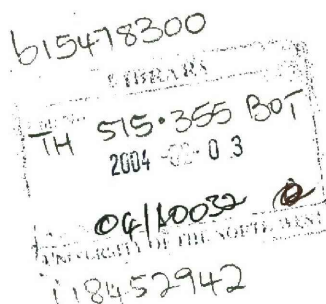


# **NON-LINEAR WAVE EQUATIONS AND THEIR INVARIANT SOLUTIONS**



by

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## Declaration

I declare that the contents of this research for a degree of Master of Science(Mathematics) at the University of North West are original except where due references have been made. It has not been submitted by me or any other person for any degree to any institution.

A handwritten signature in black ink, appearing to read 'E.W.L. Botolo', with a stylized flourish at the end.

E.W.L Botolo

# Abstract

We carry out a preliminary group classification of the following family of non-linear wave equations

$$u_{tt} = f(u_x)u_{xx} + g(u_x) + x.$$

We first reobtain the principal Lie algebra obtained by Ibragimov *et al*[3] and then construct the equivalence Lie algebra. In order to partially classify this family of wave equations, optimal systems of one-dimensional subalgebras of the equivalence Lie algebra are constructed and in so doing, two distinct equations are obtained. We furthermore determine some invariant solutions of these equations.

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# Chapter 1

## Lie 's symmetry algorithm , invariant solutions and the method of preliminary group classification

### 1.1 Introduction

A preliminary group classification of a second order partial differential equation which is a non-linear wave equation

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$$

was done by Ibragimov, Torrisi and Valenti [3]. They obtained thirty three families of equations. Each family needs further classification. In this project we focus attention on classification of one of those families: *viz*

$$u_{tt} = f(u_x)u_{xx} + g(u_x) + x.$$

The challenge of classification of families of differential equations using symmetries is an old topic. It dates back to the last century with Sophus Lie 's theory (1842–1899). Many outstanding researchers have done a great deal of work on various families of differential equations, the results of which have been captured in several literary

works. In the classification problem the principal Lie algebra is extended using the equivalence Lie algebra. For further information on this, see [3].

In chapter 1, we define one parameter group of transformations and point symmetries. We also show by an example how to determine point symmetries. In the second chapter we determine the principal Lie algebra, the equivalence Lie algebra and the extensions of the principal Lie algebra by one generator. Thus we use the extension results to obtain the preliminary group classification of the family

$$u_{tt} = f(u_x)u_{xx} + g(u_x) + x.$$

In chapter 3, we construct some invariant solutions of the nonsimilar equations obtained in chapter 2.

## 1.2 Symmetries

### 1.2.1 Point symmetries

Throughout this project, we will be using point symmetries. We now proceed to define these symmetries. Let  $x \in \mathbf{R}^n$  and  $u(x) \in \mathbf{R}$ .

**Definition 1.2.1:** The collection  $G$  of transformations  $T_a$  is called a *local one-parameter group* if

$$1) T_a T_b = T_{a+b} \in G, \quad 2) T_0 = I \in G, \quad 3) T_a^{-1} = T_{-a} \in G \quad (1.1)$$

for  $a$  and  $b$  sufficiently small real numbers, where

$$T_a : \bar{x}^i = \varphi^i(x, u, a), \quad \bar{u} = \psi(x, u, a), \quad (1.2)$$

depending on a real parameter  $a$ , such that

$$\bar{x}^i|_{a=0} = x^i, \quad \bar{u}|_{a=0} = u, \quad (1.3)$$

where  $i = 1, \dots, n$ .

$$\begin{aligned}
\xi_u^1 &= 0, \\
\xi_x^1 &= 0, \\
\xi_{uu}^1 &= 0, \\
\xi_u^2 &= -\xi_{xu}^1, \\
\xi_t^1 &= \xi_{xx}^1 + 2\xi_x^2, \\
\xi_{uu}^2 &= 0, \\
\eta_{uu} &= 2\xi_{xu}^1, \\
\xi_t^2 &= -2\eta_{xu} + \xi_{xx}^2, \\
\eta_t &= \eta_{xx}
\end{aligned}$$

(1.13)

Equations (1.13) is a system of linear partial differential equations whose solution is

$$\begin{aligned}
\eta &= \left(-\frac{1}{8}k_1x^2 - \frac{1}{2}k_4x - \frac{1}{4}k_1t + k_6\right)u + \beta(t, x), \\
\xi^1 &= \frac{1}{2}k_1t^2 + k_2t + k_3, \\
\xi^2 &= \frac{1}{2}k_1xt + \frac{1}{2}k_2x + k_4t + k_5,
\end{aligned}$$

(1.14)

where  $k_i, i = 1, \dots, 6$ , are constants of integration and  $\beta$  is an arbitrary solution of the heat equation. The Lie algebra admitted by (1.7) is therefore spanned by the following operators

$$\begin{aligned}
X_1 &= 4t^2 \frac{\partial}{\partial t} + 4xt \frac{\partial}{\partial x} - (x^2 + 2t)u \frac{\partial}{\partial u}, \\
X_2 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
X_3 &= \frac{\partial}{\partial t}, \\
X_4 &= 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}, \\
X_5 &= \frac{\partial}{\partial x},
\end{aligned}$$

To determine the Lie algebra elements, let us expand the functions  $\varphi^i$  and  $u$  in Taylor series with respect to the parameter  $a$  in the neighbourhood of  $a = 0$ . For each  $i = 1, 2, \dots, n$  and  $u$  we obtain the following approximations

$$\bar{x}^i \approx x^i + a\xi^i(x), \quad \bar{u} \approx u + a\eta(x), \quad (1.4)$$

$$\xi^i(x) = \left. \frac{\partial \varphi^i(x, u, a)}{\partial a} \right|_{a=0}, \quad \eta(x) = \left. \frac{\partial \psi}{\partial a} \right|_{a=0}. \quad (1.5)$$

The vector  $(\xi^1(x), \dots, \xi^n(x), \eta(x))$  whose components are given by (1.5) is the tangent vector to the curve, described by the transformed points  $(\bar{x}^1, \dots, \bar{x}^n, \bar{u})$ . This vector defines a *tangent vector field (Lie algebra)* of the group  $G$ . This tangent vector field is sometimes also written as a first-order differential operator

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + \eta(x) \frac{\partial}{\partial u} \quad (1.6)$$

and

$$x \in \mathbf{R}^n.$$

Now we wish to give an example of a one-parameter group of transformations with its corresponding tangent vector field.

**Example:** For the group of rotations

$$\bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a$$

the infinitesimal transformation has the form

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa$$

where  $a$  is a small real number. Its symbol is

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

### 1.3 Lie's algorithm for the calculation of symmetries of partial differential equation.

By way of example we wish to demonstrate the method of determining point symmetries.

Let us now consider the heat equation

$$u_t = u_{xx}, \quad (1.7)$$

where subscripts denote differentiation. Let

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (1.8)$$

be a point symmetry of equation (1.7). Equation (1.7) admits X if and only if

$$X^2(u_t - u_{xx})|_{u_t - u_{xx} = 0} = 0, \quad (1.9)$$

where

$$X^2 = X + \zeta^1 \frac{\partial}{\partial t} + \zeta^2 \frac{\partial}{\partial x} + \zeta^{11} \frac{\partial}{\partial u_{tt}} + \zeta^{12} \frac{\partial}{\partial u_{tx}} + \zeta^{22} \frac{\partial}{\partial u_{xx}}. \quad (1.10)$$

Thus (1.9) is equivalent to

$$[\zeta^1 - \zeta^{22}]_{u_t - u_{xx} = 0} = 0, \quad (1.11)$$

where

$$\begin{aligned} \zeta^i &= D_i(\eta) - u_t D_i(\xi^1) - u_x D_i(\xi^2), \\ \zeta^{ij} &= D_i(\zeta^j) - u_{jt} D_i(\xi^1) - u_{jx} D_i(\xi^2), \end{aligned} \quad (1.12)$$

where  $D_i$  is a total derivative.

Substituting (1.12) into (1.11) and equating the coefficients of the derivatives of  $u$  and their products to zero, we obtain the following

$$\begin{aligned}
X_6 &= u \frac{\partial}{\partial u}, \\
X_\beta &= \beta(t, x) \frac{\partial}{\partial u},
\end{aligned}
\tag{1.15}$$

where  $\beta$  is any solution of (1.7).

