Bornologies of extended quasi-metric spaces for student recruitment at the North-West University

D. Mukonda

orcid.org/0000-0003-2791-6361

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Supervisor: Prof. Olivier Olela Otafudu

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Student number: 29476364

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Abstract

In this MSc dissertation, we present the results of bornologies on extended quasi-metric spaces. Our results generalise Beer's results on bornologies of extended metric spaces to the framework of quasi-metric spaces. We construct our own universal space of extended quasi-metric spaces in sense of Beer and show that most of the results in Beer's universal space hold in our universal space too. It is proved, for example, that the constructed universal space is bicomplete and there is an embedding between an extended quasi-metric space and this universal space. The last chapter of this dissertation is devoted to bornologies of extended quasi-metric spaces, this chapter has been inspired by the fact that being bounded on quasi-metric space (X, p) does not necessarily mean bounded on the symmetrized metric space (X, p^s). For instance, a nonempty set A can be p-bounded but not p^s-bounded. Furthemore, we use the asymmetric version of Hu's theorem to prove that for an extended T_0-quasi-metric p on the bornological universe $(X, \mathcal{B})$, the bornology $\mathcal{B}$ coincides with the quasi-metric bornology $\mathcal{B}_p(X)$.

Preface

The work described in this MSc dissertation was carried out under the supervision of Prof. Olivier Olela Otafudu, Department of Mathematical Sciences, North-West University, Mafikeng compus (Present Address: University of the Witwatersand) South Africa.

The MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

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D. Mukonda (Student)

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Prof. Oliver Olela Otafudu (Supervisor)

Dedication

This MSc dissertation is dedicated to my parents, Timoth and Judith who have been a constant source of support and encouragement. I am truly thankful for having you in my life.

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Firstly, I would like to express my sincere gratitude to my supervisor Prof. Olivier Olela Otafudu for the continuous support of my MSc study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this dissertation.

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I would also like to thank my family: my parents: Timoth and Judith, my brothers and sisters: Merran, Enock, Janet, Frank and my friends Hope and Levy for supporting me throughout writing this Msc dissertation.

Finally, I thank and praise my almighty God for giving me this opportunity and the capacity to proceed successfully.

Contents

Introduction

The notion of bounded sets is very important in the theory of metric spaces and normed vector spaces. For this reason, this concept has been extended to different settings like topological vector spaces, uniform spaces or even the general context of topological spaces. In spite of the fact that there are different generalizations of the notion of bounded sets, we can observe that all of them define what is called a *bornology*. We say that a family of subsets of a set X is a bornology if it is a cover of X and it is stable under inclusions and finite unions. If we take a nonempty set X and a bornology $\mathcal{B}$ on X, then the pair $(X, \mathcal{B})$ is called a *bornological universe*.

The systematic study of bornologies in topological spaces started with papers by Hu [19] and Alexander [4] some 68 years ago. The main result of their papers was to characterize bornologies that are metric bornologies. For instance, Hu [19] investigated bornologies defined on a metric space that correspond to the bornology of bounded sets. Examples of spaces that have the property of forming bornologies include compact metric spaces. Over the last 15 years, Borwein [10] used bornologies to develop a unified theory of differentiability for functions defined on general normed spaces, characterizing various geometric properties of spaces in terms of the relationships between different bornological derivatives on various classes of functions.

In his PhD thesis, Meyer [29] noticed that, given a metrizable topological vector space, it is possible to use its von Neumann bornology or its bornology of precompact subsets to do analysis. He proved that the bornological and topological approaches are equivalent for many problems. For instance, the concepts of convergence for sequences of points or linear operators were yielded. Not only that, Meyer added that the bornological and topological versions of Grothendiecks approximation property are equivalent for Frechet spaces. These results are important for applications in noncommutative geometry.

Meroño and Garrido [18] applied the theorem of uniform metrization to a metric space with the bornology of the bounded sets in the sense of Bourbaki, that is, the subsets of X which are bounded for every uniformly continuous function. They also applied this theorem to the the bornology of all the compact subsets of a topological space X. This was done by finding a metric for the topology with the Heine-Borel property. The concept of Meroño and Garrido with the use of characteristic function. made it easy for Beer [5] to show the similar results of bornologies on an extended metric space.

Beer, Costantini and Levi [9] studied metric bornologies using d-bounded and d-totally bounded subsets. They showed that a metric space is separable if and only if the bornology of d-bounded subsets agrees with the bornology of d-totally bounded subsets with respect to same equivalent remetrization, they even characterized those bornologies of totally bounded sets as determined by some metric compatible with the topology.

The construction of universal space has been studied by many scholars in different perspectives. For example, Beer [5] produced a universal space of bounded continuous functions on the extended metric space X equipped with uniform distance. His universal space was the space of partial functions defined for any Hausdorff space. In addition, Agyingi, Haihambo and Künzi [1] used a T_0-quasi-metric to construct their universal space using a set of real-valued functions which are bounded.

More recently, Piękosz and Wajch [31] have showed that Hu's metrization theorem of bornological universe can be applied in ZF-(*Zermelo-Fraenkel*) and adopted it to a quasi-metrization theorem for bornologies in bitopological and generalized topological spaces in the sense of Delfs and Knebusch. Piękosz and Wajch also investigated the problem of uniform quasi-metrization of quasi-metric bornological universe, they showed that given a quasi-metric space X and if $\tau = \max\{\tau_0, \tau_1\}$ is a topology on X, then a bornological biuniverse $((X, \tau_0, \tau_1), \mathcal{B})$ is metrizable if and only if the bornological universe $((X, \tau), \mathcal{B})$ is metrizable.

An interesting question raised by Meroño and Garrido [18] in their work, is when a bornological universe $(X, \mathcal{B})$ is metrizable or uniformly metrizable? In [5], Beer answered this question by finding an extended metric d such that the bornological universe is metrizable or uniformly metrizable. Moreover, Beer studied bornologies of extended metric spaces and achieved the following results: (i) given an extended metric space (X, d) and a bornology $\mathcal{B}$ on X, Beer proved that the bornology $\mathcal{B} = \mathcal{B}_d(X)$. (ii) for an extended metric d on X, he used a metric $d_1 = \min\{1, d\}$ on X to prove that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$. (iii) lastly, Beer constructed an extended metric space (Δ_X, ρ_X) of partial functions and called it the universal space of an extended metric space.

However, the study of Beer's fundamental results [5] in extended metric spaces has motivated us to generalize his results to the framework of quasi-metric spaces. The main aim of this MSc dissertation is to construct the universal space of extended quasi-metric spaces in sense of Beer[5] and generalize Beer's results of extended metric bornologies to quasi-metric point of view. The following are the objectives of this MSc dissertation:

(i) whenever (X, p) is an extended T_0-quasi-metric space, we should be able to construct our own universal space on X and prove the Beer [5]'s results in our universal space.

(ii) given an extended T_0-quasi-metric space (X, p) and a bornology $\mathcal{B}$ on X, we need to prove that $\mathcal{B} = \mathcal{B}_p(X)$.

(iii) for an extended T_0-quasi-metric p on X, we are going to use an extended T_0-quasi-metric $p_1 = \min\{1, p\}$ to show that $\mathcal{B}_{p_1}(X) = \mathcal{B}_p(X)$.

Outline of the MSc dissertation

Chapter 1. In this chapter, we recall some basic concepts and definitions to be used throughout the dissertation. In the first section, we define a quasi-pseudometric space, define its topological structures and present some examples of quasi-metric spaces. In the second part of this chapter, we present some useful concepts of bitopological spaces and thereafter define the notion of convergence and bicompleteness. The last section of this chapter discusses the concept of a bornology (see Definition 1.2.1), an extension of the concept of bounded sets and later presents examples of bornologies and some of their properties.

Chapter 2. In this chapter, we summarize Beer [5]'s results on the bornology of extended metric spaces. The first section of this chapter discusses Beer's construction of the universal space of extended metric spaces. We start the construction in this chapter by recalling the definition of universal space of extended metric spaces and thereafter prove that the universal space is a complete space. Given two universal spaces an embedding between them is constructed. After providing some useful concepts on bornologies from Hu [19], we also prove in this section that if $\mathcal{B}$ is a bornology on (X, d), then there exists an extended metric d_1 on X uniformly equivalent to d, such that the bornology of d_1-bounded sets coincides with $\mathcal{B}$. Finally, we have concluded this chapter by providing the existence of the bornology on a normed linear space.

Chapter 3. In this chapter, we start our own investigations and present our own results. We generalize the construction of the universal space studied in Chapter 2 to quasi-metric settings. The first section of this chapter defines the concept of quasi-metric components (see Definition 3.1.1). We then define and present our own universal space of extended T_0-quasi-metric space (see Definition 3.2.1). In the second section, Theorem 3.2.1 has proved that there is an isometry from an extended quasi-metric space to our universal space. Furthermore, we have also embedded Beer[5]'s universal space of extended metric spaces into our own universal space. In the same section, we have also proved that the universal space of an extended T_0-quasi-metric space is bicomplete (see Theorem 3.2.2). Proposition 3.2.1 is an asymmetric version of Theorem 2.2.7 and illustrates how a Hausdorff space of an extended quasi-metric space can be embedded into our universal space. Corollary 3.2.1 has concluded this chapter by providing the comparison of the universal space of the extended T_0-quasi-metric spaces (X, p) and (X, p^{-1}).

Chapter 4. In this chapter, we continue with our own investigations on bornologies of extended quasi-metric spaces. We start this chapter by recalling some definitions of bounded sets and some other properties of bornologies. Given an extended T_0-quasi-metric space (X, p), the notion of quasi-metric bornology is defined and compared with other quasi-metric bornologies determined by p^s and p^{-1} (see Lemma 4.1.1 and Lemma 4.1.2). The last part of this section has shown that the collection of all finite unions of double balls forms a base for the quasi-metric bornology and it contains $\tau(p)$-relatively compact subsets of X (see Lemma 4.2.1). In the second section, we prove that for an extended quasi-metric p, the bornological universe $(X, \mathcal{B})$ is quasi-metrizable by p and the bornology $\mathcal{B}$ coincides with the quasi-metric bornology on X(see Theorem 4.2.1 and Theorem 4.2.2). We have ended this chapter by proving in Theorem 4.2.4 that for a given bornology $\mathcal{B}$ on an extended T_0-quasi-metric space (X, p), there exists an extended T_0-quasi-metric p_1 on X, uniformly equivalent to p, for which the bornology of p_1-bounded sets coincides with $\mathcal{B}$.

Chapter 5. In this chapter, we summarise our investigations and present some open problems to be studied in future.

1

Preliminaries

In this chapter, we present the basic concepts, definitions and results about quasi-metric spaces and bornologies to be used throughout the MSc dissertation. There is substantial number of publications about topological and uniform structures related to quasi-metric spaces in the major reviews by Cobzas [11], Kopperman [23] and Künzi [26] and on bornologies too by Beer [5, 9] and Hu [19].

1.1. Quasi-metric spaces

In this section, we recall some useful concepts from the theory of quasi-metric spaces and some of their basic topological properties.

Definition 1.1.1. [25, Definition 2.1] Let X be a nonempty set and let $p : X^2 \to [0, \infty)$ be a mapping. Then p is called a *quasi-pseudometric* on X if

(i) $p(x, x) = 0$ for all $x \in X$,

(iii) $p(x, y) \leq p(x, z) + p(z, y)$ for all x, y, $z \in X$.

The pair (X, p) is called a *quasi-pseudometric space*.

If in addition, for any $x, y \in X$,

$$p(x, y) = 0 = p(y, x) \implies x = y,$$

then p is called a T_0-quasi-metric and the pair (X, p) is called a T_0-quasi-metric space. If we replace the set $[0, \infty)$ by $[0, \infty]$, then we have the mapping p called extended T_0-quasi-metric and (X, p) is called extended T_0-quasi-metric space.

If p is a quasi-pseudometric on a set X, then $p^{-1} : X \times X \to [0, \infty)$ defined by $p^{-1}(x, y) = p(y, x)$ for every $x, y \in X$, often called the conjugate quasi-pseudometric, is also quasi-pseudometric on X. If p is a quasi-pseudometric on X, then $p^s = \max\{p, p^{-1}\}$ is a pseudometric and if (X, p) is a T_0-quasi-metric space, then p^s a metric on X.

The next remark is about the topological propeties of quasi-pseudometric spaces.

Remark 1.1.1. [25, Remark 2.2] Let (X, p) be a quasi-pseudometric space. The *open ball* of radius $\epsilon > 0$ centred at $x \in X$ is the set $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon\}$. The collection of *open balls* yields a *base for the topology* $\tau(p)$ and it is called the topology induced by p on X. Similarly, the *closed ball* of radius $\epsilon \geq 0$ centred at $x \in X$ is the set $C_p(x, \epsilon) = \{y \in X : p(x, y) \leq \epsilon\}$. If (X, p) is a quasi-pseudometric space, then the pair $[C_p(x, r) : C_{p^{-1}}(x, s)]$ where $x \in X$ and $r, s \in [0, \infty)$ is called a *double ball*. In general, $[(C_p(x_i, r_i))_{i \in I} : (C_{p^{-1}}(x_i, s_i))_{i \in I}]$, with $x_i \in X$ and $r_i, s_i \in [0, \infty)$, is called the *family of double balls*. Also note that $C_p(x, \epsilon)$ is not $\tau(p)$-closed but is $\tau(p^{-1})$-closed.

The above remark leads us to the following example.

Example 1.1.1. [11, Example 1.1.6] For $x, y \in \mathbb{R}$ define a quasi-pseudometric by $p(x, y) = y - x$ if $x \leq y$ and $p(x, y) = 1$ if $x > y$: A basis of $\tau(p)$-open neighborhoods of a point $x \in \mathbb{R}$ is formed by the family $[x, x + \epsilon) : 0 < \epsilon < 1$: The family of intervals $(x - \epsilon, x]; 0 < \epsilon < 1$; forms a basis of $\tau(p^{-1})$-open neighborhoods of x. If $p^s(x, y) = 1$ for $x \neq y$ then $\tau(p^s)$ is called a discrete topology of $\mathbb{R}$.

Below are examples of quasi-metric spaces from different literatures.

Example 1.1.2. [12] Let $u, v \in \mathbb{R}$, then the mapping $p : \mathbb{R}^2 \longrightarrow [0, \infty)$ defined by

$$p(u, v) = \begin{cases} u - v \text{ if } u \leq v \\ \\ 1 \text{ if } u > v, \end{cases}$$

is a T_0-quasi-metric on $\mathbb{R}$.

Example 1.1.3. For any $x, y \in \mathbb{R}$. If we define u by $u(x, y) = \max\{y - x, 0\}$, then u is a T_0-quasi-metric on $\mathbb{R}$. The T_0-quasi-metric u on $\mathbb{R}$ is called the standard quasi-metric of $\mathbb{R}$.

Note that the u^s of the above example is the usual metric in $\mathbb{R}$.

Example 1.1.4. [1, Example 1] Let a five point set $S = \{0, 1, 2, 3, 4\}$. Then the function p defined by the distance matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

for $p(i, j) = A_{i,j}$, whenever $i, j \in S$ is a T_0-quasi-metric on S.

Example 1.1.5. Let $n, m \in \mathbb{N}$, then the function $p : \mathbb{N} \times \mathbb{N} \longrightarrow [0, \infty]$ defined by

$$p(n, m) = \begin{cases} \frac{1}{m} & \text{if } n < m \\ \\ 0 & \text{if } n = m \\ 1 & \text{if } n > m, \end{cases}$$

is an extended T_0-quasi-metric.

Let us recall a notion of quasi-uniformity in the next definition.

Definition 1.1.2. Let X be a nonempty set. A quasi-uniformity on a set X is a filter $\mathcal{U}$ on X^2 such that:

(i) for each $U \in \mathcal{U}$, $\{(x, x) : x \in X\} \subseteq U$;

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

A quasi-uniform space is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a quasi-uniformity on X.

Definition 1.1.3. A topological space (X, τ) is *quasi-metrizable* if there exists a quasi metric p such that $\tau = \tau(p)$.

Definition 1.1.4. Let (X, p) be a T_0- quasi-metric space and $\mathscr{P}(X)$ be a set of all subsets of X. Then, for each $A \in \mathscr{P}(X)$, we write $\mathrm{cl}_{\tau(p)}(A)$ the *closure* of A with respect to $\tau(p)$ and $\mathrm{int}_{\tau(p)}(A)$ the *interior* of A with respect $\tau(p)$.

Definition 1.1.5. Let (X, p) be an extended T_0-quasi-metric space and $A \subset X$ for $\delta > 0$. We define the *δ-enlargement* of A by:

$$[A]_p^\delta = \{x \in X : p(A, x) < \delta\} = \{x \in X : \inf\{p(x, a) : a \in A\} < \delta\} = \bigcup_{a \in A} B_p(a, \delta).$$

For any quasi-pseudometric space (X, p), we denote the set of all nonempty subsets of X by $\mathscr{P}_0(X)$. Also, if $C \in \mathscr{P}_0(X)$, then $\operatorname{dist}(x, C) = \inf\{p(x, c) : c \in C\}$ and $\operatorname{dist}(C, x) = \inf\{p(c, x) : c \in C\}$ whenever $x \in X$.

For any $A, B \in \mathscr{P}_0(X)$, we set

$$p_H(A, B) = \max\left\{\sup_{b \in B} \operatorname{dist}(A, b), \sup_{a \in A} \operatorname{dist}(a, B)\right\}.$$

Then p_H is called *extended Hausdorff quasi-pseudometric* on $\mathscr{P}_0(X)$.

We now give some concepts about the convergence and completeness in quasi-metric spaces. For more details we refer the reader to Cobzas [11] and Salbany [32].

Definition 1.1.6. Let (X, p) be a quasi-pseudometric space, a sequence (x_n) is said to be:

(i) *p-convergent* to x if for each $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that $p(x_n, x) < \epsilon$ for all $n \geq N$.

(ii) *p^{-1}-convergent* to x if for each $\epsilon > 0$,there exists $N = N(\epsilon) > 0$ such that $p(x, x_n) < \epsilon$ for all $n \geq N$.

Proposition 1.1.1. *[11, Proposition 1.1.2] Let (x_n) be a sequence in a quasi-metric space (X, p) :*

i if (x_n) is $\tau(p)$-convergent to x and $\tau(p^{-1})$-convergent to y; then $p(x, y) = 0$;

ii If (x_n) is $\tau(p)$-convergent to x, and $p(y, x) = 0$; then (x_n) is also $\tau(p)$-convergent to y.

Proof. (i). Letting $n \longrightarrow \infty$ in the inequality $p(x; y) \leq p(x, x_n) + p(x_n, y)$; one obtains $p(x, y) = 0$;

(ii). Follows from the relations $p(y; x_n) \leq p(y, x) + p(x, x_n) = p(x, x_n) \longrightarrow 0$ as $n \longrightarrow 0$. $\square$

Definition 1.1.7. Let (X, p) be a quasi-pseudometric space. A sequence (x_n) is called:

(i) *Left K-Cauchy* if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{Z}^+$ such that $p(x_n, x_m) < \epsilon$ for all $m \geq n \geq n_\epsilon$.

