= (z-1)^2g + (z-3)c + \frac{4c}{z+1} + a + \frac{b-a}{z+1}. \end{aligned}$$

This implies that

$$\frac{4c+b-a}{z+1} = f - (z-1)^2g - (cz - 3c + a) \in H^p,$$

which, by Lemma 2.2.2, can only happen if $\frac{4c+b-a}{z+1} = 0$.

As $\frac{4c+b-a}{z+1} = f - (z-1)^2g - (cz-3c+a) \in H^p$ from Lemma 2.2.2 we get $\frac{4c+b-a}{z+1} = 0$. Thus $f = (z-1)^2g + (cz-3c+a) \in (z-1)^2H^p + \mathcal{P}_1$ and so $\mathrm{Dom}(T_\omega) = (z-1)^2H^p + \mathcal{P}_1$.

From Lemma 2.4.1 it follows that $\mathrm{Ker}(T_\omega) = \mathbb{C}$ and for the matrix representation with respect to $\{z^n\}_{n=0}^\infty$ note

$$\frac{z+1}{(z-1)^2}z^k = \sum_{n=0}^{k-1}(2n+1)z^{k-n-1} + \frac{(2k+3)z - (2k+1)}{(z-1)^2}$$

and so the matrix representation $[T_\omega]$ given by

$$[T_\omega] = \begin{pmatrix} 0 & 1 & 3 & 5 & 7 & \cdots \\ 0 & 0 & 1 & 3 & 5 & \cdots \\ 0 & 0 & 0 & 1 & 3 & \cdots \\ & \vdots & & \ddots & & \cdots \end{pmatrix}.$$

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Chapter 3

The spectrum

Abstract. This paper is a continuation of our study of a class of Toeplitz-like operators with a rational symbol which has a pole on the unit circle. A description of the spectrum and its various parts, i.e., point, residual and continuous spectrum, is given, as well as a description of the essential spectrum. In this case, the essential spectrum need not be connected in $\mathbb{C}$. Various examples illustrate the results. [1]

Mathematics Subject Classification (2010). Primary 47B35, 47A53; Secondary 47A68.
Keywords. unbounded Toeplitz operator, spectrum, essential spectrum.

3.1 Introduction

This chapter is a continuation of Chapter 2 where Toeplitz-like operators with rational symbols which may have poles on the unit circle were introduced. While the aim of Chapter 2 was to determine the Fredholm properties of such Toeplitz-like operators, in the current chapter we will focus on properties of the spectrum. For this purpose we further analyse this class of Toeplitz-like operators, specifically in the case where the operators are not necessarily Fredholm.

We start by recalling the definition of our Toeplitz-like operators. Let Rat denote the space of rational complex functions. Write $\mathrm{Rat}(\mathbb{T})$ and $\mathrm{Rat}_0(\mathbb{T})$ for the subspaces of Rat consisting of the rational functions in Rat with all poles on $\mathbb{T}$ and the strictly proper rational functions in Rat with all poles on the unit circle $\mathbb{T}$, respectively. For $\omega \in \mathrm{Rat}$, possibly having poles on $\mathbb{T}$, we define a Toeplitz-like operator $T_\omega(H^p \to H^p)$, for $1 < p < \infty$, as follows:

$$\mathrm{Dom}(T_\omega) = \{g \in H^p \,|\, \omega g = f + \rho \text{ with } f \in L^p, \rho \in \mathrm{Rat}_0(\mathbb{T})\}, \; T_\omega g = \mathbb{P} f. \tag{3.1.1}$$

Here $\mathbb{P}$ is the Riesz projection of L^p onto H^p.

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[1] Up to some minor modifications, this chapter has been published as
G.J. Groenewald, S. ter Horst, J. Jaftha, and A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle II: the spectrum, *Oper. Theory Adv. Appl.*, Vol. 272 (2019), 133 - 154.

In Chapter 2 it was established that this operator is a densely defined, closed operator which is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. In case the symbol ω of T_ω is in $\mathrm{Rat}(\mathbb{T})$ with no zeroes on $\mathbb{T}$, i.e., T_ω Fredholm, explicit formulas for the domain, kernel, range and a complement of the range were also obtained in Chapter 2. Here we extend these results to the case that ω is allowed to have zeroes on $\mathbb{T}$, cf., Theorem 3.2.2 below. By a reduction to the case of symbols in $\mathrm{Rat}(\mathbb{T})$, we then obtain for general symbols in Rat, in Proposition 3.2.4 below, necessary and sufficient conditions for T_ω to be injective or have dense range, respectively.

Main results Using the fact that $\lambda I_{H^p} - T_\omega = T_{\lambda-\omega}$, our extended analysis of the operator T_ω enables us to describe the spectrum of T_ω, and its various parts. Our first main result is a description of the essential spectrum of T_ω, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\lambda I_{H^p} - T_\omega$ is not Fredholm.

Theorem 3.1.1. *Let $\omega \in \mathrm{Rat}$. Then the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ of T_ω is an algebraic curve in $\mathbb{C}$ which is given by*

$$\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) := \{\omega(e^{i\theta}) \mid 0 \leq \theta \leq 2\pi,\ e^{i\theta} \text{ not a pole of } \omega\}.$$

Furthermore, the map $\lambda \mapsto \mathrm{Index}\,(T_{\lambda-\omega})$ is constant on connected components of $\mathbb{C}\backslash\omega(\mathbb{T})$ and the intersection of the point spectrum, residual spectrum and resolvent set of T_ω with $\mathbb{C}\backslash\omega(\mathbb{T})$ coincides with sets of $\lambda \in \mathbb{C}\backslash\omega(\mathbb{T})$ with $\mathrm{Index}\,(T_{\lambda-\omega})$ being strictly positive, strictly negative and zero, respectively.

Various examples, specifically in Section 3.5, show that the algebraic curve $\omega(\mathbb{T})$, and thus the essential spectrum of T_ω, need not be connected in $\mathbb{C}$.

Our second main result provides a description of the spectrum of T_ω and its various parts. Here and throughout the chapter $\mathcal{P}$ stands for the subspace of H^p consisting of all polynomials and $\mathcal{P}_k$ for the subspace of $\mathcal{P}$ consisting of all polynomials of degree at most k.

Theorem 3.1.2. *Let $\omega \in \mathrm{Rat}$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. For $\lambda \in \mathbb{C}$, define*

$$\begin{aligned} k_q &= \sharp\{\text{roots of } q \text{ inside } \overline{\mathbb{D}}\} = \sharp\{\text{poles of } \lambda - \omega \text{ inside } \overline{\mathbb{D}}\},\\ k_\lambda^- &= \sharp\{\text{roots of } \lambda q - s \text{ inside } \mathbb{D}\} = \sharp\{\text{zeroes of } \lambda - \omega \text{ inside } \mathbb{D}\},\\ k_\lambda^0 &= \sharp\{\text{roots of } \lambda q - s \text{ on } \mathbb{T}\} = \sharp\{\text{zeroes of } \lambda - \omega \text{ on } \mathbb{T}\}, \end{aligned} \tag{3.1.2}$$

where in all these sets, multiplicities of the roots, poles and zeroes are to be taken into account. Then the resolvent set $\rho(T_\omega)$, point spectrum $\sigma_{\mathrm{p}}(T_\omega)$, residual spectrum $\sigma_{\mathrm{r}}(T_\omega)$ and continuous spectrum $\sigma_{\mathrm{c}}(T_\omega)$ of T_ω are given by

$$\begin{gathered} \rho(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 = 0 \text{ and } k_q = k_\lambda^-\},\\ \sigma_{\mathrm{p}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_q > k_\lambda^- + k_\lambda^0\}, \quad \sigma_{\mathrm{r}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_q < k_\lambda^-\},\\ \sigma_{\mathrm{c}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0 \text{ and } k_\lambda^- \leq k_q \leq k_\lambda^- + k_\lambda^0\}. \end{gathered} \tag{3.1.3}$$

Furthermore, $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0\}$.

Again, in subsequent sections various examples are given that illustrate these results. In particular, examples are given where T_ω has a bounded resolvent set , even with an empty resolvent set. This is in sharp contrast to the case where ω has no poles on the unit circle $\mathbb{T}$. For in this case the operator is bounded, the resolvent set is a nonempty unbounded set and the spectrum a compact set, and the essential spectrum is connected.

Both Theorems 3.1.1 and 3.1.2 are proven in Section 3.3.

Discussion of the literature In the case of a bounded selfadjoint Toeplitz operator on ℓ^2, Hartman and Wintner in [11] showed that the point spectrum is empty when the symbol is real and rational and posed the problem of specifying the spectral properties of such a Toeplitz operator. Gohberg in [7], and more explicitly in [8], showed that a bounded Toeplitz operator with continuous symbol is Fredholm exactly when the symbol has no zeroes on $\mathbb{T}$, and in this case the index of the operator coincides with the negative of the winding number of the symbol with respect to zero. This implies immediately that the essential spectrum of a Toeplitz operator with continuous symbol is the image of the unit circle.

Hartman and Wintner in [12] followed up their earlier question by showing that in the case where the symbol, φ, is a bounded real-valued function on $\mathbb{T}$, the spectrum of the Toeplitz operator on H^2 is contained in the interval bounded by the essential lower and upper bounds of φ on $\mathbb{T}$ as well as that the point spectrum is empty whenever φ is not a constant. Halmos, after posing in [10] the question whether the spectrum of a Toeplitz operator is connected, with Brown in [1] showed that the spectrum cannot consist of only two points. Widom, in [16], established that bounded Toeplitz operators on H^2 have connected spectrum, and later extended the result for general H^p, with $1 \leq p \leq \infty$. That the essential (Fredholm) spectrum of a bounded Toeplitz operator in H^2 is connected was shown by Douglas in [5]. For the case of bounded Toeplitz operators in H^p it is posed as an open question in Böttcher and Silbermann in [2, Page 70] whether the essential (Fredholm) spectrum of a Toeplitz operator in H^p is necessarily connected. Clark, in [3], established conditions on the argument of the symbol φ in the case $\varphi \in L^q, q \geq 2$ that would give the kernel index of the Toeplitz operator with symbol φ on L^p, where $\frac{1}{p} + \frac{1}{q} = 1$, to be $m \in \mathbb{N}$.

Janas, in [13], discussed unbounded Toeplitz operators on the Bargmann-Segal space and showed that $\sigma_{\text{ess}}(T_\varphi) \subset \cap_{R>0}$ closure $\{\varphi(z) : |z| \geq R\}$.

Overview The chapter is organized as follows. Besides the current introduction, the chapter consists of five sections. In Section 3.2 we extend a few results concerning the operator T_ω from Chapter 2 to the case where T_ω need not be Fredholm. These results are used in Section 3.3 to compute the spectrum of T_ω and various of its subparts, and by doing so we prove the main results, Theorems 3.1.1 and 3.1.2. The remaining three sections contain examples that illustrate our main results and show in addition that the resolvent set can be bounded, even empty, and that the essential spectrum can be disconnected in $\mathbb{C}$.

Figures We conclude this introduction with a remark on the figures in this chapter illustrating the spectrum and essential spectrum for several examples. The color coding in these figures is as follows: the white region is the resolvent set, the black curve is the essential spectrum, and the colors in the other regions codify the Fredholm index, where red indicates index 2, blue indicates index 1, cyan indicates index -1, magenta indicates index -2.

3.2 Review and new results concerning T_ω

In this section we recall some results concerning the operator T_ω defined in (3.1.1) that were obtained in Chapter 2 and will be used in the present chapter to determine spectral properties of T_ω. A few new features are added as well, specifically relating to the case where T_ω is not Fredholm.

The first result provides necessary and sufficient conditions for T_ω to be Fredholm, and gives a formula for the index of T_ω in case T_ω is Fredholm.

Theorem 3.2.1 (Theorems 2.1.1 and 2.5.4)**.** *Let $\omega \in \mathrm{Rat}$. Then T_ω is Fredholm if and only if ω has no zeroes on $\mathbb{T}$. In case T_ω is Fredholm, the Fredholm index of T_ω is given by*

$$\mathrm{Index}\,(T_\omega) = \sharp \left\{ \begin{matrix} \textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account} \end{matrix} \right\} - \sharp \left\{ \begin{matrix} \textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account} \end{matrix} \right\},$$

and T_ω is either injective or surjective. In particular, T_ω is injective, invertible or surjective if and only if $\mathrm{Index}\,(T_\omega) \leq 0$, $\mathrm{Index}\,(T_\omega) = 0$ or $\mathrm{Index}\,(T_\omega) \geq 0$, respectively.

Special attention is given in Chapter 2 to the case where ω is in $\mathrm{Rat}(\mathbb{T})$, since in that case the kernel, domain and range can be computed explicitly; for the domain and range this was done under the assumption that T_ω is Fredholm. In the following result we collect various statements from Proposition 2.4.5 and Theorems 2.1.2 and 2.4.7 and extend to or improve some of the claims regarding the case that T_ω is not Fredholm.

Theorem 3.2.2. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+ having roots only inside, on, or outside $\mathbb{T}$. Then*

$$\begin{aligned} \mathrm{Ker}(T_\omega) &= \{ r_0/s_+ \mid \deg(r_0) < \deg(q) - \deg(s_- s_0) \}\,; \\ \mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{\deg(q)-1}; &\quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}, \end{aligned} \tag{3.2.1}$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{ r \in \mathcal{P} \mid rq = r_1 s + r_2 \textit{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1} \} \subset \mathcal{P}_{\deg(s)-1}. \tag{3.2.2}$$

Furthermore, $H^p = \overline{\mathrm{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ forms a direct sum decomposition of H^p, where

$$\widetilde{\mathcal{Q}} = \mathcal{P}_{k-1} \quad \textit{with} \quad k = \max\{ \deg(s_-) - \deg(q), 0 \}, \tag{3.2.3}$$

following the convention $\mathcal{P}_{-1} := \{0\}$.

The following result will be useful in the proof of Theorem 3.2.2.

Lemma 3.2.3. *Factor $s \in \mathcal{P}$ as $s = s_- s_0 s_+$ with s_-, s_0 and s_+ having roots only inside, on, or outside $\mathbb{T}$. Then $sH^p = s_- s_0 H^p$ and $\overline{sH^p} = s_- H^p$.*

Proof. Since s_+ has no roots inside $\overline{\mathbb{D}}$, we have $s_+ H^p = H^p$. Furthermore, s_0 is an H^∞ outer function (see, e.g., [14], Example 4.2.5) so that $\overline{s_0 H^p} = H^p$. Since s_- has all its roots inside $\mathbb{D}$, $T_{s_-} : H^p \to H^p$ is an injective operator with closed range. Consequently, we have

$$\overline{sH^p} = \overline{s_- s_0 s_+ H^p} = \overline{s_- s_0 H^p} = s_- \overline{s_0 H^p} = s_- H^p,$$

as claimed. $\square$

Proof of Theorem 3.2.2. In case T_ω is Fredholm, i.e., s_0 constant, all statements follow from Theorem 2.1.2. Without the Fredholm condition, the formula for $\mathrm{Ker}(T_\omega)$ follows from Lemma 2.4.1 and for $\mathrm{Dom}(T_\omega)$ and $\mathrm{Ran}(T_\omega)$ Proposition 2.4.5 provides

$$\begin{aligned} qH^p + \mathcal{P}_{\deg(q)-1} &\subset \mathrm{Dom}(T_\omega); \\ T_\omega(qH^p + \mathcal{P}_{\deg(q)-1}) &= sH^p + \widetilde{\mathcal{P}} \subset \mathrm{Ran}(T_\omega). \end{aligned} \tag{3.2.4}$$

Thus in order to prove (4.2.1), it remains to show that $\mathrm{Dom}(T_\omega) \subset qH^p + \mathcal{P}_{\deg(q)-1}$.

Assume $g \in \mathrm{Dom}(T_\omega)$. Thus there exist $h \in H^p$ and $r \in \mathcal{P}_{\deg(q)-1}$ so that $sg = qh + r$. Since s and q are co-prime, there exist $a, b \in \mathcal{P}$ such that $sa + qb \equiv 1$. Next write $ar = qr_1 + r_2$ for $r_1, r_2 \in \mathcal{P}$ with $\deg(r_2) < \deg(q)$. Thus $sg = qh + r = qh + qbr + sar = q(h + br + sr_1) + sr_2$. Hence $g = q(h + br + sr_1)/s + r_2$. We are done if we can show that $\widetilde{h} := (h + br + sr_1)/s$ is in H^p.

The case where g is rational is significantly easier, but still gives an idea of the complications that arise, so we include a proof. Hence assume $g \in \mathrm{Rat} \cap H^p$. Then $h = (sg - r)/q$ is also in $\mathrm{Rat} \cap H^p$, and $\widetilde{h}$ is also rational. It follows that $q(h + br + sr_1)/s = q\widetilde{h} = g - r_2 \in \mathrm{Rat} \cap H^p$ and thus cannot have poles in $\overline{\mathbb{D}}$. Since q and s are co-prime and h cannot have poles inside $\overline{\mathbb{D}}$, it follows that $\widetilde{h} = (h + br + sr_1)/s$ cannot have poles in $\overline{\mathbb{D}}$. Thus $\widetilde{h}$ is a rational function with no poles in $\overline{\mathbb{D}}$, which implies $\widetilde{h} \in H^p$.

Now we prove the claim for the general case. Assume $q\widetilde{h} + r_2 = g \in H^p$, but $\widetilde{h} = (h + br + sr_1)/s \notin H^p$, i.e., $\widetilde{h}$ is not analytic on $\mathbb{D}$ or $\int_{\mathbb{T}} |\widetilde{h}(z)|^p \mathrm{d}z = \infty$. Set $\widehat{h} = h + br + sr_1 \in H^p$, so that $\widetilde{h} = \widehat{h}/s$. We first show $\widetilde{h}$ must be analytic on $\mathbb{D}$. Since $\widetilde{h} = \widehat{h}/s$ and $\widehat{h} \in H^p$, $\widetilde{h}$ is analytic on $\mathbb{D}$ except possibly at the roots of s. However, if $\widetilde{h}$ would not be analytic at a root $z_0 \in \mathbb{D}$ of s, then also $g = q\widetilde{h} + r_2$ should not be analytic at z_0, since g is bounded away from 0 on a neighborhood of z_0, using that s and q are co-prime. Thus $\widetilde{h}$ is analytic on $\mathbb{D}$. It follows that $\int_{\mathbb{T}} |\widetilde{h}(z)|^p \mathrm{d}z = \infty$.

Since s and q are co-prime, we can divide $\mathbb{T}$ as $\mathbb{T}_1 \cup \mathbb{T}_2$ with $\mathbb{T}_1 \cap \mathbb{T}_2 = \emptyset$ and each of $\mathbb{T}_1$ and $\mathbb{T}_2$ being nonempty unions of intervals, with $\mathbb{T}_1$ containing all roots of s on $\mathbb{T}$ as interior points and $\mathbb{T}_2$ containing all roots of q on $\mathbb{T}$ as interior points. Then there exist $N_1, N_2 > 0$ such that $|q(z)| > N_1$ on $\mathbb{T}_1$ and $|s(z)| > N_2$ on $\mathbb{T}_2$. Note that

$$\int_{\mathbb{T}_2} |\widetilde{h}(z)|^p \mathrm{d}z = \int_{\mathbb{T}_2} |\widehat{h}(z)/s(z)|^p \mathrm{d}z \leq N_2^{-p} \int_{\mathbb{T}_2} |\widehat{h}(z)|^p \mathrm{d}z \leq N_2^{-p} \|\widehat{h}\|_{H^p}^p < \infty.$$

Since $\int_{\mathbb{T}} |\widetilde{h}(z)|^p \mathrm{d}z = \infty$ and $\int_{\mathbb{T}_2} |\widetilde{h}(z)|^p \mathrm{d}z < \infty$, it follows that $\int_{\mathbb{T}_1} |\widetilde{h}(z)|^p \mathrm{d}z = \infty$. However, since $|q(z)| > N_1$ on $\mathbb{T}_1$, this implies that

$$\begin{aligned}\|g - r_2\|_{H^p}^p &= \int_{\mathbb{T}} |g(z) - r_2(z)|^p \mathrm{d}z = \int_{\mathbb{T}} |q(z)\widetilde{h}(z)|^p \mathrm{d}z \geq \int_{\mathbb{T}_1} |q(z)\widetilde{h}(z)|^p \mathrm{d}z \\ &\geq N_1^p \int_{\mathbb{T}_1} |\widetilde{h}(z)|^p \mathrm{d}z = \infty,\end{aligned}$$

in contradiction with the assumption that $g \in H^p$. Thus we can conclude that $\widetilde{h} \in H^p$ so that $g = q\widetilde{h} + r_2$ is in $qH^p + \mathcal{P}_{\deg(q)-1}$.

It remains to show that $H^p = \overline{\mathrm{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ is a direct sum decomposition of H^p. Again, for the case that T_ω is Fredholm this follows from Theorem 2.1.2. By the preceding part of the proof we know, even in the non-Fredholm case, that $\mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}}$. Since $\widetilde{\mathcal{P}}$ is finite dimensional, and thus closed, we have

$$\overline{\mathrm{Ran}(T_\omega)} = \overline{sH^p} + \widetilde{\mathcal{P}} = s_- H^p + \widetilde{\mathcal{P}},$$

using Lemma 3.2.3 in the last identity. We claim that

$$\overline{\mathrm{Ran}(T_\omega)} = s_- H^p + \widetilde{\mathcal{P}} = s_- H^p + \widetilde{\mathcal{P}}_-,$$

where $\widetilde{\mathcal{P}}_-$ is defined by

$$\widetilde{\mathcal{P}}_- := \{r \in \mathcal{P} \mid qr = r_1 s_- + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{\deg(q)-1}\} \subset \mathcal{P}_{\deg(s_-)-1}.$$

Once the above identity for $\overline{\mathrm{Ran}(T_\omega)}$ is established, the fact that $\widetilde{\mathcal{Q}}$ is a complement of $\overline{\mathrm{Ran}(T_\omega)}$ follows directly by applying Lemma 2.4.8 to $s = s_-$.

We first show that $\overline{\mathrm{Ran}(T_\omega)} = s_- H^p + \widetilde{\mathcal{P}}$ is contained in $s_- H^p + \widetilde{\mathcal{P}}_-$. Let $g = s_- h + r$ with $h \in H^p$ and $r \in \widetilde{\mathcal{P}}$, say $qr = r_1 s + r_2$ with $r_1, r_2 \in \mathcal{P}_{\deg(q)-1}$. Write $r_1 s_0 s_+ = \widetilde{r}_1 q + \widetilde{r}_2$ with $\deg(\widetilde{r}_2) < \deg(q)$. Then

$$qr = r_1 s_- s_0 s_+ + r_2 = q\widetilde{r}_1 s_- + \widetilde{r}_2 s_- + r_2, \text{ so that } q(r - \widetilde{r}_1 s_-) = \widetilde{r}_2 s_- + r_2,$$

with $r_2, \widetilde{r}_2 \in \mathcal{P}_{\deg(q)-1}$. Thus $r - \widetilde{r}_1 s_- \in \widetilde{\mathcal{P}}_-$. Therefore, we have

$$g = s_-(h + \widetilde{r}_1) + (r - \widetilde{r}_1 s_-) \in s_- H^p + \widetilde{\mathcal{P}}_-,$$

proving that $\overline{\mathrm{Ran}(T_\omega)} \subset s_- H^p + \widetilde{\mathcal{P}}_-$.

For the reverse inclusion, assume $g = s_- h + r \in s_- H^p + \widetilde{\mathcal{P}}_-$. Say $qr = r_1 s_- + r_2$ with $r_1, r_2 \in \mathcal{P}_{\deg(q)-1}$. Since $s_0 s_+$ and q are co-prime and $\deg(r_1) < \deg(q)$ there exit polynomials $\widetilde{r}_1$ and $\widetilde{r}_2$ with $\deg(\widetilde{r}_1) < \deg(q)$ and $\deg(\widetilde{r}_2) < \deg(s_0 s_+)$ that satisfy the Bézout equation $\widetilde{r}_1 s_0 s_+ + \widetilde{r}_2 q = r_1$. Then

$$\widetilde{r}_1 s + r_2 = \widetilde{r}_1 s_0 s_+ s_- + r_2 = (r_1 - \widetilde{r}_2 q)s_- + r_2 = r_1 s_- + r_2 - q\widetilde{r}_2 s_- = q(r - \widetilde{r}_2 s_-).$$

Hence $r - \widetilde{r}_2 s_-$ is in $\widetilde{\mathcal{P}}$, so that $g = s_- h + r = s_-(h + \widetilde{r}_2) + (r - \widetilde{r}_2 s_-) \in s_- H^p + \widetilde{\mathcal{P}}$. This proves the reverse inclusion, and hence completes the proof of Theorem 3.2.2. □

The following result makes precise when T_ω is injective and when T_ω has dense range, even in the case where T_ω is not Fredholm.

Proposition 3.2.4. *Let $\omega \in \mathrm{Rat}$. Then T_ω is injective if and only if*

$$\sharp \left\{\begin{matrix}\textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account}\end{matrix}\right\} \leq \sharp \left\{\begin{matrix}\textit{zeroes of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account}\end{matrix}\right\}.$$

Moreover, T_ω has dense range if and only if

$$\sharp \left\{\begin{matrix}\textit{poles of } \omega \textit{ in } \overline{\mathbb{D}} \textit{ multi.} \\ \textit{taken into account}\end{matrix}\right\} \geq \sharp \left\{\begin{matrix}\textit{zeroes of } \omega \textit{ in } \mathbb{D} \textit{ multi.} \\ \textit{taken into account}\end{matrix}\right\}.$$

In particular, T_ω is injective or has dense range.

Proof. First assume $\omega \in \mathrm{Rat}(\mathbb{T})$. By Corollary 2.4.2, T_ω is injective if and only if the number of zeroes of ω inside $\overline{\mathbb{D}}$ is greater than or equal to the number of poles of ω, in both cases with multiplicity taken into account. By Theorem 3.2.2, T_ω has dense range precisely when $\widetilde{\mathcal{Q}}$ in (4.2.3) is trivial. The latter happens if and only if the number of poles of ω is greater than or equal to the number of zeroes of ω inside $\mathbb{D}$, again taking multiplicities into account. Since in this case all poles of ω are in $\mathbb{T}$, our claim follows for $\omega \in \mathrm{Rat}(\mathbb{T})$.

Now we turn to the general case, i.e., we assume $\omega \in \mathrm{Rat}$. In the remainder of the proof, whenever we speak of numbers of zeroes or poles, this always means that the respective multiplicities are to be taken into account. Recall from Lemma 2.5.1 that we can factor $\omega(z) = \omega_-(z)z^\kappa\omega_0(z)\omega_+(z)$ with $\omega_-, \omega_0, \omega_+ \in \mathrm{Rat}$, ω_- having no poles or zeroes outside $\mathbb{D}$, ω_+ having no poles or zeroes inside $\overline{\mathbb{D}}$ and ω_0 having poles and zeroes only on $\mathbb{T}$, and κ the difference between the number of zeroes of ω in $\mathbb{D}$ and the number of poles of ω in $\mathbb{D}$. Moreover, we have $T_\omega = T_{\omega_-}T_{z^\kappa\omega_0}T_{\omega_+}$ and T_{ω_-} and T_{ω_+} are boundedly invertible on H^p. Thus T_ω is injective or has closed range if and only if $T_{z^\kappa\omega_0}$ is injective or has closed range, respectively.

Assume $\kappa \geq 0$. Then $z^\kappa\omega_0 \in \mathrm{Rat}(\mathbb{T})$ and the results for the case that the symbol is in $\mathrm{Rat}(\mathbb{T})$ apply. Since the zeroes and poles of ω_0 coincide with the zeroes and poles of ω on $\mathbb{T}$, it follows that the number of poles of $z^\kappa\omega_0$ is equal to the number of poles of ω on $\mathbb{T}$ while the number of zeroes of $z^\kappa\omega_0$ is equal to κ plus the number of zeroes of ω on $\mathbb{T}$ which is equal to the number of zeroes of ω in $\overline{\mathbb{D}}$ minus the number of poles of ω in $\mathbb{D}$. It thus follows that $T_{z^\kappa\omega_0}$ is injective, and equivalently T_ω is injective, if and only if the number of zeroes of ω in $\overline{\mathbb{D}}$ is greater than or equal to the number of poles of ω in $\overline{\mathbb{D}}$, as claimed.

Next, we consider the case where $\kappa < 0$. In that case $T_{z^\kappa\omega_0} = T_{z^\kappa}T_{\omega_0}$, by Lemma 2.5.3. We prove the statements regarding injectivity and T_ω having closed range separately.

First we prove the injectivity claim for the case where $\kappa < 0$. Write $\omega_0 = s_0/q_0$ with $s_0, q_0 \in \mathcal{P}$ co-prime. Note that all the roots of s_0 and q_0 are on $\mathbb{T}$. We need to show that $T_{z^\kappa\omega_0}$ is injective if and only if $\deg(s_0) \geq \deg(q_0) - \kappa$ (recall, κ is negative).

Assume $\deg(s_0) + \kappa \geq \deg(q_0)$. Then $\deg(s_0) > \deg(q_0)$, since $\kappa < 0$, and thus T_{ω_0} is injective. We have $\mathrm{Ker}(T_{z^\kappa}) = \mathcal{P}_{|\kappa|-1}$. So it remains to show $\mathcal{P}_{|\kappa|-1} \cap \mathrm{Ran}(T_{\omega_0}) = \{0\}$. Assume $r \in \mathcal{P}_{|\kappa|-1}$ is also in $\mathrm{Ran}(T_{\omega_0})$. So, by Lemma 2.2.3, there exist $g \in H^p$ and $r' \in \mathcal{P}_{\deg(q_0)-1}$

so that $s_0 g = q_0 r + r'$, i.e., $g = (q_0 r + r')/s_0$. This shows that g is in $\mathrm{Rat}(\mathbb{T}) \cap H^p$, which can only happen in case g is a polynomial. Thus, in the fraction $(q_0 r + r')/s_0$, all roots of s_0 must cancel against roots of $q_0 r + r'$. However, since $\deg(s_0) + \kappa \geq \deg(q_0)$, with $\kappa < 0$, $\deg(r) < \deg |\kappa| - 1$ and $\deg(r') < \deg(q_0)$, we have $\deg(q_0 r + r') < \deg(s_0)$ and it is impossible that all roots of s_0 cancel against roots of $q_0 r + r'$, leading to a contradiction. This shows $\mathcal{P}_{|\kappa|-1} \cap \mathrm{Ran}(T_{\omega_0}) = \{0\}$, which implies $T_{z^\kappa \omega_0}$ is injective. Hence also T_ω is injective.

Conversely, assume $\deg(s_0) + \kappa < \deg(q_0)$, i.e., $\deg(s_0) < \deg(q_0) + |\kappa| =: b$, since $\kappa < 0$. Then

$$s_0 \in \mathcal{P}_{b-1} = q_0 \mathcal{P}_{|\kappa|-1} + \mathcal{P}_{\deg(q_0)-1}.$$

This shows there exist $r \in \mathcal{P}_{|\kappa|-1}$ and $r' \in \mathcal{P}_{\deg(q_0)-1}$ so that $s_0 = q_0 r + r'$. In other words, the constant function $g \equiv 1 \in H^p$ is in $\mathrm{Dom}(T_{\omega_0})$ and $T_{\omega_0} g = r \in \mathcal{P}_{|\kappa|-1} = \mathrm{Ker}(T_{z^\kappa})$, so that $g \in \mathrm{Ker}(T_{z^\kappa \omega_0})$. This implies T_ω is not injective.