## Chapter 2

# Preliminary group classification of the equation

The focus is on the following non-linear wave equation:

$$u_{tt} = f(u_x)u_{xx} + g(u_x) + x. \quad (2.1)$$

In this chapter we determine the principal algebra, equivalence algebra and an extension of the principal algebra by one generator. The results will be the preliminary classification of this family (2.1).

### 2.1 Principal algebra

We construct the Principal Lie algebra  $l_p$  of the non-linear wave equation (2.1). The principal Lie algebra was obtained by Ibragimov, Torrisi and Valenti [3]. We will confirm their results by determining the principal algebra from scratch.

Let the operator

$$X = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}$$

be an arbitrary element of the Principal Lie algebra of the wave equation (2.1). The

second prolongation of the operator  $X$  is given by:

$$X^2 = X + \zeta^x \frac{\partial}{\partial u_x} + \zeta^t \frac{\partial}{\partial u_t} + \zeta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{tx} \frac{\partial}{\partial u_{tx}} + \zeta^{tt} \frac{\partial}{\partial u_{tt}},$$

where

$$\begin{aligned}\zeta^i &= D_i(\eta) - u_t D_i(\tau) - u_x D_i(\xi), \\ \zeta^{ij} &= D_i(\zeta^j) - u_{jt} D_i(\tau) - u_{jx} D_i(\xi),\end{aligned}\tag{2.2}$$

and

$D_i$  is a total derivative. The invariance condition is given by

$$X^2(u_{tt} - f(u_x)u_{xx} - g(u_x) - x) |_{u_{tt}=f u_{xx}+g+x} = 0.$$

We obtain the following determining equations:

$$(\zeta^{tt} - f' \zeta^x u_{xx} - g' \zeta^x - \xi) |_{(2.1)} = 0.$$

Solving the system, one obtains the following results:

$$\xi = 2c_1, \quad \tau = c_2, \quad \eta = c_1 t^2 + c_3 t + c_4.$$

The basis of a four dimensional principal Lie algebra,

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t \frac{\partial}{\partial u}, \quad X_4 = 2 \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial u}\tag{2.3}$$

This confirms what Ibragimov *et al* obtained [3].

## 2.2 Equivalence algebra

Each family of the wave equation (2.1) can be written as a system of equations in the  $(x, t, u, u_t, u_x, f^1, f^2)$  space:

$$u_{tt} = f^1 u_{xx} + f^2 + x, \quad f_x^k = f_t^k = f_u^k = f_{u_t}^k = 0, \quad k = 1, 2.\tag{2.4}$$

Where  $f^1$  replaces  $f$  and  $f^2$  replaces  $g$ . In this space  $f^1$  and  $f^2$  are regarded as variables. We would like to determine the equivalence algebra of each family of equations. Let an arbitrary equivalence algebra element of any system be given by the generator

$$E = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \mu^k \frac{\partial}{\partial f^k},$$

where

$$\xi = \xi(x, t, u), \quad \tau = \tau(x, t, u), \quad \eta = \eta(x, t, u),$$

$$\mu^k = \mu(x, t, u, u_x, u_t, f^1, f^2)$$

and  $k = 1, 2$ .

We now introduce the following derivatives

$$\overline{D}_\alpha = \frac{\partial}{\partial \alpha} + f_\alpha^k \frac{\partial}{\partial f^k} + f_{\alpha t}^k \frac{\partial}{\partial f_t^k} + f_{\alpha x}^k \frac{\partial}{\partial f_x^k} + f_{\alpha u}^k \frac{\partial}{\partial f_u^k} + f_{\alpha u_t}^k \frac{\partial}{\partial f_{u_t}^k} + \dots$$

for

$$\alpha \in \{x, t, u, u_t\}.$$

The extension of the equivalence algebra element  $E$ , takes the form

$$\overline{E} = E + \zeta^t \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{tt} \frac{\partial}{\partial u_{tt}} + \omega_t^k \frac{\partial}{\partial f_t^k} + \omega_x^k \frac{\partial}{\partial f_x^k} + \omega_u^k \frac{\partial}{\partial f_u^k} + \omega_{u_t}^k \frac{\partial}{\partial f_{u_t}^k},$$

where  $\zeta^i, \zeta^{ij}$  are as given above for

$$i, j \in \{x, t\}$$

and

$$\omega_\alpha^k = D_\alpha(\mu^k) - f_t^k D_\alpha(\tau) - f_x^k D_\alpha(\xi) - f_u^k D_\alpha(\eta) - f_{u_t}^k D_\alpha(\zeta^t) - f_{u_x}^k D_\alpha(\zeta^x)$$

where  $k = 1, 2$ .

The invariance conditions for the system of equations (2.1) is given by

$$\bar{E}(u_{tt} - f^1 u_{xx} - f^2 - x) |_{u_{tt}=f^1 u_{xx}+f^2+x} = 0, \quad E(f_\alpha^k) = 0. \quad (2.5)$$

We obtain

$$(\zeta^{tt} - \mu^1 u_{xx} - f^1 \zeta^{xx} - \mu^2 - \xi) |_{u_{tt}=f^1 u_{xx}+f^2+x} = 0, \quad (2.6)$$

and for each

$$E(f_\alpha^k) = 0$$

in (2.5) we have

$$\varpi_\alpha^k = 0, \quad (2.7)$$

for  $\alpha \in \{x, t, u, u_t\}$ .

From equation (2.6) we have

$$(\mu^k)_\alpha = (\zeta^x)_\alpha = 0, \quad \alpha \in \{x, t, u, u_t\}$$

and  $k = 1, 2$  which implies that the  $\mu^k$  are independent of  $x, t, u, u_t$  and hence

$$\mu^k = \mu^k(u_x, f^1, f^2), \quad k = 1, 2.$$

Furthermore

$$(\zeta^x)_\alpha = 0,$$

yields

$$\xi = c_1 x + c_2 u + p(t), \quad \tau = \tau(t), \quad \eta = c_3 u + c_4 + q(t), \quad (2.8)$$

where  $c_1, c_2, c_3, c_4$  are arbitrary constants.

From the above equations, we obtain

$$\begin{aligned} \xi &= (b_3 - 2b_1)x + b_4, \\ \tau &= b_1 t + b_2, \\ \eta &= b_3 u + b_5 x + b_6 t^2 + b_7 t + b_8, \\ \mu^1 &= 2(b_3 - 2b_1)f^1, \\ \mu^2 &= 2b_6 + (b_3 - 2b_1)f^2, \end{aligned} \quad (2.9)$$

where  $b_j$  are constants for  $j = 1, 2, \dots, 8$ .

For our specific equation (2.1) we will set  $f^1 = f$  and  $f^2 = g$ .

The equivalence algebra is 8-dimensional and is spanned by

$$\begin{aligned} G_1 &= \frac{\partial}{\partial u}, & G_2 &= \frac{\partial}{\partial t}, & G_3 &= \frac{\partial}{\partial x}, & G_4 &= t \frac{\partial}{\partial u}, & G_5 &= t^2 \frac{\partial}{\partial u} + 2 \frac{\partial}{\partial g}, \\ G_6 &= x \frac{\partial}{\partial u}, & G_7 &= x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + g \frac{\partial}{\partial g} + 2f \frac{\partial}{\partial f}, \\ G_8 &= -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - 4f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}. \end{aligned} \quad (2.10)$$

### 2.3 One-dimensional optimal systems.

We prolong the equivalence operators  $G_j$  to the following operators:

$$\bar{G}_j = G_j + \zeta^x \frac{\partial}{\partial u_x},$$

for  $j = 1, 2, \dots, 8$ .

Therefore the results are operators extended to the  $(x, t, u, u_x, f, g)$ -space. We have

$$\begin{aligned} \bar{G}_j &= G_j, \quad j = 1, 2, \dots, 5, \\ \bar{G}_6 &= x \frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \\ \bar{G}_7 &= G_7, \\ \bar{G}_8 &= G_8 + 2u_x \frac{\partial}{\partial u_x}. \end{aligned} \quad (2.11)$$

We form new operators by projecting each  $\bar{G}_j$  onto the  $(u_x, f, g)$ -subspace of the  $\{x, t, u, u_x, f, g\}$ -space.

