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Abundant invariant and classical solutions with the conservation laws of a new (3+1)-dimensional fifth-order nonlinear Wazwaz equation with the third-order dispersion terms in ocean physics

Oke Davies Adeyemo *, Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, Republic of South Africa

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ABSTRACT

Ocean plays key roles in the earth's system in shaping the planet's climate as well as weather by absorbing, storing, and also transporting huge quantities of carbon-dioxide, water, heat and moisture. This paper presents the analytical examination of a new (3+1)-dimensional fifth-order nonlinear Wazwaz equation with third-order dispersion terms which is applicable in ocean physics and other nonlinear sciences. Ocean physics is the study of various conditions as well as processes that are physical within the ocean, especially the physical features and motions of ocean waters. Thus, the diverse outcomes of the investigations carried out on the model under study possesses significant applications in physical oceanography (Adeyemo, 2022). We first apply Lie group analysis to obtain various infinitesimal generators admitted by the equation. This is followed by invoking the generators to reduce the understudy equation for the purpose of achieving copious group-invariant solutions. We solve the resultant ordinary differential equations in order to obtain diverse classical solutions of the underlying equation. The outcome yields several solutions with arbitrary functions, quadratures, rational and hyperbolic functions alongside various topological kink solitons. Moreover, numerical simulations of the obtained results are performed so as to give some physical interpretations to the solutions and these yield various solitary waves as well as interactions between soliton waves of interest. Furthermore, we outline the applications of our results in ocean physics. Conclusively, conservation laws are constructed for the equation understudy with the use of Ibragimov's theorem.

1. Introduction

Lately, investigations have been intensified in examining nonlinear partial differential equations (NLNPDEs) alongside their analytic travelling wave solutions due to the fact that many physical phenomena are represented using these NLNPDEs. Nonlinearity is a captivating and enthralling element of nature. Scientists in their numbers have deemed it fit to contemplate nonlinear science as the most significant frontier for the fundamental comprehension of nature. Some of these models include a three-coupled variable-coefficient nonlinear Schrödinger system which is a nonlinear Schrödinger-type models that are usually found in fibre optics and other fields inclusive of plasma physics, ferromagnetism, Bose–Einstein condensation as well as oceanography in [1] which was studied. In [2], we have the authors study the generalized advection–diffusion equation which is a nonlinear partial differential equation in fluid mechanics, characterizing the motion of buoyancy propelled plume in a bent-on absorptive medium. Moreover,

* Corresponding author.
E-mail addresses: adeyemodaviz@gmail.com (O.D. Adeyemo), Masood.Khalique@nwu.ac.za (C.M. Khalique).

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in [3], Adeyemo studied a two-dimensional quantum Zakharov–Kuznetsov model which is derived based on an isotropic nonlinear evolution equation. The evolution model was first established for waves that are weakly nonlinear ion-acoustic that is existing in a bi-dimensional strongly magnetized plasma discovered to be lossless. Thus, the model is applicable in plasma physics. Moreover, in [4], a generalized Korteweg–de Vries–Zakharov–Kuznetsov equation was studied. This equation delineates mixtures of warm adiabatic fluid, hot isothermal as well as cold immobile background species applicable in fluid dynamics. Oceanic water-wave studies are carried out by modelling the nonlinear dispersive long gravity waves that is existing in two horizontal directions on shallow water. This is further observed on an open sea which can also possess a wide channel of finite depth, using a generalized (2 + 1)-dimensional dispersive long-wave system investigated in [5]. Moreover, in [6] waves are discovered in the atmosphere and oceans, to say the least. Water waves have been a source of attraction for researchers to investigate. Thus, in [6], a (3 + 1)-dimensional generalized nonlinear evolution equation was examined. In addition, the variable-coefficient modified Burgers–Korteweg–De Vries equation, arising when various physical phenomena in physics and diverse nonlinear sciences are modelled, is studied in [7]. the list goes on and on, see more in [7–28]. Besides, physical oceanography utilizes the laws of physics in examining both the structure as well as dynamics of ocean circulation alongside properties of water, waves, water mass formation, turbulence, tides and other physical phenomena.

It is revealed that no general and well-structured technique existed in gaining various analytic travelling wave solutions of nonlinear partial differential equations. However, various sound and efficient techniques had arisen lately to secure a lasting solution to this seemingly never-ending problem. In consequence, we present some of these techniques as $\exp(-\Phi(\eta))$-expansion technique [10], Painlevé expansion [11], bifurcation technique [12], ansatz technique [13], homotopy perturbation technique [14], extended homoclinic test approach [15], tanh-coth approach [16], generalized unified technique [17], Adomian decomposition approach [18], Cole-Hopf transformation approach [19], Bäcklund transformation [20], F-expansion technique [21], rational expansion technique [22], tan-cot method [23], Lie symmetry analysis [24,25], Hirota technique [26], extended simplest equation method [27], Darboux transformation [28], the $\left(G''/G\right)$-expansion technique [29], tanh-function technique [30], Kudryashov technique [31], sine-Gordon equation expansion technique [32] and exponential function technique [33], to name a few.

The Lie symmetry method [24,25] which is a robust technique useful in achieving analytic solutions of differential equations in a systematic structure, was pioneered by Marius Sophus Lie, who was a Norwegian mathematician. Sophus Lie established the theory of continuous symmetry and engaged it in his investigation of geometry along with differential equations. Besides, Lie to a large extent contributed to the continuous groups of transformations, and algebraic invariants theories with differential equations. Eventually, Lie algebras as well as Lie groups are named after him. Moreover, Sophus Lie did institute the theory of continuous symmetry and also utilized it to examine structures of the geometry of differential equations.

In particular, it is noteworthy to highlight the fact that soliton solutions are most astonishing in the investigation of the nonlinear physical sciences. For instance, the wave phenomena are perceived in biophysics, fluid mechanics, high-energy physics, chemical kinematics, and optical fibres to mention a few. However, nonlinear processes pose some basic challenges and are not readily easy to control, due to the fact that the nonlinear feature of the system alters sharply due to small changes in valid parameters involving time. Hence, the matter becomes more convoluted, thus compelling the ultimate solution. Solitary wave solutions to the nonlinear partial differential equation are progressively becoming an enticing field in nonlinear sciences each passing day.

In recent times, a variety of integrable systems having diverse temporal as well as spatial dimensions have been developed. Integrable models which were newly discovered have been thoroughly utilized in various applications in science alongside ocean engineering, see [34–36]. It is germane that nonlinear integrable models appear with terms u_t or u_{tt} as temporal derivatives in the first and second-order [37–39]. This is revealed in the most general fifth-order KdV equation that reads [40]

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + u_{xxxxx} = 0 \tag{1}$$

with arbitrary real constants a, b and c. The model (1) delineates long waves which are propagated in shallow water with the existence of gravity. Diverse other famous fifth-order KdV equations ensue from (1) such as the fifth-order Sawada–Kotera equation given as [41]

$$u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0 \tag{2}$$

that arises when $a = 3b$, $b = c$ where $c = 15$. Besides, when $a = 5/2b$, with $b = 5/4c$ alongside $c = 20$, one arrives at the nonlinear fifth-order Kaup–kupershmidt equation presented as [42]

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0. \tag{3}$$

However, integrable systems that are possessive of temporal dispersion which is third-order are rarely investigated. These types of models were recently examined by Wazwaz [34,35], Ashmead [43] as well as Schelte et al. [38]. In [43], the author has recounted that in relativity, space alongside time came in based on formal equivalence. In addition to that, the coordinates of these space and time under Lorentz transformations [43] rotate into each other. Schelte et al. [38] also studied the time-delayed systems having a third-order dispersion where the second-order diffusive parameter is absent. Moreover, the author in [43] reported that the appearance of time dispersion is founded on the same basis as that of spatial dispersion.

A new fifth-order nonlinear integrable equation developed by Wazwaz [34,35] involves a third-order time dispersion and it reads

$$u_{ttt} - 4u_tu_{xxx} - 8u_xu_{txx} - 12u_{xx}u_{tx} - u_{txxxx} = 0. \tag{4}$$

The author studied three-soliton solutions to the equation. Besides, the single-soliton solution constructed for (4), as well as the phase, shifts a_{ij} invoked in deriving two- and three-soliton solutions were presented respectively in the structure

$$u(t,x) = \frac{4k_1 e^{k_1 x \pm k_1^2 t}}{1 + e^{k_1 x \pm k_1^2 t}}, \text{ and } a_{ij} = \frac{(k_i - k_j)^2}{k_i^2 + k_j^2}, \ 1 \le i < j \le 3. \tag{5}$$

The solution $u(t,x)$ has been achieved after utilizing the dispersion relations $c_i = \pm k_i^2, i = 1,2,3$ with respect to the phase shifts a_{ij}. The fifth-order (4) with third-order dispersion is possessive of potential applications and has also gained the massive attention of scholars [36–39]. Furthermore, several other analytic solutions consisting of multiple solitons have been secured through the use of Painlevé analysis, copious direct algebraic methods along with Lie symmetry technique [44–46]. Interesting to note is the extension of the model (4) to other systems with the addition of more dispersive terms as we can see in [47]. Nevertheless, in [48], a new fifth-order integrable equation

$$u_{ttt} - 4u_t u_{xxx} - 8u_x u_{txx} - 12u_{xx}u_{tx} - u_{txx} - u_{txxxx} = 0 \tag{6}$$

with the introduction of new term u_{txx} to the standard equation (4). The simplified Hirota approach was implored in [48] to gain multiple soliton solutions via non-typical phase shifts.

1.1. The governing equation

In this paper, we consider the investigation of a (3 + 1)-dimensional fifth-order nonlinear Wazwaz equation (3D-WAZWAZe) with third-order dispersions given as [49]

$$u_{ttt} + \alpha u_{xxx} + \beta u_{xxy} + \gamma u_{xxz} - 4u_t u_{xxx} - 12u_{tx}u_{xx} - 8u_x u_{txx} - u_{txxxx} = 0 \tag{7}$$

with temporal dispersion term u_{ttt} and spatial dispersion terms u_{xxx}, u_{xxy} and u_{xxz}. The addition of the spatial dispersion terms to the standard equation (4) gives us the new model (7) with the real-valued constant coefficients α, β and γ. The new fifth-order (7) in three dimensions was investigated by the author who conducted the Painlevé test on it. This is done with a view to first showing that the integrability property holds for the new equation with third-order temporal as well as spatial dispersions. Thereafter, he further applied the simplified structure of Hirota's bilinear technique in constructing multiple soliton solutions of 3D-WAZWAZe (7). On a final note, he examined some particular cases of dispersion relations via the utilization of various coefficients of the spatial variables. In addition, a variety of distinct phase shifts were formally derived for the equation. Therefore, for the first time, we secure various soliton solutions such as several topological kinks, singular soliton, as well as diverse kinds of hyperbolic and algebraic results with arbitrary functions of note for high-dimensional equation (7), using the method of Lie theory of differential equations whose robust application in securing closed-form solutions of differential equations has been mentioned earlier on. In consequence, the novelty of the various results achieved in the research is guaranteed.

The research work is planned in this article as follows. In Section 2, we carry out Lie group analysis of 3D-WAZWAZe (7) systematically and the found results are utilized in the reduction of the equation. Section 3 presents the wave depictions and analysis of solutions. Further, in Section 4, we derive the conservation laws of (7) through the use of Ibragimov's theorem. In Section 5, we outline the application of our results in physics and this is followed by concluding remarks in Section 6.

2. Lie group analysis and solutions of (7)

In this section we first compute the Lie point symmetries of (7) and thereafter use them to construct analytic solutions.

2.1. Lie point symmetries of (7)

We contemplate an infinitesimal dimensional Lie algebra that is spanned by the vector fields declared as

$$\begin{aligned} Q = {} & \xi^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,u)\frac{\partial}{\partial z} \\ & + \eta(t,x,y,z,u)\frac{\partial}{\partial u}, \end{aligned}$$

with the vector field generating the symmetries of 4D-gBSe (7) and Q fulfilling Lie symmetry criteria

$$pr^{(5)}Q[u_{ttt} + \alpha u_{xxx} + \beta u_{xxy} + \gamma u_{xxz} - 4u_t u_{xxx} - 12u_{tx}u_{xx} - 8u_x u_{txx} - u_{txxxx}] = 0, \tag{8}$$

whenever $u_{ttt} + \alpha u_{xxx} + \beta u_{xxy} + \gamma u_{xxz} - 4u_t u_{xxx} - 12u_{tx}u_{xx} - 8u_x u_{txx} - u_{txxxx} = 0$, where $pr^{(5)}Q$ is regarded as the fifth prolongation of Q. Related formula for fifth prolongation $pr^{(5)}Q$ [25] is

$$\begin{aligned} pr^{(5)}Q = {} & Q + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \eta^{tx}\partial_{u_{tx}} + \eta^{xx}\partial_{u_{xx}} + \eta^{ttt}\partial_{u_{ttt}} + \eta^{txx}\partial_{u_{txx}} + \eta^{xxx}\partial_{u_{xxx}} \\ & + \eta^{xxy}\partial_{u_{xxy}} + \eta^{xxz}\partial_{u_{xxz}} + \eta^{txxxx}\partial_{u_{txxxx}}. \end{aligned} \tag{9}$$

Application of fifth prolongation $pr^{(5)}Q$ to Eq. (7), the corresponding invariant conditions is expressed as

$$\begin{aligned} & \eta^{ttt} + \alpha\eta^{xxx} + \gamma\eta^{xxz} - 4u_t\eta^{xxx} - 4u_{xxx}\eta^t - 12u_{tx}\eta^{xx} - 12u_{xx}\eta^{tx} - 8u_x\eta^{txx} \\ & + \beta\eta^{xxy} - 8u_{txx}\eta^x - \eta^{txxxx} = 0, \end{aligned} \tag{10}$$

with the coefficients of $pr^{(5)}Q$, that is, η^t, η^x, η^{xx}, η^{tx}, η^{ttt}, η^{txx}, η^{xxx}, η^{xxy}, η^{xxz}, and η^{txxxx} given as

$$\begin{aligned}
\eta^t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3) - u_z D_t(\xi^4),\\
\eta^x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4),\\
\eta^{tx} &= D_x(\eta^t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - u_{yt} D_x(\xi^3) - u_{zt} D_x(\xi^4),\\
\eta^{xx} &= D_x(\eta^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4),\\
\eta^{txx} &= D_x(\eta^{tx}) - u_{ttx} D_x(\xi^1) - u_{txx} D_x(\xi^2) - u_{txy} D_x(\xi^3) - u_{txz} D_x(\xi^4),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{txx} D_x(\xi^1) - u_{xxx} D_x(\xi^2) - u_{xxy} D_x(\xi^3) - u_{xxz} D_x(\xi^4),\\
\eta^{xxy} &= D_x(\eta^{xy}) - u_{txy} D_x(\xi^1) - u_{xxy} D_x(\xi^2) - u_{xyy} D_x(\xi^3) - u_{xyz} D_x(\xi^4),\\
\eta^{xxz} &= D_x(\eta^{xz}) - u_{txz} D_x(\xi^1) - u_{xxz} D_x(\xi^2) - u_{xyz} D_x(\xi^3) - u_{xzz} D_x(\xi^4),\\
\eta^{txxxx} &= D_x(\eta^{txxx}) - u_{ttxxx} D_x(\xi^1) - u_{txxxx} D_x(\xi^2) - u_{txxxy} D_x(\xi^3) - u_{txxxz} D_x(\xi^4).
\end{aligned} \tag{11}$$

We have the involved total derivatives with regards to t and x expressed as [24,25,50]

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{ty}\frac{\partial}{\partial u_y} + u_{tz}\frac{\partial}{\partial u_z} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + u_{xz}\frac{\partial}{\partial u_z} + \cdots.
\end{aligned} \tag{12}$$

Applying Eqs. (11) in (10), equating the differential coefficients of u alongside its derivatives to zero and consequently solving the determining equations associated with (7) using SYM, a Mathematica-based computer software, we secure the system of linear partial differential equations (LNPDE)

$$\begin{aligned}
&\eta_{ttt} = 0, \xi^1_{tt} = 0,\ \ \xi^1_{tx} = 0,\ \ \xi^1_{tu} = 0,\ \ \xi^4_t = 0,\ \ \xi^3_t = 0,\ \ \xi^2_t = 0,\ \ \eta_{xx} = 0,\\
&\xi^4_x = 0,\ \ \xi^3_x = 0,\ \ \xi^1_x = 0,\ \ \xi^4_u = 0,\ \ \xi^3_u = 0,\ \ \xi^2_u = 0,\ \ \xi^1_u = 0,\ \ \xi^1_t + 2\eta_u = 0,\\
&2\xi^2_x - \xi^1_t = 0,\ \ 8\eta_{tx} + \beta\xi^1_{ty} + \gamma\xi^1_{tz} = 0,\ \ 8\eta_x + \beta\xi^1_y + \gamma\xi^1_z = 0,\\
&8\eta_t - 3\alpha\xi^1_t + 2\beta\xi^2_y + 2\gamma\xi^2_z = 0,\ \ \beta\left(2\xi^2_x - 3\xi^1_t + \xi^3_y\right) + \gamma\xi^3_z = 0,\\
&3\gamma\xi^1_t - 2\gamma\xi^2_x - \beta\xi^4_y - \gamma\xi^4_z = 0,\ \ 2\beta\eta_{xyu} + 2\gamma\eta_{xzu} + 5\alpha\xi^2_{xxx} + \beta\xi^2_{xxy} + \gamma\xi^2_{xxz} = 0.
\end{aligned} \tag{13}$$