Finally, we turn to the proof of the dense range claim for the case $\kappa < 0$. Since $\kappa < 0$ by assumption, ω has more poles in $\overline{\mathbb{D}}$ (and even in $\mathbb{D}$) than zeroes in $\mathbb{D}$. Thus to prove the dense range claim in this case, it suffices to show that $\kappa < 0$ implies that $T_{z^\kappa \omega_0}$ has dense range. We have $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$ and T_{z^κ} is surjective. Also, $\omega_0 \in \mathrm{Rat}(\mathbb{T})$ has no zeroes inside $\mathbb{D}$. So the proposition applies to ω_0, as shown in the first paragraph of the proof, and it follows that T_{ω_0} has dense range. But then also $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$ has dense range, and our claim follows. □

3.3 The spectrum of T_ω

In this section we determine the spectrum and various subparts of the spectrum of T_ω for the general case, $\omega \in \mathrm{Rat}$, as well as some refinements for the case where $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper. In particular, we prove our main results, Theorems 3.1.1 and 3.1.2.

Note that for $\omega \in \mathrm{Rat}$ and $\lambda \in \mathbb{C}$ we have $\lambda I - T_\omega = T_{\lambda - \omega}$. Thus we can relate questions on the spectrum of T_ω to question on injectivity, surjectivity, closed rangeness, etc. for Toeplitz-like operators with an additional complex parameter. By this observation, the spectrum of T_ω, and its various subparts, can be determined using the results of Section 3.2.

Proof of Theorem 3.1.1. Since $\lambda I - T_\omega = T_{\lambda-\omega}$ and $T_{\lambda-\omega}$ is Fredholm if and only if $\lambda - \omega$ has no zeroes on $\mathbb{T}$, by Theorem 3.2.1, it follows that λ is in the essential spectrum if and only if $\lambda = \omega(e^{i\theta})$ for some $0 \leq \theta \leq 2\pi$. This shows that $\sigma_{\mathrm{ess}}(T_\omega)$ is equal to $\omega(\mathbb{T})$.

To see that $\omega(\mathbb{T})$ is an algebraic curve, let $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then $\lambda = u + iv = \omega(z)$ for $z = x + iy$ with $x^2 + y^2 = 1$ if and only if $\lambda q(z) - s(z) = 0$. Denote $q(z) = q_1(x, y) + i q_2(x, y)$ and $s(z) = s_1(x, y) + i s_2(x, y)$, where $z = x + iy$ and the functions q_1, q_2, s_1, s_2 are real polynomials in two variables. Then $\lambda = u + iv$ is on the curve $\omega(\mathbb{T})$ if and only if

$$\begin{aligned} q_1(x,y)u - q_2(x,y)v &= s_1(x,y), \\ q_2(x,y)u + q_1(x,y)v &= s_2(x,y), \\ x^2 + y^2 &= 1. \end{aligned}$$

Solving for u and v, this is equivalent to

$$\begin{aligned} (q_1(x,y)^2+q_2(x,y)^2)u-(q_1(x,y)s_1(x,y)+q_2(x,y)s_2(x,y))&=0,\\ (q_1(x,y)^2+q_2(x,y)^2)v-(q_1(x,y)s_2(x,y)-q_2(x,y)s_1(x,y))&=0,\\ x^2+y^2&=1. \end{aligned}$$

This describes an algebraic curve in the plane.

For λ in the complement of the curve $\omega(\mathbb{T})$ the operator $\lambda I - T_\omega = T_{\lambda-\omega}$ is Fredholm, and according to Theorem 3.2.1 the index is given by

$$\text{Index}\,(\lambda I - T_\omega) = \sharp\{\text{ poles of } \omega \text{ in } \overline{\mathbb{D}}\} - \sharp\{\text{zeroes of } \lambda-\omega \text{ inside } \mathbb{D}\},$$

taking the multiplicities of the poles and zeroes into account. Indeed, $\lambda-\omega=\frac{\lambda q-s}{q}$ and since q and s are co-prime, $\lambda q - s$ and q are also co-prime. Thus Theorem 3.2.1 indeed applies to $T_{\lambda-\omega}$. Furthermore, $\lambda-\omega$ has the same poles as ω, i.e., the roots of q. Likewise, the zeroes of $\lambda-\omega$ coincide with the roots of the polynomial $\lambda q - s$. Since the roots of this polynomial depend continuously on the parameter λ, the number of them is constant on connected components of the complement of the curve $\omega(\mathbb{T})$.

That the index is constant on connected components of the complement of the essential spectrum in fact holds for any unbounded densely defined operator (see [15, Theorem VII.5.2]; see also [4, Proposition XI.4.9] for the bounded case; for a much more refined analysis of this fact see [6]).

Finally, the relation between the index of $T_{\lambda-\omega}$ and λ being in the resolvent set, point spectrum or residual spectrum follows directly by applying the last part of Theorem 3.2.1 to $T_{\lambda-\omega}$. $\square$

Next we prove Theorem 3.1.2 using some of the new results on T_ω derived in Section 3.2.

Proof of Theorem 3.1.2. That the two formulas for the numbers k_q, k_λ^- and k_λ^0 coincides follows from the analysis in the proof of Theorem 3.1.1, using the co-primeness of $\lambda q - s$ and q. By Theorem 3.2.1, $T_{\lambda-\omega}$ is Fredholm if and only if $k_\lambda^0=0$, proving the formula for $\sigma_{\text{ess}}(T_\omega)$. The formula for the resolvent set follows directly from the fact that the resolvent set is contained in the complement of $\sigma_{\text{ess}}(T_\omega)$, i.e., $k_\lambda^0=0$, and that it coincides there with the set of λ's for which the index of $T_{\lambda-\omega}$ is zero, together with the formula for Index$\,(T_{\lambda-\omega})$ obtained in Theorem 3.2.1.

The formulas for the point spectrum and residual spectrum follow by applying the criteria for injectivity and closed rangeness of Proposition 3.2.4 to $T_{\lambda-\omega}$ together with the fact that $T_{\lambda-\omega}$ must be either injective or have dense range.

For the formula for the continuous spectrum, note that $\sigma_{\text{c}}(T_\omega)$ must be contained in the essential spectrum, i.e., $k_\lambda^0>0$. The condition $k_\lambda^- \leq k_q \leq k_\lambda^- + k_\lambda^0$ excludes precisely that λ is in the point or residual spectrum. $\square$

For the case where $\omega \in \text{Rat}(\mathbb{T})$ is proper we can be a bit more precise.

Theorem 3.3.1. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$ be proper, say $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Thus $\deg(s) \leq \deg(q)$ and all roots of q are on $\mathbb{T}$. Let a be the leading coefficient of q and b the coefficient of s corresponding to the monomial $z^{\deg(q)}$, hence $b = 0$ if and only if ω is strictly proper. Then $\sigma_{\mathrm{r}}(T_\omega) = \emptyset$, and the point spectrum is given by*

$$\sigma_{\mathrm{p}}(T_\omega) = \omega(\mathbb{C}\backslash\overline{\mathbb{D}}) \cup \{b/a\}.$$

Here $\omega(\mathbb{C}\backslash\overline{\mathbb{D}}) = \{\omega(z) \mid z \in \mathbb{C}\backslash\overline{\mathbb{D}}\}$. In particular, if ω is strictly proper, then $0 = b/a$ is in $\sigma_{\mathrm{p}}(T_\omega)$. Finally,

$$\sigma_{\mathrm{c}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0 \text{ and all roots of } \lambda q - s \text{ are in } \overline{\mathbb{D}}\}.$$

Proof. Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ be proper with $s, q \in \mathcal{P}$ co-prime. Then $k_q = \deg(q)$. Since $\deg(s) \leq \deg(q)$, for any $\lambda \in \mathbb{C}$ we have

$$k_\lambda^- + k_\lambda^0 \leq \deg(\lambda q - s) \leq \deg(q) = k_q.$$

It now follows directly from (3.1.3) that $\sigma_{\mathrm{r}}(T_\omega) = \emptyset$ and $\sigma_{\mathrm{c}}(T_\omega) = \{\lambda \in \mathbb{C} \mid k_\lambda^0 > 0, k_\lambda^- + k_\lambda^0 = \deg(q)\}$. To determine the point spectrum, again using (3.1.3), one has to determine when strict inequality occurs. We have $\deg(\lambda q - s) < \deg(q)$ precisely when the leading coefficient of λq is cancelled in $\lambda q - s$ or if $\lambda = 0$ and $\deg(s) < \deg(q)$. Both cases correspond to $\lambda = b/a$. For the other possibility of having strict inequality, $k_\lambda^- + k_\lambda^0 < \deg(\lambda q - s)$, note that this happens precisely when $\lambda q - s$ has a root outside $\overline{\mathbb{D}}$, or equivalently $\lambda = \omega(z)$ for a $z \notin \overline{\mathbb{D}}$. $\square$

3.4 The spectrum may be unbounded, the resolvent set empty

In this section we present some first examples, showing that the spectrum can be unbounded and the resolvent set may be empty.

Example 3.4.1. Let $\omega(z) = \frac{z-\alpha}{z-1}$ for some $1 \neq \alpha \in \mathbb{C}$, say $\alpha = a + ib$, with a and b real. Let $L \subset \mathbb{C}$ be the line given by

$$L = \{z = x + iy \in \mathbb{C} \mid 2by = (a^2 + b^2 - 1) + (2 - 2a)x\}. \tag{3.4.1}$$

Then we have

$$\rho(T_\omega) = \omega(\mathbb{D}), \quad \sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T}) = L = \sigma_{\mathrm{c}}(T_\omega),$$
$$\sigma_{\mathrm{p}}(T_\omega) = \omega(\mathbb{C}\backslash\overline{\mathbb{D}}), \quad \sigma_{\mathrm{r}}(T_\omega) = \emptyset.$$

Moreover, the point spectrum of T_ω is the open half-plane determined by L that contains 1 and the resolvent set of T_ω is the other open half-plane determined by L, see Figure 3.1.

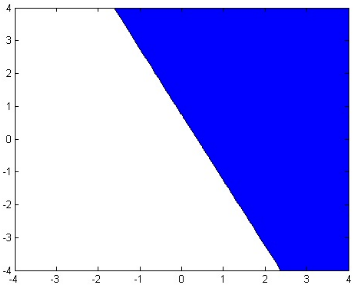

Figure 3.1: Spectrum of T_ω where $\omega(z) = \frac{z-\alpha}{z-1}$, with $\alpha = -\frac{i}{2}$.

To see that these claims are true, note that for $\lambda \neq 1$,

$$\lambda - \omega(z) = \frac{z(\lambda-1)+\alpha-\lambda}{z-1} = \frac{1}{\lambda-1}\frac{z+\frac{\alpha-\lambda}{\lambda-1}}{z-1},$$

while for $\lambda = 1$ we have $\lambda - \omega(z) = \frac{\alpha-\lambda}{z-1}$. Thus $\lambda = 1 \in \sigma_{\rm p}(T_\omega)$ for every $1 \neq \alpha \in \mathbb{C}$ as in that case $k_q = 1 > 0 = k_\lambda^- + k_\lambda^0$. For $\lambda \neq 1$, $\lambda - \omega$ has a zero at $\frac{\alpha-\lambda}{\lambda-1}$ of multiplicity one. For $\lambda = x + iy$ we have $|\alpha - \lambda| = |\lambda - 1|$ if and only if $(a-x)^2 + (b-y)^2 = (x-1)^2 + y^2$, which in turn is equivalent to $2by = (a^2 + b^2 - 1) + (2 - 2a)x$. Hence the zero of $\lambda - \omega$ is on $\mathbb{T}$ precisely when λ is on the line L. This shows $\sigma_{\rm ess}(T_\omega) = L$. One easily verifies that the point spectrum and resolvent set correspond to the two half-planes indicated above and that these coincide with the images of ω under $\mathbb{C}\backslash\overline{\mathbb{D}}$ and $\mathbb{D}$, respectively. Since $\lambda - \omega$ can have at most one zero, it is clear from Theorem 3.1.2 that $\sigma_{\rm r}(T_\omega) = \emptyset$, so that $\sigma_{\rm c}(T_\omega) = L = \sigma_{\rm ess}(T_\omega)$, as claimed. $\square$

Example 3.4.2. Let $\omega(z) = \frac{1}{(z-1)^k}$ for some positive integer $k > 1$. Then

$$\sigma_{\rm p}(T_\omega) = \sigma(T_\omega) = \mathbb{C}, \quad \sigma_r(T_\omega) = \sigma_{\rm c}(T_\omega) = \rho(T_\omega) = \emptyset,$$

and the essential spectrum is given by

$$\sigma_{\rm ess}(T_\omega) = \omega(\mathbb{T}) = \{(it - \tfrac{1}{2})^k \mid t \in \mathbb{R}\}.$$

For $k = 2$ the situation is as in Figure 3.2; one can check that the curve $\omega(\mathbb{T})$ is the parabola $\mathrm{Re}(z) = \frac{1}{4} - \mathrm{Im}(z)^2$. (Recall that different colors indicate different Fredholm indices, as explained at the end of the introduction.)

To prove the statements, we start with the observation that for $|z| = 1$, $\frac{1}{z-1}$ is of the form $it - \frac{1}{2}, t \in \mathbb{R}$. Thus for $z \in \mathbb{T}$ with $\frac{1}{z-1} = it - \frac{1}{2}$ we have

$$\omega(z) = \frac{1}{(z-1)^k} = (z-1)^{-k} = (it - \tfrac{1}{2})^k.$$

This proves the formula for $\sigma_{\rm ess}(T_\omega)$. For $\lambda = re^{i\theta} \neq 0$ we have

$$\lambda - \omega(z) = \frac{\lambda(z-1)^k - 1}{(z-1)^k}.$$

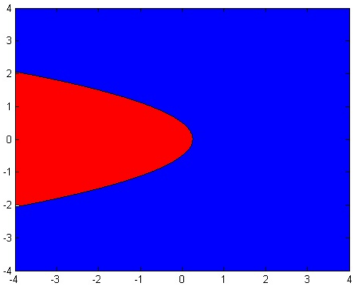

Figure 3.2: Essential spectrum of T_ω where $\omega(z) = \frac{1}{(z-1)^2}$

Thus $\lambda - \omega(z) = 0$ if and only if $(z-1)^k = \lambda^{-1}$, i.e., $z = 1 + r^{-1/k} e^{i(\theta+2\pi l)/k}$ for $l = 0, \ldots, k-1$. Thus the zeroes of $\lambda - \omega$ are k equally spaced points on the circle with center 1 and radius $r^{-1/k}$. Clearly, since $k > 1$, not all zeroes can be inside $\overline{\mathbb{D}}$, so $k_q > k_\lambda^0 + k_\lambda^-$, and thus $\lambda \in \sigma_{\rm p}(T_\omega)$. It follows directly from Theorem 3.1.2 that $0 \in \sigma_{\rm p}(T_\omega)$. Thus $\sigma_{\rm p}(T_\omega) = \mathbb{C}$, as claimed. The curve $\omega(\mathbb{T})$ divides the plane into several regions on which the index is a positive constant integer, but the index may change between different regions.

$\square$

3.5 The essential spectrum need not be connected

For a continuous function ω on the unit circle, it is obviously the case that the curve $\omega(\mathbb{T})$ is a connected and bounded curve in the complex plane, and hence the essential spectrum of T_ω is connected in this case. It was proved by Widom [16] that also for ω piecewise continuous the essential spectrum of T_ω is connected, and it is the image of a curve related to $\omega(\mathbb{T})$ (roughly speaking, filling the jumps with line segments). Douglas [5] proved that even for $\omega \in L^\infty$ the essential spectrum of T_ω as an operator on H^2 is connected. In [2] the question is raised whether or not the essential spectrum of T_ω as an operator on H^p is always connected when $\omega \in L^\infty$.

Returning to our case, where ω is a rational function possibly with poles on the unit circle. Clearly when ω does have poles on the unit circle it is not a-priori necessary that $\sigma_{\rm ess}(T_\omega) = \omega(\mathbb{T})$ is connected. We shall present examples that show that indeed the essential spectrum need not be connected, in contrast with the case where $\omega \in L^\infty$.

Consider $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ with real coefficients. In that case $\overline{\omega(z)} = \omega(\overline{z})$, so that the essential spectrum is symmetric with respect to the real axis. In particular, if $\omega(\mathbb{T}) \cap \mathbb{R} = \emptyset$, then the essential spectrum is disconnected. The converse direction need not be true, since the essential spectrum can consist of several disconnected parts on the real axis, as the following example shows.

Example 3.5.1. Consider $\omega(z) = \frac{z}{z^2+1}$. Then

$$\sigma_{\rm ess}(T_\omega) = \omega(\mathbb{T}) = (-\infty, -1] \cup [1, \infty) = \sigma_{\rm c}(T_\omega), \quad \sigma_{\rm p}(T_\omega) = \mathbb{C} \backslash \omega(\mathbb{T}),$$

and thus $\sigma_{\rm r}(T_\omega) = \rho(T_\omega) = \emptyset$. Further, for $\lambda \notin \omega(\mathbb{T})$ the Fredholm index of T_ω is 1.

Indeed, note that for $z = e^{i\theta} \in \mathbb{T}$ we have

$$\omega(z) = \frac{1}{z + z^{-1}} = \frac{1}{2\,\mathrm{Re}(z)} = \frac{1}{2\cos(\theta)} \in \mathbb{R}.$$

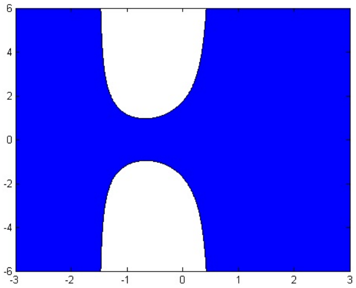

Figure 3.3: Spectrum of T_ω, where $\omega(z) = \frac{z^3+3z+1}{z^2-1}$

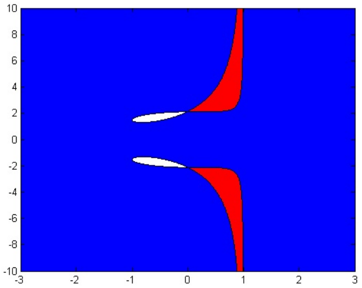

Figure 3.4: Spectrum of T_ω, where $\omega(z) = \frac{z^4+3z+1}{z^2-1}$

Letting θ run from 0 to 2π, one finds that $\omega(\mathbb{T})$ is equal to the union of $(-\infty, -1]$ and $[1, \infty)$, as claimed. Since ω is strictly proper, $\sigma_{\rm r}(T_\omega) = \emptyset$ by Theorem 3.3.1. Applying Theorem 3.2.1 to T_ω we obtain that T_ω is Fredholm with index 1. Hence T_ω is not injective, so that $0 \in \sigma_{\rm p}(T_\omega)$. However, since $\mathbb{C}\backslash\omega(\mathbb{T})$ is connected, it follows from Theorem 3.1.1 that the index of $T_{\lambda-\omega}$ is equal to 1 on $\mathbb{C}\backslash\omega(\mathbb{T})$, so that $\mathbb{C}\backslash\omega(\mathbb{T}) \subset \sigma_{\rm p}(T_\omega)$. However, for λ on $\omega(\mathbb{T})$ the function $\lambda - \omega$ has two zeroes on $\mathbb{T}$ as well as two poles on $\mathbb{T}$. It follows that $\omega(\mathbb{T}) = \sigma_{\rm c}(T_\omega)$, which shows all the above formulas for the spectral parts hold.

As a second example we specify q to be $z^2 - 1$ and determine a condition on s that guarantees $\sigma_{\rm ess}(T_\omega) = \omega(\mathbb{T})$ is not connected.

Example 3.5.2. Consider $\omega(z) = \frac{s(z)}{z^2-1}$ with $s \in \mathcal{P}$ a polynomial with real coefficients. Then for $z \in \mathbb{T}$ we have

$$\omega(z) = \frac{\overline{z}s(z)}{z - \overline{z}} = \frac{\overline{z}s(z)}{2i\,\mathrm{Im}(z)} = \frac{-i\overline{z}s(z)}{2\,\mathrm{Im}(z)}, \quad \text{so that} \quad \mathrm{Im}(\omega(z)) = \frac{-\mathrm{Re}(\overline{z}s(z))}{2\,\mathrm{Im}(z)}.$$

Hence $\mathrm{Im}(\omega(z)) = 0$ if and only if $\mathrm{Re}(\overline{z}s(z)) = 0$. Say $s(z) = \sum_{j=0}^{k} a_j z^j$. Then for $z \in \mathbb{T}$ we have

$$\mathrm{Re}(\overline{z}s(z)) = \sum_{j=0}^{k} a_j \mathrm{Re}(z^{j-1}).$$

Since $|\mathrm{Re}(z^j)| \leq 1$, we obtain that $|\mathrm{Re}(\overline{z}s(z))| > 0$ for all $z \in \mathbb{T}$ in case $2|a_1| > \sum_{j=0}^{k} |a_j|$. Hence in that case $\omega(\mathbb{T}) \cap \mathbb{R} = \emptyset$ and we find that the essential spectrum is disconnected in $\mathbb{C}$.

We consider two concrete examples, where this criteria is satisfied.

Firstly, take $\omega(z) = \frac{z^3+3z+1}{z^2-1}$. Then

$$\omega(e^{i\theta}) = \frac{1}{2}(2\cos\theta - 1) - \frac{i}{2}\frac{2(\cos\theta + 1/4)^2 + 7/4}{\sin\theta},$$

which is the curve given in Figure 3.3, that also shows the spectrum and resolvent set as well as the essential spectrum.

Secondly, take $\omega(z) = \frac{z^4+3z+1}{z^2-1}$. Figure 3.4 shows the spectrum and resolvent set and the essential spectrum. Observe that this is also a case where the resolvent set is a bounded set.

3.6 A parametric example

In this section we take $\omega_k(z) = \frac{z^k+\alpha}{(z-1)^2}$ for $\alpha \in \mathbb{C}, \alpha \neq -1$ and for various integers $k \geq 1$. Note that the case $k = 0$ was dealt with in Example 3.4.2 (after scaling with the factor $1 + \alpha$). The zeroes of $\lambda - \omega_k$ are equal to the roots of

$$p_{\lambda,\alpha,k}(z) = \lambda q(z) - s(z) = \lambda(z-1)^2 - (z^k + \alpha).$$

Thus, λ is in the resolvent set $\rho(T_{\omega_k})$ whenever $p_{\lambda,\alpha,k}$ has at least two roots in $\mathbb{D}$ and no roots on $\mathbb{T}$. Note that Theorem 3.3.1 applies in case $k = 1, 2$. We discuss the first of these two cases in detail, and then conclude with some figures that contain possible configurations of other cases.

Example 3.6.1. Let $\omega(z) = \omega_1(z) = \frac{z+\alpha}{(z-1)^2}$ for $\alpha \neq -1$. Then

$$\sigma_{\text{ess}}(T_\omega) = \omega(\mathbb{T}) = \{(it - \tfrac{1}{2}) + (1+\alpha)(it - \tfrac{1}{2})^2 \mid t \in \mathbb{R}\}. \tag{3.6.1}$$

Define the circle

$$\mathbb{T}(-\tfrac{1}{2}, \tfrac{1}{2}) = \{z \in \mathbb{C} \mid |z + \tfrac{1}{2}| = \tfrac{1}{2}\},$$

and write $\mathbb{D}(-\frac{1}{2}, \frac{1}{2})$ for the open disc formed by the interior of $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ and $\mathbb{D}^c(-\frac{1}{2}, \frac{1}{2})$ for the open exterior of $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$.

For $\alpha \notin \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ the curve $\omega(\mathbb{T})$ is equal to the parabola in $\mathbb{C}$ given by

$$\begin{aligned}\omega(\mathbb{T}) &= \{-(\alpha+1)(x(y) + iy) \mid y \in \mathbb{R}\}, \qquad \text{where} \\ x(y) &= \frac{|\alpha+1|^4}{(|\alpha|^2 + \text{Re}(\alpha))^2} y^2 + \frac{(\text{Re}(\alpha)+1)|\alpha+1|^2 \text{Im}(\alpha)}{(|\alpha|^2 + \text{Re}(\alpha))^2} y + \frac{|\alpha|^2(1-|\alpha|^2)}{(|\alpha|^2 + \text{Re}(\alpha))^2},\end{aligned}$$

while for $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ the curve $\omega(\mathbb{T})$ becomes the half-line given by

$$\omega(\mathbb{T}) = \left\{-(\alpha+1)r - \frac{(\alpha+1)(1+2\overline{\alpha})}{4(1-|\alpha|^2)} \mid r \geq 0\right\}.$$

As ω is strictly proper, we have $\sigma_{\text{r}}(T_\omega) = \emptyset$. For the remaining parts of the spectrum we consider three cases.

(i) For $\alpha \in \mathbb{D}(-\frac{1}{2}, \frac{1}{2})$ the points $-\frac{1}{2}$ and 0 are separated by the parabola $\omega(\mathbb{T})$ and the connected component of $\mathbb{C}\backslash\omega(\mathbb{T})$ that contains $-\frac{1}{2}$ is equal to $\rho(T_\omega)$, while the connected component that contains 0 is equal to $\sigma_{\text{p}}(T_\omega)$. Finally, $\sigma_{\text{ess}}(T_\omega) = \omega(\mathbb{T}) = \sigma_{\text{c}}(T_\omega)$.

(ii) For $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ we have

$$\rho(T_\omega) = \emptyset, \quad \sigma_{\text{c}}(T_\omega) = \omega(\mathbb{T}) = \sigma_{\text{ess}}(T_\omega), \quad \sigma_{\text{p}}(T_\omega) = \mathbb{C}\backslash\omega(\mathbb{T}),$$

and for each $\lambda \in \omega(\mathbb{T})$, $\lambda - \omega$ has two zeroes on $\mathbb{T}$.

(iii) For $\alpha \in \mathbb{D}^c(-\frac{1}{2}, \frac{1}{2})$ we have $\sigma_{\text{p}}(T_\omega) = \mathbb{C}$, and hence $\rho(T_\omega) = \sigma_{\text{c}}(T_\omega) = \emptyset$.

The proof of these statements will be separated into three steps.

Step 1. We first determine the formula of $\omega(\mathbb{T})$ and show this is a parabola. Note that

$$\omega(z) = \frac{z+\alpha}{(z-1)^2} = \frac{z-1}{(z-1)^2} + \frac{1+\alpha}{(z-1)^2} = \frac{1}{z-1} + (\alpha+1)\frac{1}{(z-1)^2}.$$

Let $|z| = 1$. Then $\frac{1}{z-1}$ is of the form $it - \frac{1}{2}$ with $t \in \mathbb{R}$. So $\omega(\mathbb{T})$ is the curve

$$\omega(\mathbb{T}) = \{(it - \tfrac{1}{2}) + (\alpha+1)(it - \tfrac{1}{2})^2 \mid t \in \mathbb{R}\}.$$

Thus (3.6.1) holds. Now observe that

$$\begin{aligned}
&(it - \tfrac{1}{2}) + (\alpha+1)(it - \tfrac{1}{2})^2 = \\
&\quad = -t^2(\alpha+1) + t(i - (\alpha+1)i) + (-\tfrac{1}{2} + \tfrac{1}{4}(\alpha+1)) \\
&\quad = -t^2(\alpha+1) + (-\alpha i)t + (-\tfrac{1}{4} + \tfrac{1}{4}\alpha) \\
&\quad = -(\alpha+1)\left(t^2 + t\frac{\alpha i}{\alpha+1} - \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)\right).
\end{aligned}$$

The pre-factor $-(1+\alpha)$ acts as a rotation combined with a real scalar multiplication, so $\omega(\mathbb{T})$ is also given by

$$\omega(\mathbb{T}) = -(\alpha+1)\left\{t^2 + t\left(\frac{\alpha i}{\alpha+1}\right) - \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right) \mid t \in \mathbb{R}\right\}. \tag{3.6.2}$$

Thus if the above curve is a parabola, so is $\omega(\mathbb{T})$. Write

$$\begin{aligned}
x(t) &= \mathrm{Re}\left(t^2 + t\frac{\alpha i}{1+\alpha} - \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)\right), \\
y(t) &= \mathrm{Im}\left(t^2 + t\frac{\alpha i}{1+\alpha} - \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)\right).
\end{aligned}$$

Since

$$\frac{\alpha i}{\alpha+1} = \frac{-\mathrm{Im}(\alpha) + i(|\alpha|^2 + \mathrm{Re}(\alpha))}{|\alpha+1|^2} \quad \text{and} \quad \frac{\alpha-1}{\alpha+1} = \frac{(|\alpha|^2-1) + 2i\mathrm{Im}(\alpha)}{|\alpha+1|^2}$$

we obtain that

$$x(t) = t^2 - \frac{\mathrm{Im}(\alpha)}{|\alpha+1|^2}t - \frac{|\alpha|^2-1}{4|\alpha+1|^2}, \quad y(t) = \frac{|\alpha|^2 + \mathrm{Re}(\alpha)}{|\alpha+1|^2}t - \frac{\mathrm{Im}(\alpha)}{2|\alpha+1|^2}.$$

Note that $|\alpha + \frac{1}{2}|^2 = |\alpha|^2 + \mathrm{Re}(\alpha) + \frac{1}{4}$. Therefore, we have $|\alpha|^2 + \mathrm{Re}(\alpha) = 0$ if and only if $|\alpha + \frac{1}{2}| = \frac{1}{2}$. Thus $|\alpha|^2 + \mathrm{Re}(\alpha) = 0$ holds if and only if α is on the circle $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$.

In case $\alpha \notin \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$, i.e., $|\alpha|^2 + \mathrm{Re}(\alpha) \neq 0$, we can express t in terms of y, and feed this into the formula for $x(t)$. One can then compute that

$$x(y) = \frac{|\alpha+1|^4}{(|\alpha|^2 + \mathrm{Re}(\alpha))^2}y^2 + \frac{(\mathrm{Re}(\alpha)+1)|\alpha+1|^2\mathrm{Im}(\alpha)}{(|\alpha|^2 + \mathrm{Re}(\alpha))^2}y + \frac{|\alpha|^2(1-|\alpha|^2)}{(|\alpha|^2 + \mathrm{Re}(\alpha))^2}.$$

Inserting this formula into (3.6.2), we obtain the formula for $\omega(\mathbb{T})$ for the case where $\alpha \notin \mathbb{T}(-\frac{1}{2},\frac{1}{2})$.

In case $\alpha \in \mathbb{T}(-\frac{1}{2},\frac{1}{2})$, i.e., $|\alpha|^2+\mathrm{Re}(\alpha)=0$, we have

$$|\alpha+1|^2 = 1-|\alpha|^2 = 1+\mathrm{Re}(\alpha), \quad \mathrm{Im}(\alpha)^2 = |\alpha|^2(1-|\alpha|^2)$$

and using these identities one can compute that

$$y(t) = \frac{-2\mathrm{Im}(\alpha)}{4(1-|\alpha|^2)} \quad \text{and} \quad x(t) = \left(t - \frac{\mathrm{Im}(\alpha)}{2(1-|\alpha|^2)}\right)^2 + \frac{1+2\mathrm{Re}(\alpha)}{4(1-|\alpha|^2)}.$$

Thus $\{x(t)+iy(t) \mid t \in \mathbb{R}\}$ determines a half-line in $\mathbb{C}$, parallel to the real axis and starting in $\frac{1+2\overline{\alpha}}{4(1-|\alpha|^2)}$ and moving in positive direction. It follows that $\omega(\mathbb{T})$ is the half-line

$$\omega(\mathbb{T}) = \left\{-(\alpha+1)r - \frac{(\alpha+1)(1+2\overline{\alpha})}{4(1-|\alpha|^2)} \mid r \geq 0\right\},$$

as claimed.

Step 2. Next we determine the various parts of the spectrum in $\mathbb{C}\backslash\omega(\mathbb{T})$. Since ω is strictly proper, Theorem 3.3.1 applies, and we know $\sigma_{\mathrm{r}}(T_\omega)=\emptyset$ and $\sigma_{\mathrm{p}} = \omega(\mathbb{C}\backslash\overline{\mathbb{D}}) \cup \{0\}$.