(ii) *Right K-Cauchy* if for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{Z}^+$ such that $p(x_m, x_n) < \epsilon$ for all $m \geq n \geq n_\epsilon$.

Proposition 1.1.2. *[11, Proposition 1.2.4] Let (x_n) be a left K-Cauchy sequence in a quasi-pseudometric space (X, p) :*

(i) *If* (x_n) *has a subsequence which is* $\tau(p)$*-convergent to to* x*; then* (x_n) *is* $\tau(p)$*-convergent to* x :

(ii) *If* (x_n) *has a subsequence which is* $\tau(p^{-1})$*-convergent to to* x*; then* (x_n) *is* $\tau(p^{-1})$*-convergent to* x*:*

(iii) *If* (x_n) *has a subsequence which is* $\tau(p^s)$*-convergent to to* x*; then* (x_n) *is* $\tau(p^s)$*-convergent to* x.

Definition 1.1.8. [32] A quasi-pseudometric space (X, p) is *bicomplete* if every Cauchy sequence (x_n) converges with respect to $\tau(p)$ and with respect to $\tau(p^{-1})$ to a point x_0 .

Note that (x_n) is a Cauchy sequence in (X, p) if and only if (x_n) is a Cauchy sequence in the pseudometric space (X, p^s)

Proposition 1.1.3. *[32] A quasi-pseudometric space* (X, p) *is bicomplete if and only if the metric space* (X, p^s) *is complete.*

Definition 1.1.9. Let (X, p) be a an extended quasi-metric space. Then the space X is said to be *locally compact* with respect to $\tau(p)$, if every point of X has a $\tau(p)$-compact neighbourhood.

Definition 1.1.10. Let (X, p) be an extended T_0-quasi-metric space. The space X is said to be *first-countable* with respect to $\tau(p)$ if each point has a countable neighbourhood basis (local base). That is, for each point x in X there exists a sequence $N_1, N_2, \ldots$ of neighbourhoods of x such that for any neighbourhood N of x there exists an integer i with N_i contained in N.

Definition 1.1.11. A topological space (X, τ) is *quasi-metrizable* if there exists a quasi metric p such that $\tau = \tau(p)$.

We now turn our attention to the concept of bitopological spaces.

As a space with two topologies, τ_0 and τ_1, a quasi-metric space can be viewed as a bitopological space in the sense of Kelly [20], and so all the results valid for bitopological spaces apply to a quasi-metric space. *A bitopological space* is simply a set X endowed with two topologies τ_0 and τ_1.

Definition 1.1.12. [11] A bitopological space (X, τ_0, τ_1) is *quasi-metrizable* if there exists a quasi-metric p on X such that $\tau_0 = \tau(p)$ and $\tau_1 = \tau(p^{-1})$.

Definition 1.1.13. Let (X, τ_0, τ_1) be a bitopological space. Two extended quasi-metrics on X will be called bitopologically equivalent if they define the same left and right topologies on X.

The following describe some properties of maps between two quasi-metric spaces.

Definition 1.1.14. Let (X, p) and (X_1, p_1) be extended T_0-quasi-metric space, A map $h : X \longrightarrow X_1$ is called *nonexpansive* provided that $p_1(h(x), h(y)) \leq p(x, y)$ whenever $x, y \in X$.

Definition 1.1.15. Let (X, p) and (X_1, p_1) be extended quasi metric spaces, then the mapping $h : X \longrightarrow X_1$ is an *isometric mapping* if for each $x, y \in X$, $p_1(h(x), h(y)) = p(x, y)$.

Two extended quasi metric spaces X and X_1 are called *isometric* if there exist a bijective isometry between them.

Let (X, p) and (X_1, p_1) be extended T_0-quasi-metric space. then the mapping $h : X \longrightarrow X_1$ is called *quasi-uniformly continuous* provided that for each $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$, $p(x, y) < \epsilon$ implies that $p_1(h(x), h(y)) < \delta$.

We end this section by recalling the definiton of the disjoint unions on a nonempty set.

Definition 1.1.16. Let $\{A_j : j \in I\}$ be a collection of sets indexed by I. Then the family

$$\bigcup_{j \in I} A_j = \bigcup_{j \in I} \{(x, j) : x \in A\}$$

is called the *disjoint union or free union* of $\{A_j : j \in I\}$.

We now turn our attention to the concept of a bornology in the next section.

1.2. Bornology

This section is devoted to giving some basic concepts of bornologies on a nonempty set X. For more information see [5, 19].

Definition 1.2.1. [5, Definition 1.1] Let X be a nonempty set. A collection $\mathcal{B}$ of subsets of X is called a *bornology* on X if the following properties are satisfied:

(i) $\mathcal{B}$ forms a cover of X, i.e. $X = \bigcup_{B \in \mathcal{B}} B$;

(ii) $\mathcal{B}$ is hereditary under inclusion, i.e, whenever $B \in \mathcal{B}$ and A is a subset of X contained in B, then $A \in \mathcal{B}$;

(iii) $\mathcal{B}$ is stable under finite union, i.e. if $B_1, B_2, ..., B_n \in \mathcal{B}$ then $\bigcup_{i=1}^{n} B_j \in \mathcal{B}$

A pair $(X, \mathcal{B})$ where X is a nonempty set and $\mathcal{B}$ is a bornology on X is called a *bornological universe*.

Let us look at the following examples of bornologies from the recent literature.

Example 1.2.1. [18, Example 1] For a nonempty set X the collection of all subsets in X, is a bornology on X called the trivial Bornoloy. The power set $\mathcal{B} = 2^X$ of X is its largest bornology.

Example 1.2.2. Let (X, τ) be a topological space and $\mathcal{A}$ be the collection of all compact subsets of a topological space X, then it generates a bornology called the compact bornology which is denoted by $K\mathcal{B}$.

Example 1.2.3. Let $\mathcal{A}$ be a family of subsets of a set X such that it forms a cover of X and let $\mathcal{B}$ be a family of subsets of X which consists of the totality of the subsets of the finite unions of the family $\mathcal{A}$. Then, the family $\mathcal{B}$ is the bornology generated by the cover $\mathscr{A}$.

Example 1.2.4. [18, Example 3] Let $\mathcal{F}$ be a family of all the finite subsets of a set X. Then $\mathcal{F}$ is a smallest bornology on X called a finite bornology.

Remark 1.2.1. [18, P:286] Given a non empty set X and bornology $\mathcal{B}$ on X, a family $\mathcal{C} \subset \mathcal{B}$ is said to be a *base* for $\mathcal{B}$ if for every $B \in \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $B \in C$, i.e., every set in $\mathcal{B}$ is contained in some set of $\mathcal{C}$. For instance, the (countable) family $\mathcal{C} = \{B(x_0, m) : m \in \mathbb{Z}^+\}$, of the open balls of center $x_0 \in X$ and radius $m \in \mathbb{Z}^+$, is a base for the bornology $\mathcal{B}_d(X)$ of the metric bounded sets in the metric space (X, d).

Let $(X, \mathcal{B})$ be a bornological universe. Then $\mathcal{B}$ is said to have a *countable base* if it possesses a base consisting of a sequence of bounded sets. The bornology $\mathcal{B}$ is said to be *second countable* if its base is countable.

Definition 1.2.2. [19, Definition 4.5] Let $\mathcal{B}$ and $\mathcal{B}'$ be two bornologies in a set X. Then, $\mathcal{B}$ and $\mathcal{B}'$ are said to be *bonological equivalent* if $\mathcal{B} = \mathcal{B}'$, i.e. if for every $B \in \mathcal{B}$, there exists $B' \in \mathcal{B}'$ such that $B \subseteq B'$ and for every $B' \in \mathcal{B}'$ there exists $B \in \mathcal{B}$ such that $B' \subseteq B$.

Definition 1.2.3. [19, Definition 2.1] A bornology $\mathcal{B}_2$ on a set X is a finer (stroger) bornology than a bornology $\mathcal{B}_1$ on X (or $\mathcal{B}_1$ is *a coarser (weaker)* bornology than $\mathcal{B}_2$) if $\mathcal{B}_1 \subset \mathcal{B}_2$.

We recall the following results about bornologies on a topological space in Hu[19]'s paper.

Remark 1.2.2. [19] Let (X, τ) be a topological space. For any arbitrary bornology $\mathcal{B}$ on a toplogical space X, we have $\text{int}(\mathcal{B}) \subset \mathcal{B} \subset \text{cl}(\mathcal{B})$

Definition 1.2.4. Let (X, τ_0, τ_1) be a bitopological space. Then, a bornology $\mathcal{B}$ in X is called *$\tau(p)$-proper* if and only if for every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\text{cl}_{\tau_1}(A) \subseteq \text{int}_{\tau_0}(B)$.

Remark 1.2.3. [19] Let (X, τ) be a topological space. For any two bornologies $\mathcal{B}_1$ and $\mathcal{B}_2$ on a topological space X, we have $\mathcal{B}_1 \subset \mathcal{B}_2$ implies $\text{cl}(\mathcal{B}_1) \subset \text{cl}(\mathcal{B}_2)$ and $\text{int}(\mathcal{B}_1) \subset \text{int}(\mathcal{B}_2)$.

The next theorem illustrates the existence of a closed base of a bornology.

Theorem 1.2.1. *[19, Theorem 3.5] Let (X, τ) be a topological space. For any given bornology $\mathcal{B}$ on a topological space X, the following conditions are equivalent:*

(i) $\mathcal{B}$ is closed;

(ii) $\mathcal{B}$ is generated by its subfamily of the bounded closed sets (it has a closed base);

(iii) The closure of every bounded set is a bounded set.

Proof. $(i) \Rightarrow (ii)$: Let $\mathcal{B}$ be a bornology on a topological space X and $\mathcal{B}_1 \subset \mathcal{B}$ be a subfamily of bounded sets in X. We have $\text{cl}(\mathcal{B}) = \mathcal{B}$ therefore, $\text{cl}(\mathcal{B}_1) \subset \mathcal{B}$. On the other hand, if B is a bounded set, then $B = \text{cl}(B) \in \mathcal{B}_1$. Hence, $\mathcal{B}_1$ is the subfamily of the bounded closed sets, and $\mathcal{B} = \text{cl}(\mathcal{B})$ is generated by $\text{cl}(\mathcal{B}_1)$.

$(ii) \Rightarrow (iii)$: Let B be an arbitrary bounded set of $\mathcal{B}$. Since $\mathcal{B}$ is generated by its subfamily $\text{int}(\mathcal{B}_2) = \mathcal{C}$ of bounded closed sets, B is a subset of a finite union of bounded closed sets. Hence, there exists a bounded closed set C such that $B \subset C$. This implies (iii).

$(iii) \Rightarrow (i)$: Since every bounded set is contained in the closure of some bounded set, then $\mathcal{B} \subset \text{cl}(\mathcal{B})$ and $\text{cl}(\mathcal{B}) \subset \mathcal{B}$, thus $\mathcal{B} = \text{cl}(\mathcal{B})$ and $\mathcal{B}$ is closed. □

The following result proves the existence of an open base of a bornology.

Theorem 1.2.2. *[19, Theorem 3.6] Let (X, τ) be a topological space. For any given bornology $\mathcal{B}$ in a toplogical space X, the following conditions are equivalent:*

(i) $\mathcal{B}$ is open;

(ii) $\mathcal{B}$ is generated by its subfamily of the bounded open sets (it has an open base);

(iii) Every bounded set is contained in the interior of some bounded set.

Proof. $(i) \Rightarrow (ii)$: Let $\mathcal{B}$ be a bornology in a topological space X and $\mathcal{B}_2 \subset \mathcal{B}$ be a subfamily of bounded sets in X. Since $\mathcal{B}$ is open it means that, $\text{int}(\mathcal{B}) = \mathcal{B}$; therefore, $\text{int}(\mathcal{B}_2) \subset \mathcal{B}$. However, if B is a bounded set, then $B = \text{int}(B) \in \mathcal{B}_2$. Hence, $\mathcal{B}_2$ is the subfamily of the bounded open sets. Thus, $\text{int}\mathcal{B} = \mathcal{B}$ is generated by subfamily $\text{int}(\mathcal{B}_2)$.

$(ii) \Rightarrow (iii)$: Let A be an arbitrary bounded set of $\mathcal{B}$. Since $\mathcal{B}$ is generated by its subfamily $\text{int}(\mathcal{B}_2) = \mathcal{G}$ of bounded open sets. A is a subset of a finite union of bounded open sets. Hence there exists a bounded open set G such that $A \subset G$. Hence (iii).

$(iii) \Rightarrow (i)$: Since every bounded set is contained in the interior of some bounded set, then $\mathcal{B} \subset \text{int}(\mathcal{B})$ and by Remark 1.2.2, $\text{int}(\mathcal{B}) \subset \mathcal{B}$ thus $\mathcal{B} = \text{int}(\mathcal{B})$ and $\mathcal{B}$ is open. □

Corollary 1.2.1. *Let (X, τ) be a topological space. For any given bornology $\mathcal{B}$ in a toplogical space X, the following conditions are equivalent:*

(i) $\mathcal{B}$ is proper;

(ii) $\mathcal{B}$ is generated by its subfamily of the bounded open sets and its family of bounded closed sets (it has both open and closed base);

Proof. It is a consequence of Theorem 1.2.1 and Theorem 1.2.2. □

We end this chapter by the notion of the characteristic function that Meroño and Garrido [18] also gave in order to obtain their metrization theorem for bornological universes.

Definition 1.2.5. [18, Definition 2.1] Let $(X, \mathcal{B})$ be a bornological universe. A *characteristic function* of $(X, \mathcal{B})$ is a continuous function $\chi : X \longrightarrow [0, \infty)$ such that a subset $M \subset X$ is bounded if and only if $\chi(M)$ is bounded in $[0, \infty)$, that is,

$$\mathcal{B} = \{M \subset X : \exists\delta > 0, \chi(M) \leq \delta\}.$$

Proposition 1.2.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X. Then the bornological universe $(X, \mathcal{B})$ admits a continuous characteristic function if and only if $\mathcal{B}$ has a countable base $\{B_m : m \in \mathbb{Z}^+\}$ such that for some $\delta > 0$,*

$$B_m^{\delta} \subset B_{m+1}, \textit{ for every } m \in \mathbb{Z}^+.$$

Proof. (See [18, Proposition 2.2]). □

2

Bornology of extended metric spaces

In this chapter, we summarize Beer [5]'s results on bornology of extended metric spaces. We will study and present his results on the construction of the universal space of an extended metric space and the metrizability of the bornological universe. Except when otherwise stated, most of the material in this chapter can be found in [5, 7, 8, 18].

2.1. Universal space of extended metric spaces

This section provides Beer's construction of the universal space for extended metric spaces.

Definition 2.1.1. Let X be an extended metric space. Then for any $x, y, \in X$, we define a relation $\mathcal{E}_d$ on X by $x\, \mathcal{E}_d\, y$ provided $d(x, y) < \infty$.

Lemma 2.1.1. *If (X, d) is an extended metric space, then the relation $\mathcal{E}_d$ is an equivalence relation on X.*

Proof. Let $d(x, y) < \infty$ and let $x \in X$. Since d is a metric, we get $d(x, x) = 0 < \infty$ and $x\, \mathcal{E}_d\, x$, implying that $\mathcal{E}_d$ is reflexive.

The fact that $\mathcal{E}_d$ is symmetric can be easily seen.

To show transitivity, suppose $x\, \mathcal{E}_d\, y$ and $y\, \mathcal{E}_d\, z$ for $x, y, z, \in X$, then $d(x, y) < \infty$ and $d(y, z) < \infty$ implies $d(x, z) < \infty$ by triangle inequality.

This means that $p(x, z) < \infty$, thus $x\, \mathcal{E}_d\, z$, hence $\mathcal{E}_d$ is transitive.

□

Definition 2.1.2. [5] Let (X, d) be an extended metric space and $x \in X$. Then the equivalence class of x on X, denoted by $mc_d(x)$, is called *metric component* of X.

Let $\mathscr{P}_0(X)$ be the set of all nonempty subsets of X and $C(A)$ be the set of continous functions on a nonempty set $A \in \mathscr{P}_0(X)$.

Definition 2.1.3. [5] Let (X, d) be an extended metric space and suppose $\Delta_X = \{(g, B) : g \in C(B)\}$ and $B \in \mathscr{P}_0(X)$ and let us equip the set Δ_X with a metric ρ_X defined by

$$\rho_X((g, B), (h, D)) = \begin{cases} \sup_{b\in B} |g(b) - h(b)| \text{ if B=D} \\ \\ \infty \text{ if } B \neq D. \end{cases}$$

The pair (Δ_X, ρ_X) is called the *universal space* of X

The following theorem proves the completeness of the universal space defined above.

Proposition 2.1.1. *[5, Proposition 3.1] Let (X, d) be an extended metric space. Then, the set $(\Delta_X; \rho_X)$ is an extended metric space that is complete.*

Proof. For us to show that (Δ_X, ρ_X) is a complete extended metrix space, we need to show that every Cauchy sequence in (Δ_X, ρ_X) converges. Let (g_n, An) be a Cauchy sequence in (Δ_X, ρ_X). By the definition of a Cauchy sequence and Definition 4.2.1, there exists an $\sigma > 0$ and $N \in \mathbb{N}$ such that for $n, j \geq N$, $\rho_X((g_n, A_n), (g_j, A_j)) < \sigma$. Assuming that all the partial functions have the common domain, i.e. $A_n = A_j = A_i$ for $n = j \neq i \geq N$ we have,

$$\rho_X((g_n, A_i), (g_j, A_i)) = \sup_{b\in A_i}\{|g_n(b) - g_j(b)|\} < \sigma.$$

Since $|g_n(b) - g_j(b)| < \sigma$, it means that the sequence is uniformly Cauchy (Every uniformly Cauchy sequence converges) on A_i and, thus $((g_n, A_i) \longrightarrow (g, A))$. □

Theorem 2.1.1. *[5, Proposition 3.3] Let (X, d) be an extended metric space. Then for an extended metric d on X, $(\mathscr{P}_0(X), H_d)$ can be embedded into (Δ_X, ρ_X).*

Proof. (See [5, Proposition 3.3]). □

The next result illustrates an embedding between a metric space and the universal space.

Theorem 2.1.2. *[5, Theorem 3.2] Let (X, d) be an extended metric space. Then, the function $\pi : (X, d) \longrightarrow (\Delta_X, \rho_X)$ defined by $\pi(x) = (d(x, \cdot)|_{mc_d(x)}, mc_d(x))$, for $x \in X$ is an isometry.*

Proof. In this proof, we need to show that the function π is an isometry. Let $x_1 \neq x_2$ be points in X. If $d(x_1, x_2) = \infty$, then $mc_d(x_1) \neq mc_d(x_2)$ and so by definition of ρ_X, one gets,

$$\begin{aligned}\rho_X(\pi(x_1), \pi(x_2)) &= \rho_X[(d(x_1, \cdot)|_{mc_d(x_1)}, mc_d(x_1)), (d(x_2, \cdot)|_{mc_d(x_2)}, mc_d(x_2))] \\ &= \infty \\ &= d(x_1, x_2).\end{aligned}$$

On the other hand, if $d(x_1, x_2) \neq \infty$, then $mc_d(x_1) = mc_d(x_2)$. We must show that the $\sup_{x \in [mc_d(x_1)]} |d(x_1, x) - d(x_2, x)| = d(x_1, x_2)$.