We obtain:

$$\begin{aligned} N_1 &= pr(\bar{G}_5) = 2 \frac{\partial}{\partial g}, \\ N_2 &= pr(\bar{G}_6) = \frac{\partial}{\partial u_x}, \end{aligned}$$

$$\begin{aligned}
N_3 &= pr(\overline{G}_7) = 2f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \\
N_4 &= pr(\overline{G}_8) = -4f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g} + 2u_x \frac{\partial}{\partial u_x}.
\end{aligned} \tag{2.12}$$

Their non zero commutators are

$$[N_2, N_4] = 2N_2, \quad [N_1, N_3] = N_1, \quad [N_1, N_4] = -2N_1.$$

For this 4-dimensional Lie algebra  $L_4$ , we wish to determine the optimal system of one-dimensional subalgebras.

## 2.4 The generators of adjoint algebra

The commutator table of the generators  $N_1, N_2, N_3$  and  $N_4$  of the Lie algebra  $L_4$  is:

| $[,]$ | $N_1$  | $N_2$   | $N_3$ | $N_4$   |
|-------|--------|---------|-------|---------|
| $N_1$ | 0      | 0       | $N_1$ | $-2N_1$ |
| $N_2$ | 0      | 0       | 0     | $2N_2$  |
| $N_3$ | $-N_1$ | 0       | 0     | 0       |
| $N_4$ | $2N_1$ | $-2N_2$ | 0     | 0       |

Now we wish to construct generators of the adjoint algebra as follows

$$A_i = [N_i, N_j] \frac{\partial}{\partial N_j}, \quad i, j = 1, 2, 3, 4.$$

From our commutator table, we obtain the following adjoint algebra generators:

$$\begin{aligned}
A_1 &= N_1 \frac{\partial}{\partial N_3} - 2N_1 \frac{\partial}{\partial N_4}, \\
A_2 &= 2N_2 \frac{\partial}{\partial N_4}, \\
A_3 &= -N_1 \frac{\partial}{\partial N_1}, \\
A_4 &= 2N_1 \frac{\partial}{\partial N_1} - 2N_2 \frac{\partial}{\partial N_2},
\end{aligned} \tag{2.13}$$

In order to obtain the elements of the adjoint group  $G^A$  or the group of inner automorphisms of the algebra  $L_4$ , we integrate the equations (2.13) and we obtain:

$$\begin{aligned}
A_1 : \quad \overline{N_1} &= N_1, \quad \overline{N_2} = N_2, \quad \overline{N_3} = N_1 a_1 + N_3, \quad \overline{N_4} = -2N_1 a_1 + N_4, \\
A_2 : \quad \overline{N_1} &= N_1, \quad \overline{N_2} = N_2, \quad \overline{N_3} = N_3, \quad \overline{N_4} = 2N_2 a_2 + N_4, \\
A_3 : \quad \overline{N_1} &= N_1 a_3^{-1}, \quad \overline{N_2} = N_2, \quad \overline{N_3} = N_3, \quad \overline{N_4} = N_4, \\
A_4 : \quad \overline{N_1} &= N_1 a_4^2, \quad \overline{N_2} = N_2 a_4^{-2}, \quad \overline{N_3} = N_3, \quad \overline{N_4} = N_4,
\end{aligned} \tag{2.14}$$

where  $a_1, a_2, a_3, a_4$  are non-zero constant parameters.

## 2.5 Matrix representation of adjoint operators

We construct matrix representation of all four inner automorphisms. The matrix representation of the inner automorphism generated by  $A_1$  is

$$M(a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & 1 & 0 \\ -2a_1 & 0 & 0 & 1 \end{pmatrix}$$

The matrix representation of the inner automorphism generated by  $A_2$  is

$$M(a_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2a_2 & 0 & 1 \end{pmatrix}$$

Similarly  $A_3$  yields

$$M(a_3) = \begin{pmatrix} a_3^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(a_4) = \begin{pmatrix} a_4^2 & 0 & 0 & 0 \\ 0 & a_4^{-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now the product of matrix representation is given by

$$M = M(a_1)M(a_2)M(a_3)M(a_4), \quad (2.15)$$

$$M = \begin{pmatrix} a_3^{-1}a_4^2 & 0 & 0 & 0 \\ 0 & a_4^{-2} & 0 & 0 \\ a_1a_3^{-1}a_4^2 & 0 & 1 & 0 \\ -2a_1a_3^{-1}a_4^2 & 2a_2a_4^{-2} & 0 & 1 \end{pmatrix}.$$

If we let  $N \in L_4$  be given by

$$N = e^1N_1 + e^2N_2 + e^3N_3 + e^4N_4 \quad (2.16)$$

then

$$\bar{e} = Me,$$

where

$$e = (e^1, e^2, e^3, e^4)$$

defines an equivalence relation in  $L_4$  and hence subdivides this algebra into equivalence classes. The components of  $e$  mapped under  $M$  are

$$\begin{aligned} \bar{e}^1 &= a_3^{-1}a_4^2e^1, \\ \bar{e}^2 &= a_4^{-2}e^2, \\ \bar{e}^3 &= a_1a_3^{-1}a_4^2e^1 + e^3, \\ \bar{e}^4 &= -2a_1^{-1}a_4^2e^1 + 2a_2^2a_4^{-2}e^2 + e^4, \end{aligned} \quad (2.17)$$

where  $a_i, i = 1, 2, \dots, 4$ , are as above.

From the system of equations (2.17), we have the following possibilities

$$a_3^{-1}a_4^2 = 1, -1,$$

thus  $e^1$  and  $e^2$  are invariant.

From the system of equations (2.17) we consider

$$(e^1, e^2) = 0 \text{ or } (e^1, e^2) \neq 0.$$

Let us start with

**case 1**

$$e^1 = 0, \quad e^2 = 0.$$

From the system of equations (2.17) we obtain

$$\overline{e^1} = 0,$$

$$\overline{e^2} = 0,$$

$$\overline{e^3} = e^3,$$

$$\overline{e^4} = e^4.$$

(2.18)

We now consider the following

**subcase 1(a)**

$$e^3 \neq 0, \quad e^4 \neq 0$$

to obtain

$$\overline{e} = (0, 0, \alpha, 1), \quad \alpha = \frac{e^3}{e^4} \neq 0.$$

We obtain from equation (2.17), the following

$$N^1 : N_3 + \alpha N_4, \alpha \in \mathbb{R}, \alpha \neq 0.$$

**subcase 1(b)**

$$e^3 = 0, \quad e^4 \neq 0,$$

we obtain

$$\bar{e} = (0, 0, 0, 1)$$

$$N^2 : N_4$$

subcase 1(c)

$$e^3 \neq 0, \quad e^4 = 0$$

we obtain

$$\bar{e} = (0, 0, 1, 0),$$

therefore we obtain

$$N^3 : N_3$$

We consider

case 2

$$e^1 = 0, \quad e^2 \neq 0.$$

From equations (2.17) we obtain

$$\bar{e}^1 = 0,$$

$$\bar{e}^2 = a_4^{-2} e^2,$$

$$\bar{e}^3 = e^3,$$

$$\bar{e}^4 = 2a_2^2 a_4^{-2} e^2 + e^4.$$

(2.19)

subcase 2(a)

$$e^3 \neq 0, e^4 \neq 0$$

We now let  $a_4 = \pm 1$  to obtain

$$\bar{e}^1 = 0,$$

$$\bar{e}^2 = e^2,$$

$$\bar{e}^3 = e^3,$$

$$\bar{e}^4 = 2a_2^2 e^2 + e^4.$$

(2.20)

If  $a_2^2 = -\frac{e^4}{2e^2}$  then  $\overline{e^4} = 0$ , we obtain

$$\overline{e} = (0, e^2, e^3, 0)$$

Finally

$$\overline{e} = (0, 1, \alpha, 0),$$

where  $\alpha = \frac{e^3}{e^2}$ ,  $e^2 \neq 0$ . We now have

$$N^4 : N_2 + \alpha N_3, \alpha \in R, \alpha \neq 0.$$

subcase 2(b)

$$e^3 = 0, e^4 \neq 0.$$

If we let  $a_4 = \pm 1$ , and let  $a_2^2 = -\frac{e^4}{2e^2}$ , we obtain

$$\overline{e^1} = 0, \quad \overline{e^2} = e^2, \quad \overline{e^3} = 0, \quad \overline{e^4} = 0.$$

$$\overline{e} = (0, 1, 0, 0),$$

we obtain

$$N^5 : N_2.$$

subcase 2(c)

$$e^3 \neq 0, \quad e^4 = 0,$$

then the system of equations (2.17) becomes

$$\overline{e^1} = 0,$$

$$\overline{e^2} = a_4^{-2} e^2,$$

$$\overline{e^3} = e^3,$$

$$\overline{e^4} = 2a_2^2 a_4^{-2} e^2.$$

(2.21)