For $k=1$, the polynomial $p_{\lambda,\alpha}(z) = p_{\lambda,\alpha,1}(z) = \lambda z^2 - (1+2\lambda)z + \lambda - \alpha$ has roots

$$\frac{(1+2\lambda) \pm \sqrt{1+4\lambda(1+\alpha)}}{2\lambda}.$$

We consider three cases, depending on whether α is inside, on or outside the circle $\mathbb{T}(-\frac{1}{2},\frac{1}{2})$.

Assume $\alpha \in \mathbb{D}(-\frac{1}{2},\frac{1}{2})$. Then $\omega(\mathbb{T})$ is a parabola in $\mathbb{C}$. For $\lambda = -\frac{1}{2}$ we find that $\lambda-\omega$ has zeroes $\pm i\sqrt{1+2\alpha}$, which are both inside $\mathbb{D}$, because of our assumption. Thus $-\frac{1}{2} \in \rho(T_\omega)$, so that $\rho(T_\omega) \neq \emptyset$. Therefore the connected component of $\mathbb{C}\backslash\omega(\mathbb{T})$ that contains $-\frac{1}{2}$ is contained in $\rho(T_\omega)$, which must also contain $\omega(\mathbb{D})$. Note that $0 \in \omega(\mathbb{T})$ if and only if $|\alpha|=1$. However, there is no intersection of the disc $\alpha \in \mathbb{D}(-\frac{1}{2},\frac{1}{2})$ and the unit circle $\mathbb{T}$. Thus 0 is in $\sigma_{\mathrm{p}}(T_\omega)$, but not on $\omega(\mathbb{T})$. Hence 0 is contained in the connected component of $\mathbb{C}\backslash\omega(\mathbb{T})$ that does not contain $-\frac{1}{2}$. This implies that the connected component containing 0 is included in $\sigma_{\mathrm{p}}(T_\omega)$. This proves our claims for the case $\alpha \in \mathbb{D}(-\frac{1}{2},\frac{1}{2})$.

Now assume $\alpha \in \mathbb{T}(-\frac{1}{2},\frac{1}{2})$. Then $\omega(\mathbb{T})$ is a half-line, and thus $\mathbb{C}\backslash\omega(\mathbb{T})$ consists of one connected component. Note that the intersection of the disc determined by $|\alpha+\frac{1}{2}|<\frac{1}{2}$ and the unit circle consists of -1 only. But $\alpha \neq -1$, so it again follows that $0 \notin \omega(\mathbb{T})$. Therefore the $\mathbb{C}\backslash\omega(\mathbb{T}) = \sigma_{\mathrm{p}}(T_\omega)$. Moreover, the reasoning in the previous case shows that $\lambda = -\frac{1}{2}$ is in $\sigma_{\mathrm{c}}(T_\omega)$ since both zeroes of $-\frac{1}{2}-\omega$ are on $\mathbb{T}$.

Finally, consider the case where α is in the exterior of $\mathbb{T}(-\frac{1}{2},\frac{1}{2})$, i.e., $|\alpha+\frac{1}{2}|>\frac{1}{2}$. In this case, $|\alpha|=1$ is possible, so that $0 \in \sigma_{\mathrm{p}}(T_\omega)$ could be on $\omega(\mathbb{T})$. We show that $\alpha = \omega(0) \in \omega(\mathbb{D})$ is in $\sigma_{\mathrm{p}}(T_\omega)$. If $\alpha=0$, this is clearly the case. So assume $\alpha \neq 0$. The zeroes of $\alpha-\omega$ are then equal to 0 and $\frac{1+2\alpha}{\alpha}$. Note that $|\frac{1+2\alpha}{\alpha}| > 1$ if and only if $|1+2\alpha|^2 - |\alpha|^2 > 0$. Moreover, we have

$$|1+2\alpha|^2 - |\alpha|^2 = 3|\alpha|^2 + 4\mathrm{Re}(\alpha) + 1 = 3|\alpha + \tfrac{2}{3}|^2 - \tfrac{1}{3}.$$

Thus, the second zero of $\alpha - \omega$ is outside $\overline{\mathbb{D}}$ if and only if $|\alpha + \frac{2}{3}|^2 > \frac{1}{9}$. Since the disc indicated by $|\alpha + \frac{2}{3}| \leq \frac{1}{3}$ is contained in the interior of $\mathbb{T}(-\frac{1}{2}, \frac{1}{2})$, it follows that for α satisfying $|\alpha + \frac{1}{2}| > \frac{1}{2}$ one zero of $\alpha - \omega$ is outside $\overline{\mathbb{D}}$, and thus $\omega(0) = \alpha \in \sigma_{\mathrm{p}}(T_\omega)$. Note that

$$\mathbb{C} = \omega(\mathbb{C}) = \omega(\mathbb{D}) \cup \omega(\mathbb{T}) \cup \omega(\mathbb{C}\backslash\overline{\mathbb{D}}),$$

and that $\omega(\mathbb{D})$ and $\omega(\mathbb{C}\backslash\overline{\mathbb{D}})$ are connected components, both contained in $\sigma_{\mathrm{p}}(T_\omega)$. This shows that $\mathbb{C}\backslash\omega(\mathbb{T})$ is contained in $\sigma_{\mathrm{p}}(T_\omega)$.

Step 3. In the final part we prove the claim regarding the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T})$. Let $\lambda \in \omega(\mathbb{T})$ and write z_1 and z_2 for the zeroes of $\lambda - \omega$. One of the zeroes must be on $\mathbb{T}$, say $|z_1| = 1$. Then $\lambda \in \sigma_{\mathrm{p}}(\mathbb{T})$ if and only if $|z_1 z_2| = |z_2| > 1$. From the form of $p_{\lambda,\alpha}$ determined above we obtain that

$$\lambda z^2 - (1+2\lambda)z + \lambda - \alpha = \lambda(z - z_1)(z - z_2).$$

Determining the constant term on the right hand sides shows that $\lambda z_1 z_2 = \lambda - \alpha$. Thus

$$|z_2| = |z_1 z_2| = \frac{|\lambda - \alpha|}{|\lambda|}.$$

This shows that $\lambda \in \sigma_{\mathrm{p}}(T_\omega)$ if and only if $|\lambda - \alpha| > |\lambda|$, i.e., λ is in the half-plane containing zero determined by the line through $\frac{1}{2}\alpha$ perpendicular to the line segment from zero to α.

Consider the line given by $|\lambda - \alpha| = |\lambda|$ and the parabola $\omega(\mathbb{T})$, which is a half-line in case $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$. We show that $\omega(\mathbb{T})$ and the line intersect only for $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$, and that in the latter case $\omega(\mathbb{T})$ is contained in the line. Hence for each value of $\alpha \neq -1$, the essential spectrum consists of either point spectrum or continuous spectrum, and for $\alpha \in \mathbb{T}(-\frac{1}{2}, \frac{1}{2})$ both zeroes of $\lambda - \omega$ are on $\mathbb{T}$, so that $\omega(\mathbb{T})$ is contained in $\sigma_{\mathrm{c}}(T_\omega)$.

As observed in (3.6.1), the parabola $\omega(\mathbb{T})$ is given by the parametrization $(it - \frac{1}{2})^2(\alpha + 1) + (it - \frac{1}{2})$ with $t \in \mathbb{R}$, while the line is given by the parametrization $\frac{1}{2}\alpha + si\alpha$ with $s \in \mathbb{R}$. Fix a $t \in \mathbb{R}$ and assume the point on $\omega(\mathbb{T})$ parameterized by t intersects with the line, i.e., assume there exists a $s \in \mathbb{R}$ such that:

$$(it - \tfrac{1}{2})^2(\alpha + 1) + (it - \tfrac{1}{2}) = \tfrac{1}{2}\alpha + si\alpha,$$

Thus

$$(-t^2 - it + \tfrac{1}{4})(\alpha + 1) + (it - \tfrac{1}{2}) = \tfrac{1}{2}\alpha + si\alpha,$$

and rewrite this as

$$i(-t(\alpha + 1) + t - \alpha s) + ((-t^2 + \tfrac{1}{4})(\alpha + 1) - \tfrac{1}{2} - \tfrac{1}{2}\alpha) = 0,$$

which yields

$$-\alpha i(t+s)+(\alpha+1)(-t^2-\tfrac{1}{4})=0.$$

Since $t^2+\frac{1}{4}>0$, this certainly cannot happen in case $\alpha=0$. So assume $\alpha\neq 0$. Multiply both sides by $-\overline{\alpha}$ to arrive at

$$|\alpha|^2 i(t+s)+(|\alpha|^2+\overline{\alpha})(t^2+\tfrac{1}{4})=0.$$

Separate the real and imaginary part to arrive at

$$(|\alpha|^2+\mathrm{Re}(\alpha))(t^2+\tfrac{1}{4})+i(|\alpha|^2(t+s)-(t^2+\tfrac{1}{4})\mathrm{Im}(\alpha))=0.$$

Thus

$$(|\alpha|^2+\mathrm{Re}(\alpha))(t^2+\tfrac{1}{4})=0 \quad\text{and}\quad |\alpha|^2(t+s)=(t^2+\tfrac{1}{4})\mathrm{Im}(\alpha).$$

Since $t^2+\frac{1}{4}>0$, the first identity yields $|\alpha|^2+\mathrm{Re}(\alpha)=0$, which happens precisely when $\alpha\in\mathbb{T}(-\frac{1}{2},\frac{1}{2})$. Thus there cannot be an intersection when $\alpha\notin\mathbb{T}(-\frac{1}{2},\frac{1}{2})$. On the other hand, for $\alpha\in\mathbb{T}(-\frac{1}{2},\frac{1}{2})$ the first identity always holds, while there always exists an $s\in\mathbb{R}$ that satisfies the second equation. Thus, in that case, for any $t\in\mathbb{R}$, the point on $\omega(\mathbb{T})$ parameterized by t intersects the line, and thus $\omega(\mathbb{T})$ must be contained in the line.

We conclude by showing that $\omega(\mathbb{T})\subset\sigma_{\mathrm{p}}(T_\omega)$ when $|\alpha+\frac{1}{2}|>\frac{1}{2}$ and that $\omega(\mathbb{T})\subset\sigma_{\mathrm{c}}(T_\omega)$ when $|\alpha+\frac{1}{2}|<\frac{1}{2}$. Recall that the two cases correspond to $|\alpha|^2+\mathrm{Re}(\alpha)>0$ and $|\alpha|^2+\mathrm{Re}(\alpha)<0$, respectively. To show that this is the case, we take the point on the parabola parameterized by $t=0$, i.e., take $\lambda=\frac{1}{4}(\alpha+1)-\frac{1}{2}=\frac{1}{4}(\alpha-1)$. Then $\lambda-\alpha=-\frac{1}{4}(3\alpha+1)$. So

$$|\lambda-\alpha|^2=\tfrac{1}{16}(9|\alpha|^2+6\mathrm{Re}(\alpha)+1) \quad\text{and}\quad |\lambda|^2=\tfrac{1}{16}(|\alpha^2|-2\mathrm{Re}(\alpha)+1).$$

It follows that $|\lambda-\alpha|>|\lambda|$ if and only if

$$\tfrac{1}{16}(9|\alpha|^2+6\mathrm{Re}(\alpha)+1)>|\lambda|^2=\tfrac{1}{16}(|\alpha^2|-2\mathrm{Re}(\alpha)+1),$$

or equivalently,

$$8(|\alpha|^2+\mathrm{Re}(\alpha))>0.$$

This proves our claim for the case $|\lambda+\frac{1}{2}|>\frac{1}{2}$. The other claim follows by reversing the directions in the above inequalities.

Figure 3.5 presents some illustrations of the possible situations.

□

The case $k=2$ can be dealt with using the same techniques, and very similar results are obtained in that case.

The next examples deal with other cases of ω_k, now with $k>2$.

Example 3.6.2. Let $\omega=\frac{z^3+\alpha}{(z-1)^2}$. Then

$$\omega(z)=\frac{z^3+\alpha}{(z-1)^2}=(z-1)+3+\frac{3}{z-1}+\frac{1+\alpha}{(z-1)^2}.$$

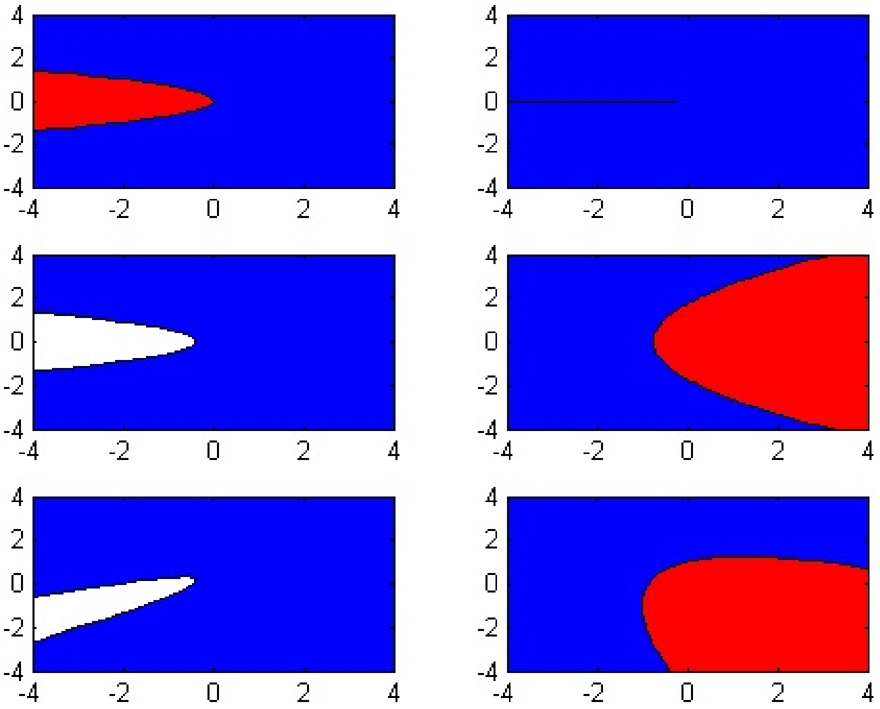

Figure 3.5: Spectrum of T_ω, where $\omega(z) = \frac{z+\alpha}{(z-1)^2}$ for some values of α, with $\alpha = 1$, and $\alpha = 0$ (top row left and right), $\alpha = 1/2$ and $\alpha = -2$ (middle row left and right), $\alpha = -\frac{1}{2} + \frac{1}{4}i$ and $\alpha = -2 + i$ (bottom row).

For $z \in \mathbb{T}$, $\frac{1}{z-1}$ has the form $-\frac{1}{2} + ti, t \in \mathbb{R}$ and so $\omega(\mathbb{T})$ has the form

$$\omega(\mathbb{T}) = \left\{ \frac{1}{-\frac{1}{2} + ti} + 3 + 3(-\frac{1}{2} + ti) + (1+\alpha)\left(-\frac{1}{2} + ti\right)^2, | \ t \in \mathbb{R} \right\}.$$

Also $\lambda - \omega(z) = \frac{\lambda(z-1)^2 - z^3 - \alpha}{(z-1)^2}$ and so for invertibility we need the polynomial $p_{\lambda,\alpha}(z) = \lambda(z-1)^2 - z^3 - \alpha$ to have exactly two roots in $\mathbb{D}$. Since this is a polynomial of degree 3 the number of roots inside $\mathbb{D}$ can be zero, one, two or three, and the index of $\lambda - T_\omega$ correspondingly can be two, one, zero or minus one. Examples are given in Figure 3.6.

Example 3.6.3. To get some idea of possible other configurations we present some examples with other values of k.

For $\omega(z) = \frac{z^4}{(z-1)^2}$ (so $k = 4$ and $\alpha = 0$) the essential spectrum of T_ω is the curve in Figure 3.7, the white region is the resolvent set, and color coding for the Fredholm index is as earlier in the chapter. For $\omega(z) = \frac{z^6+1.7}{(z-1)^2}$ (so $k = 6$ and $\alpha = 1.7$) see Figure 3.8, and as a final example Figure 3.9 presents the essential spectrum and spectrum for $\omega(z) = \frac{z^7+1.1}{(z-1)^2}$ and $\omega(z) = \frac{z^7+0.8}{(z-1)^2}$. In the latter figure color coding is as follows: the Fredholm index is -3 in the yellow region, -4 in the green region and -5 in the black region.

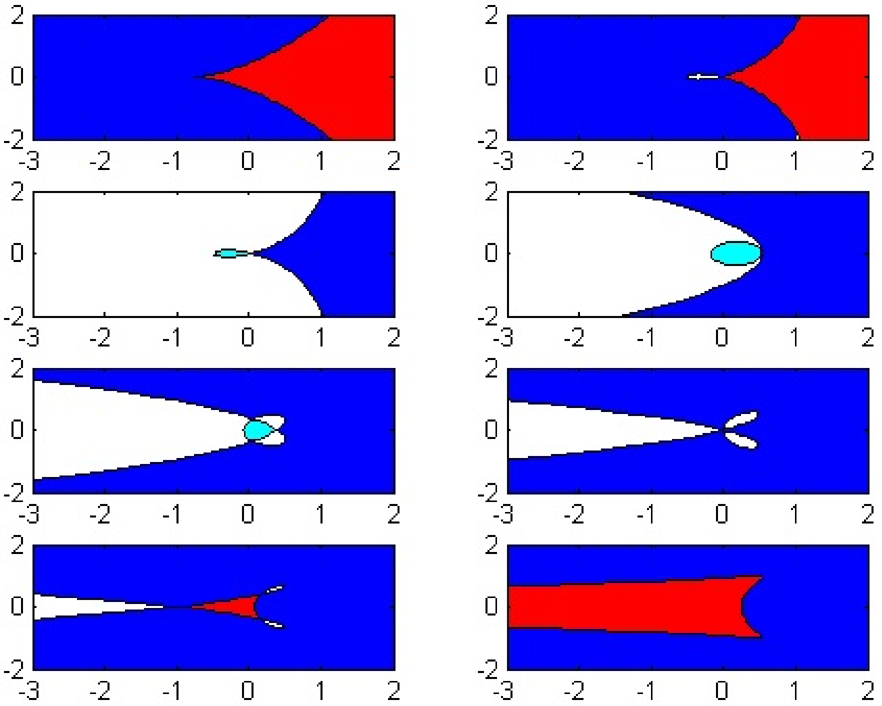

Figure 3.6: Spectrum of T_ω where $\omega(z) = \frac{z^3+\alpha}{(z-1)^2}$ for several values of α, with α being (left to right and top to bottom) respectively, $-2, -1.05, -0.95, 0.3, 0.7, 1, 1.3, 2$.

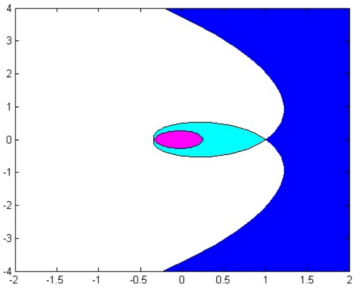

Figure 3.7: The spectrum of T_ω, with $k = 4$ and $\alpha = 0$.

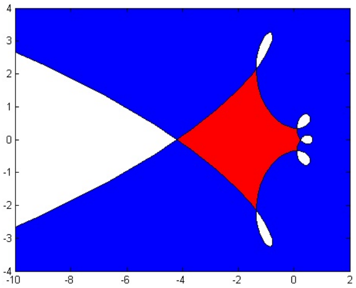

Figure 3.8: The spectrum of T_ω with $k = 6$ and $\alpha = 1.7$.

Figure 3.9: The spectrum of T_ω for $k = 7$ and $\alpha = 1.1$ (left) and $k = 7$, $\alpha = 0.8$ (right)

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Chapter 4

The adjoint

Abstract. This chapter contains a further analysis of the Toeplitz-like operators T_ω on H^p with rational symbol ω having poles on the unit circle that were previously studied in [5, 6]. Here the adjoint operator T_ω^* is described. In the case where $p = 2$ and ω has poles only on the unit circle $\mathbb{T}$, a description is given for when T_ω^* is symmetric and when T_ω^* admits a selfadjoint extension. If in addition ω is proper, it is shown that T_ω^* coincides with the unbounded Toeplitz operator defined by Sarason in [12] and studied further by Rosenfeld in [10, 11]. [1]

Mathematics Subject Classification (2010). Primary 47B35, 47A53; Secondary 47A68.
Keywords. Toeplitz operators, unbounded operators, adjoint, symmetric operators.

4.1 Introduction

In this chapter we proceed with our study of unbounded Toeplitz-like operators on H^p with rational symbols that have poles on the unit circle $\mathbb{T}$ which was initiated in Chapter 2.Our previous work on such Toeplitz-like operators focused on their Fredholm properties (in Chapter 2) and the various parts of their spectra (in Chapter 3). Here we determine properties of the adjoint operator and conditions under which the operator is symmetric and when it has a selfadjoint extension.

Before we can define our Toeplitz-like operators, some notation has to be introduced. We write Rat for the space of rational complex functions, $\mathrm{Rat}(\mathbb{T})$ for the subspace of Rat consisting of rational complex functions with poles only on the unit circle $\mathbb{T}$, and $\mathrm{Rat}_0(\mathbb{T})$ for the subspace of strictly proper functions in $\mathrm{Rat}(\mathbb{T})$. Now let $\omega \in \mathrm{Rat}$, possibly with poles on $\mathbb{T}$. As in [5], we define the Toeplitz-like operator T_ω ($H^p \to H^p$), for $1 < p < \infty$, via

$$\mathrm{Dom}(T_\omega) = \{g \in H^p \,|\, \omega g = f + \rho \text{ with } f \in L^p, \rho \in \mathrm{Rat}_0(\mathbb{T})\}, \; T_\omega g = \mathbb{P}f. \tag{4.1.1}$$

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[1] Up to some minor modifications, this chapter has been published as
G.J. Groenewald, S. ter Horst, J. Jaftha, and A.C.M. Ran, A Toeplitz-like operator with rational symbol having poles on the unit circle II: the adjoint, *Integr. Equ. Oper. Theory* **91** (2019), no. 43, https://doi.org/10.1007/s00020-019-2542-2.

Here $\mathbb{P}$ is the Riesz projection of L^p onto H^p. The operator T_ω is densely defined and closed. In case $\omega \in \mathrm{Rat}(\mathbb{T})$, explicit formulas for the domain, kernel, range, and a complement of the range were obtained in Chapter 3, as an extension of a result in Chapter 2 for the case where T_ω is Fredholm. We recall these results in Section 4.2, as they will be frequently used throughout the chapter.

If ω has no poles on $\mathbb{T}$, in fact for any $\omega \in L^\infty$, the adjoint of the Toeplitz operator T_ω on H^p can be identified with the Toeplitz operator T_{ω^*} on $H^{p'}$, with $1 < p' < \infty$ so that $1/p + 1/p' = 1$ and with ω^* defined as $\omega^*(z) = \overline{\omega(z)}$ on $\mathbb{T}$. The identification of $(H^p)'$ and $H^{p'}$ goes via the usual pairing

$$\langle f, g\rangle_{p,p'} = \tfrac{1}{2\pi} \textstyle\int_{\mathbb{T}} \overline{g(z)} f(z)\, dz \quad (f \in H^p, g \in H^{p'}).$$

In the sequel we use the same notation for the similarly defined pairing between L^p and $L^{p'}$ to identify $(L^p)'$ and $L^{p'}$, and in both cases the indices will often be omitted.

For the Toeplitz-like operators studied in this chapter the situation is more complicated than for Toeplitz operators with L^∞ symbols. However, we do obtain that T_ω^* can be identified with the restriction of the Toeplitz-like operator T_{ω^*} on $H^{p'}$ to a dense subspace of its domain. Like for the operator T_ω, in case ω is in $\mathrm{Rat}(\mathbb{T})$ we obtain a more explicit description of T_ω^*, which we present after introducing some further notation.

Throughout the chapter $\mathcal{P}$ denotes the space of complex polynomials and $\mathcal{P}_k$, for any non-negative integer k, denotes the subspace of $\mathcal{P}$ of polynomials of degree at most k. The degree of a polynomial $r \in \mathcal{P}$ is denoted as $\deg(r)$. Given $r \in \mathcal{P}$ with $\deg(r) = k$, say $r(z) = r_0 + zr_1 + \cdots + z^k r_k$, we define the polynomial $r^\sharp$ by

$$r^\sharp(z) = z^k \overline{r(1/\overline{z})} = \overline{r_0} z^k + \overline{r_1} z^{k-1} + \cdots + \overline{r_k}.$$

The following theorem is our first main result.

Theorem 4.1.1. *Let* $\omega = s/q \in \mathrm{Rat}$ *with* $s, q \in \mathcal{P}$ *co-prime and* $1 < p < \infty$. *Factor* $s = s_- s_0 s_+$ *and* $q = q_- q_0 q_+$ *with* s_-, q_- *having roots only inside* $\mathbb{T}$, s_0, q_0 *having roots only on* $\mathbb{T}$, *and* s_+, q_+ *having roots only outside* $\mathbb{T}$. *Set* $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$ $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$ *and let* $1 < p' < \infty$ *with* $1/p + 1/p' = 1$. *Then*

$$\mathrm{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \text{and} \quad T_\omega^* = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}. \tag{4.1.2}$$

Furthermore, we have

$$\begin{aligned} \mathrm{Ran}(T_\omega^*) &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} Q_{n_0+n_- - m_0 - m_-} (s_0)^\sharp H^{p'}, \\ \mathrm{Ker}(T_\omega^*) &= \left\{ \frac{(q_-)^\sharp (q_0)^\sharp r}{(s_-)^\sharp} \mid \deg(r) < n_- - m_- - m_0 \right\}. \end{aligned} \tag{4.1.3}$$

Here $Q_k = I_{H^{p'}} - P_{\mathcal{P}_{k-1}}$, *with* $P_{\mathcal{P}_{k-1}}$ *the standard projection in* $H^{p'}$ *onto* $\mathcal{P}_{k-1} \subset H^{p'}$ *to be interpreted as* 0 *if* $k \leq 0$, *i.e.,* $Q_k = I_{H^{p'}}$ *if* $k \leq 0$. *Thus, for* $n_0 + n_- \leq m_0 + m_-$ *we have* $\mathrm{Ran}(T_\omega^*) = T_{z^{m-n}/(q_+)^\sharp} (s_+ s_0)^\sharp H^{p'}$. *Moreover,*

$$\dim \mathrm{Ker}(T_\omega^*) = \max\left\{0, \#\{\textit{zeroes of } \omega \textit{ inside } \mathbb{D}\} - \#\{\textit{poles of } \omega \textit{ in } \overline{\mathbb{D}}\}\right\},$$

where the multiplicities of the zeroes and poles are taken into account. Hence, $\dim \operatorname{Ker}(T_\omega^*)$ *is the maximum of* 0 *and* $n_- - m_- - m_0$*. In particular,* T_ω^* *is injective if and only if the number of poles of* ω *inside* $\overline{\mathbb{D}}$ *is greater than or equal to the number of zeroes of* ω *inside* $\mathbb{D}$*, multiplicities taken into account.*

Before giving a proof of Theorem 4.1.1 in Section 4.4, we prove the specialization of this result for the case $\omega \in \operatorname{Rat}(\mathbb{T})$ in Section 4.3. For this purpose we first provide a description of T_{ω^*} in Section 4.2.

The injectivity result, but not the description of $\operatorname{Ker}(T_\omega^*)$, can also be derived from general theory and results on T_ω. Indeed, according to Theorem II.3.7 in [4], T_ω^* is injective if and only if T_ω has dense range, so that the claim follows from Proposition 3.2.4. More can be obtained in this way, since H^p, $1 < p < \infty$, is reflexive. By Theorem II.2.14 of [4] it follows that $T_\omega^{**} = T_\omega$, with the usual identifications of the dual spaces. Hence, applying the above to T_ω^* we find that T_ω^* has dense range if and only if T_ω is injective; see also Theorem II.4.10 in [4]. By Banach's Closed Range Theorem, cf., [14], T_ω^* has closed range if and only if T_ω has closed range. Again applying results from Chapter 3 now gives the following result.

Corollary 4.1.2. *Let* $\omega \in \operatorname{Rat}$ *and* $1 < p < \infty$*. Then* T_ω^* *has closed range if and only if* ω *has no zeroes on* $\mathbb{T}$*, or equivalently,* ω^* *has no zeroes on* $\mathbb{T}$*. Moreover,* T_ω^* *has dense range if and only if*

$$\#\begin{Bmatrix}\textit{poles of } \omega \textit{ inside } \overline{\mathbb{D}} \\ \textit{multi. taken into account}\end{Bmatrix} \leq \#\begin{Bmatrix}\textit{zeroes of } \omega \textit{ inside } \overline{\mathbb{D}} \\ \textit{multi. taken into account}\end{Bmatrix}.$$

Beyond Section 4.4, and in the remainder of this introduction, we only consider the case $p = 2$ and $\omega \in \operatorname{Rat}(\mathbb{T})$. By comparing the results on T_ω and T_ω^* it is obvious T_ω cannot be selfadjoint, except when ω has no poles on $\mathbb{T}$. In Section 4.5 we describe in terms of ω when T_ω^* is symmetric, in which case $T_\omega^* \subset T_\omega$, and whenever T_ω^* is symmetric we describe when T_{ω^*} admits a selfadjoint extension. The following theorem collects some of the main results of Section 4.5; it follows directly from Theorem 4.5.1, Corollaries 4.5.2 and 4.5.7, Propositions 4.5.4 and 4.5.9.

Theorem 4.1.3. *Let* $\omega = s/q \in \operatorname{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime. Consider* T_ω *on* H^2*. Then*

$$T_\omega^* \textit{ is symmetric} \quad \Longleftrightarrow \quad \omega(\mathbb{T}) \subset \mathbb{R}.$$

In particular, if T_ω^* *is symmetric, then* $\deg(s) \leq \deg(q) \leq 2\deg(s)$*. Furthermore, if* T_ω^* *is symmetric, then* T_ω^* *admits a selfadjoint extension if and only if the number of roots of* $s - iq$ *and* $s + iq$ *in* $\mathbb{D}$*, counting multiplicities, coincide. This happens in particular if* $\omega(\mathbb{T}) \neq \mathbb{R}$*, but cannot happen in case* $\deg(q)$ *is odd.*

Several other conditions for T_ω^* to be symmetric and/or have a selfadjoint extension are derived in Section 4.5.

In [12] Sarason introduced and studied an unbounded Toeplitz-like operator with symbol in the Smirnov class. In Section 4.6 we show that if $\omega \in \operatorname{Rat}(\mathbb{T})$ is proper, then the adjoint operator T_ω^* is precisely a Toeplitz-like operator of the type studied by Sarason. Hence in this case our Toeplitz-like operator $T_\omega = T_\omega^{**}$ coincides with the adjoint of the Toeplitz-like operator considered in [12]. Based on ideas in [12], we also show that $H(\overline{\mathbb{D}})$, the space of functions analytic on a neighborhood of $\overline{\mathbb{D}}$, is contained in $\mathrm{Dom}(T_\omega)$ and in fact is a core of T_ω.

In the last section of [12], Sarason introduces a class of closed, densely defined Toeplitz-like operators on H^2 determined by algebraic properties, which was further investigated by Rosenfeld in [10, 11]. In particular, this class of Toeplitz-like operators contains the unbounded Toeplitz-like operator studied by Sarason and is closed under taking adjoints, and hence contains our Toeplitz-like operators with proper symbols in $\mathrm{Rat}(\mathbb{T})$. In fact, we will show in Section 4.6 that T_ω is contained in the class of Toeplitz-like operators for any ω in Rat.