For each $x \in mc_d(x_1) = mc_d(x_2)$, we have $d(x_1, x_2) < \infty$. So using triangle inequality, the supremum is achieved by letting $x = x_1$, thus

$$\begin{aligned}\rho_X(\pi(x_1), (\pi(x_2)) &= \rho_X[(d(x_1, \cdot)|_{mc_d(x_1)}, mc_d(x_1)), (d(x_2, \cdot)|_{mc_d(x_2)}, mc_d(x_2))] \\ &= \sup_{x \in [mc_d(x_1)]} |d(x_1, x) - d(x_2, x)| \\ &= d(x_1, x_2).\end{aligned}$$

attained as required. ∎

The following theorem proves the existence of an isometry from one universal space to another.

Theorem 2.1.3. *[5, Theorem 3.4] Let (X, d) and (X_1, d_1) be extended metric spaces and if there exists a surjective function σ between X and Y. Then there exists an embedding between (Δ_X, ρ_X) and $(\Delta_{X_1}, \rho_{X_1})$.*

Proof. For us to show the existence of an embedding between (Δ_X, ρ_X) and $(\Delta_{X_1}, \rho_{X_1})$, let the function $\psi : (\Delta_X, \rho_X) \longrightarrow (\Delta_{X_1}, \rho_{X_1})$ be given by $\psi(g, A) = (go\sigma|_{\sigma^{-1}(A)}, \sigma^{-1}(A))$. We now prove that ψ is an isometry. Let $A_1 \neq A_2$ be sets in X_1 which are closed, and since σ is a continous surjective function, we have $\sigma^{-1}(A_1) \neq \sigma^{-1}(A_2)$ as subsets of X too. Now, if $g \in C(A_1)$ and $h \in C(A_2)$ then by the definition of ρ_X, we have

$$\begin{aligned}\rho_X(\psi(g, A_1), \psi(h, A_2)) &= \rho_X(go\sigma|_{\sigma^{-1}(A_1)}, \sigma^{-1}(A_1)), (ho\sigma|_{\sigma^{-1}(A_2)}, \sigma^{-1}(A_2))) \\ &= \infty \\ &= \rho_{X_1}((g, A_1), (h, A_2)).\end{aligned}$$

Again, let $A_1 = A_2 = A$ and $f, g \in C(A)$, since σ is a continous surjective map, we compute the suprimum as follows:

$$\begin{aligned}\rho_X(\psi(g, A_1), \psi(h, A_2)) &= \sup_{x\in\sigma^{-1}(A)} |g(\sigma(x)) - h(\sigma(x))| \\ &= \sup_{y\in A} |g(y) - h(y)| \\ &= \rho_{X_1}((f, A_1), (g, A_2)).\end{aligned}$$

□

If $\mathscr{C}_0(X)$ is a set of all closed subsets of X and $C(X)$ is a set of continuous functions on an extended metric space (X, d), then one has the following results.

Theorem 2.1.4. *[5, Theorem 3.5] Let (X, d) be an extended metric space which is normal, and let $d(g, h) = \max_{x\in X} |g(x) - h(x)|$. If we define $\mathscr{C}_0(X)$ with the an extended discrete metric, then $(g, B) \longmapsto (g|_B, B)$ is a uniformly continuous surjective function between $C(X) \times \mathscr{C}_0(X)$ and (Δ_X, ρ_X).*

Proof. Let ϕ be a function $\phi : (g, B) \to (g|_B, B)$ between $C(X) \times \mathscr{C}_0(X)$ and $(\Delta_X; \rho_X)$. To prove that ϕ is surjective, let us pick $B \in \mathscr{P}_0(X)$ and $k \in C(A)$. Since the metric space X is normal, by the Tietze extension theorem (see [13, P. 49]), there exists $g \in C(X)$ whose restriction to B is k, i.e. $\phi(g, B) = (k, B)$. Thus, ϕ is onto.

To check the uniform continuity of ϕ. Let $\delta > 0$ and if $d_1 = ((g, A), (h, D)) < \min\{1, \delta\}$, then $A = D$ and $\sup_{x\in A} |g(x) - h(x)| < \delta$.

$$\rho_X(\varphi(g, A), \phi(h, D)) = \rho_X((g|_A, A), (h|_A, A)) = \sup_{a\in A} |f(a) - g(a)| < \delta.$$

Let M be open set in the product $C(X) \times \mathscr{C}_o(X)$ and let $(s, A) \in \phi(M)$ be arbitrary. Choose $t \in C(X)$ such that $(t, A) \in M$ and s extends t. Since V is open, there exists $\gamma \in (0, 1)$ such that if $\sup_{x\in X} \mid f(x) - g(x) \mid< \gamma$, then $(g, A) \in M$. We claim that the open ρ_X-ball of radius $\frac{\gamma}{2}$ about (s, A) lies in $\phi(M)$. Let (s', A') lie in the ball: $B_{\rho_X}((s, A), \frac{\gamma}{2}) = \{(s', A') \in C(X) \times \mathscr{C}_o(X) : \rho_X((s, A), (s', A')) < \frac{\gamma}{2}\} \subset \varphi(M)$. Then $\rho_X((s, A), (s', A')) < \frac{\gamma}{2}$ which means that $A = A'$ and $\sup_{a\in A} \mid s(a) - s'(a) \mid< \frac{\gamma}{2}$.

Now, choose again by Tietze extension theorem $f_{s'} \in C(X)$ that extends $s' \in C(A')$ and define $p_{s'} \in C(X)$ by

$$p_{s'} = \sup\left\{\inf\{g_1 + \frac{\gamma}{2}, f_{h'}\}, g_1 - \frac{\gamma}{2}\right\}.$$

By this construction, $(p_{s'}, A) \in V$ and $p_{s'}$ agrees with h' on A so that ϕ maps $(p_{s'}, A)$ to $(s', A') \in \phi(M)$ as required. □

2.2. Bornologies in extended metric spaces

We study and present Beer [5]'s results on metrizability of the bornological universe in this section. Beer wanted to know when the bornological universe $(X, \mathcal{B})$ is metrizable?

Let us start with the following definitions, and thereafter present Hu[19]'s theorem .

Definition 2.2.1. Let (X, d) be an extended metric space. Then a subset A of X is called d-bounded if $A \subseteq \bigcup_{j=1}^{n} B_{d_j}(x_j, \delta_j)$ for $\delta_j > 0$ and $x_j \in X$.

Note that if (X, d) is an extended metric space, then the set of all d-bounded subsets of X denoted by $\mathcal{B}_d(X)$ is called *metric bornology.*

Definition 2.2.2. [5] If a extended metric space (X, d) can be partitioned into disjoint unions of sets $\{X_j : j \in I\}$ and each X_j is metrizable, then we can choose an extended metric p_j for X_j and define the function d defined by

$$d(v, w) = \begin{cases} d_j(v, w) \text{ if } \exists i \text{ with } \{v, w\} \subseteq X_j \\ \\ \infty \text{ otherwise.} \end{cases}$$

is an *extended metric* of X and has $\{X_j : j \in I\}$ as metric components.

The next result shows that the metric bornology is proper, locally bounded and it has a countable base.

Theorem 2.2.1. *[19, Theorem 10.1] Let (X, d) be an extended metric space. Then, the metric bornology $\mathcal{B}_d(X)$ is proper, locally bounded and has a countable base.*

Proof. Let $(X, \mathcal{B})$ be metrizable and A be an arbitrary bounded set in $\mathcal{B}$. Now, since $\operatorname{diam}_d(\operatorname{cl}(A)) = \operatorname{diam}_d(A)$. It means that the $\operatorname{cl}(A)$ is a bounded set too. Furthermore, the ϵ-neighbourhood of A is an open set that is bounded with diameter $\leq \operatorname{diam}_d(A) + 2\epsilon$. Hence $\mathcal{B}$ is proper.

Let $x \in \mathcal{B}$ be an arbitrary point. Then the ϵ-neigbourhood U_x is a bounded open set containing x. Hence $\mathcal{B}$ is locally bounded.

Now, to show that $\mathcal{B}$ has a countable base, let us choose a fixed point $x_0 \in X$ and let $G_n (n = 1, 2, 3, 4 \ldots)$ denote the n-neighbourhoods of x_0. Now we need to prove that the family $\mathscr{G} = \{G_1, G_2, \ldots\}$ forms a base for the bornology $\mathcal{B}$. Let M be an arbitrary bounded set in $\mathcal{B}$ and we choose a point $x \in M$ with $\rho = d(x_0, x_1)$ and $\delta = \text{diam}_d(M)$. Then $d(x_0, x) \leq \rho + \delta$ for each $x \in M$. Hence M is contained in G_n if $n > \rho + \delta$. $\square$

The following result is the well-known Hu's Theorem.

Theorem 2.2.2. *[17, Thorem 2] Let $((X, \tau), \mathcal{B})$ be a bornological universe. Then $\mathcal{B}$ is metrizable if and only if the following three conditions are satisfied:*

(i) The topological space X is metrizable;

(ii) $\mathcal{B}$ has a countable base, a closed base and an open base;

(iii) For each $B_1 \in \mathcal{B}$ there is a $B_2 \in \mathcal{B}$ with $cl(B_1) \subseteq int(B_2)$.

Proof. If $\mathcal{B}$ is metrizable, then axiom (i) holds. For (ii), let us pick a point $s \in X$. For each $n \in \mathbb{Z}^+$, the open balls $B_n(s) = \{x \in X : d(s, x) < n\}$ form a countable base say $\mathcal{A} = \{B_d(s, n) : n \in \mathbb{Z}^+\}$ for $\mathcal{B}$. As for every arbitrary $B \in \mathcal{B}$ there exists n such that $B \subset B_n(s)$ so, $\text{cl}(B) \subset \text{cl}(B_n(s)) \subset B_{n+1}(s)$ and we have (iii).

Conversely, assume that $(X, \mathcal{B})$ satisfies the axioms (i), (ii) and (iii). Since X is metrizable then there exists a bounded metric $\pi : X \times X \longrightarrow \mathbb{R}$ which defines the topology on X. Again by (i), (ii), (iii), we have a characteristic function $\chi : X \longrightarrow \mathbb{R}$ on the bornological universe $(X, \mathcal{B})$.

Now define the function $d : X \times X \longrightarrow \mathbb{R}$ for every $x, y \in X$ by taking

$$d(x, y) = \pi(x, y) + \mid \chi(x) - \chi(y) \mid .$$

In fact, d is a metric on X, to prove that d defines the same topology on X, it is enough to show that d and π are equivalent. To show the equivalence, consider any point $p \in X$ and any $q, t \in \mathbb{R}^+$. Since χ is continuous and since π defines the same topology of X, we have $t < \frac{q}{2}$ for which $\mid \chi(p) - \chi(x) \mid < \frac{q}{2}$ holds for every point $x \in X$ satisfying $\pi(p, x) < t$. It follows that, for every $x \in X$, $\pi(p, x) < t$ implies

$$d(p, x) = \pi(p, x) + \mid \chi(p) - \chi(x) \mid < t + \frac{q}{2} < \frac{q}{2} + \frac{q}{2} = q.$$

Now if $d(p, x) < t$ implies $\pi(p, x) < t$. So we have the topological equivalence between the metrics π and d.

To verify that the metric d defines the bornology $\mathcal{B}$. Choose $\lambda \in \mathbb{R}^+$ such that $\pi(x, y) < \lambda$ holds for every point (x, y) of $X \times X$. Let E be any subset of X. If $E \in \mathcal{B}$, then $\chi(E) < K$, $K > 0$. Hence we obtain

$$d(x, y) = \pi(x, y) + \mid \chi(x) - \chi(y) \mid < \lambda + K$$

for all $x, y \in E$. This proves that E is bounded for the metric d, implying $\mathcal{B} \subseteq \mathcal{B}_d(X)$.

Conversely, assume that $E \in \mathcal{B}_d(X)$ which is not empty. Then there exists a positive real number q such that $d(x, y) < q$ holds for all points $x, y \in E$. Again let us choose a point $p \in E$. Then we have

$$\chi(x) \leq \chi(p) + \mid \chi(p) - \chi(x) \mid \leq \chi(p) + d(p, x) < \chi(p) + q$$

for all $x \in E$. This implies $E \in \mathcal{B}$. Hence the bornologies are equivalent, that is, d defines the given bornology $\mathcal{B}$ of X and so $\mathcal{B}_d(X) \subseteq \mathcal{B}$. $\square$

We state the following lemma and its proof will be given in asymmetric settings in Chapter 4.

Lemma 2.2.1. *[5, Lemma 2.1] Given an extended metric space (X, d) and a metric bornology $\mathcal{B}_d(X)$:*

(i) the family of open balls which are all finite forms a base for the metric bornology $\mathcal{B}_d(X)$;

(ii) the metric bornology $\mathcal{B}_d(X)$ has a bornology of subsets of X which is relatively compact;

(iii) If (x_j) is a Cauchy sequence in X, then, the set $\{x_j : j \in \mathbb{Z}^+\}$ is contained in $\mathcal{B}_d(X)$.

The next two theorems prove the metrilizabiliy of the bornological universe by the extended metrics.

Theorem 2.2.3. *[5, Theorem 4.1] Let (X, d) be an extended space and let $\mathcal{B}$ be a bornology on X. Then $\mathcal{B} = \mathcal{B}_d(X)$ if and only if there exists $\mathcal{A}$ in $\mathcal{B}$ such that $\downarrow (\Sigma(\mathcal{A})) = \mathcal{B}$ and a partition $\{\mathcal{A}_j : j \in I\}$ of $\mathcal{A}$ with the following four conditions:*

(i) every partition $\mathcal{A}_j$ of $\mathcal{A}$ has a nonempty subset of X;

(ii) for all $A_1 \in \mathcal{A}_j$, there exists $A_2 \in \mathcal{A}_j$ with $cl(A_1) \subseteq int(A_2)$ for each $j \in I$;

(iii) whenever $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ for $i \neq j$, then $A_i \cap A_j = \emptyset$;

(iv) each $\mathcal{A}_j$ has a subfamily which is countable and cofinal in $\mathcal{A}_j$ with respect to set inclusion.

Proof. For necessity. Let $\mathcal{B}$ be a bornology on an extended metric space X with metric components $\{X_j : j \in I\}$. For every $j \in I$, let $\mathcal{A}_j = \{B_d(x, \delta) : x \in X_j, \delta > 0\}$ and let $\mathcal{A}$ be the collection of open balls in a metric space X. Now, since a subset of X is d-bounded if and only if it is contained in a finite union of open balls and $\mathcal{A}$ is a cover of X, we have

$$\mathcal{B}_d(X) =\downarrow \Sigma(\cup_{j\in I}\mathcal{A}_j) =\downarrow \big(\Sigma(\mathcal{A})\big).$$

Axioms (i) to (iv) hold directly with respect to the choice of $\mathcal{A}$.

Conversely, suppose the partition $\{\mathcal{A}_j : j \in I\}$ of $\mathcal{A}$ has the above conditions. For every $j \in I$, let $X_j = \cup\mathcal{A}_j$. By axiom (i), each X_j is nonempty, and by (ii), every X_j is open. Since $X_j = \cup\mathcal{A}_j$, (iii) tells us that $\{X_j : j \in I\}$ is a pairwise disjoint collection that partitions X. Since $\{X_j : j \in I\}$ are metric components and (ii) implies that it is a clopen base, we have that each X_j is closed and open. For each $j \in I$, let us put $\mathcal{B}_j :=\downarrow (\Sigma(\mathcal{A}_j))$. Because $\mathcal{A}_j$ is a cover of X_j, $\mathcal{B}_j$ is a bornology on X_j. Since the closure of a finite union is the union of the closures, property (ii) holds with $\mathcal{B}_j$ replacing $\mathcal{A}_j$. Property (iii) holds as well if $\mathcal{B}_i$ replaces $\mathcal{A}_i$ and $\mathcal{B}_j$ replaces $\mathcal{A}_j$. Finally, if the partition $\mathcal{A}_j^*$ is countable and cofinal in $\mathcal{A}_j$, then $\Sigma(\mathcal{A}_j^*)$ is countable and cofinal in $\mathcal{B}_j$ and so, (iv) holds with $\mathcal{B}_j$ replacing $\mathcal{A}_j$. Thus by Theorem 2.2.2, for each positive $j \in I$, there exists a metric d_j on X_i such that $\mathcal{B}_j = \mathcal{B}_{d_j}(X_j)$.

As $\downarrow (\Sigma(\mathcal{A}))$ is assumed to be a cover of X, we conclude that $\mathcal{A}$ is a cover of X and so $\{X_j : j \in I\}$ covered by the partition $\{\mathcal{A}_j : j \in I\}$ is a cover of X. Now considering the extended metric in Defintion 2.2.2, we get

$$\mathcal{B}_d(X) = \bigcup_{j\in I}\mathcal{B}_{d_j}(X_j) = \sum(\cup_{j\in I}\mathcal{B}_j) =\downarrow \Big(\sum(\cup_{j\in I}\mathcal{A}_j)\Big) =\downarrow \Big(\sum(\mathcal{A})\Big) = \mathcal{B}.$$

□

Theorem 2.2.4. *[5, Theorem 4.2] If $(X, \mathcal{B})$ is a bornological universe with an extended metric X. Then, the bornology $\mathcal{B} = \mathcal{B}_d(X)$ if and only if there is a family $\{\mathcal{B}_j : j \in I\}$ of collections of subsets of X with the following five conditions:*

(i) for every $j \in I$, a family $\mathcal{B}_j$ has at least two subsets of X together with $\emptyset$;

(ii) for all $B_1 \in \mathcal{B}_j$, there exists $B_2 \in \mathcal{B}_j$ with $cl(B_1) \subseteq int(B_2)$ for each $j \in I$;

(iii) suppose $B_j \in \mathcal{B}_j$ and $B_k \in \mathcal{B}_k$ for $j \neq k$, then $B_j \cap B_k = \emptyset$;

(iv) each $\mathcal{B}_j$ has a subfamily which is countable and cofinal in $\mathcal{B}_j$ with respect to set inclusion;

(v) for all $A \in \mathscr{P}(X)$, A is contained in $\mathcal{B}$ if for each $j \in I$, $A \cap (\cup\mathcal{B}_j) \subset \mathcal{B}_j$ and for all finitely many j, $A \cap (\cup\mathcal{B}_j) = \emptyset$.