Let  $a_4 = \pm 1$  and let  $a_2 = \pm 1$ , we obtain

$$\overline{e^1} = 0, \quad \overline{e^2} = e^2, \quad \overline{e^3} = e^3, \quad \overline{e^4} = 2e^2.$$

We finally obtain

$$\bar{e} = (0, e^2, e^3, 2e^2),$$

then

$$\bar{e} = (0, 1, \alpha, 2), \quad \alpha = \frac{e^3}{e^2}.$$

$$N^6 : N_2 + \alpha N_3 + 2N_4, \quad \alpha \in R, \alpha \neq 0.$$

Now consider the

subcase 2(d)

$$e^3 = 0, \quad e^4 = 0.$$

The system of equations (2.17) reduces to

$$\bar{e}^1 = 0, \quad \bar{e}^2 = a_4^{-2} e^2, \quad \bar{e}^3 = 0, \quad \bar{e}^4 = 2a_2^2 a_4^{-2} e^2,$$

we then let  $a_4 = \pm 1$ ,  $a_2 = \pm 1$ .

Therefore we obtain:

$$\bar{e} = (0, e^2, 0, 2e^2),$$

we then obtain

$$\bar{e} = (0, 1, 0, 2).$$

$$N^7 : N_2 + 2N_4.$$

Let us now consider

subcase 3

$$e^1 \neq 0, \quad e^2 = 0.$$

Equations (2.17) become

$$\bar{e}^1 = a_3^{-1} a_4^2 e^1,$$

$$\bar{e}^2 = 0,$$

$$\begin{aligned}
\overline{e^3} &= a_1 a_3^{-1} a_4^2 e^1 + e^3, \\
\overline{e^4} &= -2a_1^{-1} a_4^2 e^1 + e^4.
\end{aligned}
\tag{2.22}$$

Let us consider

**subcase 3(a)**

$$e^3 = 0, \quad e^4 \neq 0,$$

and let  $a_4 = \pm 1$ ,

then we obtain

$$\begin{aligned}
\overline{e^1} &= a_3^{-1} e^1, \\
\overline{e^2} &= 0, \\
\overline{e^3} &= a_1 a_3^{-1} e^1, \\
\overline{e^4} &= 2a_1^{-1} e^1 + e^4.
\end{aligned}
\tag{2.23}$$

We now let  $a_3^{-1}, a_1 = \pm 1$  and  $\alpha = \frac{\pm 2e^1 + e^4}{e^1}$ . We obtain

$$\bar{e} = (1, 0 \pm 1, \alpha)$$

$$N^8 : N_1 \pm N_3 + \alpha N_4.$$

Alternatively we can let  $a_1^{-1} = \frac{e^4}{e^1}$  and obtain

$$N^9 : N_1 + \alpha N_3, \quad \alpha = \frac{\pm 2(e^1)}{e^4}.$$

Now , consider

**subcase 3(b)**

$$e^3 \neq 0, \quad e^4 = 0.$$

The system of equations (2.17) becomes

$$\overline{e^1} = a_3^{-1} a_4^2 e^1,$$

$$\begin{aligned}
\overline{e^2} &= 0, \\
\overline{e^3} &= a_1 a_3^{-1} a_4^2 e^1 + e^3, \\
\overline{e^4} &= -2a_1^{-1} a_4^2 e^1.
\end{aligned}$$

(2.24)

If we let  $a_3^{-1} = \pm 1$ , we obtain

$$\overline{e^1} = \pm a_4^2 e^1, \quad \overline{e^2} = 0, \quad \overline{e^3} = \pm a_1 a_4^2 + e^3, \quad \overline{e^4} = -2a_1^{-1} a_4^2 e^1.$$

We now let  $a_4 = \pm 1$ , to obtain

$$\overline{e^1} = \pm e^1, \quad \overline{e^2} = 0, \quad \overline{e^3} = \pm a_1 e^1 + e^3, \quad \overline{e^4} = \mp 2a_1^{-1} e^1.$$

Let  $a_1 = \mp \frac{e^3}{e^1}$ ,  $e^1 \neq 0$ , we obtain

$$\overline{e^1} = \pm e^1, \quad \overline{e^2} = 0, \quad \overline{e^3} = 0, \quad \overline{e^4} = \mp \frac{2(e^1)^2}{e^3}, \quad e^3 \neq 0.$$

We now obtain

$$\begin{aligned}
\overline{e} &= (\pm 1, 0, 0, \alpha), \alpha = \pm \frac{2e^1}{e^3}. \\
N^{(10)} &: \pm N_1 + \alpha N_4.
\end{aligned}$$

**subcase 3(c)**

$$e^3 \neq 0, \quad e^4 \neq 0,$$

we obtain the same results of

$$N^{(10)},$$

Finally we consider

**case 4**

$$e^1 \neq 0, \quad e^2 \neq 0.$$

Equations (2.17) reduces to

$$\overline{e^1} = a_3^{-1} a_4^2 e^1,$$

$$\begin{aligned}
\overline{e^2} &= a_4^{-2}e^2, \\
\overline{e^3} &= a_1a_3^{-1}e^1 + e^3, \\
\overline{e^4} &= -2a_1^{-1}a_4^2e^1 + 2a_2^2a_4^{-2}e^2 + e^4.
\end{aligned}$$

(2.25)

Let us first of all consider

**subcase 4(a)**

$$e^3 \neq 0, \quad e^4 \neq 0.$$

We now let  $a_4 = \pm 1$  and  $a_3^{-1} = \pm 1, a_1 = \pm \frac{e^3}{e^1}$ .

We obtain

$$\overline{e^1} = \pm e^1, \quad \overline{e^2} = e^2, \quad \overline{e^3} = 0, \quad \overline{e^4} = \pm \frac{2(e^1)^2}{e^3} + 2a_2^2e^2 + e^4$$

There are two possibilities of  $a_2$ , i.e  $\pm 1$  Finally we have

$$\overline{e} = (\pm e^1, e^2, 0, \pm \frac{2(e^1)^2}{e^3} + 2e^2 + e^4).$$

If we let

$$\alpha = \frac{e^2}{e^1}, \quad \beta = \frac{\pm \frac{(e^1)^2}{e^3} + 2e^2 + e^4}{e^1},$$

we obtain

$$\overline{e} = (\pm 1, \alpha, 0, \beta), \quad \alpha, \quad \beta \in R, \alpha, \beta \neq 0.$$

$$N^{(11)} : \pm N_1 + \alpha N_2 + \beta N_4.$$

Now we consider

**subcase 4(b)**

$e^3 = 0, \quad e^4 = 0$ , to obtain

$$\begin{aligned}
\overline{e^1} &= a_3^{-1}a_4^2e^1, \\
\overline{e^2} &= a_4^{-2}e^2, \\
\overline{e^3} &= a_1a_3^{-1}e^1, \\
\overline{e^4} &= -2a_1^{-1}a_4^2e^1 + 2a_2^2a_4^{-2}e^2.
\end{aligned}$$

(2.26)

If we let  $a_4 = \pm 1$ ,  $a_3^{-1} = \pm 1$ , we obtain

$$\bar{e}^1 = \pm e^1, \quad \bar{e}^2 = e^2, \quad \bar{e}^3 = \pm a_1 e^1, \quad \bar{e}^4 = \pm 2a_1^{-1} e^1 + 2a_2^2 e^2.$$

Let  $a_1 = \pm 1$  and  $a_2^2 = \mp \frac{e^1}{e^2}$ .

We obtain

$$\bar{e}^1 = \pm e^1, \quad \bar{e}^2 = e^2, \quad \bar{e}^3 = \pm e^1, \quad \bar{e}^4 = 0).$$

$$\bar{e} = (\pm 1, \alpha \pm 1, 0), \quad \alpha = \frac{e^2}{e^1}.$$

We then obtain

$$N^{(12)} : \pm N_1 \alpha N_2 \pm N_3, \alpha \in R, \alpha \neq 0.$$

For the cases  $e^3 = 0$ ,  $e^4 \neq 0$ , we obtain  $N^{(12)}$  and for the case  $e^3 \neq 0$ ,  $e^4 = 0$ , we obtain  $N^{(11)}$ .