4.2 The operator T_{ω^*} for $\omega \in \mathrm{Rat}(\mathbb{T})$

In this section we recall some results from Chapter 2 and Chapter 3 on the operator T_ω for $\omega \in \mathrm{Rat}(\mathbb{T})$ that we will use in the sequel, and apply them to the operator T_{ω^*}. Hence, throughout this section let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime. We set $m = \deg(q)$ and $n = \deg(s)$. Furthermore, factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+ polynomials having roots only inside, on, or outside $\mathbb{T}$, respectively. We then recall from Theorem 3.2.2 that

$$\begin{gathered}\mathrm{Ker}(T_\omega) = \{r/s_+ \mid \deg(r) < m - \deg(s_- s_0)\}\,;\\ \mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{m-1}; \quad \mathrm{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}},\end{gathered} \tag{4.2.1}$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1}\} \subset \mathcal{P}_{n-1}. \tag{4.2.2}$$

Furthermore, $H^p = \overline{\mathrm{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ forms a direct sum decomposition of H^p, where

$$\widetilde{\mathcal{Q}} = \mathcal{P}_{k-1} \quad \text{with} \quad k = \max\{\deg(s_-) - m, 0\}, \tag{4.2.3}$$

using the convention $\mathcal{P}_{-1} := \{0\}$. Furthermore, the action of T_ω is as follows:

$$\begin{aligned}&T_\omega g = sh + \widetilde{r} \quad (g = qh + r \in qH^p + \mathcal{P}_{m-1} = \mathrm{Dom}(T_\omega)),\\ &\text{where } \widetilde{r} \in \mathcal{P}_{n-1} \text{ is such that } rs = \widetilde{r}q + r_2 \text{ for some } r_2 \in \mathcal{P}_{m-1}.\end{aligned}$$

We also recall from Lemma 2.5.3 that

$$T_{z^\kappa \omega} = T_{z^\kappa} T_\omega \ \text{ for any integer } \kappa \leq 0. \tag{4.2.4}$$

Recall that ω^* is defined as $\omega^*(z) = \overline{\omega(z)}$ on $\mathbb{T}$, i.e., $\omega^*(z) = \overline{s(z)}/\overline{q(z)}$. For $z \in \mathbb{T}$

$$\overline{q(z)} = \overline{q_0 + zq_1 + \cdots + z^m q_m} = \overline{q_0} + \overline{q_1}\frac{1}{z} + \cdots + \overline{q_m}\frac{1}{z^m} = \frac{1}{z^m} q^\sharp(z).$$

Hence $q^\sharp(z) = z^m\overline{q(z)}$, and likewise $s^\sharp(z) = z^n\overline{s(z)}$. Thus we have

$$\omega^*(z) = \frac{z^{m-n}s^\sharp(z)}{q^\sharp(z)} \text{ if } m \geq n \quad \text{and} \quad \omega^*(z) = \frac{s^\sharp(z)}{z^{n-m}q^\sharp(z)} \text{ if } m < n. \tag{4.2.5}$$

In fact, the formula $\omega^*(z) = z^{m-n}s^\sharp(z)/q^\sharp(z)$ holds in both cases, but is not always a representation as the ratio of two polynomials. Note in particular that $\omega^* \in \mathrm{Rat}(\mathbb{T})$ in case ω is proper, while this need not be the case if ω is not proper. Thus, if ω is proper, the above formulas apply directly, while for the non-proper case, using (4.2.4) we can reduce certain questions to questions concerning the Toeplitz operator $T_{s^\sharp/q^\sharp}$ with symbol $s^\sharp/q^\sharp$ which is in $\mathrm{Rat}(\mathbb{T})$.

A polynomial $r \neq 0$ is called self-inversive in case $r = \gamma r^\sharp$ for a constant $\gamma \in \mathbb{C}$, which necessarily is unimodular. In fact, γ is the ratio $r_0/\overline{r_n}$ with $r_0 = r(0)$ and r_n the leading coefficient of r. By a theorem of Cohn [2], a polynomial r has all its roots on $\mathbb{T}$ if and only if r is self-inversive and its derivative has all its roots in the closed unit disc $\overline{\mathbb{D}}$. Hence, any polynomial with roots only on $\mathbb{T}$ is self-inversive. In particular, $q = \gamma q^\sharp$ and $s_0 = \rho(s_0)^\sharp$ for unimodular constants γ and ρ.

More generally, in the transformation $r \to r^\sharp$, the nonzero roots of r (including multiplicity) transfer along the unit circle via the map $\alpha \mapsto 1/\overline{\alpha} = |\alpha|^{-2}\alpha$, while the degree decreases by the multiplicity of 0 as a root of r. Consequently, in the factorization $s^\sharp = (s_+)^\sharp(s_0)^\sharp(s_-)^\sharp$, the polynomials $(s_+)^\sharp$, $(s_0)^\sharp$ and $(s_-)^\sharp$ contain the roots of $s^\sharp$ inside, on and outside $\mathbb{T}$, respectively, taking multiplicities into account. We write $(s_+)^\sharp$ rather than $s_+^\sharp$, etc., to avoid confusion with what one may interpret as $(s^\sharp)_+$.

We now apply the above to T_{ω^*} acting on $H^{p'}$, $1 < p' < \infty$, to fit better with the remainder of the chapter.

Proposition 4.2.1. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, $m = \deg(q)$ and $n = \deg(s)$. Factor $s = s_-s_0s_+$ with s_-, s_0 and s_+ polynomials having roots only inside, on, or outside $\mathbb{T}$, respectively. Then for T_{ω^*} on $H^{p'}$, with $1 < p' < \infty$, we have*

$$\mathrm{Ker}(T_{\omega^*}) = \left\{r_0/(s_-)^\sharp \mid \deg(r_0) < \deg(s_-)\right\}, \quad \mathrm{Dom}(T_{\omega^*}) = q^\sharp H^{p'} + \mathcal{P}_{m-1}.$$

Moreover, we have

$$\begin{aligned} \mathrm{Ran}(T_{\omega^*}) &= z^{m-n}s^\sharp H^{p'} + \widetilde{\mathcal{P}}_* \quad \text{if } m \geq n, \\ \mathrm{Ran}(T_{\omega^*}) &= T_{z^{m-n}}(s^\sharp H^{p'} + \widetilde{\mathcal{P}}_*) \quad \text{if } m < n, \end{aligned} \tag{4.2.6}$$

where for $m \geq n$ the subspace $\widetilde{\mathcal{P}}_*$ is given by

$$\widetilde{\mathcal{P}}_* = \{r \in \mathcal{P} \mid rq^\sharp = z^{m-n}r_1s^\sharp + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1}\} \subset \mathcal{P}_{m-n+\deg(s^\sharp)-1},$$

while for $m < n$ we have

$$\widetilde{\mathcal{P}}_* = \{r \in \mathcal{P} \mid rq^\sharp = r_1s^\sharp + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1}\} \subset \mathcal{P}_{\deg(s^\sharp)-1}.$$

Furthermore, $\mathrm{Ran}(T_{\omega^*})$ is dense in $H^{p'}$.

Proof. We separate the cases $m \geq n$ and $m < n$.

For $m \geq n$, we have $\omega^* = \widetilde{s}/\widetilde{q} \in \mathrm{Rat}(\mathbb{T})$ with $\widetilde{s} = z^{m-n}s^\sharp$ and $\widetilde{q} = q^\sharp$. Hence $\widetilde{s}$ factors as $\widetilde{s} = (z^{m-n}(s_+)^\sharp)(s_0)^\sharp(s_-)^\sharp$, where the factors have all their roots inside, on, or outside $\mathbb{T}$, respectively. Also, $\deg(q^\sharp) = \deg(q)$ and $\deg((s_+)^\sharp) = \deg(s_+)$. So the formulas for $\mathrm{Dom}(T_{\omega^*})$ and $\mathrm{Ran}(T_{\omega^*})$ follow directly from (4.2.1), while the formula for $\mathrm{Ker}(T_{\omega^*})$ follows because the bound on the degree of r_0 can be computed as

$$m - \deg(z^{m-n}(s_+)^\sharp(s_0)^\sharp) = n - \deg((s_+)^\sharp(s_0)^\sharp) = n - \deg(s_+s_0) = \deg(s_-).$$

Finally, a complement of the closure of $\mathrm{Ran}(T_{\omega^*})$ is given by $\mathcal{P}_{k-1}$ with k the maximum of 0 and $\deg(z^{m-n}(s_+)^\sharp) - m = \deg((s_+)^\sharp) - n \leq 0$. Hence $\mathcal{P}_{-1} = \{0\}$. Thus T_{ω^*} has dense range, as claimed.

In case $m < n$, we have $T_{\omega^*} = T_{z^{m-n}}T_{s^\sharp/q^\sharp}$ and $s^\sharp/q^\sharp$ is in $\mathrm{Rat}(\mathbb{T})$. Applying the above results for T_ω to $T_{s^\sharp/q^\sharp}$ directly gives the formulas for $\mathrm{Dom}(T_{\omega^*})$ and $\mathrm{Ran}(T_{\omega^*})$.

To see that the formula for $\mathrm{Ker}(T_{\omega^*})$ holds, we follow the argumentation of the proof of Lemma 2.4.1. For $g \in \mathrm{Dom}(T_{\omega^*}) = \mathrm{Dom}(T_{s^\sharp/q^\sharp})$ to be in $\mathrm{Ker}(T_{\omega^*})$ is equivalent to $T_{s^\sharp/q^\sharp}g \in \mathcal{P}_{n-m-1}$. In other words, by Lemma 2.3.2, to $s^\sharp g = q^\sharp\widetilde{r} + r_1$ with $r_1 \in \mathcal{P}_{m-1}$ and $\widetilde{r} \in \mathcal{P}_{n-m-1}$, since then $T_{s^\sharp/q^\sharp}g = \widetilde{r}$. The latter happens precisely when $g = r/(s_-)^\sharp$ with $r \in \mathcal{P}_{\deg(s_-)-1}$. Indeed, in that case $\deg((s_+)^\sharp(s_0)^\sharp r) < n$ which in the equation $(s_+)^\sharp(s_0)^\sharp r = s^\sharp g = q^\sharp\widetilde{r} + r_1$ corresponds to $\deg(\widetilde{r}) < m-1$, as required. Finally, we note that a complement of $\overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})}$ in $H^{p'}$ is given by $\mathcal{P}_{k-1}$ with $k = \max\{0, \deg {s_+}^\sharp - m\} \leq n-m$. Let $f \in H^{p'}$ and write $z^{n-m}f = h + r \in \overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})} + \mathcal{P}_{k-1}$. Then $f = T_{z^{m-n}}z^{n-m}f = T_{z^{m-n}}(h+r) = T_{z^{m-n}}h \in T_{z^{m-n}}\overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})} \subset \overline{\mathrm{Ran}(T_{z^{m-n}}T_{s^\sharp/q^\sharp})} = \overline{\mathrm{Ran}(T_{\omega^*})}$. Thus also in this case $\mathrm{Ran}(T_{\omega^*})$ is dense in $H^{p'}$. □

We conclude this section with a lemma which will be of use in the sequel.

Lemma 4.2.2. *Let* $r_1, r_2 \in \mathcal{P}$. *Set* $n_i = \deg(r_i)$, $i = 1, 2$, *and* $n = \deg(r_1 + r_2)$. *Then*

$$(r_1 + r_2)^\sharp = z^{n-n_1}r_1^\sharp + z^{n-n_2}r_2^\sharp.$$

In case $n < \max\{n_1, n_2\}$, *then* $n_1 = n_2$ *and* 0 *is a root of* $r_1^\sharp + r_2^\sharp$ *with multiplicity* $n - n_1$, *so that the left hand side in the above identity still is a polynomial without a root at* 0.

Proof. By definition, for $z \in \mathbb{T}$ we have

$$\begin{aligned}(r_1 + r_2)^\sharp(z) &= z^n(\overline{r_1(1/\overline{z})} + \overline{r_2(1/\overline{z})}) = \\ &= z^{n-n_1}z^{n_1}\overline{r_1(1/\overline{z})} + z^{n-n_2}z^{n_2}\overline{r_2(1/\overline{z})} = z^{n-n_1}r_1^\sharp(z) + z^{n-n_2}r_2^\sharp(z).\end{aligned}$$

□

4.3 The adjoint of T_ω for $\omega \in \mathrm{Rat}(\mathbb{T})$

In this section we prove the first main result, Theorem 4.1.1, for the special case that $\omega \in \mathrm{Rat}(\mathbb{T})$. In this case, the result specializes to the following theorem, which we prove in this section.

Theorem 4.3.1. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime and* $1 < p < \infty$*. Set* $m = \deg(q)$*,* $n = \deg(s)$ *and let* $1 < p' < \infty$ *with* $1/p + 1/p' = 1$*. Then*

$$\mathrm{Dom}(T_\omega^*) = q^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^* = T_{\omega^*}|_{q^\sharp H^{p'}}. \tag{4.3.1}$$

In fact, for $g = q^\sharp v \in q^\sharp H^{p'}$ *we have* $T_\omega^* g = T_{z^{m-n}s^\sharp} v$*. Moreover, factorize* $s = s_- s_0 s_+$ *with* s_-*,* s_0 *and* s_+ *polynomials having roots only inside, on, or outside* $\mathbb{T}$*, respectively. Then*

$$\begin{aligned} \mathrm{Ran}(T_\omega^*) &= T_{z^{m-n}s^\sharp} H^{p'}, \\ \mathrm{Ker}(T_\omega^*) &= \left\{ \frac{q^\sharp r}{(s_-)^\sharp} \mid \deg(r) < \deg(s_-) - m \right\}. \end{aligned} \tag{4.3.2}$$

In particular, we have

$$\dim \mathrm{Ker}(T_\omega^*) = \max\{0, \#\{\textit{zeroes of } \omega^* \textit{ outside } \mathbb{T}\} - \#\{\textit{poles of } \omega^* \textit{ on } \mathbb{T}\}\},$$

where the multiplicities of the zeroes and poles are taken into account. Thus T_ω^* *is injective if and only if* ω *has at least as many poles inside* $\mathbb{T}$ *as zeroes inside* $\mathbb{T}$ *unequal to* 0*, multiplicities taken into account.*

We first present some auxiliary lemmas. Throughout, let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$. We will consider T_ω as an operator with domain in H^p and T_{ω^*} as an operator with domain in $H^{p'}$.

Lemma 4.3.2. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime,* $m = \deg(q)$ *and* $n = \deg(s)$*. Then*

$$q^\sharp H^{p'} \subset \mathrm{Dom}(T_\omega^*) \cap \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^*|_{q^\sharp H^{p'}} = T_{\omega^*}|_{q^\sharp H^{p'}}.$$

Moreover, for $g = q^\sharp v \in q^\sharp H^{p'}$*, with* $v \in H^{p'}$*, we have* $T_\omega^* g = T_{z^{m-n}s^\sharp} v$*, and thus* $T_\omega^*(q^\sharp H^{p'}) = T_{z^{m-n}s^\sharp} H^{p'}$*.*

Proof. The inclusion $q^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*})$ follows from Proposition 4.2.1. Let g be in $q^\sharp H^{p'}$, say $g(z) = q^\sharp(z) v(z)$ for $v \in H^{p'}$. We show that for $f \in \mathrm{Dom}(T_\omega)$ we have $\langle T_w f, g \rangle_{p,p'} = \langle f, T_{\omega^*} g \rangle_{p,p'}$. Let $f \in \mathrm{Dom}(T_\omega)$ and $h = T_\omega f \in H^p$, i.e., $sf = qh + r$ for some $r \in \mathcal{P}_{m-1}$, by [5, Lemma 2.3]. Then

$$\begin{aligned} \langle T_\omega f, g \rangle_{p,p'} &= \langle h, q^\sharp v \rangle_{p,p'} = \langle h, z^m \overline{q} v \rangle_{p,p'} = \langle qh, z^m v \rangle_{p,p'} = \langle sf - r, z^m v \rangle_{p,p'} \\ &= \langle sf, z^m v \rangle_{p,p'} \quad (\text{because } \deg(r) < m,\ v \in H^{p'}) \\ &= \langle f, z^m \overline{s} v \rangle_{p,p'} = \langle f, z^{m-n} s^\sharp v \rangle_{p,p'} = \langle f, T_{z^{m-n}s^\sharp} v \rangle_{p,p'} \quad (\text{because } f \in H^p). \end{aligned}$$

It remains to show that $T_{\omega^*} g = T_{z^{m-n}s^\sharp} v$. If $m \geq n$, then $\omega^* = z^{m-n} s^\sharp / q^\sharp$ is in $\mathrm{Rat}(\mathbb{T})$ and $\omega^* g = z^{m-n} s^\sharp v \in H^{p'}$, so that, $T_{\omega^*} g = z^{m-n} s^\sharp v = T_{z^{m-n}s^\sharp} v$, by Lemma 2.3 in [5]. If $m < n$, we have $T_{\omega^*} g = T_{z^{m-n}} T_{s^\sharp / q^\sharp} g = T_{z^{m-n}s^\sharp} v$. □

Lemma 4.3.3. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ *with* $s, q \in \mathcal{P}$ *co-prime,* $m = \deg(q)$ *and* $n = \deg(s)$. *Let* $g \in \mathrm{Dom}(T_\omega^*)$ *and* $k = T_\omega^* g \in H^{p'}$. *Then for any* $r \in \mathcal{P}_{n-1}$ *and* $r_1 \in \mathcal{P}_{m-1}$ *so that*

$$sr_1 = qr + r_2 \text{ for some } r_2 \in \mathcal{P}_{m-1} \tag{4.3.3}$$

we have

$$\langle r_1, k\rangle_{p,p'} = \langle r, g\rangle_{p,p'}.$$

Moreover, we have

$$z^{m-n}s^\sharp g - q^\sharp k \in \mathcal{P}_{m-1} \text{ if } m \geq n \text{ and } s^\sharp g - z^{n-m}q^\sharp k \in \mathcal{P}_{n-1} \text{ if } m < n. \tag{4.3.4}$$

In particular, $\mathrm{Dom}(T_\omega^*) \subset \mathrm{Dom}(T_{\omega^*})$ *and* $T_\omega^* = T_{\omega^*}|_{\mathrm{Dom}(T_\omega^*)}$.

Proof. Let $g \in \mathrm{Dom}(T_\omega^*)$ and $k = T_\omega^* g$. Hence $\langle T_\omega f, g\rangle_{p,p'} = \langle f, k\rangle_{p,p'}$ for each $f \in \mathrm{Dom}(T_\omega)$. Since $\omega \in \mathrm{Rat}(\mathbb{T})$, we have $\mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{m-1}$. Let $f = qh + r_1 \in \mathrm{Dom}(T_\omega)$, with $h \in H^p$ and $r_1 \in \mathcal{P}_{m-1}$. Then $T_\omega f = sh + r$ where $r \in \mathcal{P}_{n-1}$ is uniquely determined by (4.3.3). Thus

$$\langle sh, g\rangle + \langle r, g\rangle = \langle sh + r, g\rangle = \langle T_\omega f, g\rangle = \langle f, k\rangle = \langle qh + r_1, k\rangle = \langle qh, k\rangle + \langle r_1, k\rangle.$$

We obtain that

$$\langle sh, g\rangle - \langle qh, k\rangle = \langle r_1, k\rangle - \langle r, g\rangle.$$

However, in choosing $f \in \mathrm{Dom}(T_\omega)$ we can choose $h \in H^p$ and $r_1 \in \mathcal{P}_{m-1}$ independently, and in particular set one or the other equal to zero, so that

$$\begin{aligned} &\langle sh, g\rangle = \langle qh, k\rangle \quad (h \in H^p),\\ &\langle r_1, k\rangle = \langle r, g\rangle \quad (r \in \mathcal{P}_{n-1}, r_1 \in \mathcal{P}_{m-1} \text{ as in (4.3.3)}). \end{aligned}$$

The second identity proves the first claim of the lemma. From the first identity we obtain that

$$0 = \langle h, \overline{s}g - \overline{q}k\rangle_{p,p'} = \langle h, z^{-n}s^\sharp g - z^{-m}q^\sharp k\rangle_{p,p'} \quad (h \in H^p).$$

Thus $\mathbb{P}(z^{-n}s^\sharp g - z^{-m}q^\sharp k) = 0$. On the other hand, for $l = \max\{m, n\}$ we have

$$z^l(z^{-n}s^\sharp g - z^{-m}q^\sharp k) = z^{l-n}s^\sharp g - z^{l-m}q^\sharp k \in H^{p'}.$$

This can only occur if $z^{l-n}s^\sharp g - z^{l-m}q^\sharp k \in \mathcal{P}_{l-1}$, which proves the second claim.

To complete the proof, we show that $g \in \mathrm{Dom}(T_{\omega^*})$ and $T_{\omega^*} g = k$. For $m \geq n$ we have $\omega^* \in \mathrm{Rat}(\mathbb{T})$ and the first inclusion of (4.3.4) can be rewritten as

$$\omega^* g = \left(\frac{z^{m-n}s^\sharp}{q^\sharp}\right) g = k + \widetilde{r}/q^\sharp, \quad \text{for some } \widetilde{r} \in \mathcal{P}_{m-1}.$$

Since $\deg(q^\sharp) = \deg(q) = m$, it now follows that $g \in \mathrm{Dom}(T_{\omega^*})$ and $T_{\omega^*} g = k$. In case $m < n$ we have $T_{\omega^*} = T_{z^{m-n}} T_{s^\sharp/q^\sharp}$ and $s^\sharp/q^\sharp \in \mathrm{Rat}(\mathbb{T})$. Now the second inclusion of (4.3.4) gives

$$\left(\frac{s^\sharp}{q^\sharp}\right) g = z^{n-m} k + \widetilde{r}/q^\sharp, \quad \text{for some } \; \widetilde{r} \in \mathcal{P}_{n-1}.$$

Write $\widetilde{r} = \widetilde{r}_1 q^\sharp + \widetilde{r}_2$ with $\widetilde{r}_2 \in \mathcal{P}_{m-1}$. Then $\widetilde{r}/q^\sharp = \widetilde{r}_1 + \widetilde{r}_2/q^\sharp$ and $\deg(\widetilde{r}_1) < m - n$. Since $\widetilde{r}_2/q^\sharp \in \mathrm{Rat}_0(\mathbb{T})$ it follows that $g \in \mathrm{Dom}(T_{s^\sharp/q^\sharp}) = \mathrm{Dom}(T_{\omega^*})$ and $T_{s^\sharp/q^\sharp} g = z^{n-m} k + \widetilde{r}_1$. But then $T_{\omega^*} g = T_{z^{m-n}} T_{s^\sharp/q^\sharp} g = T_{z^{m-n}}(z^{n-m} k + \widetilde{r}_1) = k$. $\square$

A special case of the following result was proven as part of the proof of Theorem 3.2.2.

Lemma 4.3.4. *Let $r, \widetilde{r} \in \mathcal{P}$ be co-prime. Then $rH^p \cap \widetilde{r}H^p = r\widetilde{r}H^p$.*

Proof. Let $\widetilde{r} f = rg$ with $f, g \in H^p$. Then $f = r \cdot g/\widetilde{r} \in H^p$, so we should show $\widetilde{f} := g/\widetilde{r} \in H^p$, i.e., $\widetilde{f}$ analytic on $\mathbb{D}$ and $\int_{\mathbb{T}} |\widetilde{f}(z)|^p \, dz < \infty$.

Since $g \in H^p$, the function $\widetilde{f}$ can only fail to be analytic at the roots of $\widetilde{r}$ inside $\mathbb{D}$. However, if this were the case, then $f = r\widetilde{f}$ would also fail to be analytic in $\mathbb{D}$, since r and $\widetilde{r}$ are co-prime. Thus $\widetilde{f}$ is analytic on $\mathbb{D}$.

Divide $\mathbb{T}$ as $\mathbb{T}_1 \cup \mathbb{T}_2$ with $\mathbb{T}_1 \cap \mathbb{T}_2 = \emptyset$ in such a way that $\mathbb{T}_1$ and $\mathbb{T}_2$ are both nonempty finite unions of line segments of $\mathbb{T}$ so that the interior of $\mathbb{T}_1$ contains the roots of r and the interior of $\mathbb{T}_2$ the roots of $\widetilde{r}$. Then $|\widetilde{r}(z)| > N_1$ on $\mathbb{T}_1$ and $|r(z)| > N_2$ on $\mathbb{T}_2$ for some $N_1, N_2 > 0$. Note that $f = r\widetilde{f}$ and $g = \widetilde{r}\widetilde{f}$. We then obtain

$$\int_{\mathbb{T}_2} |\widetilde{f}(z)|^p \, dz = \int_{\mathbb{T}_2} |f(z)/r(z)|^p \, dz \leq N_2^{-p} \int_{\mathbb{T}_2} |f(z)|^p \, dz \leq (2\pi N_2^p)^{-1} \|f\|_{H^p}^p.$$

Using $g = \widetilde{r}\widetilde{f}$, one obtains similarly that $\int_{\mathbb{T}_1} |\widetilde{f}(z)|^p \, dz \leq (2\pi N_1^p)^{-1} \|g\|_{H^p}^p$. Thus $\int_{\mathbb{T}} |\widetilde{f}(z)|^p \, dz < \infty$. $\square$

Proof of Theorem 4.3.1. By Lemma 4.3.2, in order to prove (4.3.1), the formula for the action of T_ω^* on $q^\sharp H^{p'}$ and for the range of T_ω^* in (4.3.2), it remains to show that $\mathrm{Dom}(T_\omega^*) \subset q^\sharp H^{p'}$.

View $\mathcal{P}$ and $\mathcal{P}_k$, $k = 1, 2, \ldots$, as subspaces of H^p or $H^{p'}$, write P_k for the projection onto $\mathcal{P}_{k-1}$ and set $Q_k = I - P_k$. Also, the standard $k \times k$ compression of a Toeplitz operator T_ϕ on H^p (or $H^{p'}$) is denoted by $T_{\phi,k}$, i.e., $T_{\phi,k} = P_k T_\phi|_{\mathcal{P}_{k-1}}$. Now, the relation (4.3.3) between $r \in \mathcal{P}_{n-1}$ and $r_1 \in \mathcal{P}_{m-1}$ can be rewritten as

$$T_s r_1 - T_q r \in \mathcal{P}_{m-1},$$

or, equivalently, as

$$Q_m T_s P_m r_1 = Q_m T_s r_1 = Q_m T_q r = Q_m T_q P_n r. \tag{4.3.5}$$

We now consider the cases $m \geq n$ and $m < n$ separately.

First assume $m \geq n$. We can then decompose $Q_mT_sP_m$ and $Q_mT_qP_n$ as

$$Q_mT_sP_m = \begin{bmatrix} 0 & T^*_{s^\sharp,n}T^*_{z^{m-n}} \\ 0 & 0 \end{bmatrix} : \mathcal{P}_{m-1} = \begin{bmatrix} \mathcal{P}_{m-n} \\ T_{z^{m-n}}\mathcal{P}_{n-1} \end{bmatrix} \to \begin{bmatrix} \mathcal{P}_{n-1} \\ T^n_z H^p \end{bmatrix},$$

$$Q_mT_qP_n = \begin{bmatrix} T^*_{q^\sharp,n} \\ 0 \end{bmatrix} : \mathcal{P}_{n-1} \to \begin{bmatrix} \mathcal{P}_{n-1} \\ T_{z^n}H^p \end{bmatrix}.$$

Hence, in this case the identity in (4.3.5) can be written as

$$T^*_{s^\sharp,n}(T^*_{z^{m-n}}r_1) = T^*_{q^\sharp,n}r.$$

Since all Toeplitz matrices are upper triangular, we in fact have

$$T^*_{s^\sharp,m}T^*_{z^{m-n},m}r_1 = T^*_{q^\sharp,m}r.$$

Note that $T^*_{q^\sharp,n}$ is invertible, because q has only roots on $\mathbb{T}$ so that $q(0) \neq 0$. We obtain that for given $r_1 \in \mathcal{P}_{m-1}$, the polynomial $r \in \mathcal{P}_{n-1}$ that satisfies (4.3.3) is uniquely determined by

$$r = (T^*_{q^\sharp,m})^{-1}T^*_{s^\sharp,m}T^*_{z^{m-n},m}r_1 = T^*_{s^\sharp,m}T^{*m-n}_{z,m}(T^*_{q^\sharp,m})^{-1}r_1,$$

where the commutation of Toeplitz matrices can occur since they all have analytic symbols. Now take $r_1 \in \mathcal{P}_{m-1}$ arbitrary, and define r as above, so that (4.3.3) holds. Then, by Lemma 4.3.3, we have

$$\begin{aligned} \langle r_1, P_mk\rangle_{\mathcal{P}_{m-1}} &= \langle r_1, k\rangle_{p,p'} = \langle r, g\rangle_{p,p'} = \langle r, P_mg\rangle_{\mathcal{P}_{m-1}} \\ &= \langle T^*_{s^\sharp,m}T^{*m-n}_{z,m}(T^*_{q^\sharp,m})^{-1}r_1, P_mg\rangle_{\mathcal{P}_{m-1}} \\ &= \langle r_1, (T_{q^\sharp,m})^{-1}T^{m-n}_{z,m}T_{s^\sharp,m}P_mg\rangle_{\mathcal{P}_{m-1}}. \end{aligned}$$

Since $r_1 \in \mathcal{P}_{m-1}$ is arbitrary, we have $P_mk = (T_{q^\sharp,m})^{-1}T^{m-n}_{z,m}T_{s^\sharp,m}P_mg$, and thus

$$P_mT_{q^\sharp}k = T_{q^\sharp,m}P_mk = T^{m-n}_{z,m}T_{s^\sharp,m}P_mg = P_mT^{m-n}_zT_{s^\sharp}g.$$

This shows that $P_mq^\sharp k = P_mz^{m-n}s^\sharp g$. Together with the first inclusion in (4.3.4) we obtain that

$$q^\sharp k = z^{m-n}s^\sharp g.$$

Since $q^\sharp$ and $z^{m-n}s^\sharp$ are co-prime, we can apply Lemma 4.3.4 to conclude $g \in q^\sharp H^{p'}$.

Now assume $m < n$. By Lemma 2.2.4, we can write $\omega = \omega_0 + \omega_1$ uniquely with $\omega_0 \in \text{Rat}_0(\mathbb{T})$ and $\omega_1 \in \text{Rat}$ without poles on $\mathbb{T}$, i.e, $\omega_1 \in L^\infty(\mathbb{T})$. In fact $\omega_1 \in \mathcal{P}$, since all poles of ω are on $\mathbb{T}$, and $\omega_0 = \widetilde{s}/q$ with $\widetilde{s} \in \mathcal{P}_{m-1}$. It now follows that $\text{Dom}(T^*_{\omega_0}) = q^\sharp H^{p'}$, and since T_{ω_1} is bounded, $\text{Dom}(T^*_\omega) = \text{Dom}(T^*_{\omega_0}) = q^\sharp H^{p'}$. Furthermore, $T^*_\omega = T^*_{\omega_0} + T^*_{\omega_1}|_{q^\sharp H^{p'}} = T_{\omega_0^*}|_{q^\sharp H^{p'}} + T_{\omega_1^*}|_{q^\sharp H^{p'}} = T_{\omega^*}|_{q^\sharp H^{p'}}$.