Proof. Let $\mathcal{B} = \mathcal{B}_d(X)$ and $\{X_j : j \in I\}$ be the metric components of X. Suppose d_i is the trace of d on $X_j \times X_j$ and set $\mathcal{B}_j := \mathcal{B}_{d_j}(X_j)$, then: from the definition of a metric component of X, each X_j contains some element of X implying that each is nonempty and $\mathcal{B}_{d_j}(X_j)$ contains minimally the finite subsets of X_j and so. (i) holds. Properties (ii) and (iii) clearly hold, while (iv) is addressed in the hypothesis of this theorem.

Conversely, If $\{\mathcal{B}_j : j \in I\}$ satisfies the axioms above. Then, for all $j \in I$. Let $X_j = \cup\mathcal{B}_j$ by condition (i) each X_j is a subset of X that is not empty. By conditions (ii) and (iii), $\{X_j : j \in I\}$ partitions X into clopen subsets of X. Next, we need to show that each family $\mathcal{B}_j$ is a bornology on X_j.

Let us fix $j \in I$ and set $\mathscr{U}_j := \{B \cap X_j : B \in \mathcal{B}\}$. Since each $X_j = \cup\mathcal{B}_j$, we have by (v) that $\mathscr{U}_j$ is contained in $\mathcal{B}_j$. Thus, $\mathscr{U}_j$ is clearly a base bornology on X_j, since for each arbitrary $B \in \mathcal{B}$, there is an arbitrary $B \cap X_j \in \mathscr{U}_j$ with $B \subseteq B \cap X_j \in \mathscr{U}_j$. So. $\mathscr{U}_j$ is however a bornology on X_j. Moreover, we need to show that $\mathcal{B}_j \subseteq \mathscr{U}_j$. Let $B_j \in \mathcal{B}_j$ be arbitrary, then its enough to show that $B_j \in \mathscr{U}_j$ which suffices to show that $B_j \in \mathcal{B}$. The structure $X_j = \cup\mathcal{B}_j$ implies $B_j \cap X_i \in \mathcal{B}_i$. Taking $j \neq k$, condition (iii) gives $B_j \cap X_k = \emptyset$, by propety (i), we get $B_j \cap X_k = \emptyset \in \mathcal{B}_k$. Now. in view of axiom (v), $B_j \in \mathcal{B}$ so that $B_j = B_j \cap X_j \in \mathscr{U}_j$. Hence, the family $\mathcal{B}_j$ is established as a bornology. By Theorem 2.2.2, conditions (ii) and (iv) implies that for all $j \in I$. there exists an extended metric d_j on X_j such that $\mathcal{B}_j = \mathcal{B}_{d_j}(X_j)$. Using the metric of Definition 2.2.2 and condition (iv) gives, we get $\mathcal{B} = \mathcal{B}_d(X)$. □

We now turn our attention to the uniform metrization of a bornology in the following theorems.

Theorem 2.2.5. *[5. Theorem 4.4] Let (X, d) be an extended metric space. Then. the following two properties are equivalent:*

(i) an extended metric d induces the countable set of metric components:

(ii) given an extended metric d. there exists an extended metric $d_1 = \min\{1, d\}$ such that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$.

Proof. Firstly. we prove that if an extended metric d induces a countable set of metric components, then there is an extended metric d_1 such that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$.

If d induces the set of distinct metric components $\sum\{mc_d(x_j) : j \in I\}$ for the countable set I. then the set $\sum\{\mathcal{B}_d(x_j; m) : j \in I, m \in \mathbb{Z}^+\}$ is a countable base for $\mathcal{B}_d(X)$. For finite union of open balls, each bounded set is cotained in the interior of another, thus. by Theorem 2.2.2, there exists an extended metric $d_1 = \min\{1, d\}$ such that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$.

We now prove that given an extended metric $d_1 = \min\{1, d\}$ such that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$, there exist a set of countable metric components induced by d.

Let the distinct metric components be doneted by $\{mc_d(x_j) : j \in I\}$. If (ii) holds then by Theorem 2.2.2 again, there exists an extended metric d_1 such that $\mathcal{B}_{d_1}(X) = \mathcal{B}_d(X)$. This means, $\mathcal{B}_d(X)$ has a countable base. Hence, $\sum\{\mathcal{B}_d(x_j; m) : j \in I, m \in \mathbb{Z}^+\}$ contains a countable cofinal collection in $\mathcal{B}_d(X)$. Since $\{\{x_j\} : j \in I\}$ is a family of d-bounded sets, it implies that I is countable. $\square$

Theorem 2.2.6. *[17, Theorem 9] Let (X, d) be a metric space and $(X, \mathcal{B})$ be a bornological universe on X. Then, the following conditions are equivalent:*

(i) The bornology $\mathcal{B} = \mathcal{B}_{d'}(X)$ for some uniformly equivalent metric d' on the metric space (X, d);

(ii) The bornology $\mathcal{B}$ has a countable base $\{B_n : n \in \mathbb{Z}^+\}$ such that there exist $\epsilon > 0$, with $B_n^\epsilon \subseteq B_{n+1}$ for a natural number n.

Proof. (i) $\Rightarrow$ (ii): If $\mathcal{B} = \mathcal{B}_{d'}(X)$. Let us fix $s \in X$. For each $n \in \mathbb{N}$, we define

$$B_n = B_d(s, n) = \{x \in X : d(x, z) < n\}$$

Since for $\epsilon < \frac{1}{2}$, $[B_n]_d^{\frac{1}{2}} \subseteq B_{n+1}$, then B_n forms a countable base say $\mathcal{A} = \{B_n : n \in \Omega\}$ for $\mathcal{B}$. Hence, we infer that (ii) follows from (i).

(ii) $\Rightarrow$ (i): If $X \in \mathcal{B}$ then we are done, since $\mathcal{B} = \mathscr{P}(X)$ for $d' = \min\{1, d\}$. Now if $X \notin \mathcal{B}$ and take the bounded metric $d^* = \min\{1, d\}$, which is uniformly equivalent to d and defining a new metric $d' : X \times X \longrightarrow \mathbb{R}$ by, $d'(x, y) = d^*(x, y) + \mid \chi(x) - \chi(y) \mid$ where χ is the characteristic function associated to the base of the bornology.

If $d'(x, y) < \epsilon$ then $d^*(x, y) < \epsilon$ again using the uniform continuity of χ, for every $\sigma > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with

$$d^*(x, y) < \delta \Rightarrow d'(x, y) = d^*(x, y) + \mid \chi(x) - \chi(y) \mid < \sigma + \delta.$$

The uniform equivalence of the two metrics follows.

Next we show that $\mathcal{B} = \mathcal{B}_{d'}(X)$. If $B \in \mathcal{B}$ then it is bounded for the metric d^*. Since both d^* and d defines the some topology on X and $\chi(B)$ is bounded in $\mathbb{R}$, $B \in \mathcal{B}_{d'}(X)$ this is so because d^* and d' are equivalent. If $B \in \mathcal{B}_{d'}(X)$ then there exists a $k \in \mathbb{R}$, $k > 0$ such that $\forall x, y \in B$ then $d'(x, y) < k$. As d^* is a bounded metric then $\chi(B)$ is bounded in $\mathbb{R}$ so then $B \in \mathcal{B}$. $\square$

Theorem 2.2.7. *Let (X, d) be an extended metric space that is free union of metric components $\{X_j : j \in I\}$. Then, the following two conditions are equivalent:*

(i) each set X_j is second countable and locally compact;

(ii) there exists an extended metric d that is compact with metric components $\{X_j : j \in I\}$.

Proof. $(i) \to (ii)$: Let (X, d) be an extended metric space that is a free union of the metric components $\{X_j : j \in I\}$ and each X_j in its relative topology is second countable and locally compact. Then by Vaughans Theorem (see [33]) we can find for each $j \in I$ a metric d_j compatible with topology for X_j such that each closed d_j-ball is compact. Now let the extended metric d induced by $\{d_j : j \in I\}$ on X be defined as of Definition 2.2.2, then each closed and d-bounded subset must lie in a finite union of compact closed balls and is thus compact.

$(ii) \to (i)$: Suppose there exists a compact extended metric d with metric components $\{X_j : j \in I\}$. Now, if each X_j is closed, then the restriction of d to $X_j \times X_j$ is compact too. Again by Vaughan's theorem, we get (ii). □

We next define the notion of the weakly bounded set.

Definition 2.2.3. Let (X, d) be an extended metric space, a subset A of X is said to be *weakly bounded* if its intersection with each metric component is bounded. We denote the bornology of d-weakly bounded sets by $\mathcal{B}_d^w(X)$.

Theorem 2.2.8. *[5, Theorem 4.6] Let (X, d) be an extended metric space. Then, the following conditions are equivalent:*

(i) all the finitely many metric components for an extended metric d are d-bounded;

(ii) for every extended metric d there exists an extended metric $d_1 = \min\{1, d\}$ for which $\mathcal{B}_d^w(X) = \mathcal{B}_{d_1}(X)$.

Proof. $(i) \Rightarrow (ii)$: Suppose (X, d) is an extended metric space and $\{mc_d(x_j) : j \in I\}$ be its metric components and let $I_1 \subseteq I$ be those indices for which $mc_d(x_j) \in \mathcal{B}_d(X)$. Then, we have

$$\left\{ \cup_{j\in I_1} mc_d(x_j) \bigcup \cup_{j\in I/I_1} \mathcal{B}_d(x_j; s) : s \in \mathbb{Z}^+ \right\}$$

to be a countable base for $\mathcal{B}_d^w(X)$. The closure of each weakly bounded subset is in the interior of another. By Theorem 2.2.2, there exists an extended metric $d_1 = \min\{1, d\}$ for which $\mathcal{B}_d^w(X) = \mathcal{B}_{d_1}(X)$.

$(ii) \Rightarrow (i)$: We prove by contradiction. Suppose there exists a countably infinite set of indices $\{j_k : k \in \mathbb{Z}^+\}$ for which $mc_d(x_{j_k})$ is not d-bounded. Fix $s \in X$ to get $\{B_{d_1}(s; m) : m \in \mathbb{Z}^+\}$ so that it is cofinal in $\mathcal{B}_d^w(X)$. For each $k \in \mathbb{Z}^+$ there exists $A_k \in \mathcal{B}_d(X)$ such that $A_k \in mc_d(x_{j_k})$ and $A_k \nsubseteq B_{d_1}(s; k)$. Then $\bigcup_{k=1}^{\infty} A_k \cup \{x_j : j \in I\}$ is weakly d-bounded but it is not contained in any d_1-ball with center s. This is a contradiction, since we have a countably infinite set of indices $\{j_k : k \in \mathbb{Z}^+\}$ for which the set $mc_d(x_{j_k})$ is not d-bounded. □

Theorem 2.2.9. *[5, Theorem 4.4] Let (X, d) be an extended metric space that is a free union of $\{X_j : j \in I\}$. Then, the following properties are equivalent:*

(i) the set I is finite and the space X is locally compact and second countable;

(ii) there exists an extended metric d with metric components $\{X_j : j \in I\}$ such that each closed and weakly bounded subset is compact.

Proof. $(i) \Rightarrow (ii)$: If the set I is finite, then for any extended metric d with metric components $\{X_j : j \in I\}$, we have $\mathcal{B}_d^w(X) = \mathcal{B}_d(X)$. Each open subspace of a locally compact and second countable space has the above conditions. Thus by Theorem 2.2.7, we get (ii)

$(ii) \Rightarrow (i)$: If there exists an extended metric d with metric components $\{X_j : j \in I\}$ such that each closed and weakly bounded subset is compact, then I is finite, otherwise choosing $x_j \in X_j$, the discrete set $\{X_j : j \in I\}$ would be closed and weakly d-bounded but not compact. Again with $\mathcal{B}_d^w(X) = \mathcal{B}_d(X)$ and noting that a free union of finitely many second countable spaces is second countable, one gets (i) by Theorem 2.2.7 again. □

Bornologies also play a vital role in extended normed spaces. Let us realise this concept in the next section.

2.3. Bornologies of extended normed spaces

In this section, we summarize the concept of bornologies in extended normed spaces studied by Beer[6].

We start this section with some basic concepts of normed spaces from Beer[8]'s paper.

Definition 2.3.1. [8] Let $(Y, \|\cdot\|)$ be an extended normed space. Then for any $x, y, \in Y$, we define a relation $\mathcal{E}_n$ on Y by $x\,\mathcal{E}_n\,y$ provided $\|x-y\| < \infty$.

Lemma 2.3.1. *Suppose $(Y, \|\cdot\|)$ is an extended normed space, then the relation $\mathcal{E}_n$ is an equivalence relation on Y.*

Proof. Let $x \in Y$, since Y is a normed space, we know that $\|x-x\| = 0 < \infty$ and $x\,\mathcal{E}_n\,x$.

To show that $\mathcal{E}_p$ is symmetric: Let $x, y \in Y$ and since $\|x-y\| = \|y-x\| < \infty$ thus, $x\,\mathcal{E}_n\,y$ and $y\,\mathcal{E}_n\,x$.

Transitivity of this relation: Suppose we have $x\,\mathcal{E}_n\,y$ and $y\,\mathcal{E}_n\,z$ for $x, y, z, \in Y$ then $\|x-y\| < \infty$ and $\|y-z\| < \infty$. By the triangle inequality, it is easy to see that $\mathcal{E}_n$ is transitive.

□

Definition 2.3.2. Let $(Y, \|\cdot\|)$ be an extended normed space and $y \in Y$. Then the equivalence class of y on Y denoted by $y/\mathcal{E}_n$ is a closed and open subset of Y called *metric component* of Y.

Definition 2.3.3. Let Y be a linear vector space. Each equivalence class is a translate of a linear subspace of Y defined by $Y_{\text{fin}} = \{y \in Y : \|y\| < \infty\}$. Evidently, $(Y_{\text{fin}}, \|\cdot\|)$ is a convention normed linear space.

Let Y be a linear vector space and $A \subset Y$. We call a translate of A a *flat*. The set of all the flats which are determined by a subspace A is denoted by $\{y + A : y \in Y\}$.

Remark 2.3.1. [6] The collection of flats $\{y + Y_{\text{fin}} : y \in Y\}$ partition Y into clopen subsets of Y. For a fixed $y_0 \in Y$, the flat $y_0 + Y_{\text{fin}}$ is the equivalence class of x_0 under the equivalence relation $\mathcal{E}_n$ and we call it the *metric component* of y_0.

If $(Y, \|\cdot\|)$ is a conventional normed space and A is a closed subspace of Y, then $\|y + A\| = \inf\{\|y + a\| : a \in A\}$ defines what is called a *quotient norm* on Y/A. We say a nonempty subset A of Y is *Y^*-bounded* if $\varphi \in Y^*$ and $\varphi(A)$ is a bounded set of real numbers.

Definition 2.3.4. Let Y be a linear vector space. If A and B are two subspaces of Y, we write $Y = A \dotplus B$ provided each element of Y can be written in a unique way as $a + b$ where $a \in A$ and $b \in B$. In this case A and B are called *complementary subspaces.*

For a normed space $(Y, \|\cdot\|)$, the normed space $(Y, \|\cdot\|_{\mathcal{T}})$ is called *finitely compatible norm* on Y whenever $(Y, \|\cdot\|_{\mathcal{T}})$ is a conventional norm on Y and when restricted to the subspace Y_{fin}, $(Y, \|\cdot\|_{\mathcal{T}})$ and $(Y, \|\cdot\|)$ are equivalent conventional norms.

Definition 2.3.5. [8, Definition 4.2] Let $(Y, \|\cdot\|)$ be an extended normed space. We call a subset A of Y *bounded* if there exists $\{y_1, y_2, \ldots, y_n\} \subseteq Y$ and $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\} \subseteq (0, \infty)$ with $A \subseteq \cup_{i=1}^{n} B_d(y_i, \lambda_i)$.

The next result proves the relatively compactness of a non empty set.

Theorem 2.3.1. *[6, Theorem 4] Suppose $(Y, \|\cdot\|)$ is an extended normed space and M is a nonempty subset of Y. Then the subspace M is relatively compact if and only if there exists a normed compact subset $A \subset Y_{fin}$ and some finite subset $\{y_1, y_2, y_3 \ldots y_n\}$ of Y such that $M \subset A + \{y_1, y_2, y_3 \ldots y_n\}$.*

Proof. The forward proof can be easily seen. Conversely, let M be a nonempty subset of a normed space Y and let A_0 be a normed compact superset of M. Since A_0 is compact and using Remark 2.3.1, $\{y + Y_{\text{fin}} : y \in Y\}$ is an open cover of A_0. which means there exists a finite family $K_j = \{y_j + Y_{\text{fin}} : 1 \leq j \leq m\}$ that covers A_0 and each flat hits A_0. Now, let $C_j = K_j \cap A_0$ be normed compact set too. Then $\cup_{j=1}^{m}(-y_j + C_j)$ is a normed compact subset of Y_{fin}, thus

$$M \subset A_0 \subset + \bigcup_{j=1}^{m}(-y_j + C_j) + \{y_1, y_2. y_3 \ldots y_n\}.$$

□

Let us consider the following observation about linear functionals which are continuous.

Proposition 2.3.1. *[6, Proposition 1] Let $(Y_1, \|\cdot\|_1)$ and $(Y_2, \|\cdot\|_2)$ be extended normed spaces and $F : Y_1 \longrightarrow Y_2$ be a linear transformation, and $\|\cdot\|_T$ be finitely compatible norm on Y. If $F : (Y_1, \|\cdot\|_T) \longrightarrow (Y_2, \|\cdot\|_2)$ is continuous, then $F : (Y_1, \|\cdot\|_1) \longrightarrow (Y_2, \|\cdot\|_2)$ is also continuous.*

Proof. (See [6, Proposition 1]). □

Proposition 2.3.2. *[6, Proposition 2] Let $(Y, \|\cdot\|)$ be an extended normed space so that Y/Y_{fin} becomes a finite dimensional, and let $\|\cdot\|_T$ be a finitely compatible norm on Y. Then the following conditions are equivalent:*

(i) The norm $\|\cdot\|_{T_1}$ is equivalent to $\|\cdot\|_{T_\infty}$ on Y where $\|\cdot\|_{T_1}$ is defined by expressing Y as direct sum $Y = Y_{fin} \oplus N$ where N is finite dimensional and let us fix a norm $\|\cdot\|_N$ and then letting $\|y + n\|_{T_1} = \|y\| + \|n\|_N$ for $y \in Y_{fin}$ and $n \in N$.

(ii) Y_{fin} *is a* $\|\cdot\|_{\mathcal{T}}$*-closed set in an extended normed space* Y.

(iii) The functionals on $(Y, \|\cdot\|_{\mathcal{T}})$ *and* $(Y, \|\cdot\|)$ *are continous and coincide.*

Proof. (*i*) $\Rightarrow$ (*iii*): It is known from Proposition 2.3.1 that if ϕ is continuous on $(Y, \|\cdot\|_{\mathcal{T}}$ then it is continuous on $(Y, \|\cdot\|)$. Conversely, suppose the function $\phi \in Y'$ is continuous on $(Y, \|\cdot\|)$. Then ϕ is bounded on $\{y : \|y\| \leq 1\}$ and because ϕ is linear on the finite dimensional space N, it is bounded on $\{n \in N : \|y\|_N \leq 1\}$. Therefore, ϕ is bounded on $\{y + n : \|y\| + \|n\|_N \leq 1\}$. Consequently ϕ is continuous on $(Y, \|\cdot\|_{\mathcal{T}_1})$, and thus also continuous on $(Y, \|\cdot\|_{\mathcal{T}})$.