The solution of the system (13) gives infinitesimal generators of one-parameter Lie group of symmetries for Eq. (7) as found below;

$$\begin{aligned}
&\xi^1 = F^1(y,z) + \frac{t}{2\beta}\left(\beta F^2_y + \gamma F^2_z\right),\ \ \xi^2 = F^4(y,z) + \frac{x}{4\beta}\left(\beta F^2_y + \gamma F^2_z\right),\\
&\xi^3 = F^2(y,z),\ \ \xi^4 = \frac{\gamma}{\beta}F^2(y,z) + F^5\left(z - \frac{\gamma y}{\beta}\right)\\
&\eta = \frac{1}{16}\Bigg\{16F^3(y,z) - \frac{tx}{\beta}\left[\beta\left(2\gamma F^2_{yz} + \beta F^2_{yy}\right) + \gamma^2 F^2_{zz}\right] + t\Bigg(3\alpha F^2_y - 4\beta F^4_y + \frac{3\alpha\gamma}{\beta}F^2_z\\
&\qquad - 4\gamma F^4_z\Bigg) - \frac{4}{\beta}\left(\beta F^2_y + \gamma F^2_z\right)u - 2x\left(\beta F^1_y + \gamma F^1_z\right)\Bigg\},
\end{aligned}$$

where functions $F^i, i = 1,\dots,5$ depending on their various arguments are arbitrary. Now, by taking $F^1 = \mathbf{c}_1 y + \mathbf{c}_2 z + \mathbf{c}_3$, $F^2 = \mathbf{c}_4 y + \mathbf{c}_5 z + \mathbf{c}_6$, $F^3 = \mathbf{c}_7$, $F^4 = \mathbf{c}_8 y + \mathbf{c}_9 z + \mathbf{c}_{10}$ and $F^5 = \mathbf{c}_{11}$ with arbitrary constants $\mathbf{c}_k, k = 1,\dots,11$, we achieve an eleven-dimensional Lie algebra of infinitesimal symmetries of Eq. (7) spanned by the vector fields presented as

$$\begin{aligned}
&Q_1 = \frac{\partial}{\partial t},\quad Q_2 = \frac{\partial}{\partial x},\quad Q_3 = \frac{\partial}{\partial y},\quad Q_4 = \frac{\partial}{\partial z},\quad Q_5 = \frac{\partial}{\partial u},\\
&Q_6 = 4y\frac{\partial}{\partial x} - \beta t\frac{\partial}{\partial u},\quad Q_7 = 4z\frac{\partial}{\partial x} - \gamma t\frac{\partial}{\partial u},\\
&Q_8 = 8y\frac{\partial}{\partial t} - \beta x\frac{\partial}{\partial u},\quad Q_9 = 8z\frac{\partial}{\partial t} - \gamma x\frac{\partial}{\partial u},\\
&Q_{10} = 8\beta t\frac{\partial}{\partial t} + 4\beta x\frac{\partial}{\partial x} + 16\beta y\frac{\partial}{\partial y} + 16\gamma y\frac{\partial}{\partial z} + (3\alpha\beta t - 4\beta u)\frac{\partial}{\partial u},\\
&Q_{11} = 8\gamma t\frac{\partial}{\partial t} + 4\gamma x\frac{\partial}{\partial x} + 16\beta z\frac{\partial}{\partial y} + 16\gamma z\frac{\partial}{\partial z} + (3\alpha\gamma t - 4\gamma u)\frac{\partial}{\partial u}.
\end{aligned} \tag{14}$$

Thus, we give the succeeding theorem.

Theorem 2.1. *The fifth-order nonlinear spatially (3 + 1)-dimensional Wazwaz equation (7) admits a Lie algebra of eleven dimensions spanned by vectors $Q_1,\dots,Q_{11}$ given in (14).*

2.2. Lie symmetry reduction and invariant solutions of (7)

In this segment of the study, we present copious reductions of 3D-WAZWAZe (7) using the obtained Lie algebra to achieve abundant invariant solutions to the equation.

2.2.1. Invariant solution using infinitesimal generator $\mathcal{Q}_1$

The Lagrangian system corresponding to generator $\mathcal{Q}_1 = \partial/\partial t$ is given as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{du}{0}.$$

The system solves to give us invariants $X = x$, $Y = y$ and $Z = z$ with the group-invariant $u = U(X, Y, Z)$. Inserting this function in (7) transforms it to the PDE

$$\alpha U_{XXX} + \beta U_{XXY} + \gamma U_{XXZ} = 0. \tag{15}$$

Solving Eq. (15) and retrograding to the basic variables gives us

$$u(t, x, y, z) = f_1(y, z) + x f_2(y, z) + f_3(\alpha y - \beta x, \alpha z - \gamma x), \tag{16}$$

which is an algebraic solution of the 3D-WAZWAZe (7) in a steady-state with arbitrary functions f_1, f_2 and f_3 depending on their various arguments.

2.2.2. Invariant solution using infinitesimal generator $\mathcal{Q}_2$

The solution of the characteristic equation related to $\mathcal{Q}_2 = \partial/\partial x$ gives the function $u = U(X, Y, Z)$, where $T = t$, $Y = y$ and $Z = z$, thus transforming (7) to LNPDE $U_{TTT} = 0$. On solving the equation and reverting to the original variables, we have

$$u(t, x, y, z) = \frac{1}{2} t^2 f_1(y, z) + t f_2(y, z) + f_3(y, z), \tag{17}$$

which a non-steady state function satisfying (7) with arbitrary f_1, f_2 and f_3 depending on y and z.

2.2.3. Invariant solution using infinitesimal generator $\mathcal{Q}_3$

Lie point symmetry $\mathcal{Q}_3 = \partial/\partial y$ gives group invariant which transforms (7) to

$$U_{TTT} + \alpha U_{XXX} + \gamma U_{XXZ} - 4U_T U_{XXX} - 12U_{TX} U_{XX} - 8U_X U_{TXX} - U_{TXXXX} = 0. \tag{18}$$

The group invariant is $u = U(T, X, Z)$ where $T = t$, $X = x$ and $Z = z$. On solving NLNPDE (18), we have a solution of the equation in terms of T, X and Z, which is

$$U(T, X, Z) = A_3 \tanh\left[A_2 T + A_3 X - \frac{1}{\gamma A_3^2}\left(A_2^3 + \alpha A_3^3 - 4A_2 A_3^4\right) Z + A_1\right] + A_4,$$

where $A_i, i = 1, \ldots, 4$, are arbitrary constants. Thus, giving the tan-hyperbolic function with regards to t, x, y and z, we have a topological kink solution of (7) as

$$u(t, x, y, z) = A_3 \tanh\left[A_2 t + A_3 x - \frac{1}{\gamma A_3^2}\left(A_2^3 + \alpha A_3^3 - 4A_2 A_3^4\right) z + A_1\right] + A_4. \tag{19}$$

Further study of (18) gives a Lie symmetry which is presented as

$$\begin{aligned} X = {} & \left(\frac{1}{2} F_1'(Z) T + F_2(Z)\right) \frac{\partial}{\partial T} + \left(\frac{1}{4} F_1'(Z) X + F_3(Z)\right) \frac{\partial}{\partial X} + F_1(Z) \frac{\partial}{\partial X} \\ & + \left\{ \frac{1}{16}(3\alpha T - 4U) F_1'(Z) - \frac{1}{16} \gamma T X F_1''(Z) - \frac{1}{4} \gamma T F_3'(Z) + F_4(Z) \right. \\ & \left. - \frac{1}{8} \gamma X F_2'(Z) \right\} \frac{\partial}{\partial U}. \end{aligned}$$

We assume that $F_2(Z) = a_0$, $F_3(Z) = a_1$, $F_1(Z) = a_2$, and $F_4(Z) = a_3$. Therefore, the resulting operator gives a Lagrangian system which, when it is solved, gives

$$U(T, X, Z) = G(r, s) + \frac{a_3}{a_0} T, \text{ where } r = a_0 X - a_1 T,\ s = a_0 Z - a_2 T. \tag{20}$$

On utilizing the obtained function and variables from (20) in Eq. (18), we have

$$\begin{aligned} & a_0^3 \gamma G_{rrs} + \alpha a_0^3 G_{rrr} - a_1^3 G_{rrr} - 3a_1^2 a_2 G_{rrr} - 3a_1 a_2^2 G_{rss} - a_2^3 G_{sss} + 12 a_1 a_0^3 G_{rr}^2 \\ & + 12 a_0^3 a_2 G_{rs} G_{rr} + 12 a_0^3 a_1 G_r G_{rrr} + 4 a_0^3 a_2 G_s G_{rrr} + 8 a_0^3 a_2 G_r G_{rrs} + a_0^4 a_2 G_{rrrrs} \\ & - 4 a_0^2 a_3 G_{rrr} + a_0^4 a_1 G_{rrrrr} = 0. \end{aligned} \tag{21}$$

The solution of (21) gives no new result but Lie theoretic study of the equation furnishes four Lie point symmetries which we express in this regard as

$$\begin{aligned} & W_1 = \frac{\partial}{\partial s},\ W_2 = \frac{\partial}{\partial r},\ W_3 = \frac{\partial}{\partial G},\ W_4 = \left(\frac{1}{2} r + \frac{a_1}{2a_2} s\right) \frac{\partial}{\partial r} + s \frac{\partial}{\partial s} \\ & \quad - \left\{ \frac{1}{2} G + \frac{\gamma}{8a_2} r + \frac{3}{8 a_0 a_2^2} \left[\left(\alpha a_2 - \frac{4}{3} \gamma a_1\right) a_0 - 4 a_2 a_3\right] s \right\} \frac{\partial}{\partial G}. \end{aligned}$$

We contemplate a linear combination of the three translation symmetries as $W = W_1 + W_2 + W_3$ and so solves to give function $G(r,s) = Q(w) + r$, with $w = s - r$. Applying the function in (21) gives nonlinear ordinary differential equation (NOLODE)

$$\begin{aligned}
&\gamma a_0^3 Q'''(w) - \alpha a_0^3 Q'''(w) - 12a_0^3 a_1 Q'''(w) + 8a_0^3 a_2 Q'''(w) + 4a_0^2 a_3 Q'''(w) + a_1^3 Q'''(w)\\
&- 3a_1^2 a_2 Q'''(w) + 3a_1 a_2^2 Q'''(w) - a_2^3 Q'''(w) - 12a_0^3 a_2 Q''(w)^2 + 12a_0^3 a_1 Q'''(w)^2\\
&+ 12a_0^3 a_1 Q'(w) Q'''(w) - 12a_0^3 a_2 Q'(w) Q'''(w) - a_0^4 a_1 Q^{(5)}(w) + a_0^4 a_2 Q^{(5)}(w) = 0.
\end{aligned}$$

On investigating operator W_4, we achieve no result of interest.

2.2.4. Invariant solution using infinitesimal generator $\mathcal{Q}_4$

The characteristic equations related to Lie symmetry $\mathcal{Q}_4 = \partial/\partial z$ yields invariants

$$T = t, \quad X = x, \quad Y = y, \quad \text{with group-invariant } u = U(T,X,Y). \tag{22}$$

In consequence, we insert function in (7) and then secure the NLNPDE

$$U_{TTT} + \alpha U_{XXX} + \beta U_{XXZ} - 4U_T U_{XXX} - 12U_{TX} U_{XX} - 8U_X U_{TXX} - U_{TXXXX} = 0, \tag{23}$$

which solves to give the solution of (7) with regards to variables t, x, y and z as

$$u(t,x,y,z) = B_3 \tanh\left[B_2 t + B_3 x - \frac{1}{\beta B_3^2}\left(B_2^3 + \alpha B_3^3 - 4B_2 B_3^4\right) y + B_1\right] + B_4 \tag{24}$$

with arbitrary constants $B_i, i = 1,2,\ldots,4$. The Lie theoretic approach on (23) gives

$$W = \frac{\partial}{\partial T} + \frac{\partial}{\partial X} + \frac{\partial}{\partial Y} + \frac{\partial}{\partial U}. \tag{25}$$

We contemplate operator (25) and so achieve its group invariant solution as

$$U(T,X,Z) = G(r,s) + T, \text{ with } r = X - T, \ s = Y - T. \tag{26}$$

On utilizing invariants (26) transforms the original equation (7) to a NLNPDE

$$\begin{aligned}
&\beta G_{rrs} + \alpha G_{rrr} - 5G_{rrr} - 3G_{rrr} - 3G_{rss} - G_{sss} + 12G_{rr}^2 + 12G_{rs}G_{rr} + 12G_r G_{rrr}\\
&+ 4G_s G_{rrr} + 8G_r G_{rrs} + G_{rrrrs} + G_{rrrrr} = 0.
\end{aligned} \tag{27}$$

Thus, we solve the differential equation (27) to achieve a kink soliton solution

$$\begin{aligned}
u(t,x,y,z) = {}& A_2 \tanh\left\{ A_2(x-t) + \left[\frac{1}{6}\Delta^{1/3} + \frac{6}{\Delta^{1/3}}\left(\frac{4}{3}A_2^2 - \frac{1}{3}\beta\right) - 1\right] A_2(y-t)\right.\\
&\left. + A_1\right\} + t + A_3,
\end{aligned} \tag{28}$$

where $\Delta = 12\sqrt{\Delta_0 - 162\alpha\beta + 81\beta^2 - 648\alpha + 648\beta} - 108\beta + 108\alpha - 432$, $\Delta_0 = 1296 - 768A_2^6 - 576\beta A_2^4 - 144\beta^2 A_2^2 - 12\beta^3 + 81\alpha^2$ with arbitrary constants A_1, A_2 and A_3. On investigating equation (27) further, we secure four infinitesimal generators

$$\begin{aligned}
&W_1 = \frac{\partial}{\partial r}, \quad W_2 = \frac{\partial}{\partial s}, \quad W_3 = \frac{\partial}{\partial G}, \quad W_4 = \left(\frac{1}{2}r + \frac{1}{2}s\right)\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\\
&\quad - \left\{\frac{1}{2}G + \frac{\beta}{8}r + \frac{3}{8}\left[\left(\alpha - \frac{4}{3}\beta\right) - 4\right] s\right\}\frac{\partial}{\partial G}.
\end{aligned}$$

We combine the translation symmetries as $W = c_1 W_1 + c_2 W_2 + c_3 W_3$ and on solving the associated Lagrangian, we gain the function $G(r,s) = Q(w) + (c_3/c_1)r$, with $w = c_1 s - c_2 r$. Using the function further reduces (27) to the fifth-order NOLODE

$$\begin{aligned}
&\alpha c_2^3 c_1 Q'''(w) - \beta c_1^2 c_1^2 Q'''(w) + 12c_2^3 c_3 Q'''(w) - 8c_1 c_2^2 c_3 Q'''(w) + c_1^4 Q'''(w)\\
&- 3c_1^3 c_2 Q'''(w) + 3c_1^2 c_2^2 Q'''(w) - 5c_2^3 c_1 Q'''(w) - 12c_2^4 c_1 Q''(w)^2 + 12c_2^3 c_1^2 Q'''(w)^2\\
&+ 12c_0^3 c_1^2 Q'(w) Q'''(w) - 12c_1 c_2^4 Q'(w) Q'''(w) - c_1^2 c_2^4 Q^{(5)}(w) + c_2^5 c_1 Q^{(5)}(w) = 0.
\end{aligned}$$

No result of importance could be obtained from operator W_4.

2.2.5. Invariant solution using infinitesimal generator $\mathcal{Q}_6$

We contemplate the generator $\mathcal{Q}_6 = 4y\partial/\partial x - \beta t\partial/\partial u$ whose Lagrangian system solves to give $u = U(T,Y,Z) - \beta tx/4y$ where $T = t$, $Y = y$ and $Z = z$. Thereafter, we substitute the obtained function in 3D-WAZWAZe (7) and then we achieve $U_{TTT} = 0$. We further solve the found LNPDE and secure a solution of (7) in this regard as

$$u(t,x,y,z) = \frac{1}{2}t^2 f_1(y,z) + t f_2(y,z) + f_3(y,z) - \frac{\beta tx}{4y}, \tag{29}$$

where functions $f_i, i = 1,2,3$ are arbitrary and depending on variables y and z.

2.2.6. Invariant solution using infinitesimal generator Q_7

The Lie point symmetry $Q_7 = 4z\partial/\partial x - \gamma t\partial/\partial u$ whose characteristic equations solves to give $u = U(T,Y,Z) - \gamma tx/4z$ where $T = t$, $Y = y$ and $Z = z$. Consequently, we insert the obtained function in 3D-WAZWAZe (7) and then we secure $U_{TTT} = 0$. We further solve the found LNPDE and secure a solution of (7) in this instance as

$$u(t,x,y,z) = \frac{1}{2}t^2 f_1(y,z) + tf_2(y,z) + f_3(y,z) - \frac{\gamma tx}{4z} \tag{30}$$

with arbitrary functions $f_1(y,z)$, $f_2(y,z)$ and $f_3(y,z)$.

2.2.7. Invariant solution using infinitesimal generator Q_8

The Lagrangian system corresponding to operator $Q_8 = 8y\partial/\partial x - \beta x\partial/\partial u$ which we solve to give $u = U(X,Y,Z) - \beta tx/8y$ where $X = x$, $Y = y$ and $Z = z$. Consequently, we insert the obtained function in (7) and then we secure NLNPDE

$$3\beta U_{XX} + 2\alpha Y U_{XXX} + \beta X U_{XXX} + 2\beta Y U_{XXY} + 2\gamma Y U_{XXZ} = 0. \tag{31}$$

We further solve the found LNPDE and secure a solution of (7) in this regard as

$$\begin{aligned} u(t,x,y,z) = \frac{1}{8y^{5/2}}\Bigg\{ & 8xy\int f_1\left(\frac{\beta x - 2\alpha y}{\beta\sqrt{y}}, \frac{\beta z - \gamma y}{\beta}\right)dx + 8y^{5/2}\left(xf_3(y,z)\right. \\ & \left. + f_2(y,z)\right) - \beta txy^{3/2} - 8y\int xf_1\left(\frac{\beta x - 2\alpha y}{\beta\sqrt{y}}, \frac{\beta z - \gamma y}{\beta}\right)dx\Bigg\}, \end{aligned} \tag{32}$$

where f_1, f_2 and f_3 are arbitrary functions dependent on their arguments.