### summary of the one-dimensional optimal system

The one-dimensional optimal system is formed by

1.  $N_4$ ,
2.  $N_3 + \alpha N_4$ ,
3.  $N_3$ ,
4.  $N_2 + \alpha N_3$ ,
5.  $N_2$ ,
6.  $N_1 + N_3 + \alpha N_4$ ,
7.  $N_2 + \alpha N_3 + 2N_4$ ,
8.  $N_2 + 2N_4$ ,
9.  $N_1 + \alpha N_3$ ,
10.  $\pm N_1 + \alpha N_4$ ,
11.  $N_1 + \alpha N_2 + \beta N_4$ ,
12.  $\pm N_1 + \alpha N_2 \pm N_3$ .

## 2.6 Extension of each principal algebra by one

We will show the procedure for finding  $f$ ,  $g$  and  $X_5$ , for the first cases of subalgebras and quote the results of the remaining subalgebras.

1.  $N_4$ ,

$$-4f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g} + 2u_x \frac{\partial}{\partial u_x} \quad (2.27)$$

We obtain the characteristic equation:

$$\frac{\partial f}{-4f} = \frac{\partial u_x}{2u_x} = \frac{\partial g}{-2g} \quad (2.28)$$

We obtain

$$f = u_x^{-2}, \quad g = u_x^{-1}$$

after integrating the characteristic equation.

In order to find the extension  $X_5$ , we should recall from the eight-dimensional equivalence algebra spanned by (2.10).

$$N_4 = pr(\overline{G_8}) = -4f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g} + 2u_x \frac{\partial}{\partial u_x}, \quad (2.29)$$

and

$$G_8 = -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - 4f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}. \quad (2.30)$$

From  $G_8$ , we obtain the extension of the algebra

$$X_5 = -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}.$$

Substituting  $f$  and  $g$  in equation (2.1), we obtain

$$u_{tt} = u_x^{-2} u_{xx} + u_x^{-1} + x.$$

Using the procedure above, we obtain for

2.  $N_3 + \alpha N_4$ ,

$$f = u_x^{\frac{1-2\alpha}{\alpha}}, \quad g = u_x^{\frac{1-2\alpha}{2\alpha}}.$$

and

$$X_5 = (1 - 2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}.$$

Hence the equation corresponding to  $X_5$  will be

$$u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + u_x^{\frac{1-2\alpha}{2\alpha}} + x.$$

3.  $N_3$  yields no results, if we use the procedure mentioned above.

4.  $N_2 + \alpha N_3$ ,

we obtain

$$f = e^{2\alpha u_x}, \quad g = e^{\alpha u_x},$$

and

$$X_5 = \alpha x \frac{\partial}{\partial x} + (x + \alpha u) \frac{\partial}{\partial u}.$$

The equation thereof corresponding to the extension will be:

$$u_{tt} = e^{2\alpha u_x} u_{xx} + e^{\alpha u_x} + x.$$

5.  $N_2$

$$f = C_1, \quad g = C_2,$$

and its extension is:

$$X_5 = x \frac{\partial}{\partial u},$$

hence the equation

$$u_{tt} = C_1 u_{xx} + C_2 + x.$$

6.  $N_1 + N_3 + \alpha N_4$

yields

$$f = u_x^{\frac{1-2\alpha}{\alpha}}, \quad g = \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha},$$

and

$$X_5 = (1-2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + (t^2 + u) \frac{\partial}{\partial u},$$

hence when substituting  $f$  and  $g$  in equation (2.1) we obtain

$$u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha} + x.$$

7.  $N_2 + \alpha N_3 + 2N_4.$

We obtain the equation

$$u_{tt} = (4u_x + 1)^{\alpha-4} u_{xx} + (4u_x + 1)^{\frac{\alpha-4}{2}} + x,$$

and its corresponding

$$X_5 = (\alpha-4)x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + (x + \alpha u) \frac{\partial}{\partial u}.$$

8.  $N_2 + 2N_4$

We obtain the equation

$$u_{tt} = (4u_x + 1)^{-2} u_{xx} + (4u_x + 1)^{-1} + x$$

and its corresponding extension is

$$X_5 = -4x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial u}.$$

9.  $N_1 + \alpha N_3,$

yields no results.

10.  $\pm N_1 + \alpha N_4.$

The first possibility is given by

(i)  $N_1 + \alpha N_4.$

The equation is given by

$$u_{tt} = u_x^{-2} u_{xx} + \left( \frac{1}{\alpha} - \frac{1}{\alpha u_x} \right) + x,$$

$$X_5 = -2\alpha x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial u}.$$

$$(ii) \quad -N_1 + \alpha N_4.$$

The equation and its corresponding extension  $X_5$  will be

$$u_{tt} = u_x^{-2} u_{xx} + \left( \frac{1}{\alpha u_x} - \frac{1}{\alpha} \right) + x,$$

$$X_5 = -2\alpha x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} - t^2 \frac{\partial}{\partial u}.$$

$$11. \quad N_1 + \alpha N_2 + \beta N_4,$$

$$u_{tt} = (2\beta u_x + \alpha)^{-2\beta} u_{xx} + \frac{1}{\beta} [1 - (2\beta u_x + \alpha)^{-1}] + x,$$

$$X_5 = -2\beta x \frac{\partial}{\partial x} + \beta t \frac{\partial}{\partial t} + (t^2 + \alpha x) \frac{\partial}{\partial u}.$$

$$12. \quad (i) \quad N_1 + \alpha N_2 + N_3.$$

We will consider the above possibility as the other three possibilities (i.e.  $-N_1 + \alpha N_2 + N_3$ ,  $-N_1 + N_2 - N_3$  and  $N_1 \alpha N_2 - N_3$ ) yield similar results.

The equation for (i) is given by

$$u_{tt} = e^{\frac{2u_x}{\alpha}} u_{xx} + [e^{\frac{u_x}{\alpha}} - 2] + x,$$

$$X_5 = -2\beta x \frac{\partial}{\partial x} + \beta t \frac{\partial}{\partial t} + (x + t^2) \frac{\partial}{\partial u}.$$

We now check if the equations obtained in solving the subalgebras of the one-dimensional optimal system admits their corresponding extensions  $X_5$ .

1. The one-dimensional optimal subalgebra 1,  $N_4$  gave us the equation:

$$u_{tt} = u_x^{-2} u_{xx} + u_x^{-1} + x,$$

and the extension:

$$X_5 = -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}.$$

We prolong  $X_5$  to second order and obtain

$$X_5^2 = -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + 2u_x \frac{\partial}{\partial u_x} - u_t \frac{\partial}{\partial u_t} + 4u_{xx} \frac{\partial}{\partial u_{xx}} - 2u_{tt} \frac{\partial}{\partial u_{tt}}$$

using the prolongation formula (2.2).

$$X_5^2(u_{tt} - u_x^{-2}u_{xx} - u_x^{-1} - x) = -2(u_{tt} - u_x^{-1} - x) \neq 0.$$

Therefore

$$u_{tt} = u_x^{-2}u_{xx} + u_x^{-1} + x,$$

does not admit

$$X_5 = -2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}.$$

2.

$$u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + u_x^{\frac{1-2\alpha}{2\alpha}} + x.$$

and its extension  $X_5$  is

$$X_5 = (1 - 2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}.$$

Prolonging  $X_5$ , we obtain

$$X_5^2 = X_5 + 2\alpha u_x \frac{\partial}{\partial u_x} + (1 - \alpha)u_t \frac{\partial}{\partial u_t} + (4\alpha - 1)u_{xx} \frac{\partial}{\partial u_{xx}} + (1 - 2\alpha)u_{tt} \frac{\partial}{\partial u_{tt}}.$$

$$X_5^2(u_{tt} - u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} - u_x^{\frac{1-2\alpha}{2\alpha}} - x) \neq 0.$$

Hence the extension of 2 is not admitted by its corresponding equation. 3. No results

4. We focus on  $N_2 + \alpha N_3$ .

We obtain

$$u_{tt} = e^{2\alpha u_x} u_{xx} + e^{\alpha u_x} + x,$$

and show that it admits

$$X_5 = \alpha x \frac{\partial}{\partial x} + (x + \alpha u) \frac{\partial}{\partial u}.$$

The second prolongation is given by

$$X_5^2 = X_5 + \frac{\partial}{\partial u_x} + \alpha u_t \frac{\partial}{\partial u_t} - \alpha u_{xx} \frac{\partial}{\partial u_{xx}} + \alpha u_{tt} \frac{\partial}{\partial u_{tt}},$$

$$X_5^2(u_{tt} - e^{2\alpha u_x} u_{xx} - e^{\alpha u_x} - x) = \alpha[u_{tt} - e^{2\alpha u_x} u_{xx} - e^{\alpha u_x} - x] = 0.$$

This is case 2 in chapter 3.