(*iii*) $\Rightarrow$ (*ii*): Suppose $y_0 \in Y \backslash Y_{\text{fin}}$. Then there exists $\phi \in Y$ such that $\phi(Y_{\text{fin}}) = \{0\}$ and $\phi(y_0) = 1$. Then $\phi \in Y^*$, since $\phi|_{Y_{\text{fin}}}$ is continuous (see [8, Theorem 4.3]). Condition (iii) then implies ϕ is continuous with respect to $\|\cdot\|_{\mathcal{T}}$ and so Y_{fin} is $\|\cdot\|_{\mathcal{T}}$-closed.

(*ii*) $\Rightarrow$ (*i*): Assume (ii) holds, and set $Y = Y_{\text{fin}} \oplus N$ as in (*i*). Because $\|\cdot\|_{\mathcal{T}}$ is finitely compatible and since all norms on N are equivalent, by triangle inequality, we see that $\|\cdot\|_{\mathcal{T}} \leq \gamma\|\cdot\|_{\mathcal{T}_1}$ for some $\gamma > 0$. Now suppose there is no constant $\beta > 0$ such that $\beta\|\cdot\|_{\mathcal{T}_1} \leq \|\cdot\|_{\mathcal{T}}$. Then there exists a sequence $(a_n) \subset X$ such that $\|a_n\|_{\mathcal{T}} \longrightarrow 0$ but $\|a_n\|_{\mathcal{T}_1} = 1$ for all n. Now let us write $a_n = y_n + s_n$ where $y_n \in Y_{\text{fin}}$ and $s_n \in N$ for all n. Since $\|s_n\|_N \leq 1$ for all n, by passing to a subsequence, we may assume $(s_n) \longrightarrow y$ for some $y \in N$. Now, because Y_{fin} is $\|\cdot\|_{\mathcal{T}}$-closed, by the conventional separation theorem, we can choose a linear functional ϕ with $\|\phi\|_{\mathcal{T}} \leq 1$ so that $\phi(Y_{\text{fin}}) = 0$ and $\phi(y) > 0$. ☐

We now study the X^*-bounded subsets in the following lemma.

Lemma 2.3.2. *[6, Lemma 2] Let* $(Y, \|\cdot\|)$ *be an extended normed space and* $C \notin Y_{fin}$ *be* Y^**-bounded subset of* Y. *Then there is a maximal linearly independent subset* $\{c_1, c_2, c_3 \ldots c_n\}$ *of* C *such that for all* $i \leq n$, $\|c_i\| = \infty$ *and*

$$C \subset Y_{fin} + span(\{c_1, c_2, c_3 \ldots c_n\}) = Y_{fin} \oplus span(\{c_1, c_2, c_3 \ldots c_n\}).$$

Proof. Let $C \notin Y_{\text{fin}}$ be Y^*-bounded subset of a normed space Y and let us consider $\mathcal{B} = \{B : B \text{ is a linearly independent subset of C that span (B)} \cap Y_{\text{fin}} = \{0_y\}\}$. Since C has an element of infinite norm, $\mathcal{B}$ is nonempty and by Zorn's lemma, $\mathcal{B}$ has a maximal element B_1. If B_1 is infinite, then we can get a linear functional ϕ on Y mapping Y_{fin} to 0 and B_1 to $\mathbb{Z}^+$. Since ϕ is continuous when restricted to Y_{fin}, it means ϕ is also continuous on Y by [8, Theorem 4.3.]. Clearly, ϕ is not bounded on C, contradicting Y^*-boundedness.

Now, if $B_1 = \{c_1, c_2, c_3 \ldots c_n\}$, then by construction, we get $Y_{\text{fin}} + \text{span}(\{c_1, c_2, c_3 \ldots c_n\})$ $= Y_{\text{fin}} \oplus \text{span}(\{c_1, c_2, c_3 \ldots c_n\})$. To see that $C \subset Y_{\text{fin}} + \text{span}(\{c_1, c_2, c_3 \ldots c_n\})$, let $c \in C$

be arbitrary. It is either $c \in Y_{\text{fin}}$ or $c \in \text{span } (B_1)$. Otherwise c has infinite norm and Y_{fin} nontrivially intersects $(B_1 \cup \{c\})$ which allows us to express c as a linear combination of some nonzero element of Y_{fin} and elements of B_1. $\square$

The next theorem presents a construction of Y^*-bounded set.

Theorem 2.3.2. *[6, Theorem 5] Let $(Y, \|\cdot\|)$ be an extended normed space, and suppose we have a nonempty set $C \subset Y$. Then C is Y^*-bounded if and only if there exists a ball V in Y_{fin} and a polytope P in Y such that $C \subset V + P$.*

Proof. Suppose C is an Y^*-bounded subset of Y. If C is a subset of Y_{fin}, then we can use the conventional theory to have the results. Otherwise, by Lemma 2.3.2 we know that $C \subset X$ where the subspace X can be written as $X = Y_{\text{fin}} \oplus N$ where N is a finite dimensional subspace of Y. Next, we let X be the finitely compatible norm $\|\cdot\|_{\mathcal{T}}$ defined as follows. For $x \in X$, we write $x = y + n$ with $y \in Y_{\text{fin}}$ and $n \in N$ and let us define $\|x\|_{\mathcal{T}} = \|y\| + \|n\|_N$ where $\|\cdot\|_N$ is a norm on N. By Proposition 2.3.2, any $\Phi \in X'$ that is continuous with respect to $\|\cdot\|_{\mathcal{T}}$ can be written as $\Phi = \phi|_X$ where $\phi \in Y^*$. Thus C is $(X, \|\cdot\|_{\mathcal{T}})^*$-bounded. Let $(X_1, \|\cdot\|_{\mathcal{T}})$ be the completion of $(X, \|\cdot\|_{\mathcal{T}})$. Then $(X_1, \|\cdot\|_{\mathcal{T}})^* = (X, \|\cdot\|_{\mathcal{T}})^*$ and so C is weakly bounded in the Banach space $(X_1, \|\cdot\|_{\mathcal{T}})^*$. By the Banach-Steinhaus theorem, we get C as $\|\cdot\|_{\mathcal{T}}$-bounded. Because N is finite dimensional, any bounded subset of N lies in a polytope and $C \subset V + P$ where V is a ball in Y_{fin} and P is a polytope in N.

Conversely, suppose $C \subset V + P$ where V is considered as a ball in Y_{fin} and P is a polytope. Then C is a subset of some subspace $Y_{\text{fin}} \subset X$ such that X/Y_{fin} is finite dimensional. Next, we write $X = Y_{\text{fin}} \oplus N$ and let the finitely compatible norm be $\|\cdot\|_{\mathcal{T}}$ on $Y_{\text{fin}} \oplus N$. It follows that C is $\|\cdot\|_{\mathcal{T}}$-bounded in X. Thus, given any $\phi \in Y^*$ and using Proposition 2.3.2, we have that $\phi|_X$ is continuous with respect to $\|\cdot\|_{\mathcal{T}}$. Thus, $\phi(C)$ is bounded for every $\varphi \in Y^*$. $\square$

Theorem 2.3.3. *[6, Theorem 6] Let $(Y, \|\cdot\|)$ be an extended normed space. If A is a subset of Y which is not empty. Then the set A is relatively weakly compact if and only if there exists a weakly compact set $D \subset Y_{fin}$ and a polytope S such that $A \subset D + S$.*

Proof. If $A \subset Y$ is a relatively weakly compact. Then $A \subset U$ where U is weakly compact. As proved in Theorem 2.3.2 and Lemma 2.3.2, we know that $U \subset X$ where $X = Y_{\text{fin}} \oplus N$ and $N \subset Y$ is a finite dimensional set. Let $h_1 : X \longrightarrow X$ and $h_2 : X \longrightarrow X$ be projections such that $h_1(X) = Y_{\text{fin}}$ and $h_2(X) = N$. Using [8, Theorem 4.3], we know that h_1 and h_2 are continuous on X, this is because they are continuous at 0_X. Now, for any $\varphi \in X^*$, $\phi \circ h_1$ and $\phi \circ h_2$ are continuous linear functions on X, and then we can check that h_1 and h_2 are continuous. Therefore, $h_1(U) \subset Y_{\text{fin}}$ and $h_2(U) \subset N$ are compact sets, and $U \subset$

$h_1(U) + h_2(U)$. Lastly, $h_2(U)$ is normed compact set in N, because N is finite dimensional subspace , and so $h_2(U)$ is a subset of some polytope S. The converse part can be done using the similar argument. □

The above theorem leads us to the following corollary that we will just state without proof.

Corollary 2.3.1. *If $(Y, \|\cdot\|)$ is an extended normed space such that Y_{fin} is reflexive. Then a subset of Y is Y^*-bounded if and only if it is relatively weakly compact.*

Let us consider a bornology formed by the weakly compact bounded sets in the next theorem.

Theorem 2.3.4. *Let $(Y, \|\cdot\|)$ be an extended normed space. Then the bornology of nonempty subsets of weakly compact bounded sets is given as:*

$$\{\mathcal{B} \neq \emptyset \subset X \ \ \textit{such that} \, \exists M \subset Y_{fin} \ \textit{weakly compact and a finitely set} \ T \ \textit{with} \, \mathcal{B} \subset M + T\}.$$

Proof. If $\mathcal{B}$ is weakly compact and bounded, then $\mathcal{B}$ is contained in a finite union of balls and thus, we have $y_1 + Y_{\text{fin}}, y_2 + Y_{\text{fin}}, \ldots, y_n + X_{\text{fin}}$ where, we assume $y_i \in \mathcal{B}$ for $i \leq n$. Since flats are weakly closed, it means that for each $i \leq n$, $\mathcal{B} \cap (y_i + Y_{\text{fin}})$ is weakly compact. Put $K_i = (B \cap (y_i + Y_{\text{fin}})) - y_i$; then K_i is a weakly compact subset of Y_{fin} and we have $\mathcal{B} \subset (\cup_{i=1}^{n} K_i) + \{x_1, x_2, \ldots, x_n\}$ as required.

Conversely suppose $\mathcal{B} \subset M + T$ where M is a weakly compact subset of Y_{fin} and $T = \{x_1, x_2, \ldots, x_n\}$. For all $i \leq n, x_i + M$ is weakly compact and so $M + T = \cup_{i=1}^{n}(x_i + K)$ is weakly compact.

□

3

Universal space for extended quasi-metric spaces

In this chapter, we present our own results. We generalize the construction of the universal space of extended metric spaces in the sense of Beer [5] to extended T_0-quasi-metric settings. Our universal space is a set of (f, A) where f is a real-valued continuous function on a p^s-closed set A.

3.1. Quasi-metric components

This section presents the concept of quasi-metric components, thereafter we use them to construct our universal space in the next section.

Definition 3.1.1. Let (X, p) be an extended T_0-quasi-metric space. Then for any $x, y, \in X$, we define a relation $\mathcal{R}_p$ on X by $x\, \mathcal{R}_p\, y$ provided $p(x, y) < \infty$ and $p(y, x) < \infty$.

The next Lemma will show that the above relation is actually an equivalence relation.

Lemma 3.1.1. *If (X, p) is an extended T_0-quasi-metric space, then the relation $\mathcal{R}_p$ is an equivalence relation on X.*

Proof. Let $p(x, y) < \infty$ and $p(y, x) < \infty$ and let $x \in X$, we get $p(x, x) = 0 < \infty$ and $x\, \mathcal{R}_p\, x$, implying that $\mathcal{R}_p$ is reflexive.

To show that $\mathcal{R}_p$ is symmetric: Let $x, y \in X$ and $x\, \mathcal{R}_p\, y$ if and only if $p(x, y) < \infty$ which means $y\, \mathcal{R}_p\, x$ and $p(y, x) < \infty$ if and only if $y\, \mathcal{R}_p\, x$.

Lastly, the transitivity of this relation: Suppose we have $x\, \mathcal{R}_p\, y$ and $y \mathcal{R}_p\, z$ for $x, y, z, \in X$ then $q(x, y) < \infty$ and $p(y, x) < \infty$.

By triangle inequality, we have $x \, \mathcal{R}_p \, z$, hence $\mathcal{R}_p$ is transitive. Therefore, $\mathcal{R}_p$ is an equivalence relation on X. $\square$

Note that if $p = p^{-1}$, then the relation $\mathcal{R}_p$ is exactly the equivalence relation from Lemma 2.1.1.

Definition 3.1.2. Let (X, p) be an extended T_0-quasi-metric space and $x \in X$. Then the equivalent class of x on X denoted by $qmc_p(x)$ is called *quasi-metric component* of X.

Remark 3.1.1. For any extended quasi-pseudometric space (X, p) we have $mc_d(x) \subseteq qmc_p(x)$, where $mc_d(x)$ is a metric component mentioned in Definition 2.1.1.

We next define a quasi-metric on the set of continuous functions.

Definition 3.1.3. Let $C(X, p)$ be the set of continuous functions on (X, p), then $Q : C(X, p) \times C(X, p) \longrightarrow \mathbb{R}$ defined by

$$Q(f, g) = \sup_{x \in X} u(f(x), g(x))$$

is a T_0-quasi-metric , whenever $f, g \in C(X, p)$ and u is standard T_0-quasi-metric in Example 1.1.3.

Remark 3.1.2. Let $C^b(X, p)$ be a set of continous functions which are p^s-bounded on (X, p). If we define a restriction function by $Q' = Q|_{C^b(X,p)}$ then the space $(C^b(X, p), Q')$ is a T_0-quasi-metric space.

We define an isometric embedding from the T_0-quasi-metric space to the space $(C^b(X, p), Q')$ in the next lemma.

Lemma 3.1.2. *Let (X, p) be T_0-quasi-metric space and Let $a \in X$ be fixed and for any $b \in X$, the map $\pi : (X, p) \longrightarrow (C^b(X, p), Q')$ defined by $\pi(b) = p(b, x) - p(a, x)$ is an isometric embedding.*

Proof. We need to show that if $s, s' \in X$ then $Q'(\pi(s), \pi(s')) = p(s, s')$.

Now let $s, s' \in X$. It follows by Definition 3.1.3 that

$$\begin{aligned} Q'(\pi(s), \pi(s')) &= \sup_{x \in X} u[(p(s, x) - p(a, x)), u(p(s', x) - p(a, x))] \\ &= \sup_{x \in X} u[p(s, x), p(s', x)] \\ &= p(s, s'). \end{aligned}$$

$\square$

3.2. Construction of the universal space

We now construct and study our own universal space for an extended T_0-quasi-metric space in this section. It is important to note that our universal space generalizes the universal space of an extended metric space discussed in Chapter 2.

Let (X,p) be an extended quasi-pseudometric space and $A \subset X$, we denote the set of continuous functions on A by $C(A,p)$ and a set of closed subsets of X with respect to p^s by $\mathscr{C}_0(X,p^s)$.

Definition 3.2.1. Given a set $\Delta_{(X,p)} = \{(f,A) : f \in C(A,p) \text{ and } A \in \mathscr{C}_0(X,p^s)\}$. We define an extended quasi-metric H on $\Delta_{(X,p)}$ by

$$H((f,A),(g,B)) = \begin{cases} \sup_{x\in A} u[f(x),g(x)] \text{ if A=B} \\ \\ \infty \text{ if } A \neq B. \end{cases}$$

The set $(\Delta_{(X,p)}, H)$ is called the universal space of an extended T_0-quasi-metric space (X,p).

Recall that u used above was defined in Example 1.1.3.

Remark 3.2.1. If we replace $\sup_{x\in A} u[f(x),g(x)]$ by $\sup_{x\in A} |(f(x) - g(x)|$ in the above definition, then we get

$$H^s((f,A),(g,B)) = \begin{cases} \sup_{x\in A} |(f(x) - g(x)| \text{ if A=B} \\ \\ \infty \text{ if } A \neq B, \end{cases}$$

which is equivalent to the metric mentioned in Definition 2.1.3.

Note 3.2.1. We point out that our universal space is different from the one done by Agyingi, Haihambo and Künzi in [1]. This is because their universal space is a set of real-valued functions which are bounded, but our universal space is a set of (f,A) where f is a real-valued continuous function on p^s-closed set A.

The next result generalizes Theorem 2.1.2 to quasi-metric point of view.

Theorem 3.2.1. *(Compare Theorem 2.1.2) Let $(X;p)$ be an extended T_0-quasi-metric space. Then the function Φ from (X,p) to $(\Delta_{(X,p)}, H)$ defned by $\Phi(x) = (p(x,\cdot)|_{qmc_p(x)}, qmc_p(x))$ whenever $x \in X$ is an isometry.*

Proof. In order for us to show that Φ is an isometry (preserving function), we need to show that $H(\Phi(x), \Phi(y)) = p(x, y)$ for all cases, whenever $x, y \in X$. Let $x, y \in X$ such that $x \neq y$. Let us consider some cases:

Case 1: If $p(x, y) = \infty$, $p(y, x) = \infty$ and $qmc_p(x) \neq qmc_p(y)$. Then

$$H(\Phi(x), \Phi(y)) = H[(p(x, \cdot)|_{qmc_p(x)}, qmc_p(x)), (p(y, \cdot)|_{qmc_p(y)}, qmc_p(y))] = \infty = p(x, y).$$

Case 2: If $p(x, y) = \infty$, $p(y, x) < \infty$ and $qmc_p(x) \neq qmc_p(y)$. Then

$$H(\Phi(x), \Phi(y)) = H[(p(x, \cdot)|_{qmc_p(x)}, qmc_p(x)), (p(y, \cdot)|_{qmc_p(y)}, qmc_p(y))] = \infty = p(x, y).$$

Case 3: If $p(x, y) < \infty$, $p(y, x) = \infty$ and $qmc_p(x) \neq qmc_p(y)$. Then

$$H(\Phi(x), \Phi(y)) = H[(p(x, \cdot)|_{qmc_p(x)}, qmc_p(x))), (p(y, \cdot)|_{qmc_p(y)}, qmc_p(y))] = \infty = p(x, y).$$

Case 4: If $p(x, y) < \infty$ and $p(y, x) < \infty$, then we have $qmc_p(x) = qmc_p(y)$. We need to show that $H(\Phi(x), \Phi(y)) = p(x, y)$.