In the next part of the proof we prove the formula for $\text{Ker}(T_{\omega^*})$, without distinguishing between the proper and non-proper case. Let $g = q^\sharp v \in \text{Dom}(T^*_\omega)$ with $v \in H^{p'}$. Then

$g \in \operatorname{Ker}(T_\omega^*)$ if and only if $g \in \operatorname{Ker}(T_{\omega^*})$, i.e., $g = q^\sharp v = r_1/(s_-)^\sharp$ for $r_1 \in \mathcal{P}_{\deg(s_-)-1}$, see Proposition 4.2.1. Thus $v = r_1/((s_-)^\sharp q^\sharp) \in \mathrm{Rat} \cap H^{p'}$. Then $v \in H^{p'}$ implies $r_1 = q^\sharp r$, and $\deg(r) = \deg(r_1) - m < \deg(s_-) - m$. Hence $g = q^\sharp r/(s_-)^\sharp$ with $\deg(r) < \deg(s_-) - m$. That all such functions are in $\operatorname{Ker}(T_\omega^*) = \operatorname{Ker}(T_{\omega^*}) \cap q^\sharp H^{p'}$ follows directly from the formula for $\operatorname{Ker}(T_{\omega^*})$ obtained in Proposition 4.2.1. The formula for the dimension of $\operatorname{Ker}(T_\omega^*)$ follows directly and the condition for injectivity follows since $\deg(s_-)^\sharp$ is equal to the number of nonzero roots of s_-, counting multiplicity. $\square$

4.4 The adjoint of T_ω: General case

In the section we prove Theorem 4.1.1 in full generality. Hence let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime. As in Theorem 4.1.1, factor $s = s_- s_0 s_+$ and $q = q_- q_0 q_+$ with s_-, q_- having roots only inside $\mathbb{T}$, s_0, q_0 having roots only on $\mathbb{T}$, and s_+, q_+ having roots only outside $\mathbb{T}$. Set $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$, and $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$. By Lemma 2.5.1, and its proof, we can factor ω as $\omega = \omega_-(z^\kappa \omega_0)\omega_+$ with $\kappa = n_- - m_-$, $\omega_- = s_-/(z^\kappa q_-)$ having only poles and zeroes inside $\mathbb{T}$, $\omega_0 = s_0/q_0$ having only poles and zeroes on $\mathbb{T}$, and $\omega_+ = s_+/q_+$ having only poles and zeroes outside $\mathbb{T}$, and we have $T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}$. Moreover, T_{ω_-} and T_{ω_+} are bounded and boundedly invertible.

Note that $T_{\omega_-} T_{z^\kappa \omega_0}$ is closed and densely defined and $\operatorname{Ran}(T_{\omega_+}) = H^p$, and thus by Corollary 1 in [13]

$$T_\omega^* = T_{\omega_+}^* \left(T_{\omega_-} T_{z^\kappa \omega_0}\right)^*.$$

Furthermore, T_{ω_-} is bounded and $T_{z^\kappa \omega_0}$ is closed and densely defined. By Theorem 4 in [1] one has

$$\left(T_{\omega_-} T_{z^\kappa \omega_0}\right)^* = T_{z^\kappa \omega_0}^* T_{\omega_-}^*.$$

Combining this and using that $T_{\omega_+}^* = T_{\omega_+^*}$ and $T_{\omega_-}^* = T_{\omega_-^*}$ we see that

$$T_\omega^* = T_{\omega_+}^* T_{z^\kappa \omega_0}^* T_{\omega_-}^* = T_{\omega_+^*} T_{z^\kappa \omega_0}^* T_{\omega_-^*} \quad \text{on } \operatorname{Dom}(T_\omega^*).$$

Note that

$$\omega_-^* = \frac{(s_-)^\sharp}{(q_-)^\sharp}, \quad \omega_0^* = z^{m_0 - n_0} \frac{(s_0)^\sharp}{(q_0)^\sharp},$$
$$(z^\kappa \omega_0)^* = z^{m_0 - n_0 - \kappa} \frac{(s_0)^\sharp}{(q_0)^\sharp}, \quad \omega_+^* = z^{m_+ - n_+} \frac{(s_+)^\sharp}{(q_+)^\sharp}.$$

By construction, ω_- and $1/\omega_-$ are both anti-analytic. Consequently, ω_-^* and $1/\omega_-^*$ are both analytic functions. This implies $T_{\omega_-^*}^{\pm}(q_0)^\sharp H^{p'} \subset (q_0)^\sharp H^{p'}$, and thus $T_{\omega_-^*}(q_0)^\sharp H^{p'} = (q_0)^\sharp H^{p'}$. Since $T_{\omega_+^*}$ is invertible, to see that $\operatorname{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'}$ it suffices to show $\operatorname{Dom}(T_{z^\kappa \omega_0}^*) = (q_0)^\sharp H^{p'}$. For the case where $\kappa \geq 0$, so that $z^\kappa \omega_0 \in \mathrm{Rat}(\mathbb{T})$, this follows directly from Theorem 4.3.1. For $\kappa < 0$, note that $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$, so that $T_{z^\kappa \omega_0}^* = T_{\omega_0}^* T_{z^\kappa}^* = T_{\omega_0}^* T_{z^{-\kappa}}$, again using Theorem 4 of [1]. Then $g \in \operatorname{Dom}(T_{z^\kappa \omega_0}^*)$ holds if and only if $z^{-\kappa} g \in \operatorname{Dom}(T_{\omega_0}^*) = (q_0)^\sharp H^{p'}$. By Lemma 4.3.4 this is the same as $g \in (q_0)^\sharp H^{p'}$, since $z^{-\kappa}$ and $q_0^\sharp$ are co-prime. Thus in both cases we arrive at $\mathrm{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'}$. Moreover, we also find that $T^*_{z^\kappa\omega_0} = T_{(z^\kappa\omega_0)^*}|_{(q_0)^\sharp H^{p'}}$, so that

$$T_\omega^* = T_{\omega_+^*} T^*_{z^\kappa\omega_0} T_{\omega_-^*} = T_{\omega_+^*} T_{(z^\kappa\omega_0)^*} T_{\omega_-^*}|_{(q_0)^\sharp H^{p'}} = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}.$$

Hence (4.1.2) holds.

Next we derive the formula for $\mathrm{Ker}(T_\omega^*)$. For $\kappa \geq 0$ we have $g \in \mathrm{Ker}(T_\omega^*)$ if and only if $T_{\omega_-^*} g \in \mathrm{Ker}(T^*_{z^\kappa\omega_0}) = (q_0)^\sharp \mathcal{P}_{\kappa - m_0 - 1}$, where the last identity follows by applying Theorem 4.3.1 to $z^\kappa\omega_0$. Thus $g \in \mathrm{Ker}(T_\omega^*)$ if and only if $((s_-)^\sharp/(q_-)^\sharp)g = (q_0)^\sharp r$, i.e., $g = (q_-)^\sharp (q_0)^\sharp r/(s_-)^\sharp$, for some $r \in \mathcal{P}_{\kappa - m_0 - 1}$, as claimed. For $\kappa < 0$ we have $g \in \mathrm{Ker}(T_\omega^*)$ if and only if $z^{-\kappa}\omega_-^* g \in \mathrm{Ker}(T^*_{\omega_0})$. However, $\mathrm{Ker}(T^*_{\omega_0}) = \{0\}$, by Theorem 4.3.1, so that $\mathrm{Ker}(T_\omega^*) = \{0\}$, in line with the formula in (4.1.3). The formula for the dimension of $\mathrm{Ker}(T_\omega^*)$ follows directly.

Now we turn to the formula for $\mathrm{Ran}(T_\omega^*)$. Note that

$$\mathrm{Ran}(T_\omega^*) = T_{\omega_+^*}\mathrm{Ran}(T^*_{z^\kappa\omega_0} T_{\omega_-^*}) = T_{\omega_+^*}\mathrm{Ran}(T^*_{z^\kappa\omega_0}). \tag{4.4.1}$$

We first show that $\mathrm{Ran}(T^*_{z^\kappa\omega_0}) = T_{z^{m_0-n_0-\kappa}}(s_0)^\sharp H^{p'}$. Again, for the case $\kappa \geq 0$ this follows directly from Theorem 4.3.1. Assume $\kappa < 0$. Then $T^*_{z^\kappa\omega_0} = T^*_{\omega_0} T_{z^{-\kappa}}$. Hence,

$$\begin{aligned}\mathrm{Ran}(T^*_{z^\kappa\omega_0}) &= T^*_{\omega_0}(z^{-\kappa}H^{p'} \cap \mathrm{Dom}(T_{\omega_0})) = T^*_{\omega_0}(z^{-\kappa}H^{p'} \cap (q_0)^\sharp H^{p'}) \\ &= T^*_{\omega_0} z^{-\kappa}(q_0)^\sharp H^{p'}.\end{aligned}$$

The last identity follows by Lemma 4.3.4. Now the action of $T^*_{\omega_0}$, as described in Theorem 4.3.1, shows that $\mathrm{Ran}(T^*_{z^\kappa\omega_0}) = T_{z^{m_0-n_0}} z^{-\kappa}(s_0)^\sharp H^{p'} = T_{z^{m_0-n_0-\kappa}}(s_0)^\sharp H^{p'}$. Since $1/q_+$ is analytic, $1/(q_+)^\sharp$ is anti-analytic, and therefore, independent of the sign of $m_+ - n_+$, we have

$$T_{\omega_+^*} = T_{1/(q_+)^\sharp} T_{z^{m_+-n_+}} T_{(s_+)^\sharp}.$$

Thus

$$\mathrm{Ran}(T_\omega^*) = T_{1/(q_+)^\sharp} T_{z^{m_+-n_+}} T_{(s_+)^\sharp} T_{z^{m_0-n_0-\kappa}}(s_0)^\sharp H^{p'}.$$

Note that $T_{(s_+)^\sharp}$ and $T_{z^{m_0-n_0-\kappa}}$ need not commute, in case $m_0 - n_0 - \kappa < 0$. However, we do have $T_{(s_+)^\sharp} T_{z^{m_0-n_0-\kappa}} = T_{z^{m_0-n_0-\kappa}} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}$. Moreover, since $(s_+)^\sharp$ is analytic, $T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0} = Q_{\kappa+n_0-m_0} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}$ and we have

$$\begin{aligned}T_{z^{m_+-n_+}} T_{z^{m_0-n_0-\kappa}} Q_{\kappa+n_0-m_0} &= T_{z^{m_+-n_++m_0-n_0-\kappa}} Q_{\kappa+n_0-m_0} \\ &= T_{z^{m-n}} Q_{\kappa+n_0-m_0}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathrm{Ran}(T_\omega^*) &= T_{1/(q_+)^\sharp} T_{z^{m-n}} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}(s_0)^\sharp H^{p'} \\ &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} Q_{\kappa+n_0-m_0}(s_0)^\sharp H^{p'},\end{aligned}$$

again using that $1/(q_+)^\sharp$ is anti-analytic and $(s_+)^\sharp$ is analytic. This gives the general formula for $\mathrm{Ran}(T_\omega^*)$. In case $\kappa + n_0 - m_0 \leq 0$, we have $Q_{\kappa+n_0-m_0} = I$ and $T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}(s_0)^\sharp = (s_+ s_0)^\sharp$, as claimed.

4.5 Symmetric operators and selfadjoint extensions

For $\omega \in \mathrm{Rat}$, the second adjoint T_ω^{**} is well-defined and $T_\omega^{**} = T_\omega$, since T_ω is a closed, densely defined operator on a reflexive Banach space [8, Theorem III.5.24]. Now consider $\omega \in \mathrm{Rat}(\mathbb{T})$ and $p = 2$. From Theorem 4.1.1 it is obvious that $T_\omega \neq T_\omega^*$, except in the degenerate case where q is constant, since $\mathrm{Dom}(T_\omega) = qH^2 + \mathcal{P}_{\deg(q)-1}$ contains all polynomials while $\mathrm{Dom}(T_\omega^*) = q^\sharp H^2$ only contains the polynomials that contain $q^\sharp$ as a factor. Consequently, T_ω cannot be selfadjoint. In this section we consider the question when T_ω^* is symmetric, and, if this is the case, when does T_ω^* have a selfadjoint extension L. The first topic is addressed in the following theorem.

Theorem 4.5.1. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Set $n = \deg(s)$ and $m = \deg(q)$. Then the following are equivalent.*

(1) T_ω^* is symmetric;

(2) $\omega(\mathbb{T}) \subset \mathbb{R}$;

(3) $\omega(z) = \widetilde{\omega}(-i\frac{z+1}{z-1})$ with $\widetilde{\omega}$ a real rational function with poles only on $\mathbb{R}$;

(4) the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ of T_ω is contained in $\mathbb{R}$;

(5) ω is proper, $s = z^{m-n}\widetilde{s}$ with $\widetilde{s}$ self-inversive and $q_0\overline{s_n} = \overline{q_m}s_{m-n}$ holds, where $s(z) = \sum_{k=0}^n s_k z^k$ and $q(z) = \sum_{k=0}^m q_k z^k$.

Moreover, if T_ω^* is symmetric, then $T_\omega^* \subset T_\omega$.

Proof. We first prove the equivalence of (1) and (2), and that (1) implies $T_\omega^* \subset T_\omega$. Assume (2). For $z \in \mathbb{T}$, not a root of q, we have $\omega^*(z) = \overline{\omega(z)} = \omega(z)$. Hence $\omega^* = \omega$. Since q has only roots on $\mathbb{T}$, we have $q = \gamma q^\sharp$ for a unimodular constant γ. Hence $qH^2 = q^\sharp H^2$. This shows $T_\omega^* = T_{\omega^*}|_{q^\sharp H^2} = T_\omega|_{qH^2} \subset T_\omega$. Since $(T_\omega^*)^* = T_\omega$, it follows that T_ω^* is symmetric and $T_\omega^* \subset T_\omega$. Conversely, assume (1). Then we still have $qH^2 = q^\sharp H^2$ and $T_\omega^* \subset (T_\omega^*)^* = T_\omega$. Hence $T_\omega^* = T_\omega|_{qH^2}$. In particular, we have $\omega^* q = T_{\omega^*}q = T_\omega^* q = T_\omega q = \omega q$. This implies $\omega = \omega^*$. Hence $\omega(z) = \overline{\omega(z)}$ for $z \in \mathbb{T}$, not a root of q. Thus $\omega(\mathbb{T}) \subset \mathbb{R}$.

That (2) and (3) are equivalent follows simply because in (3) ω is the composition of $\widetilde{\omega}$ and the inverse Cayley transform, which maps the circle $\mathbb{T}$ bijectively onto $\mathbb{R}$. The fact that $\widetilde{\omega}$ is real rational, i.e., $\widetilde{\omega} = \widetilde{s}/\widetilde{q}$ with $\widetilde{s}$ and $\widetilde{q}$ real polynomials, is equivalent to $\widetilde{\omega}(\mathbb{R}) := \{\widetilde{\omega}(t)\colon t \in \mathbb{R},\ \widetilde{q}(t) \neq 0\} \subset \mathbb{R}$. Also, the equivalence of (2) and (4) is a direct consequence of the fact that $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T})$, by [6, Theorem 1.1].

Finally, we prove (2) $\Leftrightarrow$ (5). Since $q = \gamma q^\sharp$, we have

$$\omega^* = z^{m-n}\frac{s^\sharp}{q^\sharp} = z^{m-n}\gamma\frac{s^\sharp}{q}.$$

Thus, we have $\omega = \omega^*$ if and only if $z^{m-n}\gamma s^\sharp = s$. Hence (2) is equivalent to $z^{m-n}\gamma s^\sharp = s$. Now assume (2). Since $\deg(s^\sharp) \leq \deg(s)$, the identity $z^{m-n}\gamma s^\sharp = s$ can only occur if $m \geq n$, i.e., if ω is proper. The identity also shows that $s = z^{m-n}\widetilde{s}$ for $\widetilde{s} = \gamma s^\sharp$. On the other hand, $s^\sharp = (z^{m-n}\widetilde{s})^\sharp = \widetilde{s}^\sharp$. Thus $\widetilde{s} = \gamma s^\sharp = \gamma \widetilde{s}^\sharp$, which shows $\widetilde{s}$ is self-inversive, with constant γ. Note that $\gamma = q_0/\overline{q_m}$. Also, we have $s_0 = \cdots = s_{m-n-1} = 0$ and $\widetilde{s}(z) = \sum_{k=0}^{2n-m} s_{m-n+k} z^k$. Since $\widetilde{s}$ is self-inversive, $\widetilde{s} = \delta \widetilde{s}^\sharp$ with $\delta = s_{m-n}/\overline{s_n}$. But also $\delta = \gamma$, so $s_{m-n}/\overline{s_n} = q_0/\overline{q_m}$. Thus $q_0\overline{s_n} = \overline{q_m} s_{m-n}$. Hence (5) holds. Conversely, assume (5). Reversing the above argument, it follows that $q_0\overline{s_n} = \overline{q_m} s_{m-n}$ implies $\widetilde{s} = \delta \widetilde{s}^\sharp$ with $\delta = \gamma$. Thus $\gamma s^\sharp = \gamma \widetilde{s}^\sharp = \widetilde{s}$. This implies $s = z^{m-n}\widetilde{s} = z^{m-n}\gamma s^\sharp$, and hence (2). □

Corollary 4.5.2. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime.Assume T_ω^* is symmetric. Then* $\deg(s) \leq \deg(q) \leq 2\deg(s)$.

Proof. By Theorem 4.5.1 condition (5) holds with $m = \deg(q)$ and $n = \deg(s)$. Since $\widetilde{s}$ is self-inversive, we have $\widetilde{s}(0) \neq 0$. Consequently, 0 would be a non-removable singularity of $s = z^{m-n}\widetilde{s}$ in case $m < n$, which gives a contradiction. Hence $m \geq n$. Furthermore, comparing the degrees on both sides of $s = z^{m-n}\widetilde{s}$ yields, $n = m - n + \deg(\widetilde{s}) \geq m - n$. Hence $m \leq 2n$. □

When T_ω^* is symmetric, it need not be the case that T_ω^* has a selfadjoint extension. In Proposition 4.5.4 below we characterize when T_ω^* does have a selfadjoint extension. However, we first give a concrete example that shows this does not always happen.

Example 4.5.3. In [7] Helson considered the functions $\omega_k(z) = \left(-i\frac{z+1}{z-1}\right)^k$ for $k \in \mathbb{N}$. For all k we have $\omega_k(\mathbb{T}) \subset \mathbb{R}$, see Theorem 4.5.1 (3) above, hence $T_{\omega_k}^*$ is symmetric by Theorem 4.5.1. In fact, for k even $\omega_k(\mathbb{T}) = \mathbb{R}_+$, while for k odd we have $\omega_k(\mathbb{T}) = \mathbb{R}$. We show that $T_{\omega_k}^*$ does not have a selfadjoint extension for $k = 1$. In Example 4.5.8 we return to this example for general k.

For $k = 1$ we have $\omega(z) = \omega_1(z) = -i\frac{z+1}{z-1}$. Hence $\mathrm{Dom}(T_\omega) = (z-1)H^2 + \mathbb{C}$ and $\mathrm{Dom}(T_\omega^*) = (z-1)H^2$. Suppose T_ω^* has a selfadjoint extension L. Then $L = L^*$ and thus $T_\omega^* \subset L = L^* \subset T_\omega^{**} = T_\omega$. Since T_ω is not selfadjoint, the inclusions are strict. Hence $\mathrm{Dom}(T_\omega^*) \subset \mathrm{Dom}(L) \subset \mathrm{Dom}(T_\omega)$, with strict inclusions. However, the complement of $\mathrm{Dom}(T_\omega^*)$ in $\mathrm{Dom}(T_\omega)$ is one-dimensional, hence not both inclusions can be strict. Thus T_ω does not admit a selfadjoint extension.

Proposition 4.5.4. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then T_ω^* admits a selfadjoint extension if and only if the number of roots of $s - iq$ and $s + iq$ in $\mathbb{D}$, counting multiplicities, coincide.*

Proof. The operator T_ω^* is an adjoint, and hence closed, and by assumption symmetric. Following definition X.2.12 from [3] we define the deficiency subspaces of T_ω^* as the spaces

$$\mathcal{L}_+ = \mathrm{Ker}\,(T_\omega^{**} - i) = (\mathrm{Ran}\,(T_\omega^* + i))^\perp, \quad \mathcal{L}_- = \mathrm{Ker}\,(T_\omega^{**} + i) = (\mathrm{Ran}\,(T_\omega^* - i))^\perp,$$

and the deficiency indices as the integers $n_\pm = \dim \mathcal{L}_\pm$. Since $T_\omega^{**} = T_\omega$, we have

$$n_+ = \dim \mathrm{Ker}\,(T_\omega - i) \quad \text{and} \quad n_- = \dim \mathrm{Ker}\,(T_\omega + i).$$

Also, we have $T_\omega \pm i = T_{\omega\pm i}$. By item (b) of Theorem X.2.20 in [3], T_ω has a selfadjoint extension if and only if $n_+ = n_-$. Note that $\omega \pm i = (s \pm iq)/q$. We now apply Corollary 2.4.2 to $T_{\omega\pm i}$, to obtain that $n_\pm$ is equal to the maximum of 0 and the difference of m and the number of roots of $s \pm iq$ in $\overline{\mathbb{D}}$, counting multiplicities. However, since T_ω^* is symmetric, ω is proper so the number of roots cannot exceed m. Note also that $\omega(\mathbb{T}) \subset \mathbb{R}$, so $s \pm iq$ cannot have roots on $\mathbb{T}$. It thus follows that T_ω^* has a selfadjoint extension if and only if the number of roots in $\mathbb{D}$ of $s - iq$ and $s + iq$, counting multiplicities, coincide, as claimed. □

Since T_ω^* is never selfadjoint for $\omega \in \mathrm{Rat}(\mathbb{T})$ having at least one pole on $\mathbb{T}$, the formulas for $n_\pm$ in the above proof along with item (a) of Theorem X.2.20 in [3] directly give the following corollary.

Corollary 4.5.5. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then $s + iq$ or $s - iq$ must have a root in $\mathbb{D}$.*

Proposition 4.5.4 can be rephrased in terms of the index of the operators $T_{\omega\pm i}$.

Proposition 4.5.6. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then $T_{\omega+i}$ and $T_{\omega-i}$ are both Fredholm and T_ω^* admits a selfadjoint extension if and only if the Fredholm indices of $T_{\omega+i}$ and $T_{\omega-i}$ coincide.*

Proof. This follows directly from Proposition 4.5.4 and Theorem 2.1.1 applied to $\omega + i$ and $\omega - i$, using that $\omega \pm i = (s \pm iq)/q$. □

Corollary 4.5.7. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Assume $\omega(\mathbb{T}) \neq \mathbb{R}$. Then T_ω^* admits a selfadjoint extension.*

Proof. The Fredholm index of $T_{\omega-\lambda}$ is constant with respect to $\lambda \in \mathbb{C}$ on the connected components of $\mathbb{C}$ separated by the essential spectrum of T_ω, which is equal to $\omega(\mathbb{T})$; see [6, Theorem 1.1]. Hence if $\omega(\mathbb{T}) \neq \mathbb{R}$, but $\omega(\mathbb{T}) \subset \mathbb{R}$ since T_ω^* is symmetric, then i and $-i$ are in the same connected component and thus $T_{\omega+i}$ and $T_{\omega-i}$ have the same index. The conclusion now follows from Proposition 4.5.6. □

Example 4.5.8. We return to the functions $\omega_k(z) = \left(-i\frac{z+1}{z-1}\right)^k$ considered in Example 4.5.3. Since $\omega_k(\mathbb{T}) = \mathbb{R}_+$ for k even, we obtain directly from Corollary 4.5.7 that $T_{\omega_k}^*$ admits a selfadjoint extension in case k is even.

For odd values of k we have $\omega_k(\mathbb{T}) = \mathbb{R}$, and thus no conclusion can be drawn from Corollary 4.5.7. To deal with the odd case we resort to Proposition 4.5.4. Take $s(z) = (-i)^k(z+1)^k$ and $q = (z-1)^k$ and write k as $k = 2l + 1$. The polynomials $s \pm iq$ are given by

$$\begin{aligned} s(z) \pm iq(z) &= i\left((-1)^{l+1}(z+1)^{2l+1} \pm (z-1)^{2l+1}\right) \\ &= i\left((-1)^{l+1}\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j \pm \sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j(-1)^{2l+1-j}\right) \\ &= i\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j\left((-1)^{l+1} \pm (-1)^{2l+1-j}\right) \\ &= i\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j\left((-1)^{l+1} \pm (-1)^{j-1}\right). \end{aligned}$$

For odd values of l one obtains:

$$\begin{aligned} s(z) - iq(z) &= -2i\left(\binom{2l+1}{0} + \cdots + \binom{2l+1}{2l-2}z^{2l-2} + \binom{2l+1}{2l}z^{2l}\right), \\ s(z) + iq(z) &= 2i\left(\binom{2l+1}{1}z + \cdots + \binom{2l+1}{2l-1}z^{2l-1} + \binom{2l+1}{2l+1}z^{2l+1}\right) \\ &= 2iz\left(\binom{2l+1}{2l} + \cdots + \binom{2l+1}{2}z^{2-2} + \binom{2l+1}{0}z^{2l}\right) \end{aligned}$$

Observe that $s+iq$ is of the form $izp_+(z^2)$ where p_+ is a real polynomial of degree $2l$ and that $s-ig$ is of the form $ip_-(z^2)$ where p_- is a real polynomial of degree $2l$. Because p_+ and p_- are real polynomials and the fact that z^2 is the variable rather than z itself, the nonzero roots of $zp_+(z^2)$ come either in pairs (z and $-z$) for real nonzero roots or in quadruples $(z, \bar{z}, -z, -\bar{z})$ for nonreal roots, while zero appears as a simple root. Similarly, the roots of $p_-(z^2)$ come in pairs (z and $-z$) or quadruples $(z, \bar{z}, -z, -\bar{z})$ and there is no root at zero. Hence $s+iq$ has an odd number of roots inside the unit disc, and $s-iq$ has an even number of roots inside the unit disc, so that the indices n_+ and n_- can never coincide. One further observes that $p_- = p_+^\sharp$. In a similar way, for even values of l the polynomial $s+iq$ will have an even number of roots inside the unit disc and $s-iq$ will have an odd number of roots inside the unit disc. Hence, in all cases where k is odd, T_ω^* does not have a selfadjoint extension.

We now present a proposition that rephrases the criteria of Proposition 4.5.4 in terms of the roots of $s+iq$ (or $s-iq$) only. The observation that $T_{\omega_k}^*$ in Example 4.5.8 has no selfadjoint extension follows as a special case. In general, T_ω^* cannot have a selfadjoint extension whenever $\deg(q)$ is odd for any $\omega \in \mathrm{Rat}(\mathbb{T})$.

Proposition 4.5.9. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Set $l_\pm = m - \deg(s \pm iq)$ and define*

$$k_{\pm,1} = \#\left\{\begin{matrix}\textit{zeroes of } \omega \pm i \textit{ inside } \mathbb{T} \\ \textit{multi. taken into account}\end{matrix}\right\}, \quad k_{\pm,2} = \#\left\{\begin{matrix}\textit{zeroes of } \omega \pm i \textit{ outside } \mathbb{T} \\ \textit{multi. taken into account}\end{matrix}\right\}.$$

Then

$$T_\omega^* \text{ has a selfadjoint extension } \Leftrightarrow\ l_+ + k_{+,2} = k_{+,1}\ \Leftrightarrow\ l_- + k_{-,2} = k_{-,1}.$$

In particular, if T_ω^* *has a selfadjoint extension, then* $\deg(q)$ *must be even.*

The basis for the proof of Proposition 4.5.9 lies in the following lemma, which clarifies the relation between $s + iq$ and $s - iq$ under the assumption that T_ω^* is symmetric.

Lemma 4.5.10. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$*, with* $s, q \in \mathcal{P}$ *coprime, be such that* T_ω^* *is symmetric. Set* $l_\pm = \deg(q) - \deg(s \pm iq)$ *and let* γ *be the unimodular constant such that* $q = \gamma q^\sharp$*. Then*

$$s \pm iq = \gamma z^{l_\mp}(s \mp iq)^\sharp. \tag{4.5.1}$$

Moreover, we have $l_\pm = 0$ *if and only if* $\omega(0) = \pm i$*. In particular, only one of* l_+ *and* l_- *can be nonzero.*

Proof. Since T_ω^* is symmetric, by assumption, ω has the properties listed in Theorem 4.5.1. In particular, ω is proper, $m := \deg(q) \geq \deg(s) =: n$, and $s = z^{m-n}\widetilde{s}$ with $\widetilde{s}$ self-inversive and the unimodular constants that establish the self-inversiveness of $\widetilde{s}$ and q coincide (equivalently, $q_0\overline{s_n} = \overline{q_m}s_{m-n}$).

Note that $\deg(s \pm iq) \neq m$ occurs precisely when $\deg(s) = \deg(q)$ and the leading coefficients s_m and q_m of s and q, respectively, satisfy $s_m \pm iq_m = 0$, i.e., $s_m/q_m = \mp i$. Since $m = n$, the identity $q_0\overline{s_n} = \overline{q_m}s_{m-n}$ shows $\omega(0) = s_0/q_0 = \overline{s_m}/\overline{q_m}$. Hence $\deg(s \pm iq) \neq m$ holds if and only if $\omega(0) = \overline{\mp i} = \pm i$, as claimed.

We first prove (4.5.1) for the case $\omega(0) = 0$. So assume $\omega(0) = 0$, or equivalently, $s(0) = 0$. In this case $l_+ = l_- = 0$. Since $s = z^{m-n}\widetilde{s}$ and $\widetilde{s}(0) \neq 0$ (because $\widetilde{s}$ is self-inversive), we have $m > n$. Also note that $m - n$ is equal to the multiplicity of 0 as a root of s. We now employ Lemma 4.2.2, using that $\deg(s + iq) = m = \deg(iq)$, to obtain

$$\begin{aligned} \gamma(s \mp iq)^\sharp &= z^{\deg(s+iq)-\deg(s)}\gamma s^\sharp \mp (-i)\gamma q^\sharp = z^{m-n}\gamma\widetilde{s}^\sharp \pm iq \\ &= z^{m-n}\widetilde{s} \pm iq = s \pm iq. \end{aligned}$$

Hence (4.5.1) holds.

Now assume $\omega(0) \neq 0$, i.e., $s(0) \neq 0$. In that case $s = \widetilde{s}$. Hence s is self-inversive with the same constant γ that establishes the self-inversiveness of q. This also yields $m = n$. Since s and q are self-inversive with the same constant γ, we have

$$\overline{s_{m-k}}q_k = \overline{q_{m-k}s_{m-k}}\gamma = \overline{q_{m-k}}s_k \quad \text{for } k = 0, \ldots, m.$$

Hence for all k we have

$$\overline{s_{m-k}}(s_k + iq_k) = s_k(\overline{s_{m-k}} + i\overline{q_{m-k}}) \ \text{ and } \ \overline{q_{m-k}}(s_k + iq_k) = q_k(\overline{s_{m-k}} + i\overline{q_{m-k}}).$$

In case $s_{m-k} = 0$ and $q_{m-k} = 0$, also $s_k = 0$ and $q_k = 0$, since $s_k = \gamma\overline{s_{m-k}}$ and $q_k = \gamma\overline{q_{m-k}}$, and thus $s_k + iq_k = 0 = \gamma(\overline{s_{m-k}} + i\overline{q_{m-k}})$. If either $s_{m-k} \neq 0$ or $q_{m-k} \neq 0$, divide the first identity by $\overline{s_{m-k}}$ or the second identity by $\overline{q_{m-k}}$ to arrive at $s_k + iq_k = \gamma(\overline{s_{m-k}} + i\overline{q_{m-k}})$. Hence

$$s_k + iq_k = \gamma(\overline{s_{m-k} - iq_{m-k}}) \quad \text{for } k = 0, \dots, m. \tag{4.5.2}$$

Thus $s_k + iq_k = 0$ if and only if $s_{m-k} - iq_{m-k} = 0$. It follows that 0 is a root of $s \pm iq$ with multiplicity $l_\mp$. Comparing coefficients, it follows that the identities in (4.5.1) correspond to the identities in (4.5.2). Hence (4.5.1) holds. □

Proof of Proposition 4.5.9. Since T_ω^* is assumed to be symmetric, (4.5.1) holds. Together with the fact that the $\sharp$ operator reflects roots over $\mathbb{T}$, this implies that the number of roots of $s \pm iq$ inside $\mathbb{T}$ are equal to $l_\pm$ plus the number of roots of $s \mp iq$ outside $\mathbb{T}$, counting multiplicities. In other words, we have

$$k_{+,1} = l_- + k_{-,2} \quad \text{and} \quad k_{-,1} = l_+ + k_{+,2}. \tag{4.5.3}$$

By Proposition 4.5.6, T_ω^* has a selfadjoint extension if and only if $s + iq$ and $s - iq$ have an equal number of roots inside $\mathbb{T}$, again counting multiplicities, equivalently, $k_{+,1} = k_{-,1}$. Given (4.5.3), it follows that $k_{+,1} = k_{-,1}$ is equivalent to $k_{+,1} = l_+ + k_{+,2}$, and likewise to $k_{-,1} = l_- + k_{-,2}$. This proves the two criteria for T_ω^* to have a selfadjoint extension.