5. Consider  $N_2$  as the results of  $-N_2$  has the same consequences as  $N_2$  itself. We obtained

$$u_{tt} = C_1 u_{xx} + C_2 + x$$

and the extension

$$X_5 = x \frac{\partial}{\partial u}.$$

Prolonging  $X_5$  to second order, we obtain

$$X_5^2 = x \frac{\partial^2}{\partial u^2} + \frac{\partial}{\partial u_x}.$$

$$X_5^2(u_{tt} - C_1 u_{xx} - C_2 - x) = 0.$$

6. Consider the equation

$$u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha} + x.$$

which has the extension

$$X_5 = (1-2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + (t^2 + u) \frac{\partial}{\partial u}.$$

$$X_5^2(u_{tt} - u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} - \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha} - x) \neq 0.$$

Therefore  $X_5$  is not admitted by the corresponding equation.

7.

$$u_{tt} = (4u_x + 1)^{\alpha-4} u_{xx} + (4u_x + 1)^{\frac{\alpha-4}{2}} + x,$$

with its corresponding

$$X_5 = (\alpha-4)x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + (x + \alpha u) \frac{\partial}{\partial u},$$

has

$$X_5^2 = X_5 + (1 + 4u_x) \frac{\partial}{\partial u_x} + (\alpha - 2)u_t \frac{\partial}{\partial u_t} + (8 - \alpha)u_{xx} \frac{\partial}{\partial u_{xx}} + (\alpha - 4)u_{tt} \frac{\partial}{\partial u_{tt}}.$$

$$X_5^2(u_{tt} - (4u_x + 1)^{\alpha-4}u_{xx} + (4u_x + 1)^{\frac{\alpha-4}{2}} + x) \neq 0,$$

which implies  $X_5$  is not admitted by its corresponding equation. 8.

$$u_{tt} = (4u_x + 1)^{-2}u_{xx} + (4u_x + 1)^{-1} + x$$

and its corresponding extension is

$$X_5 = -4x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial u}.$$

We prolong  $X_5$  to second order and obtain

$$X_5^2 = X_5 + (1 + 4u_x) \frac{\partial}{\partial u_x} - 2u_t \frac{\partial}{\partial u_t} + 8u_{xx} \frac{\partial}{\partial u_{xx}} - 4u_{tt} \frac{\partial}{\partial u_{tt}}.$$

$$X_5^2(u_{tt} - (4u_x + 1)^{-2}u_{xx} - (4u_x + 1)^{-1} - x) = -4[u_{tt} - (4u_x + 1)^{-1} - x] \neq 0.$$

Of all the twelve cases of subalgebras in the summary of one-dimensional optimal system it is only 4 and 5 which have equations admitting their corresponding extensions of the Principal Lie algebra. The two cases are taken further by using the similar procedure used in this chapter to find their invariant solutions.

# Summary

| Sub-algebra                    | Equation                                                                                                     |
|--------------------------------|--------------------------------------------------------------------------------------------------------------|
| $N_4$                          | $u_{tt} = u_x^{-2} u_{xx} + u_x^{-1} + x$                                                                    |
| $N_3 + \alpha N_4$             | $u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + u_x^{\frac{1-2\alpha}{2\alpha}} + x$                       |
| No results                     | No results                                                                                                   |
| $N_2 + \alpha N_3$             | $u_{tt} = e^{2\alpha u_x} u_{xx} + e^{\alpha u_x} + x$                                                       |
| $N_2$                          | $u_{tt} = C_1 u_{xx} + C_2 + x$                                                                              |
| $N_1 \pm N_3 + \alpha N_4$     | $u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha} + x$ |
| $N_2 + \alpha N_3 + 2N_4$      | $u_{tt} = (4u_x + 1)^{\alpha-4} u_{xx} + (4u_x + 1)^{\frac{\alpha-4}{2}} + x$                                |
| $N_2 + 2N_2$                   | $u_{tt} = (4u_x + 1)^{-2} u_{xx} + (4u_x + 1)^{-1} + x$                                                      |
| No results                     | No results                                                                                                   |
| $N_1 + \alpha N_4$             | $u_{tt} = u_x^{-2} u_{xx} + (\frac{1}{\alpha} - \frac{1}{\alpha u_x}) + x$                                   |
| $-N_2 + \alpha N_4$            | $u_{tt} = u_x^{-2} u_{xx} + (\frac{1}{\alpha u_x} - \frac{1}{\alpha}) + x$                                   |
| $N_1 + \alpha N_2 + \beta N_4$ | $u_{tt} = (2\beta u_x + \alpha)^{-2\beta} u_{xx} + \frac{1}{\beta} [1 - (2\beta u_x + \alpha)^{-1}] + x$     |
| $N_1 + \alpha N_2 + N_3$       | $u_{tt} = e^{\frac{2u_x}{\alpha}} u_{xx} + [e^{\frac{u_x}{\alpha}} - 2] + x$                                 |

| Equation                                                                                                     | $X_5$                                                                                                                        | admits     |
|--------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------|------------|
| $u_{tt} = u_x^{-2} u_{xx} + u_x^{-1} + x$                                                                    | $-2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}$                                                            | No         |
| $u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + u_x^{\frac{1-2\alpha}{2\alpha}} + x$                       | $(1-2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$            | No         |
| No results                                                                                                   | No results                                                                                                                   | No results |
| $u_{tt} = e^{2\alpha u_x} u_{xx} + e^{\alpha u_x} + x$                                                       | $\alpha x \frac{\partial}{\partial x} + (x + \alpha u) \frac{\partial}{\partial u}$                                          | Yes        |
| $u_{tt} = C_1 u_{xx} + C_2 + x$                                                                              | $x \frac{\partial}{\partial u}$                                                                                              | Yes        |
| $u_{tt} = u_x^{\frac{1-2\alpha}{\alpha}} u_{xx} + \frac{u_x^{\frac{1-2\alpha}{2\alpha}} - 2}{1-2\alpha} + x$ | $(1-2\alpha)x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + (t^2 + u) \frac{\partial}{\partial u}$    | No         |
| $u_{tt} = (4u_x + 1)^{\alpha-4} u_{xx} + (4u_x + 1)^{\frac{\alpha-4}{2}} + x$                                | $(\alpha-4)x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + (x + \alpha u) \frac{\partial}{\partial u}$      | No         |
| $u_{tt} = (4u_x + 1)^{-2} u_{xx} + (4u_x + 1)^{-1} + x$                                                      | $-4x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial u}$                           | No         |
| No results                                                                                                   | No results                                                                                                                   | No results |
| $u_{tt} = u_x^{-2} u_{xx} + (\frac{1}{\alpha} - \frac{1}{\alpha u_x}) + x$                                   | $-2\alpha x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial u}$            | No         |
| $u_{tt} = u_x^{-2} u_{xx} + (\frac{1}{\alpha u_x} - \frac{1}{\alpha}) + x$                                   | $-2\alpha x \frac{\partial}{\partial x} + \alpha t \frac{\partial}{\partial t} - t^2 \frac{\partial}{\partial u}$            | No         |
| $u_{tt} = (2\beta u_x + \alpha)^{-2\beta} u_{xx} + \frac{1}{\beta} [1 - (2\beta u_x + \alpha)^{-1}] + x$     | $-2\beta x \frac{\partial}{\partial x} + \beta t \frac{\partial}{\partial t} + (t^2 + \alpha x) \frac{\partial}{\partial u}$ | No         |
| $u_{tt} = e^{\frac{2u_x}{\alpha}} u_{xx} + [e^{\frac{u_x}{\alpha}} - 2] + x$                                 | $-2\beta x \frac{\partial}{\partial x} + \beta t \frac{\partial}{\partial t} + (x + t^2) \frac{\partial}{\partial u}$        | No         |