For each $x \in [qmc_p(x)] = qmc_p(y)$, we get

$$H(\Phi(x), \Phi(y)) = H[(p(x, \cdot)|_{qmc_p(x)}, qmc_p(x))), (p(y, \cdot)|_{qmc_p(y)}, qmc_p(y))].$$

Now let us fix $a \in qmc_p(x) = qmc_p(y)$ and by triangle inequality the suprimum of $(p(x, a) - p(y, a))$ cannot exceed $p(x, y)$, thus

$$\begin{aligned} H(\Phi(x), \Phi(y)) &= H[(p(x, a)|_{qmc_p(x)}, qmc_p(x))), (p(y, a)|_{qmc_p(y)}, qmc_p(y))] \\ &= \sup_{a \in [qmc_p(x)]} u[(p(x, a), p(y, a))] \\ &= p(x, y). \end{aligned}$$

Thus, we have $H(\Phi(x), \Phi(y)) = p(x, y)$. In all the cases, we have been able to show that $H(\Phi(x), \Phi(y)) = p(x, y)$, thus Φ is an isometry. □

Example 3.2.1. Let (X, p) be an extended T_0-quasi-metric space. If $A \in \mathscr{P}_0(X)$ and we set $g_A(x) = \text{dist}(A, x)$ whenever $x \in X$. Then $g_A : (X, p^{-1}) \longrightarrow (\mathbb{R}, u)$ is a nonexpansive map. Furthermore, $p_H(A, B) = Q_p(g_A, g_B)$ whenever $A, B \in \mathscr{P}_0(X)$. Note that the map $p_H(A, B)$ is the extended Hausdorff quasi-pseudometric on $\mathscr{P}_0(X)$.

The following proposition is an asymmetric version of Theorem 2.2.7 and illustrates how a Hausdorff space can be embedded into the universal space.

Proposition 3.2.1. *(Compare Theorem 2.2.7) Let (X, p) be an extended T_0-quasi-metric space and $\mathcal{C}_0(X, p)$ be the set of all $\tau(p)$-closed subsets of X. Then for each extended T_0-quasi-metric p on X the space $(\mathcal{C}_0(X), p_H)$ can be embedded in $(\Delta_{(X,p)}, H)$.*

Proof. First we choose s in X, which will remain fixed throughout this proof and let A and B be sets in $\mathscr{C}_0(X)$ and let $g : (\mathscr{C}_0(X), p_H) \longrightarrow (\Delta_{(X,p),H})$ be the map defined by $g(A) = (f_a, A)$ where $f_a(y) = p(y, a) - p(y, s)$. We need to show that g is an isometry.

Case 1: Let $A \neq B$, then $H(g(A), g(B)) = \infty = p_H(A, B)$

Case 2: Now, let $y \in A = B$. Then

$$\begin{aligned} H(g(A), g(B)) &= H((f_a, A), (f_b, B)) \\ &= \sup_{y \in A} u[f_a(y), f_b(y)] \\ &= \sup_{y \in A} u[p(y, a) - p(y, s), p(y, b) - p(y, s)] \\ &= \sup_{y \in A} u[p(y, A), p(y, B)] \\ &= p_H(A, B). \end{aligned}$$

In either case, we have shown that $H(g(A), g(B)) = p_H(A, B)$, thus g is an isometry. □

The theorem below provides the bicompleteness of our universal space.

Theorem 3.2.2. *(Compare Theorem 2.1.1) Let (X, p) be a extended T_0-quasi-metric space. Then $(\Delta_{(X,p)}, H)$ is a bicomplete extended T_0-quasi-metric space.*

Proof. The extended metric space $(\Delta_{(X,p)}, H^s)$ is the universal space of Section 2 which is a complete extended metric space by Theorem 2.1.1. Now, since $H \leq H^s$ (see Definion 3.2.1 and Remark 3.2.1), we therefore conclude that the universal space $(\Delta_{(X,p)}, H)$ of an extended T_0-quasi-metric space (X, p) is bicomplete. □

The next result generalises Theorem 2.1.3 in the framework of quasi-metric settings.

Theorem 3.2.3. *(Compare Theorem 2.1.3) Let (X, p_1) and (Y, p_2) be extended T_0-quasi-metric spaces and suppose there exists a continuous surjection φ from (X, P_1) to (Y, p_2). Then there exists an isometry from $(\Delta_{(Y,p_2)}, H_2)$ into $(\Delta_{(X,p_1)}, H_1)$.*

Proof. Let $A \subseteq Y$ and $f \in C(A, p_2)$. Define the map $\Psi : (\Delta_{(Y,p_2)}, H_2) \longrightarrow (\Delta_{(X,p_1)}, H_1)$ by

$$\Psi(f, A) = (f \circ \varphi|_{\varphi^{-1}(A)}, \varphi^{-1}(A)).$$

We prove that $H_1(\Psi(f, A_1), \Psi(g, A_2)) = H_2((f, A_1), (g, A_2))$ for all the cases.

Case 1: Let $A_1 \neq A_2$ be two $\tau(p_2^s)$-closed subsets of Y and $f, g \in C(A, p_2)$, and since φ is a continous surjective map, we have $\varphi^{-1}(A_1)$ and $\varphi^{-1}(A_2)$ as two different $\tau(p_1^s)$ nonempty closed subsets of X as well. Then by Definition 3.2.1, we have

$$\begin{aligned} H_1(\Psi(f, A), \Psi(g, A)) &= H_1[((f \circ \varphi)|_{\varphi^{-1}(A)}, \varphi^{-1}(A)), ((g \circ \varphi)|_{\varphi^{-1}(A)}, \varphi^{-1}(A))] \\ &= \infty \\ &= H_2((f, A_1), (g, A_2)). \end{aligned}$$

Case 2: Now, let $A_1 = A_2 = A$ and $f, g \in C(A, p_2)$, since φ is a continous surjective map, $\varphi^{-1}(A)$ is $\tau(p_1^s)$ closed subset of X, thus by Definition 3.2.1 again we get

$$\begin{aligned} H_1((\Psi(f, A), \Psi(g, A)) &= H_1[((f \circ \varphi)|_{\varphi^{-1}(A)}, \varphi^{-1}(A)), ((g \circ \varphi)|_{\varphi^{-1}(A)}, \varphi^{-1}(A))] \\ &= \sup_{x \in \varphi^{-1}(A)} u[(f \circ \varphi)(x), (g \circ \varphi)(x)] \\ &= \sup_{y \in A} u[f(y), g(y)] \\ &= H_2((f, A), (g, A)). \end{aligned}$$

Since $H_1((\Psi(f, A_1), \Psi(g, A_2))) = H_2((f, A_1), (g, A_2))$ for all the cases. Thus, Ψ is an isometry from $(\Delta_{(Y,p_2)}, H_2)$ into $(\Delta_{(X,p_1)}, H_1)$. □

The following corollary compares the universal spaces of p and p^{-1}.

Corollary 3.2.1. *Let (X, p) be extended T_0-quasi-metric space and suppose there exists a continuous identity surjection map φ from (X, p) to (X, p^{-1}). Then there exists an isometry from $(\Delta_{(X,p)}, H_1)$ into $(\Delta_{(X,p^{-1})}, H_1^{-1})$.*

Proof. It can be easily seen by defining an identity map and letting $p = p_1$ and $p_2 = p$ in the above theorem. □

4

Bornology of extended quasi-metric spaces

In this chapter, we start our own investigations on bornologies in quasi-metric spaces. We present two separate characterizations of a bornology on quasi-metrizable and uniformly quasi-metrizable spaces. Our study has been inspired by the fact that being bounded on p or p^{-1} does not necessary mean bounded on p^s. For example, a nonempty set A can be p-bounded but not p^s-bounded (See Remark 4.1.2).

4.1. Bornology of quasi-metrically bounded subsets

In this section, we recall some definitions and prove some results of bornology of quasi-metrically bounded subsets.

Definition 4.1.1. Let (X, p) be an extended T_0-quasi-metric space. An arbitrary subset A of X is called p-bounded if and only there exists $x \in X$, $r > 0$ and $s > 0$ such that $A \subseteq B_p(x, r) \cap B_{p^{-1}}(x, s)$.

Note that in the above definition, one can replace $A \subseteq B_p(x, r) \cap B_{p^{-1}}(x, s)$ with $A \subseteq C_p(x, r) \cap C_{p^{-1}}(x, s)$.

The following definition is equivalent to Definition 4.1.1 above.

Definition 4.1.2. Let (X, p) be T_0-quasi-metric space, then a nonempty set $A \subseteq X$ is said to be *p-bounded* if for all $x, y \in A$ there exist $k > 0$ such that $p(x, y) < k$.

Remark 4.1.1. It is well known from Definition 1.2.1 that a bornology on a set X is a family $\mathcal{B}$ of subsets of X that forms a cover of X and it is stable under inclusions and finite unions. As in metric point of view, we denote $\mathcal{B}_p(X)$ the bornology of all p-bounded subsets of X and call it *quasi-metric bornology* .

Definition 4.1.3. ([30, Definition 2.1]) Let $(X,\mathcal{U})$ be a quasi-uniform space and $A \subseteq X$. The set A is said to be bounded if for any $U \in \mathcal{U}$, there exists $n \in \mathbb{N}$ and a finite subset F of X such that $A \subseteq U^n[F]$, where

$$U[F] = \{y \in X : \text{ there exists } f \in F \text{ such that } (f, y) \in U\}.$$

It has been observed in [30] that if a set is bounded in the sense of the above definition, then it is also bounded in the metric sense.

Remark 4.1.2. [30, P.370] It is easy to see that if a set is p^s-bounded, then it is p-bounded but the converse is not true. For example, let us equip $[0, 1]$ with the T_0-quasi-metric in Example1.1.2. If $\mathcal{U}$ is the quasi-uniformity generated by p on $[0, 1]$ and $\mathcal{U}^{-1}$ is the conjugate quasi-uniformity of $\mathcal{U}$ on $[0, 1]$. Then $([0, 1], \mathcal{U})$ and $([0, 1], \mathcal{U}^{-1})$ are bounded but $([0, 1], \mathcal{U}^s)$ is the discrete uniform space and it is not bounded.

From the above remark, we have the following observation.

Lemma 4.1.1. *Suppose (X, p) is an extended T_0-quasi-metric space. Then, the following statements hold:*

(i) $\mathcal{B}_{p^s}(X) \subseteq \mathcal{B}_p(X)$

(ii) $\mathcal{B}_{p^s}(X) \subseteq \mathcal{B}_{p^{-1}}(X)$

Proof. (i) Let $A \in \mathcal{B}_{p^s}(X)$, then A is p^s-bounded. This simply means that A is also p-bounded. Thus, A is a member of $\mathcal{B}_p(X)$.

(ii) By similar arguments, we have that $\mathcal{B}_{p^s}(X) \subseteq \mathcal{B}_{p^{-1}}(X)$.

□

We now compare the quasi-metric bornologies of p and p^{-1} in the next lemma.

Lemma 4.1.2. *Let (X, p) be a T_0-quasi-metric space. Then $\mathcal{B}_{p^{-1}}(X)$ is bornologically equivalent to $\mathcal{B}_p(X)$.*

Proof. Since any subset A of X is p-bounded if and only if it is p^{-1} bounded. □

Remark 4.1.3. The metric bornology $\mathcal{B}_{p^s}(X)$ is equivalent to a metric bornology reviewed in Section 2.2. However, we are going to focus on $\mathcal{B}_p(X)$ since $\mathcal{B}_{p^{-1}}(X)$ is bornologically equivalent to $\mathcal{B}_p(X)$ by Lemma 4.1.2. Therefore, every result of $\mathcal{B}_p(X)$ is applied to $\mathcal{B}_{p^{-1}}(X)$.

Definition 4.1.4. Let (X, p) be a T_0-quasi-metric space and $\mathcal{B}_p(X)$ be its quasi-metric bornology. We define $\mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$ and int $_{\tau(p)}\big(\mathcal{B}_p(X)\big)$ by

$$\mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big) = \Big\{\mathrm{cl}_{\tau(p)}B : B \in \mathcal{B}_p(X)\Big\} \text{ and } \mathrm{int}_{\tau(p)}\big(\mathcal{B}_p(X)\big) = \Big\{\mathrm{int}_{\tau(p)}B : B \in \mathcal{B}_p(X)\Big\}$$

respectively.

Definition 4.1.5. (Compare [19, Definition 3.1]) Let (X, p) be a T_0-quasi-metric space and $\mathcal{B}_p(X)$ be its quasi-metric bornology. Then we say

(i) $\mathcal{B}_p(X)$ is $\tau(p)$-closed if $\mathcal{B}_p(X) = \mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$;

(ii) $\mathcal{B}_p(X)$ is $\tau(p)$-open if $\mathcal{B}_p(X) = \mathrm{int}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$.

The above definition leads us to the following propositions.

Proposition 4.1.1. *Let (X, p) be a quasi-metric space. Then the following conditions are equivalent:*

(ii) $\mathcal{B}_p(X)$ is $\tau(p)$-closed;

(ii) $\mathcal{B}_p(X)$ is generated by $cl_{\tau(p)}\big(\mathcal{B}_p(X)\big)$;

(iii) If $B \in \mathcal{B}_p(X)$, then $cl_{\tau(p)}B \in cl_{\tau(p)}\big(\mathcal{B}_p(X)\big)$.

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{B}_p(X)$ is $\tau(p)$-closed, then by Definition 4.1.5 $\mathcal{B}_p(X) = \mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$. Now if B is $\tau(p)$-closed and $\tau(p)$-bounded, then $B = \mathrm{cl}_{\tau(p)}B \in \mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$. Morever,

$$B = \mathrm{cl}_{\tau(p)}B \in \Big\{\mathrm{cl}_{\tau(p)}B : B \in \mathcal{B}_p(X)\Big\},$$

which follows that $\mathcal{B}_p(X)$ is generated by $\mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big)$.

$(ii) \Rightarrow (iii)$: Suppose (ii) holds. Let $B \in \mathcal{B}_p(X)$, then there exists a $\tau(p)$-closed and $\tau(p)$-bounded set A such that $B \subseteq A$. Hence $\mathrm{cl}_{\tau(p)}B \subseteq A \in \mathcal{B}_p(X)$.

$(iii) \Rightarrow (i)$: We only need to show that $\mathrm{cl}_{\tau(p)}B \in \mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big) \subseteq \mathcal{B}_p(X)$ since the other inclusion is obvious. But since every $\tau(p)$-closure of every $\tau(p)$-bounded is $\tau(p)$-bounded, thus we have $\mathrm{cl}_{\tau(p)}B \in \mathrm{cl}_{\tau(p)}\big(\mathcal{B}_p(X)\big) \subseteq \mathcal{B}_p(X)$. $\square$

Definition 4.1.6. [31, Definition 1.7] A bornological bi-universe $(X, \tau_0, \tau_1, \mathcal{B})$ is called quasi-metrizable if there exists a quasi-metric p on X such that $\tau_0 = \tau(p)$, $\tau_1 = \tau(p^{-1})$ and $\mathcal{B} = \mathcal{B}_p(X)$.

Recall that if (X, d) is an extended metric space and $\mathcal{B}$ is a bornology in X, then a bornological universe $(X, \tau, \mathcal{B})$ is *metrizable* if there exists a metric d on X such that $\mathcal{B} = \mathcal{B}_d(X)$ where $\mathcal{B}_d(X)$ is a metric bornology. Now in quasi-metric settings, we say that $(X, \tau, \mathcal{B})$ is *quasi-metrizable* if there exists a quasi-metric p on X such that $\mathcal{B} = \mathcal{B}_p(X)$.

When we consider $\mathbb{R}$. The topology $u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$ in $\mathbb{R}$ is called the *upper topology* and the topology $l = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ in $\mathbb{R}$ is called the *lower topology.*

The following is an example of bornologies in $\mathbb{R}$.

Example 4.1.1. [31] Consider the real line $\mathbb{R}$. Then

$$UB(\mathbb{R}) = \{A \subseteq \mathbb{R} : \text{ There exists } r \in \mathbb{R} \text{ such that } A \subseteq (-\infty, r)\}$$

and

$$LB(\mathbb{R}) = \{A \subseteq \mathbb{R} : \text{ There exists } r \in \mathbb{R} \text{ such that } A \subseteq (r, +\infty)\}$$

are bornologies in $\mathbb{R}$.

Definition 4.1.7. [31, Definition 4.1] Let (X, τ_0^X, τ_1^X) and (Y, τ_0^Y, τ_1^Y) be two bitopological spaces. A function $f : X \to Y$ is called *bicontinuous* with respect to $(\tau_0^X, \tau_1^X, \tau_0^Y, \tau_1^Y)$ if

$$\{f^{-1}(A) : A \in \tau_i^Y\} \subseteq \tau_i^X$$

whenever $i \in \{0, 1\}$.

Definition 4.1.8. [31, Definition 4.2] Let (X, τ_0, τ_1) be a bitopological space. Then a (τ_0, τ_1)-characteristic function for a bornological universe $(\mathcal{B}, X)$, is a bicontinuous function $g : (X, \tau_0, \tau_1) \longrightarrow ([0, +\infty), u, l)$ such that

$$\mathcal{B} = \big\{A \subseteq X : \max\{g(x) : x \in A\} < +\infty\big\}.$$

We give the following example to illustrate the above definitions.

Example 4.1.2. ([31, Fact 4.3]) Let (Y, p) be a T_0-quasi-metric space and let $c \in Y$. Then the function $g : (Y, \tau(p), \tau(p^{-1})) \longrightarrow ([0, +\infty), u, l)$ defined by $g(x) = p(c, x)$ whenever $x \in Y$ is bicontinuous.

Let (X, p) be a extended T_0-quasi-metric space. Then p is said to induce a bornological bi-universe $(X, \tau_0, \tau_1, \mathcal{B})$ if $\tau_0 = \tau(p)$, $\tau_1 = \tau(p^{-1})$ and $\mathcal{B} = \mathcal{B}_p(X)$.

Definition 4.1.9. Let (Y, τ_0, τ_1) be a bitopological space. A bornology $\mathcal{B}$ on Y will be called (τ_0, τ_1)-proper if for each $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\mathrm{cl}_{\tau_0}(A) \subseteq \mathrm{int}_{\tau_1}(B)$.

Proposition 4.1.2. *Let (Y, τ_0, τ_1) be a bitopological space and $((Y, \tau_0, \tau_1), \mathcal{B})$ be a bornological bi-universe. If $((Y, \tau_0, \tau_1), \mathcal{B})$ is quasi-metrizable, then there exists a (τ_0, τ_1)-characteristic function for the bornology $\mathcal{B}$.*

Proof. Let $a \in Y$ and any extended T_0-quasi-metric p such that p induces the bornological biuniverse $((Y, \tau_0, \tau_1), \mathcal{B})$. Then, by Example 4.1.2, a (τ_0, τ_1)-characteristic function for $\mathcal{B}$ is the function $g : Y \longrightarrow \mathbb{R}$ where $g(x) = p(a, x)$ for $x \in Y$. □

Proposition 4.1.3. *Let (Y, τ_0, τ_1) be a bitopological space and $((Y, \tau_0, \tau_1), \mathcal{B})$ be a bornological bi-universe. If the bornology $\mathcal{B}$ has a (τ_0, τ_1)-characteristic function, then $\mathcal{B}$ is (τ_0, τ_1)-proper and second countable.*

Proof. Let g be any (τ_0, τ_1)-characteristic function for the bornology $\mathcal{B}$. For $s \in \omega$, let $B_s = g^{-1}((-\infty, s])$. Then the collection $\{B_s : s \in \omega\}$ is a countable base for $\mathcal{B}$ such that $\mathrm{cl}_{(\tau_0)}(B_s) \subseteq \mathrm{int}_{(\tau_1)}(B_{s-1})$. □

The following result is an asymmetric version of the well-known Hu's Theorem.