By Lemma 4.5.10, either $l_+ = 0$ or $l_- = 0$. Say $l_+ = 0$. Since $s + iq$ cannot have roots on $\mathbb{T}$, we have $\deg(q) = \deg(s + iq) = k_{+,1} + k_{+,2}$. If T_ω^* admits a selfadjoint extension, then we have $k_{+,1} = l_+ + k_{+,2} = k_{+,2}$. Hence $\deg(q) = 2k_{+,1}$ is even. For $l_- = 0$ the arguments goes similarly. □

Combining the fact that T_ω^* cannot have a selfadjoint extension in case $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, s, q co-prime, and $\deg(q)$ odd with Corollary 4.5.7 immediately yields the following result.

Corollary 4.5.11. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, be such that T_ω^* is symmetric and $\deg(q)$ is odd. Then $\omega(\mathbb{T}) = \mathbb{R}$.*

The next example shows that also with $\deg(q)$ even it can occur that T_ω^* does not admit a selfadjoint extension.

Example 4.5.12. Let $\omega = s/q$ with

$$s(z) = i(1 + az + z^2), \quad \text{for some } 0 \neq a \in \mathbb{R}\ , \quad \text{and} \quad q(z) = 1 - z^2.$$

Then $m = n$ and

$$s^\sharp = -s, \quad q^\sharp = -q.$$

So T_ω^* is symmetric by Theorem 4.5.1 (5). Also, we have

$$(s + iq)(z) = i(2 + az) \quad \text{and} \quad (s - iq)(z) = iz(a + 2z).$$

Hence the number of roots of $s - iq$ inside $\mathbb{D}$ is 1 if $|a| \geq 2$ and 2 if $0 \neq |a| < 2$, while the number of roots of $s + iq$ inside $\mathbb{D}$ is 1 if $|a| > 2$ and 0 if $0 \neq |a| \leq 2$. Thus T_ω^* admits a selfadjoint extension if and only if $|a| > 2$.

4.6 Comparison with the unbounded Toeplitz operator defined by Sarason

The Smirnov class N^+ consists of quotients $\frac{b}{a}$ with a and b H^∞-functions such that the denominator a is an outer function. The function $\varphi = \frac{b}{a} \in N^+$ is said to be in *canonical form* if $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. By Proposition 3.1 of [12], every function $\varphi \in N^+$ can be uniquely written in canonical form.

In [12], Sarason investigated an unbounded Toeplitz operator T_φ^{Sa} with symbol φ in N^+, which is defined by

$$\mathrm{Dom}(T_\varphi^{\mathrm{Sa}}) = \{f \in H^2 : \varphi f \in H^2\}, \quad T_\varphi^{\mathrm{Sa}} f = \varphi f \quad (f \in \mathrm{Dom}(T_\varphi^{Sa})).$$

More generally, T_φ^{Sa} can be defined in this way for any holomorphic function φ on $\mathbb{D}$, but for T_φ^{Sa} to be densely defined, φ must be in N^+ [12, Lemma 5.2].

Let $\varphi = \frac{b}{a} \in N^+$ be the canonical representation of φ. Then it is shown in Proposition 5.3 of [12] that $\mathrm{Dom}(T_\varphi^{\mathrm{Sa}}) = aH^2$. The adjoint of the operator T_φ^{Sa} is motivated by the action of the conjugate transpose of the matrix representation of T_φ^{Sa}, which is lower triangular. The domain of the adjoint operator is shown to contain the space $H(\overline{\mathbb{D}})$ of functions that are analytic on some neighborhood of the closed unit disc $\overline{\mathbb{D}}$, and the adjoint is equal to the closure of the operator on $H(\overline{\mathbb{D}})$; see [12, Lemmas 6.1 and 6.4].

Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Set $n = \deg(s)$ and $m = \deg(q)$. Assume ω is proper, i.e., $n \leq m$. Then $\omega^*(z) = z^{m-n} s^\sharp / q^\sharp \in \mathrm{Rat}(\mathbb{T})$. Since $q^\sharp$ has zeroes only on $\mathbb{T}$ it is outer and thus $\omega^* \in N^+$. While in general T_ω and T_ω^{Sa} are different, the following proposition shows that T_ω coincides with $T_{\omega^*}^{\mathrm{Sa}}$, and hence $T_\omega = T_\omega^{**} = T_{\omega^*}^{\mathrm{Sa}}$. Without the properness assumption, ω^* is not in N^+, because ω^* has a pole at 0, and hence $T_{\omega^*}^{\mathrm{Sa}}$ is not defined.

Proposition 4.6.1. *Let* $\widetilde{\omega} = \widetilde{s}/\widetilde{q} \in \mathrm{Rat}(\mathbb{T})$ *with* $\widetilde{s}, \widetilde{q} \in \mathcal{P}$ *co-prime. Then* $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = \widetilde{q}H^2$ *and* $T_{\widetilde{\omega}}^{\mathrm{Sa}} = T_{\widetilde{\omega}}|_{\widetilde{q}H^2}$. *In particular, if* $\omega \in \mathrm{Rat}(\mathbb{T})$ *is proper, then* $T_\omega^* = T_{\omega^*}^{\mathrm{Sa}}$.

Proof. We first show $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = \widetilde{q}H^2$. Let $\widetilde{\omega} = a/b$ be the canonical form of $\widetilde{\omega}$. As noted above, $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = aH^2$. By the Fejér-Riesz Theorem there is a polynomial r such that on $\mathbb{T}$ we have $|r|^2 = |\widetilde{s}|^2 + |\widetilde{q}|^2$, r has no roots in $\mathbb{D}$ and $\arg(r(0)) = \arg(\widetilde{q}(0))$. The latter is possible since $\widetilde{q}(0) \neq 0$ and implies $\widetilde{q}(0)/r(0) > 0$. Note that r also has no roots on $\mathbb{T}$, since $\widetilde{s}$ and $\widetilde{q}$ are co-prime. It follows that $\widetilde{q}/r$ and $\widetilde{s}/r$ are both H^∞-functions, $\widetilde{q}/r$ is outer and $\widetilde{q}(0)/r(0) > 0$. Hence $a = \widetilde{q}/r$ and $b = \widetilde{s}/r$, by the uniqueness of the canonical form. Also, since all the roots of r are outside $\mathbb{T}$, $r^{-1}H^2 = H^2$, so that $aH^2 = \widetilde{q}H^2$.

Now let $f \in \mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}})$, say $f = \widetilde{q}h$ with $h \in H^2$. Then $T_{\widetilde{\omega}}^{\mathrm{Sa}} f = \widetilde{\omega} f = \widetilde{s}h$. On the other hand, the fact that $\widetilde{\omega} f = \widetilde{s}h$ and $\widetilde{s}h \in H^2$ shows $T_{\widetilde{\omega}} f = \mathbb{P}\widetilde{s}h = \widetilde{s}h$. Hence $T_{\widetilde{\omega}}^{\mathrm{Sa}} = T_{\widetilde{\omega}}|_{\widetilde{q}H^2}$. $\square$

Next we employ some of the ideas from [12] to derive the following result. Recall that for a Hilbert space operator $T : \mathrm{Dom}(T) \to \mathcal{H}$ a linear submanifold $\mathcal{D} \subset \mathrm{Dom}(T)$ is called a *core* in case the graph $G(T|_\mathcal{D})$ of $T|_\mathcal{D}$ is dense in the graph $G(T)$ of T; cf., page 166 in [8].

Theorem 4.6.2. *Let* $\omega \in \mathrm{Rat}(\mathbb{T})$*. Then* $H(\overline{\mathbb{D}})$ *is contained in* $\mathrm{Dom}(T_\omega)$*. If* ω *is proper, then* $H(\overline{\mathbb{D}})$ *is a core of* T_ω*.*

Proof of $H(\overline{\mathbb{D}}) \subset \mathrm{Dom}(T_\omega)$. Write $\omega = \frac{s}{q} \in \mathrm{Rat}_0(\mathbb{T})$ with $s, q \in \mathcal{P}$ coprime. Let $f \in H(\overline{\mathbb{D}})$. Then there exists a $R > 1$ such that f is still analytic on an open neighborhood of the closed disc with radius R. Set $\widetilde{f}(z) = f(Rz)$, $\widetilde{q}(z) = q(Rz)$ and $\widetilde{s}(z) = s(Rz)$. Then $\widetilde{f} \in H^2$ and $\widetilde{q}$ is a polynomial with no roots on $\mathbb{T}$ and $\deg(q) = \deg(\widetilde{q})$. By Theorem 2.3.1, $H^2 = \widetilde{q}H^2 + \mathcal{P}_{\deg(q)-1}$. Thus $\widetilde{s}\widetilde{f} = \widetilde{q}\widetilde{h} + \widetilde{r}$ for some $\widetilde{h} \in H^2$ and $\widetilde{r} \in \mathcal{P}$ with $\deg(\widetilde{r}) < \deg(q)$. Now set $r(z) = \widetilde{r}(z/R)$ and $h(z) = \widetilde{h}(z/R)$. Then $r \in \mathcal{P}$ with $\deg(r) = \deg(\widetilde{r}) < \deg(q)$ and $h \in H^2$, even $h \in H(\overline{\mathbb{D}})$. Also, we have $sf = qh + r$. Thus $f \in \mathrm{Dom}(T_\omega)$. □

Before proving the second claim of Theorem 4.6.2 it is useful to consider the value of T_ω when applied to the evaluation functional or reproducing kernel element $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$, where $\lambda \in \mathbb{D}$. Note that $k_\lambda \in H(\overline{\mathbb{D}})$, hence $k_\lambda \in H^2$, and k_λ has the reproducing kernel property for H^2:

$$\mathrm{span}\{k_\lambda \colon \lambda \in \mathbb{D}\} \text{ dense in } H^2 \quad \text{and} \quad \langle h, k_\lambda \rangle = h(\lambda) \quad (h \in H^2,\ \lambda \in \mathbb{D}).$$

See [9] for a recent account of the theory of reproducing kernel Hilbert spaces and further references.

Lemma 4.6.3. *Let* $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$*, with* $s, q \in \mathcal{P}$ *co-prime, be proper. Then*

$$T_\omega k_\lambda = \overline{\omega^*(\lambda)} k_\lambda \quad (\lambda \in \mathbb{D}).$$

Proof. Suppose $g = T_\omega k_\lambda$ then $s(z)(1 - \overline{\lambda}z)^{-1} = q(z)g(z) + r(z)$, where $r \in \mathcal{P}_{m-1}$. Here $m = \deg(q)$. Hence $(1 - \overline{\lambda}z)g = (s + (1 - \overline{\lambda}z)r)/q$ is in $\mathrm{Rat}(\mathbb{T})$ as well as in H^2. This can only occur if $(1 - \overline{\lambda}z)g$ is a polynomial, i.e., $g = k_\lambda \widetilde{r}$ for some $\widetilde{r} \in \mathcal{P}$. Thus $s + (1 - \overline{\lambda}z)r = q\widetilde{r}$. Since ω is proper, the degree of the left hand side is at most m. But then $\widetilde{r}$ is constant, say with value $\widetilde{c}$. This shows $T_\omega k_\lambda = \widetilde{c} k_\lambda$.

To determine $\widetilde{c}$ we evaluate the identity $s + (1 - \overline{\lambda}z)r = q\widetilde{c}$ at $1/\overline{\lambda}$. This gives $s(1/\overline{\lambda}) = q(1/\overline{\lambda})\widetilde{c}$. Note that

$$s^\sharp(\lambda) = \lambda^n \overline{s(1/\overline{\lambda})} \quad \text{and} \quad q^\sharp(\lambda) = \lambda^m \overline{q(1/\overline{\lambda})},$$

where $n = \deg(s)$. Hence

$$s(1/\overline{\lambda}) = \overline{\lambda}^{-n} \overline{s^\sharp(\lambda)} \quad \text{and} \quad q(1/\overline{\lambda}) = \overline{\lambda}^{-m} \overline{q^\sharp(\lambda)}.$$

This gives

$$\widetilde{c} = \frac{\overline{\lambda}^{-n} \overline{s^\sharp(\lambda)}}{\overline{\lambda}^{-m} \overline{q^\sharp(\lambda)}} = \overline{\left(\frac{\lambda^{m-n} s^\sharp(\lambda)}{q^\sharp(\lambda)} \right)} = \overline{\omega^*(\lambda)}. \qquad \square$$

Proof of Theorem 4.6.2. It remains to prove that $H(\overline{\mathbb{D}})$ is a core for T_ω in case ω is proper. So, assume ω is proper. We need to show that the graph of $T_\omega|_{H(\overline{\mathbb{D}})}$ is dense in the graph of T_ω. In other words, let $f, g \in H^2$ with (f, g) perpendicular to $G(T_\omega|_{H(\overline{\mathbb{D}})})$, then we need to show (f, g) is perpendicular to $G(T_\omega)$. Since $k_\lambda \in H(\overline{\mathbb{D}})$, for $\lambda \in \mathbb{D}$, we have

$$0 = \langle (f, g), (k_\lambda, T_\omega k_\lambda)\rangle = \langle f, k_\lambda\rangle + \langle g, \overline{\omega^*(\lambda)} k_\lambda\rangle = f(\lambda) + \omega^*(\lambda) g(\lambda) \ (\lambda \in \mathbb{D}).$$

Hence $\omega^* g = -f$. In particular, $\omega^* g \in H^2$. Thus $g \in \mathrm{Dom}(T_{\omega^*}^{\mathrm{Sa}}) = \mathrm{Dom}(T_\omega^*)$ and $T_\omega^* g = -f$, by Proposition 4.6.1. For any $h \in \mathrm{Dom}(T_\omega)$ we have

$$\langle (f, g), (h, T_\omega h)\rangle = \langle (-T_\omega^* g, g), (h, T_\omega h)\rangle = -\langle T_\omega^* g, h\rangle + \langle g, T_\omega h\rangle = 0. \qquad \Box$$

In Section 8 of [12], Sarason introduced the class of closed, densely defined operators T on H^2 which satisfy

(1) $T_z \,\mathrm{Dom}(T) \subset \mathrm{Dom}(T)$;

(2) $T_z^* T T_z = T$;

(3) $f \in \mathrm{Dom}(T)$, $f(0) = 0 \Rightarrow T_z^* f \in \mathrm{Dom}(T)$.

This class of operators was further studied by Rosenfeld in [11, 10] in which he referred to such operators as Sarason-Toeplitz operators. The operators T_φ^{Sa}, for $\varphi \in N^+$, are Sarason-Toeplitz operators, and the class of operators is closed under taking adjoints, by Proposition 2.1 in [11]. Hence, by Proposition 4.6.1, T_ω is a Sarason-Toeplitz operator whenever $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper. We show that in fact T_ω is a Sarason-Toeplitz operator for any $\omega \in \mathrm{Rat}$.

Proposition 4.6.4. *Let $\omega \in \mathrm{Rat}$. Then T_ω on H^2 is a Sarason-Toeplitz operator.*

Proof. First consider $\omega \in \mathrm{Rat}(\mathbb{T})$. That T_ω satisfies (1) and (2) was proved in Lemma 2.2.3. We claim that $T_z^* \,\mathrm{Dom}(T_\omega) \subset \mathrm{Dom}(T_\omega)$. Write $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then $\mathrm{Dom}(T_\omega) = qH^2 + \mathcal{P}_{\deg(q)-1}$. Let $f = qh + r \in \mathrm{Dom}(T_\omega)$ with $h \in H^2$ and $r \in \mathcal{P}$, $\deg(r) < \deg(q)$. Then $T_z^* f = qT_z^* h + h(0) T_z^* q + T_z^* r$, which is in $qH^2 + \mathcal{P}_{\deg(q)-1} = \mathrm{Dom}(T_\omega)$. Hence T_ω is a Sarason-Toeplitz operator in case $\omega \in \mathrm{Rat}(\mathbb{T})$.

Now take $\omega \in \mathrm{Rat}$ arbitrarily. By Lemma 2.5.1, see also Section 4.4 above, $\omega = \omega_- z^\kappa \omega_0 \omega_+$ with $\kappa \in \mathbb{Z}$, and ω_-, ω_0 and ω_+ in Rat with zeroes and poles only inside, on or outside $\mathbb{T}$, respectively. In particular, $\omega_0 \in \mathrm{Rat}(\mathbb{T})$, ω_- and ω_-^{-1} are both anti-analytic, and ω_+ and ω_+^{-1} are both analytic. Also, $T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}$. Note that $z^\kappa \omega_0 \in \mathrm{Rat}(\mathbb{T})$ in case $\kappa \geq 0$ and $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$ in case $\kappa < 0$ (by Lemma 2.5.3). In both cases it now easily follows that $T_{z^\kappa \omega_0}$ is a Sarason-Toeplitz operator. The claim for T_ω follows since $T_{\omega_+}^{\pm 1} T_z = T_z T_{\omega_+}^{\pm 1}$ and $T_{\omega_-}^{\pm 1} T_z^* = T_z^* T_{\omega_-}^{\pm 1}$. $\Box$

In fact, by the same arguments one can show that T_ω on H^p, $1 < p < \infty$, satisfies (1)-(3) in case T_z^* is replaced by $T_{z^{-1}}$.

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Chapter 5

Matrix symbols

Abstract. This chapter concerns the analysis of a Toeplitz-like operator generated by a rational matrix function having poles on $\mathbb{T}$. It extends the analysis of such operators generated by scalar rational functions with poles on $\mathbb{T}$ found in [7], [8] and [9]. A Wiener-Hopf type factorization of rational matrix functions with poles and zeroes on $\mathbb{T}$ is introduced and then used to analyse the Fredholm properties of Toeplitz-like operators. A formula for the index, based on the factorization, is given. Furthermore, it is shown that the determinant having no zeroes on $\mathbb{T}$ is not sufficient for the operator being Fredholm, which is in contrast to the classical case.

5.1 Introduction

This chapter is a continuation of Chapters 2, 3 and 4 where Toeplitz-like operators with rational symbols which may have poles on the unit circle, $\mathbb{T}$, were introduced. Whilst the aim of the preceding chapters was to analyze such Toeplitz-like operators with scalar symbols, in this chapter we will focus on such Toeplitz-like operators with matrix symbol.

Let $\mathrm{Rat}^{m\times n}$ denote the space of $m \times n$ rational matrix functions. For positive integers m and n we denote by $\mathrm{Rat}^{m\times n}(\mathbb{T})$ the set of all $m \times n$ rational matrix functions with poles only on $\mathbb{T}$ and by $\mathrm{Rat}_0^{m\times n}(\mathbb{T})$ all strictly proper $m \times n$ rational matrix functions with poles only on the unit circle $\mathbb{T}$. The column vectors $\mathrm{Rat}^{m\times 1}(\mathbb{T})$ and $\mathrm{Rat}_0^{m\times 1}(\mathbb{T})$ will be written as $\mathrm{Rat}^{m}(\mathbb{T})$ and $\mathrm{Rat}_0^{m}(\mathbb{T})$, respectively. For the special case $m = 1 = n$, i.e. the scalar case, as in the earlier chapters we will use Rat, $\mathrm{Rat}(\mathbb{T})$ and $\mathrm{Rat}_0(\mathbb{T})$ for rational functions, rational functions with poles only on $\mathbb{T}$ and strictly proper rational functions with poles only on $\mathbb{T}$, respectively.

The space of $m \times n$ matrix polynomials will be denoted by $\mathcal{P}^{m\times n}$ and the column vector polynomials $\mathcal{P}^{m\times 1}$ will be denoted by $\mathcal{P}^{m}$ and, as before, the polynomials by $\mathcal{P}$. By $\mathcal{P}_k^{m\times n}$ we shall mean all $m \times n$ matrix polynomials of degree at most k. As before in Section 1.1.8, by L_m^p and H_m^p we shall mean functions with m-components and each component is in L^p and H^p, respectively.

A pole of rational matrix function is a pole of any of its entries. The zero of a rational matrix function, $\Omega(z)$, is a pole of its inverse, $\Omega^{-1}(z)$, see for example [3]. However, the zeroes of a matrix polynomial $P(z)$, are the $z \in \mathbb{C}$ where $\det P(z) = 0$.

Let $\Omega \in \mathrm{Rat}^{m\times m}$ with possibly poles on $\mathbb{T}$ and $\det \Omega(z) \not\equiv 0$. Define $T_\Omega\ (H^p_m \to H^p_m)$ by

$$\mathrm{Dom}(T_\Omega) = \left\{ f \in H^p_m : \begin{array}{c} \Omega f = h + r \text{ where } h \in L^p_m(\mathbb{T}), \\ \text{and} \quad r \in \mathrm{Rat}^m_0(\mathbb{T}) \end{array} \right\}$$

$$T_\Omega f = \mathbb{P}h \text{ where } \mathbb{P} \text{ is the Riesz projection of } L^p_m(\mathbb{T}) \text{ onto } H^p_m.$$

For simplicity we will consider the square case only but many of the results in this chapter extend to the non-square case, i.e., $m \neq n$.

The aim of this chapter is the determination of Fredholm-properties of Toeplitz-like operators with matrix symbol having poles on the unit circle. For the scalar case, the Fredholm-properties of Toeplitz-like operators received attention in Chapter 2. For the classical case, i.e., bounded block Toeplitz operators with rational symbol having no poles on the unit circle, the Fredholm-properties appear in Chapters XXIII and XXIV of [6] and are summarised in Section 1.1.8. In the classical case, the bounded block Toeplitz operator is Fredholm exactly when the determinant of the symbol has no zeroes on $\mathbb{T}$ (Theorem XXIII.4.3 in [6]). As we will see later, cf. Section 5.6, this is not the case for Toeplitz-like operators having poles on the unit circle due to possible pole-zero cancellation.

The Wiener-Hopf factorization of matrices with no poles on the unit circle (see for example Theorem 1.1.12 and Theorem XXIV.3.1 of [6]) allows one to determine invertibility conditions and Fredholm-properties of the block Toeplitz operator (see for example Theorem 1.1.13 and Theorem XXIV.4.1 and Theorem XXIV.4.2 of [6]).

Main Results. In our first main result, using an adaptation of the construction in [2] we prove a Wiener-Hopf type factorization for a rational matrix function with poles on the unit circle. As before in Section 1.1.8, we call an $m \times m$ rational matrix function a plus function if it has no poles on the closed unit disk $\overline{\mathbb{D}}$ and we call it a minus function if it no poles on $|z| \geq 1$ with the point at infinity included.

Theorem 5.1.1. *Let* $\Omega \in \mathrm{Rat}^{m\times m}$ *with* $\det \Omega \not\equiv 0$. *Then* $\Omega = z^{-k}\Omega_-\Omega_0 P_0 \Omega_+$ *for some* $k > 0$, Ω_- *and* $(\Omega_-)^{-1}$ *are minus functions ,* Ω_+ *and* $(\Omega_+)^{-1}$ *are plus functions,* $\Omega_0 = \mathrm{Diag}^m_{j=1}(\phi_j)$ *with* ϕ_j *a scalar rational function with poles and zeroes only on* $\mathbb{T}$ *and* P_0 *is a lower triangular matrix polynomial with* $\det(P_0(z)) = z^N$ *for some* $N \geq 0$.

Note that, if we choose P_0 to be lower triangular then the degree of the entries on the diagonal need not be in any order (increasing or decreasing), and conversely, if we choose to have either increasing or decreasing order of the degree of the entries on the diagonal, P_0 is not necessarily lower triangular. This is in sharp contrast to the classical Wiener-Hopf factorization result in Theorem 1.1.13 where the entries on the diagonal has increasing degree.

As is the case of rational matrix functions, the Wiener-Hopf type factorization of rational matrix functions, $\Omega(z)$, allows us to first factorise the Toeplitz-like operator T_Ω, and then determine its Fredholm properties by reducing it to the case of a diagonal matrix followed by a polynomial matrix function with determinant z^N, for some $N \geq 0$. Our second main result is the factorization of Toeplitz-like operator based on the Wiener-Hopf type factorization below.

Theorem 5.1.2. *Let $\Omega = z^{-k}\Omega_-\Omega_0 P_0 \Omega_+$ be the Wiener-Hopf type factorization of Ω as in Theorem 5.1.1. Then*

$$T_\Omega = T_{\Omega_-} T_{z^{-k}} T_{\Omega_0} T_{P_0} T_{\Omega_+}.$$

Our third main result is using the factorization to reduce the question on Fredholm properties to the case where the operator is generated by a matrix that is the product of a diagonal rational matrix function, $\Omega_0(z) \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$ with zeroes only on $\mathbb{T}$, and a polynomial matrix $P_0(z)$ with determinant z^N, $N \geq 0$.

Theorem 5.1.3. *Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ be the Wiener-Hopf type factorization of Ω as in Theorem 5.1.1. Then T_Ω is Fredholm if and only if T_{Ω_0} is Fredholm, which happens exactly when each of the entries ϕ_j of Ω_0 has no zeroes on $\mathbb{T}$. In case T_Ω is Fredholm,*

$$\begin{aligned} \mathrm{Index}\, T_\Omega &= mk + \mathrm{Index}\, T_{\Omega_0} + \mathrm{Index}\, T_{P_0} \\ &= mk + \sum_{j=1}^{m} \deg q_j - \sum_{j=1}^{m} k_j, \end{aligned}$$

where $\phi_j = \frac{s_j}{q_j}$ and q_j has roots only on $\mathbb{T}$ and k_j are the powers of z on the diagonal of P_0.

Comparison with literature. There are not many known results concerning unbounded (or closed) Toeplitz operators with matrix symbols. We use a factorization based on the Smith form for polynomial matrices (Theorem 1 and Theorem 2 of Gantmacher, chapter VI [4]) which is related to the Wiener-Hopf factorization for rational matrix functions without poles on the unit circle appearing in Theorem 2.1 of [2]. The proof of Theorem 2.1 in [2], in fact, uses the Smith form as a strating point.

Factorization of matrices, however, have a long tradition. Wiener-Hopf type factorization of matrices with no poles on the contour, for example, appears in Gohberg and Krein [11] and Clancey and Gohberg [2]; factorization of matrix functions as solutions to barrier problems in complex function theory appear as early as 1908 in Plemelj [12], whereas factorization of rational matrix functions relative to the unit circle appears as spectral factorization in electrical engineering in Belevitch [1] and Youla [14]. In all of these, however, there is a restriction of no poles on the contour or the unit circle.

The Wiener-Hopf type factorization for a rational matrix function with poles on the unit circle we use is based on the approach found in [2]. However, there is a slight oversight in the proof of Theorem 2.1 in Chapter 1 of [2]. An application of Lemma 5.3.2 below would eliminate this. It is not true that $ED(z)P(z) = D(z)EP(z)$ where E is a lower triangular elementary matrix with ones on the main diagonal and only one row of nonzero entries off the main diagonal, $D(z) = \mathrm{Diag}(z^{k_j})$ is a diagonal matrix with $k_1 \geq k_2 \geq \ldots \geq k_m$ and $P(z)$ a polynomial matrix function with $\det P(z)$ having a zero at $z = 0$. Applying Lemma 5.3.2 we write $L(z)D(z)P(z) = D(z)EP(z)$ where $L(z)$ is a lower triangular minus matrix function with ones on the main diagonal. And so the result of Theorem 2.1 in Chapter 1 of [2] follows using this adaptation.

The factorization allows us to consider the Fredholm properties of T_Ω by reducing to the case $\Omega \in \mathrm{Rat}(\mathbb{T} \cup \{0\})$. Using this, we determine the index, dimension of the kernel as well as the codimension of the range in case the operator is Fredholm.

There is some divergence from the situation where Ω has no poles on $\mathbb{T}$. In that case, from Theorem 4.3 in [6] we have that T_Ω is Fredholm if and only if $\det \Omega(z) \neq 0$ for $z \in \mathbb{T}$. When poles on $\mathbb{T}$ are allowed there could be pole-zero cancellation in the determinant of $\Omega(z)$ for $z \in \mathbb{T}$, for example $\Omega(z) = \mathrm{Diag}(\frac{z+1}{z-1}, \frac{z-1}{z+1})$ has $\det \Omega \equiv 1$. However, T_Ω is not Fredholm. Thus, there are cases where $\det \Omega(z) \neq 0$ for $z \in \mathbb{T}$ but T_Ω is not Fredholm, which does not happen in the case where Ω has no poles on $\mathbb{T}$.

Overview. The chapter is organized as follows: Besides the current introduction, the chapter consists of five sections. In Section 5.2 we prove basic results concerning the Toeplitz-like operator T_Ω. In the following section, Section 5.3, we prove the Wiener-Hopf type factorization of rational matrix functions with poles on the unit circle, Theorem 5.1.1. The next section, Section 5.4, is devoted to an example that illustrates the factorization. In the section that follows, Section 5.5, we prove the factorization of the Toeplitz-like operator, Theorem 5.1.2. This is then used to establish the shift invariance of the Toeplitz-like operator as stated in **??** below . Finally, Section 5.6 is devoted to the Fredholm properties of T_Ω, and in particular, we prove Theorem 5.1.3.

5.2 Basic properties of T_Ω

Using similar arguments as in the scalar case, cf., Chapter 2, we determine various basic properties of the Toeplitz-like operator T_Ω. Some of these results can be derived by restricting to the entries of Ω, in which case we give minimal details of the proof. We start with an analogue of Proposition 2.2.1.

Proposition 5.2.1. *Let $\Omega \in \mathrm{Rat}^{m\times m}$, possibly with poles on $\mathbb{T}$. Then T_Ω is a well-defined, closed, densely defined linear operator on H^p_m. More specifically, $\mathcal{P}^m \subset \mathrm{Dom}(T_\Omega)$. Moreover,*

$$S_- T_\Omega S_+ f = T_\Omega f \qquad \text{for all } f \in \mathrm{Dom}(T_\Omega), \tag{5.2.1}$$

where $S_+ = T_{zI_m}$ and $S_- = T_{z^{-1}I_m}$ on H^p_m.

Proof that T_Ω is well-defined, has dense domain and is a closed linear operator. The proof will follow by direct generalization of the arguments in Chapters 2 and 3, sometimes reducing to results for $T_{\omega_{ij}}$, where $\Omega = [\omega_{ij}]_{i,j=1}^m$.

For $\rho \in \mathrm{Rat}_0^m(\mathbb{T})$, using a similar argument as in Lemma 2.2.2 on its entries, it follows that ρ is identically zero whenever $\rho \in L_m^p$. Now following the argument in Proposition 2.2.1 shows that T_Ω is well-defined.