2.2.8. Invariant solution using infinitesimal generator Q_9

We consider the generator $Q_9 = 8z\partial/\partial t - \gamma x\partial/\partial u$ which when solved produces group invariant solution $u = U(X,Y,Z) - \beta tx/8y$ where $X = x$, $Y = y$ and $Z = z$. Consequently, we substitute the obtained function in (7) and then achieves NLNPDE

$$3\beta U_{XX} + 2\alpha Y U_{XXX} + \beta X U_{XXX} + 2\beta Y U_{XXY} + 2\gamma Y U_{XXZ} = 0. \tag{33}$$

On solving the secured LNPDE, one secures a solution of (7) in this instance as

$$\begin{aligned} u(t,x,y,z) = \frac{1}{8\beta z\sqrt{\beta z}}\Bigg\{ & 8x\int f_1\left(\frac{\beta z - \gamma y}{\beta}, \frac{\gamma x - 2\alpha z}{\gamma\sqrt{\beta z}}\right)dx - \beta\sqrt{\beta z}\left(-8zf_2(y,z)\right. \\ & \left. + x\left(\gamma t - 8zf_3(y,z)\right)\right) - 8\int f_1\left(\frac{\beta z - \gamma y}{\beta}, \frac{\gamma x - 2\alpha z}{\gamma\sqrt{\beta z}}\right)xdx\Bigg\}, \end{aligned} \tag{34}$$

with arbitrary functions f_1, f_2 and f_3 dependent on their various arguments.

2.2.9. Invariant solution using infinitesimal generator Q_{10}

The characteristic equations associated to the Lie point symmetry $Q_{10} = 8\beta t\partial/\partial t + 4\beta x\partial/\partial x + 16\beta y\partial/\partial y + 16\gamma y\partial/\partial z + (3\alpha\beta t - 4\beta u)\partial/\partial u$ is presented as

$$\frac{dt}{8\beta t} = \frac{dx}{4\beta x} = \frac{dy}{16\beta y} = \frac{dz}{16\gamma y} = \frac{du}{3\alpha\beta t - 4\beta u}.$$

Thus, group invariant and the constituent invariants are computed accordingly as

$$u = \frac{1}{\sqrt{t}}U(X,Y,Z) + \frac{1}{4}\alpha t, \text{ and } X = x/\sqrt{t}, \ Y = y/t^2, \ Z = (\beta z - \gamma y)/\beta. \tag{35}$$

Inserting the invariants from (35) in Eq. (7), we achieve the NLNPDE

$$\begin{aligned} & 8\beta U_{XXY} - X^3 U_{XXX} - 12X^3 Y U_{XXY} - 48XY^2 U_{XYY} - 64Y^3 U_{YYY} - 300YU_Y \\ & + 48XU_X U_{XXX} + 48XU_{XX}^2 + 192YU_{XX}U_{XY} - 12X^2 U_{XX} + 128YU_X U_{XXY} \\ & + 64YU_Y U_{XXX} - 132XYU_{XY} - 336Y^2 U_{YY} + 16UU_{XXX} + 192U_X U_{XX} - 33XU_X \\ & + 4XU_{XXXXX} - 15U + 20U_{XXXX} + 16YU_{XXXXY} = 0. \end{aligned} \tag{36}$$

Further investigation of the equation yields no outcome of importance.

2.2.10. Invariant solution using infinitesimal generator Q_{11}

The similarity solution associated with operator $Q_{11} = 8\gamma t\partial/\partial t + 4\gamma x\partial/\partial x + 16\beta z\partial/\partial y + 16\gamma z\partial/\partial z + (3\alpha\gamma t - 4\gamma u)\partial/\partial u$ through the involvement of invariants calculated as

$$X = x/\sqrt{t}, \ Y = z/t^2, \ Z = (\gamma y - \beta z)/\gamma \text{ with group invariant } u = \frac{1}{\sqrt{t}}U(X,Y,Z) + \frac{1}{4}\alpha t.$$

transforms 3D-WAZWAZe (7) to the NLNPDE

$$\begin{aligned}&8\gamma U_{XXY}-X^3U_{XXX}-12X^3YU_{XXY}-48XY^2U_{XYY}-64Y^3U_{YYY}-300YU_Y\\&+48XU_XU_{XXX}+48XU_{XX}^2+192YU_{XX}U_{XY}-12X^2U_{XX}+128YU_XU_{XXY}\\&+64YU_YU_{XXX}-132XYU_{XY}-336Y^2U_{YY}+16UU_{XXX}+192U_XU_{XX}-33XU_X\\&-15U+20U_{XXXX}+4XU_{XXXXX}+16YU_{XXXXY}=0.\end{aligned}$$

2.2.11. Invariant solution using infinitesimal generators $Q_1, Q_2, \ldots, Q_4$

The solution to the Lagrangian system related to the linear combination of $Q_1, \ldots, Q_4$ is

$$u=U(X,Y,Z),\ \text{ where }\ X=x-t,\ \ Y=y-t,\ \ Z=z-t. \tag{37}$$

Application of the invariants from (37) in Eq. (7) transforms it further to

$$\begin{aligned}&\alpha U_{XXX}+\beta U_{XXY}+\gamma U_{XXZ}+12U_{XX}^2-U_{XXX}-3U_{XXY}-3U_{XXZ}-3U_{XYY}\\&-6U_{XYZ}-3U_{XZZ}-U_{YYY}-3U_{YYZ}-3U_{YZZ}-U_{ZZZ}+12U_XU_{XXX}+4U_YU_{XXX}\\&+4U_ZU_{XXX}+8U_XU_{XXY}+8U_XU_{XXZ}+12U_{XX}U_{XY}+12U_{XX}U_{XZ}+U_{XXXXX}\\&+U_{XXXXY}+U_{XXXXZ}=0.\end{aligned} \tag{38}$$

Thus, we achieve a solution of (38) and on reverting to the original variables gives

$$\begin{aligned}u(t,x,y,z)=&\,2A_3\sinh\left[A_1+A_2(x-t)-\frac{A_2}{\gamma-\beta}\left((\gamma-\alpha)(y-t)+(\alpha-\beta)(z-t)\right)\right]^2\\&\times\left\{\sinh\left[2A_2(x-t)-\frac{2A_2}{\gamma-\beta}\left((\gamma-\alpha)(y-t)+(\alpha-\beta)(z-t)\right)\right.\right.\\&\left.\left.+2A_1\right]\right\}^{-1}+A_4,\end{aligned} \tag{39}$$

where constants A_1, A_2, A_3 and A_4 are arbitrary. Further, (39) admits symmetries

$$W_1=\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}+\frac{\partial}{\partial Z}\ \text{ and }\ W_2=\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}+\frac{\partial}{\partial Z}+\frac{\partial}{\partial G}.$$

The study carried out on vector W_1 yields the function $U(X,Y,Z)=G(r,s)$, where $r=Y-X$ and $s=Z-X$. Inserting the function in (7) produces the NLNPDE

$$\gamma G_{rrs}+2\gamma G_{rss}+\gamma G_{sss}-\alpha G_{rrr}+\beta G_{rrr}-3\alpha G_{rrs}+2\beta G_{rrs}-3\alpha G_{rss}+\beta G_{rss}-\alpha G_{sss}=0. \tag{40}$$

Eq. (40) solves to give us a solution of 3D-WAZWAZe (7) given as

$$u(t,x,y,z)=f_1(z-y)+(y-x)f_2(z-y)+f_3\left((\beta-\alpha)(z-x)-(\gamma-\alpha)(y-x)\right), \tag{41}$$

where functions f_1, f_2 and f_3 are arbitrary. Further investigation reveals that

$$P_1=\frac{\partial}{\partial r},\ \ P_2=\frac{\partial}{\partial s},\ \text{ and }\ P_1=\frac{\partial}{\partial G},$$

are admitted by (41). We linearly combine the symmetries as $P=c_0P_1+c_1P_2+c_2P_3$. Thus, we obtain the solution to its characteristic equation as $G(r,s)=Q(w)+(c_2/c_0)r$, where $w=c_0s-c_1r$. Substituting the new expression in (40) gives a third-order linear ordinary differential equation (LNODE) $Q'''(w)=0$, which solves to give algebraic solution of (7) as

$$\begin{aligned}u(t,x,y,z)=&\,\frac{1}{2}A_0(c_0(z-x)-c_1(y-x))^2+\frac{c_2}{c_0}(y-x)+A_1(c_0(z-x)\\&-c_1(y-x))+A_2,\end{aligned} \tag{42}$$

where constants A_0, A_1 and A_2 are arbitrary. Moreover, in the same vein, from W_2, we obtain $U(X,Y,Z)=G(r,s)+X$ with $r=Y-X$ and $s=Z-X$. Invoking the secured function in Eq. (7), reduces it to a partial differential equation (PDE)

$$\begin{aligned}&\gamma G_{rrs}+2\gamma G_{rss}+\gamma G_{sss}-\alpha G_{rrr}+\beta G_{rrr}-3\alpha G_{rrs}+2\beta G_{rrs}-3\alpha G_{rss}+\beta G_{rss}\\&-\alpha G_{sss}-4G_{rrr}-12G_{rrs}-12G_{rss}-4G_{sss}=0,\end{aligned} \tag{43}$$

which solves to give a function whose back-substitution to original variables gives

$$\begin{aligned}u(t,x,y,z)=&\,f_1(z-y)+f_3\left((\beta-\alpha-4)(z-x)-(\gamma-\alpha-4)(y-x)\right)\\&+(y-x)f_2(z-y)+x-t,\end{aligned} \tag{44}$$

with functions f_1, f_2 and f_3 arbitrary. Further investigation on PDE (43) produces

$$P_1=\frac{\partial}{\partial r},\ \ P_2=\frac{\partial}{\partial s},\ \text{ and }\ P_1=\frac{\partial}{\partial G},$$

which we combine linearly as earlier demonstrated. The resulting group-invariant solution transforms (44) to a third-order LNODE which solves to give the function

$$u(t,x,y,z) = x - t + \frac{c_2}{c_0}(y-x) + \frac{1}{2}A_0(c_0(z-x) - c_1(y-x))^2 \\ + A_1(c_0(z-x) - c_1(y-x)) + A_2, \tag{45}$$

which is a solution of (7) where parameters A_0, A_1 and A_2 are integration constants.

2.2.12. Invariant solution using infinitesimal generators $Q_1, Q_2, Q_3, \ldots, Q_5$

The linear combination of $Q_1, \ldots, Q_5$ as $Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5$ gives function

$$u = t + U(X,Y,Z), \text{ where we have } X = x - t,\ Y = y - t,\ Z = z - t. \tag{46}$$

Utilization of the group-invariant (46) in Eq. (7) reduces it to NLNPDE

$$\alpha U_{XXX} + \beta U_{XXY} + \gamma U_{XXZ} + 12U_{XX}^2 - 5U_{XXX} - 3U_{XXY} - 3U_{XXZ} - 3U_{XYY} \\ - 6U_{XYZ} - 3U_{XZZ} - U_{YYY} - 3U_{YYZ} - 3U_{YZZ} - U_{ZZZ} + 12U_X U_{XXX} + 4U_Y U_{XXX} \\ + 4U_Z U_{XXX} + 8U_X U_{XXY} + 8U_X U_{XXZ} + 12U_{XX}U_{XY} + 12U_{XX}U_{XZ} + U_{XXXXX} \\ + U_{XXXXY} + U_{XXXXZ} = 0. \tag{47}$$

The primitive outcome of Eq. (47) gives the solution of 3D-WAZWAZe (7) as

$$u(t,x,y,z) = A_3 \tanh\left[A_0 + A_1(x-t) - \frac{A_1}{\gamma - \beta}\left\{(\gamma - \alpha + 4)(y-t) \right.\right. \\ \left.\left. + (\alpha - \beta - 4)(z-t)\right\}\right] + A_2 \tag{48}$$

with arbitrary constants A_0, A_1, A_2 and A_3. Further, (39) admits symmetries

$$W_1 = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z} \text{ and } W_2 = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z} + \frac{\partial}{\partial G}.$$

In consequence, symmetry W_1 gives the function $U(X,Y,Z) = G(r,s)$, where $r = Y - X$ and $s = Z - X$. Substitution of the function in (7) produces the equation

$$12G_{rrs} + 12G_{rss} + 4G_{sss} + 4G_{rrr} + \gamma G_{rrs} + 2\gamma G_{rss} + \gamma G_{sss} - \alpha G_{rrr} + \beta G_{rrr} \\ - 3\alpha G_{rrs} + 2\beta G_{rrs} - 3\alpha G_{rss} + \beta G_{rss} - \alpha G_{sss} = 0. \tag{49}$$

The solution to Eq. (49) satisfies 3D-WAZWAZe (7) and it is obtained as

$$u(t,x,y,z) = t + f_1(z-y) + f_3\left((\beta - \alpha + 4)(z-x) - (\gamma - \alpha + 4)(y-x)\right) \\ + (y-x)f_2(z-y) \tag{50}$$

with arbitrary functions f_1, f_2 and f_3. On investigating (49) further, we have

$$P_1 = \frac{\partial}{\partial r},\ P_2 = \frac{\partial}{\partial s} \text{ and } P_1 = \frac{\partial}{\partial G}.$$

We linearly combine the symmetries as $P = a\partial/\partial r + b\partial/\partial s + c\partial/\partial G$. Thus, we secure the solution to Lagrangian system related to P as $G(r,s) = Q(w) + (c/a)r$, where $w = as - br$. On inserting the new expression of $G(r,s)$ in (49) gives the third-order LNODE which when solved returns the algebraic solution of (7) as

$$u(t,x,y,z) = t + \frac{c}{a}(y-x) + \frac{1}{2}A_0(a(z-x) - b(y-x))^2 + A_1(a(z-x) \\ - b(y-x)) + A_2, \tag{51}$$

with arbitrary constants A_0, A_1 and A_2. Furthermore, we investigate scaling symmetry W_2 to achieve $U(X,Y,Z) = X + G(r,s)$, where $r = Y - X$ and $s = Z - X$. Invoking the secured function in (7), the equation further transforms to PDE

$$\gamma G_{rrs} + 2\gamma G_{rss} + \gamma G_{sss} - \alpha G_{rrr} + \beta G_{rrr} - 3\alpha G_{rrs} + 2\beta G_{rrs} - 3\alpha G_{rss} + \beta G_{rss} \\ - \alpha G_{sss} = 0. \tag{52}$$

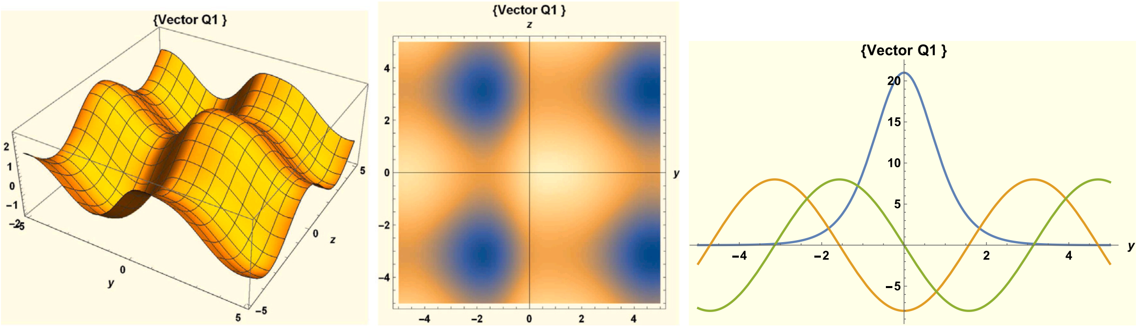

Fig. 1. Wave interactions, depicting algebraic solution (16) at $t = 0$ and $x = 0$.

Solving Eq. (52) and returning to the basic variables yields a solution of (7) as

$$u(t,x,y,z) = x + f_1(z-y) + f_3\left((\beta-\alpha)(z-x) - (\gamma-\alpha)(y-x)\right) + (y-x)f_2(z-y), \tag{53}$$

where f_1, f_2 and f_3 are arbitrary functions. Further study on PDE (52) gives

$$P_1 = \frac{\partial}{\partial r},\quad P_2 = \frac{\partial}{\partial s}\quad \text{and}\quad P_1 = \frac{\partial}{\partial G}.$$

On combining the translation symmetries linearly as earlier exhibited, the resulting function reduces (52) to a LNODE which we solve to obtain

$$u(t,x,y,z) = x + \frac{c}{a}(y-x) + \frac{1}{2}A_0(a(z-x) - b(y-x))^2 + A_1(a(z-x) - b(y-x)) + A_2, \tag{54}$$

which satisfies Eq. (7) with integration constants A_0, A_1 as well as A_2.

3. Solitary wave depictions and analysis of results

We explore the dynamical behaviour of some of the secured solutions for 3D-WAZWAZe (7) through graphical interpretations which are revealed via numerical simulation of the solutions. Thus, the investigation of the novel solitary wave structures of the established results is performed in this section. We apply these simulations by taking various suitable choices of arbitrary functions as well as constants appearing in the selected ones among the achieved nineteen classical solutions of (7). The nineteen solutions include (16), (17), (19), (24), (28), (29), (32), (30), (34), (39), (41), (42), (44), (45), (48), (50), (51), (53) and (54), comprising diverse topological kink solitons, algebraic as well as hyperbolic functions. We now select (16), (17), (19) together with (29) and perform their numerical simulations as representatives of the nineteen results. The portrayal of these solutions is done with 3D, 2D as well as density plots using computer software. Therefore, as we have earlier noted, by letting $x = 0$, we take the resulting arbitrary function in solution (16) as $f(y,z) = \mathrm{Sech}^2(y) + \cos(z) + \sin(y)$ with $-5 \le y,z \le 5$. In consequence, we observe soliton and double periodic interactions along (y,z)-axes as presented in Fig. 1. In Fig. 2 with the same assigned function but $-10 \le y,z \le 10$, we notice a stronger collisions between the waves. Further, we examine the dynamics of the solution in Fig. 3 by considering $f(y,z) = \mathrm{Sech}^2(y) + \cos^2(z) + \sin^2(y)$ where $-10 \le y,z \le 10$ which gives soliton interactions propagating at variant amplitudes. The next five Figures give various soliton collisions, depicting solution (17) at $t = 0$. We first take $f_3(y,z) = \mathrm{Sech}(y) + \cos^2(z) + \sin(y)$ for $-5 \le y,z \le 5$ which produces periodic and 1-soliton interactions in Fig. 4. Moreover, we assign $f_3(y,z) = \cos^2(z) + \mathrm{Si}(y)$ which gives the soliton interactions observed in Figs. 5 and 6 respectively with soliton propagating at different amplitudes and travelling at variant frequencies at different intervals of y and z. In Fig. 7, we let $f_3(y,z) = \cos(z) + \mathrm{Si}(5y)$ where $-5 \le y,z \le 5$ and in the same vein, we assume $f_3(y,z) = \cos(y) + \mathrm{Si}(z)$ with the same interval of variables y and z in Fig. 8, which reveal various wave propagations in the Figures. The topological kink soliton solution (19) is numerically simulated as depicted in Fig. 9 with dissimilar parametric values $A_1 = 1$, $A_2 = 2$, $A_3 = 1$, $\gamma = 1$, $\alpha = 1$ with $-10 \le t,z \le 10$, where variables $x = 1$ and $y = 0$. Obviously, the Figure reveals a kink-shaped wave. Furthermore, the dynamical wave behaviour of solution (29) with constant $\beta = 0.1$ at $t = 1$ and $y = 20$, is represented in the remaining three Figures where Figs. 10 and 11 are plotted with function $f(y,z) = \cos(y)\,\mathrm{Sech}(z) + \sin(y)$, $\left(f(y,z) = 1/2 f_1(y,z) + f_2(y,z) + f_3(y,z)\right)$, for $-10 \le x,z \le 10$ and $-5 \le x,z \le 5$ accordingly. In the case of Fig. 12, we assign $f(y,z) = \cos(y) + \mathrm{Sech}^5(z) + \sin(y)$ in the solution with the same values of β and t but $y = 5$. We notice that the three Figures exhibit bright soliton waves which are pertinent in the study of nonlinear wave equations.