## Chapter 3

### Some invariant solutions

#### 3.1 Solutions of case 1.

In this chapter we determine the invariant solutions of the two distinct waves obtained from chapter 2. Let us focus attention on case 1 which has the equation:

$$u_{tt} = C_1 u_{xx} + C_2 + x \quad (3.1)$$

Consider equation (3.1). It admits a five dimensional Lie algebra  $L_5$ , which is generated by

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t \frac{\partial}{\partial u}, \quad X_4 = 2 \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial u}, \quad X_5 = x \frac{\partial}{\partial u}. \quad (3.2)$$

The commutator table is given by:

| $[,]$ | $X_1$ | $X_2$   | $X_3$ | $X_4$   | $X_5$  |
|-------|-------|---------|-------|---------|--------|
| $X_1$ | 0     | 0       | 0     | 0       | 0      |
| $X_2$ | 0     | 0       | $X_1$ | $2X_3$  | 0      |
| $X_3$ | 0     | $-X_1$  | 0     | 0       | 0      |
| $X_4$ | 0     | $-2X_3$ | 0     | 0       | $2X_1$ |
| $X_5$ | 0     | 0       | 0     | $-2X_1$ | 0      |

We now determine the generators of the adjoint algebra. The generators of the

adjoint algebra are:

$$\begin{aligned}
A_1 &= 0, \\
A_2 &= X_1 \frac{\partial}{\partial X_3} + 2X_3 \frac{\partial}{\partial X_4}, \\
A_3 &= -X_1 \frac{\partial}{\partial X_2}, \\
A_4 &= -2X_3 \frac{\partial}{\partial X_2} + 2X_1 \frac{\partial}{\partial X_5}, \\
A_5 &= -2X_1 \frac{\partial}{\partial X_4}.
\end{aligned} \tag{3.3}$$

These generators provides the following adjoint-group elements :

$$\begin{aligned}
A_1 : \quad \overline{X}_1 &= X_1, \quad \overline{X}_2 = X_2, \quad \overline{X}_3 = X_3, \quad \overline{X}_4 = X_4, \quad \overline{X}_5 = X_5, \\
A_2 : \quad \overline{X}_1 &= X_1, \quad \overline{X}_2 = X_2, \quad \overline{X}_3 = X_1 a_2 + X_3, \quad \overline{X}_4 = X_1 a_2^2 + 2X_3 a_2 + X_4, \quad \overline{X}_5 = X_5, \\
A_3 : \quad \overline{X}_1 &= X_1, \quad \overline{X}_2 = -X_1 a_3 + X_2, \quad \overline{X}_3 = X_3, \quad \overline{X}_4 = X_4, \quad \overline{X}_5 = X_5, \\
A_4 : \quad \overline{X}_1 &= X_1, \quad \overline{X}_2 = -2X_3 a_2 + X_2, \quad \overline{X}_3 = X_3, \quad \overline{X}_4 = X_4, \quad \overline{X}_5 = 2X_1 a_4 + X_5, \\
A_5 : \quad \overline{X}_1 &= X_1, \quad \overline{X}_2 = X_2, \quad \overline{X}_3 = X_3, \quad \overline{X}_4 = -2X_1 a_5 + X_4, \quad \overline{X}_5 = X_5.
\end{aligned}$$

Let us now determine the one-dimensional optimal system of case 1. We define

$$\bar{e} = Me,$$

where  $e = (e^1, e^2, e^3, e^4, e^5)$  in this case.

$$\begin{aligned}
\bar{e}^1 &= e^1 - a_3 e^2 + a_2 e^3 + [a_2^2 - 2a_5] e^4 + 2a_4 e^5, \\
\bar{e}^2 &= e^2, \\
\bar{e}^3 &= e^3 + 2a_2 e^4, \\
\bar{e}^4 &= e^4, \\
\bar{e}^5 &= e^5.
\end{aligned} \tag{3.5}$$

There are nine subcases obtained from case 1. Using the algorithm used in chapter 2 we obtain the following one-dimensional optimal system.

$$X^{(1)} : X_1,$$

$$\begin{aligned}
X^{(2)} &: X_3, \\
X^{(3)} &: X_5, \\
X^{(4)} &: \alpha X_3 + X_5, \\
X^{(5)} &: \alpha X_1 + \beta X_4 + X_5, \\
X^{(6)} &: \alpha X_4 + X_5, \\
X^{(7)} &: \alpha X_2 + X_5, \\
X^{(8)} &: \alpha X_1 + \beta X_2 + X_5, \\
X^{(9)} &: \alpha X_2 + \beta X_3 + X_5, \\
X^{(10)} &: \alpha X_2 + X_4, \\
X^{(11)} &: \alpha X_1 + \beta X_2 + X_4, \\
X^{(12)} &: X_4, \\
X^{(13)} &: \alpha X_1 + X_4, \\
X^{(14)} &: \alpha X_2 + X_3, \\
X^{(15)} &: X_2, \\
X^{(16)} &: \alpha X_2 + \beta X_4 + X_5.
\end{aligned} \tag{3.6}$$

We determine the invariant solutions of the sixteen above-mentioned cases. Of all the sixteen cases only nine yields solutions. The reason being that all generators that have

$$\frac{\partial}{\partial u}$$

as a generator cannot workout. The nine cases will be identified and solutions will be provided. It will be of great importance to outline the algorithm used to obtain the overall exact solution of the equation we considered, for the first case and the rest solutions will just be provided. **subcase 1**

$$X^{(5)} : (\alpha X_1 + \beta X_4 + X_5)I = 0$$

The invariants obtained are

$$I_1 = t$$

and

$$I_2 = u - \frac{\alpha + \beta t^2}{2\beta}x - \frac{x^2}{4\beta}.$$

We then use the equation,

$$I_2 = \phi(I_1). \quad (3.7)$$

To obtain the general solution:

$$u = \phi(t) + \frac{\alpha + \beta t^2}{2\beta}x + \frac{x^2}{4\beta}.$$

We substitute in equation (3.1)

$$u_t = \phi'(t) + tx, \quad u_x = \frac{\alpha + \beta t^2}{2\beta} + \frac{x}{2\beta}$$

$$u_{tt} = \phi''(t) + x, \quad u_{xx} = \frac{1}{2\beta}.$$

Substituting in the considered equation, we obtain the solution

$$\phi(t) = \left[ \frac{C_1}{2\beta} + C_2 \right] \frac{t^2}{2} + C_3 t + C_4.$$

Thefore the solution will be:

$$u(t, x) = \left[ \frac{C_1}{2\beta} + C_2 \right] \frac{t^2}{2} + C_3 t + C_4 + \frac{\alpha + \beta t^2}{2\beta}x + \frac{x^2}{4\beta}.$$

**Subcase 2**

$$X^{(6)} : (\alpha X_4 + X_5)I = 0,$$

The solution will be given by :

$$u(t, x) = \left[ \frac{C_1}{2\alpha} + C_2 + x \right] \frac{t^2}{2} + C_3 t + \frac{x^2}{4\alpha} + C_4.$$

**Subcase 3**

$$X^{(7)} : (\alpha X_2 + X_5)I = 0,$$

the solution is

$$u(t, x) = -\frac{C_2 x^2}{2C_1} - \frac{x^3}{6C_1} + C_3 x + C_4 + \frac{xt}{\alpha}$$

**Subcase 4**

$$X^{(8)} : (\alpha X_1 + \beta X_2 + X_5)I = 0,$$

we obtain the exact solution

$$u(t, x) = -\frac{C_2 x^2}{2C_1} - \frac{x^3}{6C_1} + C_3 x + C_4 + \left[\frac{\alpha + x}{\beta}\right]t$$

**subcase 5**

$$X^{(9)} : (\alpha X_2 + \beta X_3 + X_5)I = 0,$$

the solution is given by:

$$u(t, x) = -\left[C_2 + \frac{x}{3} - \frac{\beta}{\alpha}\right]\frac{x^2}{2C_1} + \left[C_3 + \frac{t}{\alpha}\right]x + \frac{\beta t^2}{2\alpha} + C_4.$$

**Subcase 6**

$$X^{(15)} : X_2 I = 0,$$

the solution is :

$$u(t, x) = -\left[C_2 + \frac{x}{3}\right]\frac{x^2}{2C_1} + C_3 x + C_4.$$

**Subcase 7**

$$X^{(14)} : (\alpha X_2 + X_3)I = 0,$$

the solution is given by :

$$u(t, x) = -\left[C_2 + \frac{x}{3} - \frac{1}{\alpha}\right]\frac{x^2}{2C_1} + C_3 x + \frac{t^2}{2\alpha} + C_4.$$