Theorem 4.1.1. *[31, Theorem 4.7] Let (X, τ_0, τ_1) be a bitopological space. If (X, τ_0, τ_1) is quasi-metrizable and $\mathcal{B}$ is a bornology on X. Then the following properties are equivalent:*

(i) the bornological bi-universe $((X, \tau_0, \tau_1), \mathcal{B})$ is quasi-metrizable;

(ii) the bornology $\mathcal{B}$ has a (τ_0, τ_1)-characteristic function;

(ii) the bornology $\mathcal{B}$ is (τ_0, τ_1)-proper and it has countable base.

Proof. Let (X, p) be a quasi-metric space and let us consider ρ to be any quasi-metric on X such that $\tau_0 = \tau(p)$, $\tau_1 = \tau(p^{-1})$. Since X is quasi-metrizable, there exists a bounded quasi-metric $p : X \times X \longrightarrow \mathbb{R}$ which defines the same topology on X. Let $p(x, y) = \min\{\rho(x, y), 1\}$ for $x, y \in X$. if $X \in \mathcal{B}$ then X is bounded and (i) and (ii) hold directly.

If $X \notin \mathcal{B}$, by Proposition 4.1.2 (i) implies (ii). if (ii)holds and suppose that g is a (τ_0, τ_1) -characteristic function for $\mathcal{B}$. Now for $x, y \in X$, lets define the function $\sigma : X \times X \longrightarrow \mathbb{R}$ by taking

$$\sigma(x, y) = \rho(x, y) + \max\{f(y) - f(x), 0\}.$$

The quasi-metric σ induces the bornological biuniverse $((X,\tau_0,\tau_1),\mathscr{B})$. Hence (ii) implies (i). Now, assuming (iii) and since $X \notin \mathcal{B}$, it follows by [31, Proposition 3.5] that there is a base $\mathcal{A} = \{A_n : n \in \Omega\}$ for $\mathcal{B}$ such that $\mathrm{cl}_{\tau_1} A_n \subset \mathrm{int}_{\tau_0} A_{n+1}$ for all $n \in \Omega$. Again let us assume $A_o = \emptyset$. For $n \in \Omega - \{0\}$ and $x \in X$, let $f_n(x) = p(\mathrm{cl}_{\tau_1} A_n, x)$ and $g_n(x) = p(x, X - \mathrm{int}_{\tau_1} A_{n+1})$. Then $p_n : (X,\tau_0,\tau_1) \longrightarrow ([0,+\infty), u, l)$ and $g_n : (X,\tau_0,\tau_1) \longrightarrow ([0,+\infty), l, u)$ are bicontinuous. For all $x \in X$, we get $p_n(x) + g_n(x) \neq 0$, so we can put

$$h_n(x) = \frac{p_n(x)}{p_n(x) + g_n(x)}.$$

Moreover, we define $h_o(x) = 1$ for each $x \in X$. It is easy to check that the function $h_n : (X,\tau_0,\tau_1) \longrightarrow ([0,1], u, l)$ is bicontinuous for each $n \in \Omega$.

Let $\psi(x) = h_n(x) + n$ when $x \in \mathrm{int}_{\tau_1} A_{n+1} - \mathrm{int}_{\tau_1} A_n$. We are going to prove that the function ψ is bicontinuous with respect to (τ_0,τ_1,u,l).

Let $x \in \mathrm{int}_{\tau_0} A_{n+1} - \mathrm{int}\tau_0 A_n$ and $y \in \mathrm{int}_{\tau_0} A_{m+1} - \mathrm{int}_{\tau_0} A_m$. Now let real numbers $r > 0$, $s > 0$ such that $r < \psi(x) < s$ and assume $n \neq 0$. Then there exists $U_s \in \tau_1$ such that

$x \in U_s \subseteq \mathrm{int}_{\tau_0} A_{n+1}$ and if $y \in U_s$, then $h_n(y) + n < s$. There exists $V_r \in \tau_1$ such that $x \in V_r \subseteq X - \mathrm{cl}_{\tau_1} A_{n-1}$ and if $y \in V_r$, then $h_n(y) + n > r$. Of course, if $m = n$, then $\psi(y) < s$ when $y \in U_s$, while $\psi(y) > r$ when $y \in V_r$.

Let us assume that $m \neq n$. Suppose that $y \in U_s$. Then $m < n$, so $\psi(y) \leq 1 + m \leq n \leq \psi(x) < s$. Suppose that $y \in V_r$. If $m > n$, we have $\psi(y) \geq m \geq 1 + n \geq \psi(x) > r$.

Let $m < n$. Since $y \notin \mathrm{int}_{\tau_0} A_{n-1}$, we have $m + 1 \geq n$. As $m + 1 \leq n$, we have $m + 1 = n$. If $x \notin \mathrm{cl}_{\tau_1} A_n$ we could take $V_r^* = V_r \cap (X - cl_{\tau_2} A_n) \in \tau_1$ and observe that if $y \in V_r^*$, then $m \geq n$ and $\psi(y) > r$. If $m < n$ and $x \in \mathrm{cl}_{\tau_1} A_n$. Then $\psi(x) = n$. We take a positive real number ϵ such that $n - \epsilon > r$. Since $h_{n-1}(x) = 1$, there exists $W_\epsilon \in \tau_1$ such that $x \in W_\epsilon$ and $h_{n-1}(t) > 1 - \epsilon$ for each $t \in W_\epsilon$. If $y \in W_\epsilon \cap V_r$ and $m + 1 = n$, then $\psi(y) = h_{n-1}(y) + n - 1 > 1 - \epsilon + n - 1 = n - \epsilon > r$. The case when $n = 0$ is also obvious now. This completes the proof that ψ is bicontinuous with respect to (τ_0,τ_1,u,l).

It is easy to check that $\mathcal{B} = \{A \subseteq X : \sup \psi(A) < +\infty\}$, so ψ is a (τ_0,τ_1) -characteristic function for $\mathcal{B}$. Hence (ii) follows from (iii). $\square$

The above theorem leads us to the following Corollary.

Corollary 4.1.1. *Let (X,p) be an extended T_0-quasi-metric space and $((X,\tau),\mathcal{B})$ be a bornological universe. Then the bornological biuniverse $((X,\tau_0,\tau_1),\mathcal{B})$ is quasi-metrizable if and only if the bornological universe $((X,\tau),\mathcal{B})$ is metrizable.*

Proof. It is enough to prove that if there exists a quasi-metric which induces $((X, \tau_0, \tau_1), \mathcal{B})$, then $((X, \tau), \mathcal{B})$ is metrizable. Note that $\tau = \{\tau_0, \tau_1\}$.

Let p be a quasi-metric in X such that $\tau_0 = \tau(p)$, $\tau_1 = \tau(p^{-1})$ and $\mathcal{B} = \mathcal{B}_p(X)$. Define $\rho = \max\{p, p^{-1}\}$. Then ρ is a metric in X such that $\tau(\rho) = \tau$. Moreover, by Theorem 4.1.1, the bornology $\mathcal{B}$ is second-countable and (τ_0, τ_1)-proper and so is τ-proper; hence, the bornological universe $((X, \tau), \mathcal{B})$ is metrizable by Theorem 4.1.1. □

4.2. Quasi-metrization of the bornological universe

Our interest in this section is to know when there exists an extended quasi-metric on the bornological universe $(X, \mathcal{B})$ such that $\mathcal{B}$ is quasi-metrizable and uniformly quasi-metrizable.

We first recall the following definitions.

Definition 4.2.1. Suppose X is a nonempty set and if $\mathcal{A} \subseteq \mathscr{P}(X)$, then $\downarrow(\mathcal{A})$ and $\sum(\mathcal{A})$ are defined respectively as:

$$\downarrow(\mathcal{A}) = \{B : B \subseteq A, \text{for some } A \in \mathcal{A}\} \text{ and } \sum(\mathcal{A}) = \left\{\bigcup_{i=1}^{n} A_i : n \in \mathbb{N}, i \leq n, A_i \in \mathcal{A}\right\}.$$

If $\mathcal{B}$ is a bornology on X. Then we say that a family $\mathcal{B}_0$ of subsets of X is a *base* of $\mathcal{B}$ if $\downarrow(\mathcal{B}_0) = \mathcal{B}$.

Definition 4.2.2. Let (X, p) be an extended T_0-quasi-metric space and D be a subset of X. We say that D is $\tau(p)$- *relatively compact* if D is $\tau(p^s)$-relatively compact.

Note that the collection of $\tau(p)$-relatively compact subsets on an extended metric space (X, p) is a natural bornology.

The following lemma is an asymmetric version of Lemma 2.2.1.

Lemma 4.2.1. *(Compare [5, Lemma 2.1]) Let (X, p) be T_0-quasi-metric space:*

(i) the family of all finite unions of double balls forms a base for the quasi-metric bornology $\mathcal{B}_p(X)$;

(ii) the quasi-metric bornology $\mathcal{B}_p(X)$ contains the bornology of $\tau(p)$-relatively compact subsets of X;

(iii) if (x_n) is p-convergent and p^{-1}-convergent sequence in X, then the set $\{x_n : n \in \mathbb{Z}^+\}$ is contained in a quasi-metric bornology $\mathcal{B}_p(X)$.

Proof. (i) Let $\mathcal{F}_B$ be a family of all finite unions of double open balls that are defined by

$$\mathcal{F}_B := \left\{ \bigcup_{i=1}^{n} B_p(x_i, s_i) \cap B_{p^{-1}}(x_i, \delta_i) : x_i \in X \text{ and } s_i, \delta_i \in (0, \infty) \text{ whenever } n \in \mathbb{N} \right\}.$$

We need to prove that $\downarrow (\mathcal{F}_B) = \mathcal{B}_p(X)$.

Let $A \in \mathcal{F}_B$ then $A = \bigcup_{i=1}^{n} B_p(x_i, s_i) \cap B_{p^{-1}}(x_i, \delta_i)$. Therefore, $B \in \mathcal{B}_p(X)$ as B is p-bounded. Thus $\downarrow (\mathcal{F}_B) \subseteq \mathcal{B}_p(X)$.

Conversely, if $B \in \mathcal{B}_p(X)$ then there exists $x \in X$ and $r, s > 0$ such that $B \subseteq B_p(x, r) \cap B_{p^{-1}}(x, s)$. Since $B_p(x, r) \cap B_{p^{-1}}(x, s) \in \mathcal{F}_B$, it follows that $B \in \downarrow (\mathcal{F}_B)$.

(ii) Let A be a $\tau(p)$-relatively compact subset of X. Then by Definition 4.2.2 A is p^s-bounded. Hence $A \in \mathcal{B}_p(X)$ by Remark 4.1.2.

(iii) Suppose that the sequence (x_n) is p-convergent to $x \in X$. Then for $\epsilon = 1$ there exists $N' \in \mathbb{N}$ and $n \geq N'$ such that if $r = \max\{1, p(x_0, x), p(x_1, x), \ldots, p(x_{N'}, x)\}$, then $p(x_n, x) < r$ whenever $n \in \mathbb{Z}^+$.

By similar arguments. Suppose (x_n) is p^{-1}-convergent to $x \in X$, then $p(x, x_n) < s$ whenever $n \in \mathbb{Z}^+$. Therefore, $(x_n) \in B_p(x, r) \cap B_{p^{-1}}(x, s)$ whenever $n \in \mathbb{Z}^+$ and $r, s > 0$.

□

The following definition extends the Definition 2.2.2 to quasi-metric spaces.

Definition 4.2.3. (Compare Definition 2.2.2) Let (X, p) be an extended T_0-quasi-metric space. If X can be partitioned into disjoint unions of sets $\{X_j : j \in I\}$ and each X_j is an extended T_0-quasi-metric space, then choosing an extended quasi-metric p_j for X_j, the function p on X defined by

$$p(u, v) = \begin{cases} p_j(u, v) & \text{if } \exists i \text{ with } \{u, v\} \subseteq X_j \\ \\ \infty & \text{otherwise.} \end{cases}$$

is an *extended T_0-quasi-metric* and has $\{X_j : j \in I\}$ as quasi-metric components.

We provide the asymmetric version of Theorem 2.2.3 in the next results.

Theorem 4.2.1. *(Compare Theorem 2.2.3) Let (X, p) be an extended T_0-quasi-metric space and let $(X, \mathcal{B})$ be a bornological universe. Then $\mathcal{B} = \mathcal{B}_p(X)$ if and only if there exists $\mathcal{A} \subseteq \mathcal{B}$ for which $\downarrow (\Sigma\mathcal{A})) = \mathcal{B}$ and a partition $\{\mathcal{A}_j : j \in I\}$ of $\mathcal{A}$ with the following four properties:*

(i) the partition $\mathcal{A}_j$ contains a nonempty subset of X whenever $j \in I$;

(ii) for all $A_1 \in \mathcal{A}_j$, there exists $A_2 \in \mathcal{A}_j$ with $cl_{\tau(p^{-1})}(A_1) \subseteq int_{\tau(p)}(A_2)$ whenever $j \in I$;

(iii) whenever $A_j \in \mathcal{A}_j$ and $A_k \in \mathcal{A}_k$ for $j \neq k$, then $A_j \cap A_k = \emptyset$;

(iv) each partition $\mathcal{A}_j$ has a subfamily which is countable and cofinal in $\mathcal{A}_j$ with respect to set inclusion.

Proof. Let p be a T_0-quasi-metric on X and $\{X_j : j \in I\}$ be quasi-metric components of X. Consider $\mathcal{A}_j = \{B_p(x,\delta) \cap B_{p^{-1}}(x,r) : x \in X_j, r, \delta > 0\}$ and let A be the collection of double open balls in X. Since any subset of X is p-bounded if and only if it is a subset of a finite union of double open balls, it follows that

$$\mathcal{B}_p(X) =\downarrow (\Sigma(\cup_{j\in I}\mathcal{A}_j)) =\downarrow (\Sigma(\mathcal{A}))\,.$$

Whenever $j \in I$, if $A \in \mathcal{A}_j$ then A is double open ball. Therefore $A \neq \emptyset$. With a choice of $\mathcal{A}_j$ we have (ii), (iii) and (iv).

Conversely, suppose the partition $\{\mathcal{A}_j : j \in I\}$ satisfies conditions (i), (ii), (iii) and (iv). Then for each $j \in I$, let $X_j = \cup\mathcal{A}_j$. Since $\{\mathcal{A}_j : j \in I\}$ satisfies (i) we have that $X_j \neq \emptyset$. Furthermore, from (ii) and (iii) we know that X_j is $\tau(p)$-open and $\{X_j : j \in I\}$ is a pairwise disjoint family. It follows that each is $\tau(p)$-closed and $\tau(p)$-open.

For each $j \in I$, we set $\mathcal{B}_j =\downarrow (\Sigma(\mathcal{A}_j))$. Since $\mathcal{A}_j$ is a cover of X_j, we have that $\mathcal{B}_j$ is a bornology on X_j.

Since the $\tau(p^{-1})$-closure of a finite union is the union of $\tau(p^{-1})$-closure, property (ii) holds by replacing $\mathcal{A}_j$ with $\mathcal{B}_j$. Similarly property (3) holds if we replace $\mathcal{A}_j$ by $\mathcal{B}_j$ and $\mathcal{A}_k$ by $\mathcal{B}_k$.

Finally, if $\mathcal{B}_j^*$ is countable and cofinal in $\mathcal{A}_i$, then $\Sigma(\mathcal{B}_j^*)$ is countable and cofinal in $\mathcal{B}$, that is, (iv) holds with replacing $\mathcal{A}_j$ by $\mathcal{B}_j$. Then by Theorem 4.1.1, we can find an extended T_0-quasi-metric p_j on X_j such that $\mathcal{B}_j = \mathcal{B}_{p_j}(X)$.

As $\downarrow (\Sigma(\mathcal{A}))$ is considered to be a cover of X, we can conclude that $\mathcal{A}$ and $\{X_j : j \in I\}$ cover X. By Definition 4.2.3 we have

$$\mathcal{B}_p(X) = \bigcup_{j\in I}(B_{p_j}(X_j)) = \sum(\cup_{j\in I}\mathcal{B}_j) =\downarrow \left(\sum(\cup_{j\in I}\mathcal{A}_j)\right) =\downarrow \left(\sum(\mathcal{A})\right) = \mathcal{B}.$$

□

The next result is a generalization of Theorem 2.2.4.

Theorem 4.2.2. *(Compare [5, Theorem 4.2]) Let (X, p) be an extended T_0-quasi-metric space and $(X, \mathcal{B})$ be a bornological universe. Then the bornology $\mathcal{B} = \mathcal{B}_p(X)$ if and only if there exists a collection $\{\mathcal{B}_j : j \in I\}$ of families of subsets of X with the following five properties:*

(i) for all $j \in I$, the collection $\mathcal{B}_j$ has at least two subsets of X as elements including $\emptyset$;

(ii) for every $B_1 \in \mathcal{B}_i$, $\exists B_2 \in \mathcal{B}_j$ such that $cl_{\tau(p^{-1})}(B_1) \subseteq int_{\tau(p)}(B_2)$ for each $j \in I$;

(iii) whenever $B_j \in \mathcal{B}_j$ and $B_k \in \mathcal{B}_k$ for $j \neq k$, then $B_j \cap B_k = \emptyset$;

(iv) each bornology $\mathcal{B}_j$ has a subfamily which is countable and cofinal in $\mathcal{B}_j$ with respect to set inclusion;

(v) for every A in $\mathscr{P}(X)$, the set A is in $\mathcal{B}$ if and only if $\forall j \in I$, $A \cap (\cup \mathcal{B}_j) \subset \mathcal{B}_j$ and for all but finitely many j, $A \cap (\cup \mathcal{B}_j) = \emptyset$.

Proof. Let $\mathcal{B} = \mathcal{B}_p(X)$ and $\{X_j : j \in I\}$ be the quasi-metric components of X and let p_j be the trace of p on X_j^2. We set $\mathcal{B}_j = \mathcal{B}_{p_j}(X_j)$, then property (i) holds because each X_j as a quasi-metric component is not empty and each contains some element of X and $\mathcal{B}_{p_j}(X_j)$ contains minimally the finite subsets of X_j as it is a space of p_j-bounded sets in X_j including the empty set.

For (ii),(iii) and (iv) similar arguments as in the above theorem give us the desired result.