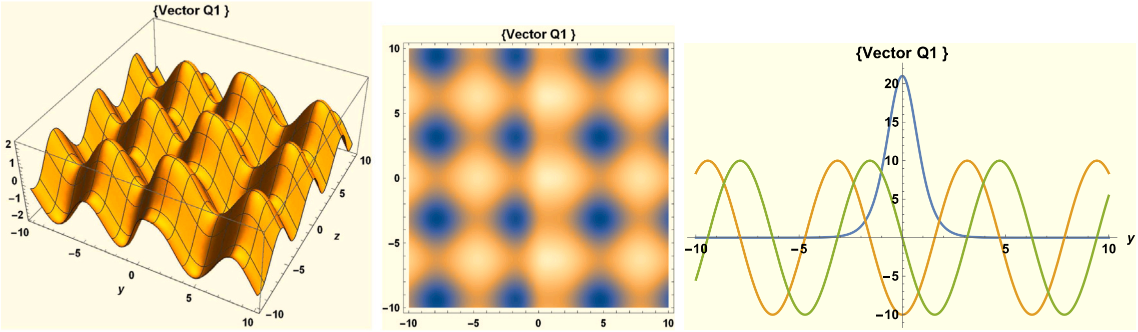

Fig. 2. Wave interactions, depicting algebraic solution (16) at $t = 0$ and $x = 0$.

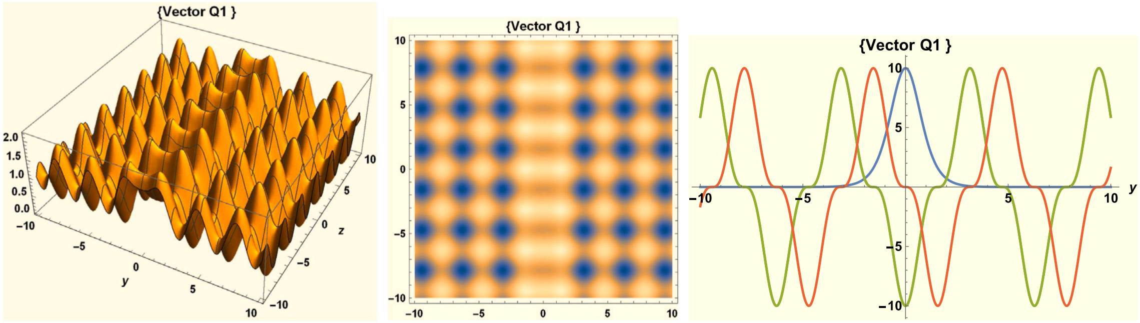

Fig. 3. Wave interactions, depicting algebraic solution (16) at $t = 0$ and $x = 0$.

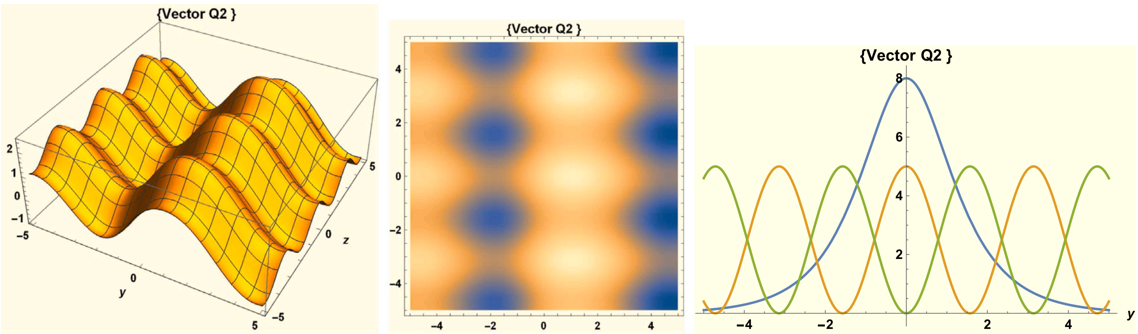

Fig. 4. Wave interactions, depicting algebraic solution (17) at $t = 0$ and $x = 0$.

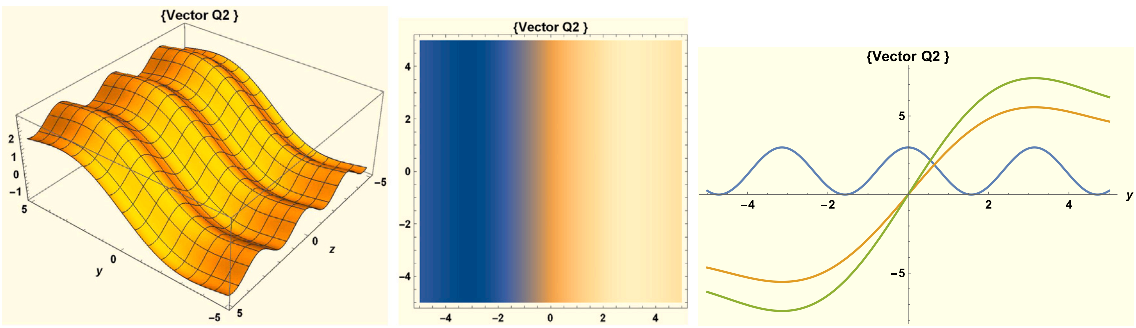

Fig. 5. Wave collisions, representing algebraic solution (17) at $t = 0$ and $x = 0$.

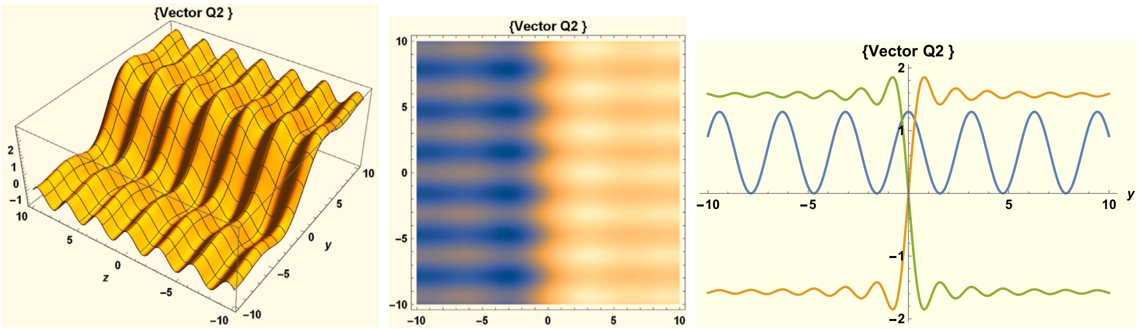

Fig. 6. Wave collisions, representing algebraic solution (17) at $t = 0$ and $x = 0$.

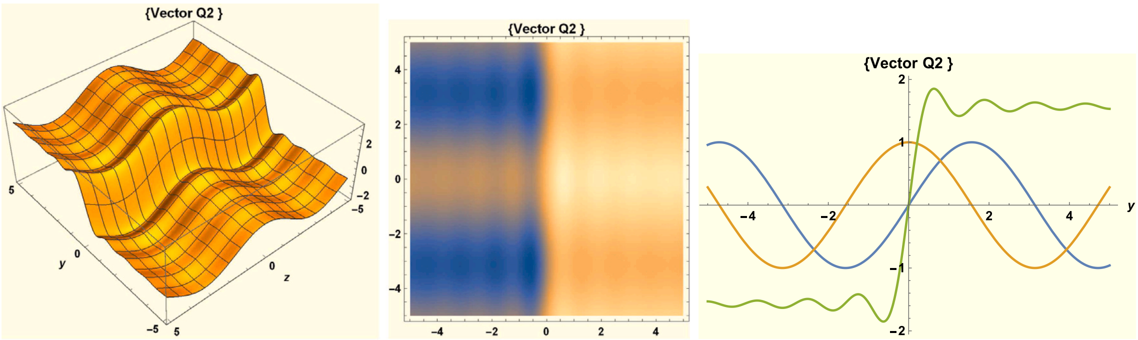

Fig. 7. Wave collisions, representing algebraic solution (17) at $t = 0$ and $x = 0$.

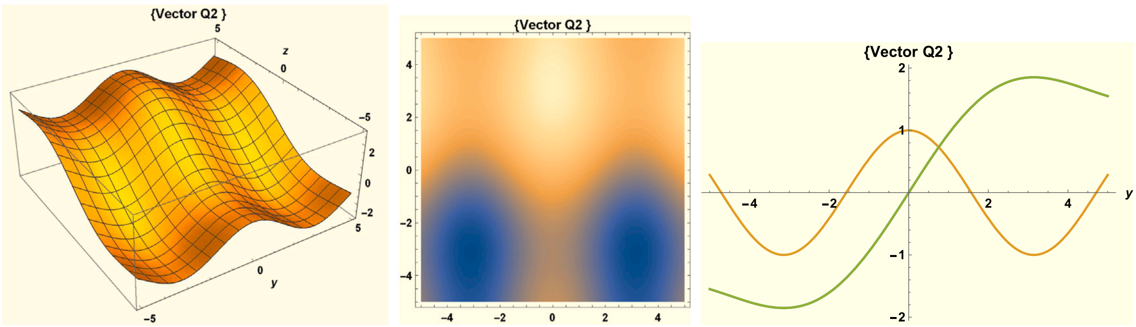

Fig. 8. Wave collisions, representing algebraic solution (17) at $t = 0$ and $x = 0$.

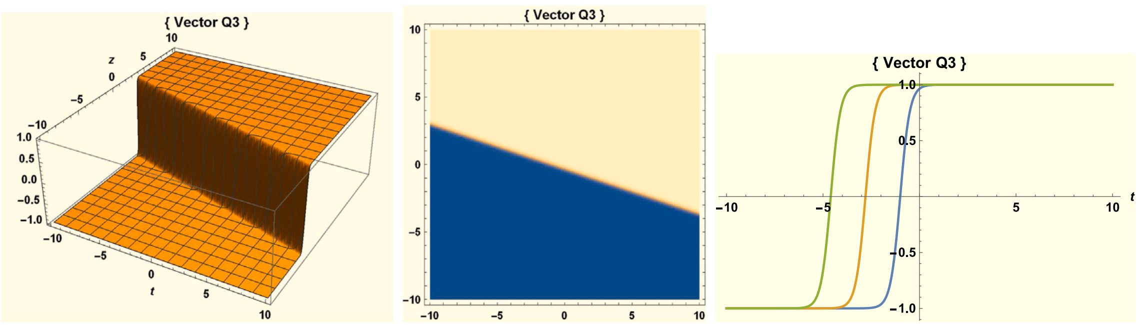

Fig. 9. Kink wave depiction of topological soliton (19) at $x = 1$ and $y = 0$.

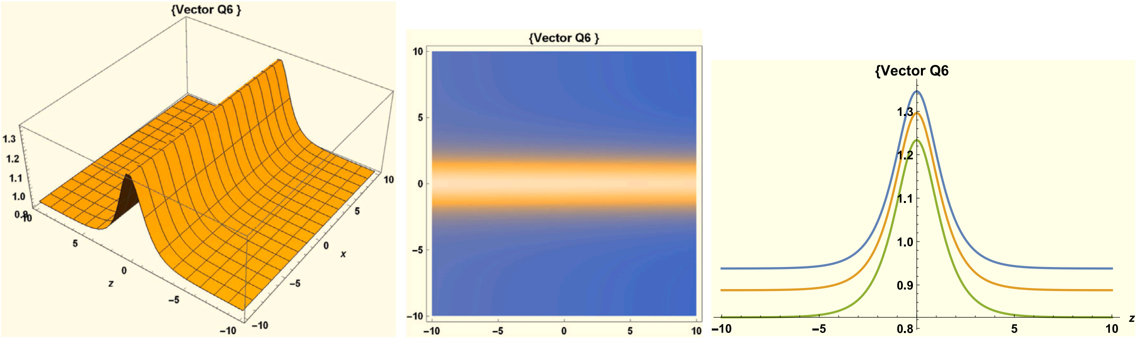

Fig. 10. Wave interactions, portraying algebraic solution (29) at $t = 1$ and $y = 5$.

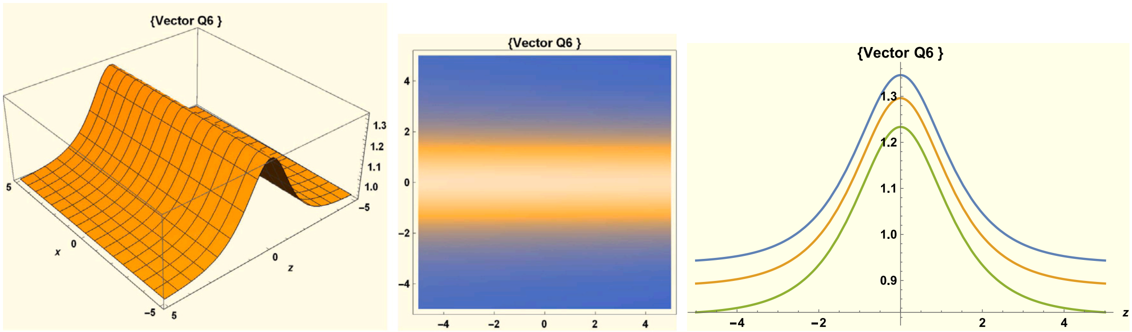

Fig. 11. Wave interactions, portraying algebraic solution (29) at $t = 1$ and $y = 5$.

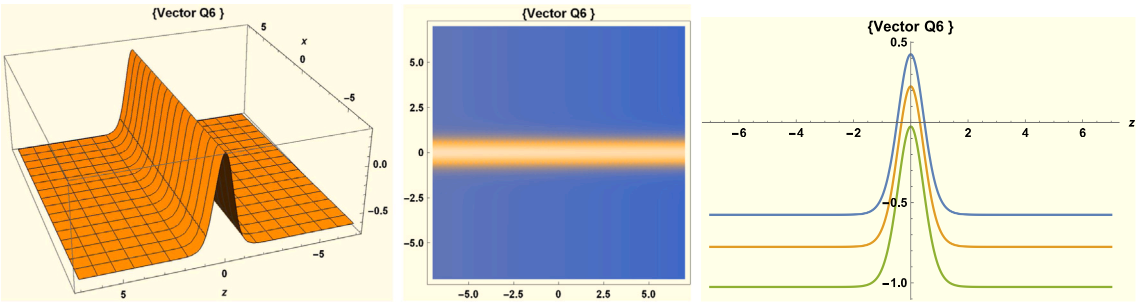

Fig. 12. Wave interactions, portraying algebraic solution (29) at $t = 1$ and $y = 5$.