Let $\Omega \in \mathrm{Rat}^{m\times m}$. By entrywise application of Lemma 2.2.4 we can write $\Omega = \Omega_1 + \Omega_2$, where $\Omega_1 \in L_\infty^{m\times m}$ and $\Omega_2 \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$. Then $T_\Omega = T_{\Omega_1} + T_{\Omega_2}$ and the domains of T_Ω and T_{Ω_2} coincide. To see this, note that $f \in \mathrm{Dom}(T_\Omega)$ if and only if $f \in \mathrm{Dom}(T_{\Omega_2})$ and that the latter is the case if and only if $\Omega_2 f = h_2 + \rho$ where $h_2 \in L_2^m$ and $\rho \in \mathrm{Rat}_0^m(\mathbb{T})$. Now for such a function f consider $\Omega f = \Omega_1 f + \Omega_2 f$. Since $\Omega_1 \in L_\infty^{m\times m}$, also $\Omega_1 f \in L_2^m$. Moreover, we have

$$\Omega f = \Omega_1 f + \Omega_2 f = (\Omega_1 f + h_2) + \rho = h + \rho,$$

where $h = \Omega_1 f + h_2$. Now

$$\mathbb{P}h = \mathbb{P}(\Omega_1 f) + \mathbb{P}h = T_{\Omega_1} f + T_{\Omega_2} f = T_\Omega f$$

as desired. From this it follows that we only need to look at the case $\Omega \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$.

Let $\Omega \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$. Then we can write $\Omega = q^{-1}P$ where $q \in \mathcal{P}_\ell$ has zeros only on $\mathbb{T}$ and $P \in \mathcal{P}_{\ell-1}^{m\times m}$ for some $\ell \in \mathbb{N}$. Using a similar argument as in the proof of Lemma 2.2.3 we can show that $f \in \mathrm{Dom}(T_\Omega)$ if and only if $\Omega f = h + q^{-1}r$, where $h \in H_m^p$ and $r \in \mathcal{P}_{\ell-1}^m$. Moreover, r and h are unique, and in that case $T_\Omega f = h$. Now using a similar argument as in the proof of Proposition 2.2.1 it follows that T_Ω is closed and that the domain of T_Ω contains all the polynomials and so T_Ω is densely defined.

The proof of 5.2.1 is postponed to Section 5.5. □

For further basic properties we restrict to the case where $\Omega \in \mathrm{Rat}^{m\times m}(\mathbb{T})$, i.e., all poles on $\mathbb{T}$, or $\Omega \in \mathrm{Rat}^{m\times m}(\mathbb{T}\cup\{0\})$, where also a pole at 0 is allowed. In order to determine the Fredholm properties of T_Ω, via the factorization of Section 5.3 below, we reduce to the case of a diagonal matrix function in $\mathrm{Rat}^{m\times m}(\mathbb{T})$, with zeroes all on $\mathbb{T}$. Therefore we will not attempt here to give an explicit description of the kernel, range and domain for the case $\Omega \in \mathrm{Rat}^{m\times m}(\mathbb{T})$ in the form of an analogue of Theorem 3.2.2. For the diagonal matrix case, results are easily obtained by reduction to the scalar case. Here, and in the sequel, we shall identify the orthogonal sum $H^p \oplus \cdots \oplus H^p$ of m copies of H^p with H_m^p.

Proposition 5.2.2. *Suppose that* $\Omega(z) \in \mathrm{Rat}^{m\times m}(\mathbb{T})$ *with*

$$\Omega(z) = \begin{pmatrix} \omega_1(z) & & & \\ & \omega_2(z) & & \\ & & \ddots & \\ & & & \omega_m(z) \end{pmatrix}$$

and $\omega_j(z) \in \mathrm{Rat}(\mathbb{T}), j = 1, 2, \ldots m$. *Then*

1. $\mathrm{Dom}(T_\Omega) = \mathrm{Dom}(T_{\omega_1}) \oplus \mathrm{Dom}(T_{\omega_2}) \oplus \cdots \oplus \mathrm{Dom}(T_{\omega_m})$.

2. $\operatorname{Ran}(T_\Omega) = \operatorname{Ran}(T_{\omega_1}) \oplus \operatorname{Ran}(T_{\omega_2}) \oplus \cdots \oplus \operatorname{Ran}(T_{\omega_m})$.

3. $\operatorname{Ker}(T_\Omega) = \operatorname{Ker}(T_{\omega_1}) \oplus \operatorname{Ker}(T_{\omega_2}) \oplus \cdots \oplus \operatorname{Ker}(T_{\omega_m})$.

Proof. Suppose that $f_j \in \operatorname{Dom}(T_{\omega_j})$ then $\omega_j f_j = h_j + \rho_j$ with $h_j \in H^p$ and $\rho_j \in \operatorname{Rat}_0(\mathbb{T})$. But then

$$\Omega f = h + \rho$$

where

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{pmatrix},$$

and so $f \in H^p_m$, $h \in H^p_m$ and $\rho \in \operatorname{Rat}^m_0(\mathbb{T})$, showing that $f \in \operatorname{Dom}(T_\Omega)$ from which it follows that $\operatorname{Dom}(T_{\omega_1}) \oplus \operatorname{Dom}(T_{\omega_2}) \oplus \cdots \oplus \operatorname{Dom}(T_{\omega_m}) \subset \operatorname{Dom}(T_\Omega)$.

To show the converse, suppose that $f \in \operatorname{Dom}(T_\Omega)$ then there are $h \in H^p_m$ and $\rho \in \operatorname{Rat}^m_0(\mathbb{T})$ with $\Omega f = h + \rho$. But then $\omega_j f_j = h_j + \rho_j$ for each j showing that $f_j \in \operatorname{Dom}(T_{\omega_j})$.

Clearly, the fact that the domains coincide will also imply that the ranges coincide.

Suppose now that $f \in \operatorname{Ker}(T_\Omega)$, then $\Omega f = \rho$ where $\rho \in \operatorname{Rat}^m_0(\mathbb{T})$. But then $\omega_j f_j = \rho_j$ for $\rho_j \in \operatorname{Rat}_0(\mathbb{T})$ from which it follows that $f_j \in \operatorname{Ker}(T_{\omega_j})$. The converse is immediate. $\square$

Getting an explicit formulation of the domain of T_Ω beyond the diagonal case or the case where $\Omega \in \operatorname{Rat}^{m\times m}_0(\mathbb{T})$, i.e., where the entries of Ω are proper rational functions with poles only on $\mathbb{T}$, is much more complicated than in the scalar case. Let $\Omega \in \operatorname{Rat}^{m\times m}_0(\mathbb{T})$ and write $\Omega = q^{-1}P$ where q is a scalar polynomial and $P \in \mathcal{P}^{m\times m}$. Using a similar argument as in the proof of Lemma 2.2.3 we have

$$\operatorname{Dom}(T_\Omega) = \left\{ g \in H^p_m : \Omega g = h + q^{-1}\rho \right\}$$

where $h \in H^p_m$ and $\rho \in \mathcal{P}_{\deg(q)-1}$ and $T_\Omega g = h$.

Lemma 5.2.3. *Let* $\Omega \in \operatorname{Rat}^{m\times m}$ *and write* $\Omega = \Omega_1 + \Omega_2$ *where* $\Omega_1 \in L^{m\times m}_\infty$ *and* $\Omega_2 \in \operatorname{Rat}^{m\times m}_0(\mathbb{T})$. *Then* $\operatorname{Dom}(T_\Omega) = \operatorname{Dom}(T_{\Omega_2})$. *Let* $\Omega_2 = q^{-1}P$ *for some polynomial* q *with zeroes only on* $\mathbb{T}$ *and* P *a matrix polynomial. Suppose* $\Omega_2 = \left[\frac{s_{ij}}{q_{ij}}\right]$ *and let* $q_j = LCM^m_{i=1}\{q_{ij}\}$, *i.e., the least common multiple of the denominators in the* j*-th column of* Ω_2. *Then*

$$qH^p_m + \mathcal{P}^m_{\deg q - 1} \subset \oplus_j \left(q_j H^p + \mathcal{P}_{\deg q_j - 1}\right) \subset \operatorname{Dom}(T_\omega) \tag{5.2.2}$$

and both inclusions can be strict.

Proof. The first inclusion follows from the fact that zeroes of each q_j will be included in the zeroes of q and $qH^p + \mathcal{P}_{\deg q-1} \subset q_jH^p + \mathcal{P}_{\deg q_j-1}$ follows by extending the argument in Lemma 2.3.4 as follows: If $r, s \in \mathcal{P}$ and r divides s then $sH^p + \mathcal{P}_{\deg(s)-1} \subset rH^p + \mathcal{P}_{\deg(r)-1}$.

Let

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

and suppose $f_j = q_jh_j + r_j \in q_jH^p + \mathcal{P}_{\deg q_j-1}$. Then $q_j = u_{ij}q_{ij}$ for some polynomial u_{ij} and so $\frac{s_{ij}}{q_{ij}} = \frac{u_{ij}s_{ij}}{q_j}$ and

$$\begin{aligned} f_j &= q_jh_j + r_j \\ &= u_{ij}q_{ij}h_j + q_{ij}r_{ij} + \widetilde{r_j} \end{aligned}$$

for some $\widetilde{r_j} \in \mathcal{P}_{\deg q_{ij}-1}$ and $r_{ij} \in \mathcal{P}$.

Since the i-th entry in $\Omega_2 f$ is given by $\sum_{j=1}^m \frac{s_{ij}}{q_{ij}} f_j$ we have

$$\begin{aligned} (\Omega_2 f)_i = \sum_{j=1}^n \frac{s_{ij}}{q_{ij}} f_j &= \sum_{j=1}^m \frac{s_{ij}}{q_{ij}} \left(u_{ij}q_{ij}h_j + q_{ij}r_{ij} + \widetilde{r_j}\right) \\ &= \sum_{j=1}^m \left(u_{ij}s_{ij}h_j + s_{ij}r_{ij}\right) + \sum_{j=1}^m \frac{s_ij\widetilde{r_j}}{q_{ij}}. \end{aligned}$$

Writing $s_{ij}\widetilde{r_j} = k_jq_{ij} + \widehat{r_j}$ for some $k_j \in \mathcal{P}$ and $\widehat{r_j} \in \mathcal{P}_{\deg(q_{ij})-1}$ we arrive at

$$(\Omega_2 f)_i = \sum_{j=1}^m \left(u_{ij}s_{ij}h_j + s_{ij}r_{ij}\right) + \sum_{j=1}^m \frac{s_ij\widetilde{r_j}}{q_{ij}} = \widetilde{h_i} + \rho_i$$

where $\rho_i = \frac{r}{q_i} \in \mathrm{Rat}_0(\mathbb{T})$ and $\widetilde{h_i} \in H^p$. From this it follows that if f is as described, then $f \in \mathrm{Dom}(T_{\Omega_2})$. $\square$

We give an example that shows that the second inclusion is strict i.e., it is not necessarily true that $\mathrm{Dom}(T_\Omega)$ is the direct sum $\oplus_{j=1}^m \mathrm{Dom}(T_{\frac{1}{q_j}})$ where q_j is the LCM of the denominators of the j-th column of Ω. Take, for example, $\Omega(z) = \begin{pmatrix} 1+\frac{1}{z-1} & \frac{1}{z-1} \\ \frac{1}{z+1} & 1+\frac{1}{z+1} \end{pmatrix}$. Then $\det\Omega(z) = \frac{z^2+2z-1}{z^2-1} \not\equiv 0$ and $f = (f_1, -f_1)^T$ is in the domain of T_Ω for each $f_1 \in H^p$. Also $\frac{1}{z^2-1}$ is the LCM for the denominators in both columns of $\Omega(z)$. Suppose

$$\begin{pmatrix} f_1 \\ -f_1 \end{pmatrix} \in \mathrm{Dom}(T_{\frac{1}{z^2-1}}) \oplus \mathrm{Dom}(T_{\frac{1}{z^2-1}})$$

then $f_1 \in \mathrm{Dom}(T_{\frac{1}{z^2-1}})$. But for $f_1 \notin \mathrm{Dom}(T_{\frac{1}{z^2-1}})$ we still have $\begin{pmatrix} f_1 \\ -f_1 \end{pmatrix} \in \mathrm{Dom}(T_\Omega)$ and so

$$\mathrm{Dom}(T_\Omega) \neq \mathrm{Dom}(T_{\frac{1}{z^2-1}}) \oplus \mathrm{Dom}(T_{\frac{1}{z^2-1}}).$$

For $q(z) = z^2 - 1$, from Theorem 2.3.3 there exists an $f \in (z-1)H^p$ with $f \notin qH^p + \mathcal{P}_1$. Thus $(f, 0) \notin qH_2^p + \mathcal{P}_1^2$ and so $qH_2^p + \mathcal{P}_1^2 \neq ((z-1)H^p + \mathbb{C}) \oplus ((z+1)H^p + \mathbb{C})$. From this it follows that the first inclusion in 5.2.2 can also be strict.

5.3 Matrix Function Factorization

In this section we prove the construction of the Wiener-Hopf type factorization, Theorem 5.1.1, with respect to $\mathbb{T}$ for $m \times m$ rational matrix functions which are allowed to have poles on $\mathbb{T}$.

The construction relies strongly on the Smith decomposition of matrix polynomials; see Gantmacher [4] for a proof. The following theorem is an easy consequence of the Smith form.

Theorem 5.3.1. *Let $R(z) \in \mathcal{P}^{m\times m}$. Then we can write*

$$R(z) = E(z)D(z)F(z)$$

where E, F are polynomial matrices with nonzero constant determinants and D a diagonal polynomial matrix that factors as

$$D(z) = D_-(z)D_0(z)D_+(z)$$

with D_-, D_0 and D_+ also diagonal polynomial matrices with the entries of $D_-(z), D_0(z)$ and $D_+(z)$ having their zeroes in $\mathbb{D}$, on $\mathbb{T}$ and outside $\overline{\mathbb{D}}$ respectively.

The next result will be used to repair the oversight in the construction in [2].

Lemma 5.3.2. *Let $F(z) \in \mathcal{P}^{m\times m}$ with $\det F(z) = z^k$. Then*

$$F(z) = L(z)\begin{pmatrix} z^{n_1} & & & 0 \\ & z^{n_2} & & \\ & & \ddots & \\ 0 & & & z^{n_m} \end{pmatrix} R(z)$$

where $R(z)$ is a matrix polynomial with nonzero constant determinant, $n_j \geq 0$ with $\sum_{i=1}^m n_i = k$ and $L(z)$ is a lower triangular matrix where each entry in the strictly lower part is a polynomial in z^{-1} and z while the main diagonal entries are one. In addition, $L(z)\mathrm{Diag}(z^{n_j}) \in \mathcal{P}^{m\times m}$.

Proof. First write

$$F(z) = \begin{pmatrix} z^{n_1} & & & \\ & z^{n_2} & & \\ & & \ddots & \\ & & & z^{n_m} \end{pmatrix} R_1(z)$$

where $R_1(z)$ is a matrix polynomial and n_i is the highest power of z dividing all the entries in row i. Then $\det R_1(z) = z^{k'}$ with $k' = k - \sum_{i=1}^m n_i$. If $\sum_{i=1}^m n_i = k$ we are done and $L(z) = I$. Otherwise, write

$$R_1(z) = \begin{pmatrix} r_1(z) \\ r_2(z) \\ \vdots \\ r_m(z) \end{pmatrix}.$$

Then $r_1(0), r_2(0), \dots, r_m(0)$ are linearly dependent. Let p be the smallest integer such that $r_1(0), \dots, r_p(0)$ are linearly dependent. Then there are numbers $\alpha_1, \alpha_2, \dots, \alpha_{p-1}$ such that

$$\alpha_1 r_1(0) + \alpha_2 r_2(0) + \cdots + \alpha_{p-1} r_{p-1}(0) + r_p(0) = 0.$$

Put

$$E = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & & & & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} & 1 & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ - p^{\text{th}} \text{ row}. \\ \\ \\ \\ \end{matrix}$$

Then

$$F(z) = \begin{pmatrix} z^{n_1} & & & \\ & z^{n_2} & & \\ & & \ddots & \\ & & & z^{n_m} \end{pmatrix} E^{-1} E R_1(z).$$

Now

$$E R_1(z) = \begin{pmatrix} r_1(z) \\ \vdots \\ r_{p-1}(z) \\ \sum_{i=1}^p \alpha_i r_i(z) \\ r_{p+1}(z) \\ \vdots \\ r_m(z) \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & z^l & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} R_2(z)$$

for some $l \geq 1$ and for a matrix polynomial $R_2(z)$, since $\sum_{i=1}^m \alpha_i r_i(z)$ has a zero at zero.

Note that

$$E^{-1} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & & & & \\ -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{p-1} & 1 & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} - p^{\text{th}} \text{ row}$$

and by direct computation it follows that

$$\begin{pmatrix} z^{n_1} & & & \\ & z^{n_2} & & \\ & & \ddots & \\ & & & z^{n_m} \end{pmatrix} E^{-1} = L(z) \begin{pmatrix} z^{n_1} & & & \\ & z^{n_2} & & \\ & & \ddots & \\ & & & z^{n_m} \end{pmatrix}.$$

where

$$L(z) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & & & & \\ -\alpha_1 z^{n_p-n_1} & -\alpha_2 z^{n_p-n_2} & \cdots & -\alpha_{p-1} z^{n_p-n_{p-1}} & 1 & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} - p^{\text{th}} \text{ row}.$$

We now have

$$F(z) = L(z) \begin{pmatrix} z^{n_1} & & & & & & \\ & \ddots & & & & & \\ & & z^{n_{p-1}} & & & & \\ & & & z^{n_p+l} & & & \\ & & & & z^{n_{p+1}} & & \\ & & & & & \ddots & \\ & & & & & & z^{n_m} \end{pmatrix} R_2(z),$$

and

$$\det R_2(z) = z^{k-l-\sum_{i=1}^m n_i}.$$

Put $\widetilde{n}_j = n_j$ for $j \neq p$ and $\widetilde{n}_p = n_p + l$. Note that $\sum_{i=1}^m \widetilde{n}_i = (\sum_{i=1}^m n_i) + l$ and

$$\widetilde{F}(z) = \begin{pmatrix} z^{\widetilde{n}_i} & & \\ & \ddots & \\ & & z^{\widetilde{n}_m} \end{pmatrix} R_2(z)$$

is a matrix polynomial with $\det \widetilde{F}(z) = \det F(z) = z^k$. If $\sum_{i=1}^m \widetilde{n}_i = k$, then we are done, otherwise repeat the argument. $\square$

We are now ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. The proof is an adaptation of the proof of Theorem 2.1 in [2]. Firstly, write

$$q(z)\Omega(z) = P_1(z),$$

where P_1 is a polynomial matrix and q a scalar polynomial function. Then we can write the scalar rational function $q^{-1}(z) = \omega_-(z)\omega_0(z)z^\kappa\omega_+(z)$ as in Lemma 2.5.1 where ω_- and ω_-^{-1} are minus functions, ω_+ and ω_+^{-1} are plus functions and ω_0 has zeroes and poles only on $\mathbb{T}$.

As in [2], by Theorem 5.3.1, the Smith form for polynomial matrices, we can write

$$P_1(z) = E_1(z)D_1^-(z)D_1^0(z)D_1^+(z)F_1(z)$$

where E_1, F_1 are polynomial matrices with nonzero constant determinants, $D_1^- = \mathrm{Diag}(d_j^-)$ with d_j^- having roots only inside $\mathbb{D}$, $D_1^0 = \mathrm{Diag}(d_j^0)$ having roots only on $\mathbb{T}$ and $D_1^+ = \mathrm{Diag}(d_j^+)$ with d_j^+ having roots only outside $\overline{\mathbb{D}}$ and so $D_1^+F_1$ is invertible in $\mathbb{D}$. Then $\det\big(E_1(z)D_1^-(z)D_1^0(z)\big)$ has its zeroes only in $\overline{\mathbb{D}}$. Let $N > 0$ be such that

$$P_2(z) = z^N E_1(z^{-1})D_1^-(z^{-1})D_1^0(z^{-1})$$

is a polynomial matrix then the zeroes of P_2 can only be on $\mathbb{T}$, outside $\mathbb{D}$ except at ∞ or at zero. And so

$$P_1(z) = z^N P_2(z^{-1})D_1^+(z)F_1(z).$$

As above, using the Smith form for polynomial matrices, we can write

$$P_2(z) = E_2(z)D_2^-(z)D_2^0(z)D_2^+(z)F_2(z)$$

where E_2, F_2 are polynomial matrices with nonzero constant determinants, $D_2^- = \mathrm{Diag}(e_j)$ with e_j having roots only outside $\mathbb{D}$, $D_2^0 = \mathrm{Diag}(\psi_j)$ with ψ_j having roots only on $\mathbb{T}$ and $D_2^+ = \mathrm{Diag}(z^{n_j})$ with $n_j \geq 0$.

Since $D_2^0(z)$ is a diagonal matrix with each of its entries a polynomial with roots only on $\mathbb{T}$, we can write $D_2^0(\frac{1}{z}) = \widetilde{D}_2^0(z)\widetilde{D}_2^-(z)$ where $\widetilde{D}_2^0(z)$ is a diagonal matrix with each of its entries a polynomial with roots only on $\mathbb{T}$ and $\widetilde{D}_2^-(z)$ a diagonal matrix with each of its entries of the form z^{η_j} and $\eta_j \leq 0$. For some integer $K > 0$, $P_3(z) = z^K\widetilde{D}_2^-(z)D_2^+(z^{-1})F_2(z^{-1})$ is a polynomial matrix function such that $\det P_3(z)$ is zero only at zero. And so

$$P_1(z) = z^{N-K}E_2(z^{-1})D_2^-(z^{-1})\widetilde{D}_2^0(z)P_3(z)D_1^+(z)F_1(z).$$

From Lemma 5.3.2 it follows that we can write $P_3(z) = L(z)D_3(z)F_3(z)$ where $L(z)$ is a lower triangular matrix function where each entry in the strictly lower part is a polynomial in z^{-1} and z while the diagonal entries are one, $D_3(z) = \mathrm{Diag}(z^{n_j})$ with $n_j \geq 0$ and $F_3(z)$ is a polynomial matrix with constant determinant.

Note that

$$E_1(z^{-1})D_1^-(z^{-1})D_1^0(z^{-1}) = z^{-N}P_2(z) = z^{-N}E_2(z)D_2^-(z)D_2^0(z)D_2^+(z)F_2(z)$$

and so

$$\begin{aligned}
P_1(z) &= E_1(z)D_1^-(z)D_1^0(z)D_1^+(z)F_1(z)\\
&= z^N P_2(z^{-1})D_1^+(z)F_1(z)\\
&= z^N E_2(z^{-1})D_2^-(z^{-1})D_2^0(z^{-1})D_2^+(z^{-1})F_2(z^{-1})D_1^+(z)F_1(z)\\
&= z^N E_2(z^{-1})D_2^-(z^{-1})\widetilde{D}_2^0(z)\widetilde{D}_2^-(z)D_2^+(z^{-1})F_2(z^{-1})D_1^+(z)F_1(z)\\
&= z^N E_2(z^{-1})D_2^-(z^{-1})\widetilde{D}_2^0(z)z^{-K}P_3(z)D_1^+(z)F_1(z)\\
&= z^{N-K}E_2(z^{-1})D_2^-(z^{-1})\widetilde{D}_2^0(z)L(z)D_3(z)F_3(z)D_1^+(z)F_1(z).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\Omega(z) &= q^{-1}(z)P_1(z)\\
&= z^{N-K+\kappa}\omega_-(z)E_2(z^{-1})D_2^-(z^{-1})\omega_0(z)\widetilde{D}_2^0(z)L(z)D_3(z)\omega_+(z)F_3(z)D_1^+(z)F_1(z)\\
&= z^M\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z),
\end{aligned}$$

where $\Omega_-(z) = z^\eta\omega_-(z)E_2(z^{-1})D_2^-(z^{-1})$ is such that Ω_- and its inverse Ω_-^{-1} are minus functions for a suitable η, $M + \eta = N - K + \kappa$, $\Omega_+(z) = \omega_+(z)F_3(z)D_1^+(z)F_1(z)$ and its inverse are plus functions, $\Omega_0(z) = \omega_0(z)\widetilde{D}_2^0(z)$ and $P_0(z) = L(z)D_3(z)$. $\square$

Corollary 5.3.3. *Let* $\Omega \in \mathrm{Rat}^{m\times m}$ *with* $\det\Omega \not\equiv 0$ *and suppose*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

is the factorization of Ω *as in Theorem 5.1.1. Then the zeroes and poles of* Ω *on* $\mathbb{T}$ *correspond to the zeroes and poles of* ψ_j *where* $\Omega_0(z) = \mathrm{Diag}\,(\psi_j(z))_{j=1}^m$.

5.4 An example

We present an example illustrating the factorisation procedure. Let $\Omega(z) = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}$ then $q(z) = z - 1$ and $q(z)\Omega(z) = \begin{pmatrix} z-1 & 1 \\ 0 & z-1 \end{pmatrix}$. The Smith decomposition of $q\Omega$ is given by

$$\begin{aligned}
q(z)\Omega(z) &= E_1(z)D_1(z)F_1(z)\\
&= E_1(z)D_1^-(z)D_1^0(z)D_1^+(z)F_1(z)\\
&= \begin{pmatrix} 0 & 1 \\ -1 & z-1 \end{pmatrix}\begin{pmatrix} (z-1)^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix}
\end{aligned}$$

where $D_1^-(z) = I_2 = D_1^+(z)$, the 2×2 identity matrix. Then

$$E_1(z)D_1^-(z)D_1^0(z) = \begin{pmatrix} 0 & 1 \\ -(z-1)^2 & z-1 \end{pmatrix}$$

and so

$$P_2(z) = z^2 E_1(\frac{1}{z})D_1^-(\frac{1}{z})D_1^0(\frac{1}{z}) = z^2 \begin{pmatrix} 0 & 1 \\ -(\frac{1}{z}-1)^2 & \frac{1}{z}-1 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ -(z-1)^2 & z-z^2 \end{pmatrix}.$$

The Smith decomposition of $P_2(z)$ is given by

$$\begin{aligned} P_2(z) &= E_2(z)D_2(z)F_2(z) \\ &= \begin{pmatrix} -z-1 & -z^2 \\ 1 & z-1 \end{pmatrix} \begin{pmatrix} (z-1)^2z^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (z-1)^2(z+1) & z(z^2-z-1) \end{pmatrix} \end{aligned}$$

and so

$$D_2^+(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_2^0(z) = \begin{pmatrix} (z-1)^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } D_2^-(z) = I_2.$$

Put

$$D_2^0(\frac{1}{z}) = \widetilde{D}_2^0(z)\widetilde{D}_2^-(z) = \begin{pmatrix} \left(\frac{1}{z}-1\right)^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1-z)^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix}$$

Now

$$\begin{aligned} P_3(z) &= z^K \widetilde{D}_2^-(z)D_2^+(\frac{1}{z})F_2(\frac{1}{z}) \\ &= z^4 \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ \left(\frac{1}{z}-1\right)^2\left(\frac{1}{z}+1\right) & \frac{1}{z}\left(\frac{1}{z^2}-\frac{1}{z}-1\right) \end{pmatrix} \end{aligned}$$

from which it follows that

$$P_3(z) = \begin{pmatrix} -1 & -1 \\ z(1-z)^2(1+z) & z(1-z-z^2) \end{pmatrix}$$

and $\det P_3(z) = z^4$.

Now

$$P_3(z) = D(z)R(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^3-z^2-z+1 & 1-z-z^2 \end{pmatrix}$$

Since the sum of the row multiplicities $= 1 < 4$ with $r_1(0) = [-1, -1]$ and $r_2(0) = [1, 1]$ we put $E = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. So

$$\begin{aligned} P_3(z) &= D(z)E^{-1}ER(z) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^3 - z^2 - z + 1 & 1 - z - z^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^3 - z^2 - z & -z - z^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^2 - z - 1 & -1 - z \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^2 - z - 1 & -1 - z \end{pmatrix} \\ &= L'(z)D'(z)R_2(z) \end{aligned}$$

The sum of the row multiplicities $= 2 < 4$ with $r'_1(0) = [-1, -1]$ and $r'_2(0) = [-1, -1]$ we put $E' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

$$\begin{aligned} P_3(z) &= L'(z)D'(z)R_2(z) = L'(z)D'(z)(E')^{-1}E'R_2(z) \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^2 - z - 1 & -1 - z \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^2 - z & -z \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^2 - z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z - 1 & -1 \end{pmatrix} \\ &= L''(z)D''(z)R_2(z). \end{aligned}$$

The sum of the row multiplicities is still less than 4 and so we continue the process. Now $r''_1(0) = [-1, -1] = r''_2(0)$ and so we put $E'' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. So

$$\begin{aligned} P_3(z) &= L''(z)D''(z)R_2(z) = L''(z)D''(z)(E'')^{-1}E''R_2(z) \\ &= \begin{pmatrix} 1 & 0 \\ z^2 - z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z - 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^2 - z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^4 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= L'''(z)D'''(z)R_3(z). \end{aligned}$$

From the above we see that

$$L(z) = \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & 1 \end{pmatrix}, \quad D_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^4 \end{pmatrix}, \quad F_3(z) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} P_1(z) &= z^{N-K} E_2(\tfrac{1}{z}) D_2^-(\tfrac{1}{z}) \widetilde{D}_2^0(z) L(z) D_3(z) F_3(z) \\ &= z^{-2} \begin{pmatrix} -\frac{1}{z}-1 & -\frac{1}{z^2} \\ 1 & \frac{1}{z}-1 \end{pmatrix} \begin{pmatrix} (1-z)^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^3+z^2-z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^4 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Since $q(z) = z-1$, $\omega_-(z) = 1, \omega_+(z) = 1, \kappa = 0$ and $\omega_0(z) = \frac{1}{z-1}$ we have

$$\Omega(z) = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix} = z^{-k} \Omega_-(z) \Omega_0(z) P_0(z) \Omega_+(z)$$

where $k = 2$,

$$\Omega_-(z) = z^{\kappa} \omega_- E_2(\frac{1}{z}) D_2^-(\frac{1}{z}) = \begin{pmatrix} -\frac{1}{z}-1 & -\frac{1}{z^2} \\ 1 & \frac{1}{z}-1 \end{pmatrix},$$

$$\Omega_+(z) = F_3(z) F_1(z) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix} = \begin{pmatrix} -z & -1 \\ 1 & 0 \end{pmatrix},$$

$$\Omega_0(z) = \omega_0(z) \widetilde{D}_2^0(z) = \begin{pmatrix} z-1 & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} \quad \text{and}$$

$$P_0(z) = L(z) D_3(z) = \begin{pmatrix} 1 & 0 \\ z^3+z^2-z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^3+z^2-z & z^4 \end{pmatrix}.$$

5.5 Factorization of the Toeplitz operator

In this section we prove Theorem 5.1.2 by using the Wiener-Hopf type factorization of $\Omega \in \mathrm{Rat}^{m\times m}$ with possible poles on $\mathbb{T}$. We first prove some technical lemmas.

Theorem 5.5.1. *Suppose that $\Omega \in \mathrm{Rat}^{m\times m}$, U and its inverse U^{-1} are $m \times m$ minus rational matrix functions and V is an analytic $m \times m$ rational matrix function. Then $T_{\Omega V} = T_\Omega T_V$ and $T_{U\Omega} = T_U T_\Omega$.*

Proof. Let $f \in \mathrm{Dom}(T_{\Omega V})$. By Proposition 5.2.1, we have $\Omega V f = h + \rho \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$ and $T_{\Omega V} f = \mathbb{P} h$. Now since V is analytic, $T_V f = V f$. Put $g = V f$, then by the above relation we have $g \in \mathrm{Dom}(T_\Omega)$ and $T_\Omega g = \mathbb{P} h$. This just says that $T_\Omega T_V f = T_{\Omega V} f$. So $\mathrm{Dom}(T_{\Omega V}) \subset \mathrm{Dom}(T_\Omega T_V)$, and on $\mathrm{Dom}(T_{\Omega V})$ the two operators coincide.

Conversely, suppose that $f \in \mathrm{Dom}(T_\Omega T_V)$. Then $T_V f = V f \in \mathrm{Dom}(T_\Omega)$. Since V is analytic we have $\Omega V f = h + \rho \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$, and so $f \in \mathrm{Dom}(T_{\Omega V})$.