Suppose $A \in \mathscr{P}(X)$ is such that $A \in \mathcal{B}$. As $\{\mathcal{B}_j : j \in I\}$ partitions $\mathcal{B}$, $A \in \mathcal{B}_p(X)$ must be a p-bounded set of X in some $\mathcal{B}_i$. Since each $\mathcal{B}_j$ is a bornology on each quasi-metric component $X_i \subset X$. $A \cap \mathcal{B}_j$ is an intersection of $\mathcal{B}_j$ and those p-balls formed by elements of X_j in A and so, $A \cap (\cup \mathcal{B}_j) = \cup (A \cap \mathcal{B}_j) \subset \mathcal{B}_j$. Since A is arbitrary, we have by conditions (i) and (iii) for all but finitely many indices $j \in I$ that, $A \cap \mathcal{B}_j = \emptyset$ and so, $A \cap (\cup \mathcal{B}_j) = \cup_j (A \cap \mathcal{B}_j) = \emptyset$.

Conversely, suppose $\{\mathcal{B}_j : j \in I\}$ satisfies the properties above. For each $j \in I$ and $\delta > 0$, let each $\mathcal{B}_j = \{B_{p_j}(x, \delta) : x \in X_j\}$ and put $X_j := \cup \mathcal{B}_j$ which by condition (i) is a nonempty subset of X. By condition (ii), each family $\mathcal{B}_j$ has a $\tau(p)$-closed base and $\tau(p)$-open base, implying that $\{X_j : j \in I\}$ is a family of $\tau(p)$-clopen subsets in X. By (iii), $\{X_j : j \in I\}$ partitions X. We next need to show that each family $\mathcal{B}_i$ is a bornology on X_j.

Fix $j \in I$ and set $\mathcal{F}_j := \{B \cap X_i : B \in \mathcal{B}\}$. Since each $X_j = \cup \mathcal{B}_j$, it is evident that $\mathcal{F}_j$ is a bornology on X_j which, by condition (v), is contained in $\mathcal{B}_i$. Indeed, if $A \subseteq B$ then $A \in \mathcal{B}$ and clearly $A \subseteq A \cap X_j \subseteq B \cap X_j \in \mathcal{F}_j$. So. $\mathcal{F}_j$ is a base-bornology on X_j. Conversely, we need to show that $\mathcal{B}_j \subseteq \mathcal{F}_j$. Let $B_j \in \mathcal{B}_j$ be arbitrary. Then, it is enough to show that

$B_j \in \mathcal{F}_j$ which suffices to show that $B_j \in \mathcal{B}$. Clearly, since $X_j = \cup\mathcal{B}_j$, we have $B_j \cap X_j \in \mathcal{B}_j$ by condition (v). Taking $j \neq k$, condition (iii) gives $B_j \cap X_k = \emptyset$ and so, by condition (i) we have $B_j \cap X_k = \emptyset \in \mathcal{B}_j$. Now, in view of condition (v), $B_j \in \mathcal{B}$ so that $B_j = B_j \cap X_j \in \mathcal{F}_j$. Hence, the family $\mathcal{B}_i$ is established as a bornology.

Furthermore, using propeties (ii) and (iv) and Theorem 4.1.1, for each $j \in I$, there exists a quasi-metric p_j on X_j such that $\mathcal{B}_j = \mathcal{B}_{p_j}(X_j)$ that is to say $\mathcal{B}_j$ is a quasi-metric bornology on X_j induced by some metric p_j. With the quasi-metric of Definition 4.2.3, we have by condition (iv) that

$$\mathcal{B} = \sum(\cup_{j\in I}\mathcal{B}_j) = \bigcup_{j\in I}(B_{p_j}(X_j)) = \mathcal{B}_p(X)$$

$\square$

Before we prove the following results on the uniformly quasi-metrizability of a bornology, it is important to state the following definitions.

Definition 4.2.4. Let (X, p_1) and (X, p_2) be extended T_0-quasi-metric spaces. The quasi-metrics p_1, p_2 are said to be *uniformly equivalent* if there exists $\delta_1, \delta_2 > 0$ such that $\forall x, y \in X$, $p_1(x, y) < \delta_1 \Rightarrow p_2(x, y) < \delta_2$.

Definition 4.2.5. Let (X, p) be an extended T_0-quasi-metric space. An extended T_0-quasi-metric in the form $p_1 : X^2 \longrightarrow [0, \infty]$ defined by $p_1(x, y) = \min\{1, p(x, y)\}$ is called a *bone-fide extended quasi-metric* of p.

Let $(X, \mathcal{B})$ be a bornological universe on a quasi-metric space X. The bornology $\mathcal{B}$ is said to be *uniformly quasi-metrizable* with respect to p if there exists a quasi-metric $p_1 = \min\{1, p\}$ in X such that p and p_1 are uniformly equivalent and $\mathcal{B} = \mathcal{B}_{p_1}(X)$, where $\mathcal{B}_{p_1}(X)$ is the quasi-metric bornology of p_1- bounded sets.

Theorem 4.2.3. *[31, Theorem 6.5] Let (X, p) be an extended T_0-quasi-metric space and $(X, \mathcal{B})$ be its bornological universe. Then the following conditions are all equivalent:*

(i) there exists an extended quasi-metric $p_1 = \min\{1, p\}$ such that $\mathcal{B} = \mathcal{B}_{p_1}(X)$;

(ii) the bornology $\mathcal{B}$ has a base $\{B_m : m \in \omega\}$ such that, for some $\epsilon \in (0; +\infty)$ and for each $m \in \omega$ the inclusion $[B_m]_p^\epsilon \subseteq B_{m+1}$ holds;

(iii) there exists a uniformly continuous $(\tau(p), \tau(p^{-1}))$-characteristic function for $\mathcal{B}$.

Proof. (See [31, Theorem 6.5]). □

Theorem 4.2.4. *(Compare Theorem 2.2.5) If (X, p) is an extended T_0-quasi-metric space and $\mathcal{B}$ is a bornology on X. Then the following properties are equivalent:*

(i) the set of quasi-metric components induced by p is countable.

(ii) there exists an extended T_0-quasi-metric p_1 uniformly equivalent to p such that $\mathcal{B}_{p_1}(X) = \mathcal{B}_p(X)$.

Proof. *(i)* $\rightarrow$ *(ii)*: Let $\sum\{qmc_p(x_j) : j \in I\}$ be the set of distinct quasi-metric components induced by a quasi-metric p where I is a countable set. Since the set I is countable, it means that the set $\sum\{\mathcal{B}_p(x_j; m) : j \in I, m \in \mathbb{Z}^+\}$ is a countable base for $\mathcal{B}_p(X)$. Since for the finite union of double balls, the $\tau(p)$-closure of each p-bounded set is contained in the $\tau(p)$-interior of another. By Theorem 4.1.1, there exists an extended quasi-metric p_1 such that $\mathcal{B}_{p_1}(X) = \mathcal{B}_p(X)$.

(ii) $\rightarrow$ *(i)*: Let us denote distinct metric quasi-components by $\{qmc_p(x_j) : j \in I\}$, if $\mathcal{B}_{p_1}(X) = \mathcal{B}_p(X)$, then a base for $\mathcal{B}_p(X)$ is countable and thus $\sum\{\mathcal{B}_p(x_j; m) : j \in I, m \in \mathbb{Z}^+\}$ must contain a countable cofinal family in $\mathcal{B}_p(X)$. Since $\{\{x_j\} : j \in I\}$ is a family of p-bounded sets, it implies that I is a countable set. Hence the proof. □

Definition 4.2.6. Let (X, p) be an extended quasi-metric space, a nonempty set $A \subseteq X$ is said to *weakly bounded* if for each $j \in I$, we have $A \cap qmc_p(x_j) \in \mathcal{B}_p(X)$. We denote a set of weakly bounded subsets of X by $\mathcal{B}_p^w(X)$.

Theorem 4.2.5. *(Compare [5, Theorem 4.6]) Let (X, p) be an extended T_0-quasi-metric space. The following conditions are equivalent:*

(i) all the finitely many quasi-metric components for an extended quasi-metric p are p-bounded.

(ii) there exists an extended quasi-metric $p_1 = \min\{1, p\}$ such that $\mathcal{B}_{p_1}(X) = \mathcal{B}_p^w(X)$.

Proof. Throughout the proof, we denote the quasi-metric components of p by $\{qmc_p(x_j) : j \in I\}$.

(i) $\Rightarrow$ (ii): Asume (i) holds, let I_1 be a subset of I and be those indices for which $qmc_p(x_j) \in \mathcal{B}_p(X)$. To show that (ii) holds, we need to find a countable base for $\mathcal{B}_p^w(X)$. For each $m \in \mathbb{Z}^+$, we define $\mathcal{A}$ by

$$\mathcal{A} = \left\{\cup_{j\in I_1} qmc_p(x_j) \cup_{j\in I/I_1} \mathcal{B}_p(x_j, m) : m \in \mathbb{Z}^+\right\}.$$

Since each $qmc_p(x_j)$ for $j \in I_1$ is p-bounded, $\cup_{j\in I_1} qmc_p(x_j)$ is also p-bounded and does not intersects $\cup_{j\in I/I_1} \mathcal{B}_p(x_j, m)$ for all finitely many indices j. But $\left(\cup_{j\in I_1} qmc_p(x_j)\right) \bigcap \left(\cup_{j\in I/I_1} \mathcal{B}_p(x_j, n)\right) \in \mathcal{B}_p(X) = \mathcal{B}_{p_1}(X)$, which implies that $\cup_{j\in I_1} qmc_p(x_j)$ and $qmc_p(x_j)$ are weakly bounded. Hence, the collection $\mathcal{A}$ forms a countable base for $\mathcal{B}_p^w(X)$. Furthermore, $\{qmc_p(x_j) : j \in I_1\} \subseteq \{B_p(x_i, m) : j \in I_1, m \in \mathbb{Z}^+\}$ and so,

$$\begin{aligned} \mathrm{cl}_{\tau(p^{-1})}[\{qmc_p(x_j) : j \in I_1\}] &\subseteq \{qmc_p(x_j) : j \in I/I_1\} \\ &\subseteq \{B_p(x_j, m) : j \in I/I_1, m \in \mathbb{Z}^+\} \\ &\subset \mathrm{int}_{\tau(p)}[\{B_p(x_j, m) : j \in I/I_1, m \in \mathbb{Z}^+\}]. \end{aligned}$$

Since the $\tau(p)$-closure of each weakly bounded subset is in the $\tau(p)$-interior of another. By Theorem 4.1.1, there exists a $p_1 = \min\{1, p\}$ such that $\mathcal{B}_{p_1}(X) = \mathcal{B}_p^w(X)$.

(ii) $\Rightarrow$ (i): Suppose a quasi-metric $p_1 = \min\{1, p\}$ exists with $\mathcal{B}_{p_1}(X) = \mathcal{B}_p^w(X)$, for some countably infinite set of indices $\{j_k : k \in \mathbb{Z}^+\}$ such that $qmc_p(x_{j_k})$ is not p-bounded. Let us fix $s \in X$ and for each $m \in \mathbb{Z}^+$, put $B_m = \{B_{p_1}(s, m) : m \in \mathbb{Z}^+\}$ and let $\delta \in (0, 1)$ for $\epsilon < \frac{1}{2}$. Since $[B_m]^{\frac{1}{2}}_{p_1} \subseteq B_{m+1}$ and $[B_m]^{\frac{1}{2}}_{p_1} \cap B_{p_1}(s, m) \in \mathcal{B}_{p_1}(X)$, B_m must be cofinal in $\mathcal{B}_p^w(X)$.

Now for each $j \in \mathbb{Z}^+$, there exists $A_j \in \mathcal{B}_p(X)$ such that $A_j \subseteq qmc_p(x_{j_k})$ and such that $A_j \nsubseteq B_{p_1}(p, j)$. Then $\cup_{j=1}^{\infty} A_j \cup \{\{x_j\} : j \in I\}$ is weakly p-bounded but is not contained in any p_1-ball with center s, which is a contradiction sincewe assumed $\mathcal{B}_p(X) = \mathcal{B}_{p_1}(X)$. Hence, $qmc_p(x_{j_k})$ must be p-bounded for some countably infinite set of indices $\{j_k : k \in \mathbb{Z}^-\}$ and so $\{qmc_p(x_j) : j \in I\}$ is p-bounded. $\square$

5

Conclusion

In this MSc dissertation, we have successfully generalised the results of Beer [5] on bornologies of extended metric spaces from metric point of view to quasi-metric settings. Before presenting our own investigations, we have first summarized Beer's results on bornologies of extended metric spaces.

In the second part of this work, we have started our own investigations. We have generalized the construction of the Beer's universal space of extended metric spaces to our universal space of extended quasi-metric spaces. In order to achieve this construction, we have defined the concept of quasi-metric components which later led us to the definition of our universal space (see Definition 3.2.1).

We have proved (see Theorem 3.2.1) that there is always an isometry between extended quasi-metric space and its universal space. In the same manner, it has also been proved that the universal space of an extended T_0-quasi-metric space is bicomplete (see Theorem 3.2.2). Corollary 3.2.1 has concluded this part of our investigations by providing the comparison of the universal space of the extended T_0-quasi-metrics p and p^{-1}.

The last part of our investigations was to generalize the concept of bornologies of extended metric spaces to bornologies of extended T_0-quasi-metric spaces. We have presented two separate characterizations of a bornology on quasi-metrizable and uniformly quasi-metrizable spaces. Before we presented these characterizations, we first compared the quasi-metric bornologies determined by p, p^s and p^{-1} (see Lemma 4.1.1 and Lemma 4.1.2) and then went on to show that the collection of all finite unions of double balls forms a base for the quasi-metic bornology and it contains $\tau(p)$-relatively compact subsets of X (see Lemma 4.2.1).

The first characterization has been presented by finding an extended quasi-metric p on $(X, \mathcal{B})$ such that $\mathcal{B}$ coincide with the quasi-metric bornology $\mathcal{B}_p(X)$ on X(see Theorem 4.2.1 and

Theorem 4.2.2). In the second characterization, we have used an extended quasi-metric $p_1 = \min\{1, p\}$ on X to prove that the bornology of p_1-bounded sets coincides with the bornology $\mathcal{B}$ (see Theorem 4.2.5).

Our conclusion leads us to list some open problems encountered throughout the present investigations. We hope to study these problems in future work.

Problem 1. *Is it possible to extend our results on bornologies of extended quasi-metric spaces to quasi-uniform point of view?*

Problem 2. *To what extent can we generalize the work of Beer[6] on bornologies of normed spaces to asymmetric settings?*

Problem 3. *Is it possible to extend the results of Beer, Costantini and Levi [9] on metric bornologies of d-bounded and d-totally bounded subsets to quasi-metric settings?*

References

[1] C.A. Agyingi, P.Haihambo and H.-P.A.Künzi, Endpoints in T_0-quasi-metric spaces. Topol. Appl, 168 (2014) 82-93.

[2] A.M. Aminpour, Some Results in Asymmetric Metric Spaces. Maths. Aeterna. 2 (2012) 533-540.

[3] E. Alemany and S. Romaguera. On half-completion and bicompletion of quasi-metric spaces. Comment. Math. Univer. Carol. 37 (1996) 749-756.

[4] J.W.Alexander. On concept of a topological space.Proc.Nat. Acand. 25 (1939) 52-54.

[5] G. Beer. The Structure of Extended Real-valued Metric Spaces. Set-Valued Var. Anal. 21 (2013) 591-602.

[6] G. Beer, Structural Properties of Extended Normed Spaces. Set-Valued Var.Anal 23 (2015) 613-630.

[7] G. Beer. Topologies on Closed and Closed Convex Sets. Kluwer. Dordrecht (1993).

[8] G. Beer, Norms with infinite values. J. Conv Anal. 22 (2015) 35-58.

[9] G. Beer, C. Costantini and S. Levi, Total boundedness in metrizable spaces. Houston J. Math. 37 (2011) 1347-1362.

[10] J. Borwein. M. Fabian and J. Vanderwerff, Locally Lipschitz functions and bornological derivatives, CECM Report No. 93:012.

[11] S. Cobzas. Functional analysis in asymmetric normed spaces, Frontiers in Mathematics. Springer. Basel. 2013.

[12] N. V. Dung, Remarks on quasi-Metric spaces. Miskolc Maths Notes. 15 (2014) 1787-2413.

[13] *J. Dugundji, Topology, Wm. C. Brown Publishers, Dubuque, 1989.*

[14] *R. Espinola and M. A. Khamsi, Introduction to hyperconvex spaces, in: Handbook of metric fixed point theory, Kluwer. Dordrecht. (2001) 391-435.*

[15] *R. Espínola and W. A Kirk, Set-valued contractions and fixed points, Nonlinear Anal. 54 (2002) 485-494.*

[16] *M. Frigon, Fixed point results for generalised contractions in gauge spaces and applications, Proc. Amer. Math. Soc. 128 (2002) 2957-2965.*

[17] *Meroño A. S. and I. C. Garrido, Bornologies in metric spaces. Thesis- Proyecto Fin de Máster en. Investigación Matemática, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 2009-2010.*

[18] *I. C. Garrido and A. S. Meroño, Uniformly metrizable bornologies. J. Conv Anal. 20 (2013) 285-299.*

[19] *S.T. Hu. Boundedness in a topological space J. Math. Pures Appl. 228 (1949) 287-320. 12 (1976).*

[20] *J. C Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963), 71-89.*

[21] *E. Kemajou, H.-P. A. Künzi and O. Olela Otafudu, The Isbell-hull of a di-space. Topol. Appl. 159 (2012) 2463-2475.*

[22] *R. D. Kopperman, Which topologies are quasi-metrizable? Topol. Appl. 52 (1993) 99-107.*

[23] *R. D. Kopperman. Asymmetry and duality in topology. Topol. Appl. 66 (1995) 1-39.*

[24] *E. Klein and A. Thompson, Theory of correspondences, Wiley, New York, (1984).*

[25] *H.-P. A. Künzi and O. Olela Otafudu, q-Hyperconvexity in quasi-pseudometric spaces and fixed point theorems, J. Func. Spaces Appl. 10 (2012) 1-18.*

[26] *H.P. Künzi : Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology. In Handbook of the history of general topology, Vol. 3. volume 3 of Hist. Topol., pages 853–968. Kluwer Acad. Publ., Dordrecht, 2001.*

[27] *A. Lechicki, S. Levi and A. Spakowski, Bornological convergences. J. Math. Anal. Appl. 297 (2004) 751-770.*

[28] J. Marín, Weak bases and quasi-pseudo-metrization of bi-spaces. Topol. Appl. 156 (2009) 3070-3076.

[29] R. Meyer, Analytic cyclic cohomology, PhD. Thesis, 1999, arXiv:math.KT/9906205.

[30] M.G. Murdeshwar and K.K. Theckedath, Boundedness in a quasi-uniform space, Canada. Math. Bull. 13 (1970) 367-370.

[31] A. Piękosz and E, Wajch, Quasi-metrizability of bornological biuniverses in ZF, J. Conv Anal. 22 (2015) 1041-1060.

[32] S. Salbany, Bitopological spaces, compactifications and completions, Math. Monographs, no. 1, University of Cape Town, Depart. Math. 1974

[33] H. Vaughan, On locally compact metrizable spaces. Bull. Am. Math. Soc. 43(1937) 532-535.

[34] W.A. Wilson, On quasi-metric spaces, Amer. J. Math. 53 (1931) 675-684.