**Subcase 8**

$$X^{(13)} : (\alpha X_1 + X_4)I = 0,$$

we obtain the following solution:

$$u(t, x) = \frac{C_2 t^2}{2} + \left[\frac{\alpha + t^2}{2}\right]x + C_3 t + C_4.$$

**Subcase 9**

$$X^{(12)} : X_4 I = 0,$$

the solution is given by:

$$u(t, x) = \left[C_2 + x\right]\frac{t^2}{2} + C_3 t + C_4.$$

### 3.2 Solutions of case 2

We now focus attention on the equation:

$$u_{tt} = e^{2\alpha u} u_{xx} + e^{\alpha u} + x. \quad (3.8)$$

Equation (3.8) admits the same generators

$$X_1, X_2, X_3$$

and

$$X_4$$

in case 1, in addition to that it admits the extension:

$$X_5 = x \frac{\partial}{\partial x} + (x + u) \frac{\partial}{\partial u}$$

The commutator table is given by:

| $[\cdot, \cdot]$ | $X_1$         | $X_2$   | $X_3$         | $X_4$                  | $X_5$               |
|------------------|---------------|---------|---------------|------------------------|---------------------|
| $X_1$            | 0             | 0       | 0             | 0                      | $\alpha X_1$        |
| $X_2$            | 0             | 0       | $X_1$         | $2X_3$                 | 0                   |
| $X_3$            | 0             | $-X_1$  | 0             | 0                      | $\alpha X_3$        |
| $X_4$            | 0             | $-2X_3$ | 0             | 0                      | $\alpha X_4 + 2X_1$ |
| $X_5$            | $-\alpha X_1$ | 0       | $-\alpha X_3$ | $-(\alpha X_4 + 2X_1)$ | 0                   |

(3.8)

The adjoint algebra of case 2 is given by:

$$\begin{aligned}
 A_1 &= \alpha X_1 \frac{\partial}{\partial X_5}, \\
 A_2 &= X_1 \frac{\partial}{\partial X_3} + 2X_3 \frac{\partial}{\partial X_4}, \\
 A_3 &= -X_1 \frac{\partial}{\partial X_2} + \alpha X_3 \frac{\partial}{\partial X_5}, \\
 A_4 &= -2X_3 \frac{\partial}{\partial X_4} + (\alpha X_4 + 2X_1) \frac{\partial}{\partial X_5}, \\
 A_5 &= -[\alpha X_1 \frac{\partial}{\partial X_1} + \alpha X_3 \frac{\partial}{\partial X_3} + (\alpha X_4 + 2X_1) \frac{\partial}{\partial X_4}].
 \end{aligned} \quad (3.10)$$

From the adjoint algebra above we obtain the inner-automorphism also known as the adjoint group. The inner-automorphism of the above adjoint algebra will be given by:

$$\begin{aligned}
A_1 : \bar{X}_1 &= X_1, \quad \bar{X}_2 = X_2, \quad \bar{X}_3 = X_3, \quad \bar{X}_4 = X_4, \quad \bar{X}_5 = \alpha X_1 a_1 + X_5, \\
A_2 : \bar{X}_1 &= X_1, \quad \bar{X}_2 = X_2, \quad \bar{X}_3 = X_1 a_2 + X_3, \quad \bar{X}_4 = X_1 a_2^2 + 2X_3 a_2 + X_4, \quad \bar{X}_5 = X_5, \\
A_3 : \bar{X}_1 &= X_1, \quad \bar{X}_2 = -X_1 a_3 + X_2, \quad \bar{X}_3 = X_3, \quad \bar{X}_4 = X_4, \quad \bar{X}_5 = \alpha X_3 a_3 + X_5, \\
A_4 : \bar{X}_1 &= X_1, \quad \bar{X}_2 = -2X_3 a_4 + X_2, \quad \bar{X}_3 = X_3, \quad \bar{X}_4 = X_4, \quad \bar{X}_5 = \alpha X_4 a_4 + 2X_1 a_4 + X_5, \\
A_5 : \bar{X}_1 &= \alpha X_1 a_5^{-1}, \quad \bar{X}_2 = X_2, \quad \bar{X}_3 = \alpha X_3 a_5^{-1}, \quad \bar{X}_4 = -2X_1 + (\alpha X_4 + 2X_1) a_5^{-1}, \quad \bar{X}_5 = X_5.
\end{aligned} \tag{3.11}$$

We now obtain the vector  $e$  transposed.

$$\begin{aligned}
\bar{e}^1 &= \alpha a_5^{-1} e^1 \\
\bar{e}^2 &= -\alpha a_3 a_5^{-1} e^1 + e^2 - 2\alpha a_4 a_5^{-1} e^3 \\
\bar{e}^3 &= \alpha a_2 a_5^{-1} e^1 + \alpha a_5^{-1} e^3 \\
\bar{e}^4 &= [\alpha a_2^2 a_5^{-1} + (-2 + 2a_5^{-1})] e^1 + 2\alpha a_2 a_5^{-1} e^3 + \alpha a_5 e^4 \\
\bar{e}^5 &= [\alpha^2 (a_1 + 2\alpha a_4) a_5^{-1} + [-2\alpha + 2\alpha a_5^{-1} a_4]] e^1 + \alpha^2 a_3 a_5^{-1} e^3 + \alpha^2 a_4 a_5^{-1} e^4 + e^5.
\end{aligned} \tag{3.12}$$

subcase 1

$$Z^{(1)} : \alpha X_2 + X_4$$

Subcase 2

$$Z^{(2)} : \gamma X_2 + X_5, \gamma \neq 0$$

Subcase 3

$$Z^{(3)} : X_2$$

subcase 4

$$Z^{(4)} : X_4.$$

Subcases 1 and 2 do not yield meaningfull results for the equation in discussion . We will however focus on subcases 3 and 4. Let us consider subcase 4

$$X_4 I = 0.$$

Note that

$$X_4 = 2 \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial u}.$$

The invariants are

$$I_1 = t,$$

and

$$I_2 = u - \frac{t^2}{2}x.$$

Using the equation

$$I_2 = \phi(I_1),$$

we obtain the general solution:

$$u = \phi(t) + \frac{t^2}{2}x$$

We determine the function  $\phi$  by substituting the general solution into the equation (3.8), we obtain a second order ordinary differential equation:

$$\phi''(t) = e^{\frac{\alpha t^2}{2}}.$$

Therefore

$$\phi(t) = \int (dt) \int (e^{\frac{\alpha t^2}{2}} dt) + C_1 t + C_2.$$

Finally the invariant solution is given by:

$$u(t, x) = \int (dt) \int (e^{\frac{\alpha t^2}{2}} dt) + C_1 t + C_2 + \frac{t^2}{2}x.$$

**Subcase 3**

$$X_2 I = 0,$$

and

$$X_2 = \frac{\partial}{\partial t}.$$

Using the similar algorithm as the one used in the above case we arrive at the general solution

$$u = \varphi(x),$$

then we arrive at the second order Ordinary differential equation

$$\varphi''(x) = -e^{-\alpha\varphi'(x)} - x e^{-2\alpha\varphi'(x)}. \quad (3.13)$$

The above second order ordinary differential equation admits the following generators/algebra

$$X_1 = \frac{\partial}{\partial \varphi}$$

and

$$X_2 = x \frac{\partial}{\partial x} + (x + \varphi) \frac{\partial}{\partial \varphi}$$

The Lie bracket of the two generators is given by

$$[X_1, X_2] = X_1$$

Incorporating the reduction of order by method of canonical variables in [4]. We now have

$$\bar{x} = x, \bar{\varphi} = c\varphi$$

The generator  $X_2$  becomes

$$X'_2 = x \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$$

upon changing the variables  $z = \varphi'$ , and is admitted by the equation (3.14)

$$z' = -e^{-\alpha z} - x e^{-2\alpha z} \quad (3.14)$$

### 3.3 Conclusion

A non-linear wave equation (3.1) has some invariant solutions as shown in this paper. The method of preliminary group classification as used above is inadequate. We have seen instances where the one-dimensional extension is not a symmetry of the equation. This method yields symmetries which may not possibly be the extension. The remaining thirty three families of equations as determined by Ibragimov, Torrisi and Valenti [3] need to be explored further. Other types of symmetries could possibly be used in future.

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