4. Conserved currents using Ibragimov's theorem

It is known that Ibragimov's theorem for conserved quantity recommends that for every Lie point symmetry of a differential equation, one can achieve a distinctive conservation law of the differential equation. Here, we invoke the Lie algebra of (7) in Section 2 for the construction of new conserved currents via the theorem of Ibragimov [51–53]. Thus, we give a theorem to that effect, viz:

Theorem 4.1. *Given the Euler operator $\delta/\delta u$ which is defined as,*

$$\begin{aligned}\frac{\delta}{\delta u} = &-D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} - D_z\frac{\partial}{\partial u_z} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_y\frac{\partial}{\partial u_{xy}} - D_x^2D_z\frac{\partial}{\partial u_{xxz}}\\ &- D_{xy}D_t\frac{\partial}{\partial u_{txy}} - D_x^3\frac{\partial}{\partial u_{xxx}} - D_x^2D_y\frac{\partial}{\partial u_{xxy}} - D_x^4D_y\frac{\partial}{\partial u_{xxxxy}} + \dots,\end{aligned} \tag{55}$$

the adjoint equation of 3D-WAZWAZe (7) [51] can be expressed through the relation

$$\begin{aligned}E^* \equiv &\frac{\delta}{\delta u}\left[v(u_{ttt} - 4u_tu_{xxx} - 12u_{xx}u_{tx} - 8u_xu_{txx} - u_{txxxx} + \alpha u_{xxx} + \beta u_{xxy} + \gamma u_{xxz})\right]\\ &= 0.\end{aligned} \tag{56}$$

Further expansion of (56) secures

$$E^* \equiv v_{txxxx} - v_{xxx}\left(\alpha - 4u_t\right) + 8v_{xx}u_{tx} + 4u_{xx}v_{tx} + 8u_x v_{txx} - v_{ttt} - \beta v_{xxy} - \gamma v_{xxz} = 0. \tag{57}$$

The formal Lagrangian of 3D-WAZWAZe (7) together with its adjoint presented in (57) is expressed in the format

$$\mathcal{L} = v\left(u_{ttt} - 4u_t u_{xxx} - 12u_{xx}u_{tx} - 8u_x u_{txx} - u_{txxxx} + \alpha u_{xxx} + \beta u_{xxy} + \gamma u_{xxz}\right). \tag{58}$$

Hence, one constructs the associated conserved vectors $T^i, i = (t,x,y,z)$ for the expressed Lagrangian by utilizing the relation purveyed as [51]

$$T^i = \xi^i \mathcal{L} + W^\alpha\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots\right] + D_j D_k(W^\alpha)\frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots, \tag{59}$$

for $\alpha = 1,2$ and $j = 1,2,3,4$, the Lie characteristic function W^α is $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

We are aware in Section 2 the Lie algebra of 3D-WAZWAZe (7) comprises eleven members, we reckon the conserved vectors [53,54] for (7) using Theorem 4.1 alongside information provided in the references. Thus, we have

$$\begin{aligned}
T_1^t ={}& \gamma v u_{xxz} + \beta v u_{xxy} + \alpha v u_{xxx} - \frac{2}{3}v_x u_{xx}u_t + \frac{8}{3}u_x v_{xx}u_t - \frac{10}{3}v u_{xxx}u_t + \frac{1}{5}v_{xxxx}u_t \\
&- \frac{8}{3}u_x v_x u_{tx} - \frac{26}{3}v u_{xx}u_{tx} - \frac{1}{5}v_{xxx}u_{tx} - \frac{16}{3}v u_x u_{txx} + \frac{1}{5}v_{xx}u_{txx} - \frac{1}{5}v_x u_{txxx} \\
&- \frac{4}{5}v u_{txxxx} + v_t u_{tt} - u_t v_{tt}, \\
T_1^x ={}& 4v_{xx}u_t^2 - \frac{2}{3}\gamma v_{xz}u_t - \frac{2}{3}\beta v_{xy}u_t - \alpha v_{xx}u_t - \frac{2}{3}u_{xx}v_t u_t - \frac{8}{3}v_x u_{tx}u_t + \frac{16}{3}u_x v_{tx}u_t \\
&+ \frac{10}{3}v u_{txx}u_t + \frac{4}{5}v_{txxx}u_t + \frac{16}{3}v u_{tx}^2 + \frac{1}{3}\gamma v_x u_{tz} + \frac{1}{3}\beta v_x u_{ty} + \frac{1}{3}\gamma v_z u_{tx} + \frac{1}{3}\beta v_y u_{tx} \\
&+ \alpha v_x u_{tx} - \frac{8}{3}u_x v_t u_{tx} - \frac{2}{3}\gamma v u_{txz} - \frac{2}{3}\beta v u_{txy} - \alpha v u_{txx} + \frac{2}{5}v_{tx}u_{txx} - \frac{3}{5}u_{tx}v_{txx} \\
&- \frac{1}{5}v_t u_{txxx} - \frac{8}{3}u_x v_x u_{tt} + \frac{10}{3}v u_{xx}u_{tt} - \frac{1}{5}v_{xxx}u_{tt} + \frac{16}{3}v u_x u_{ttx} + \frac{2}{5}v_{xx}u_{ttx} \\
&- \frac{3}{5}v_x u_{ttxx} + \frac{4}{5}v u_{ttxxx}, \\
T_1^y ={}& \frac{1}{3}\beta v_x u_{tx} - \frac{1}{3}\beta v_{xx}u_t - \frac{1}{3}\beta v u_{txx}, \\
T_1^z ={}& \frac{1}{3}\gamma v_x u_{tx} - \frac{1}{3}\gamma v_{xx}u_t - \frac{1}{3}\gamma v u_{txx}; \\
T_2^t ={}& \frac{8}{3}v_{xx}u_x^2 - \frac{10}{3}v_x u_{xx}u_x + \frac{10}{3}v u_{xxx}u_x + \frac{1}{5}v_{xxxx}u_x - v_{tt}u_x + \frac{10}{3}v u_{xx}^2 \\
&+ \frac{1}{5}v_{xx}u_{xxx} - \frac{1}{5}u_{xx}v_{xxx} - \frac{1}{5}v_x u_{xxxx} + \frac{1}{5}v u_{xxxxx} + v_t u_{tx} - v u_{ttx}, \\
T_2^x ={}& \frac{16}{3}v_{tx}u_x^2 - \frac{2}{3}\gamma v_{xz}u_x - \frac{2}{3}\beta v_{xy}u_x - \alpha v_{xx}u_x + 4v_{xx}u_t u_x - \frac{10}{3}u_{xx}v_t u_x \\
&- \frac{4}{3}v_x u_{tx}u_x - \frac{10}{3}v u_{txx}u_x + \frac{4}{5}v_{txxx}u_x + \frac{1}{3}\gamma v_x u_{xz} + \frac{1}{3}\beta v_x u_{xy} + \frac{1}{3}\gamma v_z u_{xx} \\
&+ \frac{1}{3}\beta v_y u_{xx} + \alpha v_x u_{xx} + \frac{1}{3}\gamma v u_{xxz} + \frac{1}{3}\beta v u_{xxy} - 4v_x u_{xx}u_t - \frac{1}{5}u_{xxxx}v_t \\
&- \frac{10}{3}v u_{xx}u_{tx} - \frac{1}{5}v_{xxx}u_{tx} + \frac{2}{5}u_{xxx}v_{tx} + \frac{2}{5}v_{xx}u_{txx} - \frac{3}{5}u_{xx}v_{txx} - \frac{3}{5}v_x u_{txxx} \\
&- \frac{1}{5}v u_{txxxx} + v u_{ttt}, \\
T_2^y ={}& \frac{1}{3}\beta v_x u_{xx} - \frac{1}{3}\beta u_x v_{xx} - \frac{1}{3}\beta v u_{xxx}, \\
T_2^z ={}& \frac{1}{3}\gamma v_x u_{xx} - \frac{1}{3}\gamma u_x v_{xx} - \frac{1}{3}\gamma v u_{xxx}; \\
T_3^t ={}& \frac{10}{3}v u_{xx}u_{xy} - \frac{8}{3}u_x v_x u_{xy} - \frac{1}{5}v_{xxx}u_{xy} - \frac{2}{3}u_y v_x u_{xx} + \frac{8}{3}u_y u_x v_{xx} + \frac{8}{3}v u_x u_{xxy} \\
&+ \frac{2}{3}v u_y u_{xxx} - \frac{1}{5}v_x u_{xxxy} + \frac{1}{5}u_y v_{xxxx} + \frac{1}{5}v u_{xxxxy} + v_t u_{ty} - u_y v_{tt} - v u_{tty} \\
&+ \frac{1}{5}v_{xx}u_{xxy}, \\
T_3^x ={}& \frac{1}{3}\gamma u_{yz}v_x + \frac{1}{3}\beta u_{yy}v_x + \alpha u_{xy}v_x - 4u_{xy}u_t v_x - \frac{8}{3}u_x u_{ty}v_x + \frac{4}{3}u_y u_{tx}v_x
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{5}u_{txxy}v_x-\frac{2}{3}\gamma u_y v_{xz}+\frac{1}{3}\gamma v_z u_{xy}+\frac{1}{3}\beta v_y u_{xy}-\frac{2}{3}\beta u_y v_{xy}-\frac{2}{3}\gamma v u_{xyz}\\
&-\frac{2}{3}\beta v u_{xyy}-\alpha u_y v_{xx}-\alpha v u_{xxy}+4u_y v_{xx}u_t+4v u_{xxy}u_t-\frac{8}{3}u_x u_{xy}v_t\\
&-\frac{2}{3}u_y u_{xx}v_t-\frac{1}{5}u_{xxxy}v_t+\frac{10}{3}v u_{xx}u_{ty}-\frac{1}{5}v_{xxx}u_{ty}+\frac{16}{3}v u_{xy}u_{tx}+\frac{2}{5}u_{xxy}v_{tx}\\
&+\frac{16}{3}u_y u_x v_{tx}+\frac{16}{3}v u_x u_{txy}+\frac{2}{5}v_{xx}u_{txy}-\frac{2}{3}v u_y u_{txx}-\frac{3}{5}u_{xy}v_{txx}+\frac{4}{5}u_y v_{txxx}\\
&+\frac{4}{5}v u_{txxxy},\\
T_3^y=\;&\frac{1}{3}\beta v_x u_{xy}-\frac{1}{3}\beta u_y v_{xx}+\frac{2}{3}\beta v u_{xxy}+\alpha v u_{xxx}-4v u_{xxx}u_t-12v u_{xx}u_{tx}\\
&+\gamma v u_{xxz}-8v u_x u_{txx}-v u_{txxxx}+v u_{ttt},\\
T_3^z=\;&\frac{1}{3}\gamma v_x u_{xy}-\frac{1}{3}\gamma u_y v_{xx}-\frac{1}{3}\gamma v u_{xxy};\\
T_4^t=\;&\frac{10}{3}v u_{xx}u_{xz}-\frac{8}{3}u_x v_x u_{xz}-\frac{1}{5}v_{xxx}u_{xz}-\frac{2}{3}u_z v_x u_{xx}+\frac{8}{3}u_z u_x v_{xx}+\frac{8}{3}v u_x u_{xxz}\\
&+\frac{1}{5}v_{xx}u_{xxz}+\frac{2}{3}v u_z u_{xxx}-\frac{1}{5}v_x u_{xxxz}+\frac{1}{5}u_z v_{xxxx}+\frac{1}{5}v u_{xxxxz}+v_t u_{tz}\\
&-u_z v_{tt}-v u_{ttz},\\
T_4^x=\;&\frac{1}{3}\gamma u_{zz}v_x+\frac{1}{3}\beta u_{yz}v_x+\alpha u_{xz}v_x-4u_{xz}u_t v_x-\frac{8}{3}u_x u_{tz}v_x+\frac{4}{3}u_z u_{tx}v_x\\
&-\frac{3}{5}u_{txxz}v_x+\frac{1}{3}\gamma v_z u_{xz}+\frac{1}{3}\beta v_y u_{xz}-\frac{2}{3}\gamma u_z v_{xz}-\frac{2}{3}\gamma v u_{xzz}-\frac{2}{3}\beta u_z v_{xy}\\
&-\frac{2}{3}\beta v u_{xyz}-\alpha u_z v_{xx}-\alpha v u_{xxz}+4u_z v_{xx}u_t+4v u_{xxz}u_t-\frac{8}{3}u_x u_{xz}v_t\\
&-\frac{2}{3}u_z u_{xx}v_t-\frac{1}{5}u_{xxxz}v_t+\frac{10}{3}v u_{xx}u_{tz}-\frac{1}{5}v_{xxx}u_{tz}+\frac{16}{3}v u_{xz}u_{tx}\\
&+\frac{16}{3}u_z u_x v_{tx}+\frac{2}{5}u_{xxz}v_{tx}+\frac{16}{3}v u_x u_{txz}+\frac{2}{5}v_{xx}u_{txz}-\frac{2}{3}v u_z u_{txx}\\
&-\frac{3}{5}u_{xz}v_{txx}+\frac{4}{5}u_z v_{txxx}+\frac{4}{5}v u_{txxxz},\\
T_4^y=\;&\frac{1}{3}\beta v_x u_{xz}-\frac{1}{3}\beta u_z v_{xx}-\frac{1}{3}\beta v u_{xxz},\\
T_4^z=\;&\frac{1}{3}\gamma v_x u_{xz}-\frac{1}{3}\gamma u_z v_{xx}+\frac{2}{3}\gamma v u_{xxz}+\beta v u_{xxy}+\alpha v u_{xxx}-4v u_{xxx}u_t\\
&-12v u_{xx}u_{tx}-8v u_x u_{txx}-v u_{txxxx}+v u_{ttt};\\
T_5^t=\;&v_{tt}+\frac{2}{3}u_{xx}v_x-\frac{8}{3}u_x v_{xx}-\frac{1}{5}v_{xxxx}-\frac{2}{3}u_{xxx}v,\\
T_5^x=\;&\frac{2}{3}u_{txx}v-4u_t v_{xx}+\frac{2}{3}v_t u_{xx}-\frac{4}{3}v_x u_{tx}-\frac{16}{3}u_x v_{tx}\\
&-\frac{4}{5}v_{txxx}+\alpha v_{xx}+\frac{2}{3}\beta v_{xy}+\frac{2}{3}\gamma v_{xz},\\
T_5^y=\;&\frac{1}{3}\beta v_{xx},\\
T_5^z=\;&\frac{1}{3}\gamma v_{xx};\\
T_6^t=\;&\frac{32}{3}y v_{xx}u_x^2-\frac{40}{3}y v_x u_{xx}u_x+\frac{8}{3}\beta t v_{xx}u_x+\frac{40}{3}y v u_{xxx}u_x+\frac{4}{5}y v_{xxxx}u_x\\
&+\frac{40}{3}y v u_{xx}^2-\frac{2}{3}\beta t v_x u_{xx}+\frac{2}{3}\beta t v u_{xxx}+\frac{4}{5}y v_{xx}u_{xxx}-4y v_{tt}u_x\\
&-\frac{4}{5}y u_{xx}v_{xxx}-\frac{4}{5}y v_x u_{xxxx}+\frac{1}{5}\beta t v_{xxxx}+\frac{4}{5}y v u_{xxxxx}+\beta v_t+4y v_t u_{tx}\\
&-\beta t v_{tt}-4y v u_{ttx},\\
T_6^x=\;&\frac{4}{3}y v_x u_{xy}\beta-\frac{2}{3}t v_{xy}\beta^2-\frac{4}{3}u_x v_x\beta-\frac{2}{3}t\gamma v_{xz}\beta-\frac{8}{3}y u_x v_{xy}\beta+\frac{2}{3}v u_{xx}\beta\\
&+\frac{4}{3}y v_y u_{xx}\beta-t\alpha v_{xx}\beta+\frac{4}{3}y v u_{xxy}\beta-\frac{1}{5}v_{xxx}\beta+4t v_{xx}u_t\beta-\frac{2}{3}t u_{xx}v_t\beta\\
&+\frac{4}{3}t v_x u_{tx}\beta+\frac{16}{3}t u_x v_{tx}\beta-\frac{2}{3}t v u_{txx}\beta+\frac{4}{5}t v_{txxx}\beta+\frac{4}{3}y\gamma v_x u_{xz}\\
&-\frac{8}{3}\gamma y u_x v_{xz}+\frac{4}{3}y\gamma v_z u_{xx}+4y\alpha v_x u_{xx}-4y\alpha u_x v_{xx}+\frac{4}{3}y\gamma v u_{xxz}\\
&-16y v_x u_{xx}u_t+16y u_x v_{xx}u_t-\frac{40}{3}y u_x u_{xx}v_t-\frac{4}{5}y u_{xxxx}v_t-\frac{16}{3}y u_x v_x u_{tx}\\
&-\frac{40}{3}y v u_{xx}u_{tx}-\frac{4}{5}y v_{xxx}u_{tx}+\frac{64}{3}y u_x^2 v_{tx}+\frac{8}{5}y u_{xxx}v_{tx}-\frac{40}{3}y v u_x u_{txx}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{5}yv_{xx}u_{txx}-\frac{12}{5}yu_{xx}v_{txx}-\frac{12}{5}yv_xu_{txxx}+\frac{16}{5}yu_xv_{txxx}-\frac{4}{5}yvu_{txxxx}\\
&+4yvu_{ttt},\\
T_6^y=&\ \frac{4}{3}yv_xu_{xx}\beta-\frac{1}{3}tv_{xx}\beta^2-\frac{4}{3}yu_xv_{xx}\beta-\frac{4}{3}yvu_{xxx}\beta,\\
T_6^z=&\ \frac{4}{3}\gamma yv_xu_{xx}-\frac{1}{3}\beta\gamma tv_{xx}-\frac{4}{3}\gamma yu_xv_{xx}-\frac{4}{3}y\gamma vu_{xxx};\\
T_7^t=&\ \frac{32}{3}zv_{xx}u_x^2-\frac{40}{3}zv_xu_{xx}u_x+\frac{8}{3}\gamma tv_{xx}u_x+\frac{40}{3}zvu_{xxx}u_x+\frac{4}{5}zv_{xxxx}u_x\\
&-4zv_{tt}u_x+\frac{40}{3}zvu_{xx}^2-\frac{2}{3}\gamma tv_xu_{xx}+\frac{2}{3}\gamma tvu_{xxx}+\frac{4}{5}zv_{xx}u_{xxx}\\
&-\frac{4}{5}zu_{xx}v_{xxx}-\frac{4}{5}zv_xu_{xxxx}+\frac{1}{5}\gamma tv_{xxxx}+\frac{4}{5}zvu_{xxxxx}+\gamma v_t\\
&+4zv_tu_{tx}-\gamma tv_{tt}-4zvu_{ttx},\\
T_7^x=&\ \frac{4}{3}\gamma zv_xu_{xz}-\frac{2}{3}\gamma tv_{xz}^2-\frac{4}{3}u_xv_x\gamma-\frac{8}{3}\gamma zu_xv_{xz}-\frac{2}{3}\gamma\beta tv_{xy}+\frac{2}{3}\gamma vu_{xx}\\
&+\frac{4}{3}\gamma zv_zu_{xx}-\alpha\gamma tv_{xx}+\frac{4}{3}\gamma zvu_{xxz}-\frac{1}{5}\gamma v_{xxx}+4\gamma tv_{xx}u_t-\frac{2}{3}tu_{xx}v_t\gamma\\
&+\frac{4}{3}tv_xu_{tx}\gamma+\frac{16}{3}tu_xv_{tx}\gamma-\frac{2}{3}tvu_{txx}\gamma+\frac{4}{5}tv_{txxx}\gamma\\
&+\frac{4}{3}z\beta v_xu_{xy}-\frac{8}{3}z\beta u_xv_{xy}+\frac{4}{3}z\beta v_yu_{xx}+4z\alpha v_xu_{xx}-4z\alpha u_xv_{xx}\\
&+\frac{4}{3}z\beta vu_{xxy}-16zv_xu_{xx}u_t+16zu_xv_{xx}u_t-\frac{40}{3}zu_xu_{xx}v_t-\frac{4}{5}zu_{xxxx}v_t\\
&-\frac{16}{3}zu_xv_xu_{tx}-\frac{40}{3}zvu_{xx}u_{tx}-\frac{4}{5}zv_{xxx}u_{tx}+\frac{64}{3}zu_x^2v_{tx}+\frac{8}{5}zu_{xxx}v_{tx}\\
&-\frac{40}{3}zvu_xu_{txx}+\frac{8}{5}zv_{xx}u_{txx}-\frac{12}{5}zu_{xx}v_{txx}-\frac{12}{5}zv_xu_{txxx}+\frac{16}{5}zu_xv_{txxx}\\
&-\frac{4}{5}zvu_{txxxx}+4zvu_{ttt},\\
T_7^y=&\ \frac{4}{3}\beta zv_xu_{xx}-\frac{1}{3}\beta\gamma tv_{xx}-\frac{4}{3}\beta zu_xv_{xx}-\frac{4}{3}\beta zvu_{xxx},\\
T_7^z=&\ \frac{4}{3}\gamma zv_xu_{xx}-\frac{1}{3}\gamma^2tv_{xx}-\frac{4}{3}\gamma zu_xv_{xx}-\frac{4}{3}\gamma zvu_{xxx};\\
T_8^t=&\ \frac{10}{3}\beta vu_{xx}-\frac{8}{3}\beta u_xv_x-\frac{2}{3}x\beta u_{xx}v_x-\frac{16}{3}yu_{xx}u_tv_x-\frac{64}{3}yu_xu_{tx}v_x\\
&+\frac{8}{3}x\beta u_xv_{xx}+8y\gamma vu_{xxz}+8y\beta vu_{xxy}+8y\alpha vu_{xxx}+\frac{2}{3}x\beta vu_{xxx}-\frac{1}{5}\beta v_{xxx}\\
&+\frac{1}{5}x\beta v_{xxxx}+\frac{64}{3}yu_xv_{xx}u_t-\frac{80}{3}yvu_{xxx}u_t+\frac{8}{5}yv_{xxxx}u_t-\frac{208}{3}yvu_{xx}u_{tx}\\
&-\frac{8}{5}yv_{xxx}u_{tx}-\frac{128}{3}yvu_xu_{txx}+\frac{8}{5}yv_{xx}u_{txx}-\frac{32}{5}yvu_{txxxx}+8yv_tu_{tt}-\beta xv_{tt}\\
&-\frac{8}{5}yu_{txxx}v_x-8yu_tv_{tt},\\
T_8^x=&\ \frac{1}{3}v_y\beta^2-\frac{2}{3}xv_{xy}\beta^2+\frac{1}{3}\gamma v_z\beta+\alpha v_x\beta-\frac{2}{3}x\gamma v_{xz}\beta-x\alpha v_{xx}\beta-\frac{4}{3}v_xu_t\beta\\
&-\frac{16}{3}yv_{xy}u_t\beta+4xv_{xx}u_t\beta-\frac{8}{3}u_xv_t\beta-\frac{2}{3}xu_{xx}v_t\beta+\frac{8}{3}yv_xu_{ty}\beta+\frac{8}{3}yv_yu_{tx}\beta\\
&+\frac{4}{3}xv_xu_{tx}\beta+\frac{16}{3}xu_xv_{tx}\beta-\frac{16}{3}yvu_{txy}\beta-\frac{2}{3}xvu_{txx}\beta-\frac{3}{5}v_{txx}\beta+\frac{4}{5}xv_{txxx}\beta\\
&+32yv_{xx}u_t^2+\frac{128}{3}yvu_{tx}^2-\frac{16}{3}y\gamma v_{xz}u_t-8y\alpha v_{xx}u_t-\frac{16}{3}yu_{xx}u_tv_t+\frac{8}{3}y\gamma v_xu_{tz}\\
&+\frac{8}{3}y\gamma v_zu_{tx}+8y\alpha v_xu_{tx}-\frac{64}{3}yv_xu_tu_{tx}-\frac{64}{3}yu_xv_tu_{tx}+\frac{128}{3}yu_xu_tv_{tx}\\
&-\frac{16}{3}y\gamma vu_{txz}-8y\alpha vu_{txx}+\frac{80}{3}yvu_tu_{txx}+\frac{16}{5}yv_{tx}u_{txx}-\frac{24}{5}yu_{tx}v_{txx}\\
&-\frac{8}{5}yv_tu_{txxx}+\frac{32}{5}yu_tv_{txxx}-\frac{64}{3}yu_xv_xu_{tt}+\frac{80}{3}yvu_{xx}u_{tt}-\frac{8}{5}yv_{xxx}u_{tt}\\
&+\frac{128}{3}yvu_xu_{ttx}+\frac{16}{5}yv_{xx}u_{ttx}-\frac{24}{5}yv_xu_{ttxx}+\frac{32}{5}yvu_{ttxxx},\\
T_8^y=&\ \frac{1}{3}\beta^2v_x-\frac{1}{3}\beta^2xv_{xx}-\frac{8}{3}\beta yv_{xx}u_t+\frac{8}{3}\beta yv_xu_{tx}-\frac{8}{3}\beta yvu_{txx},\\
T_8^z=&\ \frac{1}{3}\beta\gamma v_x+\frac{8}{3}y\gamma u_{tx}v_x-\frac{1}{3}x\beta\gamma v_{xx}-\frac{8}{3}y\gamma v_{xx}u_t-\frac{8}{3}y\gamma vu_{txx};\\
T_9^t=&\ \frac{10}{3}\gamma vu_{xx}-\frac{8}{3}\gamma u_xv_x-\frac{2}{3}x\gamma u_{xx}v_x-\frac{16}{3}zu_{xx}u_tv_x-\frac{64}{3}zu_xu_{tx}v_x-\frac{8}{5}zu_{txxx}v_x
\end{aligned}
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\begin{aligned}
&+\frac{8}{3}x\gamma u_x v_{xx}+8z\gamma v u_{xxz}+8z\beta v u_{xxy}+8z\alpha v u_{xxx}+\frac{2}{3}x\gamma v u_{xxx}-\frac{1}{5}\gamma v_{xxx}\\
&+\frac{1}{5}x\gamma v_{xxxx}+\frac{64}{3}zu_x v_{xx}u_t-\frac{80}{3}zvu_{xxx}u_t+\frac{8}{5}zv_{xxxx}u_t-\frac{208}{3}zvu_{xx}u_{tx}\\
&-\frac{8}{5}zv_{xxx}u_{tx}-\frac{128}{3}zvu_x u_{txx}+\frac{8}{5}zv_{xx}u_{txx}-\frac{32}{5}zvu_{txxxx}+8zv_t u_{tt}-x\gamma v_{tt}\\
&-8zu_t v_{tt},\\
T_9^x=&\ \frac{1}{3}\gamma^2 v_z-\frac{2}{3}\gamma^2 x v_{xz}+\frac{1}{3}\gamma\beta v_y+\alpha\gamma v_x-\frac{2}{3}\gamma x\beta v_{xy}-\alpha\gamma x v_{xx}-\frac{4}{3}\gamma v_x u_t\\
&-\frac{16}{3}\gamma z v_{xz}u_t+4\gamma x v_{xx}u_t-\frac{8}{3}\gamma u_x v_t-\frac{2}{3}\gamma x u_{xx}v_t+\frac{8}{3}\gamma z v_x u_{tz}+\frac{8}{3}zv_z u_{tx}\gamma\\
&+\frac{4}{3}xv_x u_{tx}\gamma+\frac{16}{3}xu_x v_{tx}\gamma-\frac{16}{3}zvu_{txz}\gamma-\frac{2}{3}xvu_{txx}\gamma-\frac{3}{5}v_{txx}\gamma+\frac{4}{5}xv_{txxx}\gamma\\
&+32zv_{xx}u_t^2+\frac{128}{3}zvu_{tx}^2-\frac{16}{3}z\beta v_{xy}u_t-8z\alpha v_{xx}u_t-\frac{16}{3}zu_{xx}u_t v_t+\frac{8}{3}z\beta v_x u_{ty}\\
&+\frac{8}{3}z\beta v_y u_{tx}+8z\alpha v_x u_{tx}-\frac{64}{3}zv_x u_t u_{tx}-\frac{64}{3}zu_x v_t u_{tx}+\frac{128}{3}zu_x u_t v_{tx}\\
&-\frac{16}{3}z\beta v u_{txy}-8z\alpha v u_{txx}+\frac{80}{3}zvu_t u_{txx}+\frac{16}{5}zv_{tx}u_{txx}-\frac{24}{5}zu_{tx}v_{txx}\\
&-\frac{8}{5}zv_t u_{txxx}+\frac{32}{5}zu_t v_{txxx}-\frac{64}{3}zu_x v_x u_{tt}+\frac{80}{3}zvu_{xx}u_{tt}-\frac{8}{5}zv_{xxx}u_{tt}\\
&+\frac{128}{3}zvu_x u_{ttx}+\frac{16}{5}zv_{xx}u_{ttx}-\frac{24}{5}zv_x u_{ttxx}+\frac{32}{5}zvu_{ttxxx},\\
T_9^y=&\ \frac{1}{3}\beta\gamma v_x+\frac{8}{3}z\beta u_{tx}v_x-\frac{1}{3}x\beta\gamma v_{xx}-\frac{8}{3}z\beta v_{xx}u_t-\frac{8}{3}z\beta v u_{txx},\\
T_9^z=&\ \frac{1}{3}\gamma^2 v_x-\frac{1}{3}\gamma^2 x v_{xx}-\frac{8}{3}\gamma z v_{xx}u_t+\frac{8}{3}\gamma z v_x u_{tx}-\frac{8}{3}\gamma z v u_{txx};\\
T_{10}^t=&\ 8tvu_{xxy}\beta^2+\frac{40}{3}xvu_{xx}^2\beta-\frac{64}{3}u_x^2 v_x\beta-\frac{128}{3}yu_x v_x u_{xy}\beta+\frac{176}{3}vu_x u_{xx}\beta\\
&+2t\alpha v_x u_{xx}\beta-\frac{8}{3}uv_x u_{xx}\beta-\frac{32}{3}yu_y v_x u_{xx}\beta-\frac{40}{3}xu_x v_x u_{xx}\beta+\frac{32}{3}xu_x^2 v_{xx}\beta\\
&-8t\alpha u_x v_{xx}\beta+\frac{32}{3}uu_x v_{xx}\beta+\frac{128}{3}yu_y u_x v_{xx}\beta+\frac{160}{3}yvu_{xy}u_{xx}\beta+\frac{12}{5}u_{xx}v_{xx}\beta\\
&+8t\gamma v u_{xxz}\beta+\frac{128}{3}yvu_x u_{xxy}\beta+\frac{16}{5}yv_{xx}u_{xxy}\beta+6t\alpha v u_{xxx}\beta+\frac{8}{3}uvu_{xxx}\beta\\
&+\frac{32}{3}yvu_y u_{xxx}\beta+\frac{40}{3}xvu_x u_{xxx}\beta-\frac{16}{5}v_x u_{xxx}\beta+\frac{4}{5}xv_{xx}u_{xxx}\beta-\frac{8}{5}u_x v_{xxx}\beta\\
&-\frac{16}{5}yu_{xy}v_{xxx}\beta-\frac{4}{5}xu_{xx}v_{xxx}\beta-\frac{16}{5}yv_x u_{xxxy}\beta+4vu_{xxxx}\beta-\frac{4}{5}xv_x u_{xxxx}\beta\\
&-\frac{3}{5}t\alpha v_{xxxx}\beta+\frac{4}{5}uv_{xxxx}\beta+\frac{16}{5}yu_y v_{xxxx}\beta+\frac{4}{5}xu_x v_{xxxx}\beta+\frac{16}{5}yvu_{xxxxy}\beta\\
&+\frac{4}{5}xvu_{xxxxx}\beta-\frac{16}{3}tv_x u_{xx}u_t\beta+\frac{64}{3}tu_x v_{xx}u_t\beta-\frac{80}{3}tvu_{xxx}u_t\beta-3\alpha v_t\beta\\
&+12u_t v_t\beta+\frac{8}{5}tv_{xxxx}u_t\beta+16yv_t u_{ty}\beta-\frac{64}{3}tu_x v_x u_{tx}\beta-\frac{208}{3}tvu_{xx}u_{tx}\beta\\
&-\frac{8}{5}tv_{xxx}u_{tx}\beta+4xv_t u_{tx}\beta-\frac{128}{3}tvu_x u_{txx}\beta+\frac{8}{5}tv_{xx}u_{txx}\beta-\frac{8}{5}tv_x u_{txxx}\beta\\
&-\frac{32}{5}tvu_{txxxx}\beta-20vu_{tt}\beta+8tv_t u_{tt}\beta+3t\alpha v_{tt}\beta-4uv_{tt}\beta-16yu_y v_{tt}\beta\\
&-8tu_t v_{tt}\beta-16yvu_{tty}\beta-4xvu_{ttx}\beta-\frac{128}{3}y\gamma u_x v_x u_{xz}-\frac{32}{3}y\gamma u_z v_x u_{xx}\\
&-4xu_x v_{tt}\beta+\frac{160}{3}y\gamma v u_{xz}u_{xx}+\frac{128}{3}y\gamma u_z u_x v_{xx}+\frac{128}{3}y\gamma v u_x u_{xxz}\\
&+\frac{16}{5}y\gamma v_{xx}u_{xxz}+\frac{32}{3}y\gamma v u_z u_{xxx}-\frac{16}{5}y\gamma u_{xz}v_{xxx}-\frac{16}{5}y\gamma v_x u_{xxxz}\\
&+\frac{16}{5}y\gamma u_z v_{xxxx}+\frac{16}{5}y\gamma v u_{xxxxz}+16y\gamma v_t u_{tz}-16y\gamma u_z v_{tt}-16y\gamma v u_{ttz},\\
T_{10}^x=&\ 3t\beta v_{xx}\alpha^2+16\beta u_x v_x\alpha+16y\gamma v_x u_{xz}\alpha+2t\beta\gamma v_{xz}\alpha+16y\beta v_x u_{xy}\alpha\\
&+2t\beta^2 v_{xy}\alpha-22\beta v u_{xx}\alpha+4x\beta v_x u_{xx}\alpha-4\beta u v_{xx}\alpha-16y\gamma u_z v_{xx}\alpha\\
&-16y\beta u_y v_{xx}\alpha-4x\beta u_x v_{xx}\alpha-16y\gamma v u_{xxz}\alpha-16y\beta v u_{xxy}\alpha+\frac{3}{5}\beta v_{xxx}\alpha\\
&-20t\beta v_{xx}u_t\alpha+2t\beta u_{xx}v_t\alpha+4t\beta v_x u_{tx}\alpha-16t\beta u_x v_{tx}\alpha-6t\beta v u_{txx}\alpha\\
&-\frac{12}{5}t\beta v_{txxx}\alpha+32t\beta v_{xx}u_t^2+\frac{128}{3}t\beta v u_{tx}^2+\frac{8}{3}\beta\gamma v_z u_x+\frac{8}{3}\beta^2 v_y u_x\\
&+\frac{20}{3}\beta\gamma u_z v_x+\frac{16}{3}y\gamma^2 u_{zz}v_x+\frac{20}{3}\beta^2 u_y v_x+\frac{32}{3}y\beta\gamma u_{yz}v_x+\frac{16}{3}y\beta^2 u_{yy}v_x
\end{aligned}
$$