Note that for every $f \in L^p_m$ we have $\mathbb{P} U (I - \mathbb{P}) f = 0$ because U is analytic outside $\overline{\mathbb{D}}$. Let $f \in \mathrm{Dom}(T_{U\Omega})$, then $U \Omega f = h + \rho$, with $h \in L^p_m$ and $\rho \in \mathrm{Rat}_0(\mathbb{T})$. Then $\Omega f = U^{-1} h + U^{-1} \rho$. Now, since U^{-1} is a rational matrix function analytic outside $\overline{\mathbb{D}}$, the function $U^{-1}\rho$ can be written as $h_1 + \rho_1$ with $h_1 \in L^p_m$ a rational function with poles only in the unit disc and $\rho_1 \in \mathrm{Rat}^m_0(\mathbb{T})$. In particular, $\mathbb{P} h_1 = 0$. So, $\Omega f = U^{-1} h + h_1 + \rho_1$, which shows that

$f \in \mathrm{Dom}(T_\Omega)$ and $T_\Omega f = \mathbb{P}(U^{-1}h + h_1) = \mathbb{P}(U^{-1}h)$. Then $T_U T_\Omega f = \mathbb{P}U\mathbb{P}(U^{-1}h)$. Since $\mathbb{P}U(I-\mathbb{P})(U^{-1}h) = 0$, we have

$$T_U T_\Omega f = \mathbb{P}U\mathbb{P}(U^{-1}h) = \mathbb{P}U\mathbb{P}(U^{-1}h) + \mathbb{P}U(I-\mathbb{P})(U^{-1}h) = \mathbb{P}UU^{-1}h = \mathbb{P}h = T_{U\Omega}f$$

as was claimed. □

Lemma 5.5.2. *Let* $\Omega \in \mathrm{Rat}^{m\times m}(\mathbb{T})$. *Then for* $k \geq 0$, $T_{z^{-k}\Omega} = T_{z^{-k}}T_\Omega$.

Proof. We need to show that $\mathrm{Dom}(T_\Omega) = \mathrm{Dom}(T_{z^{-k}\Omega})$ and that $T_{z^{-k}\Omega}$ and $T_{z^{-k}}T_\Omega$ coincides on $\mathrm{Dom}(T_\Omega)$. Suppose that $f \in \mathrm{Dom}(T_{z^{-k}\Omega})$. Then $z^{-k}\Omega f = h + \rho \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$ and so $\Omega f = z^k h + z^k \rho \in L^p_m + \mathrm{Rat}^m(\mathbb{T})$. Apply the Euclidian algorithm to write $z^k\rho = \rho_1 + \rho_2 \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$. Then $\Omega f = (z^k h + \rho_1) + \rho_2 \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$ from which it follows that $f \in \mathrm{Dom}(T_\Omega)$.

Conversely suppose $f \in \mathrm{Dom}(T_\Omega)$. Then $\Omega f = h + \rho \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T})$ so $\Omega z^{-k} f = z^{-k}h + z^{-k}\rho \in L^p_m + \mathrm{Rat}^m_0(\mathbb{T}\cup\{0\})$. Now write $z^{-k}\rho = \rho_1 + \rho_2$ with $\rho_2 \in \mathrm{Rat}^m_0(\mathbb{T})$ and $\rho_1 \in \mathrm{Rat}^m$, with only a pole at 0. Then $\rho_1 \in L^p_m$ and $\mathbb{P}\rho_1 = 0$. We now have $z^{-k}\Omega f = (z^{-k}h + \rho_1) + \rho_2 \in L^2_m + \mathrm{Rat}^m_0(\mathbb{T})$ and so $f \in \mathrm{Dom}(T_{z^{-k}\Omega})$. From this it follows that $\mathrm{Dom}(T_\Omega) = \mathrm{Dom}(T_{z^{-k}\Omega})$.

Now, suppose $f \in \mathrm{Dom}(T_\Omega)$. Then, from the above and the fact that $\mathbb{P}z^{-k}(I-\mathbb{P}) = 0$, we have

$$T_{z^{-k}\Omega}f = \mathbb{P}(z^{-k}h + \rho_1) = \mathbb{P}(z^{-k}h) = \mathbb{P}z^{-k}\mathbb{P}h = T_{z^{-k}}\mathbb{P}h = T_{z^{-k}}T_\Omega f.$$

□

Proof of Theorem 5.1.2. Factor $\Omega = z^{-k}\Omega_-\Omega_0 P_0 \Omega_+$ as in Theorem 5.1.1. Then from Theorem 5.5.1 we can write

$$T_\Omega = T_{\Omega_-}T_{z^{-k}\Omega_0}T_{P_0}T_{\Omega_+}$$

and applying Lemma 5.5.2 we have

$$T_\Omega = T_{\Omega_-}T_{z^{-k}}T_{\Omega_0}T_{P_0}T_{\Omega_+}.$$

□

Proof of 5.2.1 in Proposition 5.2.1. Since $z^{-1}I_m$ is a minus function, $S_- = T_{z^{-1}I_m}$ commutes with T_{Ω_-} and since zI_m is a plus function, $S_+ = T_{zI_m}$ commutes with $T_{P_0}T_{\Omega_+}$. Applying Theorem 5.1.2 along with the factorization $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ from Theorem 5.1.1, on $\mathrm{Dom}(T_\Omega)$, we get

$$\begin{aligned}
S_-T_\Omega S_+ &= T_{z^{-1}I_m}T_{\Omega_-}T_{z^{-k}}T_{\Omega_0}T_{P_0}T_{\Omega_+}T_{zI_m} \\
&= T_{\Omega_-}T_{z^{-k}}T_{z^{-1}I_m}T_{\Omega_0}T_{zI_m}T_{P_0}T_{\Omega_+} \\
&= T_{\Omega_-}T_{z^{-k}}T_{z^{-1}I_m}T_{z\Omega_0}T_{P_0}T_{\Omega_+} \qquad \text{follows by applying Theorem 5.5.1 to } T_{z\Omega_0} \\
&= T_{\Omega_-}T_{z^{-k}}T_{z^{-1}z\Omega_0}T_{P_0}T_{\Omega_+} \qquad \text{follows by applying Lemma 5.5.2 to } T_{z^{-1}z\Omega_0} \\
&= T_{\Omega_-}T_{z^{-k}}T_{\Omega_0}T_{P_0}T_{\Omega_+} \\
&= T_\Omega.
\end{aligned}$$

This completes the proof of Proposition 5.2.1. □

5.6 Fredholm properties

Using the Wiener-Hopf type factorization for $m \times m$ rational matrix functions with poles (and zeroes) on $\mathbb{T}$, we are able to reduce the Fredholm properties to the diagonal case, in particular we obtain Theorem 5.1.3. We start with the diagonal case.

Theorem 5.6.1. *Suppose that* $\Omega(z) \in \mathrm{Rat}^{m\times m}(\mathbb{T})$ *with*

$$\Omega(z) = \begin{pmatrix} \omega_1(z) & & & \\ & \omega_2(z) & & \\ & & \ddots & \\ & & & \omega_m(z) \end{pmatrix}$$

and $\omega_j(z) \in \mathrm{Rat}(\mathbb{T}), j = 1, 2, \ldots, m$*. Then* T_Ω *is Fredholm if and only if* T_{ω_j} *is Fredholm for each* $j = 1, \ldots m$*, i.e., if and only if* ω_j *has no zeroes on* $\mathbb{T}$ *for each* $j = 1, \ldots, m$*. In case* T_Ω *is Fredholm we have*

$$\mathrm{Index}\, T_\Omega = \sum_{j=1}^{m} \mathrm{Index}\, T_{\omega_j}.$$

Proof. From Proposition 5.2.2 we have $\mathrm{Ran}(T_\Omega) = \mathrm{Ran}(T_{\omega_1}) \oplus \mathrm{Ran}(T_{\omega_2}) \oplus \cdots \oplus \mathrm{Ran}(T_{\omega_m})$ and so $\mathrm{Ran}(T_\Omega)$ is closed if and only if $\mathrm{Ran}(T_{\omega_j})$ is closed for each j. It follows, thus, that T_Ω is Fredholm if and only if each T_{ω_j} is Fredholm. Given that we also have $\mathrm{Ker}(T_\Omega) = \mathrm{Ker}(T_{\omega_1}) \oplus \mathrm{Ker}(T_{\omega_2}) \oplus \cdots \oplus \mathrm{Ker}(T_{\omega_m})$ it now follows that $\mathrm{Index}\, T_\Omega = \sum_{j=1}^{m} \mathrm{Index}\, T_{\omega_j}$. $\square$

Note that, in contrast to the case when Ω has no poles on the unit circle, it is not true that T_Ω is Fredholm if and only if $\det \Omega$ is not zero on $\mathbb{T}$. Take, for example, $\Omega(z) = \mathrm{Diag}(\frac{z+1}{z-1}, \frac{z-1}{z+1})$ then $\det \Omega(z) = 1$, but T_Ω is not Fredholm.

We are now ready to prove Theorem 5.1.3.

Proof of Theorem 5.1.3. Applying Theorem 5.1.2 allows us to write

$$T_\Omega = T_{\Omega_-} T_{z^{-k}} T_{\Omega_0} T_{P_0} T_{\Omega_+}.$$

Given that $\Omega_-(z)$ and its inverse are minus functions, T_{Ω_-} is invertible with $T_{\Omega_-}^{-1} = T_{\Omega_-^{-1}}$. Similarly T_{Ω_+} is invertible and $T_{\Omega_+}^{-1} = T_{\Omega_+^{-1}}$, as $\Omega_+(z)$ and its inverse are plus functions. Thus T_{Ω_-}, $T_{\Omega_-^{-1}}$, T_{Ω_+} and $T_{\Omega_+^{-1}}$ are Fredholm and

$$T_{z^{-k}} T_{\Omega_0} T_{P_0} = T_{\Omega_-^{-1}} T_\Omega T_{\Omega_+^{-1}}.$$

Applying Theorems IV.2.7 and IV.2.10 from [5] (see also [10]) it now follows that T_Ω is Fredholm if and only if $T_{z^{-k}} T_{\Omega_0} T_{P_0}$ is Fredholm and in that case we have

$$\mathrm{Index}\, T_\Omega = \mathrm{Index}\,(T_{z^{-k}} T_{\Omega_0} T_{P_0}).$$

Since Ω_0 and $z^{-k}\Omega_0$ are diagonal matrices, the question on Fredholm properties of $T_{z^{-k}\Omega_0}$ viz-a-viz $T_{z^{-k}}T_{\Omega_0}$ reduces to the scalar case. From Theorem 5.6.1 it now follows that $T_{z^{-k}\Omega_0}$ is Fredholm if and only if T_{Ω_0} is Fredholm, and in case T_{Ω_0} is Fredholm we have

$$\operatorname{Index} T_{z^{-k}\Omega_0} = \operatorname{Index} T_{z^{-k}} + \operatorname{Index} T_{\Omega_0}.$$

Suppose T_{Ω_0} is Fredholm. Since $P_0(z)$ is a polynomial matrix function with zeroes only at $z = 0$, T_{P_0} is a bounded Fredholm operator. Then, again from [5], we have that $T_{z^{-k}\Omega}T_{P_0}$ is Fredholm.

Conversely, suppose $T_{z^{-k}\Omega}T_{P_0}$ is Fredholm. By Theorem 3.4 of [13] we have that $T_{z^{-k}\Omega}$ is Fredholm. From this it now follows that T_ω is Fredholm if and only if T_{Ω_0} is Fredholm.

Given that $P_0(z) = L(z)P_2(z)$ is a polynomial matrix function with zero as the only root of $\det P_0$ and that $\det L(z) = 1$ it follows that $\operatorname{Index} T_{P_0} = \operatorname{Index} T_{P_2} = -\sum_{j=1}^m k_j$ where k_j are the powers of z on the diagonal of $P_2(z)$. This can be seen from the fact that if $L(z)P_2(z) = M(z)D_+(z)P(z)$ is a Wiener-Hopf factorization of $L(z)P_2(z)$ then $\det D_+(z) = z^N$ with $N = \sum_{j=1}^m k_j$.

Suppose now that T_{Ω_0} is Fredholm with $\Omega_0 = \operatorname{Diag}(\phi_j)$ where $\phi_j = \frac{s_j}{q_j}$ then without loss of generality $s_j = 1$ and so $\phi_j = \frac{1}{q_j}$. Then, as $\operatorname{Index} T_{\Omega_-} = \operatorname{Index} T_{\Omega_+} = 0$ and $\operatorname{Index} T_{z^{-k}} = mk$,

$$\begin{aligned}
\operatorname{Index} T_\Omega &= \operatorname{Index} T_{\Omega_-} + \operatorname{Index} T_{z^{-k}} + \operatorname{Index} T_{\Omega_0} + \operatorname{Index} T_{P_0} + \operatorname{Index} T_{\Omega_+} \\
&= \operatorname{Index} T_{z^{-k}} + \operatorname{Index} T_{\Omega_0} + \operatorname{Index} T_{P_2} \\
&= mk + \sum_{j=1}^m \text{ zeroes of } q_j \text{ mult. counted} - \sum_{j=1}^m k_j \\
&= mk + \sum_{j=1}^m \deg q_j - \sum_{j=1}^m k_j.
\end{aligned}$$

This concludes the proof of Theorem 5.1.3. □

Remark 5.6.2. It now follows from the decomposition of $\Omega(z) = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}$ in the example in Section 5.4 that T_Ω is not Fredholm.

Proposition 5.6.3. *Let*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

be the Wiener-Hopf type factorization of Ω *as in Theorem 5.1.1 and put* $\Xi(z) = z^{-k}\Omega_0(z)P_0(z)$. *Suppose that* T_Ω *is Fredholm then*

$$\dim \operatorname{Ker} T_\Xi = \dim \operatorname{Ker} T_\Omega = \sum_{j=1}^m \max(k + \deg q_j - k_j, 0)$$

and

$$\operatorname{codim}\operatorname{Ran}T_{\Xi} = \operatorname{codim}\operatorname{Ran}T_{\Omega} = \sum_{j=1}^{m} \max(k_j - \deg q_j - k, 0)$$

where $\phi_j = \frac{1}{q_j}$ *are the entries of* Ω_0*,* q_j *has roots only on* $\mathbb{T}$ *and* k_j *are the powers of* z *on the diagonal of* $P_0(z)$*.*

Proof. Since T_{Ω} is Fredholm, $\Omega_0 = \operatorname{Diag}(q_j^{-1})$ where q_j is a polynomial with roots only on $\mathbb{T}$. From Lemma 5.3.2, we can write $P_0(z) = L(z)D_+(z)$ where $D_+(z) = \operatorname{Diag}(z^{k_j})$ and L is a lower triangular polynomial matrix function with ones on the diagonal and its entries polynomials in z and z^{-1}. So

$$\begin{aligned}\Xi(z) &= z^{-k}\Omega_0(z)P_0(z) = z^{-k}\Omega_0(z)L(z)D_+(z)\\ &= z^{-k}\begin{pmatrix} \frac{1}{q_1} & & & \\ & \frac{1}{q_2} & & \\ & & \ddots & \\ & & & \frac{1}{q_m}\end{pmatrix}\begin{pmatrix} 1 & & & \\ \alpha_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ \alpha_{m1} & \alpha_{m2} & \cdots & 1\end{pmatrix}\begin{pmatrix} z^{k_1} & & & \\ & z^{k_2} & & \\ & & \ddots & \\ & & & z^{k_m}\end{pmatrix}.\end{aligned}$$

If $f \in \operatorname{Ker}T_{\Xi}$ then $\Xi f = h + \rho$ where $\mathbb{P}h = 0$ and $\rho \in \operatorname{Rat}_0^m(\mathbb{T})$. Also

$$z^k(\Omega_0)^{-1}(h+\rho) = LD_+f \in H_m^p.$$

Suppose $f = (f_1, f_2, \cdots, f_m)^T$, $h = (h_1, h_2, \cdots, h_m)^T$ and $\rho = (\rho_1, \rho_2, \cdots, \rho_m)^T$.

Now let us look at the components of f. It follows that $z^{k_1}f_1 = z^k q_1(h_1 + \rho_1)$. As $z^k q_1$ is a polynomial, $\mathbb{P}h_1 = 0$ and $\rho_1 \in \operatorname{Rat}_0(\mathbb{T})$ with $z^k q_1(h_1+\rho_1) \in H^p$ we have that $z^k q_1(h_1+\rho_1)$ is a polynomial of degree at most $k + \deg q_1 - 1$. But then $z^{k_1}f_1$ is a polynomial of degree at most $k + \deg q_1 - 1$ from which it follows that f_1 is a polynomial of degree at most $(k + \deg q_1 - 1) - k_1$. If, however, $k + \deg q_1 \leq k_1$ then $f_1 \equiv 0$ and so the degree of freedom we have in the choice for f_1 is $\max(k + \deg q_1 - k_1, 0)$.

For the second component we have

$$\alpha_{21}z^{k_1}f_1 + z^{k_2}f_2 = z^k q_2(h_2 + \rho_2) \in H^p.$$

As above, we have that $z^k q_2(h_2+\rho_2)$ is a polynomial of degree at most $k + \deg q_2 - 1$ from which it follows that $\alpha_{21}z^{k_1}f_1 + z^{k_2}f_2$ is a polynomial of degree at most $k + \deg q_2 - 1$. We now have two cases, namely $\deg(\alpha_{21}z^{k_1}f_1) \leq k + \deg q_2 - 1$ and $\deg(\alpha_{21}z^{k_1}f_1) > k + \deg q_2 - 1$.

Suppose now that $\deg(\alpha_{21}z^{k_1}f_1) \leq k + \deg q_2 - 1$. In this case we need $z^{k_2}f_2$ a polynomial of degree at most $k + \deg q_2 - 1$ and hence f_2 is a polynomial of degree at most $(k + \deg q_2 - 1) - k_2$.

If, however, $\deg(\alpha_{21}z^{k_1}f_1) > k + \deg q_2 - 1$ then $z^{k_2}f_2$ is a polynomial of degree at most $\deg(\alpha_{21}z^{k_1}f_1)$ but it needs to cancel the terms of the polynomial $\alpha_{21}z^{k_1}f_1$ that are of power higher than $k + \deg q_2 - 1$. So f_2 is a polynomial of degree at most $\deg(\alpha_{21}z^{k_1}f_1) - k_2$ with

$\alpha_{21}z^{k_1}f_1 + z^{k_2}f_2$ a polynomial of degree at most $k + \deg q_2 - 1$. Thus we have freedom only in the first $(k + \deg q_2) - k_2$ terms of the polynomial f_2.

So in both cases we have only $\max(k + \deg q_2 - k_2, 0)$ degrees of freedom in the choice for f_2.

Let us now look at the third component. In this case we have

$$\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2 + z^{k_3}f_3 = z^k q_3(h_3 + \rho_3) \in H^p.$$

As above, $z^k q_3(h_3+\rho_3)$ is a polynomial of degree at most $k+\deg q_3 - 1$ from which it follows that $\alpha_{31}z^{k_1}f_1+\alpha_{32}z^{k_2}f_2+z^{k_3}f_3$ is a polynomial of degree at most $k+\deg q_3-1$. We now have two cases, namely $\deg(\alpha_{31}z^{k_1}f_1+\alpha_{32}z^{k_2}f_2) \leq k+\deg q_3-1$ and $\deg(\alpha_{31}z^{k_1}f_1+\alpha_{32}z^{k_2}f_2) > k+\deg q_3 - 1$.

In case $\deg(\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2) \leq k + \deg q_3 - 1$ then it would suffice for $z^{k_3}f_3$ to be a polynomial of degree at most $k + \deg q_3 - 1$ and thus f_3 is a polynomial of degree at most $(k + \deg q_3 - 1) - k_3$.

If, however, $\deg(\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2) > k + \deg q_3 - 1$ then $z^{k_3}f_3$ is a polynomial of degree at most $\deg(\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2)$ but it needs to cancel the terms of the polynomial $\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2$ that is of power higher than $k + \deg q_3 - 1$. So f_3 is a polynomial of degree at most $\deg(\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2) - k_3$ with $\alpha_{31}z^{k_1}f_1 + \alpha_{32}z^{k_2}f_2 + z^{k_3}f_3$ a polynomial of degree at most $k + \deg q_3 - 1$. Thus we have freedom only in the first $(k + \deg q_3) - k_3$ terms of the polynomial f_3.

Again, in both cases we have $\max(k + \deg q_3 - k_3, 0)$ degrees of freedom in the choice for f_3.

Continuing in this way, we find f_n to be a polynomial of degree at most $\max(\deg(\alpha_{n1}z^{k_1}f_1 + \alpha_{n2}z^{k_2}f_2 + \cdots + \alpha_{n(n-1)}z^{k_{n-1}}f_{n-1}), k+\deg q_n - 1) - k_n$ in which we have freedom in the terms f_n up to $(k + \deg q_n) - k_n$. Thus we have $\max(k + \deg q_n - k_n, 0)$ degrees of freedom in the choice for f_n.

From this it follows that

$$\dim \mathrm{Ker} T_\Xi = \sum_{j=1}^{m} \max(k + \deg q_j - k_j, 0).$$

Given that T_Ξ is Fredholm, from Theorem 5.1.3 we have

$$\begin{aligned}
\mathrm{Index}\, T_\Xi &= \dim \mathrm{Ker} T_\Xi - \mathrm{codim}\, \mathrm{Ran} T_\Xi \\
&= \sum_{j=1}^{m} \max(k + \deg q_j - k_j, 0) - \mathrm{codim}\, \mathrm{Ran} T_\Xi \\
&= mk + \sum_{j=1}^{m} \deg q_j - \sum_{j=1}^{m} k_j \\
&= \sum_{j=1}^{m} (k + \deg q_j - k_j)
\end{aligned}$$

and so it follows that

$$\operatorname{codim}\operatorname{Ran}T_\Xi = \sum_{j=1}^{m} \max(k_j - \deg q_j - k, 0).$$

□

Lemma 5.6.4. *Let*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

be the Wiener-Hopf type factorization of Ω *as in Theorem 5.1.1 and suppose* T_Ω *is Fredholm. Then* T_Ω *is injective exactly when* $k + \deg q_j \leq k_j$ *for each* j*.*

Proof. Let Ξ, Ω_0, L and D_+ be as in the proof of Proposition 5.6.3 above and suppose $f \in \operatorname{Ker}T_\Xi$. Then f_1 is a polynomial of degree at most $k+\deg q_1-1-k_1$. So, if $k+\deg q_1 \leq k_1$ then $f_1 \equiv 0$. Also, f_2 is a polynomial of degree at most $k+\max\{\deg(\alpha_{21}z^{k_1}f_1), \deg q_2-1\}-k_2$ and $\alpha_{21}z^{k_1}f_1+z^{k_2}f_2$ is a polynomial of degree at most $k+\deg q_2-1$. Suppose that $k+\deg q_2 \leq k_2$. As $f_1 \equiv 0$ we have f_2 is a polynomial of degree at most $k + \deg q_2 - 1 - k_2$ and so $f_2 \equiv 0$.

Continuing in this way we see that if $k + \deg q_j \leq k_j$ for each $j = 1, \cdots, m$ then T_{Ω_0} is injective. We can reverse the argument and so we can conclude that T_{Ω_0} is injective exactly when $k + \deg q_j \leq k_j$ for each $j = 1, \cdots, m$. □

Corollary 5.6.5. *Let*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

be the Wiener-Hopf type factorization of Ω *as in Theorem 5.1.1 and suppose* T_Ω *is Fredholm. Then* T_Ω *is invertible exactly when* $k + \deg q_j = k_j$ *for each* j*.*

Proof. For T_Ω to be invertible we need $\operatorname{Index} T_\Omega = 0$ as well as T_Ω injective. For T_Ω injective we have $k + \deg q_j \leq k_j$ and so $\sum_{j=1}^{m}(k_j - k - \deg q_j) \geq 0$ and for each of the terms we have $k_j - k - \deg q_j \geq 0$. But $\operatorname{Index} T_\Omega = \sum_{j=1}^{m}(k_j - k - \deg q_j) = 0$ from which it follows that each of the terms are zero, or equivalently $k_j = k + \deg q_j$, $j = 1, 2, \ldots, m$. □

Using a similar argument as found on page 590 in [6] we can show the following.

Proposition 5.6.6. *Let* $\Omega \in \mathrm{Rat}^{m\times m}$ *with* $\det\Omega \not\equiv 0$ *and suppose*

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

is a Wiener-Hopf type factorization of Ω *w.r.t.* $\mathbb{T}$ *as in Theorem 5.1.1 and assume that* T_Ω *is Fredholm. Then the number of times* $r \in \mathbb{Z}_+$ *appears in the sequence of indices* $k_j - \deg q_j - k$ *is equal to* $\operatorname{codim}\operatorname{Ran}(T_{z^{1-r}\Omega}) - 2\operatorname{codim}\operatorname{Ran}(T_{z^{-r}\Omega}) + \operatorname{codim}\operatorname{Ran}(T_{z^{-r-1}\Omega})$*.*

Proof. Let $\Omega \in \mathrm{Rat}^{m\times m}$ with $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ be the Wiener-Hopf type factorisation of Ω w.r.t. $\mathbb{T}$. Then, for $r \in \mathbb{N}$, $\Xi_r(z) = z^{-r}(z^{-k}\Omega_0(z)P_0)$ can be written as

$$\Xi_r(z) = z^{-(k+r)}\mathrm{Diag}(\phi_j)L(z)\mathrm{Diag}(z^{k_j}).$$

Applying Proposition 5.6.3 we have

$$d(T_{\Xi_r}) := \operatorname{codim} \operatorname{Ran}(T_{\Xi_r}) = \sum_{k_j - \deg(q_j) - (k+r) \geq 0} k_j - \deg(q_j) - (k+r)$$

and so

$$\begin{aligned}
&d(T_{\Xi_r}) - d(T_{\Xi_{r+1}}) \\
&= \sum_{k_j - \deg(q_j) - (k+r) \geq 0} k_j - \deg(q_j) - (k+r) \\
&\quad - \sum_{k_j - \deg(q_j) - (k+r+1) \geq 0} k_j - \deg(q_j) - (k+r+1) \\
&= \sum_{k_j - \deg(q_j) - (k+r) \geq 1} k_j - \deg(q_j) - (k+r) \\
&\quad - \sum_{k_j - \deg(q_j) - (k+r+1) \geq 1} k_j - \deg(q_j) - (k+r+1) \\
&= \sum_{k_j - \deg(q_j) - (k+r) \geq 1} 1 \\
&= \sharp\{j | k_j - \deg(q_j) - k \geq r+1\}.
\end{aligned}$$

From this we get

$$\sharp\{j | k_j - \deg(q_j) - k = r\} = d(T_{\Xi_{r-1}}) - 2d(T_{\Xi_r}) + d(T_{\Xi_{r+1}})$$

and so the result follows. □

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List of Symbols

$(H^p)^*$	Dual space of H^p.
$[T_a]$	Matrix representation of T_a with respect to the usual basis.
$\mathbb{C}$	Complex plane.
$\mathbb{D}$	Open unit disk in $\mathbb{C}$.
$\mathbb{P}$	Projection of L^p onto H^p due to M. Riesz, not the Riesz projection in spectral operator theory, due to F. Riesz.
$\mathbb{R}$	Real numbers.
$\mathbb{T}$	The unit circle in the complex plane.
$\mathbb{T}^\circ$	$\mathbb{T}\backslash\{-1\}$
$\mathbb{Z}$	Integers
$\mathcal{C}$	Space of continuous complex valued functions.
$\mathcal{P}$	Space of polynomials.
$\mathcal{P}_k$	Space of polynomials of degree at most k.
$\mathcal{W}$	The Wiener class of functions.
$\mathcal{W}_+$	Analytic functions in the Wiener class.
$\mathrm{Dom}(A)$	Domain of the operator A.
$\mathrm{Ker}(A)$	Kernel or Null space of the operator A.
$\omega(\mathbb{T})$	Image of $\mathbb{T}$ under ω.
$\overline{\mathbb{D}}$	Closed unit disk in $\mathbb{C}$.
$\mathrm{Ran}(A)$	Range of the operator A.
Rat	Space of rational functions.
$\mathrm{Rat}(\mathbb{T})$	Space of rational functions with poles only on $\mathbb{T}$.
$\mathrm{Rat}^{m\times n}$	Space of $m \times m$ rational matrix valued functions.
$\mathrm{Rat}_0(\mathbb{T})$	Space of strictly proper rational functions with poles only on $\mathbb{T}$.
$\mathrm{Rat}_0^{m\times n}$	Space of strictly proper $m\times m$ rational matrix valued functions.
$\mathrm{Rat}_0^{m\times n}(\mathbb{T})$	Space of strictly proper $m \times m$ rational matrix valued functions with poles only on $\mathbb{T}$.
$\rho(A)$	Resolvent set of the operator A.
$\sigma(A)$	Spectrum of the operator A.
$\sigma_c(A)$	Continuous spectrum of the operator A.
$\sigma_p(A)$	Point spectrum of the operator A.
$\sigma_r(A)$	Residual spectrum of the operator A.
$\sigma_{ess}(A)$	Essential (Fredholm) spectum of the operator A.
$a_+(z)$	Right Wiener-Hopf factor in the non canonical Wiener-Hopf factorization.

$a_-(z)$	Left Wiener-Hopf factor in the non canonical Wiener-Hopf factorization.
$e_n(z)$	$e_n(z) = z^n, z \in \mathbb{T}$
f_n	n-th Fourier coefficient of the function f.
H^p	Hardy space of p summable functions on $\mathbb{T}$.
$H^{p'}$	Hardy space of p' summable functions where $\frac{1}{p} + \frac{1}{p'} = 1$.
H^p_m	Space of functions with m components where each component is in H^p.
K^p	Sub space of anti-analytic functions in L^p.
L^∞	Space of essentially bounded functions on $\mathbb{T}$.
L^p	Space of p summable functions on $\mathbb{T}$.
L^p_m	Space of functions with m components where each component is in L^p.
M_a	Multiplication operator generated by the function a
M_z	Bilateral shift or multiplication operator generated by the function $a(z) = z$.
N^+	Smirnov class of analytic functions.
$O(j^N)$	Sequence of numbers bounded by polynomial of degree N.
$qH^p + \mathcal{P}_k$	Space of functions of the type $qh + r$ where $h \in H^p$ and $r \in \mathcal{P}_k$.
qH^p	Space of functions of the type qh where $h \in H^p$.
$r^\sharp$	For a polynomial r, $r^\sharp(z) = \overline{r(\frac{1}{\bar{z}})}$.
s_+	The factor of the polynomial s that have roots outside $\mathbb{T}$.
s_-	The factor of the polynomial s that have roots inside $\mathbb{D}$.
s_0	The factor of the polynomial s that have roots on $\mathbb{T}$.
$T^{\mathrm{cl,B}}_\omega$	Classical Toeplitz operator on the Bargmann-Segal space.
T^{cl}_ω	Classical Toeplitz operator
T^{He}_ω	Unbounded Toeplitz operator defined by H. Helson and D. Sarason.
T^{J}_ω	Unbounded Toeplitz operator on the Bargmann-Segal space defined by Janas.
T^{Sa}_ϕ	Unbounded Toeplitz operator defined by D. Sarason.
T_a	Toeplitz operator generated by the function a.
T^*_a	Adjoint operator of the Toeplitz operator T_a.
T^{Hr}_f	Unbounded Toeplitz operator defined by P. Hartman and A. Wintner
T_z	Unilateral shift or Toeplitz operator generated by the function $a(z) = z$.
$T_{z^{-1}}$	Unilateral backward shif or the Toeplitz operator generated by the function $a(z) = z^{-1}$.
$wind(a)$	winding number of the complex valued function a with respect to 0.

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