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\begin{aligned}
&- 16\beta\gamma v u_{xz} + \frac{16}{3} y\gamma^2 v_z u_{xz} + \frac{16}{3} y\beta\gamma v_y u_{xz} + \frac{4}{3} x\beta\gamma v_x u_{xz} - \frac{8}{3}\beta\gamma u v_{xz} \\
&- \frac{32}{3} y\gamma^2 u_z v_{xz} - \frac{32}{3} y\beta\gamma u_y v_{xz} - \frac{8}{3} x\beta\gamma u_x v_{xz} - \frac{32}{3} y\gamma^2 v u_{xzz} - 16\beta^2 v u_{xy} \\
&+ \frac{16}{3} y\beta\gamma v_z u_{xy} + \frac{16}{3} y\beta^2 v_y u_{xy} + \frac{4}{3} x\beta^2 v_x u_{xy} - \frac{8}{3}\beta^2 u v_{xy} - \frac{32}{3} y\beta\gamma u_z v_{xy} \\
&- \frac{32}{3} y\beta^2 u_y v_{xy} - \frac{8}{3} x\beta^2 u_x v_{xy} - \frac{64}{3} y\beta\gamma v u_{xyz} - \frac{32}{3} y\beta^2 v u_{xyy} + \frac{4}{3} x\beta\gamma v_z u_{xx} \\
&+ \frac{4}{3} x\beta^2 v_y u_{xx} + \frac{4}{3} x\beta\gamma v u_{xxz} + \frac{4}{3} x\beta^2 v u_{xxy} - 64\beta u_x v_x u_t - 64 y\gamma v_x u_{xz} u_t \\
&- \frac{16}{3} t\beta\gamma v_{xz} u_t - 64 y\beta v_x u_{xy} u_t - \frac{16}{3} t\beta^2 v_{xy} u_t + 88\beta v u_{xx} u_t - 16 x\beta v_x u_{xx} u_t \\
&+ 16\beta u v_{xx} u_t + 64 y\gamma u_z v_{xx} u_t + 64 y\beta u_y v_{xx} u_t + 16 x\beta u_x v_{xx} u_t + 64 y\gamma v u_{xxz} u_t \\
&+ 64 y\beta v u_{xxy} u_t - \frac{12}{5}\beta v_{xxx} u_t - \frac{64}{3}\beta u_x^2 v_t - \frac{128}{3} y\gamma u_x u_{xz} v_t - \frac{128}{3} y\beta u_x u_{xy} v_t \\
&- \frac{8}{3}\beta u u_{xx} v_t - \frac{32}{3} y\gamma u_z u_{xx} v_t - \frac{32}{3} y\beta u_y u_{xx} v_t - \frac{40}{3} x\beta u_x u_{xx} v_t - \frac{16}{5}\beta u_{xxx} v_t \\
&- \frac{16}{5} y\gamma u_{xxxz} v_t - \frac{16}{5} y\beta u_{xxxy} v_t - \frac{4}{5} x\beta u_{xxxx} v_t - \frac{16}{3} t\beta u_{xx} u_t v_t + \frac{8}{3} t\beta\gamma v_x u_{tz} \\
&- \frac{128}{3} y\gamma u_x v_x u_{tz} + \frac{160}{3} y\gamma v u_{xx} u_{tz} - \frac{16}{5} y\gamma v_{xxx} u_{tz} - \frac{128}{3} y\beta u_x v_x u_{ty} \\
&+ \frac{8}{3} t\beta^2 v_x u_{ty} + \frac{160}{3} y\beta v u_{xx} u_{ty} - \frac{16}{5} y\beta v_{xxx} u_{ty} + \frac{8}{3} t\beta\gamma v_z u_{tx} + \frac{8}{3} t\beta^2 v_y u_{tx} \\
&+ 128\beta v u_x u_{tx} + \frac{16}{3}\beta u v_x u_{tx} + \frac{64}{3} y\gamma u_z v_x u_{tx} + \frac{64}{3} y\beta u_y v_x u_{tx} - \frac{16}{3} x\beta u_x v_x u_{tx} \\
&+ \frac{256}{3} y\gamma v u_{xz} u_{tx} + \frac{256}{3} y\beta v u_{xy} u_{tx} - \frac{40}{3} x\beta v u_{xx} u_{tx} + \frac{32}{5}\beta v_{xx} u_{tx} + \frac{64}{3} x\beta u_x^2 v_{tx} \\
&- \frac{4}{5} x\beta v_{xxx} u_{tx} - \frac{64}{3} t\beta v_x u_t u_{tx} - \frac{64}{3} t\beta u_x v_t u_{tx} + \frac{64}{3}\beta u u_x v_{tx} + \frac{256}{3} y\gamma u_z u_x v_{tx} \\
&+ \frac{256}{3} y\beta u_y u_x v_{tx} + \frac{24}{5}\beta u_{xx} v_{tx} + \frac{32}{5} y\gamma u_{xxz} v_{tx} + \frac{32}{5} y\beta u_{xxy} v_{tx} + \frac{8}{5} x\beta u_{xxx} v_{tx} \\
&+ \frac{128}{3} t\beta u_x u_t v_{tx} - \frac{16}{3} t\beta\gamma v u_{txz} + \frac{256}{3} y\gamma v u_x u_{txz} + \frac{32}{5} y\gamma v_{xx} u_{txz} - \frac{16}{3} t\beta^2 v u_{txy} \\
&+ \frac{256}{3} y\beta v u_x u_{txy} + \frac{32}{5} y\beta v_{xx} u_{txy} - \frac{8}{3}\beta u v u_{txx} - \frac{32}{3} y\gamma v u_z u_{txx} - \frac{32}{3} y\beta v u_y u_{txx} \\
&- \frac{40}{3} x\beta v u_x u_{txx} - 12\beta v_x u_{txx} + \frac{8}{5} x\beta v_{xx} u_{txx} + \frac{80}{3} t\beta v u_t u_{txx} + \frac{16}{5} t\beta v_{tx} u_{txx} \\
&- \frac{24}{5}\beta u_x v_{txx} - \frac{48}{5} y\gamma u_{xz} v_{txx} - \frac{48}{5} y\beta u_{xy} v_{txx} - \frac{12}{5} x\beta u_{xx} v_{txx} - \frac{24}{5} t\beta u_{tx} v_{txx} \\
&- \frac{48}{5} y\gamma v_x u_{txxz} - \frac{48}{5} y\beta v_x u_{txxy} + \frac{96}{5}\beta v u_{txxx} - \frac{12}{5} x\beta v_x u_{txxx} - \frac{8}{5} t\beta v_t u_{txxx} \\
&+ \frac{16}{5}\beta u v_{txxx} + \frac{64}{5} y\gamma u_z v_{txxx} + \frac{64}{5} y\beta u_y v_{txxx} + \frac{16}{5} x\beta u_x v_{txxx} + \frac{32}{5} t\beta u_t v_{txxx} \\
&+ \frac{64}{5} y\gamma v u_{txxxz} + \frac{64}{5} y\beta v u_{txxxy} - \frac{4}{5} x\beta v u_{txxxx} - \frac{64}{3} t\beta u_x v_x u_{tt} + \frac{80}{3} t\beta v u_{xx} u_{tt} \\
&- \frac{8}{5} t\beta v_{xxx} u_{tt} + \frac{128}{3} t\beta v u_x u_{ttx} + \frac{16}{5} t\beta v_{xx} u_{ttx} - \frac{24}{5} t\beta v_x u_{ttxx} + 4x\beta v u_{ttt} \\
&+ \frac{32}{5} t\beta v u_{ttxxx},
\end{aligned}
$$

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\begin{aligned}
T_{10}^y = {}& \frac{8}{3} u_x v_x \beta^2 + \frac{16}{3} y v_x u_{xy}\beta^2 - 4 v u_{xx}\beta^2 + \frac{4}{3} x v_x u_{xx}\beta^2 + t\alpha v_{xx}\beta^2 - \frac{4}{3} u v_{xx}\beta^2 \\
&- \frac{16}{3} y u_y v_{xx}\beta^2 - \frac{4}{3} x u_x v_{xx}\beta^2 + \frac{32}{3} y v u_{xxy}\beta^2 - \frac{4}{3} x v u_{xxx}\beta^2 - \frac{8}{3} t v_{xx} u_t \beta^2 \\
&+ \frac{8}{3} t v_x u_{tx}\beta^2 - \frac{8}{3} t v u_{txx}\beta^2 + \frac{16}{3} y\gamma v_x u_{xz}\beta - \frac{16}{3} y\gamma u_z v_{xx}\beta + \frac{32}{3} y\gamma v u_{xxz}\beta \\
&- 64 y v u_{xxx} u_t \beta - 192 y v u_{xx} u_{tx}\beta - 128 y v u_x u_{txx}\beta - 16 y v u_{txxxx}\beta + 16 y v u_{ttt}\beta \\
&+ 16 y\alpha v u_{xxx}\beta,
\end{aligned}
$$

$$
\begin{aligned}
T_{10}^z = {}& \frac{16}{3} y v_x u_{xz}\gamma^2 - \frac{16}{3} y u_z v_{xx}\gamma^2 + \frac{32}{3} y v u_{xxz}\gamma^2 + \frac{8}{3}\beta u_x v_x \gamma + \frac{16}{3} y\beta v_x u_{xy}\gamma \\
&- 4\beta v u_{xx}\gamma + \frac{4}{3} x\beta v_x u_{xx}\gamma + t\alpha\beta v_{xx}\gamma - \frac{4}{3}\beta u v_{xx}\gamma - \frac{16}{3} y\beta u_y v_{xx}\gamma \\
&- \frac{4}{3} x\beta u_x v_{xx}\gamma + \frac{32}{3} y\beta v u_{xxy}\gamma + 16 y\alpha v u_{xxx}\gamma - \frac{4}{3} x\beta v u_{xxx}\gamma - \frac{8}{3} t\beta v_{xx} u_t \gamma \\
&- 64 y v u_{xxx} u_t \gamma + \frac{8}{3} t\beta v_x u_{tx}\gamma - 192 y v u_{xx} u_{tx}\gamma - \frac{8}{3} t\beta v u_{txx}\gamma - 128 y v u_x u_{txx}\gamma \\
&- 16 y v u_{txxxx}\gamma + 16 y v u_{ttt}\gamma;
\end{aligned}
$$

$$
\begin{aligned}
T_{11}^{t} = {} & 8tvu_{xxz}\gamma^2 + \frac{40}{3}xvu_{xx}^2\gamma - \frac{64}{3}u_x^2v_x\gamma - \frac{128}{3}zu_xv_xu_{xz}\gamma + \frac{176}{3}vu_xu_{xx}\gamma \\
& + 2t\alpha v_xu_{xx}\gamma - \frac{8}{3}uv_xu_{xx}\gamma - \frac{32}{3}zu_zv_xu_{xx}\gamma - \frac{40}{3}xu_xv_xu_{xx}\gamma + \frac{32}{3}xu_x^2v_{xx}\gamma \\
& - 8t\alpha u_xv_{xx}\gamma + \frac{160}{3}zvu_{xz}u_{xx}\gamma + \frac{32}{3}uu_xv_{xx}\gamma + \frac{128}{3}zu_zu_xv_{xx}\gamma + \frac{12}{5}u_{xx}v_{xx}\gamma \\
& + \frac{128}{3}zvu_xu_{xxz}\gamma + \frac{16}{5}zv_{xx}u_{xxz}\gamma + 8t\beta vu_{xxy}\gamma + 6t\alpha vu_{xxx}\gamma + \frac{8}{3}uvu_{xxx}\gamma \\
& + \frac{32}{3}zvu_zu_{xxx}\gamma + \frac{40}{3}xvu_xu_{xxx}\gamma - \frac{16}{5}v_xu_{xxx}\gamma + \frac{4}{5}xv_{xx}u_{xxx}\gamma - \frac{8}{5}u_xv_{xxx}\gamma \\
& - \frac{16}{5}zu_{xz}v_{xxx}\gamma - \frac{4}{5}xu_{xx}v_{xxx}\gamma - \frac{16}{5}zv_xu_{xxxz}\gamma + 4vu_{xxxx}\gamma - \frac{4}{5}xv_xu_{xxxx}\gamma \\
& - \frac{3}{5}t\alpha v_{xxxx}\gamma + \frac{4}{5}uv_{xxxx}\gamma + \frac{16}{5}zu_zv_{xxxx}\gamma + \frac{4}{5}xu_xv_{xxxx}\gamma + \frac{16}{5}zvu_{xxxxz}\gamma \\
& + \frac{4}{5}xvu_{xxxxx}\gamma - \frac{16}{3}tv_xu_{xx}u_t\gamma + \frac{64}{3}tu_xv_{xx}u_t\gamma - \frac{80}{3}tvu_{xxx}u_t\gamma - 3\alpha v_t\gamma \\
& + 12u_tv_t\gamma + 16zv_tu_{tz}\gamma + \frac{8}{5}tv_{xxxx}u_t\gamma - \frac{64}{3}tu_xv_xu_{tx}\gamma - \frac{208}{3}tvu_{xx}u_{tx}\gamma \\
& - \frac{8}{5}tv_{xxx}u_{tx}\gamma + 4xv_tu_{tx}\gamma - \frac{128}{3}tvu_xu_{txx}\gamma + \frac{8}{5}tv_{xx}u_{txx}\gamma - \frac{8}{5}tv_xu_{txxx}\gamma \\
& - \frac{32}{5}tvu_{txxxx}\gamma - 20vu_{tt}\gamma + 8tv_tu_{tt}\gamma + 3t\alpha v_{tt}\gamma - 4uv_{tt}\gamma - 16zu_zv_{tt}\gamma \\
& - 8tu_tv_{tt}\gamma - 16zvu_{ttz}\gamma - 4xvu_{ttx}\gamma - \frac{128}{3}z\beta u_xv_xu_{xy} - \frac{32}{3}z\beta u_yv_xu_{xx} \\
& - 4xu_xv_{tt}\gamma + \frac{160}{3}z\beta vu_{xy}u_{xx} + \frac{128}{3}z\beta u_yu_xv_{xx} + \frac{128}{3}z\beta vu_xu_{xxy} \\
& + \frac{16}{5}z\beta v_{xx}u_{xxy} + \frac{32}{3}z\beta vu_yu_{xxx} - \frac{16}{5}z\beta u_{xy}v_{xxx} - \frac{16}{5}z\beta v_xu_{xxxy} \\
& + \frac{16}{5}z\beta u_yv_{xxxx} + \frac{16}{5}z\beta vu_{xxxxy} + 16z\beta v_tu_{ty} - 16z\beta u_yv_{tt} - 16z\beta vu_{tty}, \\
T_{11}^{x} = {} & 3t\gamma v_{xx}\alpha^2 + 16\gamma u_xv_x\alpha + 16z\gamma v_xu_{xz}\alpha + 2t\gamma^2v_{xz}\alpha + 16z\beta v_xu_{xy}\alpha + 2t\beta\gamma v_{xy}\alpha
\end{aligned}
$$

$$
\begin{aligned}
& - 22\gamma vu_{xx}\alpha + 4x\gamma v_xu_{xx}\alpha - 4\gamma uv_{xx}\alpha - 16z\gamma u_zv_{xx}\alpha - 16z\beta u_yv_{xx}\alpha \\
& - 4x\gamma u_xv_{xx}\alpha - 16z\gamma vu_{xxz}\alpha - 16z\beta vu_{xxy}\alpha + \frac{3}{5}\gamma v_{xxx}\alpha - 20t\gamma v_{xx}u_t\alpha \\
& + 2t\gamma u_{xx}v_t\alpha + 4t\gamma v_xu_{tx}\alpha - 16t\gamma u_xv_{tx}\alpha - 6t\gamma vu_{txx}\alpha - \frac{12}{5}t\gamma v_{txxx}\alpha \\
& + 32t\gamma v_{xx}u_t^2 + \frac{128}{3}t\gamma vu_{tx}^2 + \frac{8}{3}\gamma^2v_zu_x + \frac{8}{3}\beta\gamma v_yu_x + \frac{20}{3}\gamma^2u_zv_x + \frac{16}{3}z\gamma^2u_{zz}v_x \\
& + \frac{20}{3}\beta\gamma u_yv_x + \frac{32}{3}z\beta\gamma u_{yz}v_x + \frac{16}{3}z\beta^2u_{yy}v_x - 16\gamma^2vu_{xz} + \frac{16}{3}z\gamma^2v_zu_{xz} \\
& + \frac{16}{3}z\beta\gamma v_yu_{xz} + \frac{4}{3}x\gamma^2v_xu_{xz} - \frac{8}{3}\gamma^2uv_{xz} - \frac{32}{3}z\gamma^2u_zv_{xz} - \frac{32}{3}z\beta\gamma u_yv_{xz} \\
& - \frac{8}{3}x\gamma^2u_xv_{xz} - \frac{32}{3}z\gamma^2vu_{xzz} - 16\beta\gamma vu_{xy} + \frac{16}{3}z\beta\gamma v_zu_{xy} + \frac{16}{3}z\beta^2v_yu_{xy} \\
& + \frac{4}{3}x\beta\gamma v_xu_{xy} - \frac{8}{3}\beta\gamma uv_{xy} - \frac{32}{3}z\beta\gamma u_zv_{xy} - \frac{32}{3}z\beta^2u_yv_{xy} - \frac{8}{3}x\beta\gamma u_xv_{xy} \\
& - \frac{64}{3}z\beta\gamma vu_{xyz} - \frac{32}{3}z\beta^2vu_{xyy} + \frac{4}{3}x\gamma^2v_zu_{xx} + \frac{4}{3}x\beta\gamma v_yu_{xx} + \frac{4}{3}x\gamma^2vu_{xxz} \\
& + \frac{4}{3}x\beta\gamma vu_{xxy} - 64\gamma u_xv_xu_t - 64z\gamma v_xu_{xz}u_t - \frac{16}{3}t\gamma^2v_{xz}u_t - 64z\beta v_xu_{xy}u_t \\
& - \frac{16}{3}t\beta\gamma v_{xy}u_t + 88\gamma vu_{xx}u_t - 16x\gamma v_xu_{xx}u_t + 16\gamma uv_{xx}u_t + 64z\gamma u_zv_{xx}u_t \\
& + 64z\beta u_yv_{xx}u_t + 16x\gamma u_xv_{xx}u_t + 64z\gamma vu_{xxz}u_t + 64z\beta vu_{xxy}u_t - \frac{12}{5}\gamma v_{xxx}u_t \\
& - \frac{64}{3}\gamma u_x^2v_t - \frac{128}{3}z\gamma u_xu_{xz}v_t - \frac{128}{3}z\beta u_xu_{xy}v_t - \frac{8}{3}\gamma uu_{xx}v_t - \frac{32}{3}z\gamma u_zu_{xx}v_t \\
& - \frac{32}{3}z\beta u_yu_{xx}v_t - \frac{40}{3}x\gamma u_xu_{xx}v_t - \frac{16}{5}\gamma u_{xxx}v_t - \frac{16}{5}z\gamma u_{xxxz}v_t - \frac{16}{5}z\beta u_{xxxy}v_t \\
& - \frac{4}{5}x\gamma u_{xxxx}v_t - \frac{16}{3}t\gamma u_{xx}u_tv_t + \frac{8}{3}t\gamma^2v_xu_{tz} - \frac{128}{3}z\gamma u_xv_xu_{tz} + \frac{160}{3}z\gamma vu_{xx}u_{tz} \\
& - \frac{16}{5}z\gamma v_{xxx}u_{tz} + \frac{8}{3}t\beta\gamma v_xu_{ty} - \frac{128}{3}z\beta u_xv_xu_{ty} + \frac{160}{3}z\beta vu_{xx}u_{ty} - \frac{16}{5}z\beta v_{xxx}u_{ty} \\
& + \frac{8}{3}t\gamma^2v_zu_{tx} + \frac{8}{3}t\beta\gamma v_yu_{tx} + 128\gamma vu_xu_{tx} + \frac{16}{3}\gamma uv_xu_{tx} + \frac{64}{3}z\gamma u_zv_xu_{tx} \\
& + \frac{64}{3}z\beta u_yv_xu_{tx} - \frac{16}{3}x\gamma u_xv_xu_{tx} + \frac{256}{3}z\gamma vu_{xz}u_{tx} + \frac{256}{3}z\beta vu_{xy}u_{tx}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{40}{3}x\gamma v u_{xx}u_{tx}+\frac{32}{5}\gamma v_{xx}u_{tx}-\frac{4}{5}x\gamma v_{xxx}u_{tx}-\frac{64}{3}t\gamma v_x u_t u_{tx}-\frac{64}{3}t\gamma u_x v_t u_{tx}\\
&+\frac{64}{3}x\gamma u_x^2 v_{tx}+\frac{64}{3}\gamma u u_x v_{tx}+\frac{256}{3}z\gamma u_z u_x v_{tx}+\frac{256}{3}z\beta u_y u_x v_{tx}+\frac{24}{5}\gamma u_{xx}v_{tx}\\
&+\frac{32}{5}z\gamma u_{xxz}v_{tx}+\frac{32}{5}z\beta u_{xxy}v_{tx}+\frac{8}{5}x\gamma u_{xxx}v_{tx}+\frac{128}{3}t\gamma u_x u_t v_{tx}-\frac{16}{3}t\gamma^2 v u_{txz}\\
&+\frac{256}{3}z\gamma v u_x u_{txz}+\frac{32}{5}z\gamma v_{xx}u_{txz}-\frac{16}{3}t\beta\gamma v u_{txy}+\frac{256}{3}z\beta v u_x u_{txy}+\frac{32}{5}z\beta v_{xx}u_{txy}\\
&-\frac{8}{3}\gamma u v u_{txx}-\frac{32}{3}z\gamma v u_z u_{txx}-\frac{32}{3}z\beta v u_y u_{txx}-\frac{40}{3}x\gamma v u_x u_{txx}-12\gamma v_x u_{txx}\\
&+\frac{8}{5}x\gamma v_{xx}u_{txx}+\frac{80}{3}t\gamma v u_t u_{txx}+\frac{16}{5}t\gamma v_{tx}u_{txx}-\frac{24}{5}\gamma u_x v_{txx}-\frac{48}{5}z\gamma u_{xz}v_{txx}\\
&-\frac{48}{5}z\beta u_{xy}v_{txx}-\frac{12}{5}x\gamma u_{xx}v_{txx}-\frac{24}{5}t\gamma u_{tx}v_{txx}-\frac{48}{5}z\gamma v_x u_{txxz}-\frac{48}{5}z\beta v_x u_{txxy}\\
&+\frac{96}{5}\gamma v u_{txxx}-\frac{12}{5}x\gamma v_x u_{txxx}-\frac{8}{5}t\gamma v_t u_{txxx}+\frac{16}{5}\gamma u v_{txxx}+\frac{64}{5}z\gamma u_z v_{txxx}\\
&+\frac{64}{5}z\beta u_y v_{txxx}+\frac{16}{5}x\gamma u_x v_{txxx}+\frac{32}{5}t\gamma u_t v_{txxx}+\frac{64}{5}z\gamma v u_{txxxz}+\frac{64}{5}z\beta v u_{txxxy}\\
&-\frac{4}{5}x\gamma v u_{txxxx}-\frac{64}{3}t\gamma u_x v_x u_{tt}+\frac{80}{3}t\gamma v u_{xx}u_{tt}-\frac{8}{5}t\gamma v_{xxx}u_{tt}+\frac{128}{3}t\gamma v u_x u_{ttx}\\
&+\frac{16}{5}t\gamma v_{xx}u_{ttx}-\frac{24}{5}t\gamma v_x u_{ttxx}+\frac{32}{5}t\gamma v u_{ttxxx}+4x\gamma v u_{ttt},
\end{aligned}
$$

$$
\begin{aligned}
T_{11}^{y}=&\ \frac{16}{3}z v_x u_{xy}\beta^2-\frac{16}{3}z u_y v_{xx}\beta^2+\frac{32}{3}z v u_{xxy}\beta^2+\frac{8}{3}\gamma u_x v_x\beta+\frac{16}{3}z\gamma v_x u_{xz}\beta\\
&-4\gamma v u_{xx}\beta+\frac{4}{3}x\gamma v_x u_{xx}\beta+t\alpha\gamma v_{xx}\beta-\frac{4}{3}\gamma u v_{xx}\beta-\frac{16}{3}z\gamma u_z v_{xx}\beta\\
&-\frac{4}{3}x\gamma u_x v_{xx}\beta+\frac{32}{3}z\gamma v u_{xxz}\beta+16z\alpha v u_{xxx}\beta-\frac{4}{3}x\gamma v u_{xxx}\beta-\frac{8}{3}t\gamma v_{xx}u_t\beta\\
&-64z v u_{xxx}u_t\beta+\frac{8}{3}t\gamma v_x u_{tx}\beta-192z v u_{xx}u_{tx}\beta-\frac{8}{3}t\gamma v u_{txx}\beta-128z v u_x u_{txx}\beta\\
&-16z v u_{txxxx}\beta+16z v u_{ttt}\beta,
\end{aligned}
$$

$$
\begin{aligned}
T_{11}^{z}=&\ \frac{8}{3}u_x v_x\gamma^2+\frac{16}{3}z v_x u_{xz}\gamma^2-4v u_{xx}\gamma^2+\frac{4}{3}x v_x u_{xx}\gamma^2+t\alpha v_{xx}\gamma^2-\frac{4}{3}u v_{xx}\gamma^2\\
&-\frac{16}{3}z u_z v_{xx}\gamma^2-\frac{4}{3}x u_x v_{xx}\gamma^2+\frac{32}{3}z v u_{xxz}\gamma^2-\frac{4}{3}x v u_{xxx}\gamma^2-\frac{8}{3}t v_{xx}u_t\gamma^2\\
&+\frac{8}{3}t v_x u_{tx}\gamma^2-\frac{8}{3}t v u_{txx}\gamma^2+\frac{16}{3}z\beta v_x u_{xy}\gamma-\frac{16}{3}z\beta u_y v_{xx}\gamma+\frac{32}{3}z\beta v u_{xxy}\gamma\\
&+16z\alpha v u_{xxx}\gamma-64z v u_{xxx}u_t\gamma-192z v u_{xx}u_{tx}\gamma-128z v u_x u_{txx}\gamma+16z v u_{ttt}\gamma\\
&-16z v u_{txxxx}\gamma.
\end{aligned}
$$

5. Application of solitary waves and conservation laws in physics of the ocean

We are aware that various solitary waves, as well as hyperbolic functions, are obtained in this study, hence the need to outline their significance. In physics as well as mathematics, a soliton usually called solitary wave depicts a self-reinforcing packet of waves propagating at a constant velocity with its shape sustained. Solitons are occasioned through the neutralization of dispersive alongside nonlinear effects in the involved medium. The envelope of solitary waves is identified with unique global peaks and decays which is isolated from the peak. Solitary waves emerge in diverse contexts which consist of the optical fibres' light intensity as well as the water surface elevation. A soliton which is also a nonlinear solitary wave possesses the additional feature that after colliding with another soliton wave still keeps back its permanent structure. For instance, two solitons generated in opposite directions pass through each other effectively without breaking [55]. The impacts of the extreme internal solitary waves on deep-ocean currents as well as thermal structures is highly notable [56] (see Figs. 13–15).

Moreover, solitary waves emerge in many structures. Perhaps the most recognizable of these forms is the tsunami, which arises consequently to the disruption on the ocean floor which can as well travel, inexorably for hundreds of miles at high speeds [57]. Physicists have demonstrated the first acoustic solitary waves existing in the airwaves that can move through long distances without altering their shape. Solitary waves in air, which according to them could result in the undistorted transmission of heat as well as other energy forms and the extermination of disturbing shock waves from air compressors and train tunnels [58].

Mathematical functions such as trigonometric, as well as hyperbolic, arise in physical sciences. An example of this is the case of hyperbolic cosine functions that possess the shape of a catenary which by extension is applied in delineating satellite rings' formed around planets. Furthermore, they are also applicable in the theory of special relativity. Besides, in quantum physics, tan-hyperbolic functions emerge in calculating the rapidity of special relativity and magnetic moment. In the laminar jet's profile, secant hyperbolic functions appear. Moreover, in the Langevin function for polarization of magnets, cotangent hyperbolic functions come to fore [59]. The Langevin and Brillouin functions are a pair of special functions that surfaces when investigating an idealized paramagnetic material that exists in statistical mechanics.

Conservation laws [2,53,60,61] on the other hand, are a subject of pertinent engrossment in physics, inclusive of theoretical as well as quantum mechanics. This study investigates the conserved quantities of Eq. (7) where the results reveal that the model

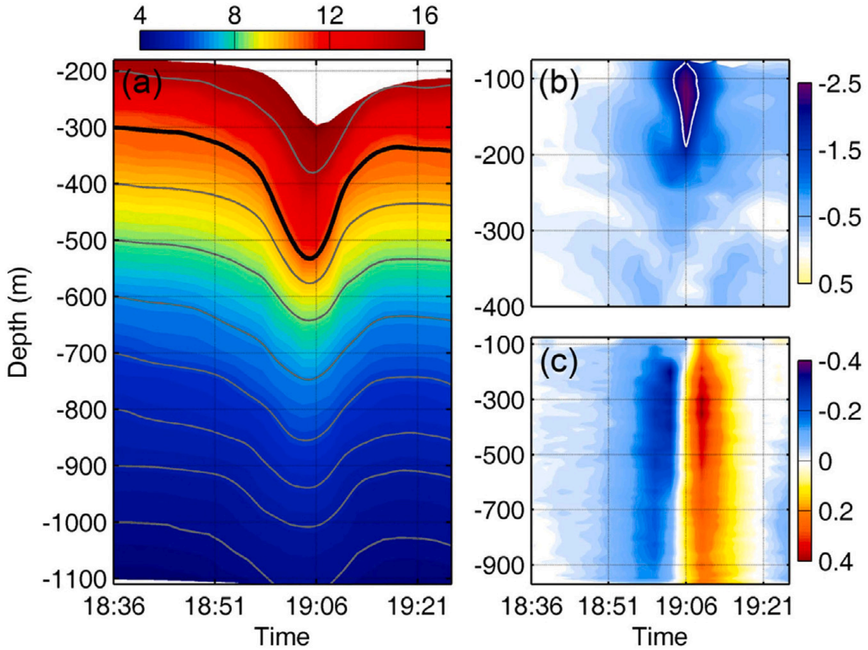

Fig. 13. The extreme internal solitary wave event captured at M10 on 4 December 2013. (a) Shadings indicate the temperatures (in degree Celsius) measured by the thermistor chains, and the lines represent the isothermal displacements at every 100 m between 200 and 1000 m. (b) Shadings indicate the zonal velocity anomalies (m/s) measured by the upward-looking Acoustic Doppler Current Profiler, and the white line shows the shape of velocity anomalies with magnitude exceeding 2 m/s. (c) Shadings indicate vertical velocity (m/s) measured by the upward and downward-looking Acoustic Doppler Current Profilers [56].

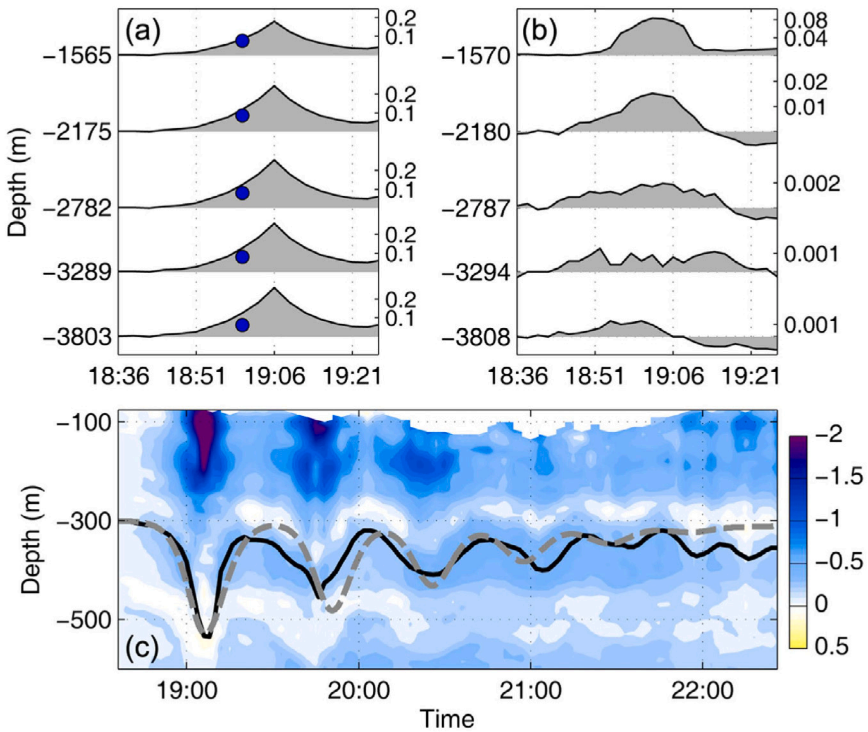

Fig. 14. Measurements of the extreme internal solitary wave in the deep ocean and its trailing waves. (a) Zonal velocity anomalies (m/s) during the passage of the extreme internal solitary wave at depths of 1565 m, 2175 m, 2782 m, 3289 m, and 3803 m, respectively. The blue dots denote the zonal velocity anomalies recorded by the recording current meters 6 min prior to the wave trough arrival, and the shaded areas are the zonal velocity anomalies estimated from the mode analysis method. (b) Temperature fluctuations (in degree Celsius) measured by the conductivity-temperature-depths at depths of 1570 m, 2180 m, 2787 m, 3294 m, and 3808 m, respectively. (c) Measurements of the trailing waves following the extreme internal solitary wave. Shadings indicate the zonal velocity anomalies (m/s) in the upper 600 m, and the bold line denotes the isotherm displacement at 300 m. The dashed line is the analytical waveform calculated by the dnoidal solution to the KdV equation [56].

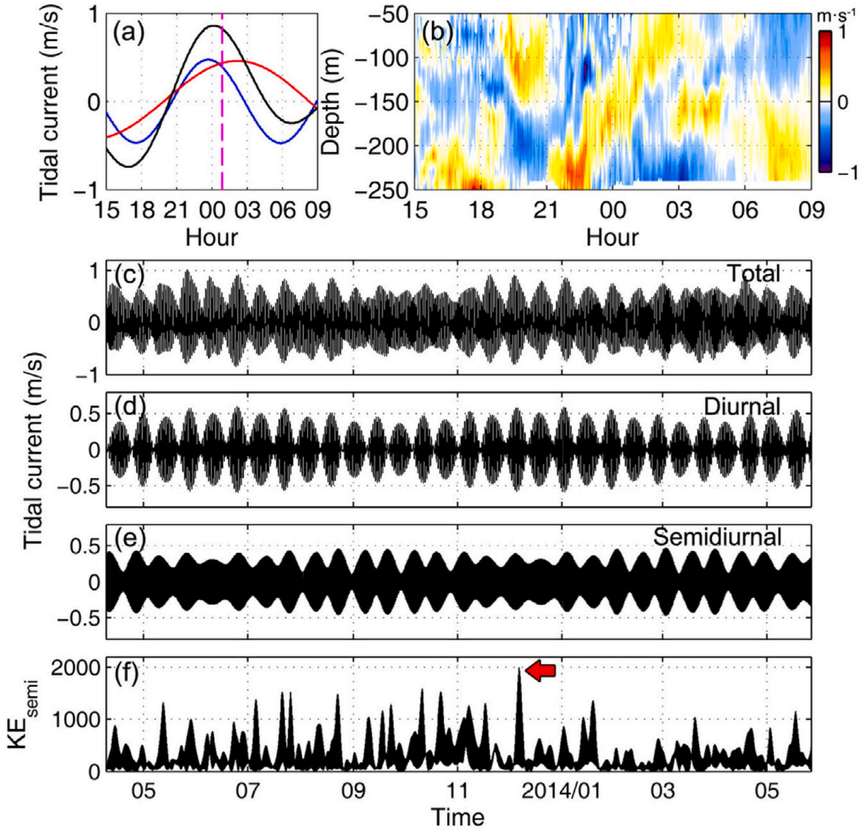

Fig. 15. Mooring measurements at internal wave near Batan Island. (a) Time series of total barotropic tidal current (black line) around 00:00 4 December 2013 at IW1 and its semidiurnal (blue line) and diurnal (red line) components. The magenta dashed line denotes the estimated generation time of the extreme internal solitary wave. (b) Zonal baroclinic current between 50 and 250 m around 00:00 4 December 2013 at internal wave. (c–e) Time series of total barotropic tidal current, diurnal tidal current and semidiurnal tidal current at internal wave from April 2013 to June 2014, respectively. (f) Depth-integrated kinetic energy density (J/m) of the semidiurnal internal tide [56]. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

possesses conservation of momentum and energy. In isolated systems physical quantities such as charge, mass, energy, and angular momentum along linear momentum are conserved. Conserved quantities can be invoked in carrying out integrability checks on differential equations. In addition, we have among others, the fact that conserved quantities play a key part in the establishment of existence as well as the uniqueness of solutions and linearization mappings. They are also utilized in the global behaviour of solutions and stability analysis.

6. Concluding remarks

In this research paper, the studies carried out on a new (3 + 1)-dimensional fifth-order nonlinear Wazwaz equation (7) with third-order dispersion terms are presented. Using the technique of Lie group analysis, an eleven-dimensional Lie algebra of the equation was achieved where we engage the involved subalgebras to reduce the equation and secure abundant group-invariant solutions. Moreover, the ordinary differential equations gained via the reductions are further solved to secure various classical solutions. In addition, many of the results obtained consist of various arbitrary, hyperbolic and rational functions with diverse quadratures of interest. We also gained copious topological soliton solutions which were graphically depicted to reveal the physical meaning of these solitons. Further, by assigning various mathematical functions to the arbitrary functions in the solutions, several solitary wave interactions of interest were demonstrated and discussed. Subsequently, we examined the conservation laws of the understudy equation with the application of Ibragimov's theorem using its formal Lagrangian. As a result, we achieved eleven conserved quantities corresponding to the basis of the eleven-dimensional Lie algebra. In consequence, we have among others, the conservation of momentum and energy. We further outlined the applications of our results in physics with various diagrams which can be of immense use for researchers in the fields of ocean engineering and physical sciences. The physical applications of the secured results in the ocean and sea [3] coupled with the significance of the various trigonometric and hyperbolic functions solutions in the real life situations reveals plainly that nonlinear partial differential equations (especially the higher dimensional equations) are very important in modelling life circumstances.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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